Analysis I: Exercises
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4.3 Absolute value and exponentiation

Exercise 4.3.3

Let x, y be rational numbers and let n, m be natural numbers.

b. Suppose $n > 0$. Then we have $x^n = 0$ if and only if $x = 0$.

Proof. We prove by induction.

Suppose $n = 1$ as a base case. Since $x = x^1$, $x = 0 \iff x^n = 0$.

Now suppose $x = 0 \iff x^n = 0$. We have $x^{n+1} = x^n * x$. Now $x = 0$ or $x \neq 0$.

If $x = 0$, $x^n = 0$ by inductive assumption. Then $x^{n+1} = 0 * 0 = 0$, so $x = 0 \implies x^{n+1} = 0$.

If instead $x \neq 0$, $x^n \neq 0$ we have $x^{n+1} \neq 0$, since the product of two nonzero rationals is a nonzero rational. (Extension of 4.1.8 that I haven't proved)

Thus we have $x^{n+1} = 0 \iff x = 0$, closing the induction, and so for all x, for all $n > 0$, we have $x^n = 0$ if and only if $x = 0$. $\square$