# Solar Radiation Pressure N-Plates Math Specification

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**Abstract**

This document provides the mathematical specifications Solar Radiation Pressure force , as implemented in GMAT.

## Contents

# 1 Solar Radiation Pressure (SRP) Force

Solar radiation pressure (SRP) force applying to an object in inertial frame is generally expressed as the equation below. In this study, the object is a spacecraft.

$$\mathbf{F}_I = -\nu.\frac{\Phi}{c}.\left[\frac{1AU}{\|\mathbf{s}_I\|}\right]^2.\mathbf{A}_I \tag{1}$$

| | |
|---|---|
| $\mathbf{F}_I$ | SRP force vector in the spacecraft's inertial frame |
| $\mathbf{A}_I$ | reflectance vector in the spacecraft's inertial frame |
| $c$ | speed of light |
| $\mathbf{s}_I$ | vector from the spacecraft to the Sun (briefly the Sun vector) in inertial frame |
| $\Phi$ | mean solar radiation pressure flux at 1AU |
| $\nu$ | percentage of of sunlight due to umbra, penumbra, or eclipse |

Where:

$$\mathbf{s}_I = \mathbf{r}\left(t_0\right)_{Sun} - \mathbf{r}\left(t_1\right)_{sc} \tag{2}$$

| | |
|---|---|
| $\mathbf{r}\left(t_0\right)_{Sun}$ | Sun's position at time $t_0$when the photons beam departs from the Sun |
| $\mathbf{r}\left(t_1\right)_{sc}$ | spacecraft's position at time $t_1$when the photons beam arrives to the spacecraft surface |

Convert reflectance vector from inertial frame to the spacecraft's body fixed frame

$$\mathbf{A}_I = [M]^T\mathbf{A} \tag{3}$$

| | |
|---|---|
| $\mathbf{A}$ | reflectance vector in the spacecraft's body fixed frame |
| $[M]$ | rotation matrix from the inertial frame to the spacecraft's body fixed frame |
| $[M]^T$ | transpose of [M], rotation matrix from the body fixed frame to the inertial frame |

The spacecraft's body fixed frame (B frame) has its origin at the center of mass of the spacecraft and its axes are defined by a set of three unit vectors in the spacecraft's inertial frame $< \hat{\mathbf{x}}_B, \hat{\mathbf{y}}_B, \hat{\mathbf{z}}_B >$.

$$\begin{aligned}\hat{\mathbf{z}}_B &= -\frac{\mathbf{r}}{\|\mathbf{r}\|}\\ \hat{\mathbf{y}}_B &= \hat{\mathbf{z}}_B \times \left(\frac{\mathbf{v}}{\|\mathbf{v}\|}\right)\\ \hat{\mathbf{x}}_B &= \hat{\mathbf{y}}_B \times \hat{\mathbf{z}}_B\end{aligned} \tag{4}$$

Rotation matrix $[M]^T$ is specified based on vectors $\hat{\mathbf{x}}_B$, $\hat{\mathbf{y}}_B$, and $\hat{\mathbf{z}}_B$.

$$[M]^T = \begin{bmatrix}\hat{\mathbf{x}}_{B_{3x1}} & \hat{\mathbf{y}}_{B_{3x1}} & \hat{\mathbf{z}}_{B_{3x1}}\end{bmatrix}_{3x3} \tag{5}$$

Where: $\mathbf{r}$ and $\mathbf{v}$ are position and velocity of the spacecraft in the inertial frame.
Let:

$$K = -\nu.\frac{\Phi}{c}.\left[\frac{1AU}{\|\mathbf{r}_S\|}\right]^2 \tag{6}$$

Substitute equations 6 and 3 to equation 1 to obtain SRP force with the spacecraft's reflectance vector presenting in the spacecraft's body fixed frame.

$$\mathbf{F}_I = K[M]^T\mathbf{A} \tag{7}$$

## 2 SRP Force Partial Derivative

Apply Derivative by Parts rule to SRP force in equation 7, we obtain

$$\frac{\partial \mathbf{F}_I}{\partial \mathbf{X}} = \left[\frac{\partial K}{\partial \mathbf{X}}[M]^T + K\frac{\partial [M]^T}{\partial \mathbf{X}}\right]\mathbf{A} + K[M]^T\frac{\partial \mathbf{A}}{\partial \mathbf{X}} \tag{8}$$

Where: $\mathbf{X}$ is a solve-for vector. Element in solve-for vector could be the state of spacecraft, drag or pressure coefficients, etc. More details will be shown in section 4.

The quantity K in this equation is a function of the Sun vector $\mathbf{r}_S$ and percentage of sunlight $\nu$, so value of K and its partial derivative $\dfrac{\partial K}{\partial \mathbf{X}}$ are specified from $\mathbf{r}_S$ and $\nu$. Matrix $[M]^T$ and partial derivative $\dfrac{\partial [M]^T}{\partial \mathbf{X}}$ are calculated based on equation 5 and equation 9 respectively.

The spacecraft's reflectance vector $\mathbf{A}$ in the body fixed frame and its partial derivative $\dfrac{\partial \mathbf{A}}{\partial \mathbf{X}}$ are specified as shown in section 3 and section 4 respectively.

### 2.1 Derivative of matrix $[M]^T$

Matrix $[M]^T$ is a function of spacecraft state. Therefore its partial derivative with respect to any variable other than position and velocity of spacecraft has to be zero. Therefore, we only need to specify its partial derivative w.r.t. spacecraft position $\mathbf{r}$ and velocity $\mathbf{v}$. The derivative of matrix $[M]^T$ is a 3-dimension matrix (Appendix A.6).

$$\frac{\partial}{\partial \mathbf{X}}[M]^T = \left[\frac{\partial \hat{\mathbf{x}}_B}{\partial \mathbf{X}} \quad \frac{\partial \hat{\mathbf{y}}_B}{\partial \mathbf{X}} \quad \frac{\partial \hat{\mathbf{z}}_B}{\partial \mathbf{X}}\right] \tag{9}$$

Where $\dfrac{\partial \hat{\mathbf{x}}_B}{\partial \mathbf{X}}$, $\dfrac{\partial \hat{\mathbf{y}}_B}{\partial \mathbf{X}}$, and $\dfrac{\partial \hat{\mathbf{z}}_B}{\partial \mathbf{X}}$ are 2-dimension matrices. They are specified by the following terms.

$$\frac{\partial \hat{\mathbf{z}}_B}{\partial \mathbf{r}} = -\frac{\partial}{\partial \mathbf{r}}\left(\frac{\mathbf{r}}{\|\mathbf{r}\|}\right) = -\frac{1}{\|\mathbf{r}\|}.[I]_{3x3} + \frac{1}{\|\mathbf{r}\|^3}.\left(\mathbf{r}.\mathbf{r}^T\right) \tag{10}$$

$$\frac{\partial \hat{\mathbf{z}}_B}{\partial \mathbf{v}} = -\frac{\partial}{\partial \mathbf{v}}\left(\frac{\mathbf{r}}{\|\mathbf{r}\|}\right) = [0]_{3x3} \tag{11}$$

$$\frac{\partial \hat{\mathbf{y}}_B}{\partial \mathbf{r}} = \frac{\partial}{\partial \mathbf{r}}\left[\hat{\mathbf{z}}_\mathbf{B} \times \left(\frac{\mathbf{v}}{\|\mathbf{v}\|}\right)\right] = \frac{\partial \hat{\mathbf{z}}_B}{\partial \mathbf{r}} \times \left(\frac{\mathbf{v}}{\|\mathbf{v}\|}\right) + \hat{\mathbf{z}}_\mathbf{B} \times \frac{\partial}{\partial \mathbf{r}}\left(\frac{\mathbf{v}}{\|\mathbf{v}\|}\right) = \frac{\partial \hat{\mathbf{z}}_B}{\partial \mathbf{r}} \times \left(\frac{\mathbf{v}}{\|\mathbf{v}\|}\right) \tag{12}$$

$$\frac{\partial \hat{\mathbf{y}}_B}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}}\left[\hat{\mathbf{z}}_\mathbf{B} \times \left(\frac{\mathbf{v}}{\|\mathbf{v}\|}\right)\right] = \frac{\partial \hat{\mathbf{z}}_B}{\partial \mathbf{v}} \times \left(\frac{\mathbf{v}}{\|\mathbf{v}\|}\right) + \hat{\mathbf{z}}_\mathbf{B} \times \frac{\partial}{\partial \mathbf{v}}\left(\frac{\mathbf{v}}{\|\mathbf{v}\|}\right) = \hat{\mathbf{z}}_B \times \left[\frac{1}{\|\mathbf{v}\|}.[I]_{3x3} - \frac{1}{\|\mathbf{v}\|^3}.\left(\mathbf{v}.\mathbf{v}^T\right)\right] \tag{13}$$

$$\frac{\partial \hat{\mathbf{x}}_B}{\partial \mathbf{r}} = \frac{\partial}{\partial \mathbf{r}}\left(\hat{\mathbf{y}}_B \times \hat{\mathbf{z}}_B\right) = \frac{\partial \hat{\mathbf{y}}_B}{\partial \mathbf{r}} \times \hat{\mathbf{z}}_B + \hat{\mathbf{y}}_B \times \frac{\partial \hat{\mathbf{z}}_B}{\partial \mathbf{r}} \tag{14}$$

$$\frac{\partial \hat{\mathbf{x}}_B}{\partial \mathbf{v}} = \frac{\partial}{\partial \mathbf{v}}\left(\hat{\mathbf{y}}_B \times \hat{\mathbf{z}}_B\right) = \frac{\partial \hat{\mathbf{y}}_B}{\partial \mathbf{v}} \times \hat{\mathbf{z}}_B + \hat{\mathbf{y}}_B \times \frac{\partial \hat{\mathbf{z}}_B}{\partial \mathbf{v}} \tag{15}$$

## 3 Spacecraft Reflectance Vector

Figure 1 shows a spacecraft. It has a main body stationed to the spacecraft's body fixed frame (B-frame) and n different moving arms. In this case, each arm is a solar panel. Each panel generally has ability to move (translation movement) with respect to the main body of the spacecraft or rotate about its own rotation axis. In a design of a spacecraft, the translation movement of a solar panel sometimes is needed to avoid the shadow of the main body or other panels.

Total reflectance of incoming photons on a panel only depends on its rotation. For a translation movement of the panel with respect to the spacecraft's main body, it causes very small change in sunlight direction to the panel so the changes of reflectance is neglect able.

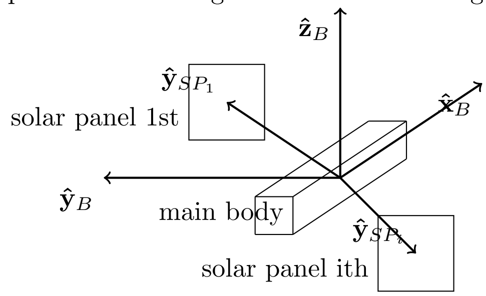


Figure 1: spacecraft has a main body and n moving arms

We define solar panel frame $SP_i$ for the ith panel $< \hat{\mathbf{x}}_{SP_i}, \hat{\mathbf{y}}_{SP_i}, \hat{\mathbf{z}}_{SP_i} >$ presenting in the spacecraft's body fixed frame such as:

$\hat{\mathbf{y}}_{SP_i}$ is the unit vector pointing along the ith panel's rotation axis from the spacecraft main body to the panel. The value of this vector is recorded in SRP N-Plates file which is used to store physical and design information about a plate on spacecraft's main body or solar panels.

$\hat{\mathbf{x}}_{SP_i}$ and $\hat{\mathbf{z}}_{SP_i}$ are specified by the below equations

$$\begin{aligned} \hat{\mathbf{x}}_{SP_i} &= \hat{\mathbf{y}}_{SP_i} \times \hat{\mathbf{s}}_B \\ \hat{\mathbf{z}}_{SP_i} &= \hat{\mathbf{x}}_{SP_i} \times \hat{\mathbf{y}}_{SP_i} \end{aligned} \tag{16}$$

Where: $\hat{\mathbf{s}}_B$ is the Sun's unit vector $\hat{\mathbf{s}}$ presenting in the spacecraft's body fixed frame

$$\hat{\mathbf{s}} = \frac{\mathbf{s}}{\|\mathbf{s}\|} \tag{17}$$

In N-plates model, we assume that spacecraft surface is covered by flat plates. The spacecraft effective are vector is the accumulation of effective area of all plates on spacecraft's body.

$$\begin{aligned} \mathbf{A} =& Cp_B \sum \mathbf{A}_B + \\ & Cp_{SP_1}[T_1]^T \sum \mathbf{A}_{SP_1} + Cp_{SP_2}[T_2]^T \sum \mathbf{A}_{SP_2} + ... + Cp_{SP_n}[T_n]^T \sum \mathbf{A}_{SP_n} \end{aligned} \tag{18}$$

Where:

| | |
|---|---|
| $\mathbf{A}$ | spacecraft's reflectance vector in the spacecraft's body fixed frame |
| $\mathbf{A}_B$ | spacecraft's reflectance vector of a plate on the spacecraft's main body. This vector is presented in the spacecraft's body fixed frame |
| $Cp_B$ | pressure efficient associated to the plates in the spacecraft's main body |
| $\mathbf{A}_{SP_i}$ | spacecraft's reflectance vector of a plate on the ith panel. This vector is presented in the ith panel frame $SP_i$ |
| $Cp_{SP_i}$ | pressure efficient associated to the plates in the ith panel |
| $[T_i]$ | rotation matrix from the body frame B to the ith panel frame $SP_i$ |

The first term on the left hand side of the equation 18 is the reflectance vector associated with the main body of the spacecraft. The successive terms are panel's (or arm's) reflectance vectors. Solar radiation pressure coefficients for the spacecraft main body $Cp_B$ and for panels $Cp_1$, $Cp_2$, ..., $Cp_n$ are numbers in range of (0,1]. When those values are assumed to be the same, they are similar to Cr value in Solar Radiation Pressure spherical shape model. In N-plates model, generally those coefficients are considered to be different. Rotation matrix $[T_i]^T$ in this equation is specified based on the set of vectors $< \hat{\mathbf{x}}_{SP_i}, \hat{\mathbf{y}}_{SP_i}, \hat{\mathbf{z}}_{SP_i} >$.

$$[T_i]^T = \begin{bmatrix} \hat{\mathbf{x}}_{SP_i} & \hat{\mathbf{y}}_{SP_i} & \hat{\mathbf{z}}_{SP_i} \end{bmatrix} \tag{19}$$

### 3.1 Plate reflectance vector

In equation 18, $\mathbf{A}_B$ and $\mathbf{A}_{SP_i}$ are plate's reflectances. Plate reflectance is vector quantity showing the effect of specular reflection and diffuse reflection of incoming photons. Those portion of photons generate pressure on the plate surface. For the portion of incoming photons absorbed by the surface. It does not generate a pressure but it is converted to other kinds of energy such as heat or electric power. The portion of incoming photons transmitted through the surface (for transparent or semitransparent body) may generate a pressure that depends on the direction shift and/or the scatter of out-coming photons. In this study, we do not count transmitted effect and assume that all plate is completely opaque.

For each plate, reflectance vector $\mathbf{A}$ is specified by

$$\mathbf{A} = \begin{cases} A.\left(\hat{\mathbf{s}}^T.\hat{\mathbf{n}}\right).\left[\left(1-\rho\right).\hat{\mathbf{s}} + 2\left(\dfrac{\delta}{3} + \rho.\left(\hat{\mathbf{s}}^T.\hat{\mathbf{n}}\right)\right).\hat{\mathbf{n}}\right] \text{ for } \hat{\mathbf{s}}^T.\hat{\mathbf{n}} > 0 \\ \mathbf{0} \text{ for } \hat{\mathbf{s}}^T.\hat{\mathbf{n}} \leq 0 \end{cases} \tag{20}$$

Where:

| | |
|---|---|
| $\mathbf{A}$ | plate's reflectance vector |
| $A$ | area of the portion of the plate without shadow |
| $\hat{\mathbf{s}}$ | column unit vector pointing from spacecraft to the Sun |
| $\hat{\mathbf{n}}$ | the plate surface's normal unit vector |
| $\rho$ | fraction of sunlight specularly reflected off the plate |
| $\delta$ | fraction of sunlight diffusely reflected off the plate |

To simplify this problem, we assume that no self-shadowing occurs so A is the area of the plate. If we want to count self-shadowing in the calculation, we need to choose an algorithm to specify the shaded portion of the plate and eliminate it from the calculation of refectance vector.

Note that: $\mathbf{A}$, $\hat{\mathbf{n}}$, and $\hat{\mathbf{s}}$ have to be presented in the same frame. The reflectance vector $\mathbf{A}_B$ for plates on the main body of the spacecraft is presented in B-frame, so vectors $\hat{\mathbf{s}}$ and $\hat{\mathbf{n}}$ in equation 20 have to be specified in B-frame. Similarly, The reflectance vector $\mathbf{A}_{SP_i}$ for plates on the ith panel is presented in $SP_i$-frame, so vectors $\hat{\mathbf{s}}$ and $\hat{\mathbf{n}}$ have to be specified in $SP_i$-frame as well.

Notation $\mathbf{A}$ in equation 20 and equation 18 are completely different. In equation 18, $\mathbf{A}$ is the spacecraft's reflectance vector. Meanwhile, in equation 20, it is the reflectance vector of a plate. In order to calculate vector $\mathbf{A}$ in equation 18, we need to specify reflectance vector $\mathbf{A}$ for each plate as shown in equation 20, then accumulate all of those values by using in equation 18.

### 3.2 SRP N-plates File

SRP N-Plates file stores all information related to material properties and design parameters of a plate. plate's material properties involve fractions of sunlight specularly and diffusely reflected off the plate ($\rho$ and $\delta$). Plate's design parameters involve information about its shape and dimension, position, and orientation. The following are information in the file.

**Case1: No self-shadowing and no change of orientation with respect to the plate's panel frame**
.Name of frame: the value could be B, $SP_1$, ..., $SP_n$
.Fraction of specular reflection
.Fraction of diffused reflection
.Plate's area vector
.Unit vector of y-axis $\hat{\mathbf{y}}_{SP_i}$ of its solar panel's frame

**Case2: Having self-shadowing and no change of orientation with respect to the plate's panel frame** In this case plate's area vector is not enough information to specify shade portion on each plate. We need to have information vertices of the plates with assumption the plate is a polygon. The file should contain following information. .Name of frame: the value could be B, $SP_1$, ..., $SP_n$
.Fraction of specular reflection
.Fraction of diffused reflection
.List of vertices presenting in its frame
.Unit vector of y-axis $\hat{\mathbf{y}}_{SP_i}$ of its solar panel's frame

**Case3: No self-shadowing and having orientation change with respect to the plate's panel frame** .Name of frame: the value could be B, $SP_1$, ..., $SP_n$
.Fraction of specular reflection
.Fraction of diffused reflection
.Plate's area vector
.Plate's rotation history file
.Unit vector of y-axis $\hat{\mathbf{y}}_{SP_i}$ of its solar panel's frame

**Case2: Having self-shadowing and change of orientation with respect to the plate's panel frame** In this case plate's area vector is not enough information to specify shade portion on each plate. We need to have information vertices of the plates with assumption the plate is a polygon. The file should contain following information. .Name of frame: the value could be B, $SP_1$, ..., $SP_n$
.Fraction of specular reflection
.Fraction of diffused reflection
.List of vertices presenting in its frame
.Plate's rotation history file
.Unit vector of y-axis $\hat{\mathbf{y}}_{SP_i}$ of its solar panel's frame

# 4 Spacecraft Reflectance Vector Partial Derivative

The solve-for vector $\mathbf{X}$ may contains spacecraft position and/or velocity, pressure coefficient vector [$Cp_B$, $Cp_{SP_1}$,..., $Cp_{SP_n}$], and other variables. Therefore, we need to specified partial derivative with respect to those variables.

## 4.1 Partial Derivative w.r.t. Pressure Coefficient Vector

$$\begin{aligned}\frac{\partial \mathbf{A}}{\partial Cp_B} &= \frac{\partial}{\partial Cp_B}\left[Cp_B.\sum \mathbf{A}_B + \sum_{i=1}^{n}\Big(Cp_{SP_i}.[T_i]^T.\sum \mathbf{A}_{SP_i}\Big)\right] \\ &= \sum \mathbf{A}_B\end{aligned} \tag{21}$$

$$\begin{aligned}\frac{\partial \mathbf{A}_B}{\partial Cp_{SP_i}} &= \frac{\partial}{\partial Cp_{SP_i}}\left[Cp_B.\sum \mathbf{A}_B + \sum_{i=1}^{n}\Big(Cp_{SP_i}.[T_i]^T.\sum \mathbf{A}_{SP_i}\Big)\right] \\ &= [T_i]^T.\sum \mathbf{A}_{SP_i}\end{aligned} \tag{22}$$

### 4.2 Partial Derivative w.r.t. the Spacecraft's Position and Velocity

$$\begin{aligned}\frac{\partial \mathbf{A}}{\partial \mathbf{X}} &= \frac{\partial}{\partial \mathbf{X}}\left[Cp_B.\sum \mathbf{A}_B + \sum_{i=1}^{n}\left(Cp_{SP_i}.[T_i]^T.\sum \mathbf{A}_{SP_i}\right)\right] \\ &= Cp_B.\sum \frac{\partial \mathbf{A}_B}{\partial \mathbf{X}} + \\ &\sum_{i=1}^{n}\left(Cp_{SP_i}.\left[\left(\frac{\partial}{\partial \mathbf{X}}[T_i]^T\right).\sum \mathbf{A}_{SP_i} + [T_i]^T.\sum \frac{\partial \mathbf{A}_{SP_i}}{\partial \mathbf{X}}\right]\right)\end{aligned} \tag{23}$$

In equation 23, $\mathbf{A}_{SP_i}$ is specified by equation 20. $[T_i]^T$ is calculated from equation 19. $\dfrac{\partial}{\partial \mathbf{X}}[T_i]^T$ matrix is specified by equation 47.

Plate reflectance partial derivatives w.r.t. spacecraft state $\dfrac{\partial \mathbf{A}_B}{\partial \mathbf{X}}$ and $\dfrac{\partial \mathbf{A}_{SP_i}}{\partial \mathbf{X}}$ are specified by equations in section 4.3.

### 4.3 Plate Reflectance Partial Derivative

For a plate, reflectance vector is specified by equation 20. It is a function of the Sun unit vector $\hat{\mathbf{s}}$ and the plate surface's unit normal vector $\hat{\mathbf{n}}$. Applying Chain Rule,

$$\frac{\partial \mathbf{A}}{\partial \mathbf{X}} = \frac{\partial \mathbf{A}}{\partial \hat{\mathbf{s}}}.\frac{\partial \hat{\mathbf{s}}}{\partial \mathbf{X}} + \frac{\partial \mathbf{A}}{\partial \hat{\mathbf{n}}}.\frac{\partial \hat{\mathbf{n}}}{\partial \mathbf{X}} \tag{24}$$

In this equation, $\mathbf{A}$, $\hat{\mathbf{s}}$, and $\hat{\mathbf{n}}$ have to present in the same frame. If $\mathbf{A}$ is presented in the spacecraft's body fixed frame B for case $\mathbf{A}_B$ in equation 20, $\hat{\mathbf{s}}$ and $\hat{\mathbf{n}}$ have to be in that frame. If $\mathbf{A}$ is presented in the ith panel frame $SP_i$ for case $\mathbf{A}_{SP_i}$ in equation 20, $\hat{\mathbf{s}}$ and $\hat{\mathbf{n}}$ have to present in $SP_i$ frame.

Let:

$$D = \hat{\mathbf{s}}_{1x3}^T.\hat{\mathbf{n}}_{3x1} \tag{25}$$

$$\mathbf{C} = (1-\rho).\hat{\mathbf{s}} + 2\left[\frac{\delta}{3} + \rho.D\right].\hat{\mathbf{n}} \tag{26}$$

Substitute equations 26 and 25 to equation 20, we obtain

$$\mathbf{A} = A.[D.\mathbf{C}] \tag{27}$$

Apply equations in Appendix A.4, partial derivative of plate reflectance vector $\mathbf{A}$ w.r.t. $\hat{\mathbf{s}}$ and $\hat{\mathbf{n}}$ are specified as 3x3 matrices as shown below:

$$\frac{\partial \mathbf{A}}{\partial \hat{\mathbf{s}}} = A.\left[\mathbf{C}.\frac{\partial D}{\partial \hat{\mathbf{s}}} + D.\frac{\partial \mathbf{C}}{\partial \hat{\mathbf{s}}}\right] \tag{28}$$

$$\frac{\partial \mathbf{A}}{\partial \hat{\mathbf{n}}} = A.\left[\mathbf{C}.\frac{\partial D}{\partial \hat{\mathbf{n}}} + D.\frac{\partial \mathbf{C}}{\partial \hat{\mathbf{n}}}\right] \tag{29}$$

Where: $\dfrac{\partial \mathbf{A}}{\partial \hat{\mathbf{s}}}$ and $\dfrac{\partial \mathbf{A}}{\partial \hat{\mathbf{n}}}$ are matrices of size 3x3.

Partial derivatives of scalar term D w.r.t. $\hat{\mathbf{s}}$ and $\hat{\mathbf{n}}$ are row 1x3 vectors (Appendix A.2)

$$\left[\frac{\partial D}{\partial \hat{\mathbf{s}}}\right]_{1x3} = \frac{\partial}{\partial \hat{\mathbf{s}}}\left[\hat{\mathbf{s}}^T.\hat{\mathbf{n}}\right] = \hat{\mathbf{n}}^T \tag{30}$$

$$\left[\frac{\partial D}{\partial \hat{\mathbf{n}}}\right]_{1x3} = \frac{\partial}{\partial \hat{\mathbf{n}}}\left[\hat{\mathbf{s}}^T.\hat{\mathbf{n}}\right] = \hat{\mathbf{s}}^T \tag{31}$$

Substitute $\mathbf{C}$ from equation 26 to $\dfrac{\partial \mathbf{C}}{\partial \hat{\mathbf{s}}}$ and apply the equation in Appendix A.4.

$$\begin{aligned}
\frac{\partial \mathbf{C}}{\partial \hat{\mathbf{s}}} &= \frac{\partial}{\partial \hat{\mathbf{s}}}\left[(1-\rho).\hat{\mathbf{s}} + 2\left[\frac{\delta}{3} + \rho.D\right].\hat{\mathbf{n}}\right] \\
&= (1-\rho).\frac{\partial \hat{\mathbf{s}}}{\partial \hat{\mathbf{s}}} + 2.\frac{\partial}{\partial \hat{\mathbf{s}}}\left[\left(\frac{\delta}{3} + \rho.D\right).\hat{\mathbf{n}}\right] \\
&= (1-\rho).[I]_{3x3} + 2.\left[\hat{\mathbf{n}}.\frac{\partial}{\partial \hat{\mathbf{s}}}\left(\frac{\delta}{3} + \rho.D\right) + \left(\frac{\delta}{3} + \rho.D\right).\frac{\partial \hat{\mathbf{n}}}{\partial \hat{\mathbf{s}}}\right] \\
&= (1-\rho).[I]_{3x3} + 2.\left[\hat{\mathbf{n}}.\left(\rho.\frac{\partial D}{\partial \hat{\mathbf{s}}}\right) + \left(\frac{\delta}{3} + \rho.D\right).[0]_{3x3}\right] \\
&= (1-\rho).[I]_{3x3} + 2.\rho.\hat{\mathbf{n}}_{3x1}.\hat{\mathbf{n}}^T_{1x3}
\end{aligned} \tag{32}$$

Substitute $\mathbf{C}$ from equation 26 to $\dfrac{\partial \mathbf{C}}{\partial \hat{\mathbf{n}}}$ and apply the equation in Appendix A.4.

$$\begin{aligned}
\frac{\partial \mathbf{C}}{\partial \hat{\mathbf{n}}} &= \frac{\partial}{\partial \hat{\mathbf{n}}}\left[(1-\rho).\hat{\mathbf{s}} + 2\left[\frac{\delta}{3} + \rho.D\right].\hat{\mathbf{n}}\right] \\
&= (1-\rho).\frac{\partial \hat{\mathbf{s}}}{\partial \hat{\mathbf{n}}} + 2.\frac{\partial}{\partial \hat{\mathbf{n}}}\left[\left(\frac{\delta}{3} + \rho.D\right).\hat{\mathbf{n}}\right] \\
&= (1-\rho).[0]_{3x3} + 2.\left[\hat{\mathbf{n}}.\frac{\partial}{\partial \hat{\mathbf{n}}}\left(\frac{\delta}{3} + \rho.D\right) + \left(\frac{\delta}{3} + \rho.D\right).\frac{\partial \hat{\mathbf{n}}}{\partial \hat{\mathbf{n}}}\right] \\
&= 2.\left[\hat{\mathbf{n}}.\left(\rho.\frac{\partial D}{\partial \hat{\mathbf{n}}}\right) + \left(\frac{\delta}{3} + \rho.D\right).[I]_{3x3}\right] \\
&= 2.\left[\rho.\hat{\mathbf{n}}.\hat{\mathbf{s}}^T + \left(\frac{\delta}{3} + \rho.D\right).[I]_{3x3}\right]
\end{aligned} \tag{33}$$

## 4.4 Reflectance Derivative for Plates on the Spacecraft's Main Body

For a plate on the spacecraft's main body, its unit normal $\hat{\mathbf{n}}$ is constant due to the plate does not rotate with respect to B-frame. So the term $\dfrac{\partial \hat{\mathbf{n}}}{\partial \mathbf{X}}$ in equation 24 equals zero 3x3 matrix. As a result,

$$\frac{\partial \mathbf{A}}{\partial \mathbf{X}} = \frac{\partial \mathbf{A}}{\partial \hat{\mathbf{s}}_B}.\frac{\partial \hat{\mathbf{s}}_B}{\partial \mathbf{X}} \tag{34}$$

The first term on the right hand side of the equation is specified by equation 28. The second term is specified by equation 41 and equation 42.

## 4.5 Reflectance Derivative for Plates on the Spacecraft's Solar Panels

For a plate on the $i^th$ solar panel, its unit normal vector $\hat{\mathbf{n}}$ may or may not a constant that depends on the direction of the plate's surface is changed or not.

If the plate has no rotation with respect to $SP_i$ frame, $\hat{\mathbf{n}}$ is a constant vector. As s result,

$$\frac{\partial \mathbf{A}}{\partial \mathbf{X}} = \frac{\partial \mathbf{A}}{\partial \hat{\mathbf{s}}_{SP_i}}.\frac{\partial \hat{\mathbf{s}}_{SP_i}}{\partial \mathbf{X}} \tag{35}$$

The first term on the right hand side of the equation is specified by equation 28. The second term is specified by equation 44 and equation 46.

If the plate rotates with respect to $SP_i$ frame, the unit normal $\hat{\mathbf{n}}$ is no longer a constant. In this case, term $\dfrac{\partial \hat{\mathbf{n}}}{\partial \mathbf{X}}$ in equation 24 is specified from variables $\hat{\mathbf{n}}$ and $\mathbf{X}$ by using finite difference method. The term $\dfrac{\partial \mathbf{A}}{\partial \hat{\mathbf{n}}}$ is specified by equation 29.

### 4.6 Derivative of the Sun unit vector

The frames used in this math specification document are inertial frame, spacecraft's body fixed frame, and solar panel frames. Therefore, we need to specify the Sun unit vector and its derivative in each frame.

#### 4.6.1 The Sun unit vector and its derivative in inertial frame

In an inertial frame, the Sun vector is the vector from the spacecraft's location at time $t_1$ when the photon beam arrived to the Sun's location at time $t_0$ when the photon beam emitted as shown in equation 2.

The Sun unit vector in inertial frame is

$$\hat{\mathbf{s}}_I = \frac{\mathbf{s}_I}{\|\mathbf{s}_I\|} = \frac{\mathbf{r}_{Sun} - \mathbf{r}}{\|\mathbf{r}_{Sun} - \mathbf{r}\|} \tag{36}$$

Its derivative w.r.t. position $\mathbf{r}$ is

$$\frac{\partial \hat{\mathbf{s}}_I}{\partial \mathbf{r}} = \frac{\partial}{\partial \mathbf{r}} \left( \frac{\mathbf{r}_{Sun} - \mathbf{r}}{\|\mathbf{r}_{Sun} - \mathbf{r}\|} \right) = -\frac{1}{\|\mathbf{s}_I\|}.[I]_{3x3} + \frac{1}{\|\mathbf{s}_I\|^3}.\left(\mathbf{s}_I.\mathbf{s}_I^T\right) \tag{37}$$

Due to $\hat{\mathbf{s}}_I$ is not a function of spacecraft's velocity $\mathbf{v}$ so its derivative w.r.t. $\mathbf{v}$ is a zero matrix.

$$\frac{\partial \hat{\mathbf{s}}_I}{\partial \mathbf{v}} = [0]_{3x3} \tag{38}$$

#### 4.6.2 The Sun unit vector and its derivative in the spacecraft's body fixed frame

The relation between Sun unit vectors in the inertial frame and the spacecraft's body fixed frame is

$$\hat{\mathbf{s}}_I = [M]^T.\hat{\mathbf{s}}_B \tag{39}$$

From equation 39, we take derivative w.r.t. the spacecraft's position $\mathbf{r}$ both sides of this equation.

$$\frac{\partial \hat{\mathbf{s}}_I}{\partial \mathbf{r}} = \frac{\partial [M]^T}{\partial \mathbf{r}}.\hat{\mathbf{s}}_B + [M]^T.\frac{\partial \hat{\mathbf{s}}_B}{\partial \mathbf{r}} \tag{40}$$

Subtract both sides by $\frac{\partial [M]^T}{\partial \mathbf{r}}.\hat{\mathbf{s}}_B$ and multiply by $[M]$, we have

$$\frac{\partial \hat{\mathbf{s}}_B}{\partial \mathbf{r}} = [M].\left[\frac{\partial \hat{\mathbf{s}}_I}{\partial \mathbf{r}} - \frac{\partial [M]^T}{\partial \mathbf{r}}.\hat{\mathbf{s}}_B\right] \tag{41}$$

Where matrix $[M]$ is the transpose of $[M]^T$. $\frac{\partial \hat{\mathbf{s}}_I}{\partial \mathbf{r}}$, $\frac{\partial [M]^T}{\partial \mathbf{r}}$, and $\hat{\mathbf{s}}_B$ are calculated from equation 37, 9, and 39 respectively.

Due to $\hat{\mathbf{s}}_B$ is not a function of spacecraft's velocity $\mathbf{v}$ so its derivative w.r.t. $\mathbf{v}$ is a zero matrix.

$$\frac{\partial \hat{\mathbf{s}}_B}{\partial \mathbf{v}} = [0]_{3x3} \tag{42}$$

#### 4.6.3 The Sun unit vector and its derivative in the spacecraft's solar panel frame

The relation between Sun unit vectors in the inertial frame and the spacecraft's solar panel frame is

$$\hat{\mathbf{s}}_I = [T.M]^T.\hat{\mathbf{s}}_{SP_i} \tag{43}$$

Do the same step as done for the spacecraft's body fixed frame. We obtains

$$\frac{\partial \hat{\mathbf{s}}_{SP_i}}{\partial \mathbf{r}} = [T.M].\left[\frac{\partial \hat{\mathbf{s}}_I}{\partial \mathbf{r}} - \frac{\partial [T.M]^T}{\partial \mathbf{r}}.\hat{\mathbf{s}}_{SP_i}\right] \tag{44}$$

Where matrix $[T.M]$ is the transpose of $[M]^T.[T]^T$. Both terms $\dfrac{\partial \hat{\mathbf{s}}_I}{\partial \mathbf{r}}$ and $\hat{\mathbf{s}}_{SP_i}$ are calculated from equation 37 and 43 respectively. Derivative $\dfrac{\partial [T.M]^T}{\partial \mathbf{r}}$ is specified by following equation.

$$\frac{\partial [T.M]^T}{\partial \mathbf{r}} = \frac{\partial}{\partial \mathbf{r}}\left([M]^T.[T]^T\right) = \frac{\partial [M]^T}{\partial \mathbf{r}}.[N]^T + [M]^T.\frac{\partial [N]^T}{\partial \mathbf{r}} \tag{45}$$

Due to $\hat{\mathbf{s}}_{SP_i}$ is not a function of spacecraft's velocity $\mathbf{v}$ so its derivative w.r.t. $\mathbf{v}$ is a zero matrix.

$$\frac{\partial \hat{\mathbf{s}}_{SP_i}}{\partial \mathbf{v}} = [0]_{3x3} \tag{46}$$

## 4.7 Derivative of matrix $[T]^T$ w.r.t. spacecraft's state

Derivative of matrix $[T]^T$ is a 3-dimension matrix (appendix A.6). and it is specified by the equation below.

$$\frac{\partial}{\partial \mathbf{X}}\left([T]^T\right) = \frac{\partial}{\partial \mathbf{X}}\begin{bmatrix}\hat{\mathbf{x}}_{SP_i} & \hat{\mathbf{y}}_{SP_i} & \hat{\mathbf{z}}_{SP_i}\end{bmatrix} = \begin{bmatrix}\dfrac{\partial \hat{\mathbf{x}}_{SP_i}}{\partial \mathbf{X}} & \dfrac{\partial \hat{\mathbf{y}}_{SP_i}}{\partial \mathbf{X}} & \dfrac{\partial \hat{\mathbf{z}}_{SP_i}}{\partial \mathbf{X}}\end{bmatrix} \tag{47}$$

Three 2-dimension matrices $\dfrac{\partial \hat{\mathbf{x}}_{SP_i}}{\partial \mathbf{X}}$, $\dfrac{\partial \hat{\mathbf{y}}_{SP_i}}{\partial \mathbf{X}}$, and $\dfrac{\partial \hat{\mathbf{z}}_{SP_i}}{\partial \mathbf{X}}$ are specified by their sub matrices as shown below.

$$\begin{aligned}
\frac{\partial \hat{\mathbf{y}}_{SP_i}}{\partial \mathbf{r}} &= [0]_{3x3} \\
\frac{\partial \hat{\mathbf{x}}_{SP_i}}{\partial \mathbf{r}} &= \frac{\partial}{\partial \mathbf{r}}\left(\hat{\mathbf{y}}_{SP_i} \times \hat{\mathbf{s}}_B\right) \\
&= \hat{\mathbf{y}}_{SP_i} \times \frac{\partial \hat{\mathbf{s}}_B}{\partial \mathbf{r}} \\
\frac{\partial \hat{\mathbf{z}}_{SP_i}}{\partial \mathbf{r}} &= \frac{\partial}{\partial \mathbf{r}}\left(\hat{\mathbf{x}}_{SP_i} \times \hat{\mathbf{y}}_{SP_i}\right) \\
&= \frac{\partial \hat{\mathbf{x}}_{SP_i}}{\partial \mathbf{r}} \times \hat{\mathbf{y}}_{SP_i} \\
\frac{\partial \hat{\mathbf{x}}_{SP_i}}{\partial \mathbf{v}} &= \frac{\partial \hat{\mathbf{y}}_{SP_i}}{\partial \mathbf{v}} = \frac{\partial \hat{\mathbf{z}}_{SP_i}}{\partial \mathbf{v}} = [0]_{3x3}
\end{aligned}$$

$\dfrac{\partial \hat{\mathbf{x}}_{SP_i}}{\partial \mathbf{r}}$ and $\dfrac{\partial \hat{\mathbf{z}}_{SP_i}}{\partial \mathbf{r}}$ are derived from $\hat{\mathbf{y}}_{\mathbf{SP_i}}$ and $\dfrac{\partial \hat{\mathbf{s}}_B}{\partial \mathbf{r}}$.

# 5 Appendix

Let: **a**, **b**, **c**, and **x** are column vectors. $\alpha$ is a scalar. [M] [N] are matrices.

A.1. Derivative of a vector w.r.t. a scalar is a mx1 row vector:

$$\frac{\partial \mathbf{a}}{\partial \alpha} \triangleq \begin{bmatrix} \frac{\partial a_1}{\partial \alpha} \\ \frac{\partial a_2}{\partial \alpha} \\ \cdots \\ \frac{\partial a_m}{\partial \alpha} \end{bmatrix}_{mx1}$$

A.2. Derivative of a scalar w.r.t. a vector is a 1xn row vector:

$$\frac{\partial \alpha}{\partial \mathbf{x}} \triangleq \begin{bmatrix} \frac{\partial \alpha}{\partial x_1} & \frac{\partial \alpha}{\partial x_2} & \cdots & \frac{\partial \alpha}{\partial x_n} \end{bmatrix}_{1xn}$$

A.3. Derivative of a vector w.r.t. a vector is a mxn matrix:

$$\frac{\partial \mathbf{a}}{\partial \mathbf{x}} \triangleq \begin{bmatrix} \frac{\partial \mathbf{a}}{\partial x_1} & \frac{\partial \mathbf{a}}{\partial x_2} & \cdots & \frac{\partial \mathbf{a}}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial a_1}{\partial \mathbf{x}} \\ \frac{\partial a_2}{\partial \mathbf{x}} \\ \cdots \\ \frac{\partial a_m}{\partial \mathbf{x}} \end{bmatrix}_{mxn}$$

A.4. Derivative by parts rule for a product of a scalar and a vector:

If $\mathbf{a} = \alpha.\mathbf{b}$ then

$$\frac{\partial \mathbf{a}}{\partial \mathbf{x}} = \mathbf{b}.\frac{\partial \alpha}{\partial \mathbf{x}} + \alpha.\frac{\partial \mathbf{b}}{\partial \mathbf{x}}$$

Note that: in this equation, the order of each term in the left hand side of equation has to be followed restrictively as shown in the equation.

When $\alpha$ is a constant w.r.t. $\mathbf{x}$, $\frac{\partial \alpha}{\partial \mathbf{x}} = 0$, as a result

$$\frac{\partial \mathbf{a}}{\partial \mathbf{x}} = \alpha.\frac{\partial \mathbf{b}}{\partial \mathbf{x}}$$

A.5. Derivative of a matrix w.r.t. a scalar is a matrix with a same size:

$$\frac{\partial [M]}{\partial \alpha} \triangleq \begin{bmatrix} \frac{\partial M_{1,1}}{\partial \alpha} & \frac{\partial M_{1,2}}{\partial \alpha} & \cdots & \frac{\partial M_{1,p}}{\partial \alpha} \\ \frac{\partial M_{2,1}}{\partial \alpha} & \frac{\partial M_{2,2}}{\partial \alpha} & \cdots & \frac{\partial M_{2,p}}{\partial \alpha} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial M_{m,1}}{\partial \alpha} & \frac{\partial M_{m,2}}{\partial \alpha} & \cdots & \frac{\partial M_{m,p}}{\partial \alpha} \end{bmatrix}$$

A.6. Derivative of a matrix w.r.t. a vector is a 3-dimension matrix:

$$\frac{\partial[M]}{\partial\mathbf{x}} \triangleq \begin{bmatrix} \frac{\partial\,[M]}{\partial x_1} & \frac{\partial\,[M]}{\partial x_2} & ... & \frac{\partial\,[M]}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \frac{\partial M_{i,j}}{\partial x_k} \end{bmatrix}$$

Each element $\frac{\partial\,[M]}{\partial x_i}$ is a 2-dimension matrix with the same size of matrix [M].

A.7. Product of a matrix and a matrix derivative:

$$\frac{\partial[M]}{\partial\mathbf{x}}.[N] \triangleq \begin{bmatrix} \frac{\partial\,[M]}{\partial x_1}.[N] & \frac{\partial\,[M]}{\partial x_2}.[N] & ... & \frac{\partial\,[M]}{\partial x_n}.[N] \end{bmatrix}$$

$$[N].\frac{\partial[M]}{\partial\mathbf{x}} \triangleq \begin{bmatrix} [N].\frac{\partial\,[M]}{\partial x_1} & [N].\frac{\partial\,[M]}{\partial x_2} & ... & [N].\frac{\partial\,[M]}{\partial x_n} \end{bmatrix}$$

A.8. Derivative of a product of 2 matrices w.r.t. a scalar:

$$\frac{\partial}{\partial\alpha}\left([M].[N]\right) = \frac{\partial[M]}{\partial\alpha}.[N] + [M].\frac{\partial[N]}{\partial\alpha}$$

A.9. Derivative of a product of 2 matrices w.r.t. a vector:

$$\begin{aligned} \frac{\partial}{\partial\mathbf{x}}\left([M].[N]\right) &= \begin{bmatrix} \frac{\partial[M]}{\partial x_1}.[N] + [M].\frac{\partial[N]}{\partial x_1} & \frac{\partial[M]}{\partial x_2}.[N] + [M].\frac{\partial[N]}{\partial x_2} & ... & \frac{\partial[M]}{\partial x_n}.[N] + [M].\frac{\partial[N]}{\partial x_n} \end{bmatrix} \\ &= \begin{bmatrix} \frac{\partial[M]}{\partial x_1}.[N] & \frac{\partial[M]}{\partial x_2}.[N] & ... & \frac{\partial[M]}{\partial x_n}.[N] \end{bmatrix} + \begin{bmatrix} [M].\frac{\partial[N]}{\partial x_1} & [M].\frac{\partial[N]}{\partial x_2} & ... & [M].\frac{\partial[N]}{\partial x_n} \end{bmatrix} \\ &= \frac{\partial[M]}{\partial\mathbf{x}}.[N] + [M].\frac{\partial[N]}{\partial\mathbf{x}} \end{aligned}$$

B.1. Cross product of 2 vectors:

Cross product of 2 vectors $\mathbf{a} = \begin{bmatrix} a_1 & a_2 & a_3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 & b_2 & b_3 \end{bmatrix}$ is defined as

$$\mathbf{a} \times \mathbf{b} = \begin{bmatrix} a_2.b_3 - a_3.b_2 \\ a_3.b_1 - a_1.b_3 \\ a_2.b_3 - a_3.b_2 \end{bmatrix}$$

B.2. Cross product of a matrix and a vector:

Cross product of a matrix $[C]_{3xn} = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & ... & \mathbf{c}_n \end{bmatrix}$ and a vectors $\mathbf{b}$ is defined as

$$[C]_{3xn} \times \mathbf{b} = \begin{bmatrix} \mathbf{c}_1 \times \mathbf{b} & \mathbf{c}_2 \times \mathbf{b} & ... & \mathbf{c}_n \times \mathbf{b} \end{bmatrix}_{3xn}$$

$$\mathbf{b} \times [C]_{3xn} = \begin{bmatrix} \mathbf{b} \times \mathbf{c}_1 & \mathbf{b} \times \mathbf{c}_2 & ... & \mathbf{b} \times \mathbf{c}_n \end{bmatrix}_{3xn}$$

Where: $\mathbf{b}, \mathbf{c}_1, \mathbf{c}_2, ..., \mathbf{c}_n$ are vectors of size 3.

B.3. Derivative of a cross product:

B.3.1. Derivative of a cross product of 2 vectors w.r.t. a scalar:

If $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ with $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors of size 3 then

$$\frac{\partial \mathbf{c}}{\partial \alpha} = \left(\frac{\partial \mathbf{a}}{\partial \alpha} \times \mathbf{b}\right) + \left(\mathbf{a} \times \frac{\partial \mathbf{b}}{\partial \alpha}\right)$$

Where: $\alpha$ is a scalar variable

B.3.2. Derivative of a cross product of 2 vectors w.r.t. a vector:

If $\mathbf{c} = \mathbf{a} \times \mathbf{b}$ with $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors of size 3 then

$$\left[\frac{\partial \mathbf{c}}{\partial \mathbf{x}}\right]_{3xn} = \left(\left[\frac{\partial \mathbf{a}}{\partial \mathbf{x}}\right]_{3xn} \times \mathbf{b}\right) + \left(\mathbf{a} \times \left[\frac{\partial \mathbf{b}}{\partial \mathbf{x}}\right]_{3xn}\right)$$

Where: $\mathbf{x}$ is a vector of size n.

If $\mathbf{a}$ is a constant vector w.r.t. $\mathbf{x}$ then

$$\left[\frac{\partial \mathbf{c}}{\partial \mathbf{x}}\right]_{3xn} = \mathbf{a} \times \left[\frac{\partial \mathbf{b}}{\partial \mathbf{x}}\right]_{3xn}$$

If $\mathbf{b}$ is a constant vector w.r.t. $\mathbf{x}$ then

$$\left[\frac{\partial \mathbf{c}}{\partial \mathbf{x}}\right]_{3xn} = \left[\frac{\partial \mathbf{a}}{\partial \mathbf{x}}\right]_{3xn} \times \mathbf{b}$$