Analysis I: Exercises
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April 30, 2023

Chapter 3

Set Theory

3.3 Functions

Exercise 3.3.1

Let $f : X \to Y$ and $g : Y \to Z$ be functions. Show that if f and g are both injective, then so is $g \circ f$; similarly show that if f and g are both surjective, then so is $g \circ f$.

Proof. Suppose f and g are injective. We need to show that for each x and x' in X, $x \neq x'$ implies $(g \circ f)(x) \neq (g \circ f)(x')$. The assumption that f is injective tells us that for every x and x' in X, $x \neq x' \implies f(x) \neq f(x')$. Since each $f(x)$ and $f(x')$ is in Y, the assumption that g is injective also tells us that for every $f(x)$ and $f(x')$ in Y, $f(x) \neq f(x') \implies g(f(x)) \neq g(f(x'))$. Thus for each x and x' in X, $x \neq x'$ implies $(g \circ f)(x) \neq (g \circ f)(x')$ as desired.

Now suppose f and g are surjective. We need to show that for each z in Z, there exists x in X such that $(g \circ f)(x) = z$. The assumption that f is surjective tells us that for each y in Y, there exists x in X such that $f(x) = y$. Since each $f(x)$ is in Y, the assumption that g is surjective tells us that for each z in Z, there exists $f(x)$ in Y, and thus x in X, such that $g(f(x)) = z$. Then for each z in Z, there exists x in X such that $(g \circ f)(x) = z$, as desired. □

Exercise 3.3.2

When is the empty function into a given set injective? surjective? bijective?

Answer. The empty function is only bijective into the empty set.

Proof. For the empty function to be injective into a given set, for each x and x' in $\emptyset$, $x \neq x'$ implies $empty(x) \neq empty(x')$. This is vacuously true as neither x nor x' exist. Thus the empty function is injective into all sets.

For the empty function to be surjective onto a given set, for each y in some set Y there needs to be some x in $\emptyset$ such that $empty(x) = y$. But there are no x in $\emptyset$, so this is false if there are any y in Y. Thus the empty function is only surjective onto the empty set, when it is vacuously true.

For the empty function to be bijective into a given set, it must be injective into and surjective onto that set. Since the empty function is only surjective onto the empty set, it is only bijective into the empty set. □

Exercise 3.3.4

Let $f : X \to Y$ and $g : Y \to Z$ be functions.

a. Show that if $g \circ f$ is injective, then f must be injective.

Proof. For the sake of contradiction suppose $g \circ f$ is injective but f is not. Then there exists some x and x' in X that go to the same y in Y. By the definition of a function, when we apply g each y in Y goes to exactly one z in Z, so when we apply $g \circ f$, any x and x' that go to the same y go to the same z, which contradicts the assumption that $g \circ f$ is injective. Thus if $g \circ f$ is injective f must also be. □

b. Is it true that g must also be injective?

Answer. No.

Proof. A counterexample shows that g is not necessarily injective when $(g \circ f)$ is. Suppose X is the empty set, Y is $\mathbb{N}$, Z is $\mathbb{N}$, and suppose g is the function $x \mapsto 0$. We can see g is not injective from Y to Z. (To pick one counterexample, $g(1)$ and $g(2)$ are equal.) But since any function from $\emptyset$ is an empty function, which is injective, $(g \circ f)(x)$ is injective. □

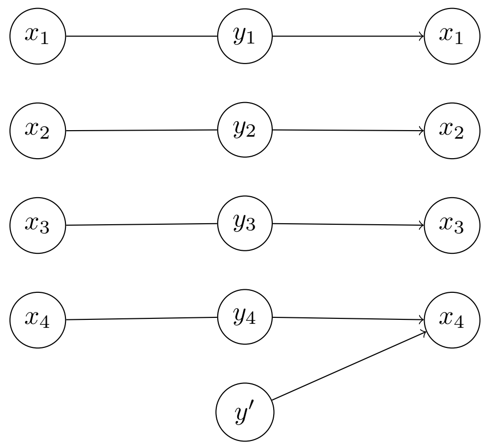
c. Show that if $g \circ f$ is surjective, then g must be surjective.

Proof. For the sake of contradiction, suppose $g \circ f$ is surjective but g is not. If g is not surjective, there exists some z in Z for which there is no y in Y such that $g(y) = z$. Then as all $f(x)$ are in Y there is also no $f(x)$ such that $g(f(x)) = z$. By the definition of a function if $f(x)$ does not exist then x cannot exist. So there is some z in Z for which there is no x in x such that $g(f(x)) = z$. But this contradicts our supposition that $g \circ f$ is surjective. Thus if $g \circ f$ is surjective then g is too. □

d. Is it true that f must also be surjective?

Answer. No.

Proof. To provide a counterexample, let $g : Y \to Z$ be a function which is surjective but for which there is some $y \in Y$ and some $y' \in Y$ such that for some z' in Z, $z' = g(y) = g(y')$, i.e. g is not injective. Also let $f : X \to Y$ be a function for which there is some x such that $f(x) = y$ for all $y \in Y$ except y', in which case f is not surjective (but would be if y' were left out). See the diagram below for one example of such a function:



We can verify that $(g \circ f)$ is still surjective, since for each z in Z there is at least one $f(x)$ such that $g(f(x)) = z$. In particular, for the z' which is equal to $g(y')$, since it is also equal to $g(y)$, and $f(x) = y$ for all $y \in Y$ except y', there is some x for which $(g \circ f)(x) = z'$.

Thus it is not the case that if f is not surjective that $(g \circ f)$ is also not, or to remove the contrapositive, it is not the case that if $(g \circ f)$ is surjective that f is also. $\square$

Exercise 3.3.7

If X is a subset of Y, let $\iota_{X \to Y} : X \to Y$ be *the inclusion map from X to Y*, defined by mapping $x \mapsto x$ for all $x \in X$, i.e., $\iota_{X \to Y}(x) := x$ for all $x \in X$. The map $\iota_{X \to X}$ is in particular called the *identity map* on X.

a. If $X \subseteq Y \subseteq Z$ then $\iota_{Y \to Z} \circ \iota_{X \to Y} = \iota_{X \to Z}$.

Proof. To prove equality, we need to show the domains and codomains of $\iota_{Y \to Z} \circ \iota_{X \to Y}$ and $\iota_{X \to Z}$ agree, and that for all x in their common domain $\iota_{Y \to Z} \circ \iota_{X \to Y}(x) = \iota_{X \to Z}(x)$.

Let $X \subseteq Y \subseteq Z$. Since $X \subseteq Y$, $\iota_{X \to Y} : X \to Y$ and since $Y \subseteq Z$, $\iota_{Y \to Z} : Y \to Z$, by the definition of composition (Definition 3.3.13) $\iota_{Y \to Z} \circ \iota_{X \to Y} : X \to Z$. Likewise, since $X \subseteq Z$, $\iota_{X \to Z} : X \to Z$, thus the codomains of the two functions agree.

Since $Y \subseteq Z$, we know $\iota_{Y \to Z}(y) := y$ for all $y \in Y$. In particular, since $X \subseteq Y$, $\iota_{Y \to Z}(\iota_{X \to Y}(x)) := \iota_{X \to Y}(x)$, which is x for all X. Likewise, since $X \subseteq Z$, $\iota_{X \to Z}(x) := x$ for all X.Thus for all $x \in X$, the output of $\iota_{Y \to Z}(\iota_{X \to Y}(x))$ is equal to the output of $\iota_{X \to Z}(x)$.

Therefore the two functions are equal. $\square$

b. Show that if $f : A \to B$ is any function, then $f = f \circ \iota_{A \to A} = \iota_{B \to B} \circ f$.

Proof. To prove equality, we need to show the domains and codomains of f, $f \circ \iota_{A \to A}$, and $\iota_{B \to B} \circ f$ all agree, and that for all x in their common domain, $f(x) = f \circ \iota_{A \to A}(x) = \iota_{B \to B} \circ f(x)$.

Let $f : A \to B$ be any function from A to B. Since $A \subseteq A$, we are given $\iota_{A \to A} : A \to A$. Then by the function composition rule (Definition 3.3.13) $f \circ \iota_{A \to A} : A \to B$ as well. Likewise, since $B \subseteq B$, we are given $\iota_{B \to B} : A \to B$. So composing $\iota_{B \to B} \circ f$ gives us a signature of $A \to B$ once again. Thus the domains and codomains all agree.

We now only need the outputs are equal. Since $f : A \to B$, for every element $a \in A$, for some element $b \in B$ we have $f(a) = b$.

Since $A \subseteq A$, for each element $a \in A$, we have $\iota_{A \to A}(a) = a$. So the function $(f \circ \iota_{A \to A}) : A \to B$ first sends each a to itself, then each a to b. Thus for every element $a \in A$, for some element $b \in B$ we again have $(f \circ \iota_{A \to A})(a) = b$.

Likewise, since $B \subseteq B$, for each element $b \in B$, we have $\iota_{B \to B}(b) = b$. So the function $\iota_{B \to B} \circ f$. first sends each a to b, then each b to itself. Thus for every element $a \in A$, for some element $b \in B$ we yet again have $\iota_{B \to B} \circ f(a) = b$.

Thus the outputs are all equal, and the two functions are equal. □

c. Show that if $f : A \to B$ is a bijective function, then $f \circ f^{-1} = \iota_{B \to B}$ and $f^{-1} \circ f = \iota_{A \to A}$.

Proof. The axiom of set equality states that

$$f : X \to Y = g : X' \to Y' \iff \Big(X = X' \quad Y = Y' \quad \forall x \in X \; f(x) = g(x) \Big)$$

Let $f : A \to B$ be a bijective function, which by Remark 3.3.27 has an inverse function denoted $f^{-1} : B \to A$. We separate the claims by conjunction elimination, and first prove $f \circ f^{-1} = \iota_{B \to B}$.

By the function composition rule (Definition 3.3.13), $f \circ f^{-1}$ is a function from $B \to B$. Since by definition $\iota_{B \to B}$ is from $B \to B$ as well, their domains and codomains agree.

Since f is bijective, for all $a \in A$, there exists exactly one $b \in B$ such that f $a \mapsto b$. We defined f^{-1} as the function $b \mapsto a$. Then $f \circ f^{-1}$ is the function $(b \mapsto a) \mapsto b$, in other words $f \circ f^{-1}(b) = b$. We also know $\iota_{B \to B}(b) = b$. Thus $f \circ f^{-1} = \iota_{B \to B}$.

The other part is very similar. $f^{-1} \circ f$ is a function from $A \to A$, which we also know is true of $\iota_{A \to A}$. Then $f^{-1} \circ f$ is the function $(a \mapsto b) \mapsto a$,in other words $f^{-1} \circ f(a) = a$. Likewise $\iota_{A \to A}(a) = a$. Thus $f^{-1} \circ f = \iota_{A \to A}$, and we have completed our proof. □

d. Show that if X and Y are disjoint sets, and $f : X \to Z$ and $g : Y \to Z$ are functions, then there is a unique function $h : X \cup Y \to Z$ such that $h \circ \iota_{X \to X \cup Y} = f$ and $h \circ \iota_{Y \to X \cup Y} = g$.

Proof. Let X and Y be disjoint sets, and $f : X \to Z$ and $g : Y \to Z$ be functions. We need to show that there is some function $h : X \cup Y \to Z$ which satisfies $h \circ \iota_{X \to X \cup Y} = f$ and $h \circ \iota_{Y \to X \cup Y} = g$, and then show that it is unique.

By pairwise union, (Axiom 3.5) there exists $X \cup Y$. By the definition of disjunction, X and Y have no common elements, so $a \in X \cup Y$ is exclusively in X or Y. Thus we can construct the following:

Let $h : X \cup Y \to Z = \begin{cases} f(a) & \text{if } a \in X \\ g(a) & \text{if } a \in Y \end{cases}$

Using this h, suppose $h \circ \iota_{X \to X \cup Y}$. Since $h : X \cup Y \to Z$ and $\iota_{X \to X \cup Y} : X \to X \cup Y$, their composition $h \circ \iota_{X \to X \cup Y}$ is a function from $X \to Z$. Thus it has the same domain and codomain as f. For all $x \in X$, $\iota_{X \to X \cup Y}(x) = x$. This means $h \circ \iota_{X \to X \cup Y}(x) = h(x)$. Since x in X, $h(x) = f(x)$. Therefore we have satisfied both conditions to prove $h \circ \iota_X \to X \cup Y = f$.

Likewise, suppose $h \circ \iota_{Y \to X \cup Y}$. Since $h : X \cup Y \to Z$ and $\iota_{Y \to X \cup Y} : Y \to X \cup Y$, their composition $h \circ \iota_{Y \to X \cup Y}$ is a function from $Y \to Z$. Thus it has the same domain and codomain as g. For all $y \in Y$, $\iota_{Y \to X \cup Y}(y) = y$. This means $h \circ \iota_{Y \to X \cup Y}(y) = h(y)$. Since y in Y, $h(y) = g(y)$. Now we have also satisfied both conditions to prove $h \circ \iota_X \to X \cup Y = g$.

We now prove uniqueness. Suppose there exists another function $h : X \cup Y \to Z$ and $h' : X \cup Y \to Z$ such that $h' \circ \iota_{X \to X \cup Y} = f$ and $h' \circ \iota_{Y \to X \cup Y} = g$ as well. We know that the domains and codomains of h and h' are the same.

Since $h' \circ \iota_{X \to X \cup Y} = f$, and $\iota_{X \to X \cup Y}$ is an inclusion map, $h' = f$ for all $x \in X$. Likewise since $h' \circ \iota_{Y \to X \cup Y} = g$, and $\iota_{Y \to X \cup Y}$ is an inclusion map, $h' = g$ for all $y \in Y$. This means h' has the same value as h for $a \in X \cup Y$, so they are identical and h is unique. □

e. Show that the hypothesis that X and Y are disjoint can be dropped in (d) if one adds the additional hypothesis that $f(x) = g(x)$ for all $x \in X \cap Y$.

Proof. Suppose $f(x) = g(x)$ for all $x \in X \cap Y$, and X and Y are not disjoint. If the relation

$$h : X \cup Y \to Z = \begin{cases} f(x) & \text{if } x \in X \\ g(x) & \text{if } x \in Y \end{cases}$$

is a function, it satisfies the conditions in Exercise 3.3.7.d. A relation h is a function if and only if each x only has one corresponding $h(x)$. We previously required that x be exclusively in X or Y, which meant that $f(x)$ and $g(x)$ were never defined for the same x. We check if it is still a function under our current assumption. If x in $X \cup Y$, then one of the three following hold:

(a) $x \in X; x \notin Y$

(b) $x \in Y; x \notin X$

(c) $x \in X \cap Y$

If cases **a** or **b**, then h is a function because either $f(x)$ is defined or $g(x)$ is, but not both, as before. If case **c**, since we have assumed $f(x) = g(x)$ for all $x \in X \cap Y$, we know $h(x) = f(x) = g(x)$ and therefore h has only one value. Thus in all cases h is a function. □

3.4 Images and inverse images

Exercise 3.4.1

Let $f\colon X \to Y$ be a bijective function, and let $f^{-1}\colon Y \to X$ be its inverse. Let V be any subset of Y. Prove that the forward image of V under f^{-1} is the same set as the inverse image of V under f; thus the fact that both sets are denoted by $f^{-1}(V)$ will not lead to any inconsistency.

Proof. Suppose $f\colon X \to Y$ is a bijective function, and $f^{-1}\colon Y \to X$ is its inverse, where V is any subset of Y. Let $f^{-1}(V)$ denote the inverse image of V, and let $(f^{-1})(V)$ denote the forward image of V under f^{-1}. We define

$$f^{-1}(V) = \{\, x \in X \mid f(x) \in V \,\}$$
$$(f^{-1})(V) = \{\, f^{-1}(y) \mid y \in V \,\}$$

1. First we show $f^{-1}(V) \subseteq (f^{-1})(V)$.
2. Let $z \in f^{-1}(V)$.
3. Then $z \in X$ and $f(z) \in V$.
4. Since f is bijective, for all y in $V \subseteq Y$, $y = f(x) = f(f^{-1}(y))$.
5. Thus $f(z) \in V \implies y \in V$.
6. Since f is bijective, for all x in X, $x = f^{-1}(y) = f^{-1}(f(x))$.
7. Thus $z \in X \implies z = f^{-1}(y)$.

□

Exercise 3.4.2

Let $f\colon X \to Y$ be a function from one set X to another set Y, let S be a subset of X, and let U be a subset of Y.

a. What, in general, can one say about $f^{-1}(f(S))$ and S?

Answer. S is a subset of $f^{-1}(f(S))$, but S may not be equal to $f^{-1}(f(S))$.

Proof. (informal) Let x be an element of X. We have $f(S) = \{\, f(x) \mid x \in S \,\}$, and therefore $f^{-1}(f(S)) = \{\, x \in X \mid f(x) \in f(S) \,\}$.

Suppose $x \in S$, then $x \in X$ and $f(x) \in f(s)$, thus $x \in f^{-1}(f(S))$ for all $x \in S$, so S is a subset of $f^{-1}(f(S))$. Now instead suppose $x \notin S$. Since we have not stated that f is injective, it is still possible that $f(x) \in f(S)$. Once again $x \in X$ and $f(x) \in f(s)$, thus for some x not in S, x may still be in $x \in f^{-1}(f(S))$. Thus $f^{-1}(f(S))$ may contain more members of X than S does, so they may not be equal. □ □

b. What about $f(f^{-1}(U))$ and U?

Answer. $f(f^{-1}(U))$ is a subset of U, but the two sets may not be equal.

Proof. (informal) Let x be an element of X. We have $f^{-1}(U) = \{ x \in X \mid f(x) \in U \}$. Then $f(f^{-1}(U)) = \{ f(x) \mid x \in f^{-1}(U) \}$. Since f is not stated to be surjective, there may be some y in U for which $y \neq f(x)$ for all x. So when we take the forward image of $f^{-1}(U)$, every element of $f^{-1}(U)$ is in U, but there may be some y in U that are not in $f^{-1}(U)$. □

c. What about $f^{-1}(f(f^{-1}(U)))$ and $f^{-1}(U)$?

Answer.

Proof. (informal) As before we have $f^{-1}(U) = \{ x \in X \mid f(x) \in U \}$, and $f(f^{-1}(U)) = \{ f(x) \mid x \in f^{-1}(U) \}$.

$$\begin{aligned} f^{-1}(f(f^{-1}(U))) &= \{ x \in X \mid f(f^{-1}(U)) \in U \} \\ &= x \in X \text{ and } f(f^{-1}(U)) \in U \\ &= x \in X \text{ and } \{ f(x) \mid x \in f^{-1}(U) \} \in U \\ &= x \in X \text{ and } (\exists x \text{ such that } y = f(x) \text{ and } x \in f^{-1}(U)) \in U \end{aligned}$$

$$\begin{aligned} f^{-1}(f(f^{-1}(U))) &= \{ x \in X \mid f(f^{-1}(U)) \in U \} \\ &= \{ x \in X \mid \{ f(x) \mid x \in f^{-1}(U) \} \in U \} \\ &= \{ x \in X \mid \{ f(x) \mid x \in \{ x \in X \mid f(x) \in U \} \} \in U \} \\ &= (x \in X) \text{ and } (f(x) \text{ is true and } (x \in (x \in X \text{ and } f(x) \in U)) \in U). \\ &= x \in X \text{ and } f(x) \in U (\textit{incomplete}) \end{aligned}$$

(good lord...)

□

Exercise 3.4.3

Let A, B be two subsets of a set X, and let $f : X \to Y$ be a function. Show that

a. $f(A \cap B) \subseteq f(A) \cap f(B)$**,**

Proof. We prove this statement by showing every element of $f(A \cap B)$ is an element of $f(A) \cap f(B)$.

1. Let y be an arbitrary element of $f(A \cap B)$.
2. $A \subseteq X$ and $B \subseteq X \implies A \cap B \subseteq X$.
3. By definition the image of $A \cap B$ under f is $\{ f(x) \mid x \in A \cap B \}$.
4. By the axiom of replacement (3.7) $y = f(x)$ for some $x \in A \cap B$.
5. $x \in A \cap B \implies x \in A$
6. $y = f(x)$ for some $x \in A$

7. $x \in A \cap B \implies x \in B$
8. $y = f(x)$ for some $x \in B$
9. $y = f(x)$ for some $x \in A$ and $y = f(x)$ for some $x \in B$
10. $y \in \{ f(x) \mid x \in A \}$ and y $\in \{ f(x) \mid x \in B \}$
11. $y \in f(A) \cap f(B)$, as desired.

$\square$

b. $f(A) \setminus f(B) \subseteq f(A \setminus B)$**,**

Proof. We prove this statement by showing every element of $f(A) \setminus f(B)$ is an element of $f(A \setminus B)$.

1. Let $y \in f(A) \setminus f(B)$ be arbitrary. Conditional introduction
2. $y \in f(A)$ and $y \notin f(B)$.
3. $\exists x \in A \; y = f(x)$
4. Suppose x such that $x \in A$ and $y = f(x)$
   4.1. $x \in A$
   4.2. $y = f(x)$
   4.3. $\forall z \in B \; y \neq f(z)$
   4.4. $\forall z \; z \in B \implies y \neq f(z)$
   4.5. $\forall z \; y = f(z) \implies z \notin B$
   4.6. $y = f(x) \implies x \notin B$
   4.7. $x \notin B$
   4.8. $x \in A$, $x \notin B$, and $y = f(x)$.
   4.9. $y = f(x)$ and $x \in A \setminus B$.
   4.10. $y \in \{ y \mid y = f(x) \text{ for } x \in A \setminus B \}$.
5. $y \in f(A \setminus B)$ Existential elimination
6. $y \in f(A) \setminus f(B) \implies y \in f(A \setminus B)$ Conditional elimination

Thus $f(A) \setminus f(B) \subseteq f(A \setminus B)$. $\square$

c. $f(A \cup B) = f(A) \cup f(B)$**.**

Proof. We prove this statement by showing every element of $f(A \cup B)$ is an element of $f(A) \cup f(B)$ and vice versa. First we do the forward direction:

1. Let $y \in f(A \cup B)$ be arbitrary.
2. $A \in X$
3. $B \in X$
4. $A \cup B \in X$
5. $y \in \{ f(x) \mid x \in A \cup B \}$
6. $\exists x$ such that $x \in A \cup B$ and $y = f(x)$
7. Suppose x such that $x \in A \cup B$ and $y = f(x)$
   7.1. $y = f(x)$
   7.2. $x \in A \cup B$
   7.3. $x \in A$ or $x \in B$
   7.4. $(x \in A$ and $y = f(x))$ or $(x \in B$ and $y = f(x))$

7.4.1. test

7.5. $y \in \{ f(x) \mid x \in A \}$ or $y \in \{ y = f(x) \mid x \in B \}$

7.6. $y \in f(A)$ or $y \in f(B)$

7.7. $y \in f(A) \cup f(B)$

8. $y \in f(A \cup B) \implies y \in f(A) \cup f(B)$
9. $f(A \cup B) \subseteq f(A) \cup f(B)$

Now in the backwards direction.

1. Let $y \in f(A) \cup f(B)$ be arbitrary.
2. $y \in f(A)$ or $y \in f(B)$
3. Case $y \in f(A)$

3.1. $y \in \{ f(x) \mid x \in A \}$

3.2. $\exists x$ such that $(x \in A$ and $y = f(x))$

3.3. Suppose x such that $(x \in A$ and $y = f(x))$

3.3.1. $x \in A$ and $y = f(x)$

4. Case $y \in f(B)$

4.1. $y \in \{ y = f(x) \mid x \in B \}$

4.2. $\exists x$ such that $(x \in B$ and $y = f(x))$

4.3. Suppose x such that $(x \in B$ and $y = f(x))$

4.3.1. $x \in B$ and $y = f(x)$

5. $(x \in B$ and $y = f(x))$ or $(x \in A$ and $y = f(x))$
6. $y = f(x)$ and $(x \in A$ or $x \in B)$ Distributivity
7. $y = f(x)$ and $(x \in A \cup B)$
8. $y \in \{ fx \mid x \in A \cup B \}$
9. $y \in f(A) \cup f(B) \implies y \in \{ f(x) \mid x \in A \cup B \}$
10. $f(A) \cup f(B) \subseteq f(A \cup B)$

Thus we have $f(A \cup B) = f(A) \cup f(B)$. □

For the first two statements, is it true that the $\subseteq$ relation can be improved to $=$?

Answer.

Proof. I want to first try to prove $f(A \cap B) = f(A) \cap f(B)$. Since I already have $f(A \cap B) \subseteq f(A) \cap f(B)$, I just need $f(A) \cap f(B) \subseteq f(A \cap B)$.

1. Suppose $y \in f(A) \cap f(B)$
2. $y \in f(A)$ and $y \in f(B)$
3. $y \in \{ f(x) \mid x \in A \}$
4. $\exists x$ st. $y = f(x)$ and $x \in A$
5. Suppose x st. $y = f(x)$ and $x \in A$
6. $y \in \{ f(x) \mid x \in B \}$
7. $\exists x$ st. $y = f(x)$ and $x \in A$

Next I'm going to try to prove $f(A) \setminus f(B) = f(A \setminus B)$. I already have $f(A) \setminus f(B) \subseteq f(A \setminus B)$ and I just need $f(A \setminus B) \subseteq f(A) \setminus f(B)$.

1. Suppose $y \in f(A \setminus B)$.
2. $\exists x$ such that $y = f(x)$ and $x \in A \setminus B$.
3. Suppose x such that $y = f(x)$ and $x \in A \setminus B$.
   - 3.1. $y = f(x)$
   - 3.2. $x \in A \setminus B$
   - 3.3. $x \in A$ and $x \notin B$
   - 3.4. $y = f(x)$ and $x \in A$
   - 3.5. $y \in \{ f(x) \mid x \in A \}$
   - 3.6. $y \in f(A)$
   - 3.7. $y = f(x)$ and $x \notin B$
   - 3.8. $\cancel{y \in \{ f(x) \mid x \notin B \}}$ (not useful!)

not sure where to go from here

□

Exercise 3.4.5

Let $f\colon X \to Y$ be a function from one set X to another set Y.

a. Show that $f(f^{-1}(S)) = S$ for every $S \subseteq Y$ if and only if f is surjective.

Proof. □

b. Show that $f^{-1}(f(S)) = S$ for every $S \subseteq X$ if and only if f is injective.

Proof. □

Exercise 3.4.9

Show that if β and β' are two elements of a set I, and to each $\alpha \in I$ we assign a set A_α, then

$$\{x \in A_\beta : x \in A_\alpha \textbf{ for all } \alpha \in I\} = \{x \in A_{\beta'} : x \in A_\alpha \textbf{ for all } \alpha \in I\},$$

and so the definition of $\bigcap_{\alpha \in I} A_\alpha$ defined in (3.3) does not depend on β.

Proof. □

Also explain why (3.4) is true.

Proof. □

Exercise 3.4.10

Suppose that I and J are two sets, and for all $\alpha \in I \cup J$ let A_α be a set. Show that

$$\bigcup_{\alpha\in I} A_\alpha \cup \bigcup_{\alpha\in J} A_\alpha = \bigcup_{\alpha\in I\cup J} A_\alpha.$$

Proof. We need to show that every element of $\bigcup_{\alpha\in I} A_\alpha \cup \bigcup_{\alpha\in J} A_\alpha$ is also in $\bigcup_{\alpha\in I\cup J} A_\alpha$ and vice versa. We begin in the forward direction.

1.

Now in reverse:

1.

□

If I and J are non-empty, show that

$$\bigcap_{\alpha\in I} A_\alpha \cap \bigcap_{\alpha\in J} A_\alpha = \bigcap_{\alpha\in I\cup J} A_\alpha.$$

Proof.

□

3.5 Cartesian Products

Exercise 3.5.2

Suppose we define an ordered n-tuple to be a surjective function $x : i \in \mathbb{N} : 1 \leq i \leq n \to X$ whose codomain is some arbitrary set X (so different ordered n-tuples are allowed to have different ranges); we then write x_i for $x(i)$ and also write x as $(x_i)1 \leq i \leq n$. Using this definition, verify that we have $(x_i)1 \leq i \leq n = (y_i)1 \leq i \leq n$ if and only if $x_i = y_i$ for all $1 \leq i \leq n$.

Proof. □

Also, show that if $(X_i)1 \leq i \leq n$ are an ordered n-tuple of sets, then the Cartesian product, as defined in Definition 3.5.6, is indeed a set. (Hint: use Exercise 3.4.7 and the axiom of specification.)

Proof. □

Exercise 3.5.4

Let A, B, C be sets. Show that:

a. $A \times (B \cup C) = (A \times B) \cup (A \times C)$**,**

Proof. □

b. $A \times (B \cup C) = (A \times B) \cup (A \times C)$**,** $A \times (B \cap C) = (A \times B) \cap (A \times C)$**,**

Proof. □

c. and $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$**. (One can of course prove similar identities in which the roles of the left and right factors of the Cartesian product are reversed.)**

Proof. □

Exercise 3.5.7

Let X and Y be sets, and let $\pi_{X\times Y\to X} : X \times Y \to X$ and $\pi_{X\times Y\to Y} : X \times Y \to Y$ be the maps $\pi_{X\times Y\to X}(x, y) := x$ and $\pi_{X\times Y\to Y}(x, y) := y$; these maps are known as the coordinate functions on $X \times Y$. Show that for any functions $f : Z \to X$ and $g : Z \to Y$, there exists a unique function $h : Z \to X \times Y$ such that $\pi_{X\times Y\to X} \circ h = f$ and $\pi_{X\times Y\to Y} \circ h = g$. (Compare this to the last part of Exercise 3.3.8, and to Exercise 3.1.7.) This function h is known as the pairing of f and g and is denoted $h = (f, g)$.

Proof. □

Exercise 3.5.8

Let $X_1, \ldots, X_n$ be sets. Show that the Cartesian product $\prod_{i=1}^{n} X_i$ is empty if and only if at least one of the X_i is empty.

Proof. □

Exercise 3.5.9

Suppose that I and J are two sets, and for all $\alpha \in I$ let A_α be a set, and for all $\beta \in J$ let B_β be a set. Show that

$$\left(\bigcup_{\alpha \in I} A_\alpha\right) \cap \left(\bigcup_{\beta \in J} B_\beta\right) = \bigcup_{(\alpha,\beta) \in I \times J} (A_\alpha \cap B_\beta)\,.$$

Proof. □

What happens if one interchanges all the union and intersection symbols here?

Answer.

Proof. □

Exercise 3.5.10

If $f : X \to Y$ is a function, define the graph of f to be the subset of $X \times Y$ defined by $\{(x, f(x)) : x \in X\}$.

a. **Show that two functions $f : X \to Y$, $\tilde{f} : X \to Y$ are equal if and only if they have the same graph.**

   Proof. □

b. **Conversely, if G is any subset of $X \times Y$ with the property that for each $x \in X$, the set $\{y \in Y : (x, y) \in G\}$ has exactly one element (or in other words, G obeys the vertical line test), show that there is exactly one function $f : X \to Y$ whose graph is equal to G.**

   Proof. □

c. **Suppose we define a function f to be an ordered triple $f = (X, Y, G)$, where X, Y are sets, and G is a subset of $X \times Y$ that obeys the vertical line test. We then define the domain of such a triple to be X, the codomain to be Y and for every $x \in X$, we define $f(x)$ to be the unique $y \in Y$ such that $(x, y) \in G$. Show that this definition is compatible with Definition 3.3.1 in the sense that every choice of domain X, codomain Y, and property $P(x, y)$ obeying the vertical line test produces a function as defined here that obeys all the properties required of it in that definition, and is also similarly compatible with Definition 3.3.8.**

   Proof. □

3.6 Cartesian Products