Analysis I: Exercises
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4.2 The rationals

Exercise 4.2.1

Show that the definition of equality for the rational numbers is reflexive, symmetric, and transitive. (Hint: for transitivity, use Corollary 4.1.9.)

Proof. We first prove reflexivity, which holds if $a//b = a//b$, $b \neq 0$. $ab = ab$ because reflexivity holds for the integers, and $ab = ab \iff a//b = a//b$ by definition.

Next we tackle symmetry, which holds if $a//b = c//d \iff c//d = a//b$. We have $a//b = c//d \iff ad = bc$, $b \neq 0$ and $d \neq 0$. We also have $c//d = a//b \iff cb = da$. By commutativity of multiplication on the integers $ad = da$ and $bc = cb$. So by symmetry for the integers $ad = bc \iff cb = ad$, and thus $a//b = c//d \iff c//d = a//b$.

Finally for transitivity, which holds if $a//b = c//d$ and $c//d = e//f$ implies $a//b = e//f$. We have $a//b = c//d \iff ad = bc$, we also have $c//d = e//f \iff cf = de$, where b, d, f are all nonzero. By the cancellation rule (Lemma 4.1.9), $adf = bcf$. Also by 4.1.9, $bcf = bde$. By transitivity for the integers, $adf = bde$, and by 4.1.9 again $af = be$, so $a//b = e//f$ as desired. □

Exercise 4.2.3

Prove the remaining components of Proposition 4.2.4.

I will only prove a subset of these, using $x = a//b$, $y = c//d$, and $z = e//f$ for integers a, c, e and nonzero integers b, d, f.

a. $(xy)z = x(yz)$

Proof.

$$\begin{aligned}
(xy)z &= ((a//b)(c//d))(e//f) \\
&= (ac//bd)(e//f) \\
&= (ace//bdf) \\
&= (a//b)(ce//df) \\
&= (a//b)((c//d)(e//f)) \\
&= x(yz)
\end{aligned}$$

$\square$

b. $x(y+z) = xy + xz$

Proof.

$$\begin{aligned}
xy + xz &= (a//b)(c//d) + (a//b)(e//f) \\
&= (ac//bd) + (ae//bf) \\
&= (acbf + aebd)//(bdbf) \\
&= ((acf + aed)//bdf) \\
&= (a(cf + ed)//bdf) \\
&= (a//b)((cf + de)//(df)) \\
&= (a//b)((c//d) + (e//f)) \\
&= x(y+z)
\end{aligned}$$

$\square$

Exercise 4.2.5

Proposition 4.2.9 (Basic properties of order on the rationals) Let x, y, z be rational numbers. Then the following properties hold.

Let $x = a/b$, $y = c/d$, and $z = e/f$ for integers a, b, c, d, e, f where b, d, f are nonzero.

a. (Order trichotomy) Exactly one of the three statements $x = y$, $x < y$, or $x > y$ is true.

Proof. $x < y \implies x \neq y$

$x > y \implies x \neq y$

Then by contrapositive $x = y$ implies both $x < y$ and $x > y$ are false.

$x > y \iff x - y$ is positive

$x < y \iff x - y$ is negative

By the trichotomy of rationals $x - y$ cannot be both positive and negative. So $x > y$ and $x < y$ are exclusive. $\square$

b. (Order is antisymmetric) One has $x < y$ if and only if $y > x$.

Proof. We have two directions to prove.

Suppose $x < y$. By def of order on rationals $x - y$ is negative, i.e $(ad - bc)/bd$ is negative. Then by def of negative on rationals $-(ad - bc)$ and bd are both positive integers. $-(ad - bc) = (cb - da)$ by negation and commutativity. Therefore $(cb - da)/bd$ is a positive rational, which is equal to $(y - x)$. Thus, $y > x$.

Suppose $y > x$. Then $y - x$ is positive, i.e. $(cb - da)/db$ is positive. Then $(cb - da)$ and (db) are both positive integers. If $(cb - da)$ is positive, its negation $-(cb - da)$ is negative. We know $-(cb-da) = (ad-bc)$ by negation and commutativity. Then $(ad-bc)/db$ is a negative rational. Then so is $(x - y)$. Thus, $x < y$. □

c. (Order is transitive) If $x < y$ and $y < z$, then $x < z$.

Lemma. First we need to prove $(-a)/b = a/(-b)$, so we can write the proof without loss of generality. $(-a)/b = a/(-b) \iff -a(-b) = ab$. We can prove this in either direction.

$$\begin{aligned}
-a(-b) &= (-1)(-1)(ab) \\
&= (0 - - - 1)(0 - - - 1)(ab) \\
&= ((0 * 0 + 1 * 1) - (0 * 1 + 1 * 0))(ab) \\
&= ((0 + 1) - (0 + 0))(ab) \\
&= (1 - 0)(ab) \\
&= 1(ab) \\
&= ab.
\end{aligned}$$

□

Lemma. We wish to prove $a/b < c/d \iff ad < bc$, and $b, d \neq 0$. From above, we can assume without loss of generality that $b > 0$ and $d > 0$ (otherwise choose $-a/-b$ and $-c/-d$). Suppose $a/b < c/d$. We know $a/b < c/d \iff (a/b-c/d)$ is a negative rational number. By definition of subtration $(ad-bc)/bd$ is negative. By our generality assumption, bd is positive, so $(ad - bc)$ is negative. Suppose $ad = bc$, then $(ad - bc) = 0$, so by contradiction $ad \neq bc$. There must be some m such that $(ad - bc) + m = 0$, so there exists some nonzero m such that $ad + m = bc$. Thus by the previous two statements, $ad < bc$. Suppose $ad < bc$. Then $ad + m = bc$, $m \neq 0$, and $ad - bc = 0 - m$, or $(ad - bc)$ is negative. Since we have assumed bd is positive, the rational $(ad - bc)/bd$ is negative. $(ad - bc)/bd = (a/b - c/d)$, so the latter is negative too. Thus $a/b < c/d$.

□

Proof. Without loss of generality, assume $b, d, f > 0$. (If $b > 0$, then set $a' = a, b' = b$; otherwise set $a' = -a, b' = -b$, and note that in either case $x = a'/b'$ and $b' > 0$.) Likewise for y, z.

1. Suppose $x < y$ and $y < z$.
2. Then $ad < bc$ by above lemma.
3. Likewise, $cf < de$.
4. $adf < bcf$ since we assume f is positive, and Lemma 4.1.11c (positive multiplication preserves order on integers).
5. $bcf < deb$ since we assume b is positive.
6. $adf < deb$ by transitivity of order on integers.
7. $af < eb$ since we assume d is positive.
8. $x < z$.

$\square$

d. (Addition preserves order) If $x < y$, then $x + z < y + z$.

e. (Positive multiplication preserves order) If $x < y$ and z is positive, then $xz < yz$.