Analysis I: Exercises
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3.4 Images and inverse images

Exercise 3.4.1

Let $f: X \to Y$ be a bijective function, and let $f^{-1}: Y \to X$ be its inverse. Let V be any subset of Y. Prove that the forward image of V under f^{-1} is the same set as the inverse image of V under f; thus the fact that both sets are denoted by $f^{-1}(V)$ will not lead to any inconsistency.

Proof. Suppose $f: X \to Y$ is a bijective function, and $f^{-1}: Y \to X$ is its inverse, where V is any subset of Y. Let $f^{-1}(V)$ denote the inverse image of V, and let $(f^{-1})(V)$ denote the forward image of V under f^{-1}. We define

$$f^{-1}(V) = \{\, x \in X \mid f(x) \in V \,\}$$
$$(f^{-1})(V) = \{\, f^{-1}(y) \mid y \in V \,\}$$

1. First we show $f^{-1}(V) \subseteq (f^{-1})(V)$.
2. Let $z \in f^{-1}(V)$.
3. Then $z \in X$ and $f(z) \in V$.
4. Since f is bijective, for all y in $V \subseteq Y$, $y = f(x) = f(f^{-1}(y))$.
5. Thus $f(z) \in V \implies y \in V$.
6. Since f is bijective, for all x in X, $x = f^{-1}(y) = f^{-1}(f(x))$.
7. Thus $z \in X \implies z = f^{-1}(y)$.

□

Exercise 3.4.2

Let $f: X \to Y$ be a function from one set X to another set Y, let S be a subset of X, and let U be a subset of Y.

a. What, in general, can one say about $f^{-1}(f(S))$ and S?

Answer: S is a subset of $f^{-1}(f(S))$, but S may not be equal to $f^{-1}(f(S))$.

Proof. (informal) Let x be an element of X. We have $f(S) = \{\, f(x) \mid x \in S \,\}$, and therefore $f^{-1}(f(S)) = \{\, x \in X \mid f(x) \in f(S) \,\}$.

Suppose $x \in S$, then $x \in X$ and $f(x) \in f(s)$, thus $x \in f^{-1}(f(S))$ for all $x \in S$, so S is a subset of $f^{-1}(f(S))$. Now instead suppose $x \notin S$. Since we have not stated that f is injective, it is still possible that $f(x) \in f(S)$. Once again $x \in X$ and $f(x) \in f(s)$, thus for some x not in S, x may still be in $x \in f^{-1}(f(S))$. Thus $f^{-1}(f(S))$ may contain more members of X than S does, so they may not be equal. □ □

b. What about $f(f^{-1}(U))$ and U?

Answer: $f(f^{-1}(U))$ is a subset of U, but the two sets may not be equal.

Proof. (informal) Let x be an element of X. We have $f^{-1}(U) = \{ x \in X \mid f(x) \in U \}$. Then $f(f^{-1}(U)) = \{ f(x) \mid x \in f^{-1}(U) \}$. Since f is not stated to be surjective, there may be some y in U for which $y \neq f(x)$ for all x. So when we take the forward image of $f^{-1}(U)$, every element of $f^{-1}(U)$ is in U, but there may be some y in U that are not in $f^{-1}(U)$. □

c. What about $f^{-1}(f(f^{-1}(U)))$ and $f^{-1}(U)$?

Answer:

Proof. (informal) As before we have $f^{-1}(U) = \{ x \in X \mid f(x) \in U \}$, and $f(f^{-1}(U)) = \{ f(x) \mid x \in f^{-1}(U) \}$. This means

$$\begin{aligned}
f^{-1}(f(f^{-1}(U))) &= \{ x \in X \mid f(f^{-1}(U)) \in U \} \\
&= \{ x \in X \mid \{ f(x) \mid x \in f^{-1}(U) \} \in U \} \\
&= \{ x \in X \mid \{ f(x) \mid x \in \{ x \in X \mid f(x) \in U \} \} \in U \} \\
&= (x \in X) \text{ and } (f(x) \text{ is true and } (x \in (x \in X \text{ and } f(x) \in U)) \in U). \\
&= x \in X \text{ and } f(x) \in U (\textit{incomplete})
\end{aligned}$$

(good lord...)

□

Exercise 3.4.3

Let A, B be two subsets of a set X, and let $f : X \to Y$ be a function. Show that

a. $f(A \cap B) \subseteq f(A) \cap f(B)$,

Proof. We prove this statement by showing every element of $f(A \cap B)$ is an element of $f(A) \cap f(B)$.

1. Let y be an arbitrary element of $f(A \cap B)$.
2. $A \subseteq X$ and $B \subseteq X \implies A \cap B \subseteq X$.
3. By definition the image of $A \cap B$ under f is $\{ f(x) \mid x \in A \cap B \}$.
4. By the axiom of replacement (3.7) $y = f(x)$ for some $x \in A \cap B$.
5. $x \in A \cap B \implies x \in A$
6. $y = f(x)$ for some $x \in A$
7. $x \in A \cap B \implies x \in B$

8. $y = f(x)$ for some $x \in B$
9. $y = f(x)$ for some $x \in A$ and $y = f(x)$ for some $x \in B$
10. $y \in \{\, f(x) \mid x \in A \,\}$ and y $\in \{\, f(x) \mid x \in B \,\}$
11. $y \in f(A) \cap f(B)$, as desired.

$\square$

b. $f(A) \setminus f(B) \subseteq f(A \setminus B)$**,**

Proof. We prove this statement by showing every element of $f(A) \setminus f(B)$ is an element of $f(A \setminus B)$.

1. Let $y \in f(A) \setminus f(B)$ be arbitrary. Conditional introduction
2. $y \in f(A)$ and $y \notin f(B)$.
3. $\exists x \in A\ y = f(x)$
4. Suppose x such that $x \in A$ and $y = f(x)$
   4.1. $x \in A$
   4.2. $y = f(x)$
   4.3. $\forall z \in B\ y \neq f(z)$
   4.4. $\forall z\ z \in B \implies y \neq f(z)$
   4.5. $\forall z\ y = f(z) \implies z \notin B$
   4.6. $y = f(x) \implies x \notin B$
   4.7. $x \notin B$
   4.8. $x \in A$, $x \notin B$, and $y = f(x)$.
   4.9. $y = f(x)$ and $x \in A \setminus B$.
   4.10. $y \in \{\, y \mid y = f(x) \text{ for } x \in A \setminus B \,\}$.
5. $y \in f(A \setminus B)$ Existential elimination
6. $y \in f(A) \setminus f(B) \implies y \in f(A \setminus B)$ Conditional elimination

Thus $f(A) \setminus f(B) \subseteq f(A \setminus B)$. $\square$

c. $f(A \cup B) = f(A) \cup f(B)$**.**

Proof. We prove this statement by showing every element of $f(A \cup B)$ is an element of $f(A) \cup f(B)$ and vice versa. First we do the forward direction:

1. Let $y \in f(A \cup B)$ be arbitrary.
2. $A \in X$
3. $B \in X$
4. $A \cup B \in X$
5. $y \in \{\, f(x) \mid x \in A \cup B \,\}$
6. $\exists x$ such that $x \in A \cup B$ and $y = f(x)$
7. Suppose x such that $x \in A \cup B$ and $y = f(x)$
   7.1. $y = f(x)$
   7.2. $x \in A \cup B$
   7.3. $x \in A$ or $x \in B$
   7.4. $(x \in A \text{ and } y = f(x))$ or $(x \in B \text{ and } y = f(x))$
      7.4.1. test

7.5. $y \in \{ f(x) \mid x \in A \}$ or $y \in \{ y = f(x) \mid x \in B \}$

7.6. $y \in f(A)$ or $y \in f(B)$

7.7. $y \in f(A) \cup f(B)$

8. $y \in f(A \cup B) \implies y \in f(A) \cup f(B)$
9. $f(A \cup B) \subseteq f(A) \cup f(B)$

Now in the backwards direction.

1. Let $y \in f(A) \cup f(B)$ be arbitrary.
2. $y \in f(A)$ or $y \in f(B)$
3. Case $y \in f(A)$

3.1. $y \in \{ f(x) \mid x \in A \}$

3.2. $\exists x$ such that ($x \in A$ and $y = f(x)$)

3.3. Suppose x such that ($x \in A$ and $y = f(x)$)

3.3.1. $x \in A$ and $y = f(x)$

4. Case $y \in f(B)$

4.1. $y \in \{ y = f(x) \mid x \in B \}$

4.2. $\exists x$ such that ($x \in B$ and $y = f(x)$)

4.3. Suppose x such that ($x \in B$ and $y = f(x)$)

4.3.1. $x \in B$ and $y = f(x)$

5. ($x \in B$ and $y = f(x)$) or ($x \in A$ and $y = f(x)$)
6. $y = f(x)$ and ($x \in A$ or $x \in B$) Distributivity
7. $y = f(x)$ and ($x \in A \cup B$)
8. $y \in \{ fx \mid x \in A \cup B \}$
9. $y \in f(A) \cup f(B) \implies y \in \{ f(x) \mid x \in A \cup B \}$
10. $f(A) \cup f(B) \subseteq f(A \cup B)$

Thus we have $f(A \cup B) = f(A) \cup f(B)$. □

For the first two statements, is it true that the $\subseteq$ relation can be improved to $=$?

Answer:

Proof. □

Exercise 3.4.5

Let $f \colon X \to Y$ be a function from one set X to another set Y.

a. Show that $f(f^{-1}(S)) = S$ for every $S \subseteq Y$ if and only if f is surjective.

Proof. □

b. Show that $f^{-1}(f(S)) = S$ for every $S \subseteq X$ if and only if f is injective.

Proof. □

Exercise 3.4.9

Show that if β and β' are two elements of a set I, and to each $\alpha \in I$ we assign a set A_α, then

$$\{x \in A_\beta : x \in A_\alpha \textbf{ for all } \alpha \in I\} = \{x \in A_{\beta'} : x \in A_\alpha \textbf{ for all } \alpha \in I\},$$

and so the definition of $\bigcap_{\alpha \in I} A_\alpha$ defined in (3.3) does not depend on β.

Proof. □

Also explain why (3.4) is true.

Proof. □

Exercise 3.4.10

Suppose that I and J are two sets, and for all $\alpha \in I \cup J$ let A_α be a set. Show that

$$\bigcup_{\alpha \in I} A_\alpha \cup \bigcup_{\alpha \in J} A_\alpha = \bigcup_{\alpha \in I \cup J} A_\alpha.$$

Proof. We need to show that every element of $\bigcup_{\alpha \in I} A_\alpha \cup \bigcup_{\alpha \in J} A_\alpha$ is also in $\bigcup_{\alpha \in I \cup J} A_\alpha$ and vice versa. We begin in the forward direction.

1.

Now in reverse:

1.

□

If I and J are non-empty, show that

$$\bigcap_{\alpha \in I} A_\alpha \cap \bigcap_{\alpha \in J} A_\alpha = \bigcap_{\alpha \in I \cup J} A_\alpha.$$

Proof.

□