Beliefs about Growth and Real Bonds

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ABSTRACT

In this paper I document that since the advent of inflation protected securities there appears to be very little difference between real and nominal risk premia and argue that the dominant source bond market variation should be understood in terms of the real stochastic discount factor. To shed light on the economic determinants of real bond markets I study an economy in which multiple agents engage in speculate trade based on subjective models for the dynamics of real consumption growth. The resulting term structure solution is able to explain a number of stylised features of the data and, in addition, provides a set testable implications linking the properties of traders' beliefs to both the cross-section and time series properties of bonds. Using survey evidence on trader beliefs I provide supporting evidence for the model's theoretical predictions, suggesting that investor heterogeneity plays an important role in bond markets.

The consensus among fixed income researchers is that expected returns on long term *nominal* bonds in excess of short dated securities are predictable ahead of time. This is consistent with this view that short term interest rates are pro-cyclical, driven by the properties of expected fundamentals, while long term interest rates are counter-cyclical, driven by the properties of risk premia. Well studied forecasting factors include forward spreads (Fama and Bliss (1987)), the slope of the yield curve (Campbell and Shiller (1991)), and an affine function of the forward curve (Cochrane and Piazzesi (2005)). However, there exists little consensus about the economic source of this predictability. To shed light on this issue I study, from both a theoretical and empirical perspective, the determinants of *real* bond markets.

I consider real bond markets in an economy populated by multiple agents who hold subjective beliefs regarding the correct model for the economy. When agents hold subjective beliefs they are rationally induced to speculate. In this case, the *real* pricing measure is distorted by a component related to the distribution of wealth, and a component that depends on beliefs. This implies that the characteristics of real bond markets deviate from those emerging from a homogeneous representative agent economy. I consider an economy with stochastic disagreement, learning and non-myopic agents and derive closed-form solutions for bond prices that allow us to investigate the marginal effects of different characteristics of the economy on the real term structure.[1].

A first set of implications relates to the behavior of the real short rate. When agents are non myopic, the short term interest rate is affected by disagreement in two ways. The first is due to the effect of the wealth weighted aggregation bias of the beliefs beliefs. This effect is history-dependent: as past disagreement affects agents' speculative positions and events unfold, wealth is redistributed endogeneously over time toward the agent whose model happened to align more closely with the data generating process. The second effect is due to the optimal hedging demand and depends on whether agents are risk averse or risk tolerant. As agents rely on different models, shocks to fundamentals endogenously change the *perceived* investment opportunity set, affecting agents future speculative trading opportunities. This occurs even if fundamentals are homoskedastic as long as agents agree to disagree. When agents are risk tolerant larger differences in beliefs reduce the short term interest rate (negative hedging demand). This is because the substitution effect dominates the wealth effect and agents reduce current consumption when the investment opportunity set is impacted by beliefs shocks. The opposite holds for risk averse agents (positive hedging demand). This knife edge case for short term interest rates provides a testable restriction that makes *real* bond markets a laboratory to distinguish across alternative models.

The second set of implications relates to the shape of the real yield curve. When the endowment process follows an affine dynamics, bond yields are given by the sum of an exponentially affine function in the expected economic growth plus a quadratic function in

[1] Important contributions that discuss the asset pricing implications of heterogeneity in beliefs include Basak (2005), David (2008), (Buraschi, Trojani, and Vedolin, 2011, 2010), Bhamra and Uppal (2011), and Gallmeyer and Hollifield (2008)

disagreement. I calibrate the economy and show how wealth distribution, risk aversion and disagreement interact to affect the shape of the yield curve. A robust feature is that when $\gamma \neq 1$, the *real* yield curve can be upward sloping even when the equivalent economy with homogeneous investors would give rise to a decreasing yield curve. This is interesting given the well known difficulty for single agent economies to produce upward real term structures. This feature crucially depends on the size of the term premium and it does not hold when agents are myopic. Moreover, bond prices depend on beliefs in a history-dependent way. Since equilibrium prices depend on agents wealth distribution, which is the result of the way agents traded and shared risks in the past, distant lags of disagreement should be statistically significant to explain the dynamics of both short term interest rates and today's cross-section of yields.

The third set of implications I study are related to the properties of bond risk premia. I study bond risk premia under the measure of an unbiased econometrician, and find properties differ from those arising in homogeneous economy in several respects. There are three channels that contribute. The first one is standard and proportional to the product of risk aversion γ and the volatility of aggregate consumption. It is well known that this term requires a large risk aversion to generate a bond risk premia similar to those that are observable in the data. The second term, on the other hand, is inversely proportional to γ. Indeed, for lower levels of risk aversion agents are willing to speculate more, so that the endogenous quantity of risk that each agent faces is larger. The net effect of this non linearity is that for γ below a given threshold, the risk premium can be high even if γ is low. Moreover, an implication is that bond risk premia depend on lagged disagreement. This is due to the fact that sample periods with larger belief dispersions imply larger subsequent redistribution of wealth and more volatile consumption/wealth ratios that, in turn, implies larger ex-ante conditional bond risk premia. The third term arises from agents' demand to hedge future changes in their expected marginal utility due to future shocks to the distribution of beliefs. This term is due to the fact that future changes in beliefs (of the other agent) will necessarily impact future asset prices and thus the wealth redistribution. Overall, the economy is able to generate sizeable positive risk premia on *real* bonds for the case that agents are risk tolerant and, on-average, the distribution of wealth is skewed towards optimistic agents.

Taking the model to the data I build a rich data set on the distribution of real growth forecasts by professional market participants. This data set merges all the historical paper archives of BlueChip surveys and is unique in that it is available at a monthly frequency, covers a long history, and it is based on a large and stable cross-section of forecasters.[2] The characteristics of the pricing kernel in heterogeneous agent models depend on both current disagreement and the cross-sectional distribution of wealth. However, testing the impact of the distribution of wealth is challenging since it is unobservable. I propose to test the effect of relative wealth fluctuations using distance lags of disagreement: conditional on observing large speculative activity in the past, the econometrician should expect a re-allocation of

[2]The Survey of Professional Forecasters is available only at quarterly frequency and, especially in some periods, it has a more restricted cross-section of forecasters. Previously, the commercially available BlueChip economic digital files started only in 2007.

wealth today, thus past disagreement represents an potential proxy. Using subjective expectations from surveys I obtain a number of results regarding contemporaneous disagreement and past speculative activity (relative wealth fluctuations). In addition, I attempt to distinguish between alternative structural specifications for the real stochastic discount factor by building proxies for the risk factors that arise in economies with long run risk (conditional variance of expected fundamentals) and habit economies (consumption surplus).

First, I find that real short term interest rates are negatively related to current disagreement about real growth. Projecting short rates on real disagreement I obtain negative factor loadings with t-statistics in excess of -2.13 after controlling for expected fundamentals, conditional variance of expected fundamentals, consumption surplus, and lagged disagreement proxies. This result supports heterogeneous beliefs models with $\gamma < 1$ ($EIS > 1$) as in David (2008) in which case speculative activity drives down short term *real* borrowing costs. The results are confirmed when the test is re-stated in terms of changes in 1-year interest rates after controlling for the Fama-Bliss one-year forward spot spread and/or the Cochrane-Piazzesi factor.

Second, I find a strong effect of contemporaneous real disagreement on the slope of the real term structure. A one standard deviation shock to disagreement raises the slope of the yield curve by 0.50 standard deviations with a t-statistic of 4.95. When I test whether past beliefs contain useful information about today's cross-section of yields, I find that adding a 6-month lag of disagreement to the regression for the level of interest rates raises the R^2 to 31% then 38% and my proposed relative wealth proxy is significant at the 1% level. Controlling for structural alternatives suggested by long run risk and habit economies I confirm this is a robust finding. This result is intriguing since it suggests past speculative activity, as proxied by lagged disagreement, has a large effect of the shape of the yield curve today, consistent with the hypothesis that agents are trading on their beliefs. .

Third, examining the role of heterogeneity for risk premia I build a real (CP_t^r) return forecasting factors by adapting the approach of Cochrane and Piazzesi (2005). Real CP_t^r inherits many of the properties of nominal $CP_t^\$$: (*i*) both factors load on the forward curve with a 'tent' shaped pattern, which is increasing in magnitude across maturity; (*ii*) both factors contain substantial information about expected excess returns: a single factor explains 9% of the variance of 3-month returns for 10-year real bonds, and 15% of the variance of 3-month returns on 10-year nominal bonds; (*iii*) both CP_t^r and $CP_t^\$$ are counter-cyclical peaking in 2002 and again in 2008/2009. Excluding the crisis period the correlation between the series is 0.56, while the full sample correlation is 0.46, meaning that expected excess returns, and hence risk premia, display large co-movement across markets.

Next, I evaluate whether predictions given by heterogeneous belief models can help explain such return predictability. I consider $CP_t^{r,\$}$ as a measure of the spanned risk premium and run regressions of dispersion factors after controlling for long run risk and habit economy risk factor proxies. The results suggest that both real and nominal CP_t factors are positively and significantly related to belief risk with point estimates on CP_t^r ($CP_t^\$$) equal to 0.34 (0.32) and a t-statistic of 2.82 (2.28). Moreover, the degree of predictable variation explained is quantitatively similar across markets with R^2 equal to 10% (real) and 12% (nominal). Considering the effect of past speculative activity adding lagged disagreement raises R^2 on real CP_t^r by 5% and on nominal $CP_t^{\$}$ by 18%. Moreover, both belief risk and relative wealth proxies are jointly statistically significant with t-stats of 2.07 and 3.87, respectively. This result is robust to controlling for conditional variance and surplus factors. Interestingly, while conditional variance appears unimportant, the estimated loading on consumption surplus is strongly statistically significant and negative ($\beta_4 = -0.39$ with a t-stat $= -3.83$) and raises the $\overline{R}^2$ an additional 10%. The sign on consumption surplus is consistent with both short rate and slope regression estimates, which imply positive shocks to consumption surplus should drive down the bond risk premium.

Finally, I examine these findings over a longer sample period within a classical return forecasting framework and find a substantial degree of return predictability arising from both contemporaneous and lagged dispersion terms. In the case of 5-year bonds, the R^2 of the regression is 17% and the t-statistics disagreement factors are significant a the 1% level. A similar result holds also in the case of longer maturity bonds. These results lend support for the role of heterogeneous agent models and help to distinguish alternative structural explanations with forward looking long run risks, or backward looking habit economies where the history of consumption matters.

I. Stylised Facts of the Real Term Structures

A. Average Yield Curves

The U.S nominal term structure slopes upward, from 3.10% at 3-years to 4.5% at 10-years. This is well known. Less well known is that the real curve is also upward sloping, from 1.3% at 3-years to 2.2% at 10-years. In the U.K yield curves are much flatter between 3 and 10-year maturities, ranging between 5.6% - 6.00% for the nominal curve, and 2.25% - 2.50% for the inflation linked curve. Figure 2 plots average yield curves including proxies for ex-ante real short rates.[3] Considering this wider maturity range term structures are again upward sloping in both countries and the 10-year minus 3-month slopes are quantitively similar across real and nominal securities. The online appendix to this paper reports detailed summary statistics for a host of inflation measures. I use these estimates to gauge, in a simple way, the average inflation risk premium. The red solid line in figure 2 subtracts the (average) realised year-on-year rate of inflation for the all-urban consumer price index (U.S), and the retail price index (U.K) from nominal yields. The left panel displays U.S Treasury yields for the sample period 2000.1 - 2012.12, and the right panel U.K Gilt yields for the sample period 1990.1 - 2012.12. Considering the U.S one finds that the inflation adjusted nominal curve lies just below the average TIPS term structure implying a small *negative* inflation risk premium. In the U.K the inflation adjusted nominal curve also lies just above inflation linked GILT yields implying a small *positive* inflation risk premium.

[Insert tables I and II about here]

[3]Construction of short term real rates is discussed in a data appendix at the end of the paper.

[Insert figures 1 and 2 about here]

One needs be careful inferring inflation risk premia from TIPS yields since they are known to suffer liquidity problems for which investors should demand higher yields that mask the true inflation risk premium.[4] For example, D'Amico, Kim, and Wei (2008) argue that the TIPS liquidity premium was as high as 120 basis points in 1999 and trended down to 10 basis points in 2004. The online appendix also employs a regression based model to build hypothetical liquidity adjusted yields following Grishchenko and Huang (2012) and Pflueger and Viceira (2013). Consistent with these authors I document the existence of a liquidity premium in the early years of our sample (2000 - 2003), followed by a massive liquidity shock in excess of 100 basis points during the 2008/2009 financial crisis. However, after adjusting for liquidity premia the estimated inflation risk premium remains small. For example, for the 10-year maturity I find an inflation risk premium that varies between plus and minus 50 basis points over the sample but is on average close to zero.[5]. This stylised snapshot of government bond markets is interesting since extant literature typically studies time-variation in nominal excess returns via compensation for inflation risks. For example, using a VAR based approach Campbell and Shiller (1996) find an inflation risk premium of $70 - 100$ basis points on 5-year bonds. Using a real business cycle model, Buraschi and Jiltsov (2005) derive a closed form solution linking the inflation risk premium to the money supply and productivity shocks and estimate an inflation risk premium between $25 - 70$ basis points. Ang, Bekaert, and Wei (2008) study the inflation risk premium within a regime-switching framework estimating an inflation risk premium as large as 118 basis points at the 5-year maturity that fully accounts for an upward sloping nominal term structure. More recent evidence has utilised the availability of inflation indexed securities. Hördahl, Tristani, and Vestin (2008) solve and calibrate a general equilibrium model with habit persistence and nominal rigidities, while Grishchenko and Huang (2012) use a regression based approach to measure the inflation risk premium. Both authors report positive estimates a few basis points in magnitude consistent the findings presented here.

B. *Real versus Nominal PCs*

In this section I ask the extent to which factors responsible for time-variation in nominal yields are also common to the real term structure. Following Litterman and Scheinkman (1991), among many others, I decompose the term structures into level, slope and curvature factors via a decomposition of the covariance matrix of yields.[6] I perform this decomposition for real and nominal yields including 3-month maturities and $n = 3, \ldots, 10$-years.

[4] A second concern using inferring inflation risk premia from TIPS yields is that they have been show to mis-priced during crisis period (Fleckenstein, Longstaff, and Lustig (2013)). Since the focus of this paper is on ''low' frequency explanations of yield curve dynamics I do not attempt to address 'high' frequency arbitrage effects.

[5] I also find Treasury markets experienced a massive deflation shock of -150 during the crisis

[6] Stacking yields of different maturities, $y_t^{(n)}$, one first decomposes the covariance matrix of yields as $Q\Lambda Q^\top$. The diagonal elements of Λ contain eigenvalues and columns of Q contain eigenvectors. Ordering the eigenvalues from largest to smallest as $\lambda_1, \lambda_2, \lambda_3$ with associated eigenvectors q_1, q_2, q_3, the first three PCs are given by $x_{it} = q_i^\top y_t^{(n)}$.

Table III reports the results for U.K and U.S yield curves and figure 3 plots factor loadings from the decomposition. As expected, one finds a level factor that loads equally across maturity explains the vast majority of variation: $\sim 95\%$ for U.S real and nominal curves, and $\sim 96\%$ for U.K real and nominal curves.[7] A slope factor is also present in both term structures, loading negatively on maturities $3, 4, 5, 6$, positively on maturities $7, 8, 9, 10$. A key question I seek to shed light on is the extent to which the nominal term structure can be explained by real term structure factors. To address this question I regress nominal PCs on corresponding real counterparts:

$$
\begin{aligned}
Level_t^{\$} &= \alpha_l + \beta_l Level_t^r + \varepsilon_t^l \\
Slope_t^{\$} &= \alpha_s + \beta_s Slope_t^r + \varepsilon_t^s \\
Curvature_t^{\$} &= \alpha_c + \beta_c Curvature_t^r + \varepsilon_t^c
\end{aligned}
$$

The final rows of table III reports reports the R^2 from these regressions. Considering level projections I find 78% of the variance of U.S nominal level factor is explained by the *real* level factor, and 72% of the nominal slope factor is explained by the real slope factor. Taken together 75% of the variance of the nominal term structure is explained by the first two principle components of the real term structure. Inflation does not enter real yields so these estimates imply that inflation shocks explain at most 25% of shocks to nominal yields. This is consistent with recent evidence by Duffee (2013) who argues that just 15 percent of shocks to nominal Treasury yields are accounted for by shocks to inflation.

[Insert table III and figure 3 about here]

Figures 4 and 5 examine the time-series properties of level and slope factors. The left panel plots level factors and shows that both real and nominal yield levels were persistently declining over the sample period. Moreover, real and nominal level factors are moving in lock step, the full sample correlation in the U.S is 0.88 and excluding the crisis rises to 0.94. This suggests that the well documented decline in government yields over the last 25 years originates from a decline in either real short rates or real risk premia. The right panel of figures 4 and 5 display the time-series for real and nominal slope factors. The correlation between real and nominal slopes is striking: in the U.S the full sample correlation is 0.84. The series indicate that in the U.S both real and nominal structures were flat (or negative) between the years $2000-2001$ / $2006-2007$, steep in years $2003-2004$ / 2008, and remain elevated through the end of the sample. In the U.K, over the sample period 1990.1-2012.12, this pattern is repeated: level and slope factors display a large degree of co-movement with full-sample correlation of 0.91 and 0.65, respectively. This is intriguing because the slope of the term structure has well documented forecasting ability for expected returns and is therefore revealing about the joint dynamics of investor preferences and risk factors. Consistent with a small inflation risk premium the co-movement between real and nominal term structure factors suggests one should try to understand variation in government bond yields in terms of real rates and real risk premia.

[7] Generally, the variance explained by the level factor would be lower, between $85\% - 90\%$, but here I discard maturities less than 3-years which contribute orthogonal variation.

[Insert figure 4 and 5 about here]

C. Sharpe Ratios

I construct 3-month *excess* returns on both real and nominal bonds maturing between 5 and 10-years from [8]

$$rx_{t+0.25}^{(n)} = p_{t+0.25}^{(n-0.25)} - p_t^{(n)} - y_t^{(0.25)}$$

where $p_t^{(n)}$ is a log price of a real / nominal n-year bond, and $y_t^{r/\$,(0.25)}$ is the real / nominal 3-month rate. Figure 6 displays time series for (annualised) realised excess returns on 5-year bonds. The co-movement is large. In the U.S, excluding (including) the crisis the correlation is 0.83 (0.64) and for the U.K is 0.75 (0.46). Both real bond return series display massive negative returns in the aftermath of Lehman brothers which are not present on the nominal curves.[9] The term structure of excess returns is upward sloping in the U.S from 1.10% to 1.75% for nominal bonds, and 1.20% to 1.65% for real bonds, between 5 and 10-years. Real and nominal excess returns in the U.K are around 1% lower but again upward sloping.[10]

Figure 7 plots the term structure of Sharpe ratios for both the U.S and the U.K. U.S nominal Sharpe ratios are relatively flat from 0.44 to 0.38 and close to 0.43 across maturity on the real curve.[11] Excluding the crisis, for the sample period 2000.1 - 2008.1, the Sharpe ratio on nominal bonds was much lower at ~ 0.35 across maturity, while the Sharpe ratio on TIPS was higher at ~ 0.55 across maturity. U.K GILT sharpe ratios are around half the size of U.S Treasury sharpe ratios, at close to 0.20 and homogeneous across maturities and markets.

These descriptive statistics are important because positive Sharpe ratios on real bonds provide an important moment to benchmark competing models. For example, in Lucas tree models with power utility the risk premia on long term bonds relative to short term bonds depends on the correlation between consumption and growth rate shocks. If shocks to consumption are expected to revert in the future then short term real rates rates fall given a positive shock to consumption, in which case sharpe ratios on real bonds are positive. However, this implies short term real rates are counter-cyclical, which is counterfactual. In the long run risk model of Bansal and Yaron (2004) and Bansal, Kiku, and Yaron (2009) expected consumption growth is priced even if uncorrelated with shocks to consumption levels. Since positive shocks to expected consumption growth drive up real interest rates (down bond prices) the long-run risks model with positive risk prices for consumption growth imply a negative sharpe ratios on real bonds. Indeed, Beeler and Campbell (2009) argue the long run risks models fails along a number of dimensions when confronted with real interest

[8] Studying real versus nominal (return) risk premia requires observations on short term real rates. Unfortunately, short term inflation linked bills are not issued by either the U.S or the U.K. I proxy for the short term real rates following Campbell and Shiller (1996) as discussed above.

[9] Fleckenstein, Longstaff, and Lustig (2010) also point this out

[10] These results are not reported in any table for sake of space.

[11] Using a no-arbitrage model Duffee (2010) estimates Sharpe ratios on nominal bonds to be (statistically) downward sloping. In general, statistical comparisons of Sharpe ratios rely on assumptions about the data generating process. For a generalised method of moments approach to this problem see Christie (2005).

rates. Notably, real interest rates have little predict power for future consumption growth, and long run risk models calibrated to equity imply low yields and negative risk premia in inflation-indexed markets.

[Insert figure 6 and 7 about here]

II. Theoretical Framework

Many asset pricing models are capable of generating positive *nominal* Sharpe ratios through compensation for inflation risks. Generating positive risk compensation for holding *real* bonds is more difficult. Backus, Gregory, and Zin (1989) make this point clear within the context of a monetary Mehra and Prescott (1985) economy with CRRA preferences. To generate real Sharpe ratios as large as those observed in the U.S Treasury market risk aversion must be around 10, which is probably too large. More troubling, however, the real term structure is upward sloping only if consumption growth is negatively autocorrelated. Since quarterly consumption growth has positive autocorrelation the average risk premium for inflation protected markets should be negative. In this section I study the properties of *real* bond markets in an economy populated by agents who hold subjective models for *real* growth rate dynamics.

A. Homogeneous Agents

Consider a simple endowment economy in which a representative agent has CRRA preferences $u'(c_t) = e^{-\delta t} c_t^{-\gamma}$. The growth rate of endowment is a linear combination of a vector of factors g_t, with

$$dC_t/C_t = g_t dt + \sigma_C dW_t^C \tag{1}$$

$$dg_t = -\kappa_g (g_t - \theta) dt + \sigma_g dW_t^g, \qquad \text{with } \rho_{Cg} = E\left(dW_t^C dW_t^g\right) \tag{2}$$

When agents have common beliefs about the data generating process, it is well known that bond prices satisfy a simple representation. This solution has been studied extensively since Vasicek (1977). The equilibrium pricing kernel $\mathcal{M}_t$ is given by

$$\mathcal{M}_t = u'(C_t) \tag{3}$$

Since $d\mathcal{M}_t/\mathcal{M}_t = -r_t dt - \kappa' dW_t$, using itô's lemma one finds that $r_t = \delta + \gamma\beta g_t - \frac{1}{2}\gamma(1+\gamma)\sigma_C^2$. Thus, if growth rate are constant, i.e. $\beta g_t = g_0$, so are interest rates and the term structure is flat. When g_t is stochastic, bond prices $P(t,T)$ can be computed by solving for the Euler condition $P(t,T) = E_t\left[\mathcal{M}_T/\mathcal{M}_t\right]$. In this economy, the term structure is exponentially affine in g_t and given by

$$P(t,T) = \varrho_\tau \exp\left[A(\tau) + B(\tau) g_t\right] \tag{4}$$

where $\varrho_\tau = e^{-\delta\tau}$. Bond prices belong to the exponentially affine class as characterised by Duffie and Kan (1996) and $A(\tau)$ and $B(\tau)$ are functions of time until maturity (τ) and the structural parameters of the economy (see appendix). In this setting, instantaneous bond excess returns are equal to

$$E_t[rx^{(T)}_{t,t+dt}] = -E_t\left[\frac{dP}{P}\frac{d\mathcal{M}_t}{\mathcal{M}_t}\right] = \frac{1}{P}\frac{\partial P}{\partial g}\sigma_g\gamma\sigma_D\langle dW^g_t, dW^C_t\rangle \tag{5}$$

$$= \underbrace{\gamma\sigma_C\rho_{Cg}}_{\text{Real Price of Risk}} \times \underbrace{\sigma_g B(\tau)}_{\text{Quantity of Risk}} \tag{6}$$

When short run shocks (W^C_t) are correlated with long run shocks (W^g_t) the simple benchmark model restricts expected excess returns to be a scaled multiple of the volatility of macroeconomic fundamentals.[12] As $E(dW^C_t dW^g_t) \to 0$ the risk premium on bonds converges to zero. The tight connection between first and second conditional moments of bond returns has been discussed in the literature as a problematic feature of early models of the term structure (see Duffee (2002) and Dai and Singleton (2000)). Moreover, while yield dynamics are predictable because the conditional growth rate of the economy is time-varying, expected excess returns are constant hence completely unpredictable. To break the tight link between conditional moments and introduce time-variation in risk compensation the literature has focused on two directions: (*i*) models with time-varying quantities of risk (as in Bansal and Yaron (2004) and Bansal and Shaliastovich (2013)) or (*ii*) models with time-varying prices of risk (as in the habit models of Buraschi and Jiltsov (2007) and Wachter (2006)). In the following we study an alternative channel in which both prices and quantities of risk are time-varying.

B. Heterogeneous Agents

Mehra and Prescott (1985) assume the existence of a representative agent who consumes aggregate consumption. Since aggregate consumption is smooth, so is marginal utility. Habit and long run risk models tackle this problem by introducing time-non-seperable preferences. An alternative possibility is that aggregate consumption is a poor proxy for individual consumption. Indeed, using panel data Zeldes (1989) and Parker and Preston (2002) show that the consumption of stockholders is more volatile, and more more correlated with market returns, than non-stockholders. Attanasio and Low (2004) report that per capita volatility of consumption is up to twelve times larger than aggregate consumption.

Models with heterogeneous agents address the smoothness in aggregate consumption by allowing investors to form individual consumption policies based on their beliefs. The key insight of this literature is that, if agents can trade, the equilibrium SDF is affected by disagreement. Consider two agents, a and b, each representing its own class with separate (absolutely continuous) subjective probability measures on the data generating process, denoted as $d\mathcal{P}^a_t$ and $d\mathcal{P}^b_t$. Given a filtered probability space the difference in beliefs between the two agents can be conveniently summarized by the Radon-Nikodym derivative $\eta_t = \frac{d\mathcal{P}^b_t}{d\mathcal{P}^a_t}$, so that for any random variable X_t that is $\mathcal{F}_t$-measurable,

$$E^b(X_T|\mathcal{F}_t) = E^a\left(\eta_T X_T|\mathcal{F}_t\right), \qquad \text{with } \eta_t = 1. \tag{7}$$

[12]This is the case when C_t is assumed to be a deterministic affine transformation of g_t, i.e. $C_t = \exp(\beta'\mathbf{g}_t)$.

In the literature, the Radon-Nikodym derivative η_t is either assumed as an exogenous process or obtained as the outcome of an optimal learning problem in which agents have different prior beliefs.[13] Independent of its microfundations, disagreement among agents affects the distribution of consumption in equilibrium, for $T < \infty$. Agents trade to equate ex-ante expected marginal utility of consumption, $E^a u'(c_T^a|\mathcal{F}_t) = E^b u'(c_T^b|\mathcal{F}_t)$. Thus, using (7), any frictionless equilibrium requires that $E^a u'(c_T^a|\mathcal{F}_t) = E^a(\eta_T u'(c_T^b|\mathcal{F}_t))$ so that innovations in η_t necessarily imply a different allocation of state-contingent consumption c_t^a and c_t^b between the two agents. Optimists will trade to shift consumption to states of the world in which their subjective probabilities are the highest, in exchange for a lower consumption in those states they deem less likely.

Indeed, in equilibrium, agents must have different individual stochastic discount factors, which need to satisfy $\mathcal{M}_t^a = \eta_t \mathcal{M}_t^b$. Thus, ex-post marginal utilities will not be equal since ex-ante beliefs drive a wedge between agent specific consumption.

C. Disagreement

The drift of the consumption process is unobservable which means the objective measure is not defined on either agents' filtration. In such situations it is easy to imagine the emergence of disagreement about the correct model for the economy (for discussion along these lines see Hansen, Heaton, and Li (2008) and Pastor and Stambaugh (2000)).[14] The literature has generally focused on two channels for belief dispersion: (i) subjective priors as in Basak (2000) or Buraschi and Jiltsov (2006); and (ii) subjective models as in David (2008) or Dumas, Kurshev, and Uppal (2009). In both cases agents have common information sets and 'agree to disagree' about how to process information which, mathematically, is represented by different filtered probability spaces $\{\Omega, \mathcal{F}_t^i, \mathcal{P}^i\}$. Regardless of how heterogeneity arises, since C_t is observable, consistent perceptions of dividend innovations require that

$$\begin{aligned} dW_t^{C,i} &= \sigma_D^{-1}\left(dC_t/C_t - g_t^i dt\right) \\ &= dW_t^C + \sigma_C^{-1}\left(g_t - g_t^i\right)dt = dW_t^C + error_t^{e,i}dt \end{aligned} \tag{8}$$

where we have defined standardised forecast error of agent i as $error_t^{e,o}$. Since the above holds for both agents subjective innovations are related by

$$dW_t^{C,b} = dW_t^{C,a} + \sigma_C^{-1}\left(g_t^a - g_t^b\right)dt = dW_t^{C,a} + \psi_t dt \tag{9}$$

where the scaled disagreement process is defined as ψ_t.

[13] Scheinkman and Xiong (2003), Buraschi and Jiltsov (2006), Dumas, Kurshev, and Uppal (2009), and Buraschi and Whelan (2012), all study economies in which the process η_t arise from investors' different prior knowledge about the informativeness of signals and the dynamics of unobservable economic variables. See Kurz (1994) for a discussion on the microfundations of disagreement. Dumas, Kurshev, and Uppal (2009) label the process η_t 'sentiment'.

[14] Hansen, Heaton, and Li (2008) argue about the existence of significant measurement challenges in quantifying the long-run risk-return trade-off and that 'the same statistical challenges that plague econometricians presumably also plague market participants'. Pastor and Stambaugh (2000) discuss the statistical properties of predictive systems when the predictors are autocorrelated but κ_g is not known.

Agents have identical information sets and learn in a Bayesian fashion by filtering the state dynamics from observations of the consumption process (dC_t/C_t) and a publicly available signal (s_t) that is correlated with the stochastic growth rate of the economy $\mathcal{F}_t = \{C_\tau, s_\tau\}_{\tau=0}^t$. The filtering problem contains two measurement equations

$$
\begin{aligned}
dC/C &= g_t dt + \sigma_C dW_t^C, \\
ds_t &= \boldsymbol{\phi}_i dW_t^2 + \sqrt{1-\boldsymbol{\phi}_i{}^2} dW_t^3,
\end{aligned}
$$

where, as in Scheinkman and Xiong (2003), we have introduced a subjective parameter $\boldsymbol{\phi}_i$ that determines the (under / over) reaction of subjective expectations to the arrival of news (overconfidence). In addition to overconfidence about public signals we posit that agents have subjective models for the correlation structure of the economy. Specifically, considering a rotation of our state space by writing 1 and 2 in terms of independent Brownian motions (W_t^1, W_t^2):

$$W_t^C = W_t^1 \tag{10}$$

$$W_t^g = \boldsymbol{\rho}_i W_t^1 + \sqrt{1-\boldsymbol{\rho}_i^2} W_t^2 \tag{11}$$

the parameter $\boldsymbol{\rho}_i$ is agent specific and determines the perceived correlation between shocks to the level of consumption versus shocks to the growth rate of consumption.

Denote agent i's conditional forecast $\hat{g}_t^i = E_t^i[g_t|\mathcal{F}_t]$ and posterior variance $\nu_t^i = E_t^i[(\hat{g}_t^i - g_t)^2|\mathcal{F}_t]$. Since state dynamics are conditionally Gaussian, standard linear filtering results (Lipster and Shiryayev (1974) (theorem 12.7 Page 36)) allow a closed form solution for a posteriori means and variances given by[15]

$$
\begin{aligned}
d\hat{g}_t^i &= -\kappa_g(\hat{g}_t^i - \theta)dt + \left(\frac{\gamma_i^*}{\sigma_C} + \sigma_g \rho_i\right) d\hat{W}_t^{C,i} + \left(\sigma_g \sqrt{1-\rho_i^2}\phi_i\right) ds_t \\
&= -\kappa_g(\hat{g}_t^i - \theta)dt + \sigma_{g,C}^i d\hat{W}_t^{C,i} + \sigma_{g,s}^i ds_t
\end{aligned} \tag{12}
$$

$$\gamma_i^* = \left(-\left(\kappa_g + \rho_i \frac{\sigma_g}{\sigma_C}\right) + \sqrt{\left(\kappa_g + \rho_i \frac{\sigma_g}{\sigma_C}\right)^2 + \left(\frac{\sigma_g}{\sigma_C}\right)^2 \left[1 - \phi_i^2(1-\rho_i^2) - \rho_i^2\right]} \right) \sigma_C^2 \tag{13}$$

The diffusion for standardised disagreement can then be written as

$$
\begin{aligned}
d\psi_t &= \sigma_C^{-1} d(\hat{g}_t^a - \hat{g}_t^b) \\
&= \left[-\kappa_g \frac{\hat{g}_t^a - \hat{g}_t^b}{\sigma_C} - \frac{\sigma_{g,C}^b}{\sigma_C} \frac{\hat{g}_t^a - \hat{g}_t^b}{\sigma_C} \right] dt + \sigma_C^{-1}(\sigma_{g,C}^a - \sigma_{g,C}^b) d\hat{W}_t^{C,a} + \sigma_C^{-1}(\sigma_{g,s}^a - \sigma_{g,s}^b) ds_t \\
&= -\kappa_\psi \psi_t dt + \sigma_{\psi,C} d\hat{W}_t^{C,a} + \sigma_{\psi,s} ds_t
\end{aligned}
$$

Note that the disagreement process has a zero long run mean implying that in the long run agents agree on the state of the economy, but conditionally disagreement can take both

[15] I assume the learning problem began a long time ago so that the posterior variance has converged to its steady state.

positive and negative values, i.e., growth rate optimists can become growth rate pessimists and vice-versa.

D. *Equilibrium Consumption*

The link between equilibrium asset prices and disagreement depends on the aggregation properties of the model. In complete markets, Cuoco and He (1994) show how the competitive equilibrium solution can be obtained from the problem of a central planner. [16] A representative agent U^* can be constructed as a stochastic weighted average $[1, \lambda_t]$ of the marginal utilities of the two agents:[17]

$$\begin{aligned} U^*(c_t, \lambda) :=& \max \left\{ \varrho_t u_a(c_a(t)) + \lambda_t \varrho_t u_b(c_b(t)) \right\} \\ & s.t \ (i) \ \sum_i c_t^i = c_t \forall t \end{aligned}$$

Normalising the weight on agent a, a necessary condition for a social optimum is $u_a'(c_a(t)) = \lambda_t u_b'(c_b)$. From the first order condition of the individual agent problems this implies $\lambda_t = \frac{u_a'(c_a(t))}{u_b'(c_b(t))} = \frac{\alpha_a \mathcal{M}^a(t)}{\alpha_b \mathcal{M}^b(t)}$. Imposing the aggregate resource constraint $(c_t = c_t^a + c_t^b)$ one obtains individual consumption policies and the stochastic discount factors for the representative

$$\begin{gathered} c_t^a = \frac{c_t}{1 + \eta_t^{1/\gamma}} \qquad c_t^b = c_t \frac{\eta_t^{1/\gamma}}{1 + \eta_t^{1/\gamma}} \\ \mathcal{M}_t^a = \varrho_t c_t^{-\gamma} \mathcal{H}(\eta_t) \end{gathered}$$

where one identifies $\mathcal{H}(\eta) = \left(1 + \eta_t^{1/\gamma}\right)^{\gamma}$ as a distortion to the homogeneous power utility pricing kernel.[18] In equilibrium, agents must have different individual stochastic discount factors. This implies an ex-post wedge between marginal utilities which is generated by *beliefs* about fundamentals, rather than fundamentals themselves, resulting in a 'distortion' $\mathcal{H}(\eta_t)$ to the pricing measure with respect to the standard CRRA kernel. As in a habit economy, the local curvature of the representative investor's utility function is now time-varying and

[16] Constantinides (1982) extends Negishi (1960)'s results and proves the existence of a representative agent with heterogeneous preferences and endowments but with homogeneous beliefs. In an incomplete market setting with homogeneous agents Cuoco and He (1994) show a representative agent can be constructed from a social welfare function with stochastic weights. Basak (2000) discuss the aggregation properties in economies with heterogeneous beliefs but complete markets. He shows that a representative can be constructed from a stochastic weighted average of individuals marginal utilities.

[17] Agents choose individual consumption plans, c_t^i, by solving the following intertemporal problem:

$$\max_{c^i} E_0^i \int_0^\infty \varrho_t u(c_t^i) dt \qquad s.t \quad E_0^i \int_0^\infty \mathcal{M}_t^i \left[c_t^i - e_t^i\right] dt \leq 0$$

The first order condition of this problem imply that the optimal consumption policies are of the form $c_t^i = \left(\varrho_t / (\alpha_i \mathcal{M}_t^i)\right)^{1/\gamma}$, where α_i is the Lagrange multiplier associated with the static budget constraint of agent i.

[18] Recently, Klibanoff, Marinacci, and Mukerji (2005, 2009) show that the a smooth ambiguity kernel can also be decomposed as

$$\mathcal{M}_t^* = \mathcal{M}^{\text{CRRA}} \times \mathcal{M}^{\text{EZ}} \times \mathcal{M}^{\text{S.Amb}} \tag{14}$$

given by

$$\gamma^a(t) = -c_t \frac{U_{cc}}{U_c} = \gamma \left[1 + \left(\frac{\omega_t^b}{\omega_t^a} \right)^2 \right]$$

which is strictly larger than γ for $\omega_t^a < \infty$ and tends to γ for the degenerate case that agent a dominates the distribution of wealth. Agents effective risk aversion is therefore state dependent as a function of the history of past consumption choices and, in this sense, provides micro foundations for time-varying risk aversion as in a habit economy: agents' effective risk aversion increases given a negative shock to the consumption shares. [19]

E. The Real Short Rate

Market completeness and the absence of arbitrage guarantee the existence of a unique (possibly subjective) stochastic discount factor for each agent. An application of ito's lemma reveals the equilibrium short rate as

$$r_f = \underbrace{\rho - \frac{1}{2}\gamma(\gamma+1)\sigma_C^2}_{\text{Lucas-Tree Terms}} + \gamma\beta \underbrace{(\omega_a(\eta_t)\hat{g}_t^a + \omega_b(\eta_t)\hat{g}_t^b)}_{\text{Consensus Bias}} + \underbrace{\frac{\gamma-1}{2\gamma}\omega_a(\eta_t)\omega_b(\eta_t)\psi_t^2}_{\text{Speculative Demand}}, \tag{15}$$

where $\omega_i(\eta_t) = c_t^i/C_t$ is investor's i total consumption share.

When the wealth distribution is symmetric ($\eta_t = 1$) and disagreement is zero ($\psi_t = 0$) the short term interest rate is given by the Lucas solution. In the heterogeneous case, the short term interest rates includes two new terms. The first is $\left[\omega_a(\eta_t)\hat{g}_t^a + \omega_b(\eta_t)\hat{g}_t^b\right]$ and is due to the standard wealth effect. The larger the expected growth opportunity, the higher the demand for current consumption, the lower the demand for savings, thus the higher the interest rate. However, when $\eta_t \neq 1$, this term differs from the consensus belief $\frac{1}{2}\hat{g}_t^a + \frac{1}{2}\hat{g}_t^b$. Speculative activity undertaken in the past affects agents' relative wealth today and this term biases the short rate towards the belief of the agent who has been relatively more successful.[20] The implications of this term for the term structure are rich. For example, an immediate implication is that the short rate, and hence the entire yield curve, is path dependent even though state dynamics are Markovian.[21] The second term is due to speculative demand given by the product of $\omega_a(\eta_t)\omega_b(\eta_t)$ and ψ_t^2. To understand this term consider the diffusion for the relative wealth of agent a[22]

$$d\omega_a = \underbrace{\frac{\gamma-1}{2\gamma}\omega_a(\eta_t)\omega_b(\eta_t)\psi_t^2}_{\text{Speculative Demand}} \left[\frac{(\gamma-1) + 2\gamma\omega_b(\eta_t)}{\gamma(\gamma-1)} \right] dt + \frac{1}{\gamma}\omega_a(\eta_t)\omega_b(\eta_t)\psi_t d\hat{W}_t^C \tag{16}$$

from which we identify speculative demand in the drift of individual consumption streams. The reason is because an increase in ψ_t changes the investment opportunity set, as it increases

[19] In the context of equity markets David (2008) also points out that heterogeneous agent models contain backward looking elements similar to that in habit economies.

[20] Jouini and Napp (2006) also construct a consensus investor whose SDF prices the term structure and contains an aggregation bias.

[21] η_t is not Markovian while the couple (η_t, ψ_t) is Markovian.

[22] An analogous diffusion is obtained for agent b under his measure.

speculative opportunities between agents. The sign of the effect depends on whether γ is greater or smaller than 1. For $\gamma > 1$ the wealth effect dominates: speculation raises the drift of planned consumption, which is fixed today; thus, interest rates must rise to clear the market. When $\gamma < 1$ the substitution effect dominates: speculation increases expected returns raising the price of current consumption relative to future consumption, lowering the drift of planned consumption; thus, interest rates must fall.[23]

To visualise the interaction of risk aversion and disagreement on the short rate consider its sensitivity with respect to ψ_t[24]

$$\frac{\partial r}{\partial \psi} = -\gamma\sigma_C \left(\frac{\eta^{1/\gamma}}{1+\eta^{1/\gamma}}\right) + \left(\frac{\gamma-1}{\gamma}\right) \frac{\eta^{1/\gamma}}{(1+\eta^{1/\gamma})^2}\psi_t \tag{17}$$

Figure 8 summarizes the results. The left panel shows that when $\gamma < 1$ the substitution effect dominates and short term interest rates are negatively related to disagreement, $\partial r/\partial\psi < 0$. Moreover, in this region $\partial r/\partial\psi$ is decreasing in the level of disagreement. For $\gamma > 1$ there exists a threshold for ψ above which the wealth effect dominates so that $\partial r/\partial\psi > 0$. The right panel shows the behavior of $\partial r/\partial\psi$ as a continuous function of γ for different levels of disagreement. In economies with larger disagreement and relative risk aversions, short term interest rates are increasing in disagreement; on the other hand, in economies with low risk aversion and low relative risk aversion, interest rates are decreasing in disagreement.

[Insert figure 8 about here]

F. The Term Structure of Bond Prices

The date t price of an T-period default free (real) zero-coupon bond is :

$$P_t^T = \frac{1}{\mathcal{M}_t^i} E_t^i \left[\varrho_{t,T}\mathcal{M}_T^i\right] \tag{18}$$

$$P_t^T = E_t^i \left[\varrho_{t,T}\left(\frac{C_T}{C_t}\right)^{-\gamma}\left(\frac{1+\eta_T^{1/\gamma}}{1+\eta_t^{1/\gamma}}\right)^{\gamma}\right] \tag{19}$$

where the expectation is taken under agent i's measure.[25] Defining $x_T = \ln D_T$ and $y_T = \ln \eta_T$ solving for bond prices requires computing the joint moment generating

$$\phi_{x,y}(T; u_1, u_2) = E_t^i \left(e^{u_1 x_T + u_2 y_T}\right). \tag{20}$$

[23] Also notice small risk aversion (large risk tolerance) generates large volatile consumption streams that are maximised when agents have equal wealth. When agents have low risk aversion, they are very willing to engage in speculative trades that generates large endogenous individual consumption volatility. For very large levels of risk aversion, agents are very unwilling to trade, and the economy collapses to the degenerate single agent case.

[24] The gradient with respect to the state (g_t^a , η_t , ψ_t) is obtained by re-writing the short rate as

$$r_t = \delta - \frac{1}{2}\gamma(\gamma+1)\sigma_C^2 + \gamma g^a - \psi_t\left(\frac{\eta^{1/\gamma}}{1+\eta^{1/\gamma}}\right)\left(\gamma\sigma_D - \frac{\gamma-1}{2\gamma}\frac{1}{1+\eta^{1/\gamma}}\psi_t\right)$$

[25] In equilibrium the solution can be equivalently computed with respect to any probability measure since $P(t,T) = E^a(\mathcal{M}_T^a) = E^b(\mathcal{M}_T^b) = E^*(\mathcal{M}_T^*)$.

which is a function of the 'fundamental' system ϕ_x derived above and the 'belief' system

$$\phi_y = \phi_{x,y}(T; 0, u_2) = E_t\left(e^{u_2 y_T}\right). \tag{21}$$

Given a solution for characteristic function of (C_T, η_T) we can recover the joint density via inversion which allows us to compute the price of any contingent claim.[26] Setting $u_1 = -\gamma$ we evaluate the (inverse) bilateral Laplace transform at $iu2$ using a Fractional Fast Fourier Transform (FrFFT) using the algorithm suggested by Chourdakis (2004).[27]

Theorem 1 (Bond Prices)**.** *The term structure of bond prices is equal to the product of two deterministic functions. The first is exponentially affine in the posterior growth rate of the endowment; the second is quadratic function of differences in beliefs*

$$P(t,T) = \varrho_\tau \phi_x(\tau; -\gamma) \int_0^\infty \frac{g(y,T)}{g(y,t)} \left[\frac{1}{\pi}\int_0^\infty e^{-iu_2 y_T} e^{u_2 y_t} \phi_y(\tau; -\gamma, u_2) du_2\right] dy(T) \tag{22}$$

where

$$g(y,s) = (1 + (e^{y_s})^{\frac{1}{\gamma}})^\gamma \tag{23}$$

$$\phi_x(\tau; -\gamma) = e^{A(\tau)+B(\tau)g_t^a} \tag{24}$$

$$\phi_y(\tau; u_1, u_2) = e^{L(\tau)+M(\tau)\psi_t+N(\tau)\psi_t^2} \tag{25}$$

where $A(\tau), B(\tau), L(\tau), M(\tau), N(\tau)$ are functions of time and the structural parameters of the economy, known in closed form.

The dependence of bond prices on g_t^a is exponentially affine because the dividend process is conditionally Gaussian. Under incomplete information and learning the term structure also explicitly depends on the difference in beliefs ψ_t^g. The dependence on these factors is exponentially quadratic. For the case of integer γ we can exploit the binomial expansion to obtain an exact analytical result in terms of the state vector $[g_t^a, \eta_t, \psi_t^g]$:

$$P(t,T) = \varrho_\tau \phi_x(\tau; -\gamma)(1 + \eta_t^{1/\gamma})^{-\gamma} \sum_{j=0}^{\gamma} \binom{\gamma}{j} (\eta_t)^{j/\gamma} \phi_y(\tau; -\gamma, j/\gamma)$$

from which we see that bond prices in the heterogeneous agent economy are a wealth weighted average of quadratic term structures. The myopic (log utility) for yields is obtained as a special case:

Corollary 1 (Myopic Term Structures)**.** *When agents are myopic heterogeneous equilibrium bond prices are given as wealth weighted averages of fictitious homogeneous equilibrium bond prices. For the gamma = 1 case we see from equation 15 that squared disagreement does*

[26] The solution given by Xiong and Yan (2010) only applies to the case of log utility investors. In a portfolio selection context with irrational investors Dumas, Kurshev, and Uppal (2009) show that the joint density of C_t and η_t can be computed in semi-closed form by Fourier inversion. The spirit of this approach follows methods developed in the option pricing literature by (for example) Heston (1993), Carr and Madan (1999), and Duffie, Pan, and Singleton (2000) for equity, and Chacko and Das (2002) for interest rates.

[27] For an overview of numerical inversion recipes in finance see Kahl and Lord (2010)

not enter the term structure solution. This is because agents are myopic hence there is no speculative demand to their optimal portfolios. We then have

$$P(t,T) = \omega_t^a P^a(t,T) + \omega_t^b P^b(t,T).$$

G. The Yield Curve

When $\gamma \neq 1$, equation (22) implies that the term structure of bond yields satisfy:

$$y(t,T) = \underbrace{\delta + a(\tau) + b(\tau) g_t^a}_{\text{Affine Homogeneous Yields}} + \underbrace{\mathcal{D}(\tau, \eta_t, \psi_t^g)}_{\text{Heterogeneous Adjustment}} \tag{26}$$

The first term is equivalent to what would emerge in the homogeneous benchmark solution. The second term, $\mathcal{D}(\tau, \eta_t, \psi_t^g)$, is new and captures the impact of agents heterogeneity on the yield curve. It depends explicitly on *both* η_t and the level of dispersion in beliefs ψ_t^g. The dependence is quadratic in ψ_t^g; the role of the Radon Nykodim process η_t emerges because of the role played by the relative wealth of the two agents, which depends on past realizations bets entered by the two agents.

I calibrate the economy with parameters quantitively similar to those used in Brennan and Xia (2001). To calibrate the correlation $\rho_{c,g}^i$ between consumption shocks and growth rate shocks I compute a VAR(1) between the date t realised quarterly growth rate of real GDP, and the date $t-4$ 1-quarter forecast for GDP computed from surveys.[28] During the sample period January 1999 and December 2011 we find that the correlation between short run shocks (dW^D) and long run shocks (dW^g) is 0.23. In the calibration I choose subjective $\rho_{c,g}^a = 0.20$ and $\rho_{c,g}^b = 0.90$ to emphasise that fact that short term real rates in a standard Lucas economy should be pro-cyclical and the bond risk premium should be negative, i.e., bonds hedge consumption shocks and the term structure is downward sloping. I set the time rate of preference $\delta = 0$ since this only affects the average level of the term structure. We set the long run mean of consumption growth equal to 3% and its persistence 0.40, which implies a half life for growth rate shocks of 1.73 years, and the volatility of consumption is set equal to 5%. The table below reports the parameter set.

Calibrated Parameters

δ	γ	σ_C	κ_g	θ	σ_g	ϕ_a	ϕ_b	$\rho_{c,g}^a$	$\rho_{c,g}^b$
0	[0.5 , 1.0 , 2.0 , 3.0]	0.05	0.40	0.03	0.02	0.85	0.15	0.20	0.90

To gain insight on the implications of heterogeneity for bond markets I consider the following stylised examples. Note, for simplicity we assume that (a) the true initial level of g_t to be equal to its long run mean, i.e., $g_t = \theta = 3\%$; (b) agents beliefs are symmetric with respect to g_t, i.e., they have a mean preserving spread around θ.

[28] Real GDP is obtained from http://research.stlouisfed.org/fred2/. Survey forecasts for GDP growth are from BlueChip Financial indicators and discussed in the data section that follows.

1. $\omega_a(t) = 0.50$ (**Symmetric Economy**): $g_t^a - g_t^b \in [0\% \ : \ 3\%]$, $\gamma = [0.5 \ , 1.0 \ , \ 2.0 \ , 3.0]$
2. $\omega_a(t) = 0.25$ (**Pessimistic Economy**): $g_t^a - g_t^b \in [0\% \ : \ 3\%]$, $\gamma = [0.5 \ , 1.0 \ , \ 2.0 \ , 3.0]$
3. $\omega_a(t) = 0.75$ (**Optimistic Economy**): $g_t^a - g_t^b \in [0\% \ : 3\%]$, $\gamma = [0.5 \ , 1.0 \ , \ 2.0 \ , 3.0]$

EXAMPLE 1: SYMMETRIC ECONOMY

Figure 10 shows the shape of the term structure for a symmetric economy. The central panel shows that for the log case ($\gamma = 1$) the term structure is always downward sloping for the same reason as the $\gamma = 1$ case discussed directly above. For the $\gamma \neq 1$ case the front end of the curve crucially depends on the level of risk aversion (see also David (2008) for a related discussion). For $\gamma < 1$ the short term interest rates are very low and bond yields are generally increasing in their duration. On the other hand, when $\gamma > 1$ short term interest rates are high and the term structure can become negatively sloped for high disagreement.

Figure 11 shows how the average level (defined as $\frac{1}{n}\sum_{i=1}^{n} y_t^n$) and slope of the term structure respond to an increase of disagreement. The response of the shape of the yield curve to disagreement crucially depends on the trade off between the wealth and substitution effects. For $\gamma < 1$ as disagreement increases: (a) the level of the yield curve decreases (left panel), and (b) the slope increases (right panel). On the other hand, for $\gamma > 1$ as disagreement increases: (a) the level of the yield curve increases, *(ii)* the slope decreases.

[Insert figures 10 and 11 and about here]

EXAMPLE 2 & 4: PESSIMISTIC AND OPTIMISTIC ECONOMIES

An additional empirical implication of the model is that the shape and dynamics of the term structure depends on the interaction between the wealth distribution and the beliefs distribution. Figure 12 shows the shape of the term structure for an economy populated mainly by pessimists, so that $\omega_a(t) = 0.25$. The central panel shows that in the $\gamma = 1$ case, bond yields are decreasing in disagreement. In this economy an increase in the mean preserving spread between $\hat{g}_t^a$ and $\hat{g}_t^b$ increases the difference between the arithmetic consensus growth rate and the wealth-weighted average of the expected growth rates (i.e. $(\omega_a(\eta_t)\hat{g}_t^a + \omega_b(\eta_t)\hat{g}_t^b)$). Thus, an increase in disagreement increases also the aggregation bias toward the pessimists. As a result the level of the term structure falls (since beliefs have no intertemporal effect). In an economy populated mainly by optimists, as in figure 12 with $\omega_a(t) = 0.75$, the opposite holds: an increase in disagreement, increases the aggregation bias toward the optimist so short rates and the level rise.[29]

When $\gamma \neq 1$, one needs to account also for the trade-off between the wealth and substitution effects. Consider first an economy dominated by pessimists (see figure 12, right panel). For $\gamma > 1$ short term rates increase with disagreement so that for large disagreements ψ_t^g the slope becomes negative, while for $\gamma < 1$ short term interest rates fall and the slope

[29]In the context of equity markets Jouini and Napp (2006) argue that aggregate pessimism can lead to an increase of the market price of risk and to a decrease of the risk free rate, and therefore help to resolve the equity premium puzzle.

becomes positive. Interest rates at the long end are less sensitive to current disagreement since disagreement is expected to mean-revert: in our example the half life for $d\psi_t^g$ is 0.88 years compared with the half life for dg_t of 1.73 years. When $\gamma > 1$ one can see the effect on the short rate of the interaction between the aggregation bias and the intertemporal substitution effects. For small values of disagreement, this results in a level shift; for large values of disagreement, however, this results in a significant change of the slope. Figure 12 plots the equivalent results for the optimistic economy which follows similar logic.

H. Risk Compensation

H.1. Instantaneous Risk Premia

The instantaneous bond risk premium is defined as $E_t\left[dP(t,T)/P(t,T)\right] - rdt$. The absence of arbitrage and complete markets ensure equilibrium expected excess returns for each agent are equal to $\kappa^i(t)\sigma^T_{P,D}(t)$, where $\sigma^T_{P,C}(t)$ measures risk exposure to $d\hat{W}_t^{i,C}$, and prices of risk for agent $i = a, b$ which are reported in the appendix. [30] Notice, subjective expected returns can be opposite in sign. For example, consider the case of an economy populated mainly by pessimists: since a smaller mass of optimists must absorb the total risk transfer their risk prices are larger (in magnitude) than the pessimists. Moreover, optimists prices of risk will be *positive* while pessimist's will be *negative*, and therefore expected excess returns on long term bonds will be opposite for both agents. This is important since it means that, depending on the history of outcomes for each agent wealth process, the aggregate risk premium can change sign.

To explore in detail the dynamics of bond risk premia, let us study expected bond excess returns from the perspective of an econometrician endowed with the knowledge of the true data generating process (i.e. he knows the true g_t). In this case, the wealth weighted average risk premium is

$$\mu_e^P(t,T) - \left(\omega_t^a\mu_a^Q(t,T) + \omega_t^b\mu_b^Q(t,T)\right) = \left(\gamma\sigma_C + \sum_{i=a,b}\omega_t^i error^{e,i}(t)\right)\sigma_P(t,T) \qquad (27)$$

where $error^{e,i}(t) = [g_t - \mu_i^P(t,T)]/\sigma_P(t,T)$ [See Appendix]. Therefore, the estimated risk premium depends on the relative wealth weights ω_t^i, which in turn are a function of the full *history* of outcomes between time 0 and t and the full history of beliefs of both agents. This also implies that in general the risk premium is history dependent. Two economies with the same vector $[g_t^a, \psi_t^g]$ at time t have different risk premia depending on whether previous outcomes have increased the weight of the agent who is currently an optimist (or pessimist), as captured by η_t.

In what follows we re-visit example economies 2 (symmetric), 3 (pessimistic), and 4 (optimistic) by computing expected returns from the perspective of the unbiased econometrician.

[30] From Girsanov theorem it follows that the drift of a maturity T bond under the physical and risk neutral measure of the two agents is given by $\mu_i^P(t,T) - \mu_i^Q(t,T) = \kappa^i\sigma_{P,C}(t,T)$.

This requires computing bond sensitivities to dW_t^D and ds_t shocks

$$[\sigma_{P,D}(t,T)\ ,\ \sigma_{P,s}(t,T)] = \frac{1}{P(t,T)}\left[\frac{\partial P(t,T)}{\partial g^a}\ ,\ \frac{\partial P(t,T)}{\partial \eta}\ ,\ \frac{\partial P(t,T)}{\partial \psi}\right]\cdot\begin{bmatrix}\sigma_{g,C} & \sigma_{g,s}\\ -\psi_t & 0\\ \sigma_{\psi,C} & \sigma_{\psi,s}\end{bmatrix} \tag{28}$$

which from equation 22 are known in semi-closed form and given in the appendix. Notice, in heterogeneous agent models volatility depends explicitly on η_t and ψ_t and since the term structure is quadratic in ψ_t^g bond sensitivities to $d\hat{W}_t^C$ shocks are stochastic.

Example 1: Symmetric Economy

Figure 14 summarizes the link between instantaneous risk premia and disagreement for different levels of relative risk aversion. When the economy is equally populated by pessimists and optimists, bond risk premia are decreasing in disagreement when $\gamma \leq 1$ while they are relatively flat when $\gamma > 1$. Even for $\gamma = 3$, however, it can be noticed that the sensitivity of equilibrium risk premia to disagreement is not very large in a symmetric economy. The reason is that when agents forecast errors are symmetric, the large positive distortion in the price of risk of one agent is broadly compensated by the negative distortion a negative distortion in the price of risk of the other. The negative average risk premium is due to the assumption in the calibration that $\rho_{c,g}^i > 0$.[31] In this case bonds are hedges and always command a negative risk premium in a completely symmetric economy. The larger effect on risk premia for $\gamma < 1$ is because in this case agents are almost risk neutral and are willing to substitute intertemporally driving up bond volatilities as they engage in aggressive trading. A key implication of these results is that for belief heterogeneity to have an economically significant impact on bond risk premia one requires asymmetric wealth-weighted beliefs distribution.

[Insert figure 14 about here]

Example 2: Pessimistic and Optimistic Economies

Consider first excess returns for the pessimistic economy (figure 15). At zero disagreement the economy is populated by homogeneous investors in which bonds are hedges and risk premia slope downward across maturity. For all values of γ the risk premium is increasing (in magnitude) in disagreement: large instantaneous differences in belief predict negative returns on long term bonds. This effect comes from two channels: (i) the objective price of risk is positively skewed, i.e., biased towards the forecast error of the pessimists; (ii) bond volatility is increasing (in magnitude) in disagreement, driven by an increase in trade.

Considering now excess returns for the optimistic economy. We see for small values of disagreement bonds remain hedges, however, for intermediate and large values of disagreement bond risk premia change sign. This is an intriguing feature of the model since empirically expected returns on bonds take both positive and negative values (see, for example, Dai and Singleton (2002) or Cochrane and Piazzesi (2005)). Moreover, for large values of disagreement long term bonds command larger risk compensation regardless of the magnitude

[31] The assumption is motivated by the sample unconditional correlation in the data.

of γ. The flipping of the risk premium follows the same logic as the pessimist case: (i) the objective price of risk is negatively skewed and can become negative for large enough disagreements; (ii) bond sensitivities to dW_t^D shocks are again negative and increasing in disagreement.

[Insert figures 15 and 16 about here]

These results suggest a potential explanation of why long term bonds can generate risk premia that change sign. This stylized feature has been discussed by Duffee (2002) as an empirical puzzle and it has been interpreted as evidence that bonds can emerge as either hedges or bets against bad state of the world. In the context of our economy the result emerges due to the interaction between changes in the wealth-weighted distribution of beliefs and the level of disagreement itself.

Example 3: Belief Risk

I now demonstrate the effect of *future* belief risk for the special case of disagreement equal to zero ($\psi_t = 0$). We distinguish numerically the cases of log and non-log preferences agents. The left panel of figure 9 shows the yields curves for $\gamma = 1$, as in Xiong and Yan (2010), for consumption shares $\omega_a(t) \in [0\ ,\ 1]$. Under a log-utility specification, bond prices in the heterogeneous economy are simply a consumption share weighted average of prices in the homogeneous economies. Thus, when myopic agents believe consumption growth is positively autocorrelated, the yield curve are always downward sloping even in the heterogeneous case, as they are bounded by the yield curves in the fictitious homogeneous economies.[32]

When $\gamma > 1$, however, awareness that *future* beliefs will directly affect the level and volatility of short term interest rates has an immediate effect on the current yield curve. The right panel of 9 shows that future belief risk leads to an increase in bond yields, especially for medium and long maturity bonds even when $\psi_t = 0$. The increase is the highest for $\omega_t^a = 0.5$ when agents have equal market power. As a consequence, although yield curves in the homogeneous economies are downward-sloping, yield curves in the heterogeneous economies can be upward sloping and have a humped shape at medium maturities. Indeed, even when disagreement is zero today, because agents know they will disagree tomorrow there is an additional source of risk embedded in bond prices. This risk is captured by the second term in equation 32, which is non-zero even when $\psi_t = 0$, and captures the perceived future covariance between agents wealth shares and beliefs. In addition, from equation 32 we see that when disagreement is non-zero there exists a term related to each agents expected consumption share amplified by disagreement today.

This numerical experiment is interesting since a humped-shape yield curve is common in the data and empirical evidence suggests that concavity of the yield curve (the level of medium maturities relative to the average of short- and long-term maturities) is a good

[32] One can also infer this result from equation 16 which shows the wealth processes in the log utility are martingales under the respect measure of a and b.

proxy of risk compensation for long term bonds (Cochrane and Piazzesi (2005) , Campbell, Sunderam, and Viceira (2009)).

[Insert figure 9 about here]

EXAMPLE 3: UNDERSTANDING BELIEF RISK

In equilibrium, since bonds are in zero net supply, one agent holds a long position while another agent holds a short position. In either case, not only are long term bonds exposed to growth rate shocks, but are also exposed to disagreement (the beliefs of both agents a and b) shocks and, therefore, are more risky than in an otherwise homogeneous economy.

Indeed, the level of long term bond yields depend on state-by-state equilibrium allocations of consumption and can be decomposed into a *term premium* and the expected average short-term yield over the life of the bond. Moreover, the term premium can be computed from the stochastic discount factor and the first order condition that prices default-free T-period zero-coupon bonds:[33]

$$TP(t,T) = y_t^{(T)} - \frac{1}{T}\sum_{j=0}^{T-1} y_{t+j}^{(1)} = -\frac{1}{T}\log E_t\left[\prod_{j=1}^{T}\mathcal{M}_{t+j}\right] + \frac{1}{T}E_t\sum_{j=1}^{T}\log E_{t+j-1}\mathcal{M}_{t+j} \tag{29}$$

Therefore, given a shock today, $\{W_t^C, s_t\}$, the response of the term premium depends on the impulse response function of the stochastic discount factors $\mathcal{M}_{t+j}^i$ for all maturities j. As in discrete time, impulse responses refer to the reaction of a dynamic system in response to some exogenous shock, and have been employed extensively in the literature to study the reaction of term premia to macro-economic shocks as in Ang and Piazzesi (2003), or to monetary policy or technology shocks as in Rudebusch, Sack, and Swanson (2007). Computing continuous time impulse response functions is straightforward using Malliavin calculus, which extends standard calculus of variations to stochastic process defined on a Wiener space.[34] Denoting $\mathcal{D}_t^i X_T$ the response at time T of a process X to a unit Brownian shock dW_t^i, it is possible to show that the impulse response of agent i's stochastic discount factor is given by

$$\frac{\mathcal{D}_t\mathcal{M}_T^i}{\mathcal{M}_T^i} = \underbrace{-\gamma\frac{\mathcal{D}_tC_T}{C_T}}_{\text{A standard Lucas Term}} + \underbrace{\omega_T^i(\eta_T)\frac{\mathcal{D}_t\eta_T}{\eta_T}}_{\text{a heterogeneity term}} \tag{30}$$

From equation 29 the response of the term premium to a $\{W_t^C, s_t\}$ shock depends on the *expected* impulse response of the stochastic discount factors. In a standard Lucas economy this response is deterministic and given by

$$E_t^a\left[\gamma\frac{\mathcal{D}_tC_T}{C_T}\right] = \gamma[\sigma_C\ ,\ 0] + \gamma B(T-t)[\sigma_{gC,a}\ ,\ \sigma_{gs,a}] \tag{31}$$

[33] This relationship follows directly via log-linearisation. The definition of the term premium given here ignores a convexity term which is known to be small in the data. See, for example, Campbell, Lo, MacKinlay, and Whitelaw (1998)

[34] Specifically, given a Wiener functional $F = f(W_{t1},\dots,W_{tn})$, the Malliavin derivative $\mathcal{D}F$ computes the change in F due to a change in the path of W. For an extensive application of Malliavin calculus in finance see Detemple, Garcia, and Rindisbacher (2003).

where $\gamma\sigma_C$ is the standard price of risk and $\gamma B(T-t)\sigma_{gC,a}$ reflects the deterministic response to dW_t^C shocks, which is proportional to $\frac{1-e^{-\kappa_g(u-t)}}{\kappa_g}$. This depends on the persistence of the growth rate of the economy and the volatility of consumption: (i) The larger the volatility, the larger the initial response; (ii) the larger the persistence (smaller κ_g) the longer it takes for shocks to die off.

In a heterogeneous agent economy, on the other hand, the response contains an additional stochastic term due to the dynamics of beliefs and consumption shares. Focusing on expected long run prices of risk for $W_t^{C,a}$ shocks (under the measure of agent a) one can show that

$$E_t^a\left[\omega_T^i(\eta_T)\frac{\mathcal{D}_t\eta_T}{\eta_T}\right] = -\psi_t E_t^a[\omega^a(\eta_T)] - 1/2\sigma_{\psi C,a} E_t^a\left[\omega_T^a \int_t^T 2\psi_s e^{-\kappa_\psi(u-t)} ds\right] \qquad (32)$$

This new term implies that risk premia depend not only on the local properties of η_t but also on the stream of future values of $\eta_{s>t}$ and hence on the path of future beliefs.[35] The reason is because, given a shock today and holding all future shocks at zero, even if the path of the dividend process is deterministic, the path of the consumption sharing rule remains stochastic because of the path dependence due to future disagreement.

III. Testable Implications

The implications of heterogeneous agent models for term structure studied the previous section suggests a number of empirical tests. In the empirical sections below we focus on the following hypothesis:

- H_{01} : **Short Term Real Rate.**
  - disagreement is a state variable driving the short term interest rate due to an intertemporal hedging demand. The sign of the effect depends on the wealth vs substitution effects dominate, i.e., whether γ is bigger than or smaller than one.
- H_{02} : **The Cross-Section of Real Yields.**
  - since disagreement drives short term interest rates it necessarily affects the level and slope of the term structure. Moreover, the level and slope of the term structure inherit a wealth weighted aggregation bias from the short rate. To the extent that past differences in belief proxy for contemporaneous fluctuations in wealth, distance lags of disagreement should affect today's cross-section of yields.
- H_{03} : **Expected Returns on Real Bonds, Relative Wealths, and Future Beliefs.**

[35] John Maynard Keynes famously wrote 'It is not a case of choosing those that, to the best of one's judgment, are really the prettiest, nor even those that average opinion genuinely thinks the prettiest. We have reached the third degree where we devote our intelligences to anticipating what average opinion expects the average opinion to be. And there are some, I believe, who practice the fourth, fifth and higher degrees.' Keynes (1937).

– from the perspective of an unbiased econometrician, average optimism (pessimism) drives positive (negative) variation in the bond risk premium due to a bias in the wealth weighted belief. Since the history of speculate bets determines today's relative wealths, the history of disagreement should predict risk premia after controlling for contemporaneous disagreement. Moreover, while disagreement does not drive bond risk premia instantaneous it drives long run returns.

IV. Empirical Results

In the following section I study empirically the three testable hypotheses outlined above to learn about the properties of heterogeneous beliefs models. In addition, I try to distinguish between alternative structural specifications for the real stochastic discount factor by building proxies for the risk factors that arise in economies with long run risks, (conditional variance of expected fundamentals) and habit economies (consumption surplus). To save space the construction of disagreement (ψ_t^g), volatility (σ_t^g), and surplus ($Surp_t$) factors is relegated to a data appendix at the end of the paper.

H_{01} : Short Term Real Rate

The model predicts that ψ_t^g should drive short term interest rates in addition to the dynamics of fundamentals (g_t^i). This is a result of an optimal hedging demand as opposed to some behavioural bias. Interestingly, the link between ψ_t and r_t depends crucially on whether agents are risk tolerant ($\gamma < 1$) or risk averse ($\gamma > 1$). If $\gamma < 1$ (or $EIS > 1$) positive shocks to disagreement should map negative shocks to r_t since the substitution effect dominates, while if $\gamma > 1$ we should observe the opposite the wealth effect dominates.

I construct time-series for the 3-month real interest rate following Campbell and Shiller (1996) who estimate a VAR model including the ex post real return on a 3-month nominal bills, the nominal bill yield, and lagged annual inflation rate. Solving the VAR forward I build ex-ante forecasts for date t 3-month real rates, which are displayed with their nominal counterparts in figure 1.

[Insert figure 1 about here]

Next, I estimate OLS regressions of short term real rates on contemporaneous disagreement about real GDP growth after controlling for consensus forecasts for real GDP, conditional variance, and surplus factors.

$$\text{real } y_t^{3m} = const + \beta_1 E_t(g_t) + \beta_2 \psi_t^g + \beta_3 \sigma_t^2(g) + \beta_4 Surp_t + \beta_5 \text{real } y_{t-3}^{3m}$$

Table IV reports the results. I standardise all left and right hand variables which means the point estimates measure the standard deviation response of the left hand variable to a 1-unit shock to the right hand variables. Considering the estimates reported in the first and second rows I find real short rates are negatively correlated with ψ_t^g after controlling for expected fundamentals. The loading on real disagreement is statistically significant with a t-statistic of -2.32 and factor loading of -0.28, implying that a 1-standard deviation shock to disagreement lowers the short rate 12 basis points.[36] The third and fourth rows control for structural alternatives that arise in long run risk and habit based consumption models. While the results suggest a weak link between $\sigma^2(g)$ and the real short rate I do obtain an economically large and significant result on *Surp*: the point estimate is 0.31 with a t-statistic of 3.20. In the context of the Campbell and Cochrane (1999) habit specification a positive loading on the short rate implies that the consumption smoothing channel dominates the precautionary savings channel: when surplus is below its mean agents anticipate mean-reversion and so borrow today in order to smooth marginal utility intertemporally. Since short rate rates are rising in bad states long term bonds are a hedge and should therefore command a negative risk premium. However, encouraging for the role of heterogeneity adding long run risk and habit factors does not alter the loading on disagreement. A potential statistical concern with this regression is the near unit root behaviour of the left hand variable. To address this concern the final rows add a 1-quarter lag of the short rate to the right hand side. Indeed, controlling for the persistence of the short rate the result on consumption surplus disappears. However, in support of heterogeneous agent models, I still find a significant negative loading on disagreement with a point estimate of -0.11 and a t-stat of -2.13.

[Insert table IV about here]

A second concern with the above short rate regression is that short term nominal rates are controlled by Federal reserve policy, and to the extent that the Fed influence inflation expectations they may also have an affect short term real rates. Then, if the Fed were reacting to market uncertainty (as proxied by dispersion in beliefs) the negative loadings should not be interpreted in the context of equilibrium risk sharing, but rather the result of monetary policy. Indeed, as discussed in the data section above, our proxy for belief dispersion shares a common component with the economic policy uncertainty factor from Baker, Bloom, and Davis (2012) (figure 17). However, while the Fed does have some control over the effective Federal funds rate (FF) by setting the target rate (TR) the empirical link between target rates and yields of longer maturities is weak. For example, for the period 1982 - 2012 Fama (2013) finds that when the Fed chooses to adjust the TR there is almost no correlation with yield changes beyond 6-month maturities. Motivated by this finding I ask the extent to which disagreement affects 1-year interest rates by considering an augmented Fama and Bliss (1987) complementarity regression,

$$y_{t+1}^{(1)} - y_t^{(1)} = const + \beta_1(f_t^{1,2} - y_t^{(1)}) + \beta_2 CP_t + \beta_3 \psi_t^g + \epsilon_{t+1},$$

for 1-year yield changes in the 1-year interest rate on the 1-year forward-spot spread, the return forecasting factor (CP_t) from Cochrane and Piazzesi (2005), and differences in belief (ψ_t^g).[37] Ideally this regression would use *real* yields but unfortunately 1-year real discount rates cannot be reliably extracted from the inflation linked coupon bonds. Instead, I run the regression using a longer sample period using nominal bond rates. Table V replicates these

[36]The sample average for short rate is 0.26% with a standard deviation of 0.42%.

[37] This approach closely follows Cochrane and Piazzesi (2005) who find that, not only is there a factor in the cross-section of yields that drives risk prices, but also that this factor predicts lower short rates in the future. I construct CP_t in sample using 5-forward rates as in Cochrane and Piazzesi (2005)

results in our overlapping sample period. First, row (i) shows that the forward spot spread *does* contain some information for expected spot changes. The beta on the forward spot spread is 0.32 (the expectation hypothesis predicts 1.00) and the t-stat is significant at the 10% level. Adding CP_t I find that the Cochrane-Piazzesi factor contains marginal predictive information such that a positive shock to CP_t indicates a reduction short term rates over the following year with a standardized point estimate of -0.46 and a t-stat of -3.38. Row (iii) repeats this exercise replacing CP_t with disagreement about the real economy. Interestingly, I find an almost identical result to row (ii), with loadings (t-stats) of -0.39 (-4.05), so that not only is disagreement about real growth contemporaneously negatively correlated with short rates (as discussed above) but also forecasts declining short rates in the future. Finally, row (iv) includes the forward spot spread, the Cochrane-Piazzesi factor, and disagreement. The results show that while the effect of including CP_t and disagreement in the Fama-Bliss complementarity regression have quantitatively similar predictions, they contain largely orthogonal information: the $\overline{R}^2$ rises to 35% including all right hand variables, the economic magnitude of the loadings is unaltered, and both factors remain significant at the 1% level. In summary, the empirical results of this section support heterogeneous beliefs models with $\gamma < 1$ in which increases in disagreement reduces the demand for current consumption resulting in a decline in real short rates.

[Insert table V about here]

H_{02} : *Yield Curves, Intertemporal demand, and Wealth Weighted Beliefs*

A vast literature studies the empirical properties of bond markets from the perspective of reduced form latent factor models. For instance, in affine models date-t risk factors are completely summarised by linear combinations of date-t yields through yield curve inversion (see, for example, Duffee (2002) or Dai and Singleton (2002)). The resulting latent factors are usually labelled level, slope and curvature due to the shape of their factor loadings on yields. While this literature has made great advances in understanding the statistical properties of yields there is little consensus for what these latent factors represent. In the context of our equilibrium treatment we have a joint testable implication that both the level and slope of the term structure are a function of $\psi(t)$ and $\omega^i(t)$. This is an important but challenging test for the model since we do not observe the cross-sectional distribution of wealth.

I propose a simple test based on the model implication that variations in ω_t^i require that agents hold subjective (optimal) portfolio allocations due to *past* differences in belief. Thus, I propose to test the effect of wealth fluctuations using distant lags of differences in belief. To understand of why lagged disagreement should matter empirically consider the stylised example, depicted in figure 19:

[Insert figure 19 about here]

Assume a 3-period sample where in the first node agents have equal wealth ($\omega_0^a = \omega_0^b$) but that agent a is the relative optimist ($g_0^a > g_o^b$). If in the next period a good (up) state is revealed the wealth of the optimist will be large than the pessimist ($\omega_u^a > \omega_u^b$) since his prediction for fundamentals was ex-post more accurate than the pessimist. For simplicity assume that the realised endowment and signals are such that the optimist does not update his beliefs much, since he already predicted a good state, but that the pessimist has a large forecast revision ($g_u^b >> g_o^b$). If in the next period a bad state occurs there will be a redistribution of wealth towards the pessimist ($\omega_u^b < \omega_{ud}^b$) that will depend not only on beliefs from the previous (up) node but also on beliefs from the initial period. Consider now an alternative scenario in which a bad (down) state occurs following by a good (up) state. Initially the wealth distribution is shifted towards the pessimist ($\omega_d^a < \omega_d^b$) but then subsequently redistributed towards the optimist ($\omega_d^a < \omega_{du}^a$). In the context of our empirical tests past disagreement ($\psi_{t-\tau}$) contains important information about todays cross-sectional of yields, conditional on contemporaneous disagreement (ψ_t), since the distribution of wealth depends jointly on the path of beliefs and actual realisations of the economy.

To test this hypothesis I compute the level and slope of the term structure from the 1st and 2nd principle components of yields for maturities 3-months and 3 : 10-years, as in section I above. The left panel of figure 3 displays the eigenvector loadings across maturity and shows that the 1st principle component is a level factor while the second principle component is a slope factor. Figure 4 show the time-series properties of real-nominal level and slope factors, which as argued above, display a large degree of co-movement. To test the models prediction regarding the link between the disagreement and relative wealth fluctuations I estimate the following regressions:

$$\text{Real } Level(t) = const + \beta_1 E_t(g_t) + \beta_2 \psi^g(t) + \beta_3 \psi^g(t-6) + \beta_4 \sigma_t^2(g) + \beta_5 Surp_t + \varepsilon_t^{\text{level}}$$
$$\text{Real } Slope(t) = const + \gamma_1 E_t(g_t) + \gamma_2 \psi^g(t) + \gamma_3 \psi^g(t-6) + \gamma_4 \sigma_t^2(g) + \gamma_5 Surp_t + \varepsilon_t^{\text{slope}}$$

[Insert table VI about here]

Table VI reports the results. Considering the effect of contemporaneous disagreement (belief risk) and lagged disagreement (relative wealth fluctuations) on the level of the real term structure I find no significant relationship: the factor loadings are not significant and the R^2 almost zero. Considering the slope of the real term structure I find a strong effect in terms of statistical and economic significance. Since both left and right hand variables are standardised economic significance can be understood in terms of the size of the point estimates: a 1-unit shock to $\psi^g(t)$ raises the slope of the yield curve by 0.50 standard deviations with a t-statistic is 4.95, after controlling for the marginal contribution of expected real growth, which also has large positive statistical significant. With contemporaneous dispersion and consensus forecasts as the only explanatory variables the $\overline{R}^2$ is 31%. Adding a 6-month lag of disagreement to the regression increases the $\overline{R}^2$ to 38% with lagged disagreement significant at the 1% level. This is an intriguing result which suggests past speculative activity has a large effect of the shape of the yield curve today, consistent with the model prediction that agents are trading on their beliefs. The final rows of table VI control for alternative structural explanations. Indeed, I do find that both the conditional variance of expected GDP growth and consumption surplus are important for explaining time variation in the level of the term structure with an overall $\overline{R}^2$ of 47%. In particular, *Surp* appears to be important for level variation both in terms of economic ($\beta_5 = 0.83$) and statistical significance (t-stat = 9.48). However, neither long run risk or habit factors alter the estimated relationship between disagreement risk or relative wealth fluctuations (lagged disagreement) for the slope of the term structure: both factors retain significance at the 1% level and additional regressors leave the $\overline{R}^2$ unchanged. I note, however, that the sign on $Surp$ is negative consistent with the real short rate regressions above that imply, within the context of habit economies, real bonds should hedge marginal utility. The loading on the real short rate implies positive shocks to $Surp$ lower the risk compensation on long term bonds, and the negative sign in the final row of table VI is consistent with this finding.

H_{03} : Expected Returns, Relative Wealths, and Future Beliefs

The previous section studied the role of intertemporal demand versus the effect of relative wealth fluctuation for the cross-sectional distribution of yields. In this section I study risk premia: the third testable implication of the model is that time-variation in bond risk premia is driven by (*i*) variations in relative wealth; and (*ii*) current belief risk. This is an important additional test of the model since, while the space of yields provides information on the drift of the stochastic discount factor, the space of excess returns gives us information on its diffusion component (prices of risk). As above, I propose to test relative wealth fluctuations from lags of disagreement. The underlying assumption is that the ex-post distribution of wealth is shifted towards the agent who holds relatively more accurate forecasts. Thus, conditional on a sample period with a series of large belief dispersions one should expect a large subsequent redistribution of wealth.

I repeat briefly how long run return variation arises within multiple agent economies. Optimists insure pessimists against bad state in exchange for premium in good states. This implies optimists' prices of risk are higher than pessimists so that wealth weighted instantaneous prices of risk are positively (negatively) skewed when optimists (pessimists) dominate the economy, with respect to those in an otherwise equal homogeneous Lucas economy. However, because of future belief updating the long run impulse response function of the stochastic discount factor, as opposed to its instantaneous response (prices of risk) inherit additional stochastic terms due to 'higher order beliefs', and these represent an additional source of risk. Therefore, the model predicts that both relative wealth fluctuations (proxied by lagged disagreement) and current disagreement matter for equilibrium risk compensation. In order to test this prediction the following subsection proposes a measure of the real bond risk premium based on the forward rate return predictability as studied by Cochrane and Piazzesi (2005).

Real Cochrane and Piazzesi (2005)

Figure 6 showed the dynamics of *realised* excess returns on real and nominal bonds have large co-movements. This section examines whether the dynamics of *expected* excess returns are also co-moving. Given a date t cross-section of bond yields all information regarding future interest rates (and thus expected returns) is summarised in the shape of the term structure today. Linear combinations of yields suffice to characterise risk factors through yield curve inversion.[38] Building on this notion Cochrane and Piazzesi (2005) show that an affine function of forward rates embeds substantial information for the dynamics of excess returns. The Cochrane-Piazzesi return forecasting factor, CP_t, is a tent-shaped linear combination of forward rates that predicts excess returns on bonds with $\overline{R}^2$ statistics as high as 43% (in their sample period) and has been shown to capture $\sim$ 99% of the time-variation in expected returns.

I adapt the bond risk premium estimates proposed by Cochrane and Piazzesi (2005, 2008) for inflation protected securities. Specifically, in a 1-stage regression common factors are formed by projecting 3-month excess returns on an equally weighted portfolio of bonds of maturities $n = 5 : 10$ on 3-month forward rate spreads:

$$\frac{1}{5}\sum_{n=1}^{5} rx_{t,t+0.25}^{(n)} = \overline{rx}_{t,t+0.25} = \overline{\alpha} + \gamma'(f_t^{[5\ 7\ 10]} - y_t^{(0.25)}) + \overline{\epsilon}_{t+0.25}$$

$$CP_t = \gamma'(f_t^{[5\ 7\ 10]} - y_t^{(0.25)})$$

where $f_t^{(n)}$ is the date t 3-month forward rate for borrowing for periods $n \rightarrow n + 0.25$.

[insert figures 20 and 21 about here]

Real CP_t^r inherits many of the properties of nominal $CP_t^{\$}$: Firstly, figure 20 shows a tent shaped factor structure is clearly visible on both real and nominal curves, for both the U.S and the U.K. Interpreting the 'shape' of the factor loadings is debatable since, as argued by Dai, Singleton, and Yang (2004), including yields of close maturity on the right hand side introduces collinearity problems. More importantly, the magnitude of the tent shape is increasing in maturity implying that a single factor predicts increasing risk premia on long term bonds. Secondly, while embedding substantial information about future returns CP_t^r explains less than 1% of the variance of contemporaneous yield changes.[39] Thirdly, figure 21 plots time-series for real and nominal risk premium proxies, which both display countercyclical behaviour peaking in 2002 and again in 2008/2009. Excluding the crisis period the correlation between the series is 0.56 while the full sample correlation is 0.46. This co-movement is intriguing since observing a common tent-shaped factor structure does not imply risk premiums should be positively correlated. Indeed, differences in the forward curves could have led to differences in return predictability. This evidence means that not only are *realised* excess returns highly correlated moving in lockstep but *expected* returns are also co-moving across markets. Next, I evaluate the statistical significance of this finding by projecting excess returns onto CP_t^r and $CP_t^{\$}$ in second stage regression

[38] Specifically, assume N bond yields are measured without error. Then, stacking these yields into the vector $y^N = A^N + B^N X_t$, I can solve for the risk factors through inversion as $X_t = (B^N)^{-1}\left(y^N - A^N\right)$ so long as the matrix B^N is non-singular.

[39] I do not report this result to save space but refer the reader to table 8 of the NBER version of Cochrane and Piazzesi (2005) for details.

$$rx_{t+0.25}^{\$,(5:10)} = \alpha_n + b_n CP_t^{\$} + \epsilon_{t+0.25}^{(n)}$$
$$rx_{t+0.25}^{r,(5:10)} = \alpha_n + b_n CP_t^{r} + \epsilon_{t+0.25}^{(n)}$$

Table VII reports point estimates, t-statistics, and R^2's computed via a 2-stage GMM approach that corrects for generated regressors. For the both countries real and nominal loadings are statistically significant at the 1% level. The magnitude of the nominal loadings are close to the real loadings. In the U.S ranging from 0.71 (5-years) to 1.22 (10-years) for nominal expected returns, and from 0.81 (5-years) to 1.15 (10-years) for real expected returns. In terms of relative predictability, nominal U.S bonds appear marginally more predictable than real bonds, with R^2's of $\sim$ 16% compared to $\sim$ 9% for real bonds. In the U.K this result flips, where real bonds appears more than twice as predictable, with nominal R^2's from 4% to 10% compared to real R^2's from 19% to 21%. These findings suggest a large proportion of the variation in nominal bond risk premia is common to the real term structure, and that the dominant source of bond predictability should be interpreted in the context of the real stochastic discount factor, as opposed to the dynamics of inflation.

[insert table VII about here]

Next, I ask whether the predictions given by heterogeneous belief models can help explain such return predictability. I consider $CP_t^{r,\$}$ as a measure of the spanned risk premium and estimate the regressions

$$CP_t^{r,\$} = const + \beta_1 \psi^g(t) + \beta_2 \psi^g(t-6) + \beta_3 \sigma_t^2(g) + \beta_4 Surp_t + \varepsilon_t$$

Table VIII reports the results for a number of specifications. The top panel reports estimates of univariate regressions with date t disagreement on the right hand side. The results suggest that both real and nominal CP_t factors are positively and significantly related to ψ_t^g with point estimates (t-statistics) of 0.34 (2.82) and 0.32 (2.28), respectively. Moreover, the degree of predictable variation that disagreement can explain is quantitatively similar across markets with R^2 equal to 12% (nominal) and 10% (real). Moving to the middle panel I include a 6-month lag of disagreement, $\psi^g(t-6)$, to the right hand side. Considering the nominal risk premium factor adding lagged disagreement almost triples the $\overline{R}^2$ from 12% so 30%. Both belief risk and relative wealth proxies are jointly statistically significant with t-stats of 2.07 and 3.87, respectively. However, I note the economic importance of lagged disagreement is twice the size as date t disagreement with a loading of 0.43 compared with 0.21. Considering the real risk premium factor adding $\psi^g(t-6)$ drives out $\psi^g(t)$ whose t-statistic drops from 2.28 to 1.69, however the $\overline{R}^2$ increases by 5% with respect to the univariate specification in the top panel, suggesting some marginal importance for both factors. The bottom panel controls for second moment of expected GDP growth and surplus factors which are state variables driving time-varying risk compensation in long run risk and habit economies. Interestingly, while the volatility factor appears unimportant, the estimated loading on $Surp_t$ is strongly statistically significant and negative ($\beta_4 = -0.39$ with a t-stat $= -3.83$). The sign on $Surp_t$ is consistent with both short rate and slope regressions above, which imply positive shocks to consumption surplus should drive down the risk premium. Moreover, the $\overline{R}^2$ increases by

10% above the specification reported in the middle panel. However, considering the real risk premium regression the sign on $Surp_t$ flips but is not statistically significant. To investigate the robustness the I report projections including only $\sigma_t^2(g)$ and $Surp_t$:

$$CP_t^{\$} = const + \underbrace{-0.17}_{(-0.74)}\sigma_t^2(g) + \underbrace{-0.52}_{(-6.26)}Surp_t \qquad \overline{R}^2 = 26\%$$

$$CP_t^{r} = const + \underbrace{-0.24}_{(-1.12)}\sigma_t^2(g) + \underbrace{-0.03}_{(-0.19)}Surp_t \qquad \overline{R}^2 = 4\%$$

The results confirm $Surp_t$ does indeed contain significant information for variation in 3-month nominal risk premium, and the sign on the 3-month real risk premium is, in fact, negative. However there is no statistical link. Finally, and encouraging for belief based explanation of time-varying risk premia, the bottom panel of table VIII confirms there is significant information about expected returns embedded in distant lags of disagreement consistent with the model implication that $\omega^i(t)$ is a function of past $\psi^g(t-h)$.

[Insert table VIII about here]

There are two potential concerns with inference from the above regression results. Firstly, $CP_t^{r,\$}$ is an estimated risk premium that may contain measurement errors. Secondly, the sample period is unavoidably short due to the availability of TIPS. I re-examine the finding that excess returns are predictable by dispersion factors using a longer sample period (1990.1 - 2010.12) of nominal 1-year excess returns on $n = 2$ and 5-year zeros

$$hprx_{t,t+12}^{n=2,5} = const + \beta_1\psi^g(t) + \beta_2\psi^g(t-6) + \beta_3\sigma_t^2(g) + \beta_4 Surp_t + \varepsilon_{t,t+12}^{(n)}$$

Table IX reports point estimates and test statistics. Considering expected excess returns on 2-year bonds we find that $\psi^g(t)$ is statistically significant at the 1% for both $n = 2$ and $n = 5$ with R^2 equal to 18% and 12%, respectively. Adding lagged disagreement factors raises the $\overline{R}^2$ to to 22% and 18% and, moreover, both disagreement proxies have large individual (also joint) significance. Controlling for volatility and surplus factors over this longer sample period I find point estimates that are statistically indistinguishable from zero. In summary, I find an economically and statistically meaningful role for belief risk and relative wealth fluctuations that are consistent with the predictions from the model. Furthermore, these results are robust across markets and specifications for measuring bond risk premia.

[Insert table IX about here]

V. Appendix A: Proofs

A. Single Agent Term Structure of Interest Rates

The date t price of an T-period default free zero-coupon bond is :

$$P_t^T = \frac{1}{\mathcal{M}_t} E_t\left[\varrho_{t,T}\mathcal{M}_T\right] = E_t\left[\varrho_{t,T}\left(\frac{D_T}{D_t}\right)^{-\gamma}\right]$$

Defining $x_T = \ln D_T$ solving for bond prices requires computing the following joint moment generating:

$$\phi_x(T; u_1) = E_t\left(e^{u_1 x_T}\right). \tag{33}$$

We denote this the 'fundamental system'.

B. Fundamental System

From Feynman-Kac this function satisfies the following partial differential equation

$$0 \equiv \mathcal{D}\phi_x + \frac{\partial \phi_x}{\partial t}(D, g, t, T; u_1)$$

with initial condition $\phi_x(t; u_1) = D(t)^{u_1}$ and where $\mathcal{D}$ is the Dynkin operator for the multivariate process (D_t, g_t). Applying the operator we have

$$\frac{\partial \phi_x}{\partial D} D_t g_t - \frac{\partial \phi_x}{\partial g}\kappa_g(g_t - \theta) + \frac{1}{2}\frac{\partial^2 \phi_x}{\partial D^2} D_t^2 \sigma_D^2 + \frac{1}{2}\frac{\partial^2 \phi_x}{\partial (g)^2}\left[\sigma_{g,D}^2 + \sigma_{g,s}^2\right] + \frac{\partial^2 \phi_x}{\partial D \partial g} D_t \sigma_D \sigma_{g,D} + \frac{\partial \phi_x}{\partial t} = 0$$

Defining $\tau = T - t$ the solution has the following form

$$\phi_x = D_t^{u_1} E_t\left[\left(\frac{D_T}{D_t}\right)^{u_1}\right] = D_t^{u_1} e^{A(\tau, u_1) + B(\tau, u_1) g_t} \tag{34}$$

Taking partials with respect to D_t we obtain

$$u_1 \hat{g}_t - B(\tau)[\kappa_g(g_t - \theta)] + \frac{1}{2}u_1(u_1 - 1)\sigma_D^2 + \frac{1}{2}B^2(\tau)\sigma_g^2 + u_1 B(\tau)\rho\sigma_D\sigma_g + \left[\frac{\partial A}{\partial t} + \frac{\partial B}{\partial t}\hat{g}_t\right] = 0$$

which are separable ODE's

$$\begin{aligned}
\frac{\partial A}{\partial \tau} &= +\frac{1}{2}u_1(u_1 - 1)\sigma_D^2 + [\kappa_g\theta + u_1\rho\sigma_D\sigma_g]B(\tau) + \frac{1}{2}B(\tau)^2\sigma_g^2 \\
\frac{\partial B}{\partial \tau} &+ \kappa_g B(\tau) = u_1
\end{aligned}$$

with solutions given by

$$\begin{aligned}
A(\tau) &= \frac{1}{2}u_1(u_1 - 1)\sigma_D^2\tau + \left(\frac{u_1\theta}{\kappa_g} + \frac{u_1^2}{\kappa_g^2}\sigma_D\sigma_{g,D}\right)\left(\kappa_g\tau + e^{-\kappa_g\tau} - 1\right) \\
&\quad + \frac{1}{4}\frac{u_1^2}{\kappa_g^3}\left(\sigma_{g,D}^2 + \sigma_{g,s}^2\right)\left(2\kappa_g\tau + 4e^{-\kappa_g\tau} - e^{-2\kappa_g\tau} - 3\right) \\
B(\tau) &= \frac{u_1}{\kappa_g}\left(1 - e^{-\kappa_g\tau}\right)
\end{aligned}$$

Identifying $u_1 = -\gamma$ we obtain the functions reported in the body of the paper.

C. Learning

Denote agent i's conditional forecast $\hat{g}_t^i = E_t^i[g_t|\mathcal{F}_t]$ and posterior variance $\nu)_t = E_t^i[(\hat{g}_t^i - g_t)^2|\mathcal{F}_t]$. Writing the correlated Brownians in terms of the independent Brownian motions (W_t^1, W_t^2):

$$W_t^D = W_t^1 \tag{35}$$

$$W_t^g = \rho_i W_t^1 + \sqrt{1-\rho_i^2} W_t^2 \tag{36}$$

where ρ_i is an agent specific parameter.The filtering problem contains three independent Brownian motions and two measurement equations

$$\begin{aligned} dD/D &= g_t dt + \sigma_D dW_t^1, \\ ds_t &= \phi_i dW_t^2 + \sqrt{1-\phi_i^2} dW_t^3, \end{aligned}$$

Writing the unobservable state in terms of our rotated Brownians we have

$$dg_t = -k_g(g_t - \theta^g)dt + \sigma_g \rho dW_t^1 + \sigma_g \sqrt{1-\rho^2} dW_t^2$$

Using notation consistent with Lipster and Shiryayev (1974) we can re-write the measurement equation in vector form $dY_t = \left[\begin{array}{cc} dD_t/D_t \ , & ds_t \end{array} \right]'$ so that

$$\underbrace{\begin{bmatrix} dD/D \\ ds_t \end{bmatrix}}_{dY_t} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{A_0} dt + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{A_1} g_t dt + \underbrace{\begin{bmatrix} \sigma_D \\ 0 \end{bmatrix}}_{B_1} dW_t^1 + \underbrace{\begin{bmatrix} 0 \\ \phi_i \end{bmatrix}}_{B_2} dW_t^2 + \underbrace{\begin{bmatrix} 0 \\ \sqrt{1-\phi_i^2} \end{bmatrix}}_{B_3} dW_t^3$$

$$dY_t = (A_0 + A_1 g_t)\, dt + B_1 dW_t^1 + B_2 dW_t^2 + B_3 dW_t^3$$

At the same time, consistent with this notation we can write the state dynamics as:

$$dg_t^i = \left(\underbrace{a_0}_{=\kappa_g \theta} + \underbrace{a_1}_{=-\kappa_g} \hat{g}_t^i \right) dt + \underbrace{b_1}_{=\sigma_g \rho_i} dW_t^1 + \underbrace{b_2}_{=\sigma_g \sqrt{1-\rho_i^2}} dW_t^2 + \underbrace{b_3}_{=0} dW_t^3$$

Applying the results of Lipster and Shiryayev (1974) (theorem 12.7. Page 36) the optimal linear in terms of our original Brownians follows as reported in the body of the paper.

D. Term Structure of Interest Rates

The date t price of an T-period default free zero-coupon bond is :

$$\begin{aligned} P_t^T &= \frac{1}{\mathcal{M}_t^i} E_t^i \left[\varrho_{t,T} \mathcal{M}_T^i \right] \\ P_t^T &= E_t^i \left[\varrho_{t,T} \left(\frac{D_T}{D_t} \right)^{-\gamma} \left(\frac{1+\eta_T^{1/\gamma}}{1+\eta_t^{1/\gamma}} \right)^{\gamma} \right] \end{aligned}$$

where the expectation is taken under agent i's measure. Defining $x_T = \ln D_T$ and $y_T = \ln \eta_T$ we need to compute the following joint moment generating:

$$\phi_{x,y}(T; u_1, u_2) = E_t^i \left(e^{u_1 x_T + u_2 y_T} \right). \tag{37}$$

E. Belief System

Setting $u_1 = 0$ in equation F we obtain the following moment generating function for η_T

$$\phi_y = \phi_{x,y}(T; 0, u_2) = E_t \left(e^{u_2 y_T} \right). \tag{38}$$

Feynman-Kac implies the following partial differential equation

$$0 \equiv \mathcal{D}\phi_y + \frac{\partial \phi_y}{\partial t}(\eta, \psi, t, T; u_2)$$

with initial condition $\phi_y(t; u_2) = \eta(t)^{u_2}$. Applying the operator we have

$$-\frac{\partial \phi_y}{\partial \psi}\kappa_\psi \psi_t + 1/2\frac{\partial^2 \phi_y}{\partial \eta^2}\eta_t^2\psi_t^2 + \frac{1}{2}\frac{\partial^2 \phi_y}{\partial \psi^2}(\sigma_{\psi,D}^2 + \sigma_{\psi,s}^2) - \frac{\partial^2 \phi_y}{\partial \psi \partial \eta}\eta_t\psi_t\sigma_{\psi,D} + \frac{\partial \phi_y}{\partial t} = 0$$

whose solution is affine in the extended state vector $(\psi_t, \psi_t^2)'$

$$\phi_y = \eta_t^{u_2} E_t\left[\left(\frac{\eta_T}{\eta_t}\right)^{u_2}\right] = \eta_t^{u_2} e^{L(\tau)+M(\tau)\psi_t+N(\tau)\psi_t^2} \tag{39}$$

Taking partials with respect to η_t we obtain

$$\begin{aligned}&-[M(\tau)+2N(\tau)\psi_t]\kappa_\psi\psi_t + \frac{1}{2}u_2(u_2-1)\psi_t^2 + \frac{1}{2}(\sigma_{\psi,D}^2+\sigma_{\psi,s}^2)[2N(\tau)+M(\tau)^2+4M(\tau)N(\tau)\psi_t+4N(\tau)^2\psi_t^2]\\ &-[M(\tau)+2N(\tau)\psi_t]u_2\sigma_{\psi,D}\psi_t + \left[\frac{L(\tau)}{\partial t} + \frac{M(\tau)}{\partial t}\psi_t + \frac{N(\tau)}{\partial t}\psi_t^2\right] = 0\end{aligned}$$

Defining $c_1 = -(\kappa_\psi + u_2\sigma_{\psi,D})$, $c_2 = \frac{1}{2}u_2(u_2-1)$, $c_3 = \frac{1}{2}(\sigma_{\psi,D}^2 + \sigma_{\psi,s}^2)$ we collect the following ODEs

$$\frac{\partial L}{\partial \tau} = c_3[2N(\tau) + M(\tau)^2] \tag{40}$$

$$\frac{\partial M}{\partial \tau} = [4c_3N(\tau) + c_1]M(\tau) \tag{41}$$

$$\frac{\partial N}{\partial \tau} = 4c_3N(\tau)^2 + 2c_1N(\tau) + c_2 \tag{42}$$

The last ODE is a constant coefficient Riccati equation whose solution is given by

$$N(\tau) = \frac{q}{2c_3}\frac{1}{\tilde{c}e^{2q\tau}+1} - \left(\frac{q+c_1}{4c_3}\right)$$

$$\text{where} \quad q = \sqrt{c_1^2 - 4c_3c_2} \quad \text{and} \quad \tilde{c} = \frac{q-c_1}{q+c_1}$$

Solutions for $L(\tau)$ and $M(\tau)$ can be computed in closed form but we omit these to save space.

Solving the Constant Coefficient (scalar) Riccati Equation

We derive a general solution so the constant coefficient Riccati equations $y(x)$:

$$\frac{dy(x)}{dx} = ay(x)^2 + by(x) + c$$

which we can reduce to a second order linear equation via the substitution $y(x) = -\frac{1}{a}\frac{w'(x)}{w(x)}$ yielding

$$w'' - bw' + acw = 0.$$

The general solution to this ODE is $w(x) = C_1e^{r_1x} + C_2e^{r_2x}$ where r_1 and r_2 are the positive and negative roots

$$r_{1,2} = \frac{b \pm \sqrt{b^2-4ac}}{2} = \frac{b \pm q}{2}$$

and $C_{1,2}$ are constants determined by the boundary condition. A general solution is then

$$y(x) = -\frac{1}{a}\frac{C_1r_1e^{r_1x} + C_2r_2e^{r_2x}}{C_1e^{r_1x} + C_2e^{r_2x}} = -\frac{1}{a}\frac{\tilde{c}r_1e^{qx} + r_2}{\tilde{c}e^{qx}+1} = \frac{1}{a}\frac{q}{\tilde{c}e^{qx}+1} - \left(\frac{b+q}{2a}\right)$$

where the last step follows by some partial fractions algebra. This result is applied above with an appropriate boundary condition.

F. *Joint Distribution*

We now derive the joint moment generating function:

$$\phi_{x,y}(T; u_1, u_2) = E_t^i \left(e^{u_1 x_T + u_2 y_T} \right).$$

which from Feynman-Kac satisfies the following partial differential equation

$$0 \equiv \mathcal{D}\phi_{x,y} + \frac{\partial \phi_{x,y}}{\partial t}(D, \eta, g^a, \psi, t, T; u_1, u_2)$$

with initial condition $\phi_{x,y} t, t; \epsilon, \chi) = D(t)^{u_1} \eta(t)^{u_2}$ and where $\mathcal{D}$ is the Dynkin operator for the multivariate process $(D_t, \eta_t, g_t^a, \psi_t)$. We solve for prices under the measure of agent a and so all parameters and expectations in the following should be understood from his perspective. Applying the operator (and dropping subscripts on $\phi_{x,y}$ we have

$$\begin{aligned}
&\frac{\partial \phi}{\partial D} D_t g_t^a - \frac{\partial \phi}{\partial g^a}\kappa_g (g_t^a - \theta) - \frac{\partial \phi}{\partial \psi}\kappa_\psi \psi_t + \frac{1}{2}\frac{\partial^2 \phi}{\partial D^2} D_t^2 \sigma_D^2 + \frac{1}{2}\frac{\partial^2 \phi}{\partial \eta^2}\eta_t^2 \psi_t^2 + \frac{1}{2}\frac{\partial^2 \phi}{\partial (g^a)^2}\left[\sigma_{g,D}^2 + \sigma_{g,s}^2\right] + \frac{1}{2}\frac{\partial^2 \phi}{\partial \psi^2}[\sigma_{\psi,D}^2 + \sigma_{\psi,s}^2] \\
&- \frac{\partial^2 \phi}{\partial D \partial \eta} D_t \sigma_D \eta_t \psi_t + \frac{\partial^2 \phi}{\partial D \partial g^a} D_t \sigma_D \sigma_{g,D} + \frac{\partial^2 \phi}{\partial D \partial \psi} D_t \sigma_D \sigma_{\psi,D} - \frac{\partial^2 \phi}{\partial \eta \partial g^a}\eta_t \psi_t \sigma_{g,D} \\
&- \frac{\partial^2 \phi}{\partial \eta \partial \psi}\eta_t \psi_t \sigma_{\psi,D} + \frac{\partial^2 \phi}{\partial g^a \partial \psi}[\sigma_{g,D}\sigma_{\psi,D} + \sigma_{g,s}\sigma_{\psi,s}] + \frac{\partial \phi}{\partial t} = 0
\end{aligned}$$

The solution takes the following form

$$\begin{aligned}
&\phi(T) = D_t^{u_1} \eta_t^{u_2} E_t \left[\left(\frac{D_T}{D_t} \right)^{u_1} \right] E_t \left[\left(\frac{\eta_T}{\eta_t} \right)^{u_2} \right] = D_t^{u_1} \eta_t^{u_2} \phi_x(T; u_1) \phi_y(T; u_2) \\
&\text{where } \phi_x(T; u_1) = e^{A(\tau, u_1) + B(\tau, u_1) g_t} \text{ and } \phi_y(T; u_1, u_2) = e^{L(\tau) + M(\tau)\psi_t + N(\tau)\psi_t^2}
\end{aligned}$$

Taking partials with respect to D_t and η_t allows us to factor out terms independent of ψ_t. The solutions for $A(\tau)$ and $B(\tau)$ are then as above. Next, terms in ψ_t solve

$$\begin{aligned}
&\frac{\partial \phi_y}{\partial \psi}\left[-\kappa_\psi \psi_t + u_1 \sigma_D \sigma_{\psi,D} - u_2 \sigma_{\psi,D}\psi_t + \frac{u_1}{\kappa_g}(1 - e^{-\kappa_g \tau})(\sigma_{g,D}\sigma_{\psi,D} + \sigma_{g,s}\sigma_{\psi,s}) \right] + \frac{1}{2}\frac{\partial^2 \phi_y}{\partial \psi^2}\left[\sigma_{\psi,D}^2 + \sigma_{\psi,s}^2\right] \\
&+ \phi_y u_1 \left(-u_2 \sigma_D - \frac{1}{\kappa_g}(1 - e^{-\kappa_g \tau}) u_2 \sigma_{g,D} \right)\psi_t + \phi_y \frac{1}{2} u_2 (u_2 - 1)\psi_t^2 + \frac{\partial \phi_y}{\partial t} = 0
\end{aligned}$$

which we write compactly as

$$\frac{\partial \phi_y}{\partial \psi}\left[c_1 \psi_t + u_1 b_1(\tau)\right] + \frac{\partial^2 \phi_y}{\partial \psi^2} c_3 + \phi_y u_1 b_2(\tau)\psi_t + \phi_y c_2 \psi_t^2 + \frac{\partial \phi_y}{\partial t} = 0$$

where c_1, c_2, c_3 are as above and $b_1(\tau)$ and $b_2(\tau)$ defined appropriately. Taking partials we find

$$\begin{aligned}
&[M(\tau) + 2N(\tau)\psi_t]\left[c_1 \psi_t + u_1 b_1(\tau)\right] + c_3[2N(\tau) + M(\tau)^2 + 4M(\tau)N(\tau)\psi_t + 4N(\tau)^2 \psi_t^2] \\
&u_1 b_2(\tau)\psi_t + c_2 \psi_t^2 + \left[\frac{L(\tau)}{\partial t} + \frac{M(\tau)}{\partial t}\psi_t + \frac{N(\tau)}{\partial t}\psi_t^2 \right] = 0
\end{aligned}$$

from which we obtain the following system of ODE's

$$\frac{\partial L}{\partial \tau} = u_1 M(\tau) b_1(\tau) + c_3[2N(\tau) + M(\tau)^2] \tag{43}$$

$$\frac{\partial M}{\partial \tau} = u_1 b_2(\tau) + [4c_3 N(\tau) + c_1]M(\tau) \tag{44}$$

$$\frac{\partial N}{\partial \tau} = 4c_3 N(\tau)^2 + 2c_1 N(\tau) + c_2. \tag{45}$$

Firstly, notice that setting $u_1 = 0$ we recover equations 40 - 42 which characterise the density for $\eta(T)$. Equation 45 is a again a constant coefficient Ricatti equation whose solution is given in the solution for the 'belief system'. The solution for $M(\tau)$ is given by

$$\begin{aligned} M(\tau) = &\frac{u_1 u_2}{\kappa_g q(\kappa_g - q)(\kappa_g + q)\left(\tilde{c}e^{2q\tau} + 1\right)} e^{-\kappa_g \tau}\Big(- \tilde{c}(\kappa_g - q)(\kappa_g + q)e^{\tau(\kappa_g + 2q)} \\ &\times \Big(\kappa_g \sigma_D + \sigma_{gD,a} + \kappa_g e^{\tau(\kappa_g + q)}((\tilde{c} - 1)\sigma_D(\kappa_g - q)(\kappa_g + q) + \sigma_{gD,a}((\tilde{c} - 1)\kappa_g + \tilde{c}q + q)\Big) \\ &- \tilde{c}q\sigma_{gD,a}(\kappa_g + q)e^{2q\tau} + (\kappa_g - q)(\kappa_g + q)e^{\kappa_g \tau}(\kappa_g \sigma_D + \sigma_{gD,a}) + q\sigma_{gD,a}(q - \kappa_g)\Big) \end{aligned} \tag{46}$$

The solution for $L(\tau)$ follows by direct integration.

Solving $\gamma \in \mathbb{R}_+$

Given a solution for characteristic function of (D_T, η_T) we can recover the joint density via inversion which allows us to compute the price of any contingent claim. Setting $u_1 = -\gamma$ we recover the joint transition density via inversion by evaluating the (inverse) bilateral Laplace transform at $iu2$, which equivalent the computing the continuous time Fourier transform

$$P(t,T) = \varrho_{T-t}\phi_x(\tau; -\gamma)g(y,t)^{-1} \int_0^\infty \Big[g(y,T)\frac{1}{\pi}\int_0^\infty e^{-iu_2 y_T}\phi_y(\tau; -\gamma, u_2)du_2\Big]dy \tag{47}$$

where

$$g(y,s) = (1 + (e^{y_s})^{\frac{1}{\gamma}})^{\gamma} \tag{48}$$

$$\phi_x(\tau; u_1) = e^{A(\tau) + B(\tau)g_t} \tag{49}$$

$$\phi_y(\tau; u_1, u_2) = e^{L(\tau) + M(\tau)\psi_t + N(\tau)\psi_t^2} \tag{50}$$

with $\Big\{A(\tau), B(\tau), L(\tau), M(\tau), N(\tau)\Big\}$ reported in the main body of the paper.

Solving $\gamma \in \mathbb{N}$

For integer γ a closed form solution for bond prices can be found using the binomial expansion:

$$\begin{aligned} P_t^T &= \varrho_{t,T} E_t^a \left[\left(\frac{D_T}{D_t}\right)^{-\gamma}\left(\frac{1 + \eta_T^{1/\gamma}}{1 + \eta_t^{1/\gamma}}\right)^{\gamma}\right] \\ &= \varrho_{t,T}(\omega_t^a)^{\gamma}\sum_{j=0}^{\gamma}\binom{\gamma}{j}\left(\frac{\omega_t^b}{\omega_t^a}\right)^j \left[\left(\frac{D_T}{D_t}\right)^{-\gamma}\left(\frac{\eta_T}{\eta_t}\right)^{j/\gamma}\right] \\ &= \varrho_{t,T} F_g(g^a, t, T, -\gamma)(\omega_t^a)^{\gamma}\sum_{j=0}^{\gamma}\binom{\gamma}{j}\left(\frac{\omega_t^b}{\omega_t^a}\right)^j F_{\psi}(\psi, t, T; -\gamma, j/\gamma) \end{aligned}$$

Myopic Term Structures

When agents are myopic heterogeneous equilibrium bond prices are given as wealth weighted averages of fictitious homogeneous equilibrium bond prices:

$$\begin{aligned} P(t,T) &= \varrho_\tau \phi_x^a(\tau; -1)\omega_t^a\left(1 + \frac{\omega_t^b}{\omega_t^a}e^{L(\tau) + M(\tau)\psi_t^g}\right), \\ &= \varrho_\tau(\omega_t^a \phi_x^a(\tau; -1) + \omega_t^b e^{A(\tau) + B(\tau)g_t^a + L(\tau) + M(\tau)[g_t^a - g_t^b]/\sigma_D}), \\ &= \varrho_\tau(\omega_t^a \phi_x^a(\tau; -1) + \omega_t^b e^{[A(\tau) + L(\tau)] + B(\tau)g_t^b}), \\ &= \omega_t^a P^a(t,T) + \omega_t^b P^b(t,T). \end{aligned}$$

since $M(\tau, u_1 = -\gamma) = -\sigma_D B(\tau, u_1 = -\gamma)$.[40]

Bond Sensitivities

$$
\begin{aligned}
\frac{\partial \ln P(t,T)}{\partial g^a} &= B(\tau) \\
\frac{\partial \ln P(t,T)}{\partial \eta} &= \frac{\int_0^\infty \left[g(y,T)\int_0^\infty e^{-iu_2 y_T} u_2 e^{(u_2-1)y_t}\phi_y(\tau;-\gamma,u_2)du_2\right]dy(T)}{\int_0^\infty \left[g(y,T)\int_0^\infty e^{-iu_2 y_T} e^{u_2 y_t}\phi_y(\tau;-\gamma,u_2)du_2\right]dy(T)} - \frac{\eta_t^{(1-\gamma)/\gamma}}{1+\eta_t^{1/\gamma}} \\
\frac{\partial \ln P(t,T)}{\partial \psi} &= \frac{\int_0^\infty \left[g(y,T)\int_0^\infty e^{-iu_2 y_T} e^{u_2 y_t}\phi_y(\tau;-\gamma,u_2)\left[M(\tau)+2N(\tau)\psi_t\right]du_2\right]dy(T)}{\int_0^\infty g(y,T)\left[\int_0^\infty e^{-iu_2 y_T} e^{u_2 y_t}\phi_y(\tau;-\gamma,u_2)du_2\right]dy(T)}
\end{aligned}
$$

Subjective Versus Objective Returns

A necessary condition for existence of heterogeneous belief equilibrium is that dispersion in market risk prices coincide with the scaled disagreement: $\kappa_t^a - \kappa_t^b = \psi_t$. Assuming the existence of asset spanning $d\hat{W}_t^D$ shocks with generic diffusion

$$
\begin{aligned}
\frac{dP_t^T}{P_t^T} = \mu^b(t,T)dt + \sigma_{P,D}(t,T)d\hat{W}_t^{D,b} &= (\mu^b(t,T) + \sigma_{P,D}(t,T)\psi_t)dt + \sigma_{P,D}(t,T)d\hat{W}_t^{D,a} \\
&= \mu^a(t,T)dt + \sigma_{P,D}(t,T)d\hat{W}_t^{D,a}
\end{aligned}
$$

we see that the equilibrium differences in expected returns are given by $\mu_t^a - \mu_t^b = \sigma_{P,D}(t,T)\psi_t$. Since agents agree on the price of date t securities (a necessary condition to clear consumption markets) we can also relate subjective to objective returns via individual agent forecast errors

$$
\mu^o(t,T) = \mu^i(t,T) + \sigma_{P,D}(t,T) error_t^{i,o}. \tag{51}
$$

Combining this with equilibrium conditions that clear consumption markets implies objective Sharpe ratios are wealth weighted averages of individual agent forecast errors

$$
\frac{\mu^o(t,T) - r_t}{\sigma_{P,D}(t,T)} = \gamma\sigma_D + \sum_{i=a,b}\omega_t^i \; error_t^{i,o} = \gamma\sigma_D + \frac{1}{\sigma_D}\left[g_t - (\omega_t^a g_t^a + \omega_t^b g_t^b)\right] \tag{52}
$$

[40] An alternative proof is as follows. Myopic agents consume wealth at a rate equal to their time rate of preference: $c_t^i = \rho W_t^i$. Since agents must agree on tradable prices, $E_t^a(\frac{c_t^a}{c_T^a}P(t+1,T)) = E_t^b(\frac{c_t^b}{c_T^b}P(t+1,T))$ this implies $\eta_T = W_T^b/W_T^a$ serves as a valid change of measure. Imposing market clearing, $c_t^a + c_t^b = D_t$, we obtain $c_t^a = \frac{1}{1+\eta_t}D_t$ and $c_t^b = \frac{\eta_t}{1+\eta_t}D_t$. Substituting c_t^a into the price of a bond from the perspective of agent a, and after a change of measure we obtain the desired result.

VI. Appendix B: Data

A. Yield Data

U.S inflation protected Treasuries were first issued in 1997 which adjust to the all urban consumer price index with a 3-month lag. In the early years of issue this market suffered significant liquidity problems (see, for example Roll (2004)) and our sample focuses on the period 2000.01 - 2012.12 from which I collect nominal and TIPS zero coupon bonds estimated by Gürkaynak, Sack, and Wright (2006, 2010) (GSW). GSW construct zero coupon yields by fitting the Nelson-Siegel-Svensson functional form to market quoted coupon bonds and is publicly available from the Federal Reserve Board site. I also use short-term nominal interest rates with 3 and 6-month maturities from the Fama-Bliss T-bill files available from CRSP.

Nominal zero coupon yields are available in the U.K since 1971 while inflation linked gilts have been on issue since 1985. Both the coupon payments and the principal are adjusted to the General Index of Retail Prices with a variable lag depending on the sample period.[41] I obtain both sets of prices from the Bank of England site which are estimated with a penalty based spline (Anderson and Sleath (2001)). At the short end of the nominal curve I obtain 3 and 6 month LIBOR rates from Bloomberg.

B. Subjective Beliefs

I use an extensive dataset on the distribution of beliefs to learn about the relative importance of the channels through which heterogeneity can affect asset prices. This section discusses the data sources and construction of variables designed to proxy for the state vector (g_t, η_t, ψ_t^g).

BlueChip Financial Forecasts Indicators (BCFF) is a monthly publication providing extensive panel data on expectations by agents who are working at institutions active in financial markets. Each month, BlueChip carry out surveys of professional economists from leading financial institutions and service companies regarding all maturities of the yield curve and economic fundamentals and are asked to give point forecasts at quarterly horizons out to 5-quarters ahead (6 from January 1997). While exact timings of the surveys are not published, the survey is usually conducted between the $25th$ and $27th$ of the month and mailed to subscribers within the first 5 days of the subsequent month, thus our empirical analysis is unaffected by biases induced by staleness or overlapping observations between returns and responses.[42]

BCFF is attractive along a number of dimensions with respect to alternative commonly studied surveys. First, the number of participants in the survey is stable over time. On average 45 respondents are surveyed with standard deviations of 2.8. Only on rare occasions are survey numbers less than 40 and no business cycle patterns are visible. In the Survey of Professional Forecasters, on the other hand, the distribution of respondents displays significant variability. While the mean number of respondents is around 40, the standard deviation is 13, and in some years the number of contributors is as low as 9. [43] While in the early 70's the number of SPF forecasters was around 60, it decreased in two major steps in the mid 1970's and mid 1980's to as low as 14 forecasters in 1990 and if one restricts the attention to forecasters who participated to at least 8 surveys, this limits the number of data point considerably. Second, while our dataset is available at a monthly frequency, SPF is available only at quarterly frequency. Third, the SPF survey has been administered by different agencies over the years which have changed the form of questions. For

[41] see www.dmo.gov.uk/index.aspx?page=gilts/about_gilts for details.

[42] An exception to the general rule was the survey for the January 1996 issue when non-essential offices of the U.S. government were shut down due to a budgetary impasse and at the same time a massive snow storm covered Washington, DC: www.nytimes.com/1996/01/04/us/battle-over-budget-effects-paralysis-brought-shutdown-begins-seep-private-sector.html. As a result, the survey was delayed a week.

[43] The SPF survey has been used, among others, by Buraschi and Jiltsov (2006) and Ulrich (2010); it is available at www.philadelphiafed.org/research-and-data/real-time-center/.

a detailed discussion on the issues related to SPF see D'Amico and Orphanides (2008) and Giordani and Soderlind (2003). Other well known surveys, such as the 'University of Michigan Survey of Consumers' do not provide point estimates from individual survey respondents. Finally, BCFF survey forecasts for both macro variables and interest rates are highly competitive with respect to the out-of-sample performance of sophisticated econometric models. Recent studies documenting the quality of BCFF forecasts include Chun (2012), Faust and Wright (2012), and Buraschi, Carnelli, and Whelan (2013).

To proxy for macro economic disagreement we use 1-quarter ahead point forecasts on real GDP and the GDP deflator for consumption growth and inflation disagreement proxies, respectively. Individual forecasts allow us to proxy for belief dispersion in a number of ways. In what follows we choose to proxy for belief heterogeneity from the cross-sectional inter-quartile range and check the robustness of our findings with alternative proxies. In our empirical work we focus on implications for real disagreement (ψ_t^g) but also investigate disagreement about inflation (ψ_t^π) as an alternative determinant for bond markets as argued recently by Wright (2011), Ehling, Gallmeyer, Heyerdahl-Larsen, and Illeditsch (2013) , and Hong, Sraer, and Yu (2013).

The figure below plots the time series for our macroeconomic disagreement proxy along with an economic policy uncertainty factor (UnC_t) studied by Baker, Bloom, and Davis (2012).[44] Disagreement on the real economy has a significant business cycle component: in all previous three NBER economic recessions since 1990, ψ_g is low before the recessions and it increases to peak at the end of the recessions (this occurs in 1991, 2002, and 2009). This is interesting as large disagreement is often reported at this stage of the cycle.[45] Comparing UnC_t to our measures for belief dispersion we find large co-movement with ψ^g (correlation = 0.58), which is somewhat surprising given that the weight assigned to forecaster disagreement about inflation in this index is just 1/6. The remaining components of the index are 1/2 a broad-based news index, 1/6 a tax expiration index and 1/6 a government purchases disagreement measure. Taken together these measures suggest the existence of a common component in the formation of expectations which is important since systematic variation is required for time-variation in priced risk compensation.

[Insert figure 17 about here]

C. *Alternative Structural Risk Factor Proxies*

In long run risk economies (Bansal and Yaron (2004)) with recursive preferences predictability arises from the dynamics of second moments (economic uncertainty) of the conditional growth rate of fundamentals. Recently, Bansal and Shaliastovich (2013) discuss the implications of this class of models for bond markets. In their model, equilibrium bond prices depend on 4-factors while risk compensation is time-varying with conditional second moments depending on both inflation non-neutrally and whether agents have a preference for early or late resolution of uncertainty.

I proxy for economic uncertainty following Bansal and Shaliastovich (2013) who seeks to exploit the information about future volatility contained in date t yields using survey forecasts of gdp growth and inflation. First, I fit a bivariate VAR(1) to inflation and GDP expectations:

$$g_{t+1}^e = \underset{(0.08)}{0.63} + \underset{(0.02)}{0.86}\, g_t^e - \underset{(0.01)}{0.08}\, \pi_t^e + \epsilon_{g,t+1}$$

$$\pi_{t+1}^e = \underset{(0.12)}{0.93} - \underset{(0.03)}{0.20}\, g_t^e + \underset{(0.02)}{0.87}\, \pi_t^e + \epsilon_{\pi,t+1},$$

[44] The economic uncertainty proxy plotted here is available for down from www.policyuncertainty.com

[45] In a controversial statement that attracted substantial controversy, in 1991 Normal Lamont - Chancellor of the Exchequer of the United Kingdom - labeled the initial sign of the recovery from the S&L recession as 'green shoots'. Many years later, Ben Bernanke used the same words in a well-known 'CBS 60 Minutes' interview in 2009, which was counterpointed by Nouriel Roubini who argued his disagreement and labeled those signs as 'yellow weeds'. The data indeed confirm that macro-disagreement is usually pervasive in this phase of the cycle.

where $\pi_t^e = E_t^c[\pi_{t+12}]$ and $g_t^e = E_t^c[g_{t+12}]$ for ease of notation. Similar to Bansal and Shaliastovich (2013), I find that both processes feature high persistence (their AR(1) coefficient for inflation expectations is 0.99, however), and that inflation has a non-neutral effect on growth: the loading of consumption growth on lagged inflation is strongly significant. Next, I regress squared residuals between t and $t+12$ on time-t yields:

$$\sum_{k=1}^{12} \epsilon^2_{g,t+k} = \underset{(0.12)}{0.19} + \underset{(0.08)}{0.21}\, y_t^{(1)} - \underset{(0.35)}{0.55}\, y_t^{(3)} + \underset{(0.44)}{0.39}\, y_t^{(5)} - \underset{(0.18)}{0.04}\, y_t^{(10)} + \eta_{g,t+1} \qquad R^2 = 0.07$$

$$\sum_{k=1}^{12} \epsilon^2_{\pi,t+k} = \underset{(0.14)}{0.21} + \underset{(0.09)}{0.04}\, y_t^{(1)} + \underset{(0.35)}{0.04}\, y_t^{(3)} - \underset{(0.49)}{0.24}\, y_t^{(5)} + \underset{(0.23)}{0.16}\, y_t^{(10)} + \eta_{\pi,t+1} \qquad R^2 = 0.08,$$

and take fitted values as an estimate of conditional variances $\sigma^2(g)$ and $\sigma^2(\pi)$.

In Lucas economies with habit preferences as in **?**, predictability arises in equilibrium because of an endogenously time-varying price of risk. Shocks to the current endowment affect the wedge between consumption and habit, the consumption surplus, which induce a time-varying price of risk. I follow **?** and construct a proxy of consumption surplus $surp_t$ as a weighted average of 10 years of monthly consumption growth rates:

$$Surp_t = \sum_{j=1}^{120} \phi^j \Delta c_{t-j},$$

where the weight is set to $\phi = 0.97^{1/3}$ to match the quarterly autocorrelation of the P/D ratio in the data, as in **?**. For consumption date we obtain seasonally adjusted, real per-capita consumption of nondurables and services from the Bureau of Economic Analysis.

The resulting time-series for expected conditional variances and consumption surplus proxies are displayed in figure 18 below.

[Insert figure 18 about here]

VII. Appendix C: Tables

Table I. Summary Statistics: US Treasury Curves
The top panel reports statistics for U.S nominal zero coupon bonds, and the bottom panel reports statistics for U.S TIPS curves. Both term structures contain 144 observations for each maturity from 2000.1 - 2011.12. 3-10yr interest rates are obtained from Gürkaynak, Sack, and Wright (2006, 2010). 3-month nominal rates are from the Fama-Bliss files on CRSP. Ex-ante 3-month real rates are computed following Campbell and Shiller (1996) who estimate a VAR model including the ex post real return on a 3-month nominal bills, t the nominal bill yield, and lagged annual inflation rates.

Maturity	3 month	3 Year	5 Year	7 Year	10 Year
Mean	0.6427	3.0715	3.5665	3.9930	4.4696
Std Dev	0.5482	1.6302	1.3331	1.1119	0.9021
Min	0.0007	0.3866	0.9519	1.4430	2.0606
Max	1.7231	6.6041	6.6264	6.6539	6.7009
Skew	0.4621	0.2471	0.1424	0.0801	−0.0191
Kurtosis	1.8124	2.1488	2.4908	2.8810	3.3097
1st Lag Auto	0.9800	0.9613	0.9459	0.9284	0.9022
Mean	−0.0363	1.3211	1.7172	1.9693	2.2080
Std Dev	0.4348	1.3805	1.2231	1.0903	0.9332
Min	−0.7850	−1.4388	−0.8411	−0.4537	0.0558
Max	0.8287	4.2412	4.2825	4.3059	4.2881
Skew	0.3501	0.2635	0.1466	0.1789	0.2729
Kurtosis	2.0422	2.4407	2.5091	2.6320	2.7877
1st Lag Auto	0.9589	0.9321	0.9376	0.9362	0.9320

Table II. Summary Statistics: UK Treasury Curves

The top panel reports statistics for U.K nominal zero coupon bonds, and the bottom panel reports statistics for U.K inflation protected curves. Both term structures contain 264 observations for each maturity from 1990.1 - 2011.12. 3 - 10 yr interest rates are obtained from the Bank of England website. 3-month nominal rates are LIBOR rates from Bloomberg. Ex-ante 3-month real rates are computed following Campbell and Shiller (1996) who estimate a VAR model including the ex post real return on a 3-month nominal bills, t the nominal bill yield, and lagged annual inflation rates.

Maturity	3 month	3 Year	5 Year	7 Year	10 Year
Mean	1.4810	5.6229	5.8147	5.9296	6.0132
Std Dev	0.8229	2.6092	2.4422	2.3292	2.1997
Min	0.1356	0.5722	1.1240	1.7034	2.3494
Max	3.9511	13.3149	12.9316	12.7329	12.3681
Skew	0.9701	0.3973	0.5523	0.6989	0.8128
Kurtosis	4.5863	3.2400	2.9749	2.8459	2.6589
1st Lag Auto	0.9755	0.9773	0.9786	0.9798	0.9812
Mean	0.6893	2.2539	2.3746	2.4210	2.4605
Std Dev	0.7960	1.4664	1.2427	1.1757	1.1539
Min	−0.7841	−1.9777	−1.1797	−0.7201	−0.3408
Max	3.5028	5.2647	4.8009	4.9130	4.9452
Skew	0.9768	−1.1173	−0.7901	−0.3995	−0.0396
Kurtosis	4.8877	3.8242	3.3327	2.6938	2.1966
1st Lag Auto	0.9615	0.9654	0.9595	0.9652	0.9727

Table III. Principle Component Decomposition
Table reports eigenvalue decomposition of the covariance matrix of of yields. The first two rows report the percentages explained by each orthogonal factor for nominal and real yields, respectively. The final row report the percentage explained (R^2) from a regression of nominal factors on real factors. U.S Sample period 2000.1-2011.1 U.K Sample period 1990.1 - 2011.1

U.S Treasuries	Level	Slope	Curvature
% of $cov(y_t^{\$})$ explained	94.69	4.22	1.03
% of $cov(y_t^r)$ explained	95.14	4.41	0.43
% Nom factor explained by real factor	0.78	0.72	0.03
U.K Treasuries			
% of $cov(y_t^{\$})$ explained	95.56	3.64	0.79
% of $cov(y_t^r)$ explained	98.26	1.47	0.26
% Nom factor explained by real factor	0.83	0.39	0.27

Table IV. Short Rate Regressions
OLS projections of real ($y_t^{r(3m)}$) on disagreement (ψ^g) and consensus expectations ($E[g^{(3m)}]$), conditional variance ($\sigma^2(g)$) and surplus factors ($Surp$). Lag y^{3m} is the 1-quarter lagged short rate. t-statistics are corrected for autocorrelation and heteroskedasticity. Sample Period: 1990.1 - 2010.12

	$E(g)$	$\psi(g)$	$\sigma^2(g)$	$Surp$	Lag y^{3m}	$\overline{R}^2$
real y^{3m}	-0.23	-0.28				0.06
	(-1.22)	(-2.32)				
real y^{3m}	-0.39	-0.31	-0.04	0.31		0.12
	(-2.01)	(-2.42)	(-0.34)	(3.20)		
real y^{3m}	0.00	-0.11	-0.06	0.02	0.88	0.78
	(0.00)	(-2.13)	(-1.21)	(0.26)	(19.52)	

Table V. Short Rate Regressions
Forecasting regression of 1-year changes in the 1-year yield on the $(1, 2)$ forward spot spread, CP_t, and ψ_t^g. t-statistics are corrected for autocorrelation and heteroskedasticity. Both left and right hand variables are standardized. Sample Period: 1990.1 - 2011.12

	$f_t^{1,2}$	CP_t	ψ_t^g	$\overline{R}^2$
(i)	0.32			0.10
	1.87			
(ii)	0.54	-0.46		0.25
	(2.90)	(-3.38)		
(iii)	0.39		-0.39	0.25
	(2.45)		(-4.05)	
(iv)	0.56	-0.37	-0.32	0.35
	(3.41)	(-3.56)	(-3.00)	

Table VI. The Cross-Section of Yields
OLS regression of the level and slope of the term structure on disagreement and lagged disagreement factors, conditional variance, and surplus factors. t-statistics are corrected for autocorrelation and heteroskedasticity. Both left and right hand variables are standardized. A constant is included by not reported. Sample Period: 2000.1 - 2010.12

	$E_t(g_t)$	ψ^g_t	ψ^g_{t-6}	$\sigma^2_t(g)$	$Surp_t$	$\overline{R}^2$
Real $Level_t$	0.13	0.05				0.01
	(0.50)	(0.38)				
Real $Slope_t$	0.76	0.50				0.31
	(2.64)	(4.95)				
Real $Level_t$	0.18	0.05	0.04			0.01
	(0.86)	(0.48)	(0.43)			
Real $Slope_t$	0.81	0.41	0.28			0.38
	(3.56)	(5.29)	(4.07)			
Real $Level_t$	-0.58	0.12	0.33	0.19	0.83	0.47
	(-2.53)	(1.14)	(3.74)	(2.02)	(9.48)	
Real $Slope_t$	1.02	0.40	0.21	-0.01	-0.20	0.39
	(3.94)	(5.29)	(2.62)	(-0.10)	(-1.87)	

Table VII. Real and Nominal Cochrane-Piazzesi Regressions: US
Table reports regression of 3-month excess returns on n-year real and nominal bonds on a forecasting factor constructed from an affine combination of date t forward rates (CP_t):

$$hprx^{(n)}_t = E_t[p^{(n-3)}_{t+3}] - p^{(n)}_t - r^{3m}_t = const + \beta CP_t + \varepsilon^{(n)}_{t,t+3}.$$

t-statistics, reported in ( )'s, are corrected for autocorrelation and heteroskedasticity using the Hansen and Hodrick (1983) GMM correction with 3 Newey-West lags. Sample Period: 2000.1 - 2011.12. The left panel reports point estimates from nominal bond returns on nominal $CP^{\$}_t$ and the right panel reports real bond returns on real CP^r_t.

	$const$	$CP^{\$}_t$	$\overline{R}^2$	$const$	CP^r_t	$\overline{R}^2$
$hprx^{(5)}_t$	0.06	0.71	0.16	0.02	0.81	0.08
	(0.16)	(4.47)		(0.03)	(2.44)	
$hprx^{(6)}_t$	0.03	0.85	0.17	0.01	0.90	0.08
	(0.07)	(4.53)		(0.01)	(2.55)	
$hprx^{(7)}_t$	0.00	0.98	0.17	-0.01	0.98	0.09
	(0.00)	(4.47)		(-0.01)	(2.63)	
$hprx^{(8)}_t$	-0.02	1.08	0.17	-0.01	1.05	0.09
	(-0.04)	(4.34)		(-0.02)	(2.69)	
$hprx^{(9)}_t$	-0.03	1.16	0.16	-0.01	1.11	0.09
	(-0.05)	(4.17)		(-0.01)	(2.74)	
$hprx^{(10)}_t$	-0.03	1.22	0.15	0.00	1.15	0.09
	(-0.05)	(3.98)		(0.00)	(2.76)	

Table VIII. Real and Nominal CP_t Factors
Table reports regression of 3-month real and nominal return predictability factors (CP_t) on structural risk factor proxies

$$CP_t^{r,\$} = const + \beta_1 \psi^g(t) + \beta_2 \psi^g(t-6) + \beta_3 \sigma_t^2(g) + \beta_4 Surp_t + \varepsilon_t$$

t-statistics, reported in ( )'s, are corrected for autocorrelation and heteroskedasticity. Sample Period: 2000.1 - 2011.12.

	ψ_t^g	ψ_{t-6}^g	$\sigma_t^2(g)$	$Surp$	$\overline{R}^2$
$CP_t^{\$}$	0.34				0.12
	(2.82)				
CP_t^r	0.32				0.10
	(2.28)				
$CP_t^{\$}$	0.21	0.43			0.30
	(2.07)	(3.87)			
CP_t^r	0.23	0.26			0.15
	(1.69)	(2.19)			
$CP_t^{\$}$	0.12	0.29	-0.08	-0.39	0.40
	(0.87)	(3.89)	(-0.38)	(-3.83)	
CP_t^r	0.30	0.32	-0.19	0.24	0.23
	(3.35)	(4.28)	(-0.99)	(1.76)	

Table IX. Return Predictability Regressions
Table reports regression of 1-year excess holding period returns on structural risk factor proxies

$$hprx_{t,t+12}^{(n)} = const + \beta_1 \psi^g(t) + \beta_2 \psi^g(t-6) + \beta_3 \sigma_t^2(g) + \beta_4 Surp_t + \varepsilon_{t,t+12}^{(n)}$$

t-statistics, reported in ( )'s, are corrected for autocorrelation and heteroskedasticity using the Hansen and Hodrick (1983) GMM correction with 12 Newey-West lags. Sample Period: 1990.1 - 2011.12.

	ψ_t^g	ψ_{t-6}^g	$\sigma_t^2(g)$	$Surp_t$	$\overline{R}^2$
n = 2	0.42				0.18
	(3.97)				
n = 5	0.36				0.12
	(3.88)				
n = 2	0.32	0.24			0.22
	(3.66)	(2.81)			
n = 5	0.25	0.24			0.17
	(3.15)	(2.62)			
n = 2	0.29	0.26	0.12	0.05	0.23
	(3.48)	(2.99)	(0.90)	(0.32)	
n = 5	0.25	0.24	-0.01	-0.01	0.16
	(2.90)	(2.38)	(-0.04)	(-0.05)	

VIII. Appendix D: Figures

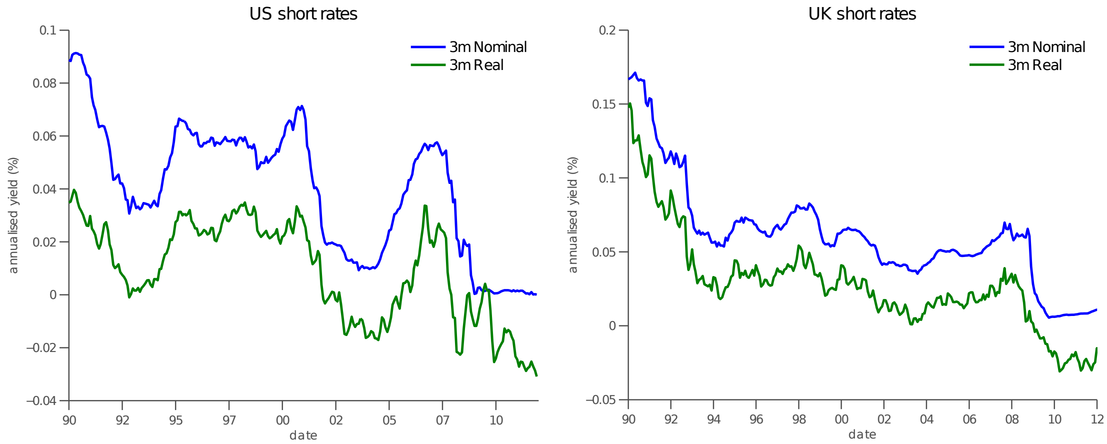

Figure 1. Real and Nominal 3 month Yields:
Figure displays time series of real and nominal 3-month risk free rates. We proxy for ex-ante real rates following Campbell and Shiller (1996) using a VAR that includes the ex post real return on a 3-month nominal bills, the nominal bill yield, and lagged annual inflation rate. Solving the VAR forward we build ex-ante forecasts for date t measurable 3-month real rates. The left panel displays estimates for the U.S while the right panel displays estimates for the U.K.

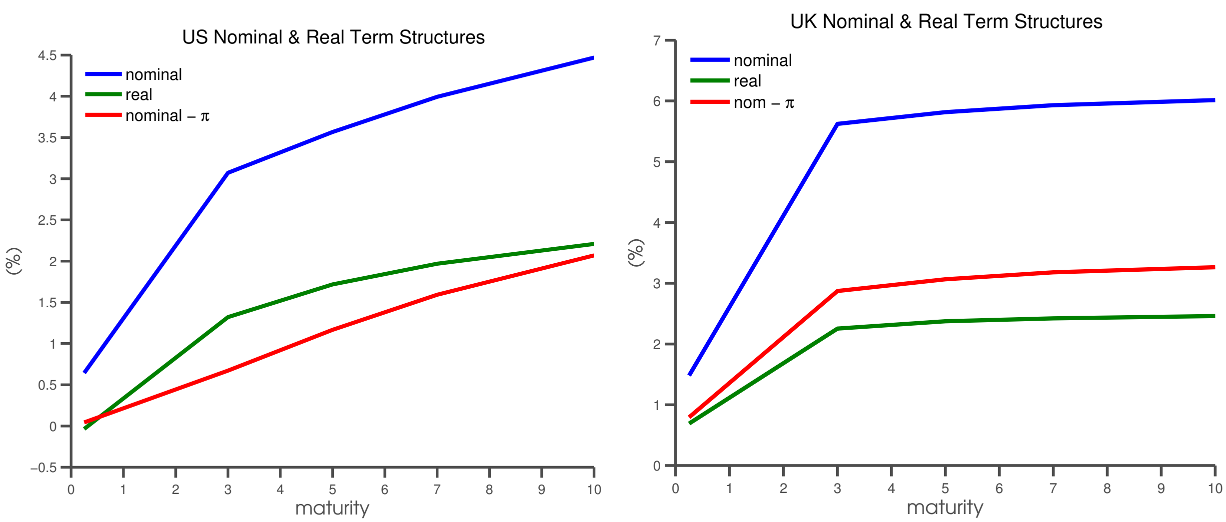

Figure 2. Average Yield Curves:
Figure displays the unconditional term structure of interest rates for nominal (blue line) and inflation protected (green line) markets. The red solid line subtracts the average realised year-on-year rate of inflation for the all-urban consumer price index (U.S), and the retail price index (U.K) from nominal yields. The left panel displays U.S Treasury yields for the sample period 2000.1 - 2012.12, and the right panel U.K Gilt yields for the sample period 1990.1 - 2012.12

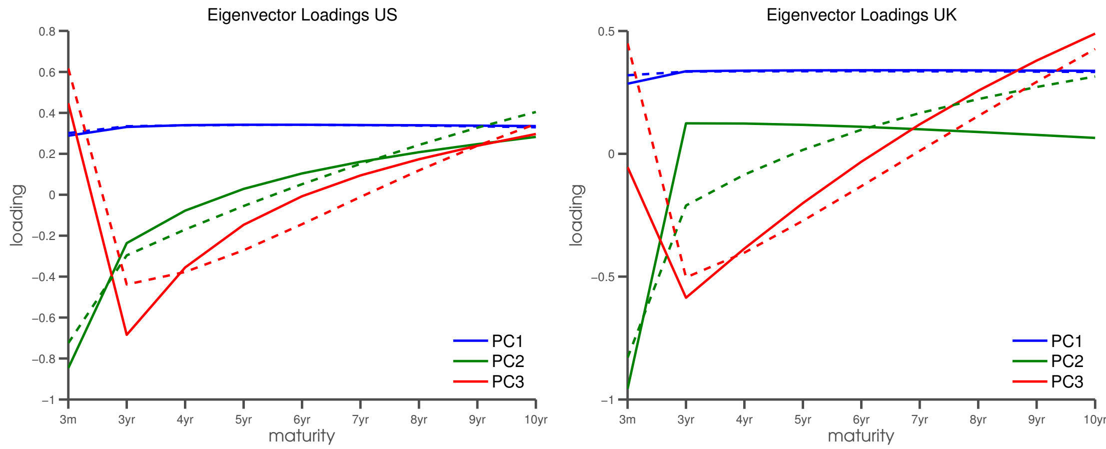

Figure 3. Principle Component Loadings:
Figure displays loadings from a principle component decomposition of the covariance matrix of real and nominal yields. The left panel reports U.S loadings and the right panel reports U.K loadings. Solid lines report loadings from nominal yields and dashed lines loadings from real yields. U.S Sample Period: 2000.1 - 2012.12. U.K Sample Period: 1990.1 - 2012.12

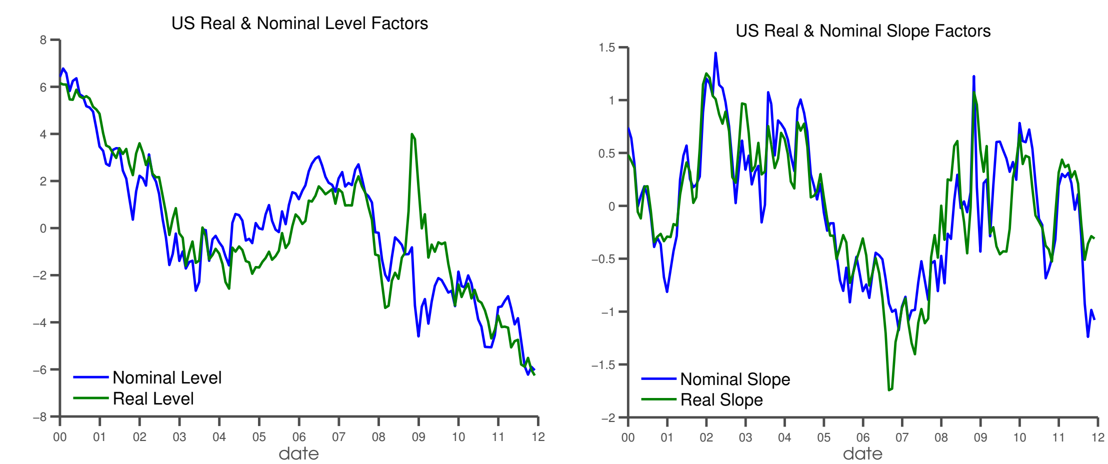

Figure 4. Level and Slope Factors: US
Figure displays time-series of the 1st (level) and 2nd (slope) principle components from an eigenvalue decomposition of the covariance matrix of real and nominal yields. The left panel shows real and nominal level factors while the right panel shows real and nominal slope factors.

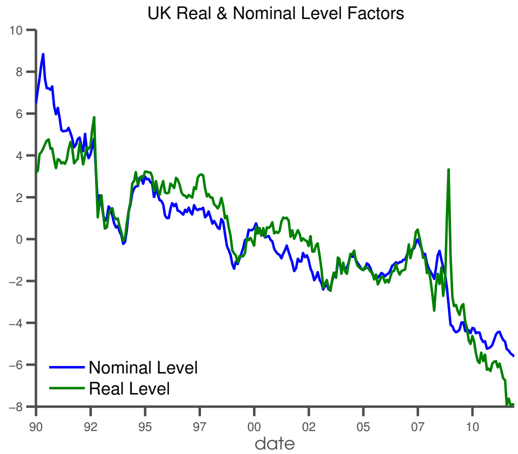

Figure 5. Level and Slope Factors: UK
Figure displays time-series of the 1st (level) and 2nd (slope) principle components from an eigenvalue decomposition of the covariance matrix of real and nominal yields. The left panel shows real and nominal level factors while the right panel shows real and nominal slope factors.

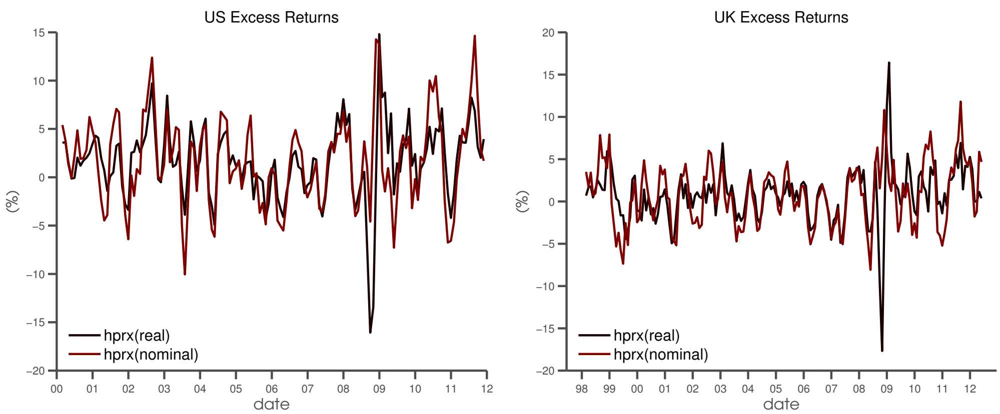

Figure 6. Real and Nominal Excess Returns:
Figure displays realised excess returns computed from 3-month holding periods returns in excess of the the 3-month risk free interest rate. The time-series records returns at the date they are realised. Black lines represents 5-year real bonds while dark red lines represent 5-year nominal bonds.

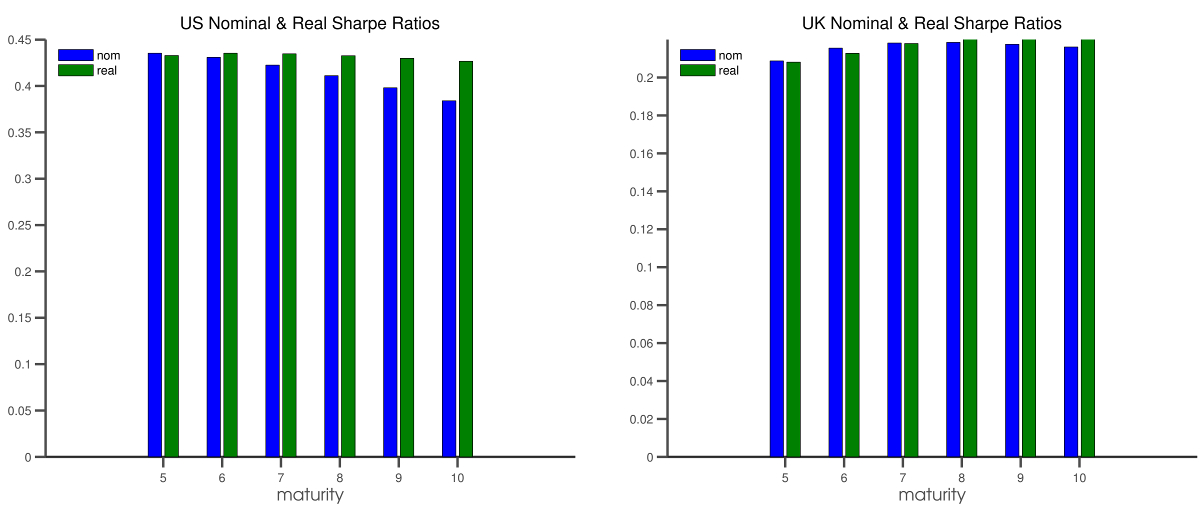

Figure 7. Real and Nominal Excess Returns:
Figure displays sharpe ratios computed from 3-month holding periods returns in excess of the the 3-month risk free interest rate.

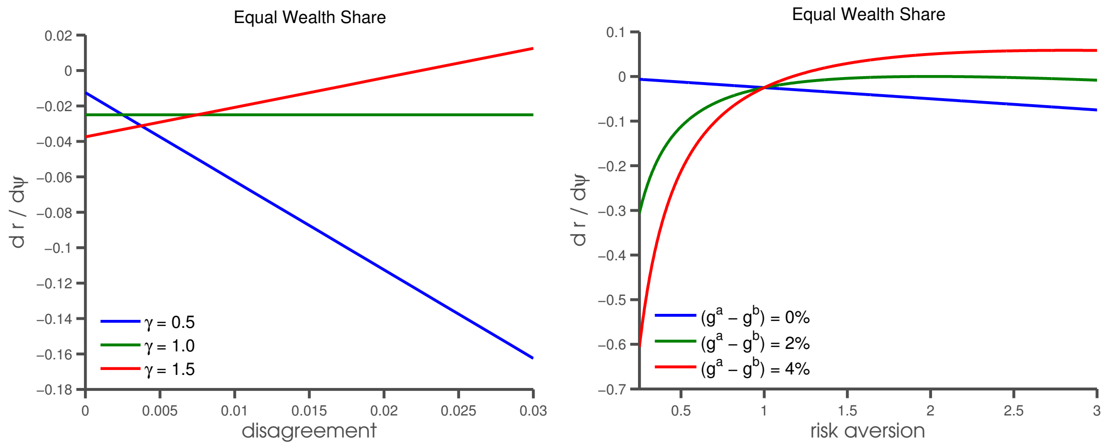

Figure 8. Short Rate Sensitivity to Disagreement:
The left panel plots the sensitivity of the short rate with respect to disagreement as a function of the level of disagreement for risk aversion levels above and below one, while the the right panel plots the short rate sensitivity with respect to disagreement as a function of risk aversion for different levels of disagreement.

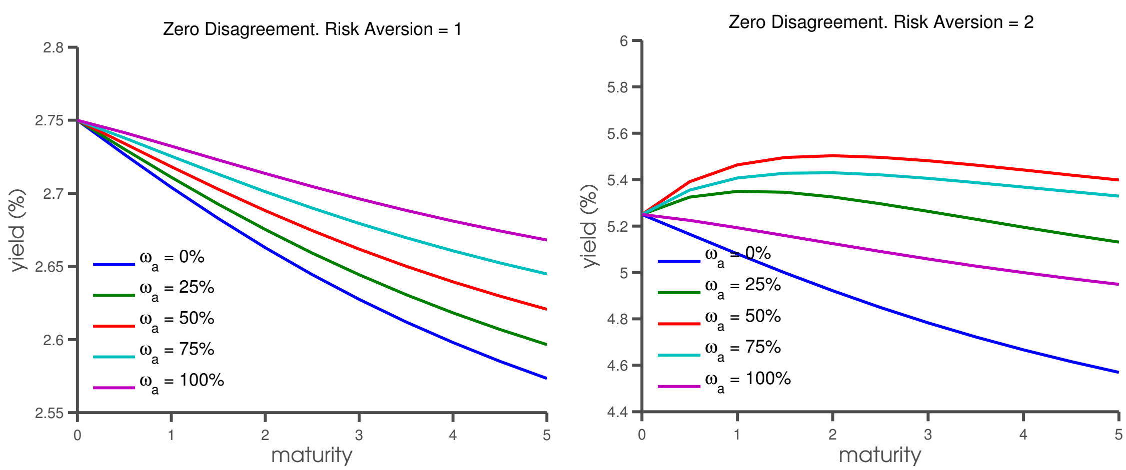

Figure 9. Term Structure Example 1: Sentiment Economy
Figure plots the term structure of interest rates for an economy with zero disagreement but varies the relative wealths between the optimistic and pessimist agents.

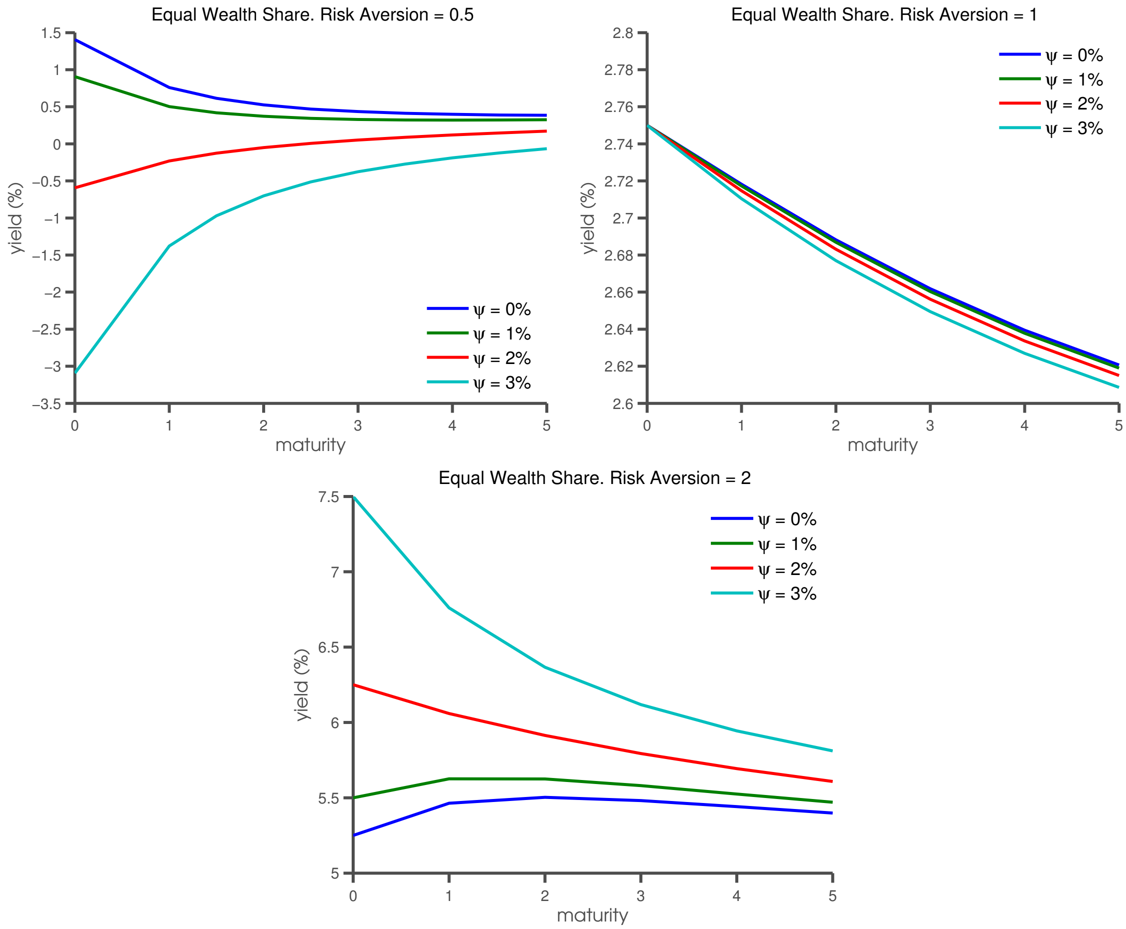

Figure 10. Term Structure Example 2: Symmetric Economy
Figure plots the term structure of interest rates for an economy with an symmetric wealth distribution $\omega_t^a = \omega_t^b = 0.50$ with $g_t^a - g_t^b = \in [0\% : 3\%]$ for various levels of risk aversion above and below 1.

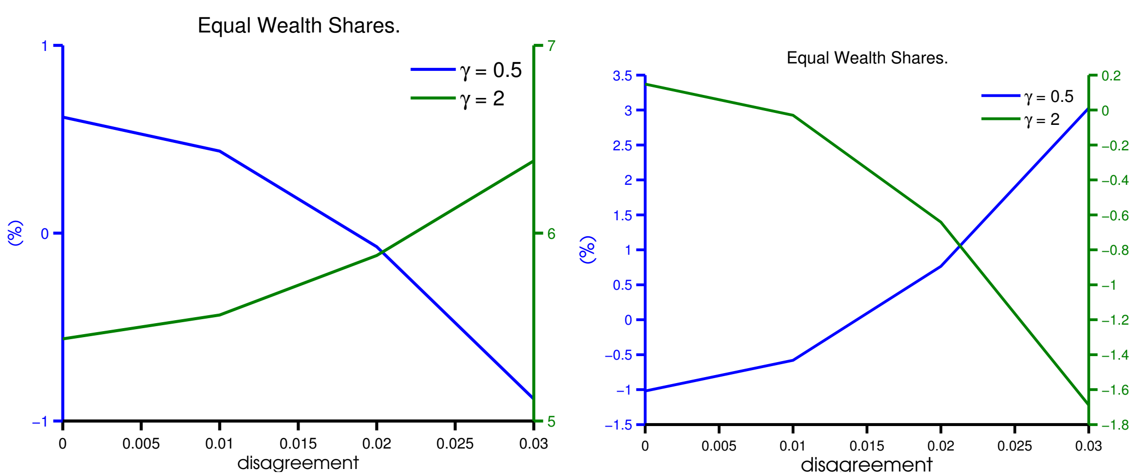

Figure 11. Term Structure Example 2: Symmetric Economy
Left Panel: Level $= \frac{1}{n}\sum_i^n y_t^n$. Right Panel: Slope $= y_t^5 - r_t$

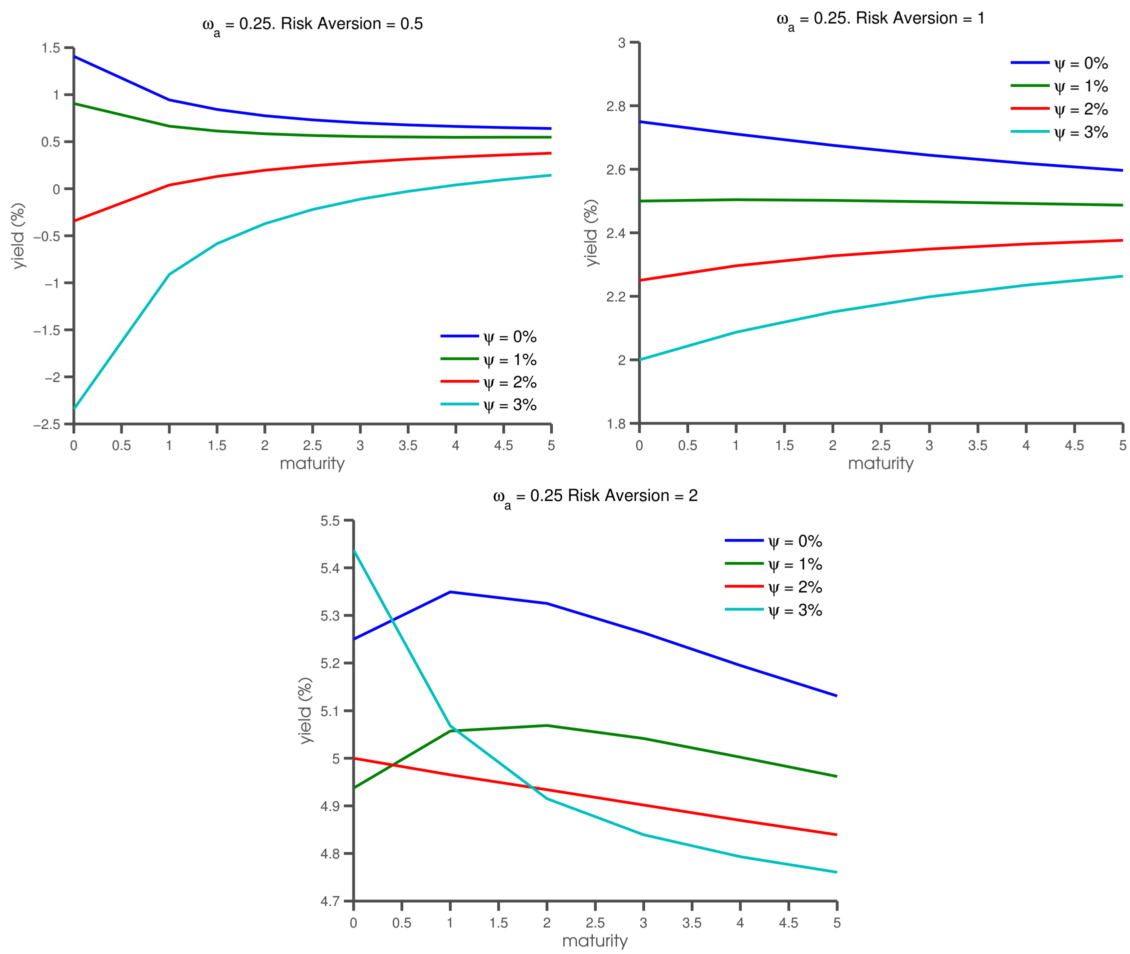

Figure 12. Term Structure Example 3: Pessimistic Economy
Figure plots the term structure of interest rates for an economy which is on average pessimist $\omega_t^a < \omega_t^b$ with $g_t^a - g_t^b = \in [0\% : 3\%]$ for various levels of risk aversion above and below 1.

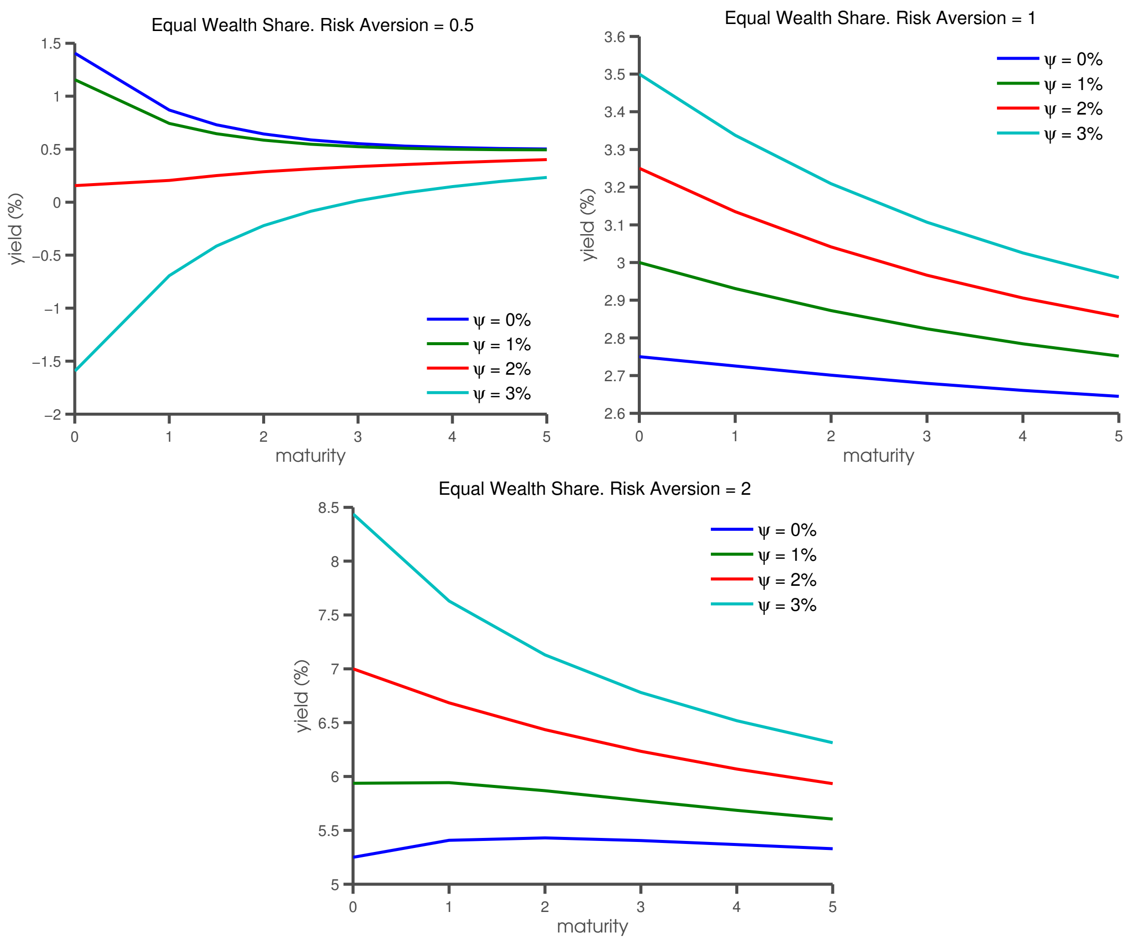

Figure 13. Term Structure Example 4: Optimistic Economy
Figure plots the term structure of interest rates for an economy which is on average optimistic $\omega_t^a > \omega_t^b$ with $g_t^a - g_t^b = \in [0\% : 3\%]$ for various levels of risk aversion above and below 1.

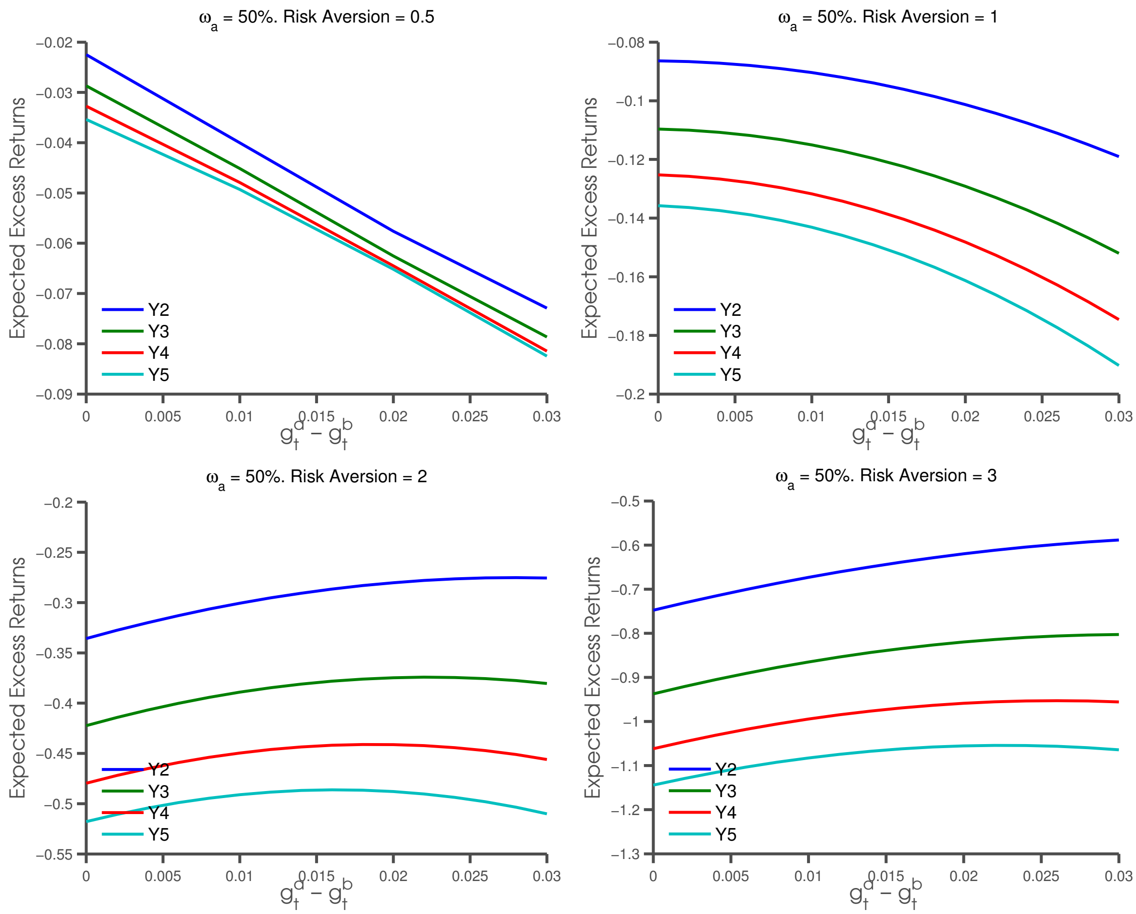

Figure 14. Risk Premia Example 1: Symmetric Economy
Figure plots the term structure of risk premia (instantaneous expected excess returns) for an economy with an symmetric wealth distribution $\omega_t^a = \omega_t^b = 0.50$ with $g_t^a - g_t^b \in [0\% : 3\%]$ for risk aversions $0.5, 1.0, 2.0$.

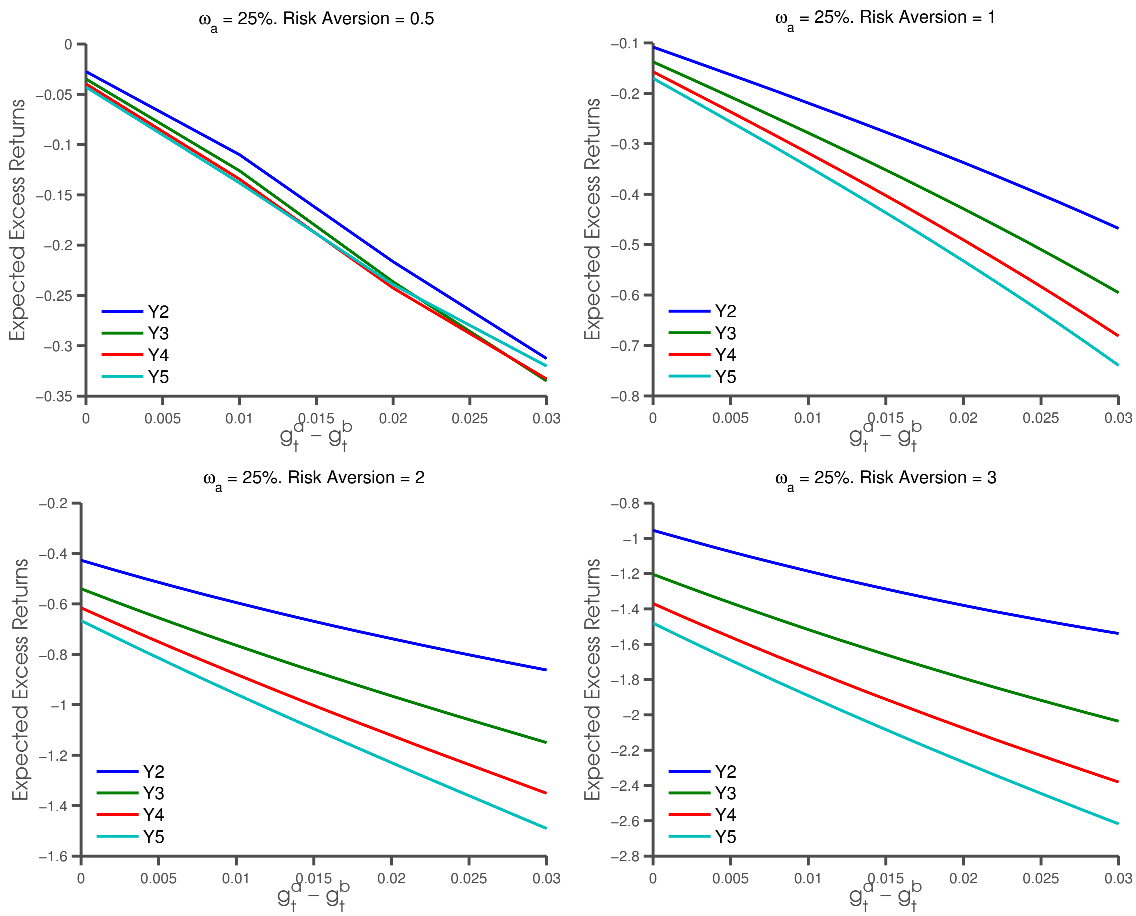

Figure 15. Risk Premia Example 2: Pessimistic Economy
Figure plots the term structure of risk premia (instantaneous expected excess returns) for an economy which is on average pessimist $\omega_t^a < \omega_t^b$ with $g_t^a - g_t^b \in [0\% : 3\%]$ for risk aversions $0.5, 1.0, 2.0$.

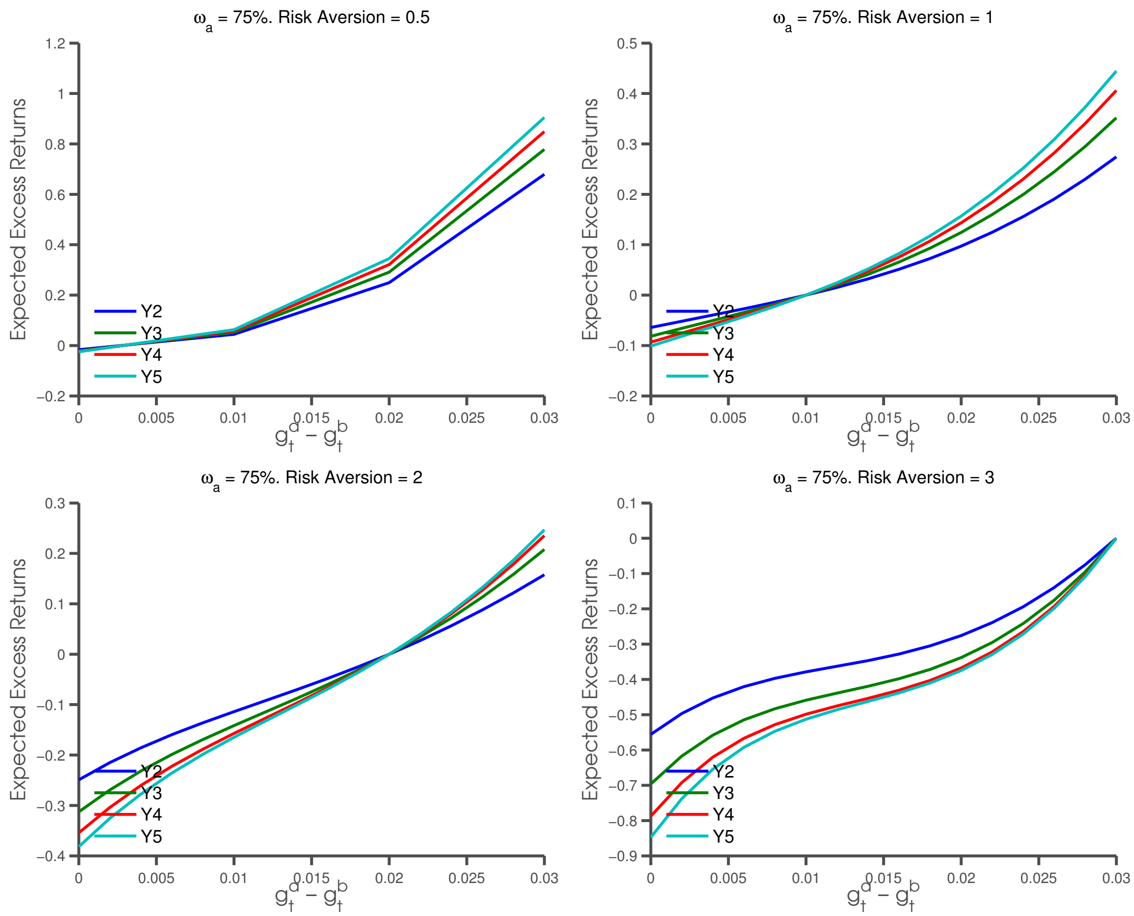

Figure 16. Risk Premia Example 2: Optimistic Economy
Figure plots the term structure of risk premia (instantaneous expected excess returns) for an economy which is on average optimistic $\omega_t^a > \omega_t^b$ with $g_t^a - g_t^b \in [0\% : 3\%]$ for risk aversions $0.5, 1.0, 2.0$.

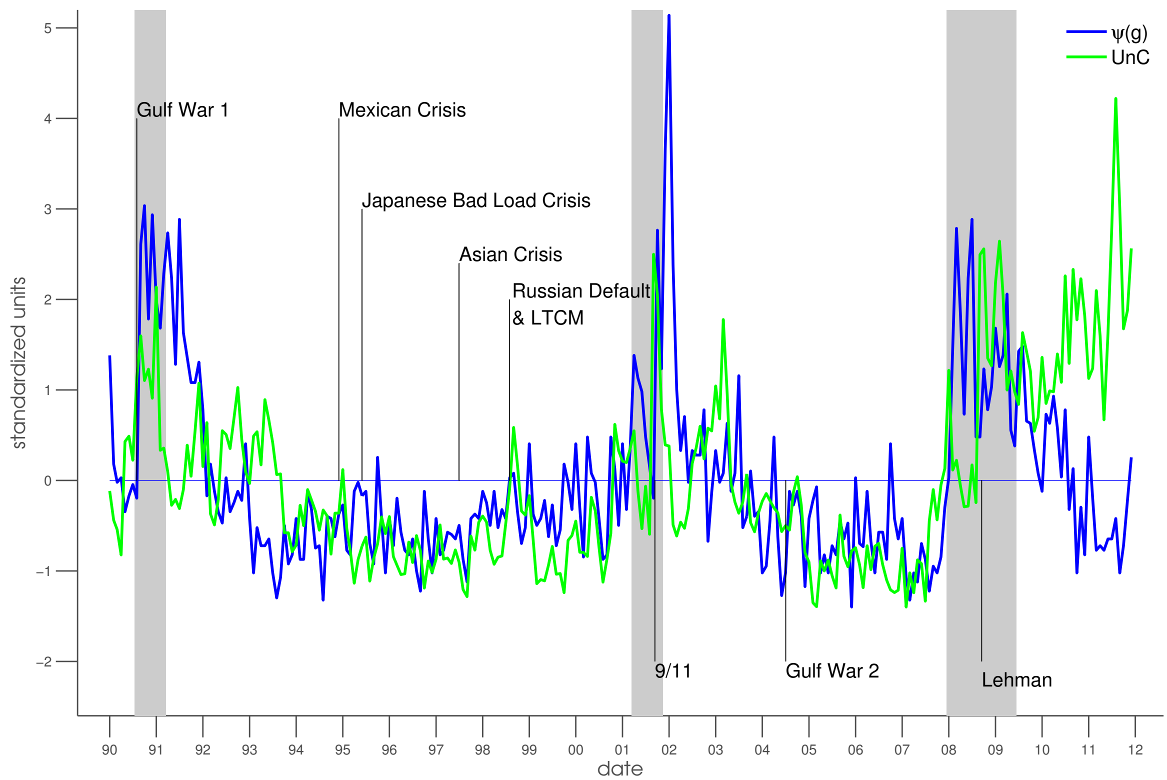

Figure 17. Differences in Belief Proxies
Figure plots time-series for differences in belief about the 1-quarter GDP growth ($\psi(g)$) along with an uncertainty proxy studied by Baker, Bloom, and Davis (2012). Sample period: January 1990 - December 2011.

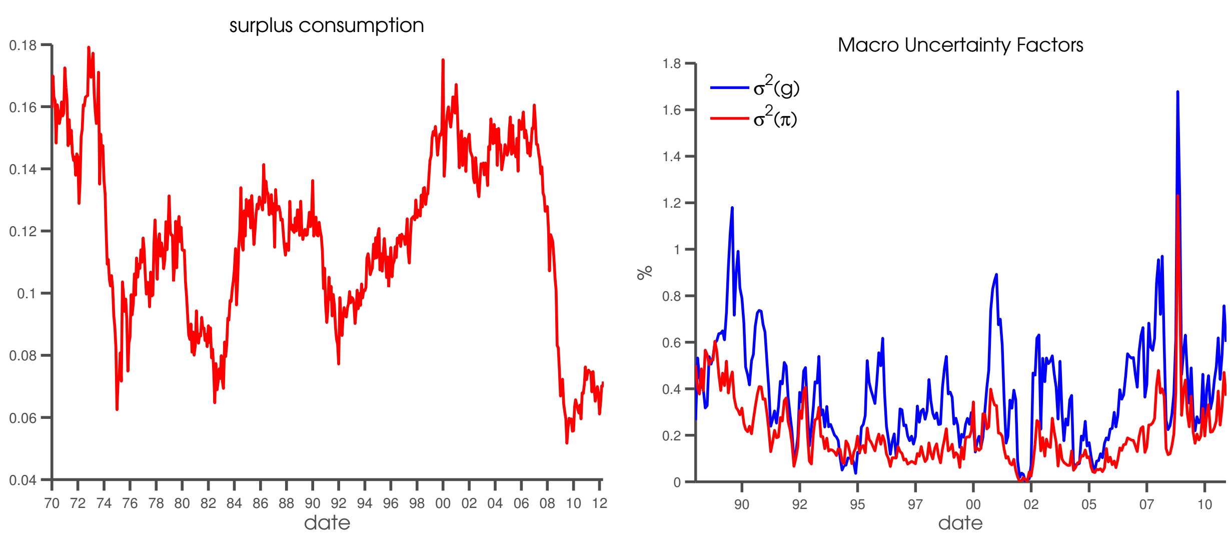

Figure 18. Alternative Risk Factors
Figure plots time-series proxies for alternative structural risk factors. The right panel plots the conditional variances of expected GDP growth ($\sigma^2(g)$) and inflation ($\sigma^2(\pi)$). The left panel plots a consumption surplus proxy. Detailed description for the construction of these factors is given in the data appendix.

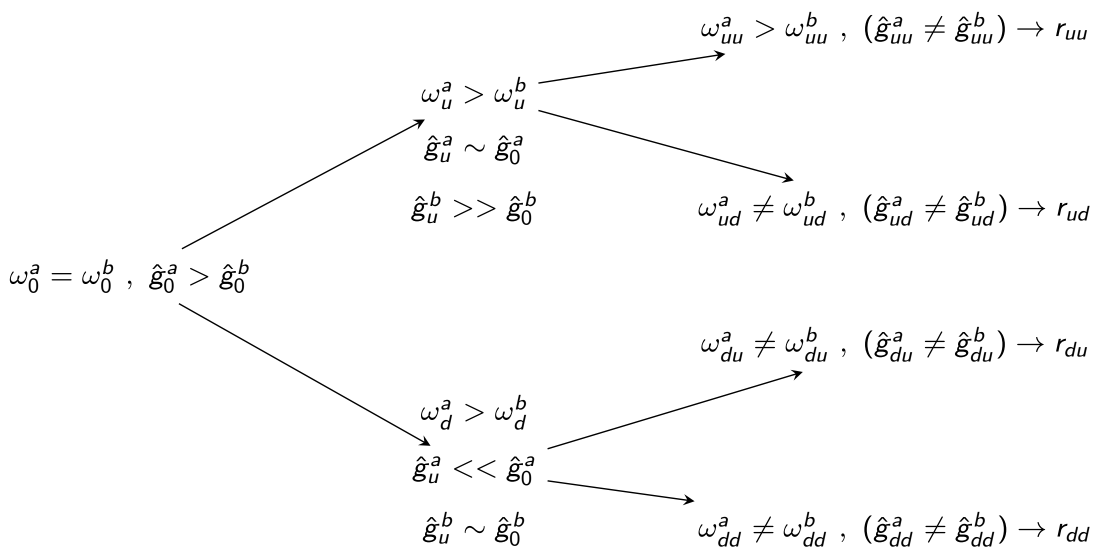

Figure 19. Path Dependence Example:
Figure plots a stylised binomial tree for the evolution of beliefs and relative wealths. The tree begins at the zeroth node where agents have equal wealths but agent a is the optimist. In subsequent periods agents revise their beliefs and there is a redistribution of wealth based on the path of beliefs from previous periods.

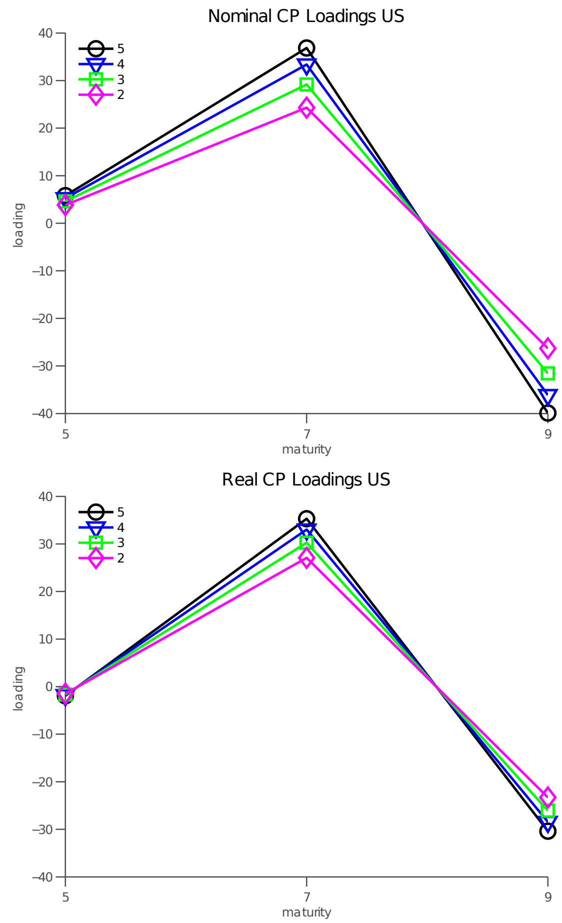

Figure 20. Real and Nominal Forecasting Factor Loadings:
Figure displays the γ's in the following regression

$$\frac{1}{3}\sum_{[5\ 7\ 10]} rx^{(n)}_{t,t+0.25} = \overline{\alpha} + \gamma'(f_t^{[5\ 7\ 10]} - y_t^{(0.25)}) + \overline{\epsilon}_{t,t+0.25}$$

where returns and forward rates are either real or nominal. Each panel displays the factor loadings used to build 3-month real and nominal Cochrane and Piazzesi (2005) forecasting factors.

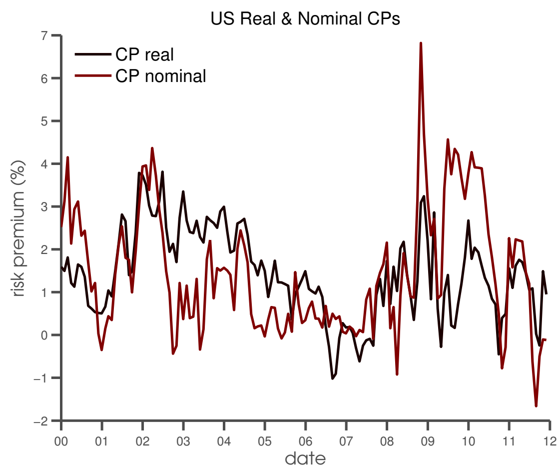

Figure 21. Real and Nominal Forward Rate Implied Risk Premia:
In a first stage the average 3-month excess returns across for maturities $5-10$ are projected on date t forward rates. The fitted value from this regression is a linear combination forward rates that predicts returns of all maturities. Red lines show the implied nominal bond risk premium ($CP_t^{\$}$) and black lines the real bond risk premium (CP_t^r).

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