Term Structure Models with Differences in Belief

Paul Whelan

paul.whelan07@imperial.ac.uk

Empirical Facts

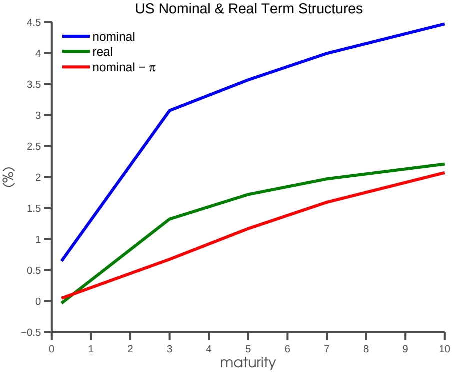
- **Expected Excess Returns on Nominal Bonds are Highly Predictable**
  - Fama and Bliss (1987), Campbell and Schiller (1991)
  - Cochrane-Piazzesi (2002), Le and Singleton (2013)

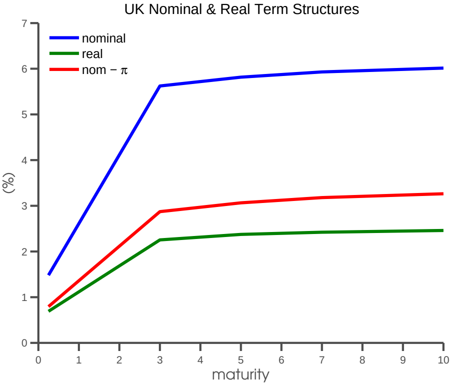
- But . . .

- **Macro-Economic Determinants**
  - growth/inflation shocks explain a small fraction of yield shocks
  - very little covariance between bond returns and macro-factors
  - weak link between monetary policy and bond markets

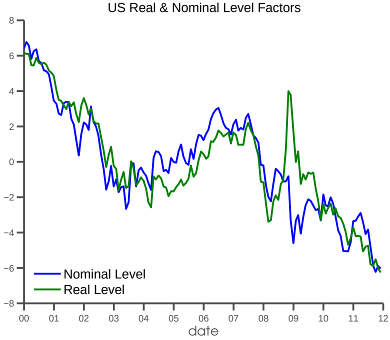
Real Bonds

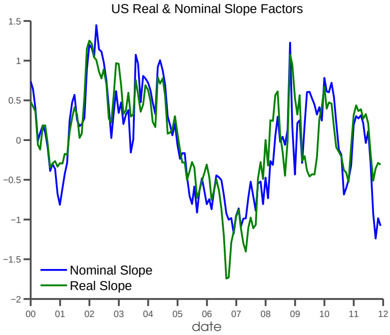
Average Yield Curves



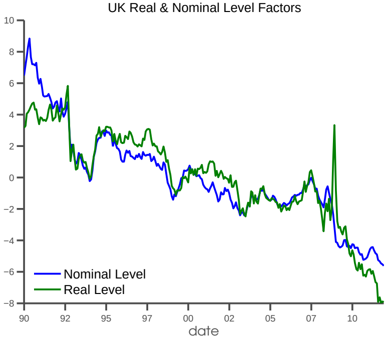


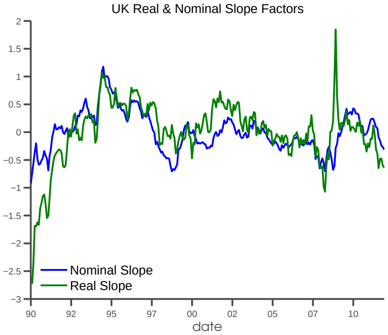
Real-Nominal PCs





Real-Nominal PCs: UK

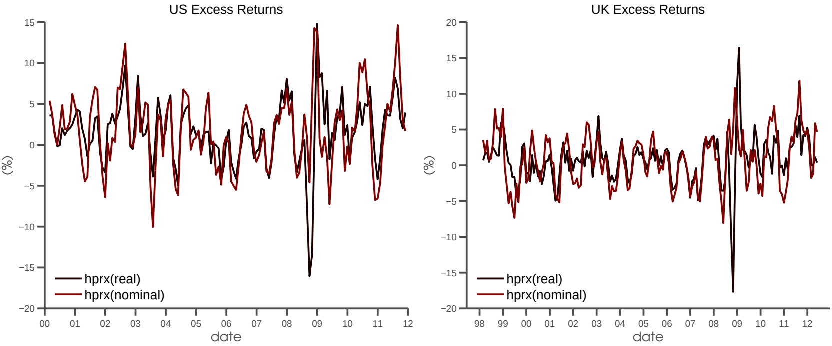




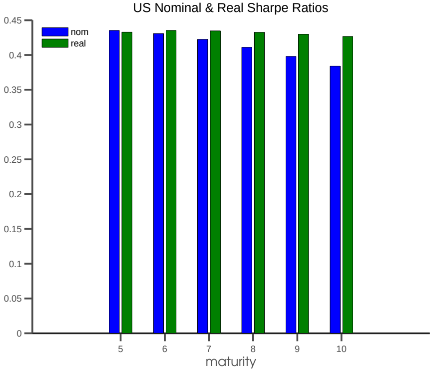
Real-Nominal PCs:

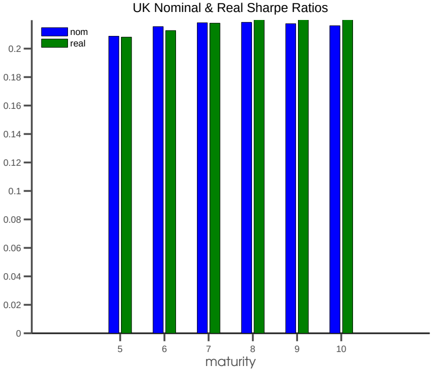
U.S Treasuries	Level	Slope	Curvature
% of $cov(y_t^{\$})$ explained	94.69	4.22	1.03
% of $cov(y_t^{r})$ explained	95.14	4.41	0.43
% Nom factor explained by real factor	78	72	3
U.K Treasuries			
% of $cov(y_t^{\$})$ explained	95.56	3.64	0.79
% of $cov(y_t^{r})$ explained	98.26	1.47	0.26
% Nom factor explained by real factor	83	39	27

Excess Returns



Term Structure of Sharpe Ratios





Models for the Real SDF

1. Time-Varying Prices of Risk: Campbell Cochrane (1999)
   - surplus drives predictable excess returns
   - bond risk premium determined by surplus covariance with short rate:
     - intertemporal demand dominates: $cov(Surp_t, r_t) > 0 \rightarrow RP < 0$
     - precautionary savings dominates: $cov(Surp_t, r_t) < 0 \rightarrow RP > 0$
2. Time-Varying Quantities of Risk: Bansal Yaron (2004)
   - expected consumption volatility drives predictable excess returns
   - calibrated to equity implies
     - downward sloping (negative) real yields.
     - negative sharpe ratios

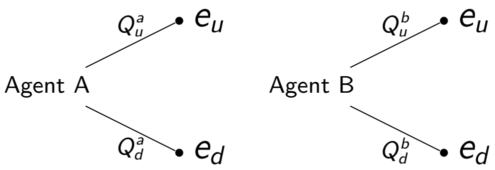
This Paper ...

- studies a model where beliefs are driving
  - agent specific risk aversion (prices of risk)
  - agent specific consumption volatility (quantities of risk)
- study theoretical and empirical explanations for the
  1. **The Short Term Real Rate:**
  2. **The Cross-Section of Real Yields:**
  3. **Expected Returns on Real Bonds:**

Theoretical Framework

Consumption Growth Forecasts

Consider two agents with subjective conditional probability measures dQ_t^a and dQ_t^b:



$$Q_u^b > Q_u^a$$

$$Q_d^b < Q_d^a$$

- In equilibrium ex-ante marginal utilities must balance

$$E^a(u_a'(C_T)|\mathcal{F}_t) = E^a\left(u_b'(C_T)|\mathcal{F}_t\right)$$

- Implication: models for consumption growth matter for equilibrium risk sharing

Which Model?

- No consensus on the correct model for consumption growth
  - Beeler and Campbell (2009) argue consumption has autocorrelations consistent with random walk
  - Bansal, Kiku, Yaron (2009) argue VAR estimates imply predictability of more than 15% at one-to-five year horizons.
- consumption asset pricing finds a wide range of estimates for autocorrelations:
  - 0.43 [NIPA]
  - 0.74 [Parker and Julliard (2005)]
  - 0.12 [Jagannathan and Wang (2007)]
  - -0.14 [Savov (2011)].

Which Model?

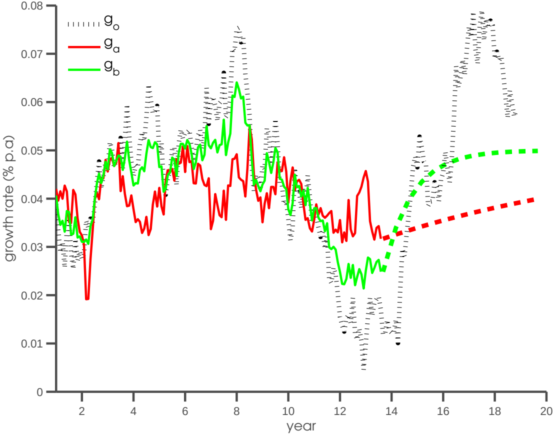
Consider an economy where agents learn about growth rates ($\mathbf{g_t}$) by observing realised consumption

$$dC/C = \mathbf{g_t^i} dt + \sigma_C dW_t^C,$$

Assume agents agree on the long run growth rate (θ) but hold dogmatic beliefs on the persistence ($\kappa_{\mathbf{g}}$) of growth shocks:

$$dg_t^i = -\kappa_{\mathbf{g}}^{\mathbf{i}}(g_t^i - \theta)dt + \sigma_g dW_t^g$$

Standard linear filtering problem generates important implications for equilibrium risk sharing.



Assume the true objective $\kappa_g^o = 0.5$, one agents who believes the economy is dominated by permanent shocks $\kappa_g^a = 0.2$, and one who believes the economy is dominated by transitory shocks $\kappa_g^a = 0.8$

Equilibrium

- Individual Problem:

$$\max_{\{c^i\}} E_0^i \int_0^\infty \varrho_t u(c_t^i) dt,$$

$$s.t.\ E_0^i \int_0^\infty \mathcal{M}_t^i \left[c_t^i - e_t^i\right] dt \leq 0 \quad \text{and} \quad \sum_i c_t^i = C_t$$

- Representative Problem:

$$U^*(C(t), \lambda) := \max_{c_a(t)+c_b(t)=C(t)} \{u_a(c_a(t)) + \lambda_t u_b(c_b(t))\}$$

- Solution:

$$c_a(t) = \frac{C_t}{1+\eta_t^{1/\gamma}} \quad , \quad c_b(t) = C_t \frac{\eta_t^{1/\gamma}}{1+\eta_t^{1/\gamma}}$$

$$\mathcal{M}_t^* = \underbrace{\varrho_t C_t^{-\gamma}}_{\text{Homogeneous CRRA SDF}} \underbrace{\left(1+\eta_t^{1/\gamma}\right)^\gamma}_{\text{Belief Distortion}}$$

where $\eta_t = \frac{d\mathcal{P}_t^a}{d\mathcal{P}_t^b}$

Effective Risk Aversion

- The local curvature of the investor's utility is time varying:

$$\gamma^a(t) = -c_t \frac{U_{cc}}{U_c} = \gamma \left[1 + \left(\frac{\omega_t^b}{\omega_t^a} \right)^2 \right]$$

- Effective risk aversion is state dependent as a function of *past* consumption choices

- The entire *history* of belief dispersion is important for equilibrium prices *today*

Stochastic Consumption

$$d\omega_a = \underbrace{\frac{\gamma - 1}{2\gamma}\omega_a(\eta_t)\omega_b(\eta_t)\psi_t^2}_{\text{Speculative Demand}} \left[\frac{(\gamma - 1) + 2\gamma\omega_b(\eta_t)}{\gamma(\gamma - 1)}\right] dt + \underbrace{\frac{1}{\gamma}\omega_a(\eta_t)\omega_b(\eta_t)\psi_t}_{\text{Stochastic Vol}} \, d\hat{W}_t^C$$

- $\omega_i(t) = c_t^i / C_t$ - investor's i's total consumption share - is stochastic
- shocks to *beliefs* about growth rates change the investment opportunity set
- speculative demand enters the drift of individual consumption streams.
  - $\gamma > 1$ the wealth effect dominates
  - $\gamma < 1$ the substitution effect dominates
- large risk tolerance generates large volatile consumption streams to due speculation

The Real Short Rate

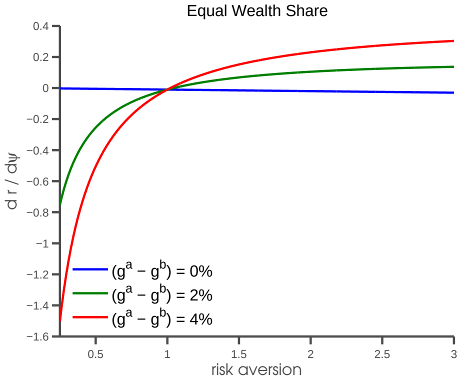
- From the drift of $d\mathcal{M}_t^*$: risk free rate

$$r_f = \rho + \gamma\beta' \underbrace{(\omega_a(t)\hat{g}_t^a + \omega_b(t)\hat{g}_t^b)}_{\text{Consensus Aggregation Bias}} - \underbrace{\frac{1}{2}\gamma(\gamma+1)\sigma_C^2}_{\text{Precautionary Savings}} + \underbrace{\frac{\gamma-1}{2\gamma}\omega_a(t)\omega_b(t)\psi_t^2}_{\text{Speculative Demand}},$$

- Implications

  1. Aggregation bias: the short rate is skewed towards the belief of the agent who has been relatively more successful.
  2. Inter-temporal component: depends on whether γ is greater or smaller than 1.

Short Rate Sensitivities



- For $\gamma > 1$ the wealth effect dominates: interest rates rise to clear the market.
- When $\gamma < 1$ the substitution effect dominates: interest rates fall to clear the market

The Term Structure of Bond Prices

Real zero-coupon bonds are given by

$$P(t,T) = \; E_t^i \left[e^{-\delta(T-t)} \left(\frac{C_T}{C_t} \right)^{-\gamma} \left(\frac{1+\eta_T^{1/\gamma}}{1+\eta_t^{1/\gamma}} \right)^{\gamma} \right]$$

Requires computing the forward density for $y_T = \ln C_T$ and $z_T = \ln \eta_T$

Define the extend state $X_t = \left(y_t, z_t, g_t, \psi_t, g_t^2, \psi_t^2, \psi_t g_t \right)$

obtain prices via inversion via inversion of the CF

$$P(t,T) = e^{-\delta(T-t)} (1+e^{\frac{1}{\gamma} z_t})^{-\gamma} \int_0^\infty \left[(1+e^{\frac{1}{\gamma} Z_T})^{\gamma} \frac{1}{\pi} \int_0^\infty e^{-iu_2 z_t} \phi_{y,z}(\tau; u) du_2 \right] dz_T$$

where

$$\phi_{y,z}(\tau; u) = e^{\alpha(\tau,u)+\beta(\tau,u)' X_t}$$

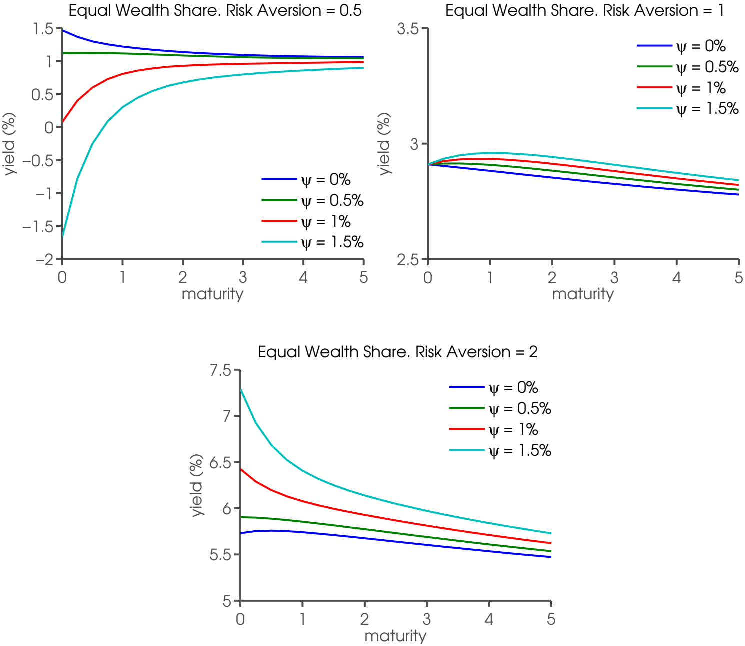
and $\beta(\tau)$ is a set of matrix valued RDEs.

Calibrated Parameters

δ	γ	σ_C	θ	σ_g	ρ_{cg}	κ_g^a	κ_g^b
0	(0.5 : 3.0)	3%	3%	1.5%	0.80	0.20	0.80

- Agent A: Growth rate shocks have a half life of $T_{1/2} \sim 3.5$ years
- Agent B: Growth rate shocks have a half life of $T_{1/2} \sim 0.90$ years
- $\rho_{cg} > 0 \rightarrow$ homogeneous real bonds have negative sharpe ratios

Term Structures: Equal Wealth Shares



Risk Premia

- The bond risk premium under the measure of each agent is

$$\mu_i^P(t,T) - \mu_i^Q(t,T) = -E_t^i\left[\frac{dP_t}{P_t}\frac{d\mathcal{M}_t^i}{\mathcal{M}_t^i}\right] = \kappa_t(t)\sigma_{P,D}(t,T)$$

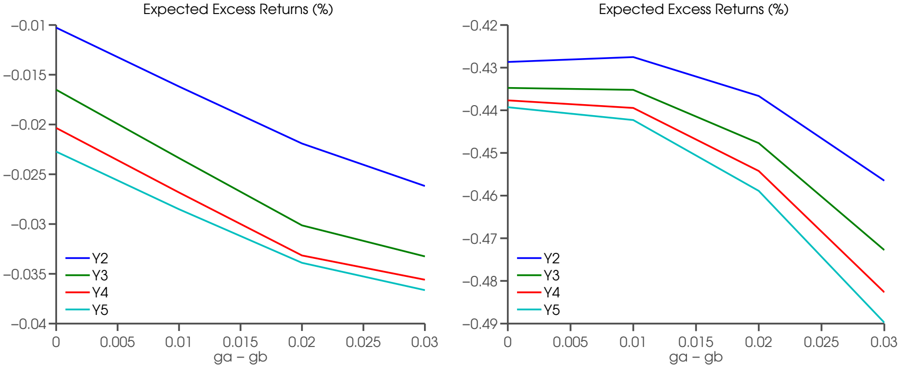
- The risk premium from the perspective of an unbiased econometrician is

$$\frac{\mu_e^P(t,T) - \left(\omega_t^a\mu_a^Q(t,T) + \omega_t^b\mu_b^Q(t,T)\right)}{\sigma_{P,D}(t,T)} = \gamma\sigma_C + \frac{1}{\sigma_C}\left[g_t - (\omega_t^a g_t^a + \omega_t^b g_t^b)\right]$$

- Bond sensitivities to shocks are

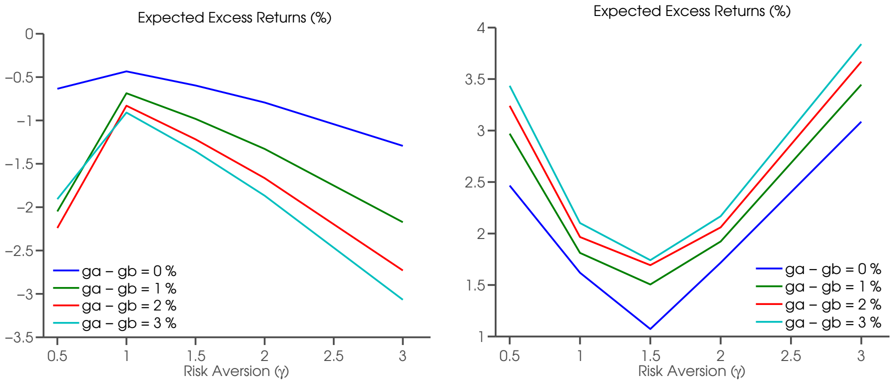
$$[\sigma_{P,D}(t,T)\ ,\ \sigma_{P,s}(t,T)] = \frac{1}{P(t,T)}\left[\frac{\partial P(t,T)}{\partial x}\right]\cdot\begin{bmatrix}\sigma_{g,D} & \sigma_{g,s}\\ -\psi_t & 0\\ \sigma_{\psi,D} & \sigma_{\psi,s}\end{bmatrix}$$

Risk Premia : Equal Wealth Shares



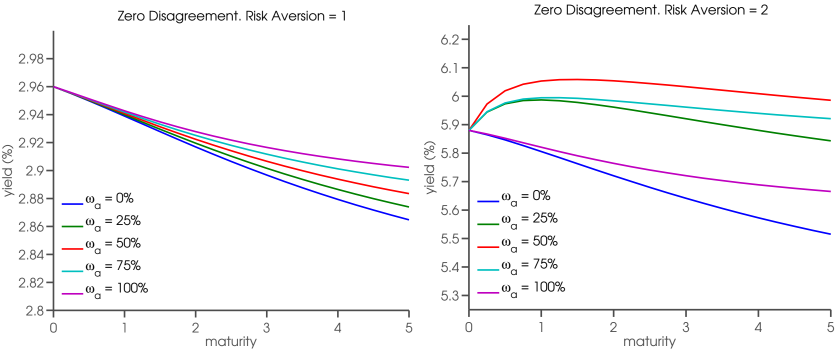
- both panels : symmetric economy ($\omega^a = \omega^b = 0.5$)

Risk Premia : Optimism Vs Pessimism



- left panel : optimistic economy ($\omega^a = 0.75$ & $\omega^b = 0.25$)
- right panel : pessimistic economy ($\omega^a = 0.25$ & $\omega^b = 0.75$)

Belief Risk



- When agents are myopic the term structure is bounded by homogeneous solutions.
- If the short rate is pro-cyclical ($\rho^i_{c,g} > 0$) term structures will be downward sloping.
- When $\gamma \neq 1$ and $\omega_a \approx \omega_b \approx 0.5$ the real term structure is upward sloping

Testable Implications

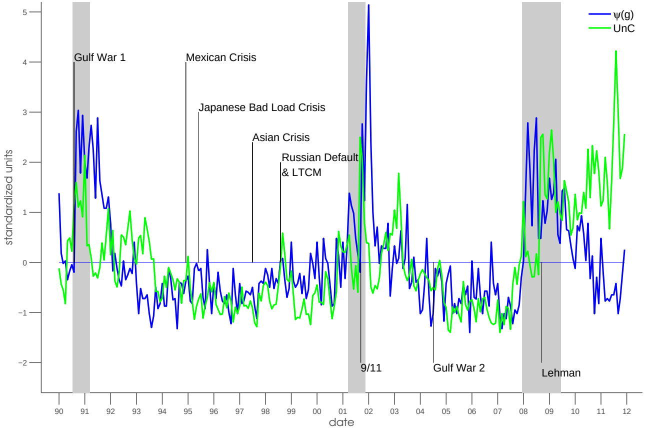
- H_{01} : **Short Term Real Rate.**
  - disagreement enters the the real short rate. The sign of its effect depends on $\gamma \gtrless 1$
- H_{02} : **The Cross-Section of Real Yields.**
  - disagreement is a state variable affecting the level and slope of the term structure. To the extent that past beliefs proxy for contemporaneous wealth fluctuations, distance lags of disagreement should affect today's cross-section of yields.
- H_{03} : **Expected Returns on Real Bonds**
  - from the perspective of an econometrician, disagreement drives positive (negative) variation in expected returns if the economy is on average pessimistic (optimistic).
  - risk compensation on intermediate to long term bonds depends on future belief risk.

Data

Subjective Expectations

- **BlueChip Financial Forecasts**:
  - Large data source of subjective expectations
  - Available at monthly frequency and out to 5-quarters
  - GDP forecast to proxy for consumption growth expectations

Dispersion Variables



Green line plots a policy uncertainty factor studied by Baker, Bloom, and Davis (2012)
Blue line plots the IQR of 1-quarter GDP forecasts.

Structural Alternatives

1. **Prices of Risk:** Wachter (2006)

   - Proxy for consumption surplus s_t as weighted average of monthly consumption growth rates:

   $$s_t = \sum_{j=1}^{120} \phi^j \Delta c_{t-j}$$

   where weight is chosen to match the quarterly autocorrelation of the P/D ratio.

2. **Quantities of Risk:** Bansal and Shaliastovich (2012)

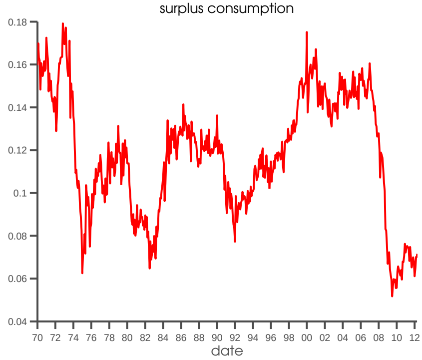
   - Fit a VAR(1) to inflation and growth expectations:

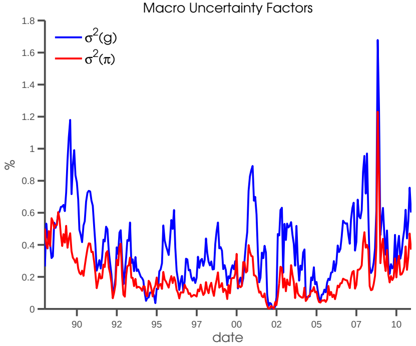
   $$g^e_{t+1} = \underset{(0.08)}{0.63} + \underset{(0.02)}{0.86} g^e_t - \underset{(0.01)}{0.08} \pi^e_t + \epsilon_{g,t+1}$$

   $$\pi^e_{t+1} = \underset{(0.12)}{0.93} - \underset{(0.03)}{0.20} g^e_t + \underset{(0.02)}{0.87} \pi^e_t + \epsilon_{\pi,t+1}$$

   - Project square root of the sum of squared residuals over the next 12 months on the date t cross-section of yields.
   - Square the fitted values which forms the uncertainty measures for date t.

Structural Alternatives





Empirical Results

H_{01} : Short Term Rate

	$E(g)$	$\psi(g)$	$\sigma^2(g)$	$Surp$	Lag y^{3m}	$\overline{R}^2$
real y^{3m}	-0.23	-0.28				0.06
	(-1.22)	(-2.32)				
real y^{3m}	-0.39	-0.31	-0.04	0.31		0.12
	(-2.01)	(-2.42)	(-0.34)	(3.20)		
real y^{3m}	0.00	-0.11	-0.06	0.02	0.88	0.78
	(0.00)	(-2.13)	(-1.21)	(0.26)	(19.52)	

Sample = 1990.1 - 2010.1

H_{02} : The Cross Section of Yields

	$E_t(g_t)$	ψ_t^g	ψ_{t-6}^g	$\sigma_t^2(g)$	$Surp_t$	$\overline{R}^2$
Real $Slope_t$	0.76	0.50				0.31
	(2.64)	(4.95)				
Real $Slope_t$	0.81	0.41	0.28			0.38
	(3.56)	(5.29)	(4.07)			
Real $Slope_t$	1.02	0.40	0.21	-0.01	-0.20	0.39
	(3.94)	(5.29)	(2.62)	(-0.10)	(-1.87)	

Sample Period : 2000.1 - 2010.1

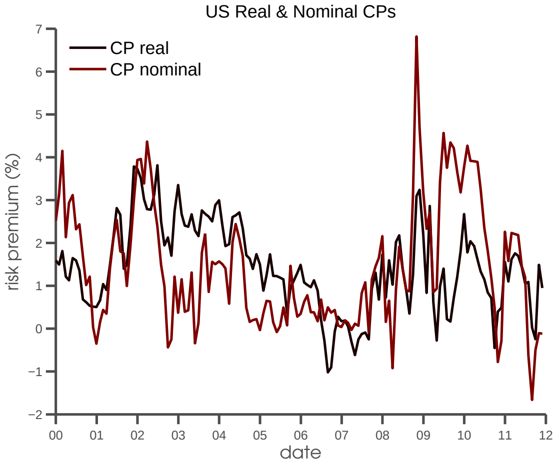
Real Cochrane-Piazzesi (2002)

- Adapt CP in order to compare real and nominal return predictability.
- Project 3-month excess returns on 3-month forward rate spreads.
- Common factors are then formed from real and nominal forward rates by factorizing the first stage regression as

$$\frac{1}{3}\sum_{n=1}^{3} rx_{t+0.25}^{n} = \overline{\alpha} + \gamma'(f_t^{[5\ 7\ 10]} - y_t^{(0.25)}) + \overline{\epsilon}_{t+0.25}$$

$$CP_t = \gamma'(f_t^{[5\ 7\ 10]} - y_t^{(0.25)})$$

Real Cochrane-Piazzesi (2002)



Real Cochrane-Piazzesi (2002)

	$const$	$CP_t^{\$}$	$\overline{R}^2$	$const$	CP_t^r	$\overline{R}^2$
$hprx_t^{(5)}$	0.06	0.71	0.16	0.02	0.81	0.08
	(0.16)	(4.47)		(0.03)	(2.44)	
$hprx_t^{(7)}$	0.00	0.98	0.17	-0.01	0.98	0.09
	(0.00)	(4.47)		(-0.01)	(2.63)	
$hprx_t^{(10)}$	-0.03	1.22	0.15	0.00	1.15	0.09
	(-0.05)	(3.98)		(0.00)	(2.76)	

$$hprx_t^{(n)} = E_t[p_{t+3}^{(n-3)}] - p_t^{(n)} - r_t^{3m} = const + \beta CP_t + \varepsilon_{t,t+3}^{(n)}.$$

H_{03} : Expected Returns

	$\psi(g)$	Lag $\psi(g)$	$\sigma^2(g)$	*Surp*	$\overline{R}^2$
CP Nom	0.34				0.12
	(2.82)				
CP Real	0.32				0.10
	2.28				
CP Nom	0.21	0.43			0.30
	(2.07)	(3.87)			
CP Real	0.23	0.26			0.15
	(1.69)	(2.19)			
CP Nom	0.12	0.29	−0.08	−0.39	0.40
	(0.87)	(3.89)	(−0.38)	(−3.83)	
CP Real	0.30	0.32	−0.19	0.24	0.23
	(3.35)	(4.28)	(−0.99)	(1.76)	

Sample Period : 2000.1 - 2010.1

H_{03} : Expected Returns

$$hprx_{t,t+12}^{(n)} = const + \beta RiskFactors_t + error_{t+12}^{(n)}$$

	$\psi(g)$	Lag $\psi(g)$	$\sigma^2(g)$	*Surp*	$\overline{R}^2$
n = 2	0.42				0.18
	(3.97)				
n = 5	0.36				0.12
	(3.88)				
n = 2	0.32	0.24			0.22
	(3.66)	(2.81)			
n = 5	0.25	0.24			0.17
	(3.15)	(2.62)			
n = 2	0.29	0.26	0.12	0.05	0.23
	(3.48)	(2.99)	(0.90)	(0.32)	
n = 5	0.25	0.24	−0.01	−0.01	0.16
	(2.90)	(2.38)	(−0.04)	(−0.05)	

Sample Period : 1990.1 - 2010.1

Conclusions

Existence of time-varying bond risk premia is one of the most interesting and challenging topic in fixed income. The weak empirical link between observable macro variables and bond returns has been a long standing puzzle.
In this study, we learn that:

1. disagreement is important for explaining time-varying in the real discount factor.
2. regressions on dispersion measures consistent with models in which agents speculate on growth rate forecasts.
3. dispersion in beliefs explains a steep yield curve through:
   - Low short term interest rates
   - Large positive bond risk premia
4. fluctuations in relative wealth (past disagreement) important for the cross-section of yields today
5. positive real sharpe ratios and $cov(DiB, hprx)$ consistent with average pessimism