Analysis I: Exercises
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3.5 Cartesian Products

Exercise 3.5.2

Suppose we define an ordered n-tuple to be a surjective function $x : \{ i \in \mathbb{N} \mid 1 \le i \le n \} \to X$ whose codomain is some arbitrary set X (so different ordered n-tuples are allowed to have different ranges); we then write x_i for $x(i)$ and also write x as $(x_i)_{1 \le i \le n}$. Using this definition, verify that we have $(x_i)_{1 \le i \le n} = (y_i)_{1 \le i \le n}$ if and only if $x_i = y_i$ for all $1 \le i \le n$.

Proof. We are asked to prove that the n-tuples x and y are equal for all $1 \le i \le n$ by showing that their components x_i and y_i are equal for all $1 \le i \le n$, and vice versa. We rewrite our definitions for clarity:

1. Let X be any set.
2. Let $x : \{ i \in \mathbb{N} \mid 1 \le i \le n \} \to X$
3. Let $x_i = x(i)$
4. Let $x = (x_i)_{1 \le i \le n}$

We proceed in the forward direction.

5. Suppose $(x_i)_{1 \le i \le n} = (y_i)_{1 \le i \le n}$.
   - 5.1. $x = y$ — line 4
   - 5.2. Suppose $1 \le i \le n$
     - 5.2.1. $i \in \{ i \in \mathbb{N} \mid 1 \le i \le n \}$
     - 5.2.2. $x(i) = y(i)$ — line 5.1
     - 5.2.3. $x_i = y_i$ — line 3
   - 5.3. $\forall i \in \mathbb{N} \quad 1 \le i \le n \implies x_i = y_i$
6. $(x_i)_{1 \le i \le n} = (y_i)_{1 \le i \le n} \implies \big(\forall i \in \mathbb{N} \quad 1 \le i \le n \implies x_i = y_i\big)$.

Now in the reverse direction:

7. Suppose $i \in \mathbb{N}$ st. $1 \le i \le n \implies x_i = y_i$.
   - 7.1. $i \in \{ i \in \mathbb{N} \mid 1 \le i \le n \}$ — line 2
   - 7.2. $x_i = y_i$ — slightly awkward MP
   - 7.3. $(x)_{1 \le i \le n}(i) = y_{1 \le i \le n}(i)$ — lines 3, 4
8. $\forall i \in \mathbb{N} \quad \big(1 \le i \le n \implies x_i = y_i\big) \implies (x)_{1 \le i \le n}(i) = y_{1 \le i \le n}(i)$

Thus they are equal. ∎

Note: Didn't use the surjectivity in the proof at all?

Also, show that if $(X_i)1 \leq i \leq n$ are an ordered n-tuple of sets, then the Cartesian product, as defined in Definition 3.5.6, is indeed a set. (Hint: use Exercise 3.4.7 and the axiom of specification.)

Proof. □

Exercise 3.5.4

Let A, B, C be sets. Show that:

a. $A \times (B \cup C) = (A \times B) \cup (A \times C)$**,**

Proof. □

b. $A \times (B \cup C) = (A \times B) \cup (A \times C)$**,** $A \times (B \cap C) = (A \times B) \cap (A \times C)$**,**

Proof. □

c. and $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$**. (One can of course prove similar identities in which the roles of the left and right factors of the Cartesian product are reversed.)**

Proof. □

Exercise 3.5.7

Let X and Y be sets, and let $\pi_{X\times Y \to X} : X \times Y \to X$ and $\pi_{X\times Y\to Y} : X \times Y \to Y$ be the maps $\pi_{X\times Y\to X}(x, y) := x$ and $\pi_{X\times Y\to Y}(x, y) := y$; these maps are known as the coordinate functions on $X \times Y$. Show that for any functions $f : Z \to X$ and $g : Z \to Y$, there exists a unique function $h : Z \to X \times Y$ such that $\pi_{X\times Y\to X} \circ h = f$ and $\pi_{X\times Y\to Y} \circ h = g$. (Compare this to the last part of Exercise 3.3.8, and to Exercise 3.1.7.) This function h is known as the pairing of f and g and is denoted $h = (f, g)$.

Proof. □

Exercise 3.5.8

Let $X_1, \ldots, X_n$ be sets. Show that the Cartesian product $\prod_{i=1}^{n} X_i$ is empty if and only if at least one of the X_i is empty.

Proof. □

Exercise 3.5.9

Suppose that I and J are two sets, and for all $\alpha \in I$ let A_α be a set, and for all $\beta \in J$ let B_β be a set. Show that

$$\left(\bigcup_{\alpha \in I} A_\alpha \right) \cap \left(\bigcup_{\beta \in J} B_\beta \right) = \bigcup_{(\alpha, \beta) \in I \times J} (A_\alpha \cap B_\beta) .$$

Proof. □

What happens if one interchanges all the union and intersection symbols here?

Answer.

Proof. □

Exercise 3.5.10

If $f : X \to Y$ is a function, define the graph of f to be the subset of $X \times Y$ defined by $\{(x, f(x)) : x \in X\}$.

a. **Show that two functions $f : X \to Y$, $\tilde{f} : X \to Y$ are equal if and only if they have the same graph.**

   Proof. □

b. **Conversely, if G is any subset of $X \times Y$ with the property that for each $x \in X$, the set $\{y \in Y : (x, y) \in G\}$ has exactly one element (or in other words, G obeys the vertical line test), show that there is exactly one function $f : X \to Y$ whose graph is equal to G.**

   Proof. □

c. **Suppose we define a function f to be an ordered triple $f = (X, Y, G)$, where X, Y are sets, and G is a subset of $X \times Y$ that obeys the vertical line test. We then define the domain of such a triple to be X, the codomain to be Y and for every $x \in X$, we define $f(x)$ to be the unique $y \in Y$ such that $(x, y) \in G$. Show that this definition is compatible with Definition 3.3.1 in the sense that every choice of domain X, codomain Y, and property $P(x, y)$ obeying the vertical line test produces a function as defined here that obeys all the properties required of it in that definition, and is also similarly compatible with Definition 3.3.8.**

   Proof. □