Analysis I: Exercises
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3.5 Cartesian Products

Exercise 3.5.2

Suppose we define an ordered n-tuple to be a surjective function $x : \{ i \in \mathbb{N} \mid 1 \le i \le n \} \to X$ whose codomain is some arbitrary set X (so different ordered n-tuples are allowed to have different ranges); we then write x_i for $x(i)$ and also write x as $(x_i)_{1 \le i \le n}$. Using this definition, verify that we have $(x_i)_{1 \le i \le n} = (y_i)_{1 \le i \le n}$ if and only if $x_i = y_i$ for all $1 \le i \le n$.

Proof. We are asked to prove that the n-tuples x and y are equal for all $1 \le i \le n$ by showing that their components x_i and y_i are equal for all $1 \le i \le n$, and vice versa. We rewrite our definitions for clarity:

1. Let X be any set.
2. Let $x : \{ i \in \mathbb{N} \mid 1 \le i \le n \} \to X$
3. Let $x_i = x(i)$
4. Let $x = (x_i)_{1 \le i \le n}$

We proceed in the forward direction.

5. Suppose $(x_i)_{1 \le i \le n} = (y_i)_{1 \le i \le n}$.
   - 5.1. $x = y$ — line 4
   - 5.2. Suppose $1 \le i \le n$
     - 5.2.1. $i \in \{ i \in \mathbb{N} \mid 1 \le i \le n \}$
     - 5.2.2. $x(i) = y(i)$ — line 5.1
     - 5.2.3. $x_i = y_i$ — line 3
   - 5.3. $\forall i \in \mathbb{N} \quad 1 \le i \le n \implies x_i = y_i$
6. $(x_i)_{1 \le i \le n} = (y_i)_{1 \le i \le n} \implies \big(\forall i \in \mathbb{N} \quad 1 \le i \le n \implies x_i = y_i\big)$.

Now in the reverse direction:

7. Suppose $i \in \mathbb{N}$ st. $1 \le i \le n \implies x_i = y_i$.
   - 7.1. $i \in \{ i \in \mathbb{N} \mid 1 \le i \le n \}$ — line 2
   - 7.2. $x_i = y_i$ — slightly awkward MP
   - 7.3. $(x)_{1 \le i \le n}(i) = y_{1 \le i \le n}(i)$ — lines 3, 4
8. $\forall i \in \mathbb{N} \quad \big(1 \le i \le n \implies x_i = y_i\big) \implies (x)_{1 \le i \le n}(i) = y_{1 \le i \le n}(i)$

Thus they are equal. □

Note: Didn't use the surjectivity in the proof at all?

Also, show that if $(X_i)1 \leq i \leq n$ are an ordered n-tuple of sets, then the Cartesian product, as defined in Definition 3.5.6, is indeed a set. (Hint: use Exercise 3.4.7 and the axiom of specification.)

Proof. □

Unsure how to approach.

Exercise 3.5.4

Let A, B, C be sets. Show that:

a. $A \times (B \cup C) = (A \times B) \cup (A \times C)$,

Proof. We need to show every element of $A \times (B \cup C)$ is also in $(A \times B) \cup (A \times C)$ and vice versa. We first prove the forward direction:

1. Suppose $a \in A \times (B \cup C)$
   1.1. $a \in \{ (x, y) \mid x \in A \text{ and } y \in (B \cup C) \}$
   1.2. $a \in \{ (x, y) \mid x \in A \text{ and } y \in B \text{ or } y \in C \}$
   1.3. $a \in \{ (x, y) \mid x \in A \text{ and } y \in B \text{ or } y \in C \}$
   1.4. $a \in \{ (x, y) \mid (x \in A \text{ and } y \in B) \text{ or } (x \in A \text{ and } y \in C) \}$
   1.5. $a \in (x, y)$ and $\big((a_1 \in A \text{ and } a_2 \in B) \text{ or } (a_1 \in A \text{ and } a_2 \in C)\big)$
   1.6. $\big(a \in (x, y) \text{ and } (a_1 \in A \text{ and } a_2 \in B)\big)$ or $\big(a \in (x, y) \text{ and } (a_1 \in A \text{ and } a_2 \in C)\big)$
   1.7. Case $a \in (x, y)$ and $(a_1 \in A \text{ and } a_2 \in B)$
      1.7.1. $a \in \{ (x, y) \mid x \in A \text{ and } y \in B \}$
      1.7.2. $a \in (A \times B)$
   1.8. Case $a \in (x, y)$ and $(a_1 \in A \text{ and } a_2 \in C)$
      1.8.1. $a \in \{ (x, y) \mid x \in A \text{ and } y \in C \}$
      1.8.2. $a \in (A \times C)$
   1.9. $a \in (A \times B)$ or $a \in (A \times C)$
   1.10. $a \in (A \times B) \cup (A \times C)$
2. $a \in A \times (B \cup C) \implies a \in (A \times B) \cup (A \times C)$
3. $A \times (B \cup C) \subseteq (A \times B) \cup (A \times C)$

Now this is mostly invertible:

4. Suppose $a \in (A \times B) \cup (A \times C)$
   4.1. $a \in (A \times B)$ or $a \in (A \times C)$
   4.2. Case $a \in (A \times C)$
      4.2.1. $a \in \{ (x, y) \mid x \in A \text{ and } y \in B \}$
      4.2.2. $a \in (x, y)$ and $(a_1 \in A \text{ and } a_2 \in B)$
   4.3. Case $a \in (A \times B)$
      4.3.1. $a \in \{ (x, y) \mid x \in A \text{ and } y \in C \}$

4.3.2. $a \in (x, y)$ and $(a_1 \in A$ and $a_2 \in C)$

4.4. $\big(a \in (x, y)$ and $(a_1 \in A$ and $a_2 \in B)\big)$ or $\big(a \in (x, y)$ and $(a_1 \in A$ and $a_2 \in C)\big)$

4.5. $a \in (x, y)$ and $\big((a_1 \in A$ and $a_2 \in B)$ or $(a_1 \in A$ and $a_2 \in C)\big)$

4.6. $a \in \{\, (x, y) \mid (x \in A$ and $y \in B)$ or $(x \in A$ and $y \in C) \,\}$

4.7. $a \in \{\, (x, y) \mid x \in A$ and $y \in B$ or $y \in C \,\}$

4.8. $a \in \{\, (x, y) \mid x \in A$ and $y \in (B \cup C) \,\}$

5. $a \in (A \times B) \cup (A \times C) \implies a \in A \times (B \cup C)$

6. $(A \times B) \cup (A \times C) \subseteq A \times (B \cup C)$

And thus we have $A \times (B \cup C) = (A \times B) \cup (A \times C)$ $\square$

A little confused, since it seems like $a \in (x, y)$ should be $a \in \{(x, y)\}$

b. $A \times (B \cup C) = (A \times B) \cup (A \times C)$,

Skipped. $\square$

c. $A \times (B \cap C) = (A \times B) \cap (A \times C)$,

Skipped. $\square$

d. and $A \times (B \setminus C) = (A \times B) \setminus (A \times C)$. (One can of course prove similar identities in which the roles of the left and right factors of the Cartesian product are reversed.)

Skipped. $\square$

Exercise 3.5.7

Let X and Y be sets, and let $\pi_{X \times Y \to X} : X \times Y \to X$ and $\pi_{X \times Y \to Y} : X \times Y \to Y$ be the maps $\pi_{X \times Y \to X}(x, y) := x$ and $\pi_{X \times Y \to Y}(x, y) := y$; these maps are known as the coordinate functions on $X \times Y$. Show that for any functions $f : Z \to X$ and $g : Z \to Y$, there exists a unique function $h : Z \to X \times Y$ such that $\pi_{X \times Y \to X} \circ h = f$ and $\pi_{X \times Y \to Y} \circ h = g$. (Compare this to the last part of Exercise 3.3.8, and to Exercise 3.1.7.) This function h is known as the pairing of f and g and is denoted $h = (f, g)$.

Proof. Rewrite the definitions:

1. Let X and Y be arbitrary sets
2. Let $\pi_{X \times Y \to X} : X \times Y \to X$ be the map $\pi_{X \times Y \to X}(x, y) := x$
3. Let $\pi_{X \times Y \to Y} : X \times Y \to Y$ be the map $\pi_{X \times Y \to Y}(x, y) := y$
4. Let $f : Z \to X$ and $g : Z \to Y$ be arbitrary functions

We first want to show $h : Z \to X \times Y$ exists.

5.

Next we want to show it is unique.

6.

Thus there exists a unique h that satisfies $\pi_{X\times Y\to X}\circ h = f$ and $\pi_{X\times Y\to Y}\circ h = g$. ☐

Exercise 3.5.8

Let $X_1, \ldots, X_n$ be sets. Show that the Cartesian product $\prod_{i=1}^n X_i$ is empty if and only if at least one of the X_i is empty.

Proof. ☐

Exercise 3.5.9

Suppose that I and J are two sets, and for all $\alpha \in I$ let A_α be a set, and for all $\beta \in J$ let B_β be a set. Show that

$$\left(\bigcup_{\alpha\in I} A_\alpha\right) \cap \left(\bigcup_{\beta\in J} B_\beta\right) = \bigcup_{(\alpha,\beta)\in I\times J} (A_\alpha \cap B_\beta).$$

Proof. ☐

What happens if one interchanges all the union and intersection symbols here?

Answer.

Proof. ☐

Exercise 3.5.10

If $f : X \to Y$ is a function, define the graph of f to be the subset of $X \times Y$ defined by $\{(x, f(x)) : x \in X\}$.

a. **Show that two functions $f : X \to Y$, $\tilde{f} : X \to Y$ are equal if and only if they have the same graph.**

   Proof. ☐

b. Conversely, if G is any subset of $X \times Y$ with the property that for each $x \in X$, the set $\{y \in Y : (x, y) \in G\}$ has exactly one element (or in other words, G obeys the vertical line test), show that there is exactly one function $f : X \to Y$ whose graph is equal to G.

Proof. $\square$

c. Suppose we define a function f to be an ordered triple $f = (X, Y, G)$, where X, Y are sets, and G is a subset of $X \times Y$ that obeys the vertical line test. We then define the domain of such a triple to be X, the codomain to be Y and for every $x \in X$, we define $f(x)$ to be the unique $y \in Y$ such that $(x, y) \in G$. Show that this definition is compatible with Definition 3.3.1 in the sense that every choice of domain X, codomain Y, and property $P(x, y)$ obeying the vertical line test produces a function as defined here that obeys all the properties required of it in that definition, and is also similarly compatible with Definition 3.3.8.

Proof. $\square$