Analysis I: Exercises

Rachel Shu

May 20, 2023

Contents

Chapter 3

Set Theory

3.3 Functions

Exercise 3.3.2

Let $f : X \to Y$ and $g : Y \to Z$ be functions. Show that if f and g are both injective, then so is $g \circ f$; similarly show that if f and g are both surjective, then so is $g \circ f$.

Proof. Suppose f and g are injective. We need to show that for each x and x' in X, $x \neq x'$ implies $(g \circ f)(x) \neq (g \circ f)(x')$. The assumption that f is injective tells us that for every x and x' in X, $x \neq x' \implies f(x) \neq f(x')$. Since each $f(x)$ and $f(x')$ is in Y, the assumption that g is injective also tells us that for every $f(x)$ and $f(x')$ in Y, $f(x) \neq f(x') \implies g(f(x)) \neq g(f(x'))$. Thus for each x and x' in X, $x \neq x'$ implies $(g \circ f)(x) \neq (g \circ f)(x')$ as desired.

Now suppose f and g are surjective. We need to show that for each z in Z, there exists x in X such that $(g \circ f)(x) = z$. The assumption that f is surjective tells us that for each y in Y, there exists x in X such that $f(x) = y$. Since each $f(x)$ is in Y, the assumption that g is surjective tells us that for each z in Z, there exists $f(x)$ in Y, and thus x in X, such that $g(f(x)) = z$. Then for each z in Z, there exists x in X such that $(g \circ f)(x) = z$, as desired. □

Exercise 3.3.3

When is the empty function into a given set injective? surjective? bijective?

Answer. The empty function is only bijective into the empty set.

Proof. For the empty function to be injective into a given set, for each x and x' in $\emptyset$, $x \neq x'$ implies $\mathit{empty}(x) \neq \mathit{empty}(x')$. This is vacuously true as neither x nor x' exist. Thus the empty function is injective into all sets.

For the empty function to be surjective onto a given set, for each y in some set Y there needs to be some x in $\emptyset$ such that $empty(x) = y$. But there are no x in $\emptyset$, so this is false if there are any y in Y. Thus the empty function is only surjective onto the empty set, when it is vacuously true.

For the empty function to be bijective into a given set, it must be injective into and surjective onto that set. Since the empty function is only surjective onto the empty set, it is only bijective into the empty set. □

Exercise 3.3.5

Let $f : X \to Y$ and $g : Y \to Z$ be functions.

a. Show that if $g \circ f$ is injective, then f must be injective.

Proof. For the sake of contradiction suppose $g \circ f$ is injective but f is not. Then there exists some x and x' in X that go to the same y in Y. By the definition of a function, when we apply g each y in Y goes to exactly one z in Z, so when we apply $g \circ f$, any x and x' that go to the same y go to the same z, which contradicts the assumption that $g \circ f$ is injective. Thus if $g \circ f$ is injective f must also be. □

b. Is it true that g must also be injective?

Answer. No.

Proof. A counterexample shows that g is not necessarily injective when $(g \circ f)$ is. Suppose X is the empty set, Y is $\mathbb{N}$, Z is $\mathbb{N}$, and suppose g is the function $x \mapsto 0$. We can see g is not injective from Y to Z. (To pick one counterexample, $g(1)$ and $g(2)$ are equal.) But since any function from $\emptyset$ is an empty function, which is injective, $(g \circ f)(x)$ is injective. □

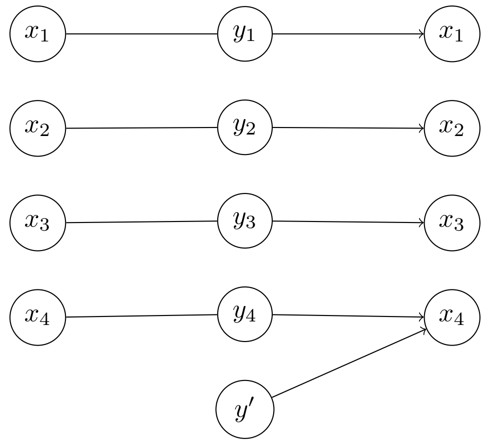
c. Show that if $g \circ f$ is surjective, then g must be surjective.

Proof. For the sake of contradiction, suppose $g \circ f$ is surjective but g is not. If g is not surjective, there exists some z in Z for which there is no y in Y such that $g(y) = z$. Then as all $f(x)$ are in Y there is also no $f(x)$ such that $g(f(x)) = z$. By the definition of a function if $f(x)$ does not exist then x cannot exist. So there is some z in Z for which there is no x in x such that $g(f(x)) = z$. But this contradicts our supposition that $g \circ f$ is surjective. Thus if $g \circ f$ is surjective then g is too. □

d. Is it true that f must also be surjective?

Answer. No.

Proof. To provide a counterexample, let $g : Y \to Z$ be a function which is surjective but for which there is some $y \in Y$ and some $y' \in Y$ such that for some z' in Z, $z' = g(y) = g(y')$, i.e. g is not injective. Also let $f : X \to Y$ be a function for which there is some x such that $f(x) = y$ for all $y \in Y$ except y', in which case f is not surjective (but would be if y' were left out). See the diagram below for one example of such a function:



We can verify that $(g \circ f)$ is still surjective, since for each z in Z there is at least one $f(x)$ such that $g(f(x)) = z$. In particular, for the z' which is equal to $g(y')$, since it is also equal to $g(y)$, and $f(x) = y$ for all $y \in Y$ except y', there is some x for which $(g \circ f)(x) = z'$.

Thus it is not the case that if f is not surjective that $(g \circ f)$ is also not, or to remove the contrapositive, it is not the case that if $(g \circ f)$ is surjective that f is also. $\square$

Exercise 3.3.8

If X is a subset of Y, let $\iota_{X \to Y} : X \to Y$ be *the inclusion map from X to Y*, defined by mapping $x \mapsto x$ for all $x \in X$, i.e., $\iota_{X \to Y}(x) := x$ for all $x \in X$. The map $\iota_{X \to X}$ is in particular called the *identity map* on X.

a. If $X \subseteq Y \subseteq Z$ then $\iota_{Y \to Z} \circ \iota_{X \to Y} = \iota_{X \to Z}$.

Proof. To prove equality, we need to show the domains and codomains of $\iota_{Y \to Z} \circ \iota_{X \to Y}$ and $\iota_{X \to Z}$ agree, and that for all x in their common domain $\iota_{Y \to Z} \circ \iota_{X \to Y}(x) = \iota_{X \to Z}(x)$.

Let $X \subseteq Y \subseteq Z$. Since $X \subseteq Y$, $\iota_{X \to Y} : X \to Y$ and since $Y \subseteq Z$, $\iota_{Y \to Z} : Y \to Z$, by the definition of composition (Definition 3.3.13) $\iota_{Y \to Z} \circ \iota_{X \to Y} : X \to Z$. Likewise, since $X \subseteq Z$, $\iota_{X \to Z} : X \to Z$, thus the codomains of the two functions agree.

Since $Y \subseteq Z$, we know $\iota_{Y \to Z}(y) := y$ for all $y \in Y$. In particular, since $X \subseteq Y$, $\iota_{Y \to Z}(\iota_{X \to Y}(x)) := \iota_{X \to Y}(x)$, which is x for all X. Likewise, since $X \subseteq Z$, $\iota_{X \to Z}(x) := x$ for all X.Thus for all $x \in X$, the output of $\iota_{Y \to Z}(\iota_{X \to Y}(x))$ is equal to the output of $\iota_{X \to Z}(x)$.

Therefore the two functions are equal. $\square$

b. Show that if $f : A \to B$ is any function, then $f = f \circ \iota_{A \to A} = \iota_{B \to B} \circ f$.

Proof. To prove equality, we need to show the domains and codomains of f, $f \circ \iota_{A \to A}$, and $\iota_{B \to B} \circ f$ all agree, and that for all x in their common domain, $f(x) = f \circ \iota_{A \to A}(x) = \iota_{B \to B} \circ f(x)$.

Let $f : A \to B$ be any function from A to B. Since $A \subseteq A$, we are given $\iota_{A \to A} : A \to A$. Then by the function composition rule (Definition 3.3.13) $f \circ \iota_{A \to A} : A \to B$ as well. Likewise, since $B \subseteq B$, we are given $\iota_{B\to B} : A \to B$. So composing $\iota_{B\to B} \circ f$ gives us a signature of $A \to B$ once again. Thus the domains and codomains all agree.

We now only need the outputs are equal. Since $f : A \to B$, for every element $a \in A$, for some element $b \in B$ we have $f(a) = b$.

Since $A \subseteq A$, for each element $a \in A$, we have $\iota_{A\to A}(a) = a$. So the function $(f \circ \iota_{A\to A}) : A \to B$ first sends each a to itself, then each a to b. Thus for every element $a \in A$, for some element $b \in B$ we again have $(f \circ \iota_{A\to A})(a) = b$.

Likewise, since $B \subseteq B$, for each element $b \in B$, we have $\iota_{B\to B}(b) = b$. So the function $\iota_{B\to B} \circ f$. first sends each a to b, then each b to itself. Thus for every element $a \in A$, for some element $b \in B$ we yet again have $\iota_{B\to B} \circ f(a) = b$.

Thus the outputs are all equal, and the two functions are equal. □

c. Show that if $f : A \to B$ is a bijective function, then $f \circ f^{-1} = \iota_{B\to B}$ and $f^{-1} \circ f = \iota_{A\to A}$.

Proof. The axiom of set equality states that

$$f : X \to Y = g : X' \to Y' \iff \Big(X = X' \quad Y = Y' \quad \forall x \in X\ f(x) = g(x) \Big)$$

Let $f : A \to B$ be a bijective function, which by Remark 3.3.27 has an inverse function denoted $f^{-1} : B \to A$. We separate the claims by conjunction elimination, and first prove $f \circ f^{-1} = \iota_{B\to B}$.

By the function composition rule (Definition 3.3.13), $f \circ f^{-1}$ is a function from $B \to B$. Since by definition $\iota_{B\to B}$ is from $B \to B$ as well, their domains and codomains agree.

Since f is bijective, for all $a \in A$, there exists exactly one $b \in B$ such that $f\ a \mapsto b$. We defined f^{-1} as the function $b \mapsto a$. Then $f \circ f^{-1}$ is the function $(b \mapsto a) \mapsto b$, in other words $f \circ f^{-1}(b) = b$. We also know $\iota_{B\to B}(b) = b$. Thus $f \circ f^{-1} = \iota_{B\to B}$.

The other part is very similar. $f^{-1} \circ f$ is a function from $A \to A$, which we also know is true of $\iota_{A\to A}$. Then $f^{-1} \circ f$ is the function $(a \mapsto b) \mapsto a$,in other words $f^{-1} \circ f(a) = a$. Likewise $\iota_{A\to A}(a) = a$. Thus $f^{-1} \circ f = \iota_{A\to A}$, and we have completed our proof. □

d. Show that if X and Y are disjoint sets, and $f : X \to Z$ and $g : Y \to Z$ are functions, then there is a unique function $h : X \cup Y \to Z$ such that $h \circ \iota_{X\to X\cup Y} = f$ and $h \circ \iota_{Y\to X\cup Y} = g$.

Proof. Let X and Y be disjoint sets, and $f : X \to Z$ and $g : Y \to Z$ be functions. We need to show that there is some function $h : X \cup Y \to Z$ which satisfies $h \circ \iota_{X\to X\cup Y} = f$ and $h \circ \iota_{Y\to X\cup Y} = g$, and then show that it is unique.

By pairwise union, (Axiom 3.5) there exists $X \cup Y$. By the definition of disjunction, X and Y have no common elements, so $a \in X \cup Y$ is exclusively in X or Y. Thus we can construct the following:

Let $h : X \cup Y \to Z = \begin{cases} f(a) & \text{if } a \in X \\ g(a) & \text{if } a \in Y \end{cases}$

Using this h, suppose $h \circ \iota_{X\to X\cup Y}$. Since $h : X \cup Y \to Z$ and $\iota_{X\to X\cup Y} : X \to X \cup Y$, their composition $h \circ \iota_{X\to X\cup Y}$ is a function from $X \to Z$. Thus it has the same domain and codomain as f. For all

$x \in X$, $\iota_{X \to X \cup Y}(x) = x$. This means $h \circ \iota_{X \to X \cup Y}(x) = h(x)$. Since x in X, $h(x) = f(x)$. Therefore we have satisfied both conditions to prove $h \circ \iota_X \to X \cup Y = f$.

Likewise, suppose $h \circ \iota_{Y \to X \cup Y}$. Since $h : X \cup Y \to Z$ and $\iota_{Y \to X \cup Y} : Y \to X \cup Y$, their composition $h \circ \iota_{Y \to X \cup Y}$ is a function from $Y \to Z$. Thus it has the same domain and codomain as g. For all $y \in Y$, $\iota_{Y \to X \cup Y}(y) = y$. This means $h \circ \iota_{Y \to X \cup Y}(y) = h(y)$. Since y in Y, $h(y) = g(y)$. Now we have also satisfied both conditions to prove $h \circ \iota_X \to X \cup Y = g$.

We now prove uniqueness. Suppose there exists another function $h : X \cup Y \to Z$ and $h' : X \cup Y \to Z$ such that $h' \circ \iota_{X \to X \cup Y} = f$ and $h' \circ \iota_{Y \to X \cup Y} = g$ as well. We know that the domains and codomains of h and h' are the same.

Since $h' \circ \iota_{X \to X \cup Y} = f$, and $\iota_{X \to X \cup Y}$ is an inclusion map, $h' = f$ for all $x \in X$. Likewise since $h' \circ \iota_{Y \to X \cup Y} = g$, and $\iota_{Y \to X \cup Y}$ is an inclusion map, $h' = g$ for all $y \in Y$. This means h' has the same value as h for $a \in X \cup Y$, so they are identical and h is unique. □

e. Show that the hypothesis that X and Y are disjoint can be dropped in (d) if one adds the additional hypothesis that $f(x) = g(x)$ for all $x \in X \cap Y$.

Proof. Suppose $f(x) = g(x)$ for all $x \in X \cap Y$, and X and Y are not disjoint. If the relation

$$h : X \cup Y \to Z = \begin{cases} f(x) & \text{if } x \in X \\ g(x) & \text{if } x \in Y \end{cases}$$

is a function, it satisfies the conditions in Exercise 3.3.8.d. A relation h is a function if and only if each x only has one corresponding $h(x)$. We previously required that x be exclusively in X or Y, which meant that $f(x)$ and $g(x)$ were never defined for the same x. We check if it is still a function under our current assumption. If x in $X \cup Y$, then one of the three following hold:

(a) $x \in X; x \notin Y$

(b) $x \in Y; x \notin X$

(c) $x \in X \cap Y$

If cases **a** or **b**, then h is a function because either $f(x)$ is defined or $g(x)$ is, but not both, as before. If case **c**, since we have assumed $f(x) = g(x)$ for all $x \in X \cap Y$, we know $h(x) = f(x) = g(x)$ and therefore h has only one value. Thus in all cases h is a function. □

3.4 Images and inverse images

Exercise 3.4.1

Let $f : X \to Y$ be a bijective function, and let $f^{-1} : Y \to X$ be its inverse. Let V be any subset of Y. Prove that the forward image of V under f^{-1} is the same set as the inverse image of V under f; thus the fact that both sets are denoted by $f^{-1}(V)$ will not lead to any inconsistency.

Proof. Suppose $f\colon X \to Y$ is a bijective function, and $f^{-1}\colon Y \to X$ is its inverse, where V is any subset of Y. Let $f^{-1}(V)$ denote the inverse image of V, and let $(f^{-1})(V)$ denote the forward image of V under f^{-1}. We define

$$f^{-1}(V) = \{\, x \in X \mid f(x) \in V \,\}$$
$$(f^{-1})(V) = \{\, f^{-1}(y) \mid y \in V \,\}$$

1. First we show $f^{-1}(V) \subseteq (f^{-1})(V)$.
2. Let $z \in f^{-1}(V)$.
3. Then $z \in X$ and $f(z) \in V$.
4. Since f is bijective, for all y in $V \subseteq Y$, $y = f(x) = f(f^{-1}(y))$.
5. Thus $f(z) \in V \implies y \in V$.
6. Since f is bijective, for all x in X, $x = f^{-1}(y) = f^{-1}(f(x))$.
7. Thus $z \in X \implies z = f^{-1}(y)$.

□

Exercise 3.4.2

Let $f\colon X \to Y$ be a function from one set X to another set Y, let S be a subset of X, and let U be a subset of Y.

a. What, in general, can one say about $f^{-1}(f(S))$ and S?

Answer. S is a subset of $f^{-1}(f(S))$, but S may not be equal to $f^{-1}(f(S))$.

Proof. (informal) Let x be an element of X. We have $f(S) = \{\, f(x) \mid x \in S \,\}$, and therefore $f^{-1}(f(S)) = \{\, x \in X \mid f(x) \in f(S) \,\}$.

Suppose $x \in S$, then $x \in X$ and $f(x) \in f(s)$, thus $x \in f^{-1}(f(S))$ for all $x \in S$, so S is a subset of $f^{-1}(f(S))$. Now instead suppose $x \notin S$. Since we have not stated that f is injective, it is still possible that $f(x) \in f(S)$. Once again $x \in X$ and $f(x) \in f(s)$, thus for some x not in S, x may still be in $x \in f^{-1}(f(S))$. Thus $f^{-1}(f(S))$ may contain more members of X than S does, so they may not be equal. □ □

b. What about $f(f^{-1}(U))$ and U?

Answer. $f(f^{-1}(U))$ is a subset of U, but the two sets may not be equal.

Proof. (informal) Let x be an element of X. We have $f^{-1}(U) = \{\, x \in X \mid f(x) \in U \,\}$. Then $f(f^{-1}(U)) = \{\, f(x) \mid x \in f^{-1}(U) \,\}$. Since f is not stated to be surjective, there may be some y in U for which $y \neq f(x)$ for all x. So when we take the forward image of $f^{-1}(U)$, every element of $f^{-1}(U)$ is in U, but there may be some y in U that are not in $f^{-1}(U)$. □

c. What about $f^{-1}(f(f^{-1}(U)))$ and $f^{-1}(U)$?

Answer.

Proof. *(informal)* As before we have $f^{-1}(U) = \{ x \in X \mid f(x) \in U \}$, and $f(f^{-1}(U)) = \{ f(x) \mid x \in f^{-1}(U) \}$.

$$\begin{aligned} f^{-1}(f(f^{-1}(U))) &= \{ x \in X \mid f(f^{-1}(U)) \in U \} \\ &= x \in X \text{ and } f(f^{-1}(U)) \in U \\ &= x \in X \text{ and } \{ f(x) \mid x \in f^{-1}(U) \} \in U \\ &= x \in X \text{ and } (\exists x \text{ such that } y = f(x) \text{ and } x \in f^{-1}(U)) \in U \end{aligned}$$

$$\begin{aligned} f^{-1}(f(f^{-1}(U))) &= \{ x \in X \mid f(f^{-1}(U)) \in U \} \\ &= \{ x \in X \mid \{ f(x) \mid x \in f^{-1}(U) \} \in U \} \\ &= \{ x \in X \mid \{ f(x) \mid x \in \{ x \in X \mid f(x) \in U \} \} \in U \} \\ &= (x \in X) \text{ and } (f(x) \text{ is true and } (x \in (x \in X \text{ and } f(x) \in U)) \in U). \\ &= x \in X \text{ and } f(x) \in U (\textit{incomplete}) \end{aligned}$$

(good lord...)

□

Exercise 3.4.3

Let A, B be two subsets of a set X, and let $f : X \to Y$ be a function. Show that

a. $f(A \cap B) \subseteq f(A) \cap f(B)$,

Proof. We prove this statement by showing every element of $f(A \cap B)$ is an element of $f(A) \cap f(B)$.

1. Let y be an arbitrary element of $f(A \cap B)$.
2. $A \subseteq X$ and $B \subseteq X \implies A \cap B \subseteq X$.
3. By definition the image of $A \cap B$ under f is $\{ f(x) \mid x \in A \cap B \}$.
4. By the axiom of replacement (3.7) $y = f(x)$ for some $x \in A \cap B$.
5. $x \in A \cap B \implies x \in A$
6. $y = f(x)$ for some $x \in A$
7. $x \in A \cap B \implies x \in B$
8. $y = f(x)$ for some $x \in B$
9. $y = f(x)$ for some $x \in A$ and $y = f(x)$ for some $x \in B$
10. $y \in \{ f(x) \mid x \in A \}$ and y $\in \{ f(x) \mid x \in B \}$
11. $y \in f(A) \cap f(B)$, as desired.

□

b. $f(A) \setminus f(B) \subseteq f(A \setminus B)$,

Proof. We prove this statement by showing every element of $f(A) \setminus f(B)$ is an element of $f(A \setminus B)$.

1. Let $y \in f(A) \setminus f(B)$ be arbitrary. Conditional introduction
2. $y \in f(A)$ and $y \notin f(B)$.
3. $\exists x \in A \; y = f(x)$
4. Suppose x such that $x \in A$ and $y = f(x)$
   - 4.1. $x \in A$
   - 4.2. $y = f(x)$
   - 4.3. $\forall z \in B \; y \neq f(z)$
   - 4.4. $\forall z \; z \in B \implies y \neq f(z)$
   - 4.5. $\forall z \; y = f(z) \implies z \notin B$
   - 4.6. $y = f(x) \implies x \notin B$
   - 4.7. $x \notin B$
   - 4.8. $x \in A$, $x \notin B$, and $y = f(x)$.
   - 4.9. $y = f(x)$ and $x \in A \setminus B$.
   - 4.10. $y \in \{ y \mid y = f(x) \text{ for } x \in A \setminus B \}$.
5. $y \in f(A \setminus B)$ Existential elimination
6. $y \in f(A) \setminus f(B) \implies y \in f(A \setminus B)$ Conditional elimination

Thus $f(A) \setminus f(B) \subseteq f(A \setminus B)$. □

c. $f(A \cup B) = f(A) \cup f(B)$.

Proof. We prove this statement by showing every element of $f(A \cup B)$ is an element of $f(A) \cup f(B)$ and vice versa. First we do the forward direction:

1. Let $y \in f(A \cup B)$ be arbitrary.
2. $A \in X$
3. $B \in X$
4. $A \cup B \in X$
5. $y \in \{ f(x) \mid x \in A \cup B \}$
6. $\exists x$ such that $x \in A \cup B$ and $y = f(x)$
7. Suppose x such that $x \in A \cup B$ and $y = f(x)$
   - 7.1. $y = f(x)$
   - 7.2. $x \in A \cup B$
   - 7.3. $x \in A$ or $x \in B$
   - 7.4. ($x \in A$ and $y = f(x)$) or ($x \in B$ and $y = f(x)$)
     - 7.4.1. test
   - 7.5. $y \in \{ f(x) \mid x \in A \}$ or $y \in \{ y = f(x) \mid x \in B \}$
   - 7.6. $y \in f(A)$ or $y \in f(B)$
   - 7.7. $y \in f(A) \cup f(B)$
8. $y \in f(A \cup B) \implies y \in f(A) \cup f(B)$
9. $f(A \cup B) \subseteq f(A) \cup f(B)$

Now in the backwards direction.

1. Let $y \in f(A) \cup f(B)$ be arbitrary.

2. $\boldsymbol{y} \in \boldsymbol{f}(\boldsymbol{A})$ or $\boldsymbol{y} \in \boldsymbol{f}(\boldsymbol{B})$
3. Case $\boldsymbol{y} \in \boldsymbol{f}(\boldsymbol{A})$
    3.1. $\boldsymbol{y} \in \{\, \boldsymbol{f}(\boldsymbol{x}) \mid \boldsymbol{x} \in \boldsymbol{A} \,\}$
    3.2. $\exists \boldsymbol{x}$ such that ($\boldsymbol{x} \in \boldsymbol{A}$ and $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$)
    3.3. Suppose $\boldsymbol{x}$ such that ($\boldsymbol{x} \in \boldsymbol{A}$ and $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$)
        3.3.1. $\boldsymbol{x} \in \boldsymbol{A}$ and $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$
4. Case $\boldsymbol{y} \in \boldsymbol{f}(\boldsymbol{B})$
    4.1. $\boldsymbol{y} \in \{\, \boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x}) \mid \boldsymbol{x} \in \boldsymbol{B} \,\}$
    4.2. $\exists \boldsymbol{x}$ such that ($\boldsymbol{x} \in \boldsymbol{B}$ and $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$)
    4.3. Suppose $\boldsymbol{x}$ such that ($\boldsymbol{x} \in \boldsymbol{B}$ and $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$)
        4.3.1. $\boldsymbol{x} \in \boldsymbol{B}$ and $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$
5. ($\boldsymbol{x} \in \boldsymbol{B}$ and $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$) or ($\boldsymbol{x} \in \boldsymbol{A}$ and $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$)
6. $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$ and ($\boldsymbol{x} \in \boldsymbol{A}$ or $\boldsymbol{x} \in \boldsymbol{B}$) — Distributivity
7. $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$ and ($\boldsymbol{x} \in \boldsymbol{A} \cup \boldsymbol{B}$)
8. $\boldsymbol{y} \in \{\, \boldsymbol{f}\boldsymbol{x} \mid \boldsymbol{x} \in \boldsymbol{A} \cup \boldsymbol{B} \,\}$
9. $\boldsymbol{y} \in \boldsymbol{f}(\boldsymbol{A}) \cup \boldsymbol{f}(\boldsymbol{B}) \implies \boldsymbol{y} \in \{\, \boldsymbol{f}(\boldsymbol{x}) \mid \boldsymbol{x} \in \boldsymbol{A} \cup \boldsymbol{B} \,\}$
10. $\boldsymbol{f}(\boldsymbol{A}) \cup \boldsymbol{f}(\boldsymbol{B}) \subseteq \boldsymbol{f}(\boldsymbol{A} \cup \boldsymbol{B})$

Thus we have $\boldsymbol{f}(\boldsymbol{A} \cup \boldsymbol{B}) = \boldsymbol{f}(\boldsymbol{A}) \cup \boldsymbol{f}(\boldsymbol{B})$. □

For the first two statements, is it true that the $\subseteq$ relation can be improved to $=$?

Answer.

Proof. I want to first try to prove $\boldsymbol{f}(\boldsymbol{A} \cap \boldsymbol{B}) = \boldsymbol{f}(\boldsymbol{A}) \cap \boldsymbol{f}(\boldsymbol{B})$. Since I already have $\boldsymbol{f}(\boldsymbol{A} \cap \boldsymbol{B}) \subseteq \boldsymbol{f}(\boldsymbol{A}) \cap \boldsymbol{f}(\boldsymbol{B})$, I just need $\boldsymbol{f}(\boldsymbol{A}) \cap \boldsymbol{f}(\boldsymbol{B}) \subseteq \boldsymbol{f}(\boldsymbol{A} \cap \boldsymbol{B})$.

1. Suppose $\boldsymbol{y} \in \boldsymbol{f}(\boldsymbol{A}) \cap \boldsymbol{f}(\boldsymbol{B})$
2. $\boldsymbol{y} \in \boldsymbol{f}(\boldsymbol{A})$ and $\boldsymbol{y} \in \boldsymbol{f}(\boldsymbol{B})$
3. $\boldsymbol{y} \in \{\, \boldsymbol{f}(\boldsymbol{x}) \mid \boldsymbol{x} \in \boldsymbol{A} \,\}$
4. $\exists \boldsymbol{x}$ st. $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$ and $\boldsymbol{x} \in \boldsymbol{A}$
5. Suppose $\boldsymbol{x}$ st. $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$ and $\boldsymbol{x} \in \boldsymbol{A}$
6. $\boldsymbol{y} \in \{\, \boldsymbol{f}(\boldsymbol{x}) \mid \boldsymbol{x} \in \boldsymbol{B} \,\}$
7. $\exists \boldsymbol{x}$ st. $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$ and $\boldsymbol{x} \in \boldsymbol{A}$

Next I'm going to try to prove $\boldsymbol{f}(\boldsymbol{A}) \setminus \boldsymbol{f}(\boldsymbol{B}) = \boldsymbol{f}(\boldsymbol{A} \setminus \boldsymbol{B})$. I already have $\boldsymbol{f}(\boldsymbol{A}) \setminus \boldsymbol{f}(\boldsymbol{B}) \subseteq \boldsymbol{f}(\boldsymbol{A} \setminus \boldsymbol{B})$ and I just need $\boldsymbol{f}(\boldsymbol{A} \setminus \boldsymbol{B}) \subseteq \boldsymbol{f}(\boldsymbol{A}) \setminus \boldsymbol{f}(\boldsymbol{B})$.

1. Suppose $\boldsymbol{y} \in \boldsymbol{f}(\boldsymbol{A} \setminus \boldsymbol{B})$.
2. $\exists \boldsymbol{x}$ such that $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$ and $\boldsymbol{x} \in \boldsymbol{A} \setminus \boldsymbol{B}$.
3. Suppose $\boldsymbol{x}$ such that $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$ and $\boldsymbol{x} \in \boldsymbol{A} \setminus \boldsymbol{B}$.
    3.1. $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$
    3.2. $\boldsymbol{x} \in \boldsymbol{A} \setminus \boldsymbol{B}$
    3.3. $\boldsymbol{x} \in \boldsymbol{A}$ and $\boldsymbol{x} \notin \boldsymbol{B}$
    3.4. $\boldsymbol{y} = \boldsymbol{f}(\boldsymbol{x})$ and $\boldsymbol{x} \in \boldsymbol{A}$

3.5. $y \in \{ f(x) \mid x \in A \}$

3.6. $y \in f(A)$

3.7. $y = f(x)$ and $x \notin B$

3.8. $\cancel{y \in \{ f(x) \mid x \notin B \}}$ (not useful!)

not sure where to go from here

□

Exercise 3.4.5

Let $f: X \to Y$ be a function from one set X to another set Y.

a. Show that $f(f^{-1}(S)) = S$ for every $S \subseteq Y$ if and only if f is surjective.

Proof. □

b. Show that $f^{-1}(f(S)) = S$ for every $S \subseteq X$ if and only if f is injective.

Proof. □

Exercise 3.4.6

a. Prove Lemma 3.4.10. (Hint: start with the set $\{0, 1\}^X$ and apply the replacement axiom, replacing each function f with the object $f^{-1}(\{1\})$. See also Exercise 3.5.11.)

(Lemma 3.4.10): Let X be a set. Then the set $\{ Y \mid Y$ is a subset of $X \}$ is a set. That is to say, there exists a set Z such that $Y \in Z \iff Y \subseteq X$.

Proof. We need to prove $\{ Y \mid Y \subseteq X \}$ exists. We construct a set that we know exists, and then prove that it is equal to $\{ Y \mid Y \subseteq X \}$.

1. Let X be an arbitrary set.
2. Let F be the set $\{0, 1\}^X$
3. $f \in F \iff f : X \to \{0, 1\}$ — power set axiom
4. Let $P(f, a)$ such that $P(f, a) \iff a = f^{-1}(\{1\})$
5. Let Z be the set $\{ a \mid P(f, a)$ is true for some $f \in F \}$
6. $z \in Z \iff P(f, z)$ is true for some $f \in F$. — axiom of replacement
7. Suppose $Y \in Z$.
   - 7.1. $P(f, Y)$ is true for some $f \in F$ — 5
   - 7.2. Suppose $g \in F$ such that $P(g, Y)$
     - 7.2.1. $Y = g^{-1}(\{1\})$ — 3
     - 7.2.2. $Y \subseteq X$ — def of inverse image
   - 7.3. $Y \subseteq X$

8. $\forall Y,\ Y \in Z \implies Y \subseteq X$

9. Suppose $Y \subseteq X$.

    9.1. Let $g : X \to \{0,1\} = x \mapsto 1$ for $x \in Y$ and $x \mapsto 0$ otherwise.

    9.2. $g \in F$ 3

    9.3. $Y = g^{-1}(\{1\})$ 7.5.1

    9.4. $P(g, Y)$ 4

    9.5. $Y \in Z$ 6

10. $\forall Y,\ Y \subseteq X \implies Y \in Z$

11. $\forall z,\ z \in Z \iff z \in \{\, Y \mid Y \subseteq X \,\}$. axiom of set equality

12. $Z = \{\, Y \mid Y \subseteq X \,\}$

13. $\{\, Y \mid Y \subseteq X \,\}$ is a set.

Let X be arbitrary. We need to prove $\{\, Y \mid Y \subseteq X \,\}$ exists. Let's proceed by creating a valid set and proving it is equal to $\{\, Y \mid Y \subseteq X \,\}$. Let F be the power set $\{0,1\}^X$. Let Z be the set $\{\, a \mid P(f,a)$ is true for some $f \in F \,\}$, where $P(f,a)$ is $a = f^{-1}(\{1\})$. In order to show $Z = \{\, Y \mid Y \subseteq X \,\}$, we need to show $z \in Z \iff z \subseteq X$ is true. We will first prove the forward direction

Suppose $Y \in Z$. Then by the axiom of specification $P(f, Y)$ is true for some $f \in F$. Choose $g \in F$ such that $P(g, Y)$ is true. Then $Y = g^{-1}(\{1\})$. Since by definition $g^{-1}(\{1\})$ is in X, $Y \in X$.

Conversely suppose $Y \subseteq X$. Let $g : X \to \{0,1\}$ be the function $x \mapsto 1$ for $x \in Y$ and $x \mapsto 0$ otherwise. Since every function $X \to \{0,1\}$ is in F, g is in F. Since $x \mapsto 1$ for $x \in Y$, we know $Y = g^{-1}(\{1\})$, which means $P(g, Y)$ is true, and thus $Y \in Z$.

We've proved $Y \in Z \iff Y \subseteq X$, so $Z = \{\, Y \mid Y \subseteq X \,\}$ is true, and since Z exists, $\{\, Y \mid Y \subseteq X \,\}$ exists. □

b. Conversely, show that Axiom 3.11 can be deduced from the preceding axioms of set theory if one accepts Lemma 3.4.10 as an axiom. (This may help explain why we refer to Axiom 3.11 as the "power set axiom".)

Proof. □

Exercise 3.4.7

Let X, Y be sets. Define a partial function from X to Y to be any function $f : X' \to Y'$ whose domain X' is a subset of X, and whose codomain Y' is a subset of Y. Show that the collection of all partial functions from X to Y is itself a set. (Hint: use Exercise 3.4.6, the power set axiom, the replacement axiom, and the union axiom.

Proof. We wish to show that ${Y'}^{X'}$ defined as $\{\, f' \mid f' : X' \to Y' \,\}$ exists.

1. Let X, Y be arbitrary sets.
2. For $X' \subseteq X$ and $Y' \subseteq Y$, $f' : X' \to Y'$
3. $\{\, f \mid f : X \to Y \,\}$ is a set. power set axiom
4. $\{\, X' \mid X' \subseteq X \,\}$ is a set. Lemma 3.4.10, see above

5. $\{\, Y' \mid Y' \subseteq Y \,\}$ is a set. Lemma 3.4.10, see above
6.
7. $\{\, f' \mid f' : X' \to Y \,\}$ is a set.
8. $\{\, f' \mid f' : X \to Y' \,\}$ is a set.
9. $\{\, f' \mid f' : X' \to Y \,\} \cup \{\, f' \mid f' : X \to Y' \,\}$ is a set. union axiom
10. $\forall a,\ a \in \{\, f' \mid f' : X' \to Y \,\} \cup \{\, f' \mid f' : X \to Y' \,\} \iff a \in X' \to Y$ and $a \in X \to Y'$ axiom of replacement
11. $a \in X' \to Y'$
12. $a \in \{\, f' \mid f' : X' \to Y' \,\}$ axiom of replacement
13. $\{\, f' \mid f' : X' \to Y' \,\}$ is a set.

□

not proved :(

Exercise 3.4.9

Show that if β and β' are two elements of a set I, and to each $\alpha \in I$ we assign a set A_α, then

$$\{x \in A_\beta : x \in A_\alpha \text{ for all } \alpha \in I\} = \{x \in A_{\beta'} : x \in A_\alpha \text{ for all } \alpha \in I\},$$

and so the definition of $\bigcap_{\alpha \in I} A_\alpha$ defined in (3.3) does not depend on β.

Proof. □

Also explain why (3.4) is true.

Proof. □

Exercise 3.4.10

Suppose that I and J are two sets, and for all $\alpha \in I \cup J$ let A_α be a set. Show that

$$\bigcup_{\alpha \in I} A_\alpha \cup \bigcup_{\alpha \in J} A_\alpha = \bigcup_{\alpha \in I \cup J} A_\alpha.$$

Proof. We need to show that every element of $\bigcup_{\alpha \in I} A_\alpha \cup \bigcup_{\alpha \in J} A_\alpha$ is also in $\bigcup_{\alpha \in I \cup J} A_\alpha$ and vice versa. We begin in the forward direction.

1.

Now in reverse:

1.

$\square$

If I and J are non-empty, show that

$$\bigcap_{\alpha \in I} A_\alpha \cap \bigcap_{\alpha \in J} A_\alpha = \bigcap_{\alpha \in I \cup J} A_\alpha.$$

Proof.

$\square$

3.5 Cartesian Products

Exercise 3.5.2

Suppose we define an ordered n-tuple to be a surjective function $x : \{ i \in \mathbb{N} \mid 1 \leq i \leq n \} \to X$ whose codomain is some arbitrary set X (so different ordered n-tuples are allowed to have different ranges); we then write x_i for $x(i)$ and also write x as $(x_i)_{1 \leq i \leq n}$. Using this definition, verify that we have $(x_i)_{1 \leq i \leq n} = (y_i)_{1 \leq i \leq n}$ if and only if $x_i = y_i$ for all $1 \leq i \leq n$.

Proof. We are asked to prove that the n-tuples x and y are equal for all $1 \leq i \leq n$ by showing that their components x_i and y_i are equal for all $1 \leq i \leq n$, and vice versa. We rewrite our definitions for clarity:

1. Let X be any set.
2. Let $x : \{ i \in \mathbb{N} \mid 1 \leq i \leq n \} \to X$
3. Let $x_i = x(i)$
4. Let $x = (x_i)_{1 \leq i \leq n}$

We proceed in the forward direction.

5. Suppose $(x_i)_{1 \leq i \leq n} = (y_i)_{1 \leq i \leq n}$.
    5.1. $x = y$ — line 4
    5.2. Suppose $1 \leq i \leq n$
        5.2.1. $i \in \{ i \in \mathbb{N} \mid 1 \leq i \leq n \}$
        5.2.2. $x(i) = y(i)$ — line 5.1
        5.2.3. $x_i = y_i$ — line 3
    5.3. $\forall i \in \mathbb{N} \quad 1 \leq i \leq n \implies x_i = y_i$
6. $(x_i)_{1 \leq i \leq n} = (y_i)_{1 \leq i \leq n} \implies \big(\forall i \in \mathbb{N} \quad 1 \leq i \leq n \implies x_i = y_i\big)$.

Now in the reverse direction:

7. Suppose $i \in \mathbb{N}$ st. $1 \leq i \leq n \implies x_i = y_i$.
    7.1. $i \in \{ i \in \mathbb{N} \mid 1 \leq i \leq n \}$ — line 2

7.2. $x_i = y_i$ slightly awkward MP

7.3. $(x)_{1 \leq i \leq n}(i) = y_{1 \leq i \leq n}(i)$ lines 3, 4

8. $\forall i \in \mathbb{N} \quad \big(1 {\leq} i {\leq} n \implies x_i = y_i\big) \implies (x)_{1 \leq i \leq n}(i) = y_{1 \leq i \leq n}(i)$

Thus they are equal. □

Also, show that if $(X_i)1 \leq i \leq n$ are an ordered n-tuple of sets, then the Cartesian product, as defined in Definition 3.5.6, is indeed a set. (Hint: use Exercise 3.4.7 and the axiom of specification.)

Proof. 1. Let

2. Suppose $(X_i)1 \leq i \leq n$.

2.1.

□

Unsure how to approach.

Exercise 3.5.4

Let A, B, C be sets. Show that:

a. $A \times (B \cup C) = (A \times B) \cup (A \times C)$,

Proof. We need to show every element of $A \times (B \cup C)$ is also in $(A \times B) \cup (A \times C)$ and vice versa. We first prove the forward direction:

1. Suppose $a \in A \times (B \cup C)$

1.1. $a \in \{\, (x, y) \mid x \in A \text{ and } y \in (B \cup C) \,\}$

1.2. $a \in \{\, (x, y) \mid x \in A \text{ and } (y \in B \text{ or } y \in C) \,\}$

1.3. $a \in \{\, (x, y) \mid x \in A \text{ and } (y \in B \text{ or } y \in C) \,\}$

1.4. $a \in \{\, (x, y) \mid (x \in A \text{ and } y \in B) \text{ or } (x \in A \text{ and } y \in C) \,\}$

1.5. $a = (x, y)$ and $\big((x \in A \text{ and } y \in B) \text{ or } (x \in A \text{ and } y \in C)\big)$ for some $x \in A$ and some $y \in B$

1.6. Let x and y st. $a = (x, y)$

1.7. 1.7.1. $(x \in A \text{ and } y \in B)$ or $(x \in A \text{ and } y \in C)$

1.7.2. Case $(x \in A \text{ and } y \in B)$

1.7.2.1. $a \in \{\, (x, y) \mid x \in A \text{ and } y \in B \,\}$

1.7.2.2. $a \in (A \times B)$

1.7.3. Case $(x \in A \text{ and } y \in C)$

1.7.3.1. $a \in \{\, (x, y) \mid x \in A \text{ and } y \in C \,\}$

1.7.3.2. $a \in (A \times C)$

1.7.4. $a \in (A \times B)$ or $a \in (A \times C)$

1.7.5. $a \in (A \times B) \cup (A \times C)$

2. $a \in A \times (B \cup C) \implies a \in (A \times B) \cup (A \times C)$
3. $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$

Now this is mostly invertible:

4. Suppose $a \in (A \times B) \cup (A \times C)$
   4.1. $a \in (A \times B)$ or $a \in (A \times C)$
   4.2. Case $a \in (A \times B)$
      4.2.1. $a \in \{ (x, y) \mid x \in A \text{ and } y \in B \}$
      4.2.2. $\exists x \in A, y \in B$ st. $a = (x, y)$
      4.2.3. Let x, y st. $a = (x, y)$
         4.2.3.1. $(x \in A \text{ and } y \in B)$
         4.2.3.2. $(x \in A \text{ and } x \in B)$ or $(x \in A \text{ and } x \in C)$
         4.2.3.3. $a = (x, y)$ and $\big((x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } y \in C)\big)$
         4.2.3.4. $a \in \{ (x, y) \mid (x \in A \text{ and } y \in B) \text{ or } (x \in A \text{ and } y \in C) \}$
   4.3. Case $a \in (A \times C)$
      4.3.1. $a \in \{ (x, y) \mid x \in A \text{ and } y \in C \}$
      4.3.2. $\exists x \in A, y \in B$ st. $a = (x, y)$
      4.3.3. Let x, y st. $a = (x, y)$
         4.3.3.1. $(x \in A \text{ and } y \in C)$
         4.3.3.2. $(x \in A \text{ and } x \in B)$ or $(x \in A \text{ and } x \in C)$
         4.3.3.3. $a = (x, y)$ and $\big((x \in A \text{ and } x \in B) \text{ or } (x \in A \text{ and } y \in C)\big)$
         4.3.3.4. $a \in \{ (x, y) \mid (x \in A \text{ and } y \in B) \text{ or } (x \in A \text{ and } y \in C) \}$
   4.4. $a \in \{ (x, y) \mid (x \in A \text{ and } y \in B) \text{ or } (x \in A \text{ and } y \in C) \}$ 4.2.3.4 and 4.3.3.4
   4.5. $a \in \{ (x, y) \mid x \in A \text{ and } (y \in B \text{ or } y \in C) \}$
   4.6. $a \in \{ (x, y) \mid x \in A \text{ and } y \in (B \cup C) \}$
   4.7. $a \in A \times (B \cup C)$
5. $a \in (A \times B) \cup (A \times C) \implies a \in A \times (B \cup C)$
6. $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$

And thus we have $A \times (B \cup C) = (A \times B) \cup (A \times C)$ □

b. $\boldsymbol{A \times (B \cap C) = (A \times B) \cap (A \times C)}$,

Skipped. We need to show that every element of $A \times (B \cap C)$ is in $(A \times B) \cap (B \times A)$, and vice versa. First we prove the forward direction. Let (a_1, a_2), such that $a_1 \in A$ and $(a_2 \in B$ and $a_2 \in C)$. So we can distribute this such that $(a_1 \in A$ and $a_2 \in B)$ and $(a_1 \in A$ and $a_2 \in C)$. Thus every element of $A \times (B \cap C)$ is in $(A \times B) \cap (A \times C)$. It remains to show the truth of the other direction.

Instead let $a \in (A \times B) \cap (A \times C)$. We have $a \in (A \times B)$ and $a \in (A \times C)$. Then $a_1 \in A$ and $a_2 \in B$ and $a_2 \in C$ as well. Then $a_2 \in B \cap C$, so $a \in A \times (B \cap C)$, completing our proof. □

c. and $\boldsymbol{A \times (B \setminus C) = (A \times B) \setminus (A \times C)}$. (One can of course prove similar identities in which the roles of the left and right factors of the Cartesian product are reversed.)

Proof. We want to first prove every element of A X (B - C) is in (A X B) - (A X C) and vice versa. We begin in the forward direction. Let a in A X (B - C). Then for the ordered pair (a1, a2), a1 is some element in A and a2 is some element in (B - C). a2 is in B, so (a1, a2) is in A X B a2 is not in C, so (a1, a2) is not in A X C. Then (a1, a2) is in A X B and not in A X C. So (a1, a2) is in (A X B) - (A x C) a is in (A X B) - (A x C) Now in the other direction Let a in (A X B) - (A X C) a in (A X B) and a not in (A X C) (a1 is some element in A and a2 is some element in B) and (not (a1 is some element in A and a2 is some element in C)) (a1 is some element in A and a2 is some element in B) and ((a1 is not in A) or a2 not in C) since a1 in A, a2 must not be in C. a1 in A, a2 in B, a2 not in C a1 in A a2 in (B - C) a in A X (B - C) We have proved both directions □

Exercise 3.5.7

Let X and Y be sets, and let $\pi_{X\times Y\to X} : X \times Y \to X$ and $\pi_{X\times Y\to Y} : X \times Y \to Y$ be the maps $\pi_{X\times Y\to X}(x, y) := x$ and $\pi_{X\times Y\to Y}(x, y) := y$; these maps are known as the coordinate functions on $X \times Y$. Show that for any functions $f : Z \to X$ and $g : Z \to Y$, there exists a unique function $h : Z \to X \times Y$ such that $\pi_{X\times Y\to X} \circ h = f$ and $\pi_{X\times Y\to Y} \circ h = g$. (Compare this to the last part of Exercise 3.3.8, and to Exercise 3.1.7.) This function h is known as the pairing of f and g and is denoted $h = (f, g)$.

Proof. Rewrite the definitions:

1. Let X and Y be arbitrary sets
2. Let $\pi_{X\times Y\to X} : X \times Y \to X$ be the map $\pi_{X\times Y\to X}(x, y) := x$
3. Let $\pi_{X\times Y\to Y} : X \times Y \to Y$ be the map $\pi_{X\times Y\to Y}(x, y) := y$
4. Let $f : Z \to X$ and $g : Z \to Y$ be arbitrary functions.

We first want to show $h : Z \to X \times Y$ exists.

5. Suppose

Next we want to show it is unique.

6.

Thus there exists a unique h that satisfies $\pi_{X\times Y\to X} \circ h = f$ and $\pi_{X\times Y\to Y} \circ h = g$. □

Unfinished

Exercise 3.5.8

Let $X_1, \ldots, X_n$ be sets. Show that the Cartesian product $\prod_{i=1}^{n} X_i$ is empty if and only if at least one of the X_i is empty.

Proof. We first show the forward direction.

1. By lemma 3.5.11 If each X_i is nonempty $\prod_{i=1}^{n} X_i$ is nonempty.
2. The contrapositive states $\prod_{i=1}^{n} X_i$ is empty implies some X_i is empty.

Let k st. X_k is empty. Suppose a in $\prod_{i=1}^{n} X_i$. Then $a_k \in X_k$. But $a_k \notin X_k$ since X_k is empty. Thus, a contradiction! Therefore for all a, we know $a \notin \prod_{i=1}^{n} X_i$. Thus $\prod_{i=1}^{n} X_i = \emptyset$. So if X_i for some i is empty, $\prod_{i=1}^{n} X_i$ is empty.

□

Exercise 3.5.9

Suppose that I and J are two sets, and for all $\alpha \in I$ let A_α be a set, and for all $\beta \in J$ let B_β be a set. Show that

$$\left(\bigcup_{\alpha\in I} A_\alpha\right) \cap \left(\bigcup_{\beta\in J} B_\beta\right) = \bigcup_{(\alpha,\beta)\in I\times J} (A_\alpha \cap B_\beta).$$

Proof. We are proving set equality, so each element of $(\bigcup_{\alpha\in I} A_\alpha)\cap(\bigcup_{\beta\in J} B_\beta)$ needs to be in $\bigcup_{(\alpha,\beta)\in I\times J}(A_\alpha\cap B_\beta)$ and vice versa.

1. Suppose $x \in (\bigcup_{\alpha\in I} A_\alpha) \cap (\bigcup_{\beta\in J} B_\beta)$
   1.1. $x \in (\bigcup_{\alpha\in I} A_\alpha)$
   1.2. $x \in (\bigcup_{\beta\in J} B_\beta)$
   1.3. Suppose $(i,j) \in I \times J$ such that $x \in A_i$ and $x \in B_j$
      1.3.1. $x \in A_i$ and $x \in B_j$
      1.3.2. $x \in (A_i \cap B_j)$
      1.3.3. $i \in I$ and $j \in J$
      1.3.4. $x \in \bigcup_{(\alpha,\beta)\in I\times J}(A_\alpha \cap B_\beta)$
   1.4. $x \in \bigcup_{(\alpha,\beta)\in I\times J}(A_\alpha \cap B_\beta)$ MP
2. $x \in (\bigcup_{\alpha\in I} A_\alpha) \cap (\bigcup_{\beta\in J} B_\beta) \implies x \in \bigcup_{(\alpha,\beta)\in I\times J}(A_\alpha \cap B_\beta)$

The reverse direction:

3. Suppose $x \in \bigcup_{(\alpha,\beta)\in I\times J}(A_\alpha \cap B_\beta)$
   3.1. $x \in (A_\alpha \cap B_\beta)$ for some $(\alpha,\beta) \in I \times J$
   3.2. Suppose $(i,j) \in I \times J$ and $x \in (A_i \cap B_j)$
      3.2.1. $x \in (A_i \cap B_j)$
      3.2.2. $x \in A_i$ and $x \in B_j$
   3.3. $x \in (\bigcup_{\alpha\in I} A_\alpha)$
   3.4. $x \in (\bigcup_{\beta\in J} B_\beta)$
   3.5. $x \in (\bigcup_{\alpha\in I} A_\alpha) \cap (\bigcup_{\beta\in J} B_\beta)$
4. $x \in \bigcup_{(\alpha,\beta)\in I\times J}(A_\alpha \cap B_\beta) \implies x \in (\bigcup_{\alpha\in I} A_\alpha) \cap (\bigcup_{\beta\in J} B_\beta)$

Then each has all the elements of the other, and they are equal. □

What happens if one interchanges all the union and intersection symbols here?

Answer.

Proof. □

Exercise 3.5.10

If $f : X \to Y$ is a function, define the graph of f to be the subset of $X \times Y$ defined by $\{(x, f(x)) : x \in X\}$.

a. Show that two functions $f : X \to Y$, $\tilde{f} : X \to Y$ are equal if and only if they have the same graph.

Proof. We begin with the forward direction.

1. Assume $f : X \to Y = g : X \to Y$.
2. The graph of $f = \{(x, f(x)) : x \in X\}$.
3. The graph of $g = \{(x, g(x)) : x \in X\}$.

We need to now show these two sets are equal, by showing all the elements of one are contained in the other and vice versa.

4. Suppose $a \in f$.
   4.1. $a \in \{(x, f(x)) : x \in X\}$
   4.2. $\exists x \in X \; a = (x, f(x))$
   4.3. Suppose $x \in X$ such that $a = (x, f(x))$
      4.3.1. $a = (x, f(x))$
      4.3.2. $f(x) = g(x)$
      4.3.3. $(x, f(x)) = (x, g(x))$
      4.3.4. $a = (x, g(x))$
   4.4. $a \in \{(x, g(x)) : x \in X\}$
   4.5. $a \in g$
5. The other way is basically the same.
6. $a \in f \implies a \in g$ and $a \in g \implies a \in f$ so they are equal.

Now the converse. We've got to show that the functions are the same, which mean they have the same domain and codomain, and each input has the same output.

7. $\forall x \in X$ ($\exists y \in Y$ such that $f(x) = y$) and ($\forall y_1, y_2, f(x) = y_1$ and $f(x) = y_2 \implies y_1 = y_2$)
8. Suppose $\{(x, f(x)) : x \in X\} = \{(x, g(x)) : x \in X\}$.
9. Let $x \in X$
   9.1. $(x, f(x)) \in \{(x, f(x)) : x \in X\}$ — by definition
   9.2. $(x, f(x)) \in \{(x, g(x)) : x \in X\}$ — line 8
   9.3. Suppose $(x, y) \in \{(x, g(x)) : x \in X\}$ and $f(x) \neq y$
      9.3.1. $g(x) = y$ — $(x, y) \in \{(x, g(x)) : x \in X\}$ and definition of graph

9.3.2. $g(x) = f(x)$ — $(x, f(x)) \in \{(x, g(x)) : x \in X\}$ and definition of graph

9.3.3. $f(x) = y$, a contradiction!

9.4. $f(x) = g(x)$

10. $\forall x\, f(x) = g(x)$

11. $f = g$

12. We already know f and g have the same domain and codomain.

13. Thus they are equal!

□

b. Conversely, if G is any subset of $X \times Y$ with the property that for each $x \in X$, the set $\{y \in Y : (x, y) \in G\}$ has exactly one element (or in other words, G obeys the vertical line test), show that there is exactly one function $f : X \to Y$ whose graph is equal to G.

Proof. □

c. Suppose we define a function f to be an ordered triple $f = (X, Y, G)$, where X, Y are sets, and G is a subset of $X \times Y$ that obeys the vertical line test. We then define the domain of such a triple to be X, the codomain to be Y and for every $x \in X$, we define $f(x)$ to be the unique $y \in Y$ such that $(x, y) \in G$. Show that this definition is compatible with Definition 3.3.1 in the sense that every choice of domain X, codomain Y, and property $P(x, y)$ obeying the vertical line test produces a function as defined here that obeys all the properties required of it in that definition, and is also similarly compatible with Definition 3.3.8.

Proof. □

3.6 Cartesian Products

Chapter 4

Integers and Rationals

4.1 The integers

Exercise 4.1.1

Verify that the definition of equality on the integers is both reflexive and symmetric.

Proof. We first prove that the definition of equality is reflexive. Let $a, b, c, d \in \mathbb{N}$. Let $(a—b) \in \mathbb{Z}$. We wish to prove $a—b = a—b$. By reflexivity for the natural numbers, $a + b = a + b$. The definition of equality for integers states $a—b = c—d \iff a + d = b + c$. Thus, $a—b = a—b$, as desired.

Next we prove the definition of equality is symmetric. Suppose $a, b, c, d \in \mathbb{N}$. We wish to prove $a—b = c—d \iff c—d = a—b$. The definition of equality for integers states $a—b = c—d \iff a + d = b + c$. We know by symmetry for the natural numbers that $a + d = b + c \iff b + c = a + d$. Thus, $a—b = c—d \iff c—d = a—b$, as desired. □

Exercise 4.1.3

Show that $(-1) \times a = -a$ for every integer a.

Proof. Let a in $\mathbb{Z}$. Allow a to be the integer $h - j$ for some h and j in $\mathbb{N}$. By integer negation, we can rewrite $(-1) \times a$ as $(0 - 1) \times (h - j)$. By integer multiplication, we have $(0h + 1j) - (0j + 1h)$, which by natural number multiplication is $(j - h)$. Meanwhile, by integer negation $-a$ is also $(j - h)$. Thus $(-1) \times a = -a$ by integer equality. □

Exercise 4.1.4

h. Let x, y, z be integers. Show that $x(y+z) = xy + xz$.

Proof. Let x be the integer $(a-b)$, y be $(c-d)$, and z be $(e-f)$.

$$\begin{aligned} x(y+z) &= (a-b)\big((c-d)+(e-f)\big) \\ &= (a-b)\big((c+e)-(d+f)\big) \\ &= (a(c+e)+b(d+f)) - (a(d+f)+b(c+e)) \\ &= (ac+ae+bd+bf) - (ad+af+bc+be); \end{aligned}$$

$$\begin{aligned} xy+xz &= (a-b)(c-d)+(a-b)(e-f) \\ &= \big((ac+bd)-(ad+bc)\big) + \big((ae+bf)-(af+be)\big) \\ &= ((ac+bd)+(ae+bf)) - ((ad+bc)+(af+be)) \\ &= (ac+ae+bd+bf) - (ad+af+bc+be) \end{aligned}$$

□

Exercise 4.1.5

Prove Proposition 4.1.8: Let a and b be integers such that $ab = 0$. Then either $a = 0$ or $b = 0$ (or both).

Proof. By the trichotomy of integers, a is equal to a positive natural number, zero, or the negation of a positive natural number. Likewise for b.

1. Suppose a is zero.
   1.1. $a = 0$ or $b = 0$
2. Suppose a is equal to a positive natural number n.
   2.1. Suppose b is zero
      2.1.1. $a = 0$ or $b = 0$
   2.2. Suppose b is equal to a positive natural number m.
      2.2.1. $n \times m = 0$
      2.2.2. $n = 0$ or $m = 0$ by Lemma 2.3.3
      2.2.3. $a = 0$ or $b = 0$
   2.3. Suppose b is the negation of a positive natural number m.
   2.4. $(n-0)(0-m) = 0$
   2.5. $(n \times 0 + 0 \times m) - (n \times m + 0 \times 0) = 0$
   2.6. $-(n \times m) = 0$
   2.7. $-(n \times m) = (0-0)$
   2.8. $(n \times m) = 0$ by Exercise 4.1.2.
   2.9. $n = 0$ or $m = 0$ by Lemma 2.3.3

2.10. $n = 0$ or $-m = 0$ by Exercise 4.1.2 again

2.11. $a = 0$ or $b = 0$

3. Suppose a is the negation of a positive natural number n.

3.1. Suppose b is zero

3.1.1. $a = 0$ or $b = 0$

3.2. Suppose b is equal to a positive natural number n.

3.2.1. By commutativity, same case as 2.3.

3.2.2. $a = 0$ or $b = 0$

3.3. Suppose b is the negation of a positive natural number m.

3.3.1. $(0 - n)(0 - m) = 0$

3.3.2. $(0 \times 0 + n \times m) - (0 \times m + 0 \times n) = 0$

3.3.3. $n \times m = 0$

3.3.4. $n = 0$ or $m = 0$

3.3.5. $(0 - n) = 0$ or $(0 - m) = 0$

3.3.6. $a = 0$ or $b = 0$

In all cases, $a = 0$ or $b = 0$.

By the trichotomy of integers a and b are either natural numbers or the negation of positive natural numbers. If a and b are equal to natural numbers, then $ab = 0 \implies a = 0$ or $b = 0$ by Lemma 2.3.3. If a is the negation of a positive natural number n and b is equal to a natural number, such that $(-n)b = 0$, then $ab = 0 \implies -n = 0$ or $b = 0$, but we said $-n \neq 0$ so $ab = 0 \implies b = 0$. Likewise if b is the negation of some positive natural number m such that $a(-m) = 0$ then $ab = 0 \implies a = 0$. If they are both negations of some positive natural numbers n and m then $nm \neq 0$ by Lemma 2.3.3, and thus $ab \neq 0$, so $ab = 0 \implies a = 0$ is vacuously true. (some mistakes around -n, need to use -n = -1(n) here)

□

Exercise 4.1.6

(Cancellation law for integers) If a, b, c are integers such that $ac = bc$ and c is non-zero, then $a = b$.

Proof. Let a, b, c be integers such that $ac = bc$ and c is non-zero.

1. $ac - bc = bc - bc$
2. $ac - bc = 0$
3. $ac + -(bc) = 0$
4. $ac + -1(bc) = 0$
5. $ac + (-1)(b)(c) = 0$
6. $(a + -1(b))c = 0$
7. $(a + -b)c = 0$
8. $(a - b)c = 0$
9. $a - b = 0$ Lemma 4.1.8, $c \neq 0$

10. $a - b + b = 0 + b$

11. $a = b$

□

Exercise 4.1.8

Show that the principle of induction (Axiom 2.5) does not apply directly to the integers. More precisely, give an example of a property $P(n)$ pertaining to an integer n such that $P(0)$ is true, and that $P(n)$ implies $P(n++)$ for all integers n, but that $P(n)$ is not true for all integers n. Thus induction is not as useful a tool for dealing with the integers as it is with the natural numbers. (The situation becomes even worse with the rational and real numbers, which we shall define shortly.)

Proof. Let $P(n)$ be the property $n \geq 0$ is true. Suppose $P(0)$, $0 = 0$ so $0 \geq 0$ is true. Assume inductively that $n \geq 0$ is true. Then there exists some $x \in \mathbb{N}$ such that $n = 0 + x$. By the definition of addition for integers $n + 1 = (n - 0) + (1 - 0)$. Then by substitution,

$$\begin{aligned} n + 1 &= ((0 + x) - 0) + (1 - 0) \\ &= (0 + x) + 1 \\ &= 0 + (x + 1) \\ &\geq 0 \end{aligned}$$

So $n \geq 0 \implies n + 1 \geq 0$ for all integers. However, if n is the negation of a positive natural number, then $-n > 0$, and thus $n < 0$, so $P(n)$ cannot be true for all n. □

Exercise 4.1.9

Show that the square of an integer is always a natural number. That is to say, prove that $n^2 \geq 0$ for every integer n.

Proof. By the trichotomy of integers, n is a positive natural number, 0, xor the negation of a positive natural number. A natural number multiplied by another natural number is a natural number. So if n is either 0 or a positive natural number, n^2 is a natural number. Each natural number is greater than or equal to 0, so $n^2 \geq 0$. Suppose n is the negation of positive natural number a, i.e. $n \times n = (0 - a) \times (0 - a)$. By applying the definition of multiplication, $(0 \times 0 + a \times a) - (0 \times a + a \times 0)$. Simplifying this we have $n^2 = (a \times a)$. Since a is a natural number, $(a \times a)$ is as well, so n^2 is a natural number again. Thus $n^2 \geq 0$ in all three cases. □

4.2 The rationals

Exercise 4.2.1

Show that the definition of equality for the rational numbers is reflexive, symmetric, and transitive. (Hint: for transitivity, use Corollary 4.1.9.)

Proof. We first prove reflexivity, which holds if $a//b = a//b$, $b \neq 0$. $ab = ab$ because reflexivity holds for the integers, and $ab = ab \iff a//b = a//b$ by definition.

Next we tackle symmetry, which holds if $a//b = c//d \iff c//d = a//b$. We have $a//b = c//d \iff ad = bc$, $b \neq 0$ and $d \neq 0$. We also have $c//d = a//b \iff cb = da$. By commutativity of multiplication on the integers $ad = da$ and $bc = cb$. So by symmetry for the integers $ad = bc \iff cb = ad$, and thus $a//b = c//d \iff c//d = a//b$.

Finally for transitivity, which holds if $a//b = c//d$ and $c//d = e//f$ implies $a//b = e//f$. We have $a//b = c//d \iff ad = bc$, we also have $c//d = e//f \iff cf = de$, where b, d, f are all nonzero. By the cancellation rule (Lemma 4.1.9), $adf = bcf$. Also by 4.1.9, $bcf = bde$. By transitivity for the integers, $adf = bde$, and by 4.1.9 again $af = be$, so $a//b = e//f$ as desired. ☐

Exercise 4.2.3

Prove the remaining components of Proposition 4.2.4.

I will only prove a subset of these, using $x = a//b$, $y = c//d$, and $z = e//f$ for integers a, c, e and nonzero integers b, d, f.

a. $(xy)z = x(yz)$

Proof.

$$\begin{aligned}
(xy)z &= ((a//b)(c//d))(e//f) \\
&= (ac//bd)(e//f) \\
&= (ace//bdf) \\
&= (a//b)(ce//df) \\
&= (a//b)((c//d)(e//f)) \\
&= x(yz)
\end{aligned}$$

☐

b. $x(y + z) = xy + xz$

Proof.

$$\begin{aligned}
xy + xz &= (a//b)(c//d) + (a//b)(e//f) \\
&= (ac//bd) + (ae//bf) \\
&= (acbf + aebd)//(bdbf) \\
&= ((acf + aed)//bdf) \\
&= (a(cf + ed)//bdf) \\
&= (a//b)((cf + de)//(df)) \\
&= (a//b)((c//d) + (e//f)) \\
&= x(y + z)
\end{aligned}$$

□

Exercise 4.2.5

Proposition 4.2.9 (Basic properties of order on the rationals) Let x, y, z be rational numbers. Then the following properties hold.

Let $x = a/b$, $y = c/d$, and $z = e/f$ for integers a, b, c, d, e, f where b, d, f are nonzero.

a. (Order trichotomy) Exactly one of the three statements $x = y$, $x < y$, or $x > y$ is true.

Proof. $x < y \implies x \neq y$

$x > y \implies x \neq y$

Then by contrapositive $x = y$ implies both $x < y$ and $x > y$ are false.

$x > y \iff x - y$ is positive

$x < y \iff x - y$ is negative

By the trichotomy of rationals $x - y$ cannot be both positive and negative. So $x > y$ and $x < y$ are exclusive. □

b. (Order is antisymmetric) One has $x < y$ if and only if $y > x$.

Proof. We have two directions to prove.

Suppose $x < y$. By def of order on rationals $x - y$ is negative, i.e $(ad - bc)/bd$ is negative. Then by def of negative on rationals $-(ad - bc)$ and bd are both positive integers. $-(ad - bc) = (cb - da)$ by negation and commutativity. Therefore $(cb - da)/bd$ is a positive rational, which is equal to $(y - x)$. Thus, $y > x$.

Suppose $y > x$. Then $y - x$ is positive, i.e. $(cb - da)/db$ is positive. Then $(cb - da)$ and (db) are both positive integers. If $(cb - da)$ is positive, its negation $-(cb - da)$ is negative. We know $-(cb - da) = (ad - bc)$ by negation and commutativity. Then $(ad - bc)/db$ is a negative rational. Then so is $(x - y)$. Thus, $x < y$. □

c. (Order is transitive) If $x < y$ and $y < z$, then $x < z$.

Lemma. First we need to prove $(-a)/b = a/(-b)$, so we can write the proof without loss of generality. $(-a)/b = a/(-b) \iff -a(-b) = ab$. We can prove this in either direction.

$$\begin{aligned} -a(-b) &= (-1)(-1)(ab) \\ &= (0 - - - 1)(0 - - - 1)(ab) \\ &= ((0 * 0 + 1 * 1) - (0 * 1 + 1 * 0))(ab) \\ &= ((0 + 1) - (0 + 0))(ab) \\ &= (1 - 0)(ab) \\ &= 1(ab) \\ &= ab. \end{aligned}$$

□

Lemma. We wish to prove $a/b < c/d \iff ad < bc$, and $b, d \neq 0$. From above, we can assume without loss of generality that $b > 0$ and $d > 0$ (otherwise choose $-a/-b$ and $-c/-d$). Suppose $a/b < c/d$. We know $a/b < c/d \iff (a/b - c/d)$ is a negative rational number. By definition of subtration $(ad - bc)/bd$ is negative. By our generality assumption, bd is positive, so $(ad - bc)$ is negative. Suppose $ad = bc$, then $(ad - bc) = 0$, so by contradiction $ad \neq bc$. There must be some m such that $(ad - bc) + m = 0$, so there exists some nonzero m such that $ad + m = bc$. Thus by the previous two statements, $ad < bc$. Suppose $ad < bc$. Then $ad + m = bc$, $m \neq 0$, and $ad - bc = 0 - m$, or $(ad - bc)$ is negative. Since we have assumed bd is positive, the rational $(ad - bc)/bd$ is negative. $(ad - bc)/bd = (a/b - c/d)$, so the latter is negative too. Thus $a/b < c/d$.

□

Proof. Without loss of generality, assume $b, d, f > 0$. (If $b > 0$, then set $a' = a, b' = b$; otherwise set $a' = -a, b' = -b$, and note that in either case $x = a'/b'$ and $b' > 0$.) Likewise for y, z.

1. Suppose $x < y$ and $y < z$.
2. Then $ad < bc$ by above lemma.
3. Likewise, $cf < de$.
4. $adf < bcf$ since we assume f is positive, and Lemma 4.1.11c (positive multiplication preserves order on integers).
5. $bcf < deb$ since we assume b is positive.
6. $adf < deb$ by transitivity of order on integers.
7. $af < eb$ since we assume d is positive.
8. $x < z$.

□

d. (Addition preserves order) If $x < y$, then $x + z < y + z$.

e. (Positive multiplication preserves order) If $x < y$ and z is positive, then $xz < yz$.

4.3 Absolute value and exponentiation

Exercise 4.3.3

Let x, y be rational numbers and let n, m be natural numbers.

b. **Suppose $n > 0$. Then we have $x^n = 0$ if and only if $x = 0$.**

Proof. We prove by induction.

Suppose $n = 1$ as a base case. Since $x = x^1$, $x = 0 \iff x^n = 0$.

Now suppose $x = 0 \iff x^n = 0$. We have $x^{n+1} = x^n * x$. Now $x = 0$ or $x \neq 0$.

If $x = 0$, $x^n = 0$ by inductive assumption. Then $x^{n+1} = 0 * 0 = 0$, so $x = 0 \implies x^{n+1} = 0$.

If instead $x \neq 0$, $x^n \neq 0$ we have $x^{n+1} \neq 0$, since the product of two nonzero rationals is a nonzero rational. (Extension of 4.1.8 that I haven't proved)

Thus we have $x^{n+1} = 0 \iff x = 0$, closing the induction, and so for all x, for all $n > 0$, we have $x^n = 0$ if and only if $x = 0$. □

4.4 Gaps in the rationals