Analysis I: Exercises

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3.4 Images and inverse images

Exercise 3.4.1

Let $f\colon X \to Y$ be a bijective function, and let $f^{-1}\colon Y \to X$ be its inverse. Let V be any subset of Y. Prove that the forward image of V under f^{-1} is the same set as the inverse image of V under f; thus the fact that both sets are denoted by $f^{-1}(V)$ will not lead to any inconsistency.

Answer:

Proof. **Scratchpad:**

$\square$

Exercise 3.4.2

Let $f\colon X \to Y$ be a function from one set X to another set Y, let S be a subset of X, and let U be a subset of Y.

- **i. What, in general, can one say about $f^{-1}(f(S))$ and S?**

  Answer:

  Proof. **Scratchpad:**

  $\square$

- **ii. What about $f(f^{-1}(U))$ and U?**

  Answer:

  Proof. **Scratchpad:**

  $\square$

iii. What about $f^{-1}(f(f^{-1}(U)))$ and $f^{-1}(U)$?

Answer:

Proof. **Scratchpad:**

□

Exercise 3.4.3

Let A, B be two subsets of a set X, and let $f\colon X \to Y$ be a function. Show that

i. $f(A \cap B) \subseteq f(A) \cap f(B)$**,**

Proof. **Scratchpad:**

□

ii. $f(A) \setminus f(B) \subseteq f(A \setminus B)$**,**

Proof. **Scratchpad:**

□

iii. $f(A \cup B) = f(A) \cup f(B)$**.**

Proof. **Scratchpad:**

□

For the first two statements, is it true that the $\subseteq$ relation can be improved to $=$?

Answer:

Proof. □

Exercise 3.4.5

Let $f\colon X \to Y$ be a function from one set X to another set Y.

i. Show that $f(f^{-1}(S)) = S$ for every $S \subseteq Y$ if and only if f is surjective.

Proof. **Scratchpad:**

□

ii. Show that $f^{-1}(f(S)) = S$ for every $S \subseteq X$ if and only if f is injective.

Proof. **Scratchpad:**

□

Exercise 3.4.9

Show that if β and β' are two elements of a set I, and to each $\alpha \in I$ we assign a set A_α, then

$$\{x \in A_\beta : x \in A_\alpha \textbf{ for all } \alpha \in I\} = \{x \in A_{\beta'} : x \in A_\alpha \textbf{ for all } \alpha \in I\},$$

and so the definition of $\bigcap_{\alpha \in I} A_\alpha$ defined in (3.3) does not depend on β.

Proof. **Scratchpad:**

□

Also explain why (3.4) is true.

Proof. **Scratchpad:**

□

Exercise 3.4.10

Suppose that I and J are two sets, and for all $\alpha \in I \cup J$ let A_α be a set. Show that

$$\bigcup_{\alpha \in I} A_\alpha \cup \bigcup_{\alpha \in J} A_\alpha = \bigcup_{\alpha \in I \cup J} A_\alpha.$$

Proof. **Scratchpad:**

□

If I and J are non-empty, show that

$$\bigcap_{\alpha \in I} A_\alpha \cap \bigcap_{\alpha \in J} A_\alpha = \bigcap_{\alpha \in I \cup J} A_\alpha.$$

Proof. **Scratchpad:**

□