Analysis I: Exercises

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Chapter 4

Integers and Rationals

4.1 The integers

Exercise 4.1.1

Verify that the definition of equality on the integers is both reflexive and symmetric.

Proof. We first prove that the definition of equality is reflexive. Let $a, b, c, d \in \mathbb{N}$. Let $(a—b) \in \mathbb{Z}$. We wish to prove $a—b = a—b$. By reflexivity for the natural numbers, $a + b = a + b$. The definition of equality for integers states $a—b = c—d \iff a + d = b + c$. Thus, $a—b = a—b$, as desired.

Next we prove the definition of equality is symmetric. Suppose $a, b, c, d \in \mathbb{N}$. We wish to prove $a—b = c—d \iff c—d = a—b$. The definition of equality for integers states $a—b = c—d \iff a+d = b+c$. We know by symmetry for the natural numbers that $a + d = b + c \iff b + c = a + d$. Thus, $a—b = c—d \iff c—d = a—b$, as desired. ☐

Exercise 4.1.3

Show that $(-1) \times a = -a$ for every integer a.

Proof. Let a in $\mathbb{Z}$. Allow a to be the integer $h - j$ for some h and j in $\mathbb{N}$. By integer negation, we can rewrite $(-1) \times a$ as $(0 - 1) \times (h - j)$. By integer multiplication, we have $(0h + 1j) - (0j + 1h)$, which by natural number multiplication is $(j - h)$. Meanwhile, by integer negation $-a$ is also $(j - h)$. Thus $(-1) \times a = -a$ by integer equality. ☐

Exercise 4.1.4

h. Let x, y, z be integers. Show that $x(y+z) = xy + xz$.

Proof. Let x be the integer $(a-b)$, y be $(c-d)$, and z be $(e-f)$.

$$\begin{aligned} x(y+z) &= (a-b)\big((c-d)+(e-f)\big) \\ &= (a-b)\big((c+e)-(d+f)\big) \\ &= (a(c+e)+b(d+f)) - (a(d+f)+b(c+e)) \\ &= (ac+ae+bd+bf) - (ad+af+bc+be); \end{aligned}$$

$$\begin{aligned} xy+xz &= (a-b)(c-d)+(a-b)(e-f) \\ &= \big((ac+bd)-(ad+bc)\big)+\big((ae+bf)-(af+be)\big) \\ &= ((ac+bd)+(ae+bf)) - ((ad+bc)+(af+be)) \\ &= (ac+ae+bd+bf) - (ad+af+bc+be) \end{aligned}$$

□

Exercise 4.1.5

Prove Proposition 4.1.8: Let a and b be integers such that $ab = 0$. Then either $a = 0$ or $b = 0$ (or both).

Proof. By the trichotomy of integers, a is equal to a positive natural number, zero, or the negation of a positive natural number. Likewise for b.

1. Suppose a is zero.
   1.1. $a = 0$ or $b = 0$
2. Suppose a is equal to a positive natural number n.
   2.1. Suppose b is zero
      2.1.1. $a = 0$ or $b = 0$
   2.2. Suppose b is equal to a positive natural number m.
      2.2.1. $n \times m = 0$
      2.2.2. $n = 0$ or $m = 0$ by Lemma 2.3.3
      2.2.3. $a = 0$ or $b = 0$
   2.3. Suppose b is the negation of a positive natural number m.
   2.4. $(n-0)(0-m) = 0$
   2.5. $(n \times 0 + 0 \times m) - (n \times m + 0 \times 0) = 0$
   2.6. $-(n \times m) = 0$
   2.7. $-(n \times m) = (0-0)$
   2.8. $(n \times m) = 0$ by Exercise 4.1.2.
   2.9. $n = 0$ or $m = 0$ by Lemma 2.3.3

2.10. $n = 0$ or $-m = 0$ by Exercise 4.1.2 again

2.11. $a = 0$ or $b = 0$

3. Suppose a is the negation of a positive natural number n.

3.1. Suppose b is zero

3.1.1. $a = 0$ or $b = 0$

3.2. Suppose b is equal to a positive natural number n.

3.2.1. By commutativity, same case as 2.3.

3.2.2. $a = 0$ or $b = 0$

3.3. Suppose b is the negation of a positive natural number m.

3.3.1. $(0 - n)(0 - m) = 0$

3.3.2. $(0 \times 0 + n \times m) - (0 \times m + 0 \times n) = 0$

3.3.3. $n \times m = 0$

3.3.4. $n = 0$ or $m = 0$

3.3.5. $(0 - n) = 0$ or $(0 - m) = 0$

3.3.6. $a = 0$ or $b = 0$

In all cases, $a = 0$ or $b = 0$.

By the trichotomy of integers a and b are either natural numbers or the negation of positive natural numbers. If a and b are equal to natural numbers, then $ab = 0 \implies a = 0$ or $b = 0$ by Lemma 2.3.3. If a is the negation of a positive natural number n and b is equal to a natural number, such that $(-n)b = 0$, then $ab = 0 \implies -n = 0$ or $b = 0$, but we said $-n \neq 0$ so $ab = 0 \implies b = 0$. Likewise if b is the negation of some positive natural number m such that $a(-m) = 0$ then $ab = 0 \implies a = 0$. If they are both negations of some positive natural numbers n and m then $nm \neq 0$ by Lemma 2.3.3, and thus $ab \neq 0$, so $ab = 0 \implies a = 0$ is vacuously true. (some mistakes around -n, need to use -n = -1(n) here)

□

Exercise 4.1.6

(Cancellation law for integers) If a, b, c are integers such that $ac = bc$ and c is non-zero, then $a = b$.

Proof. Let a, b, c be integers such that $ac = bc$ and c is non-zero.

1. $ac - bc = bc - bc$
2. $ac - bc = 0$
3. $ac + -(bc) = 0$
4. $ac + -1(bc) = 0$
5. $ac + (-1)(b)(c) = 0$
6. $(a + -1(b))c = 0$
7. $(a + -b)c = 0$
8. $(a - b)c = 0$
9. $a - b = 0$ Lemma 4.1.8, $c \neq 0$

10. $a - b + b = 0 + b$
11. $a = b$

$\square$

Exercise 4.1.8

Show that the principle of induction (Axiom 2.5) does not apply directly to the integers. More precisely, give an example of a property $P(n)$ pertaining to an integer n such that $P(0)$ is true, and that $P(n)$ implies $P(n++)$ for all integers n, but that $P(n)$ is not true for all integers n. Thus induction is not as useful a tool for dealing with the integers as it is with the natural numbers. (The situation becomes even worse with the rational and real numbers, which we shall define shortly.)

Proof. Let $P(n)$ be the property $n \geq 0$ is true. Suppose $P(0)$, $0 = 0$ so $0 \geq 0$ is true. Assume inductively that $n \geq 0$ is true. Then there exists some $x \in \mathbb{N}$ such that $n = 0 + x$. By the definition of addition for integers $n + 1 = (n - 0) + (1 - 0)$. Then by substitution,

$$\begin{aligned} n + 1 &= ((0 + x) - 0) + (1 - 0) \\ &= (0 + x) + 1 \\ &= 0 + (x + 1) \\ &\geq 0 \end{aligned}$$

So $n \geq 0 \implies n + 1 \geq 0$ for all integers. However, if n is the negation of a positive natural number, then $-n > 0$, and thus $n < 0$, so $P(n)$ cannot be true for all n. $\square$

Exercise 4.1.9

Show that the square of an integer is always a natural number. That is to say, prove that $n^2 \geq 0$ for every integer n.

Proof. By the trichotomy of integers, n is a positive natural number, 0, xor the negation of a positive natural number. A natural number multiplied by another natural number is a natural number. So if n is either 0 or a positive natural number, n^2 is a natural number. Each natural number is greater than or equal to 0, so $n^2 \geq 0$. Suppose n is the negation of positive natural number a, i.e. $n \times n = (0 - a) \times (0 - a)$. By applying the definition of multiplication, $(0 \times 0 + a \times a) - (0 \times a + a \times 0)$. Simplifying this we have $n^2 = (a \times a)$. Since a is a natural number, $(a \times a)$ is as well, so n^2 is a natural number again. Thus $n^2 \geq 0$ in all three cases. $\square$

4.2 The rationals

Exercise 4.2.1

Show that the definition of equality for the rational numbers is reflexive, symmetric, and transitive. (Hint: for transitivity, use Corollary 4.1.9.)

Proof. We first prove reflexivity, which holds if $a//b = a//b$, $b \neq 0$. $ab = ab$ because reflexivity holds for the integers, and $ab = ab \iff a//b = a//b$ by definition.

Next we tackle symmetry, which holds if $a//b = c//d \iff c//d = a//b$. We have $a//b = c//d \iff ad = bc$, $b \neq 0$ and $d \neq 0$. We also have $c//d = a//b \iff cb = da$. By commutativity of multiplication on the integers $ad = da$ and $bc = cb$. So by symmetry for the integers $ad = bc \iff cb = ad$, and thus $a//b = c//d \iff c//d = a//b$.

Finally for transitivity, which holds if $a//b = c//d$ and $c//d = e//f$ implies $a//b = e//f$. We have $a//b = c//d \iff ad = bc$, we also have $c//d = e//f \iff cf = de$, where b, d, f are all nonzero. By the cancellation rule (Lemma 4.1.9), $adf = bcf$. Also by 4.1.9, $bcf = bde$. By transitivity for the integers, $adf = bde$, and by 4.1.9 again $af = be$, so $a//b = e//f$ as desired. $\square$

Exercise 4.2.3

Prove the remaining components of Proposition 4.2.4.

I will only prove a subset of these, using $x = a//b$, $y = c//d$, and $z = e//f$ for integers a, c, e and nonzero integers b, d, f.

a. $(xy)z = x(yz)$

Proof.

$$\begin{aligned}
((a//b)(c//d))(e//f) &= \\
(ac//bd)(e//f) &= \\
(ace//bdf) & \\
(a//b)((c//d)(e//f)) &= \\
(a//b)(ce//df) &= \\
(ace//bdf) &
\end{aligned}$$

□

b. $x(y + z) = xy + xz$

Proof.

$$\begin{aligned}
(a//b)((c//d) + (e//f)) &= \\
(a//b)((cf + de)//(df)) &= \\
(a(cf + de)//bdf) &= \\
(acf + ade)//bdf & \\
(a//b)(c//d) + (a//b)(e//f) &= \\
(ac//bd) + (ae//bf) &= \\
(acbf + aebd)//(bdbf) &= \\
(acbf + aebd)//(bdbf) &= \\
((acf + aed)//bdf) * (b//b) &= \\
((acf + aed)//bdf) * (1//1) &= \\
((acf + aed)//bdf) * 1 &= \\
((acf + aed)//bdf) &= \\
((acf + ade)//bdf) &
\end{aligned}$$

□

Exercise 4.2.5

Proposition 4.2.9 (Basic properties of order on the rationals) Let x, y, z be rational numbers. Then the following properties hold.

Let $x = a/b$, $y = c/d$, and $z = e/f$ for integers a, b, c, d, e, f where b, d, f are nonzero.

a. (Order trichotomy) Exactly one of the three statements $x = y$, $x < y$, or $x > y$ is true.

Proof. $x < y \implies x \neq y$

$x > y \implies x \neq y$

Then by contrapositive $x = y$ implies both $x < y$ and $x > y$ are false.

$x > y \iff x - y$ is positive

$x < y \iff x - y$ is negative

By the trichotomy of rationals $x - y$ cannot be both positive and negative. So $x > y$ and $x < y$ are exclusive. □

b. (Order is antisymmetric) One has $x < y$ if and only if $y > x$.

Proof. We have two directions to prove.

Suppose $x < y$. By def of order on rationals $x - y$ is negative, i.e $(ad - bc)/bd$ is negative. Then by def of negative on rationals $-(ad - bc)$ and bd are both positive integers. $-(ad - bc) = (cb - da)$ by negation and commutativity. Therefore $(cb - da)/bd$ is a positive rational, which is equal to $(y - x)$. Thus, $y > x$.

Suppose $y > x$. Then $y - x$ is positive, i.e. $(cb - da)/db$ is positive. Then $(cb - da)$ and (db) are both positive integers. If $(cb - da)$ is positive, its negation $-(cb - da)$ is negative. We know $-(cb - da) = (ad - bc)$ by negation and commutativity. Then $(ad - bc)/db$ is a negative rational. Then so is $(x - y)$. Thus, $x < y$. □

c. (Order is transitive) If $x < y$ and $y < z$, then $x < z$.

Lemma. First we need to prove $(-a)/b = a/(-b)$, so we can write the proof without loss of generality.

$(-a)/b = a/(-b) \iff -a(-b) = ab.$

$-a(-b) =$

$(-1)(-1)(ab) =$

$(0\text{—}1)(0\text{—}1)(ab) =$

$((0 * 0 + 1 * 1) - (0 * 1 + 1 * 0))(ab) =$

$((0 + 1) - (0 + 0))(ab) =$

$(1 - 0)(ab) =$

$1(ab) =$

$ab.$

Thus $(-a)/b = a/(-b)$ as desired.

$\square$

Lemma. We wish to prove $a/b < c/d \iff ad < bc$, and $b, d \neq 0$. From above, we can assume without loss of generality that $b > 0$ and $d > 0$ (otherwise choose $-a/-b$ and $-c/-d$). Suppose $a/b < c/d$. We know $a/b < c/d \iff (a/b - c/d)$ is a negative rational number. By definition of subtration $(ad-bc)/bd$ is negative. By our generality assumption, bd is positive, so $(ad - bc)$ is negative. Suppose $ad = bc$, then $(ad - bc) = 0$, so by contradiction $ad \neq bc$. There must be some m such that $(ad - bc) + m = 0$, so there exists some nonzero m such that $ad + m = bc$. Thus by the previous two statements, $ad < bc$. Suppose $ad < bc$. Then $ad + m = bc$, $m \neq 0$, and $ad - bc = 0 - m$, or $(ad - bc)$ is negative. Since we have assumed bd is positive, the rational $(ad - bc)/bd$ is negative. $(ad - bc)/bd = (a/b - c/d)$, so the latter is negative too. Thus $a/b < c/d$.

$\square$

Proof. Without loss of generality, assume $b, d, f > 0$. (If $b > 0$, then set $a' = a, b' = b$; otherwise set $a' = -a, b' = -b$, and note that in either case $x = a'/b'$ and $b' > 0$.) Likewise for y, z.

1. Suppose $x < y$ and $y < z$.
2. Then $ad < bc$ by above lemma.
3. Likewise, $cf < de$.
4. $adf < bcf$ since we assume f is positive, and Lemma 4.1.11c (positive multiplication preserves order on integers).
5. $bcf < deb$ since we assume b is positive.
6. $adf < deb$ by transitivity of order on integers.
7. $af < eb$ since we assume d is positive.
8. $x < z$.

$\square$

d. (Addition preserves order) If $x < y$, then $x + z < y + z$.

e. (Positive multiplication preserves order) If $x < y$ and z is positive, then $xz < yz$.

4.3 Absolute value and exponentiation

Exercise 4.3.1

Proof. $\square$