Dedicated to my loving parents,

Whose unwavering support, boundless love, and constant encouragement have been the pillars of my strength.

This achievement reflects their sacrifices and guidance.

With deepest gratitude and affection, I dedicate this work to them.

CERTIFICATION

This is to certify that the report **"Study on Boolean Algebra"** being submitted bearing Roll No. 201618, Registration No 20200150423, Session 2019-2020, Department of Mathematics, Jahangirnagar University, Savar, Dhaka-1342, Bangladesh in some completion of the necessity for the award of degree of the Bachelor of Science in Mathematics, From the Mathematics Department, Jahangirnagar University, Savar, Dhaka- 1342 has been completed under my guidance and supervision.

To the best of my knowledge, no university or institution has requested the research works contained in this report be used in the awarding of a degree

Place: Jahangirnagar University, Savar,
Dhaka.

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Last but not least, I want to express my gratitude to my parents for their unwavering love and support. Thank you for your assistance, both directly and indirectly, throughout the preparation of my report.

ABSTRACT

Boolean algebra is a fundamental mathematical framework that underpins the core concepts of logical reasoning and digital systems. This report focuses on the essential principles of Boolean algebra, including its algebraic structure, logical operations (AND, OR, NOT), and its relationship with lattice theory and partially ordered sets (poset). The connection between Boolean algebra and lattice theory is explored, emphasizing how Boolean algebras can be viewed as distributive, complemented lattices. The report also discusses the role of ideals in poset and lattices, providing a deeper understanding of the structural properties of Boolean algebras. Key concepts such as Boolean functions, simplification techniques, and canonical forms are examined in detail. Furthermore, the application of Boolean algebra in the design and analysis of digital circuits is highlighted, demonstrating its practical utility in constructing logic gates and combinational circuits. By integrating core theoretical concepts with their use in circuit design, this report offers a comprehensive exploration of Boolean algebra and its foundational role in digital systems.

Notations in Lattice Theory

- $(\boldsymbol{P}, \leq)$ Partially Ordered Set (Poset).
- $\boldsymbol{a} \leq \boldsymbol{b}$ Partial Order Relation.
- $\boldsymbol{a} \vee \boldsymbol{b}$ Join (Least Upper Bound).
- $\boldsymbol{a} \wedge \boldsymbol{b}$ Meet (Greatest Lower Bound).
- $(\boldsymbol{L}, \vee, \wedge)$ Lattice.
- $(\boldsymbol{B}, \vee, \wedge, ', \mathbf{0}, \mathbf{1})$ Boolean Algebra,
- $\boldsymbol{a} \vee \boldsymbol{b}$ Logical OR,
- $\boldsymbol{a} \wedge \boldsymbol{b}$ Logical AND,
- $\boldsymbol{a}'$ Complement,
- $\boldsymbol{a} \oplus \boldsymbol{b}$ Logical XOR,
- $\boldsymbol{o}(\boldsymbol{A})$ Order of a Set A
- $\boldsymbol{a}|\boldsymbol{b}$ a Divides b
- $\in$ Belongs to
- $\notin$ Not Belongs to
- $\boldsymbol{\mathcal{P}}(\boldsymbol{X})$ Set of all subsets of X
- φ Is Empty Set,
- $\cong$ Isomorphic to,
- $\boldsymbol{B}$ Boolean Set,
- $\mathbb{N}$ Set of Natural Numbers
- $\mathbb{Z}$ Set of Integer
- $\subseteq$ Subset Relation.

Additional Notations

- $\cup$ Union,
- $\cap$ Intersection,
- $\Rightarrow$ Implication,
- $\Leftrightarrow$ Equivalence,
- $\forall$ Universal Quantifier,
- $\exists$ Existential Quantifier.

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Chapter 1

Introduction

Background of Lattice

George Boole's endeavor to formalize propositional logic in the middle of the nineteenth century resulted in the invention of Boolean algebra. By the end of the 1800s, ***Ernst Schröder*** and ***Charles S.*** Pierce were studying the axiomatic of Boolean algebra, and they found it useful to introduce the idea of lattice. The same innovation was produced independently by ***Richard Dedekind's*** study of algebraic number ideals. ***Dedekind*** also created a weaker version of distributive referred to as modularity. Despite some of the early findings of this mathematician and Edward Huntington being good and anything from insignificant, they failed to get the public's attention.

The work of ***Garrett Birkhoff*** in the middle of 1930 marked the beginning of the Lattice theory's comprehensive development. He demonstrated the significance of lattice theory in a spectacular collection of works. The second edition of Lattice Theory, which ***Garrett Birkhoff*** published in 1948, was a much expanded and rewritten version of the previous book.

Lattice theory began in the first part of the 1930s and has grown through multiple stages with varying techniques and expectations, but it has been expanding dramatically every decade since then.

This ongoing progress does demonstrate that a theory's place in mathematics cannot be disregarded if it finds significant applications and is a productive source of research.

Lattices are versatile mathematical structures that find applications in a wide range of fields. Here are some uses of Lattice Theory in Mathematics:

Set and Relations

Set: 1.1

In this chapter, I have discussed the definitions of sets and relations and some properties of sets and relations.

A set is a well-defined collection of objects.

Example: 1.1(a) Possibly a set of baking utensils, as in,

Baking equipment (B) = oven, pan, rack, measuring cup, spoon, and whisk.

Example: 1.1(b) Array of digits, as in N= {1,2,3,4,5....}

Definition: 1.1.1 ***An ordered pair*** is a set of numbers (x, y) written in a specific sequence.

Definition: 1.1.2 When two sets A and B are combined so that every pair in A is related to every pair in B by an ordered set of relations, we have ***a cartesian product.***

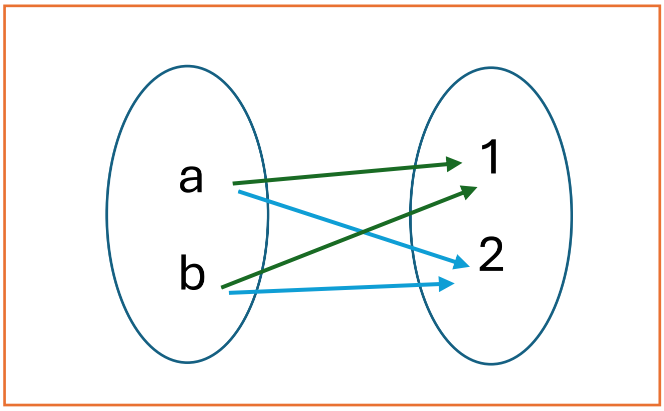


Figure: 1

As can be seen in Figure 1, the Cartesian product of sets A and B is

$$A * B = \{ (a, 1) , (a, 2), (b, 1), (b, 2)\}.$$

Relation

Definition: 1.1.3 Consider two sets, A and B. Any subset of $A \times B$ It is called a relation from A to B.

Example: 1.1.3(a) $Let\ A = \{1,2,3\}\ and\ B = \{4,5\}$

$Then\ \ A \times B = \{(1,4),(1,5),(2,4),(2,5),(3,4),(3,5)\}$

$$R_1 = \{(1,4),(1,5)\}, \quad R_2 = \{(2,5)\}, \qquad R_3 = \{(3,4),(1,5)\}$$

$if\ B = A$, We assume that R is a relation in A. $(R \subseteq A \times A)$

Some properties of relations in a set

1. **Reflexivity**: A relation R on a set A is reflexive if every element is related to itself. In other words, for every element $\boldsymbol{a} \in \boldsymbol{A}$, $(\boldsymbol{a}, \boldsymbol{a})$ It should be about.
2. **Symmetry**: A relation R on a set A is symmetric if, whenever

   $(\boldsymbol{a}, \boldsymbol{b}) \in \boldsymbol{R}, then\ (\boldsymbol{b}, \boldsymbol{a}) \in \boldsymbol{R}$.

3. **Antisymmetric**: A relation R on a set A is antisymmetric if, whenever

   $(\boldsymbol{a}, \boldsymbol{b}) \in \boldsymbol{R}\ and\ (\boldsymbol{b}, \boldsymbol{a}) \in \boldsymbol{R},$ It must follow that $\boldsymbol{a} = \boldsymbol{b}$

4. **Transitivity**: A relation R on a set A is transitive if, whenever

   $(\boldsymbol{a}, \boldsymbol{b}) \in \boldsymbol{R}\ and\ (\boldsymbol{b}, \boldsymbol{c}) \in \boldsymbol{R}, it\ follows\ that\ (\boldsymbol{a}, \boldsymbol{c}) \in \boldsymbol{R}.$

Chapter 2

POSET

Partially Ordered Set 2.1

Definition: 2.1.1 [05] A relation that is reflexive, anti-symmetric and transitive on a non-empty set is referred to as partial order relation or ***poset.***

Example: 2.1.1(a) Let X = {3, 4, 8}

The Cartesian product of X is

$$X \times X \ = \{(3,3),(3,4),(3,8),(4,3),(4,4),(4,8),(8,3),(8,4),(8,8)\}$$

Let us take a relation R, which is a subset of the Cartesian product of $X \times X$

$R \ = \{(3,3),(4,4),(8,8),(3,4),(3,8)\}$ It is a partially ordered relation.

Definition: 2.1.2 [08] If any of the following are true, a non-empty set U and a binary relation S together constitute a ***partially ordered set, or POSET:***

$$\boldsymbol{P1}: \boldsymbol{Reflexivity}: gSg \ for \ all \ g \ \in \ U$$

$$\boldsymbol{P2}: \boldsymbol{Anti-symmetricity}: if \ gSf.fSg \ then \ f \ = \ g \ forall \ g\ ,f \in \ U$$

$$\boldsymbol{P3}: \boldsymbol{Transivity}: if \ gSf,fSh \ then \ gSh; \ g,f,h \ \in U$$

The symbol " $\leq$ " It is typically used in place of S for convenience. Given that it makes the aforementioned conditions seem so natural, the cause is clear. As less than or equal to, we read (although it may have nothing to do with the usual less than or equal to that we are so familiar with).

Example: 2.1.2(a) Let G be a collection of sets A, B, C……. then G under contained in ⊆ Relation forms a poset.

Example: 2.1.2(b) Under divisibility, the set N of natural numbers forms a poset. Because of this,

$a \ \leq \ b$ denotes here $a|b$ (a divides b).

Totally ordered set (Toset or Chain) 2.2

Definition: 2.2.1 [08] U is referred to be **a totally ordered set (toset)** or a chain if it is a poset with every two members being comparable.

Consequently, let U be a chain and $j, k \in U$ then either $j \leq k$ or $k \leq j$

Example: 2.2.1(a) Under the relation (usual) ≤, the set N of natural numbers forms a chain. In a Similar way, chains are formed under relation (usual) ≤ by integers, rational numbers, and real numbers.

Example: 2.2.1(b) There is no chain formation in the set N of natural numbers under divisibility.

Hasse Diagram 2.3

A ***Hasse diagram*** is a representation of any poset P in which each relation, such as $x \leq y$ It is represented by a line or lines that progressively ascend from x to y, and unique members of P are represented by distinct points.

Hasse diagrams are also called upward drawings.

Example: 2.3 (a) We consider a poset $E = \{3, 6, 9, 12, 18, 36, 72\}$.

The Hasse diagram of this poset is as follows:

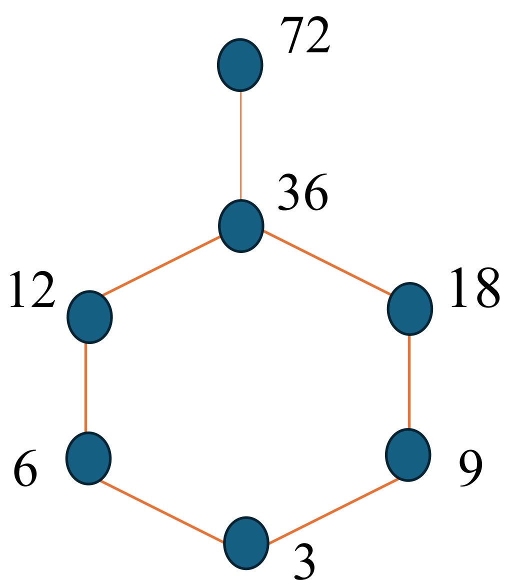


Figure 1

Example: 2.3(b) $(S = \{2,3,4,5,6,8,10\}, |)$ It is a poset. Here is the Hasse Diagram for this Poset.

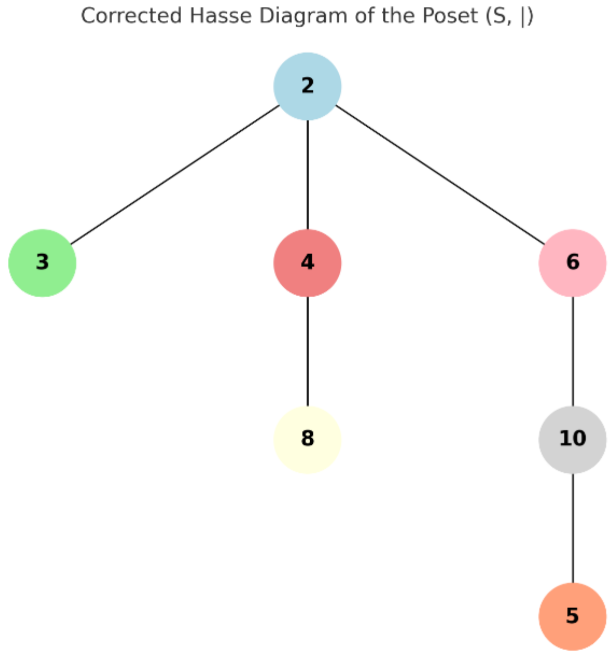


Figure 2

Example: 2.3(d) $(S = \{3,9,27\}, |)$ is a toset. The following Hasse diagram shows this toset

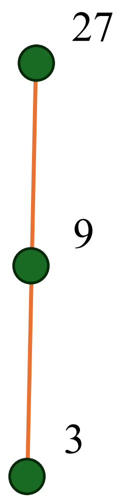


Figure 3

Definition: 2.4.1 [08] Let r be an element in a poset U is said to be ***maximal element*** of U iff

$$\exists\, no\, m \in U\, s.t.\, r < m$$

Example: 2.4.1(a) Let (A= {2, 3, 4, 5, 8, 10, 12, 24, 30}, I) is a poset. The maximal element of this poset is as follows:

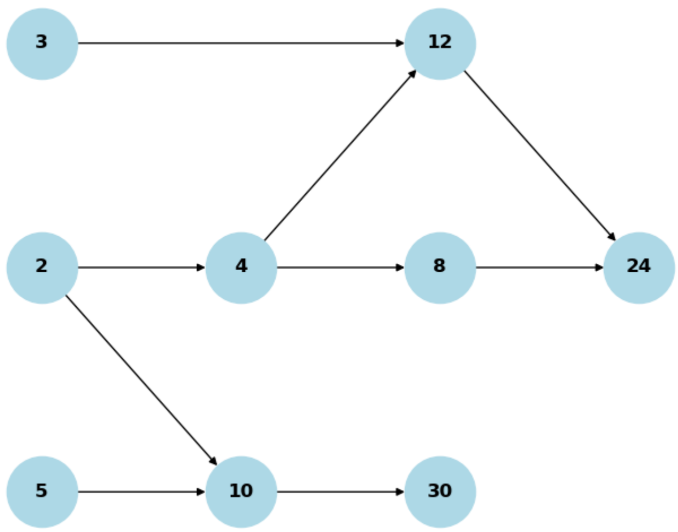


Figure 4

Maximal elements are 24,30

Definition: 2.5.1 [08] The element "m" is referred to as the ***maximum element, greatest element,*** or ***unity element*** if an element m in U such that $w \leq m; \quad \forall w \in U$

It is generally denoted by u or 1.

Example 2.5.1 (a): Let $(B = \{2,4,6,8,10,30,60,120,240), I)$ is a poset. The maximum element is as follows:

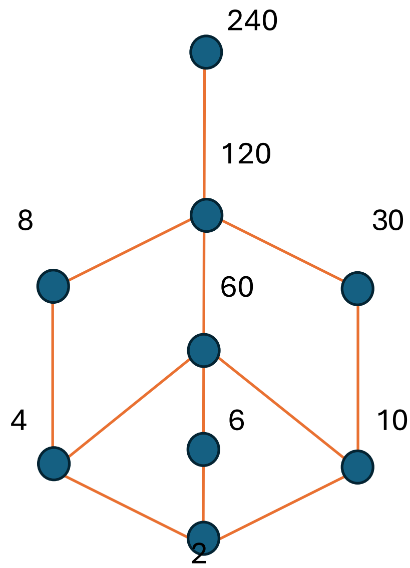


Figure-5

Here the maximum element exists and is equal to 240

Definition: 2.6.1 [08] A ***minimal element*** of a poset U is defined as an element 'j' if

$$\exists\, no\, w \in U \qquad s.t. \quad w < J$$

Example: 2.6.1(a) $Let\ (A = \{2,3,4,5,8,10,12,24,30),$ I) is a poset. The minimal element is as follows:

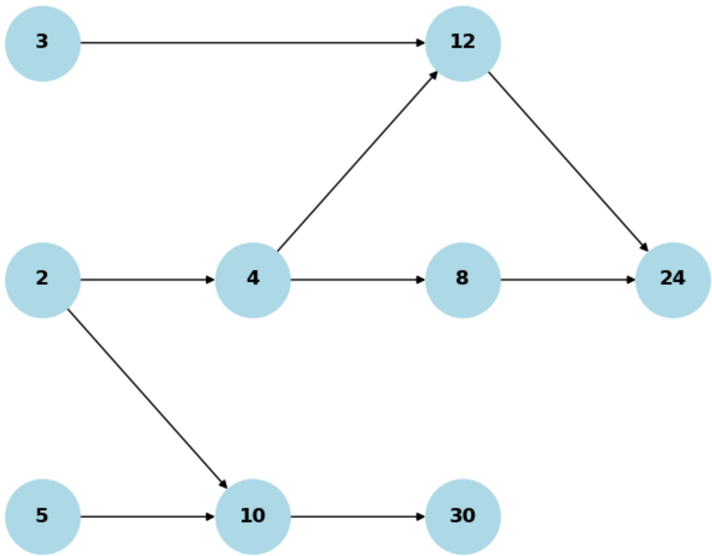


Figure: 6

Here the minimal elements are 2,3,5.

Definition: 2.7.1 [08] Assume that U is a poset. If an element 'n' in U such that $n < w, \forall w \in U$

then 'n' is referred to as the ***minimum, least*** or ***null element.***

It is denoted by the symbol 0

Example: 2.7.1 (a) Let (B= (2, 4, 6, 8, 10, 30, 60, 120, 240), I) is a poset. The minimum element is as follows:

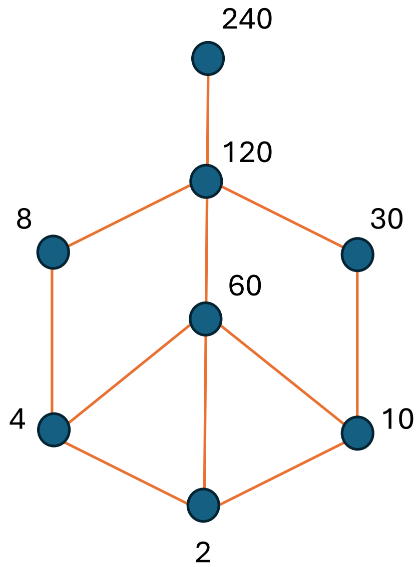


Figure-7

Here the minimum element exists and is equal to 2

Some important theorems on posets 2.8

Theorem 2.8.1: [08] A finite subset G of a poset U has maximal and minimal elements if it is non-empty.

Proof: Let all the unique elements of G be $t_1, t_2 \ldots\ldots t_n$ in any arbitrary order

We are finished if t_1 is the maximal element.

There exists some $t_k \in G\ s.t.\ t_1 < t_k$ if is not maximal. t_1

We are finished if t_k is maximal.

In case t_k is not maximal, some $t_p \in G\ s.t.\ t_k < t_p$ If we keep going in this manner, eventually some element will be maximal because G is finite.

Likewise, we may demonstrate that G has a minimal element.

Theorem 2.8.2: [08] Unity element and null element on a poset are unique (if exists)

Proof: Let unity element and null element exist on a poset (P ,<=).

If possible, suppose $(P, \leq)$ has two unity elements, say $\hat{\alpha}\ \hat{\beta}$.

$$\hat{\alpha} \text{ is unity element of } P \Rightarrow x \leq \hat{\alpha} \qquad \forall\, x \in P$$

$$\Rightarrow \hat{\beta} \leq \hat{\alpha} \ldots\ldots\ldots\ldots\ldots..(i) \qquad [\because \hat{\beta} \in P]$$

$$\hat{\beta} \text{ is unity element of } P \Rightarrow x \leq \hat{\beta} \qquad \forall\, x \in P$$

$$\Rightarrow \hat{\alpha} \leq \hat{\beta} \ldots\ldots\ldots\ldots\ldots..(ii) \qquad [\because \hat{\alpha} \in P]$$

From (i) & (ii) we get,

$$\hat{\alpha} = \hat{\beta}$$

Hence unity element is unique on poset.

Again,

if possible,

suppose (P , $\leq$) has two null elements, say $\hat{\gamma}, \hat{\delta}$, $\hat{\gamma}$ is null element

$$P \Rightarrow \hat{\gamma} \leq x, \quad \forall\, x \in P$$

$$\Rightarrow \hat{\gamma} \leq \hat{\delta}, \ldots\ldots\ldots\ldots\text{(iii)} \qquad [\hat{\delta} \in P\,]$$

$$\hat{\delta} \text{ is null element of } P \Rightarrow \hat{\delta} \leq x \qquad \forall\, x \in P$$

$$\Rightarrow \hat{\delta} \leq \hat{\gamma} \ldots\ldots\ldots\ldots\text{(iv)} \qquad [\hat{\gamma} \in P\,]$$

From (iii) & (iv) we get,

$$\Rightarrow \hat{\gamma} \leq \hat{\delta}$$

Hence null elements are unique on poset

Chapter 3

Lattices

Lattices 3.1

A lattice is just an algebraic structure that is explored in several sub-disciplines of abstract algebra and order theory. It is based on mathematics. Generally speaking, a lattice is made up of a poset where each pair of components has a unique supremum (lowest upper bound) and infimum (largest lower bound).

Definition 3.1.1(set theoretic): [08] Let $(T, \leq)$ be a poset.

For any $q, r \in T$, a poset $(T, \leq)$ is considered to constitute a lattice such that T contains both

$$inf\ \{q, r\}\ and$$

$$sup\ \{q, r\}.$$

Then, we would write,

$$sup\ \{q, r\} = q \vee r,$$

where q and r are read as 'q join r'.

$$inf\ \{q, r\} = q \wedge r,$$

where q and r are read as 'q meet r'

Example 3.1.1(a): $(P(H), \subseteq)$ is a lattice where $H = \{1,2,3\}$.

First, we will draw the Hasse diagram given as follows:

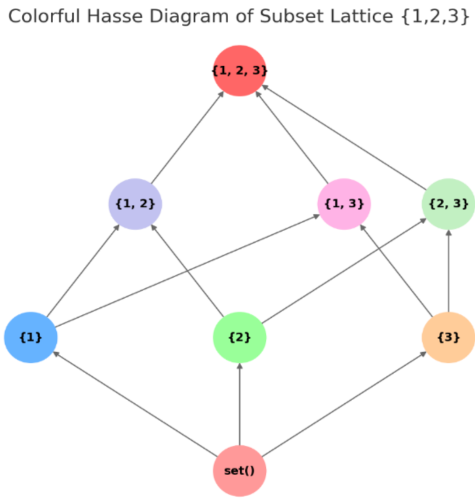


Figure: 8

The union of two subsets is known as the join, and the intersection of the subsets is known as the meet.

Such as the join:

$$\{1,2\} \vee \{2,3\} = \{1,2\} \cup \{2,3\} = \{1,2,3\};$$

And the meet:

$$\{1,2\} \wedge \{2,3\} = \{1,2\} \cap \{2,3\} = \{2\}$$

Any collection of n elements can be used to generalize this result.

Example 3.1.1(b): ($D_{60} = \{1,2,3,4,5,6,10,12,15,20,30,60\}$, I) is a lattice.

First, we will draw the Hasse diagram:

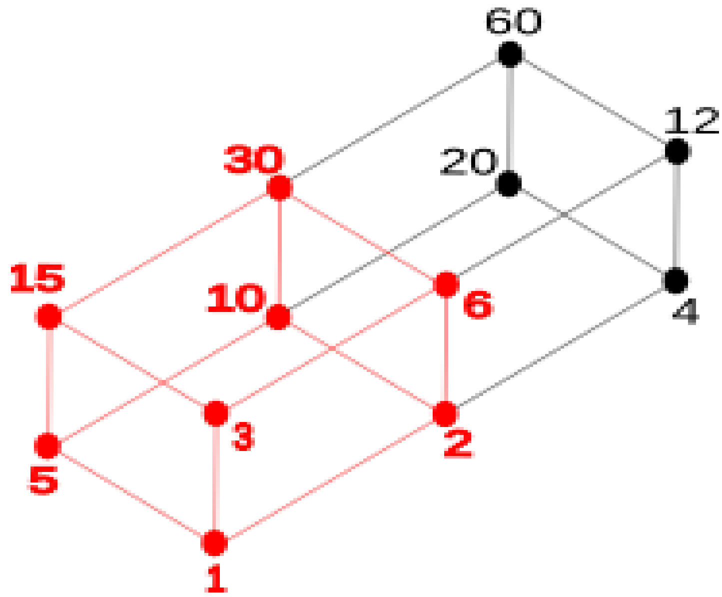


Figure 9

Here, a pair of elements, q and r have the least common multiple (lcm (q, r}) at their join, and a greatest common divisor (gcd {q,r}) at their meet.

Let the Join: 15 V 20 = lcm {6, 10) = 30;

And the Meet: 6 ∧ 10 = gcd (6,10) = 2.

Generally, natural numbers' set forms a lattice ordered by the divisibility relation "|".

Example 3.1.1(c): Suppose P = {a, b, c, d, e) is not a lattice ordered by the following diagram.

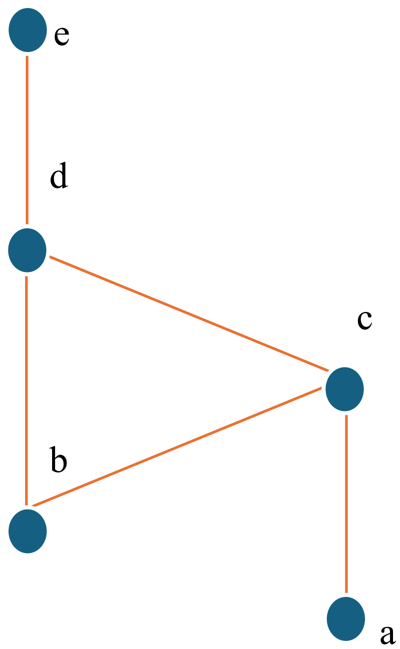


Figure - 10

In this case, there is no greatest lower limit (glb) for the elements "a" and "b" Thus, it is not a lattice but rather a poset.

Definition 3.1.2(algebraic): [08] A non-empty set U together with two binary compositions (operations), $\wedge$ and $\vee$, are said to form a lattice

if $\forall g, h, k \in U,$ the following conditions are hold:

- Idempotency: $g \wedge g = g$

  $g \vee g = g$

- Commutativity: $g \wedge h = h \wedge g$

  $g \vee h = h \vee g$

- Associativity: $g \wedge (h \wedge k) = (g \wedge h) \wedge k$

  $g \vee (h \vee k) = (g \vee h) \vee k$

- Absorption: $g \wedge (g \vee h) = g$

$$g \vee (g \wedge h) = g \quad where\ g, h, k \in U$$

Note: In this case, the two meanings of lattice are equivalent.

Some other Example:

Example 3.1.1(d): (L=(1,2,3,5,210), | **)** is a Lattice.

Hasse diagram of this lattice is given below.

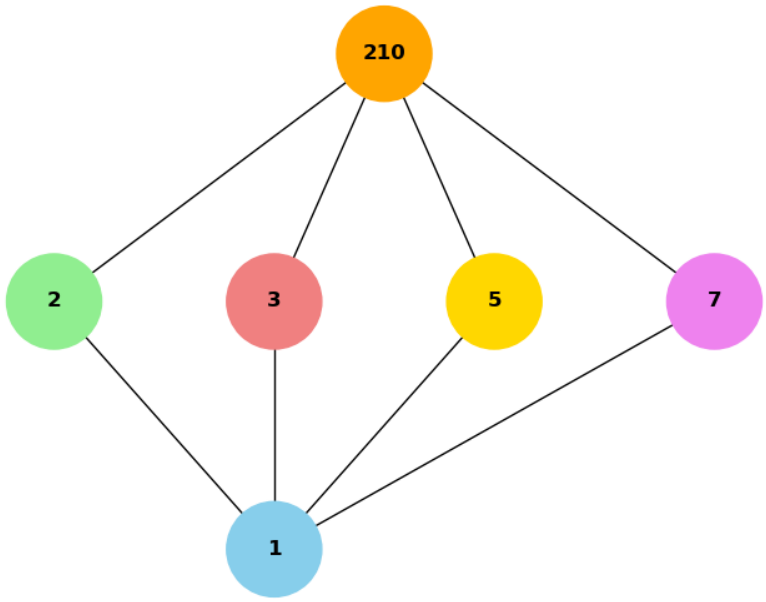


Figure:11

Now make a meet and join table in given below

∧	1	2	3	5	7	210
1	1	1	1	1	1	1
2	1	2	1	1	1	2
3	1	1	3	1	1	3
5	1	1	1	5	1	5
7	1	1	1	1	7	7
210	1	2	3	5	7	210

∨	1	2	3	5	7	210
1	1	2	3	5	7	210
2	2	2	210	210	210	210
3	3	210	3	210	210	210
5	5	210	210	5	210	210
7	7	210	210	210	7	210
210	210	210	210	210	210	210

We see that, for all $a, b \in L$

$a \wedge b$ *and* $a \vee b$ exist in L

Therefore, L form a lattice under divisibility ("|")

Example 3.1.1(e): (U = {p,q,r,s,t}, |) is a lattice,

Hasse Diagram of this lattice is given below:

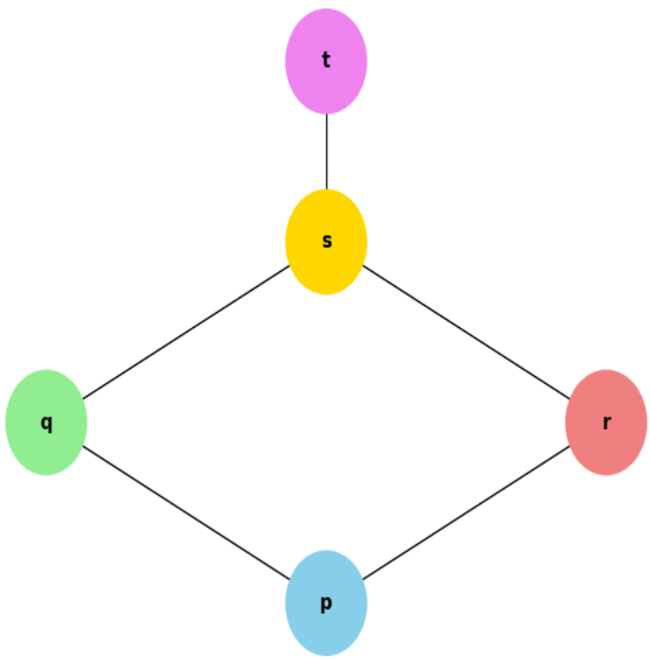


Figure: 12

Now make a meet and join table is given as follows.

∧	p	q	r	s	t
p	p	p	p	p	p
q	p	q	p	q	q
r	p	p	r	r	r
s	p	q	r	s	s
t	p	q	r	s	t

∨	p	q	r	s	t
p	p	q	r	s	t
q	q	q	s	s	t
r	r	s	r	s	t
s	s	s	s	s	t
t	t	t	t	t	t

We see, for all m, n ∈ U

$m \wedge n$ *and* $m \vee n$ *exist in* U

Therefore, U forms a lattice under divisibility ("|")

Different types of lattices 3.2

Definition 3.2.1: [08] Every non-empty subset of a lattice U that has its sup and inf in U is referred to as a ***complete lattice***.

Example 3.2.1(a): The lattice (L= {1, 2, 3, 4, 6, 12}, |) is a complete lattice.

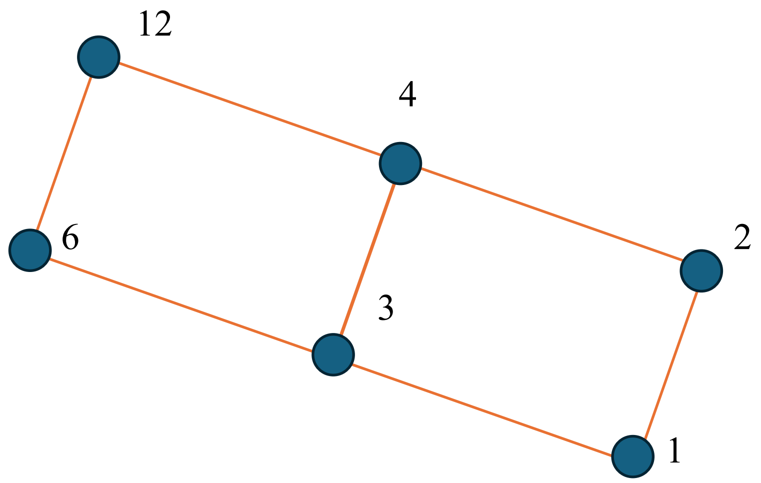


Figure 13

Since inf T = a ∧ b and sup T = a V b exist for every subset T of L, so L is a complete lattice.

Definition 3.2.2: [10] When (U ∧ V) is an algebraic structure and the constants 0,1 in U meet the following, the structure is called a bounded lattice.

i) $\forall z \in U, \quad z \wedge 1 = z \quad and \quad z \vee 1 = 1$

ii) $\forall z \in U, \quad z \wedge 0 = 0 \quad and \quad z \vee 1 = z$

1 is called the greatest element of U and 0

is called the least element of U.

Example 3.2.2(a): ($D_{18} = \{1, 2, 3, 6, 9, 18\}$. I) is a bounded lattice.

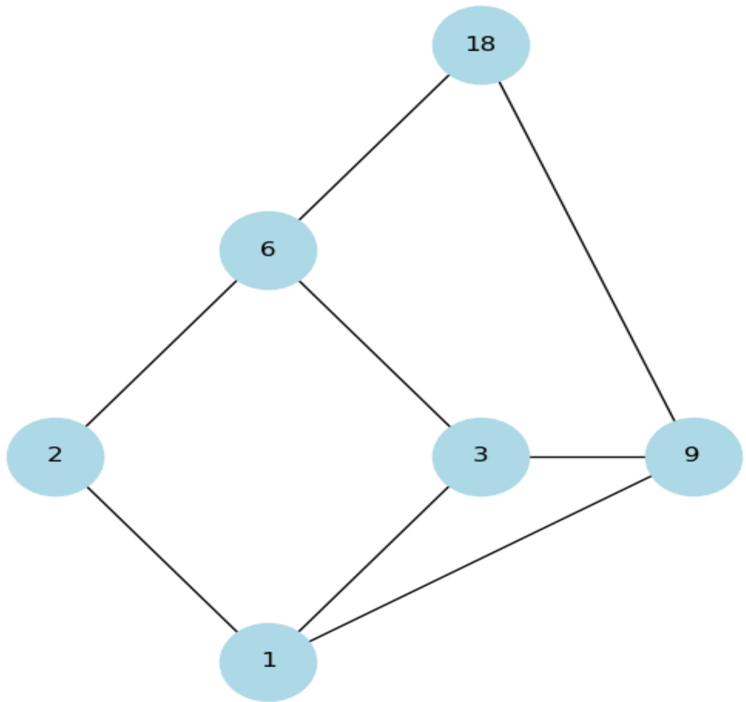


Figure: 14

Here the greatest element is 18 (1=18) and the least element is 1 (0 = 1). Since greatest element and least element both are present, it is a bounded lattice.

Properties of Bounded Lattice 3.2.3

The following identities hold for any member $b \in$ U if U is a bounded lattice:

- $b \vee 1 = 1$
- $b \wedge 1 = \mathrm{b}$
- $b \vee 0 = \mathrm{b}$
- $b \wedge 0 = 0$

Definition 3.2.4: [09] If m, n ∈ H then m An occurs in H, then a poset $(H , \leq)$ is referred to as a ***meet semilattice.***

If m, n in H sup {m, n} exists in H, then a poset $(H , \leq)$ is referred to as a ***join semilattice.***

Semilattice refers to both the meet and join of semilattices.

Example 3.2.4(a): (L= {1, 2, 3}, |) is a meet-semilattice.

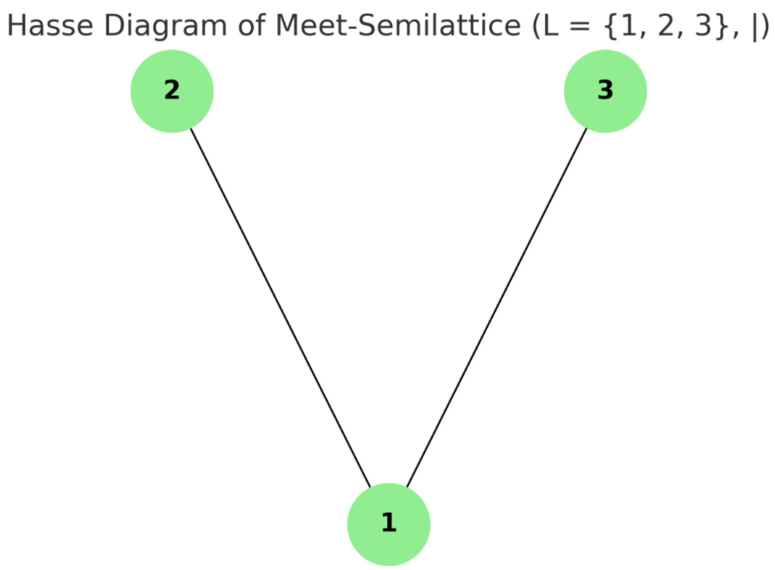


Figure: 15

Example 3.2.4(b): (L = {2, 3, 6}, |) is a join-semilattice.

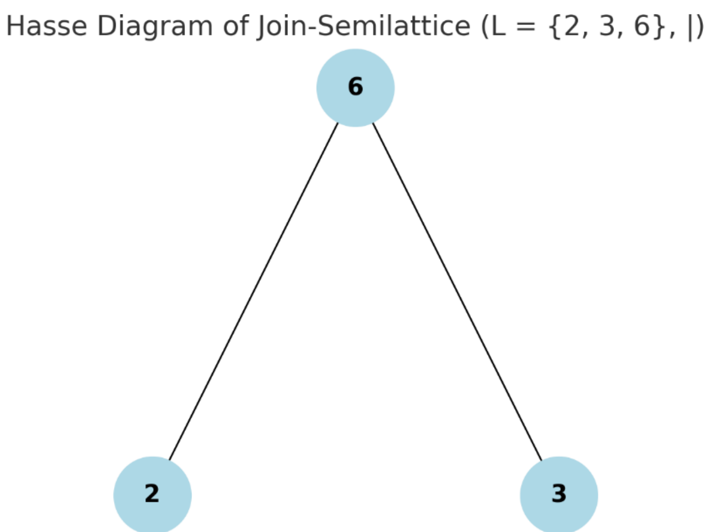


Figure: 16

Definition 3.2.5: [04] Assume that the bounded lattice (T , $\leq$) has a greatest element of 1 and a least element of 0.

Assume that m, n are in T. When

$$m \wedge n = 0 \;\; and \, m \wedge n = 1$$

then element n is referred to as the ***complement*** of m

Definition 3.2.6: [04] When each element in a bounded lattice $(T, \leq)$ has a complement, then T is said to be ***complemented.***

Specifically, 0 has a complement of 1, while 1 has a complement of 0.

Note: Here, each element should have at least one complement.

Example 3.2.6(a): The poset $(D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}, \mathrm{I})$ is a complemented lattice having complement

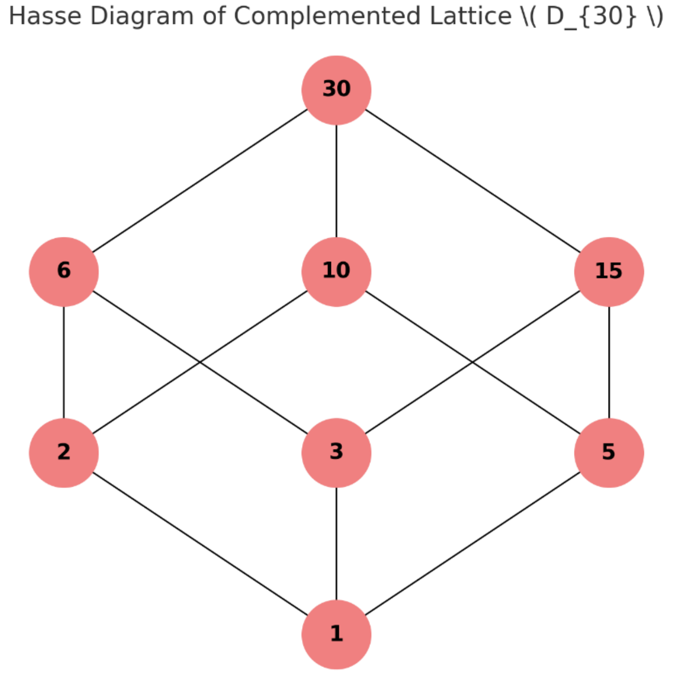


Figure: 17

Every element in D_{30} has a complement

$$1^c = 30,$$

$$2^c = 5,$$

$$3^c = 10,$$

$$5^c = 6,$$

$$6^c = 5,$$

$$10^c = 3,$$

$$15^c = 2,$$

$$30^c = 1$$

Keep in mind that not all kinds of lattices $(D_n, |)$ are complemented.

For example, the lattice (D_{20}, I) is not complemented

Example 3.2.6(b): Here (D_1= {1, 2, 3, 6), I) is a complemented lattice.

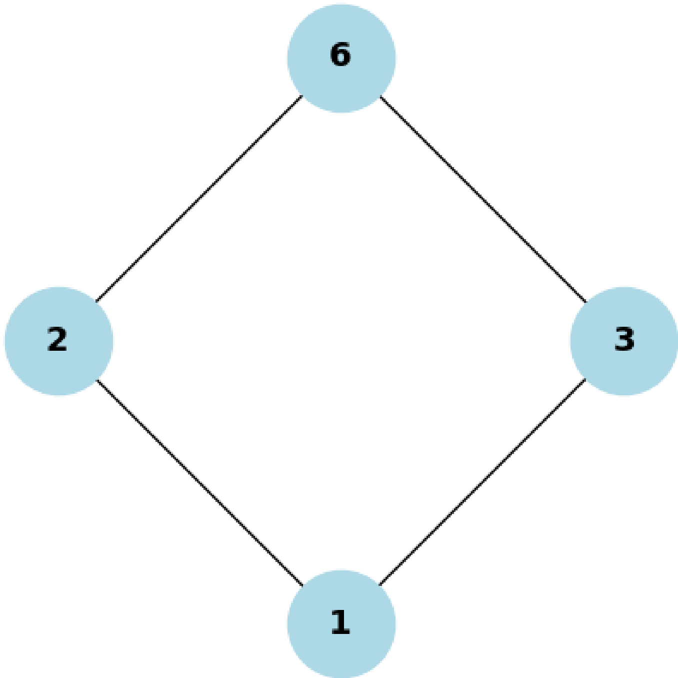


Figure: 18

In the above diagram every element has a complement.

Definition 3.2.7: [08] An interval [s, t] is said to be complemented if each member z of the interval has at least one complement with respect to [s, t].

Furthermore, a lattice is considered ***relatively complemented*** if each interval within it is complemented.

Example 3.2.7(a): Let L = ({1, 2, 3, 6), |) is a lattice

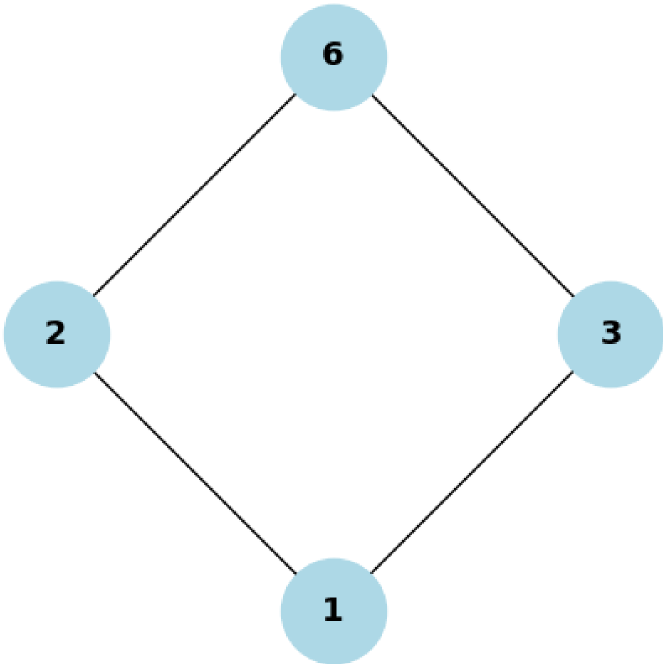


Figure: 19

Now in interval [1, 2], $1 \wedge 2 = 1$ and $1 \vee 2 = 2$

So, interval [1, 2] is relatively complemented.

Similarly, we can say it for other elements.

Definition 3.2.8: [08] Whenever a lattice (U ,≤) has the following distributive qualities, it is referred to as a ***distributive lattice***:

$$s \vee (t \wedge v) = (s \vee t) \wedge (s \vee v);$$

$$s \wedge (t \vee v) = (s \wedge t) \wedge (s \wedge v);$$

Example 3.2.8 (a): The power set of T (P(T) , ⊆) for any set T is a distributive lattice.

Example 3.2.8 (b): Suppose L = {0, a, b, c, 1} is not a distributive lattice.

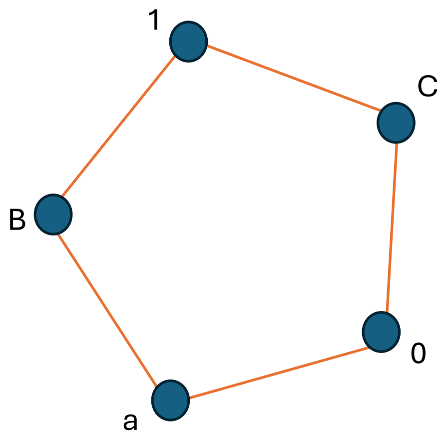


Figure: 20

We pick a, b, c as our elements and look

for the distributive law

$$b \wedge (c \vee a) = (b \wedge c) \vee (b \wedge a).$$

We now compute the identity's left and right sides.:

$$LHS = b \wedge (c \vee a) = b \wedge 1 = b$$

$$RHS = (b \wedge c) \vee (b \wedge a) = 0 \vee a = a.$$

As LHS ≠ RHS, the distributive law fails for pentagon lattice

Definition 3.2.9: [04] Let (L,) is a lattice. Then L is referred to as ***modular lattice*** iff

$$\forall s, t, v \in L \quad with \quad t \leq s$$

$$s \wedge (t \vee v) = t \vee (s \wedge v)$$

Definition 3.2.10:[4] Let (U ,≤) is a lattice. Then U is referred to as

non-modular lattice iff

$$\forall s, t, v \in U \, with \, t \leq s$$

$$s \wedge (t \vee v) \neq t \vee (s \wedge v)$$

Example 3.2.9(a): Show that M_5 is a modular lattice

Solution: First draw a Hasse diagram of M_5,

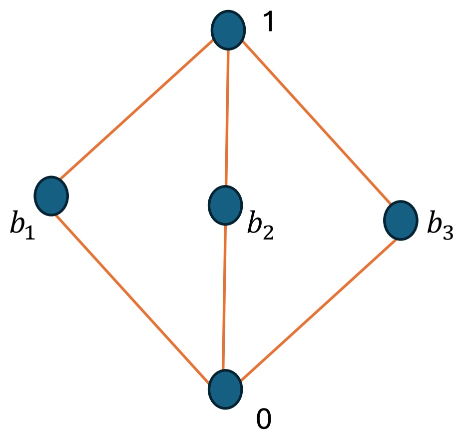


Figure: 21

Let $(L, \leq)$ be a lattice. Then L is referred to as modular lattice iff

$$\forall\, p, q, r \in L\, with\, p \leq r$$

$$p \vee (q \wedge r) = (p \vee r) \wedge r$$

Now,

- $0, b_1, b_2 \in L$

$$P \vee (q \wedge r) = (p \vee r) \wedge r$$

$$\Longrightarrow 0 \vee (b_1 \wedge b_2) = (0 \vee b_1) \wedge b_2$$

$$\Rightarrow 0 \vee 0 = b_1 \wedge b_2 \Longrightarrow 0 = 0$$

- $0, b_1, 1 \in L$

$$P \vee (q \wedge r) = (p \vee r) \wedge r$$

$$\Longrightarrow 0 \vee (b_1 \wedge 1) = (0 \vee b_1) \wedge 1$$

$$\Rightarrow 0 \vee 0 = b_1 \wedge 1 \Longrightarrow b_2 = b_2$$

$\forall\, p, q, r \in L\, with\ p \leq r,\ \ p \vee (q \wedge r) = (p \vee r) \wedge r$ is satisfied.

Thus M_5 is a modular lattice.

Some important theorems on Lattices 3.3

Theorem 3.3.1: [04] Establish the boundedness of each finite lattice

$U = \{ p_1, p_2 , p_3 \dots . p_n\}$

Proof: The finite lattice is provided by us.

$$= \{ p_1, p_2 , p_3 \dots . p_n\}$$

Lattices U's biggest element is therefore

$$=\{ p_1 \vee p_2 \vee p_3 \dots . \vee p_n\}$$

Lattice L's least element is also

$$=\{ p_1 \vee p_2 \vee p_3 \dots . \vee p_n\}$$

Since every finite lattice has both greatest and least elements.

U is hence bounded.

Theorem 3.3.2: [08] There are least and greatest elements in a finite lattice.

Proof: Assume that $U = \{ r_1, r_2 , \dots \dots \dots , r_n \}$ be the specified finite lattice.

Let us imagine that $r_1, r_2 , \dots \dots \dots , r_n$ r, are written in any arbitrary order.

Sort the following

$$s_0 = r_1$$

$$s_1 = s_0 \wedge r_1$$

$$s_2 = s_1 \wedge r_2$$

……………..

……………..

$$s_n = s_{n-1} \wedge r_{n-1}$$

$$s_n \leq s_{n-1} \leq \dots \dots \dots s_0 \; and \; s_k \leq r_k$$

$$\forall \, k = 1,2, \dots \dots \dots . n$$

i.e. $s_k \leq r_k \quad \forall \, k = 1,2, \dots \dots \dots . n$

$\Rightarrow s_n$ is least element of U

Likewise, we can determine U’s greatest element.

Theorem 3.3.4: [08] P is complete lattice if (U ,≤) is a poset with largest element u such that every non-empty subset J of U has inf

Proof: Let j be any subset of U that is not empty. We must demonstrate the existence of sup j

Given that u is U's greatest element,

$$z \leq u; \qquad \forall z \in U$$

This means that $g \leq u; \qquad \forall g \in J$

Therefore, u is an upper bound of J

Above a set M, which is the set of all upper bound of J, M is a non-empty subset of U

So,

By the given condition inf M exist.

Assume that p = inf M

Now, $g \in J \quad \Rightarrow \quad g \leq u; \ \forall z \in M$

$\Rightarrow$ Each element of J is a lower bound of M

$\Rightarrow g \leq p; \ \forall g \in J$

$\Rightarrow p \text{ } is \text{ } an \text{ } upper \text{ } bound \text{ } of \text{ } J$

P being a lower bound of M means

$p \leq z\,; \ \forall z \in M \quad that \text{ } is \text{ } p \leq z \, \forall \, upper \text{ } bound \text{ } of \text{ } J$

$\Rightarrow p = \sup J$

Thus, U us poset on which sup and inf are present in every non-empty subset

As a result, U is a complete lattice

Theorem 3.3.6: [04] A complemented modular lattice must be relatively complemented.

Proof: Suppose U is complemented modular lattice

Let k ∈ [e, f] be any element and let [e, f] be any interval on U.

So k complement say k' as U is complemented

Subsequently,

$$k \wedge k' = \hat{0} \qquad and \qquad k \vee k' = \hat{1}$$

Clearly,

$$e \leq k \leq f$$

Take

$$W = e \vee (f \wedge k')$$

Now,

$$k \wedge w = e \wedge (e \vee (f \wedge k'))$$

$$= e \vee (k \vee (f \wedge k'))$$

$$= e \vee (f \vee (k \wedge k'))$$

$$= e \vee \left(f \vee \hat{0}\right)$$

$$= e \vee \left(\hat{0}\right) = e$$

And

$$k \wedge w = k \vee (e \vee (f \wedge k'))$$

$$= (k \vee e) \vee (f \wedge k')$$

$$= k \vee (f \wedge k') \qquad [\because e \leq k]$$

$$= f(k \vee k') \quad [\because k \leq b \;\&\; U \; is \; modular \;]$$

$$= f \wedge \left(\hat{1}\right) = f$$

And so, the relative complement of k in [e, f] is w = e ∨ (f ∧ k')

Hence a complemented modular lattice is relatively complemented.

Chapter 4

Sublattices

Sublattices 4.1

Definition 4.1.1: [01] Assume of a lattice U that has a non-empty subset, F.

When $e \wedge f \in F$ $and\ e \vee f \in F$

whenever $e \in F\ and\ f \in F$

then F is referred to be a ***sublattice*** of U.

Example 4.1.1(a): The sublattices of $D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$ in which minimum four elements to be contained

1. {1, 2, 6, 30}
2. {1, 2, 3, 30}
3. {1, 5, 15, 30}
4. {1, 3, 6, 30}
5. {1, 5, 10, 30}
6. {1, 3, 15, 30}
7. {2, 6, 10, 30}

Example 4.1.1(b): $Let\ (L = \{1, 2, 3, 5, 6, 10, 15, 30\}, |)$

Figure: 22

Since $\forall\, e, f \;\in\; L\, e\; \wedge\; f and\, e\; \vee\; f\; exists\; in\, L$, hence (L, I) forms a lattice.
Now we assume a subset Here S = {1, 2, 3, 6} of lattice L.

$$1 \wedge 2 = 1 \;\in S \qquad 1 \vee 2 = 2 \;\in S$$

$$1 \wedge 3 = 1 \;\in S \qquad 1 \vee 3 = 3 \;\in S$$

$$1 \wedge 6 = 1 \;\in S \qquad 1 \vee 6 = 6 \;\in S$$

$$1 \wedge 3 = 2 \;\in S \qquad 2 \vee 3 = 6 \;\in S$$

$$1 \wedge 6 = 2 \;\in S \qquad 2 \vee 6 = 6 \;\in S$$

$$1 \wedge 6 = 3 \;\in S \qquad 3 \vee 6 = 6 \;\in S$$

Hence S is a sublattice of U.

Definition 4.1.2: [09] A sublattice J of a lattice U is said to be ***convex sublattice*** of U iff $\forall\, u\,, v\; \in\; J\,,$

we have $[\,u\; \wedge\; v\,, u\; \vee\; v] \;\subseteq\; J$

Example 4.1.2(a) Let L = {0, a, b, c, 1} is a lattice.

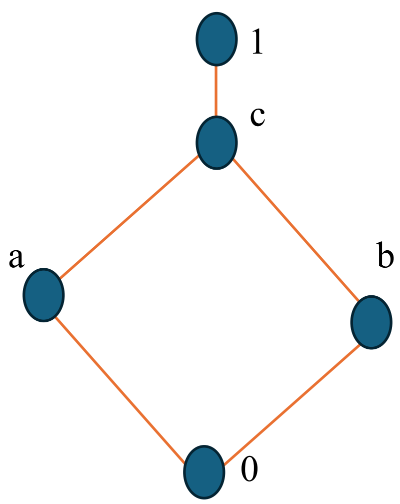


Figure 23

In this case (0, a, b, c) is a convex sublattice by figure 23

Theorem on Sub-lattices 4.2

Theorem 4.2.1: [08] Consider a sublattice j of a lattice U. It refers to convex iff

$\forall\ \hat{\alpha}, \hat{\beta}\ in\ J\ with\ \hat{\alpha}\ \leq\ \hat{\beta}, \quad [a,b] \subseteq\ J.$

Proof: Assuming/ to be a convex sublattice of U,

We find that $\forall\ \hat{\alpha}, \hat{\beta}\ \in j$

$[\ \hat{\alpha}\ \wedge\ \hat{\beta}\ , \hat{\alpha}\ \vee\ \hat{\beta}\ ] \in j$......................*(i)*

Given that, $\hat{\alpha}\ \leq\ \hat{\beta},$

$[\ \because \hat{\alpha}\ \wedge\ \hat{\beta}\ =\ \hat{\alpha}\ and\ \hat{\alpha}\ \vee \hat{\beta} =\ \hat{\beta}$

$[\ \hat{\alpha}\ \ \hat{\beta}] \subset j$

On the other hand, let us suppose

$\forall\ \hat{\alpha}, \hat{\beta}\ \in j$

$with\ \hat{\alpha}\ \leq\ \hat{\beta}, \quad [a,b] \subseteq\ J$

Currently, we demonstrate that j is convex sublattice

As j is a sublattice of u

We can write $\hat{\alpha}\ \wedge\ \hat{\beta}\ \in\ j\ and\ \hat{\alpha}\ \vee \hat{\beta}\ \in\ j$

But it is known to us that

$$\hat{\alpha}\ \wedge\ \hat{\beta}\ \leq\ \hat{\alpha}\ \vee \hat{\beta}$$

Thus, by the specified condition

$[\ \hat{\alpha}\ \wedge\ \hat{\beta}\ , \hat{\alpha}\ \vee\ \hat{\beta}\ ] \subset j$

Hence j is a convex sublattice of U.

Chapter 5

Ideal of Lattices

Ideals 5.1

Ideals are special class of lattices. They are lattices with some additional algebraic structure. Ideal lattices are Ideals.

Definition 5.1.1: [08] Suppose K be a non-empty subset of a lattice U.

Now, if K is an ***ideal*** of U, then $\forall e, f \in K$

1) $e, f \in K \; e \vee f \in K$
2) $e \in K, \quad t \in U \Rightarrow e \wedge f \in K$

Example 5.1.1(a): [08] Let (L = {1, 2, 5, 10} , |) be a lattice.

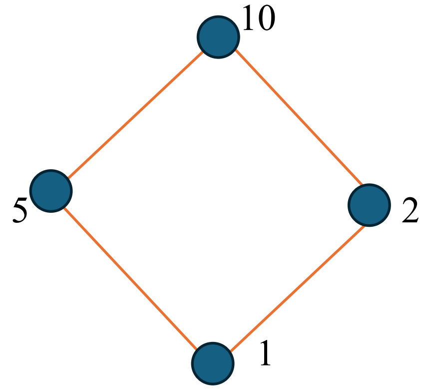


Figure- 24

Now

let $l = \{1,2\}. So\ 1,2 \in l \Rightarrow 1 \vee 2 = 2 \in l$

Again $1 \in l,\; 5 \in l \Rightarrow 1 \wedge 5 = 1 \in l$

$1 \in l,\; 10 \in l \Rightarrow 1 \wedge 10 = 1 \in l$

Hence, {1,2} is an ideal of L

Dual Ideals 5.2

Definition 5.2.1: [08] Assume a non-empty subset j of a lattice U. it is referred to as a dual ideal (or a filter) of U iff

- $\forall\, e, f \in j \Rightarrow e \wedge f \in j$
- $\forall\, e \in j, t \in U \Rightarrow e \vee t \in j$

Example 5.2.1(a):

$Let\ (l = \{1,2,5,10\}, |)$ be a lattice

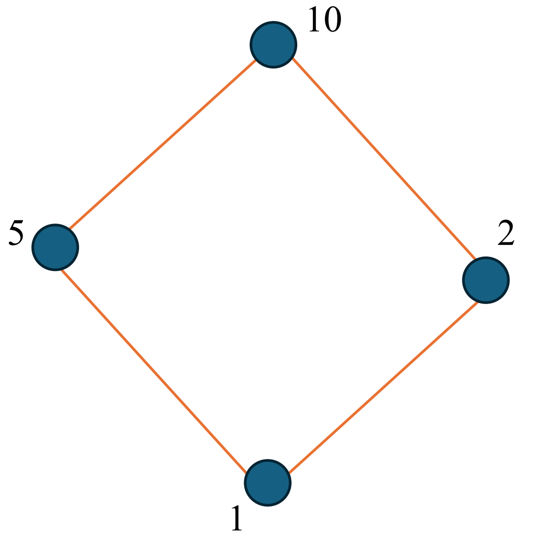


Figure: 25

Now let l = {5,10},

So $5, 10 \in l \quad \Rightarrow 5 \wedge 10 = 5 \in l$

Again $5 \in l, 1 \in l \quad \Rightarrow 1 \vee 5 = 5 \in l$

And $5 \in l, 2 \in l \quad \Rightarrow 2 \vee 5 = 10 \in l$

Also $10 \in l, 1 \in l \quad \Rightarrow 2 \vee 10 = 10 \in l$

And $10 \in l, 2 \in l \quad \Rightarrow 2 \vee 10 = 10 \in l$

Hence, {5,10} is a dual ideal of L

Prime Ideal 5.3

Definition 5.3.1: [08] if any ideal W of a lattice U is perfectly contained in U and

$$e \wedge f \in W \text{ then either } e \in W \text{ or } f \in W$$

in this case, W is said to be ***prime ideal*** of U

Example 5.3.1(a): $Let\ (l\ =\ \{1,2,5,10\}, |)$ be a lattice

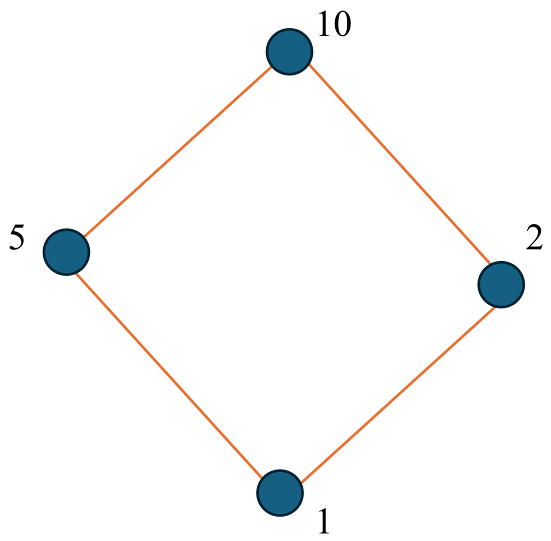


Figure: 26

Now $1\ \wedge 2 = 1$ and let A = {1} and A is properly contained in L and $1\ \in A$

Hence, A = {1,2} is a prime ideal of lattice L

Dual Prime Ideal 5.4

Definition 5.4.1: [08] if any ideal W of a lattice U is perfectly contained in U and

$e\ \vee\ f\ \in W\ then\ either\ \ e\ \in W\ or\ f \in W$ in that case, W is said to be ***dual prime ideal*** of U

Example 5.4.1(a): let {5,10} may call a dual prime ideal of the lattice U = {1,2,5,10}

Principle Ideal 5.5

Definition 5.5.1: [08] Let $\hat{\gamma}\ \in U$ be any element and let U be a lattice

Assume that $(\hat{\gamma}]\ = \{\ \hat{\alpha}\ \in U \mid \hat{\alpha}\ \leq\ \hat{\gamma}\ \}$, then $(\hat{\gamma}]$ constitutes an ideal of U. it is known as principle ideal that generates by $\hat{\gamma}$

Example 5.5.1(a): Assuming U = {0,u,v,w,1} is a lattice

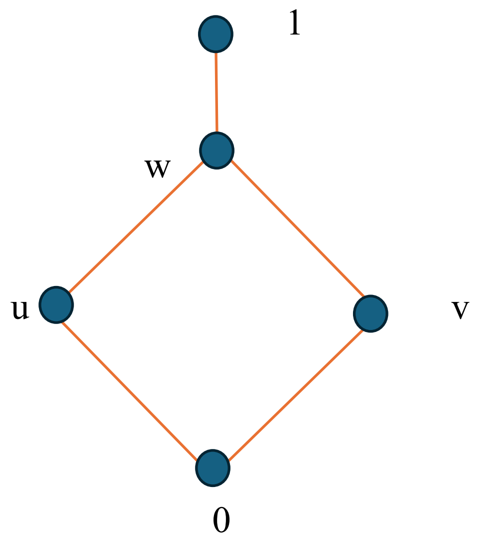


Figure: 27

Here, (u] = { $\hat{\alpha} \in U \mid \hat{\alpha} \leq u$ } = $\{0, u\}$

So $\quad 0, u \in l \qquad \Rightarrow 0 \vee a = a \in l$

Again $0 \in l, v \in L \qquad \Rightarrow 0 \wedge v = 0 \in l$

And $u \in l, v \in L \qquad \Rightarrow u \wedge v = 0 \in l$

$0 \in l, w \in L \qquad \Rightarrow 0 \wedge w = 0 \in l$

And $u \in l, w \in l \qquad \Rightarrow u \wedge w = u \in l$

So (u] is a principle ideal of L

Similarly, (0] = {0} (v] = {0, v}

(c] = {0, u, v, w}

(1] = U are all principle ideals of U

Principal Dual Ideals 5.6

Definition 5.6.1: [08] Let $\hat{\gamma} \in$ U be any element and let U be a lattice. Assuming that ($\hat{\gamma}$] = { $\hat{\alpha} \in$ U | $\hat{\gamma} \leq$ a}, then ($\hat{\gamma}$] constitutes an ideal of U.

It is known as the ***principal dual ideal*** that generates by y.

Example 5.6.1(a): Let (U = {1, 2, 5, 10}, | } be a lattice then the principal dual ideal of U that is generated by 5 is {5,10}

Chapter 6

Boolean Algebra 6.1

Boolean algebra is a symbolic form of mathematical reasoning that depicts connections between actual objects. George Boole of England created the fundamental rudder for this method in 1847, and other mathematical later improved them and used them to study set theory. Boolean algebra is important today for information theory, set geometry and probability theory. Also, it used to be the foundation for the circuit designs utilized in electronic digital computers.

Definition: 6.1.1 Assume B is a non-empty set with two binary operations. '$\wedge$' and '$\vee$' as well as unary operation '′'. Then the algebraic structure $(\boldsymbol{B}, \wedge, \vee, ')$ If the following axioms hold, the algebra is called a Boolean algebra:

$\forall$ x, y, z $\in$ B;

[B_1]: Idempotency : $x \wedge x = x; x \vee x = x$

[B_2]: Commutativity : $x \wedge y = y \wedge x ; x \vee y = y \vee x$

[B_3]: Associativity : $x \wedge (y \wedge z) = (x \wedge y) \wedge z;$

$$x \vee (y \vee z) = (x \vee y) \vee z$$

[B_4]: Absorption : $x \wedge (x \vee y) = x$

$$x \vee (x \wedge y) = x$$

[B_5]: Distributivity : $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$

$$x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$$

[B_6]: Complement : $\forall$ x $\in$ B, $\exists$ x′, $\in$ B

$$s.t: x \wedge x' = 0; x \vee x' = 1$$

Where 0 and 1 are the smallest and largest elements of B, respectively.

Basic Operations in Boolean Algebra 6.2

Boolean Algebra is a specialized branch of algebra that focuses on binary values, typically represented as true (1) and false (0). It serves as the cornerstone for designing digital circuits and understanding computer logic, providing a mathematical system to express and manipulate logical operations.

Boolean algebra involves several fundamental operations, each corresponding to basic logical functions. The core operations are:

1. **Negation (NOT Operation):**
   This operation is used to invert the value of a Boolean variable. If the value of a Boolean variable is true (1), applying the NOT operation results in false (0), and vice versa. The symbol for negation is often represented by a bar or the prime symbol (').

Example:

1. If $A=1$, then $\neg A=0$
2. $B=0$, then $\neg B=1$

2. **Conjunction (AND Operation):**
   The AND operation yields true (1) only if both operands are true (1). If any of the operands is false (0), the result of the AND operation is false (0). This operation is often symbolized by a dot (.) or the ampersand (&).

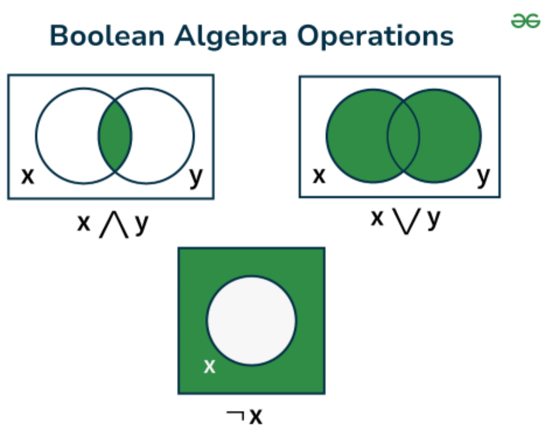
Example:

1. $A \cdot B=1$ only if both $A=1$ and $B=1$
2. $A \cdot B=0$ if $A=0$ or $B=0$

3. **Disjunction (OR Operation):**
   The OR operation results in true (1) if at least one of the operands is true (1). If both operands are false (0), then the OR operation will output false (0). It is represented by a plus symbol (+)

Example: 6.2.1(a)

- $A+B=1$ if $A=1$ or $B=1$ or both
- $A+B=0$ only if $A=0$ and $B=0$



Example: 6.2.1(b) $Let\ B = \{1,2,3,5,6,10,15,30\}$ show that under divisibility '1', $(B, 1)$is a Boolean algebra.

Solution:

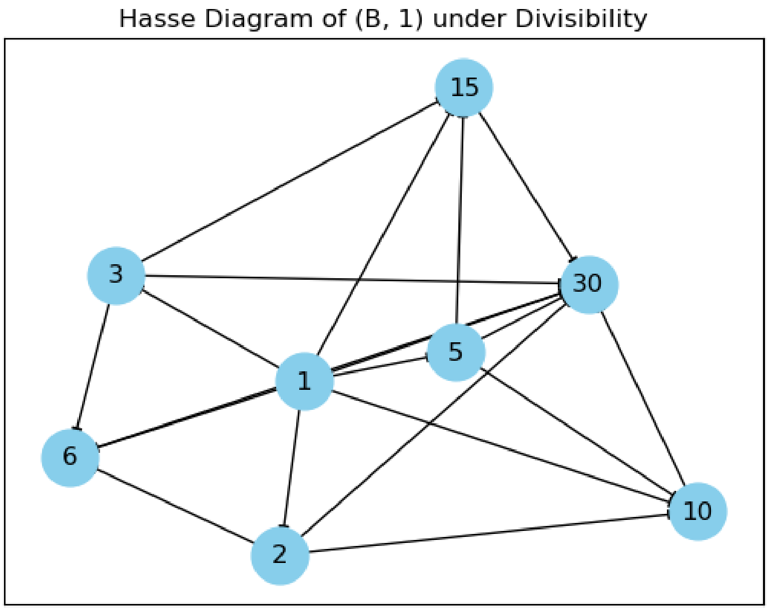
∧	1	2	3	5	6	10	15	30
1	1	1	1	1	1	1	1	1
2	1	2	1	1	2	2	1	2
3	1	1	3	1	3	1	3	3
5	1	1	1	5	1	5	5	5
6	1	2	3	1	6	2	3	6
10	1	2	1	5	2	10	5	10
15	1	1	3	5	3	5	15	15
30	1	2	3	5	6	10	15	30

Table-1

∨	1	2	3	5	6	10	15	30
1	1	2	3	5	6	10	15	30
2	2	2	6	10	6	10	30	30
3	3	6	3	15	6	30	15	30
5	5	10	15	5	30	10	15	30
6	6	6	30	6	30	30	30	30
10	10	10	30	10	30	10	30	30
15	15	30	15	15	30	30	15	30
30	30	30	30	30	30	30	30	30

/	
1	30
2	15
3	10
5	6
6	5
10	3
15	2
30	1

Table-3



Hasse Diagram of (B, 1) under Divisibility

Solution:

[B$_1$]: $\forall\, x \in B$;

$x \wedge x = x, x \vee x = x$;

It is shown by the diagonal of tab-1 and tab-2.

Hence B_1 holds.

[B$_2$]: $\forall\, x, y \in B$;

$x \wedge y = y \wedge x, x \vee y = y \vee x$;

Conjugate elements of tab-1 and tab-2 are same.

Hence B_2 holds in B.

[B$_3$]: $\forall\, x, y, z \in B$;

$x \wedge (y \wedge z) = (x \wedge y) \wedge z \&\ \ x \vee (y \vee z) = (x \vee y) \vee z.$

e.g., for $1,2,3 \in B$;

$1 \wedge (2 \wedge 3) = 1 \wedge 1 = 1$

$(1 \wedge 2) \wedge 3 = 1 \wedge 3 = 1$

Similarly

$$1 \vee (2 \vee 3) = (1 \vee 2) \vee 3$$

$\therefore$ B_3 holds in B.

[B_4]: $\forall\, x, y, \in B$;

$$x \wedge (x \vee y) = x \& \quad x \vee (x \wedge y) = x.$$

e.g., for $1,2 \in B$;

$$1 \wedge (1 \vee 2) = 1 \wedge 2 = 1$$

$$1 \vee (1 \wedge 2) = 1 \vee 1 = 1$$

$\therefore$ B_4 holds in B.

[B_5]: $\forall\, x, y, z \in B$;

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z).$$

e.g., for $1,2,3 \in B$;

$$1 \wedge (2 \vee 3) = 1 \wedge 6 = 1$$

$$(1 \wedge 2) \vee (1 \wedge 3) = 1 \vee 1 = 1$$

Hence B_5 holds in B.

[B_6]: From tab-3

$1' = 30$ *because* $1 \wedge 30 = 1 \,\&\, 1 \vee 30 = 30$;

$2' = 15$ *because* $2 \wedge 15 = 1 \,\&\, 2 \vee 15 = 30$;

$3' = 10$ *because* $3 \wedge 10 = 1 \,\&\, 3 \vee 10 = 30$;

$5' = 6$ *because* $5 \wedge 6 = 1 \,\&\, 5 \vee 6 = 30$;

$6' = 5$ *because* $6 \wedge 5 = 1 \,\&\, 6 \vee 5 = 30$;

$10' = 3$ *because* $10 \wedge 3 = 1 \,\&\, 10 \vee 3 = 30$;

$15' = 2$ *because* $15 \wedge 2 = 1 \,\&\, 15 \vee 2 = 30$;

$30' = 1$ *because* $30 \wedge 1 = 1 \,\&\, 30 \vee 1 = 30$;

$\therefore$ B_6 holds in B.

Hence $(B, \wedge, \vee, ')$ is a Boolean algebra

Boolean Lattice 6.3

Definition 6.3.1: Suppose L is a lattice. It is said to be a ***Boolean lattice*** if it is distributive and uniquely complemented.Because complements in a Boolean lattice are unique, we can think of it as an algebra with two binary operations " and " and one unary operation ". Boolean algebras are made up of Boolean lattices.

Boolean sub-algebra 6.4

Definition 6.4.1: Suppose $(B, \wedge, \vee, ')$ is a type of Boolean algebra and S is a non-empty subset of B. then S is called ***Boolean sub-algebra*** iff

$\forall$ a, b $\in$ S;

i. $a \wedge b \in S$
ii. $a \vee b \in S$
iii. $a' \in S$.

Example 6.4.1(a): Show that S is a non-empty subset of a Boolean algebra. subalgebra if and complementation are closed.

Solution: We must demonstrate that for any a, b $\in$ S, $a \wedge b \in S$

$$\text{Now} \quad (a \wedge b)' = a' \vee b' \in S$$

$$\Rightarrow (a \wedge b) = ((a \wedge b)')' \in S$$

Similarly, it is possible to demonstrate that S is a subalgebra if it is closed under and complementation.

Boolean Sub-Lattice 6.5

Definition 6.5.1: Suppose **$(B, \wedge, \vee, ')$** is a Boolean algebra and S is any non-empty subset of B. Then S is called ***Boolean sublattice*** if and only if

For all a, b $\in$ S;

i. $a \wedge b \in S$
ii. $a \vee b \in S$.

Complete Boolean Algebra 6.6

Definition 6.6.1: A Complete Boolean Algebra is a Boolean Algebra in which every expression is true.Subset has a supremum (least upper bound). It follows that every subset has an infimum (greatest upper bound).

Example 6.6.1(a):

i. Every finite Boolean algebra is complete
ii. A complete algebra is a subset of a given set
iii. A Boolean algebra is a complete if and only it's stone space of prime ideals in extremely disconnected.

Some Important Theorems on Boolean Algebra 6.7

Theorem 6.7.1: Prove that every Boolean ring with unity is Boolean algebra.

Proof: Suppose (B, +, ·) is a Boolean ring with unity.

Here '∧' & '∨' are defined though

$$a \vee b = a + b + a \cdot b$$

$$a \wedge b = a \cdot b \qquad ; \forall\, a, b \in B.$$

Let us verify the axioms of Boolean algebra:

[$B_{1\wedge}$]: $a \wedge a = a \cdot a$

$= a^2$

$= a$ [∵ B is Boolean ring]

[$B_{1\vee}$]: $a \vee a = a + a + a \cdot a$

$= (a + a) + a^2$

$= 0 + a$ [∵ a is additive inverse of a & $a^2 = a$]

$= a$ 30

∴ [B_1] holds in B.

[$B_{2\wedge}$]: $a \wedge a = a \cdot b$

$= b \cdot a$ [$\because$ multiplication is commutative]

$= b \wedge a$

[$B_{2\vee}$]: $a \vee b = a + b + a \cdot b$

$= (b + a) + b \cdot a$ [$\because$ '+' is commutative in B]

$= b + a + b \cdot a$

$= b \vee a$

$\therefore$ [B_2] holds in B.

[$B_{3\wedge}$]: $a \wedge (b \wedge c) = a \wedge (b \cdot c)$

$= a \cdot (b \cdot c)$

$= (a \cdot b) \cdot c$

$= (a \wedge b) \wedge c)$

[$B_{3\vee}$]: $a \vee (b \vee c) = a \vee (b + c + b \cdot c)$

$= a + b + c + b \cdot c + a \cdot (b + c + b \cdot c)$

$= a + b + c + b \cdot c + a \cdot b + a \cdot c + a \cdot b \cdot c$

$= a + b + c + a \cdot b + b \cdot c + a \cdot c + a \cdot b \cdot c$

And $(a \vee b) \vee c \quad = (a + b + a \cdot b) \vee c$

$= a + b + a \cdot b + c + (a + b + a \cdot b) \cdot c$

$= a + b + c + a \cdot b + a \cdot c + b \cdot c + a \cdot b \cdot c$

$\therefore$ [B_3] holds in B.

[$B_{4\wedge}$]: $a \wedge (a \vee b) = a \cdot (a \vee b)$

$= a \cdot (a + b + a \cdot b)$ 31

$= a \cdot a + a \cdot b + a \cdot a \cdot b$

$= a^2 + a \cdot b + a^2 \cdot b$

$= a + a \cdot b + a \cdot b$

$= a + 0$

$= a$

[$B_{4\vee}$]: $a \vee (a \wedge b) = a \vee (a \cdot b)$

$= a + a \cdot b + a \cdot (a \cdot b)$

$= a + a \cdot b + a \cdot a \cdot b$

$= a + a \cdot b + a^2 \cdot b$

$= a + a \cdot b + a \cdot b$

$= a + 0$

$= a$

$\therefore$ [B_4] holds in B.

[B_5]: $a \wedge (b \vee c) = a \cdot (b + c + b \cdot c)$

$= a \cdot b + a \cdot c + a \cdot b \cdot c$

Again

$(a \wedge b) \vee (a \wedge c) = (a \cdot b) \vee (a \cdot c)$

$= a \cdot b + a \cdot c + (a \cdot b) \cdot (a \cdot c)$

$= a \cdot b + a \cdot c + a \cdot (b \cdot a) \cdot c$

$= a \cdot b + a \cdot c + a \cdot (a \cdot b) \cdot c$

$= a \cdot b + a \cdot c + (a \cdot a) \cdot b \cdot c$

$= a \cdot b + a \cdot c + a^2 \cdot b \cdot c$

$= a \cdot b + a \cdot c + a \cdot b \cdot c$

$\therefore a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

$\therefore$ [B_5] holds in B.

[B_6]: Let $a \in B$ be any element,

We must demonstrate that 'a' has a complement, namely $a + 1$.
(where1is unity of ring B)

Now

$a \wedge (a + 1) = a \cdot (a + 1)$

$= a^2 + a \cdot 1$

$= a + a = 0$

$a \vee (a + 1) = a + a + 1 + a \cdot (a + 1)$

$= a + a + 1 + a \cdot a + a \cdot 1$

$= 0 + 1 + a^2 + a$

$= 1 + a + a$

$= 1 + 0$

$= 1$

Thus, we get

$a' = a+1$

Since in the ring B,

$0 \cdot a = 0 \quad ; \forall\, a \in B$

$\Rightarrow 0 \wedge a = 0 \quad ; \forall\, a \in B$

Again

$1 \cdot a = a \quad ; \forall\, a \in B$

$\Rightarrow 1 \wedge a = a \quad ; \forall\, a \in B$

Thus ‘0’ & ‘1’ B's smallest and largest elements, respectively.

∴ $[B_5]$ holds in B.

Therefore B is a Boolean algebra.

Theorem 6.7.2: Every Boolean algebra is a Boolean ring with unity

Solution : A Boolean ring is a ring that contains $x^2 = x\ ,\ \forall\, x.$

Let (A, ∧, ∨, ′) be a Boolean algebra

Define two operation + and · on A by

$a \,.\, b = a \wedge b$

$a + b = (a \wedge b') \vee (a' \wedge b) \qquad a,b \in A$

Then + and · are clearly binary operations.

To demonstrate that <A, +, >forms a Boolean ring, we check all of the conditions in the definition.

Allow, a,b,c ∈ A be each members.

$a + b = (a \wedge b') \vee (a' \wedge b) == (b \wedge a') \vee (b' \wedge a) = b+a.$

$(a+b)+c = [\ (a+b) \wedge c'] \vee [(a+b)' \wedge c]$

$= [\{\ (a \wedge b') \vee (a' \wedge b)\} \wedge c'] \vee [\{\ (a \wedge b') \vee (a' \wedge b)\}' \wedge c]$

$= [\ (a \wedge b' \wedge c') \vee (a' \wedge b \wedge c')] \vee [(a \wedge b')' \wedge\ (a' \wedge b)' \wedge c]$

$= [\ (a \wedge b' \wedge c') \vee (a' \wedge b \wedge c')] \vee [(a' \vee b) \wedge\ (a \vee b') \wedge c]$

$= [\ (a \wedge b' \wedge c') \vee (a' \wedge b \wedge c')] \vee [\{(a' \vee b) \wedge a\ \} \vee \{(a' \vee b) \wedge b'\} \wedge c]$

$= [\ (a \wedge b' \wedge c') \vee (a' \wedge b \wedge c')] \vee [\{(a' \wedge\ a) \vee (b \wedge a) \vee (a' \wedge\ b') \vee (b \wedge b'\} \wedge c]$

$= (a \wedge b' \wedge c') \vee (a' \wedge b \wedge c') \vee [\{(b \wedge\ a) \vee (a' \wedge b')\ \} \wedge c]$

$= (a \wedge b' \wedge c') \vee (a' \wedge b \wedge c') \vee [(b \wedge\ a \wedge c) \vee (a' \wedge\ b' \wedge\ c)]$

$= \ (a \wedge b' \wedge c') \vee (a' \wedge b \wedge c'\ ) \vee (a \wedge\ b \wedge c) \vee (a' \wedge\ b' \wedge c)$

Because the result is symmetric in a,b,c, it will also be equal to (b+c)+a=a+. (b+c) (due to the commutativity of +).

As a result, + is associative.

Again $\quad a+0=(a \wedge u) \vee (a' \wedge 0)=0+a$

Also $\quad a+a=(a \wedge a') \vee (a' \wedge a)=0$

As a result, (A, +) forms an abelian group.

Because $a \cdot b= a \wedge b$ and $\wedge$ is commutative and associative.

We discover that is also commutative and associative.

Again $a(b+c)=a \wedge (b+c)= a \wedge [(b \wedge c') \vee (b' \wedge c)]$

$$= (a \wedge b \wedge c') \vee (a \wedge b' \wedge c)]$$

$$\begin{aligned} ab + ac &= (a \wedge b) +(a \wedge c) \\ &= [(a \wedge b) \wedge (a \wedge c)'] \vee [(a \wedge b)' \wedge (a \wedge c)] \\ &= [(a \wedge b) \wedge (a' \wedge c')] \vee [(a' \wedge b') \wedge (a \wedge c)] \\ &= [(a \wedge b \wedge a') \vee (a \wedge b \wedge c') \vee (a \wedge c \wedge a') \vee (a \wedge c \wedge b')] \\ &= (a \wedge b \wedge c') \vee (a \wedge b' \wedge c) \end{aligned}$$

Hence , $a(b+c) = ab +ac$

Similarly, $(b+c)a =ba + ca$.

Finally,since $a \cdot u = a \wedge u =a =u \wedge a = u \cdot a$

We discover that (A,+,) forms a commutative ring with unity u.

Also as $\quad a.a = a \wedge a =a$, $\forall$ a.

We gather that A forms a Boolean ring. 34

Theorem 6.7.3 A finite Boolean algebra is isomorphic to Boolean algebra of all subsets of some finite set.

Proof: We use induction on O(L) (= number of elements in L)

If O(L) = 1 then it is isomorphic with a set with one = 2^0 elements.

If O(L) = 2, L is isomorphic with a set with 2 = 2^1 elements.

Suppose the result holds for Boolean algebras with order less than O(L).
Let O(L) > 2 then $\exists$ a$\in$ L s.t., $o < a < u$

We know $L \cong [o, a]\ [a, u] = A\times B$.

Since L is a Boolean algebra, $A\times B$ is a Boolean algebra
$\Rightarrow$ A and B are Boolean algebras and as

$O(A) < O(L)$

$O(B) < O(L)$

by assumption A is isomorphic with a set with 2^r elements and B is isomorphic with a set with 2^s elements for some r, s.

Thus $A \times B$ is isomorphic with a set with $2^r \cdot 2^s = 2^{r+s}$ elements. Take $r + s = n$ and the result is proved.

Theorem 6.7.4: Prove that In a Boolean algebra

(i) $(x \wedge y)' = x' \vee y'$

(ii) $(x \vee y)' = x' \wedge y'$

Proof(i) : Suppose B is a Boolean algebra, So B is a complemented .

Then $\forall\, x,y \in B$, Let $x', y' \in B$ s.t

$$\left.\begin{aligned} x \wedge x' &= 0 = y \wedge y' \\ x \vee x' &= u = y \vee y' \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

$$\begin{aligned} \text{Now, } (x \wedge y) \wedge (x' \vee y') &= ((x \wedge y) \wedge x') \vee ((x \wedge y) \wedge y') \\ &= ((x \wedge x') \wedge y) \vee (x \wedge (y \wedge y')) \\ &= (0 \wedge y) \vee (x \wedge 0) \\ &= 0 \vee 0 \\ &= 0 \end{aligned}$$

$$(x \wedge y) \vee (x' \vee y') = (x' \vee y') \vee (x \wedge y)$$

$$= ((x' \vee y') \vee x) \wedge ((x' \vee y') \vee y)$$

$$= (y' \vee (x' \vee x)) \wedge (x' \vee (y' \vee y))$$

$$= (y' \vee u) \wedge (x' \vee u) = u \wedge u = u$$

Hence $(x \wedge y)' = x' \vee y'$

Proof(ii): $(x \vee y) \wedge (x' \vee y') = (x' \wedge y') \wedge (x \vee y)$

$$= (x' \wedge y') \wedge x) \vee (x' \wedge y') \wedge y)$$

$$= (y' \wedge (x' \wedge x)) \vee (x' \wedge (y' \wedge y))$$

$$= (y' \wedge 0) \vee (x' \wedge 0)$$

$$= 0 \vee 0$$

$$= 0$$

$$(x \vee y) \vee (x' \wedge y') = ((x \vee y) \vee x') \wedge ((x \vee y) \vee y')$$

$$= (y \vee (x \vee x')) \wedge (x \vee (y \vee y'))$$

$$= (y \vee u) \wedge (x \vee u)$$

$$= u \wedge u$$

$$= u$$

Hence $(x \vee y)' = x' \wedge y'$

Chapter 7

Boolean Function 7.1

In this chapter we discuss the basic knowledge of Boolean function, CN form, DN form and there completed and some important problem solved.

Definition 7.1.1: Suppose (B, ∧, ∨, ′) is A Boolean algebra is a type of algebra. Boolean expressions or Boolean polynomials are any expressions that include members of B and the operations, as well as complementation. For example, x ∨ y, x, , ,x, ∧, 0 are all Boolean expressions. A Boolean function is any function that specifies these Boolean expressions.

A Boolean function's disjunctive normal form (DN form) 7.2

Definition 7.2.1: A Boolean function (expression) is said to be in disjunctive normal form if it can be written as a join of terms of the type (DN from).

$f_1(x_1) f_2(x_2) \wedge \ldots \wedge f_n(x_n)$ where $f_i(x_i) = x_i$ or x'_I for all I = 1, 2…. n and on two terms are identical. In addition, the numbers 1 and 0 are said to be in disjunctive normal form.

Some properties of D.N form:

1) Complete D.N form

If a disjunction normal form is n variable which contains 2^n minterms, it is referred to as complete D.N form in n variables.

e.g.,

i. $(x \wedge y) \vee (x \wedge y') \vee (x' \wedge y) \vee (x' \wedge y')$
   is the full form with two variables. a
ii. $(x \wedge y \wedge z) \vee (x \wedge y \wedge z') \vee (x \wedge y' \wedge z) \vee (x' \wedge y \wedge z) \vee (x \wedge y' \wedge z) \vee (x' \wedge y \wedge z') \vee (x' \wedge y' \wedge z) \vee (x' \wedge y' \wedge z')$

is the complete D. N form in three variables

2) **Complement of the D.N form** Suppose f (x_1, x_2……x_n) is expressed in complete D.N form. If g (x_1, x_2……x_n) any part of the complete D.N form of f. then complement of g i.e., g' is the left-out part of the complete D.N the form of f.

Example 7.2.1(a): Put the function

f= [(x' ∧ y) v(x ∧y∧ z') v(x ∧y' ∧z) v (x'∧ y'∧ z'∧ t')v t']' in the DN form

Solution: We have

f = (x'∧ y)'∧ (x ∧ y∧ z')' ∧ (x ∧y' ∧ z)'∧ (x' ∧ y'∧ z'∧ t')'∧t

= [(x v y') ∧ (x' v y' v z)] ∧ (x' v y v z') ∧ [(x v y v z v t') ∧ t]

= [(x∧ x') v (x∧ y') v (x ∧z) v (y'∧ x') v (y'∧y') v (y' ∧z)] ∧ (x' v y v z') ∧ [(x∧t) v (y∧t) v (z∧t) v (t ∧t')]

= [(x∧y') v(x∧ z) v(y' ∧x') v y' v (y'∧ z)] ∧ [(x' ∧ y∧ t) v(x' ∧ z∧ t)v(y ∧x∧ t) v(y∧ t) v(y∧ z∧ t) v (z'∧ x∧ t) v(z'∧ y∧t)]

= (x ∧ y'∧ z' ∧ t) v(x∧ z ∧y∧ t) v(y' ∧ x'∧ z ∧t)v (y' ∧ z ∧ t ∧ x') v(y' ∧ z'∧ x∧ t) v(y'∧ z ∧ x' ∧ t)

= (x ∧ y' ∧ z' ∧ t) v(x ∧y∧ z∧ t) v(x' ∧y' ∧ z ∧t)

A Boolean Function's Conjunctive Normal Form (CN Form) 7.3

This section discusses the conjunctive normal form (CN form), which is the inverse of the DN form. All the results that we proved for DN forms can be extended to CN form by duality. To give a formal shape we define

Definition 7.3.1: Conjunctive normal form (CN) refers to a Boolean function f. form) in n variables x1, x2,...., xn if it meets all of the terms of the type. f_1 (x_1) ∨ f_2 (x_2) ∨ f_n(x_n) where f_i(x_i) = x_i or $x_{i'}$ for all I = 1, 2,..., n, and no two terms are the same., after 0 and One is said to be in CN form.

In that case, in terms of the type

f_1(x_1) ∨ f_2(x_2) ∨ …… ∨ f_n(x_n) are called maxterms or maximal polynomials.

We leave it to the reader to state the corresponding results for CN forms, the proofs would of course, follow by duality.

Some properties of CN form 7.4

1. **Complete CN form:** Suppose *f* is a Boolean function in CN form with n variables. Then f is defined as incomplete CN form in n variables if CN form contains all the 2^n maxterms. E.g
   i. *f* (x, y) = (x ∨ y) ∧ (x′ ∨ y) ∧ (x ∨ y′) ∧ (x′ ∨ y′)

   is the full CN form with two variables

   ii. *f* (x, y, z) = (x ∨ y ∨ z) ∧ (x′ ∨ y ∨ z) ∧ (x ∨ y′ ∨ z) ∧ (x ∨ y′ ∨ z) ∧ (x ∨ y ∨ z′) ∧ (x′ ∨ y′ ∨ z) ∧ (x′ ∨ y ∨ z′) ∧ (x′ ∨ y′ ∨ z′) is the complete CN form in three variables.

2. **Complement CN form:** Suppose *f* (x_1, x_2,……, x_n) is expressed in complete CN form. If *g* (x_1, x_2, ………, x_n) is any part of the complete CN form *f*, 39
   then complement of *g* denoted by *g'* is the leftout part of the complete CN form.
   e.g.

i. *f* (x, y) = (x ∨ y) ∧ (x′ ∨ y) ∧ (x ∨ y′) ∧ (x′ ∨ y′)

   is the complete CN form of *f*.

   Suppose *g* = (x ∨ y) ∧ (x′ ∨ y)

   Then *g*′ = (x ∨ y′) ∧ (x′ ∨ y′)

Example 7.4.1(a): **:** Insert the function here.

f = {(x ∧ y′) ′ ∨ z′] ∧ (x′ ∨ z) ′ in the CN form

Solved: We have

f = [(x′ ∨ y) ∨ z′] ∧ (x ∧ z′)

= (x′ ∨ y ∨ z′) ∧ [(x ∧ z′) ∨ (y ∧ y′)]

= (x′ ∨ y ∨ z′) ∧ {[(x ∧ z′) ∨ y] ∧ [(x ∧ z′) ∨ y′]}

= (x′ ∨ y ∨ z′) ∧ [(x ∨ y) ∧ [(z′ ∨ y) ∧ (x ∨ y′) ∧ (z′ ∨ y′)]

$= (x' \vee y \vee z') \wedge [\{x \vee y \vee (z \wedge z')\} \wedge \{(z' \vee y) \vee (x \wedge x')\} \wedge \{(x \vee y') \vee (z \wedge z')\} \wedge \{(z' \vee y') \vee (x \wedge x')\}]$

$= (x' \vee y \vee z') \wedge (x \vee y \vee z) \wedge (x \vee y \vee z') \wedge (z' \vee y \vee x) \wedge (z' \vee y \vee x') \wedge (x \vee y' \vee z) \wedge (x \vee y' \vee z') \wedge (z' \vee y' \vee x) \wedge (z' \vee y' \vee x')$

$= (x \vee y \vee z) \wedge (x' \vee y \vee z') \wedge (x \vee y \vee z') \wedge (x \vee y' \vee z) \wedge (x \vee y' \vee z') \wedge (x' \vee y' \vee z')$

Example 7.4.1(b): If $P= (x \wedge y) \vee (x' \wedge y)$. Then find P'.

Solution: Here $P= (x \wedge y) \vee (x' \wedge y)$

$P'= ((x \wedge y) \vee (x' \wedge y))'$

$= (x \wedge y)' \wedge (x' \wedge y)'$ (De Morgan's law)

$= (x' \vee y') \wedge (x \vee y')$

$= ((x' \vee y') \wedge x) \vee (x' \vee y') \wedge y')$

$= ((x' \wedge x)' \quad (y \wedge x)) \vee ((x' \wedge y') \vee \ (y \wedge y'))$

$= (0 \vee (x \wedge y)) \vee ((x' \wedge y') \vee 0)$

$= (x \wedge y) \vee (x' \wedge y')$

Example 7.4.1(c): show that, $f(x ,y)=[x \wedge f(1,y)] \vee [x' \wedge f (0, y)]$.

Solution: We know that any function f (in 2 variables) in complete DN

form is

$f (x, y) = (x \wedge y) \vee (x \wedge y') \vee (x' \wedge y) \vee (x' \wedge y')$

$=[x\wedge (y \vee y')] \vee[x' \wedge (y \vee y')]$

Put $x = 1$, $x' = 0$ and we get
$f(1, y) = [1\wedge (y \vee y')] \vee[0\wedge (y \vee y')]=y \vee y'$
Again, by putting $x = 0$ $x' = 1$, we get $f(0, y) =y \vee y'$
Thus (1) gives
$(x ,y)=[x \wedge f(1,y)] \vee [x' \wedge f (0, y)]$

Chapter 8

Applications of Boolean Algebra and Lattice Theory

Application in Computer Science 8.1

Introduction: 8.1.1 Boolean algebra is a branch of mathematics that deals with binary variables and logical operations. Developed by George Boole, it forms the foundation of modern digital systems and computing. Boolean logic is applied in programming, database searches, and artificial intelligence for efficient processing. Boolean expressions help simplify complex logical conditions, making computations more efficient. It plays an important role in fields like cryptography, automation, control systems, and even everyday digital devices like smartphones and computers.

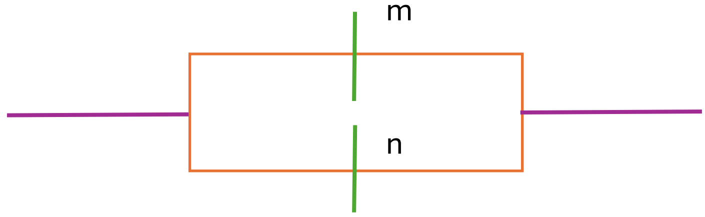
Switching circuits: 8.1.2

Definition: 8.1.3 Switching circuit is a field of application of a Boolean Algebra. Every switch has the ON (closed) and OFF (open) two states. For a switch "a”, “a" indicates that the switch is off
(or open), while "a" indicates that the switch is closed (i.e. On).
The values will be given as follows: a=0 and a'=1. There are two basics in which
switches are generally operated. These are referred to as (1) “In series” and (2) “In parallel”.

1. In series: Two switches, r and s, are used to be connected "in series," meaning that current can only flow when both are closed. If one or
Both are open, no current can flow.

It is denoted by r ∧ s.

2. In Parallel: When current flows when one or both switches are closed but not
When both are open, two switches are said to be connected "in parallel." We depict
it in the diagram below.



It is denoted by m ∨ n

Logic Gates and Digital Circuits 8.2

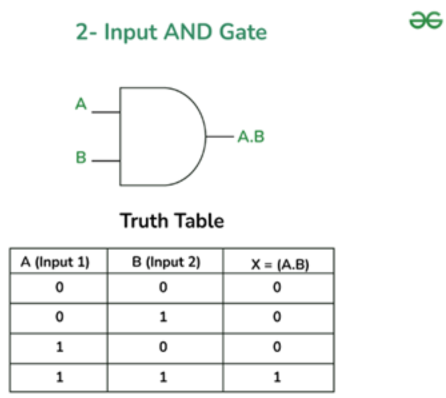
Boolean Algebra is a mathematical structure that deals with binary variables and logical operations. It includes the operations AND, OR, NOT, NAND, NOR XOR, and XNOR. The values of Boolean variables are limited to 0 (false) and 1 (true).

The key Boolean operations are defined as follows:

AND (∧): The result is 1 if both inputs are 1, otherwise, it is 0. Symbol: $A \land B$

Truth Table:

A	B	A ∧ B
0	0	0
0	1	0
1	0	0
1	1	1

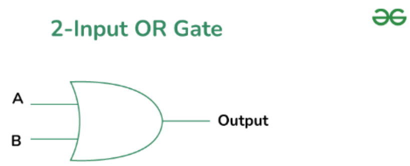


A (Input 1)	B (Input 2)	X = (A.B)
0	0	0
0	1	0
1	0	0
1	1	1

OR (∨)**:** The result is 1 if at least one input is 1, otherwise, it is 0. Symbol: $A \lor B$

Truth Table:

A	B	A ∨ B
0	0	0
0	1	1
1	0	1
1	1	1



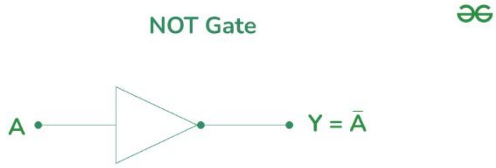
Truth Table

Input A	Input B	Output
0	0	0
0	1	1
1	0	1
1	1	1

NOT (¬)**:** The result is the inverse of the input. Symbol: $\neg A$

Truth Table:

A	¬A
0	1
1	0



Truth Table

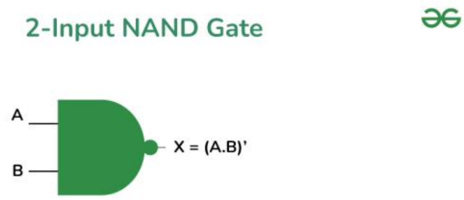
A (Input)	Y = Ā (Output)
0	1
1	0

NAND (NOT AND): The result is the negation of the AND operation.

Symbol: $A \uparrow B$

Truth Table:

A	B	A ↑ B
0	0	1
0	1	1
1	0	1
1	1	0



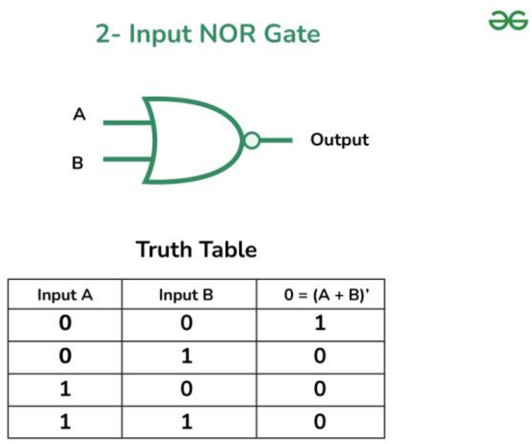
Truth Table

Input A	Input B	X = (A.B)'
0	0	1
0	1	1
1	0	1
1	1	0

NOR (NOT OR): The result is the negation of the OR operation. Symbol: $A \downarrow B$

Truth Table:

A	B	A ↓ B
0	0	1
0	1	0
1	0	0
1	1	0

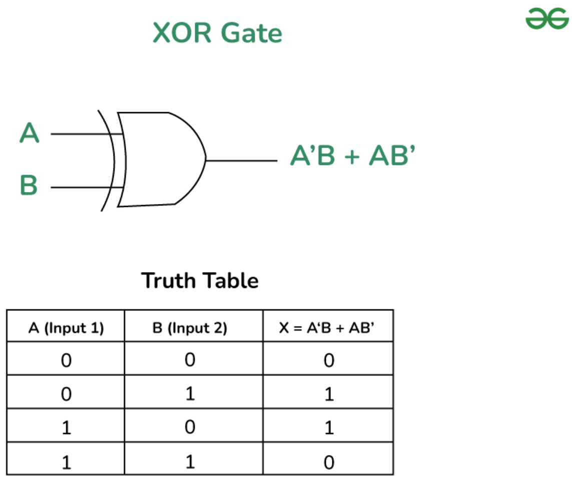


Input A	Input B	0 = (A + B)'
0	0	1
0	1	0
1	0	0
1	1	0

XOR (Exclusive OR): The result is 1 if only one input is 1, otherwise, it is 0. Symbol: $A \oplus B$

Truth Table:

A	B	A ⊕ B
0	0	0
0	1	1
1	0	1
1	1	0

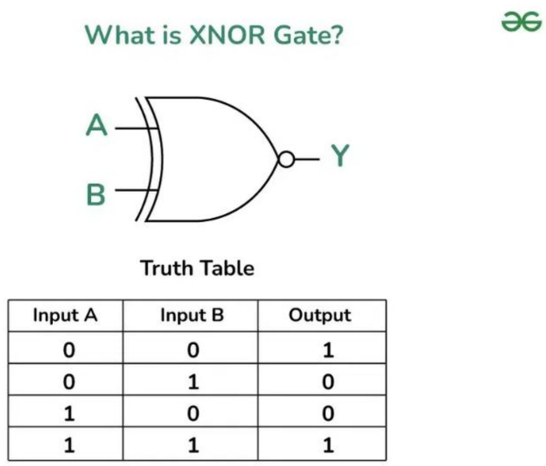


A (Input 1)	B (Input 2)	X = A'B + AB'
0	0	0
0	1	1
1	0	1
1	1	0

XNOR (Exclusive NOR): The result is the negation of XOR. Symbol: $A \equiv B$

Truth Table:

A	B	A ≡ B
0	0	1
0	1	0
1	0	0
1	1	1



Input A	Input B	Output
0	0	1
0	1	0
1	0	0
1	1	1

Digital Circuits Based on Logic Gates 8.3

A digital circuit is a network of interconnected logic gates that perform a specific task or computation. The design of digital circuits involves combining gates in different configurations to achieve desired output for specific inputs. These circuits can be combinational or sequential:

Combinational Circuits: The output depends only on the current input values. Example: adders, multiplexers, encoders.

Sequential Circuits: The output depends on both the current input and the history of previous inputs. Example: flip-flops, counters, memory units.

Common digital circuit examples based on logic gates include:

AND Gate Circuit: Performs the multiplication of inputs A and B. It outputs high (1) only when both A and B are high.

OR Gate Circuit: Outputs high (1) when at least one of the inputs is high.

NAND Gate Circuit: Outputs low (0) only when both A and B are high.

XOR Gate Circuit: Outputs high (1) only when the inputs differ (one is high, the other is low).

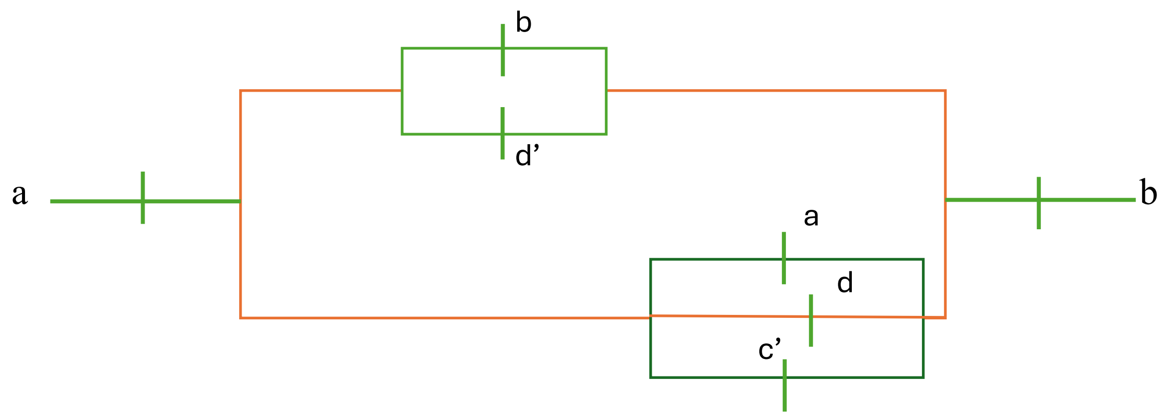
These gates are wired together to form more complex circuits. For instance, a half-adder circuit, which adds two single-bit binary numbers, uses an XOR gate for the sum and an AND gate for the carry.

Some Circuit Problem 8.3.1

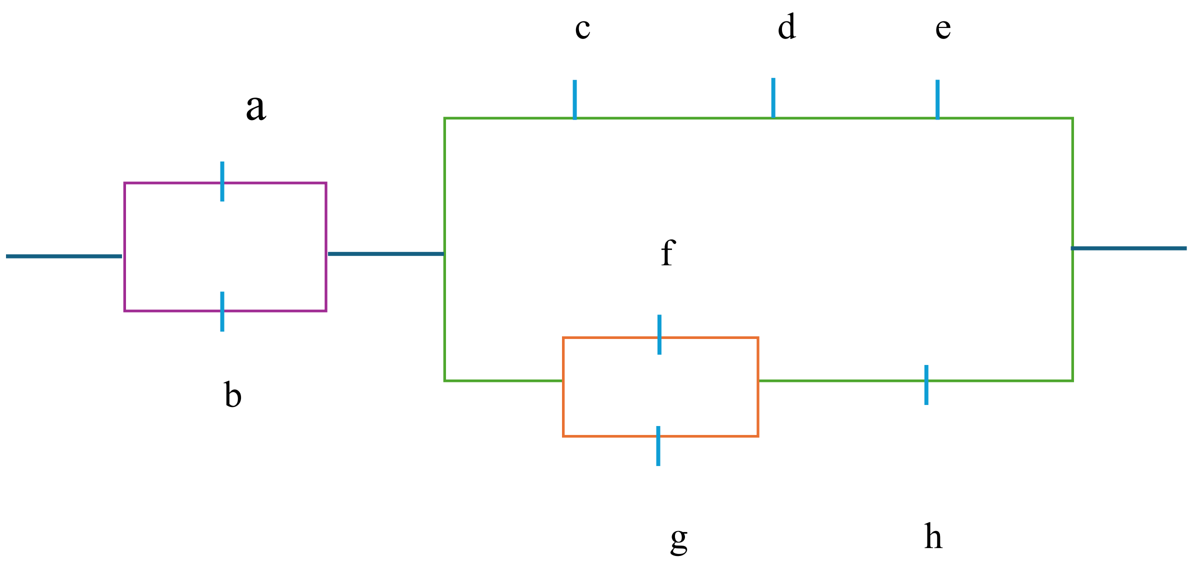
Example 8.3.1(a): Draw the circuit with realize the function

$$a \wedge [(b \vee d') \vee (c' \wedge (a \vee d \vee c'))] \wedge b$$

Solution: The circuit is given by the diagram.



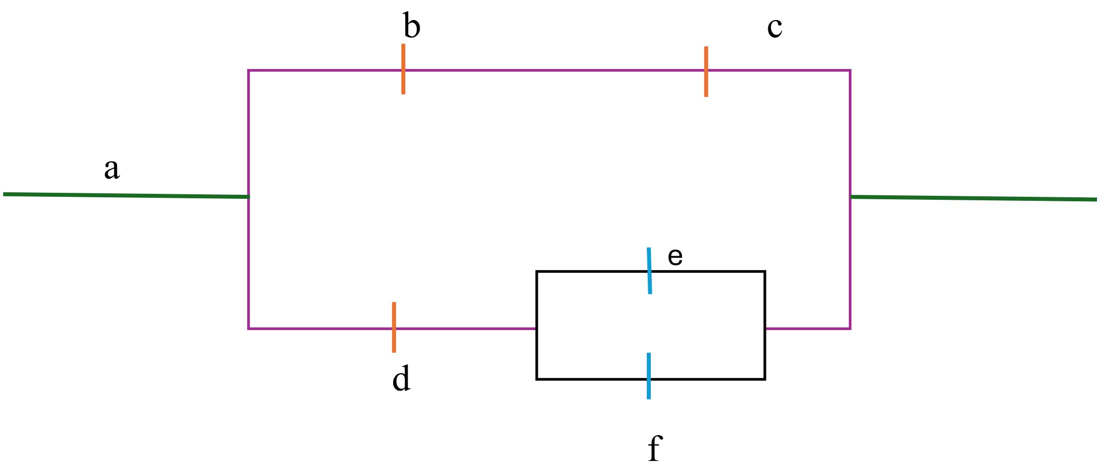
Example 8.3.1(b): Find Function f that represents the Circuit.



Solution: The Given Circuit is represented by the function

$$f = (a \vee b) \wedge [(c \wedge d \wedge e) \vee \{(f \vee g) \wedge h\}]$$

Example 8.3.1(c): Find the function that represents the circuit



Solution: The Circuit is given by the function $a \wedge [(b \wedge c] \vee \big(d \wedge (e \vee f)\big)]$

Design of n-terminal circuits 8.4

Under this section we demonstrate how different 2-terminal circuits can be combined to give a new multi-terminal circuit and where these new circuits have lesser number of switches. This is achieved by combining the given circuits in such a way that some switches in different circuits can be shared. Thus, it would involve simplification of the given circuits so that the different corresponding functions have common factors. This is normally done by simplifying the given circuits so as to get common factors or to reduce the functions in the CN forms and then look for common factors. This combining is done under the assumption that it does not lead to any other problems in the apparatus being used.

Example 8.4(a): Suppose we are given 2 circuits, specified by the functions

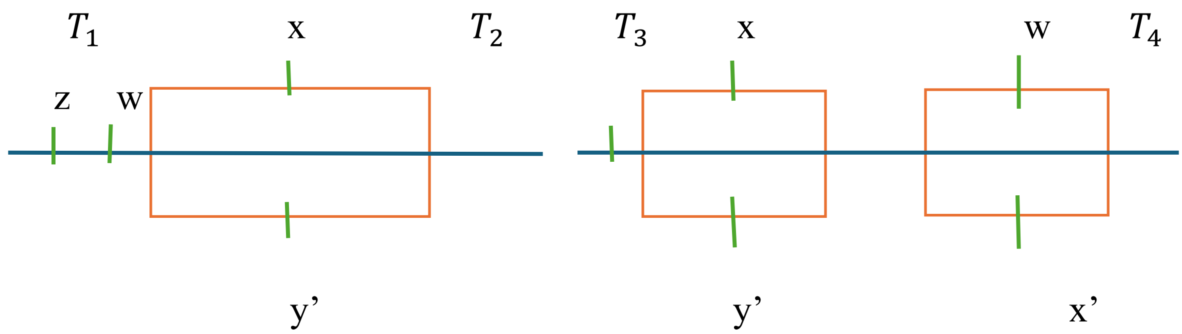
$$f = (x \wedge z \wedge w) \vee (y' \wedge \; z \; w)$$

$$g = (x \wedge z \wedge w) \vee (y' \wedge \; z \; w) \vee (x' \wedge y' \wedge z')$$

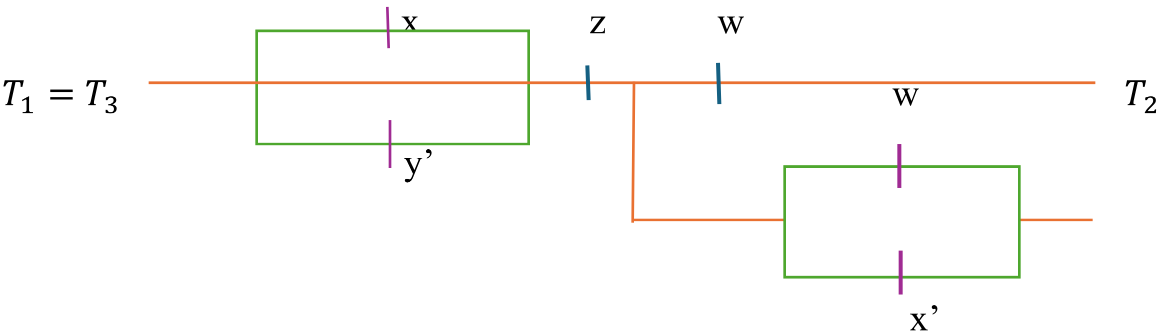
These can be written as

$$f = (z \wedge w) \vee (x \vee y')$$

$$g = (x \vee y') \wedge z \wedge (w \vee x')$$



We notice z $and\ x\ \vee\ y'$ are common in both. The combined circuit should, therefore, share these. It is given by

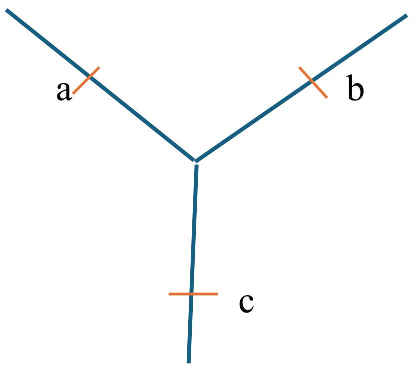


Non-series-parallel circuits 8.5

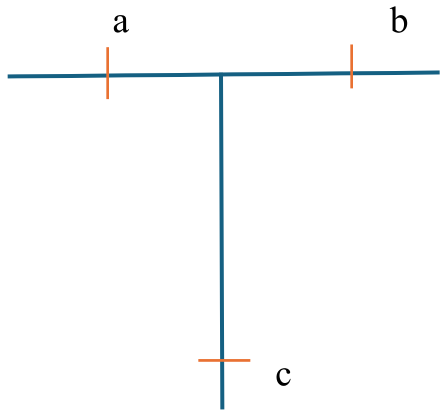
In earlier sections we've been talking about only series-parallel circuits. There are, of course, other types also. We take a brief look at these.

By a Wye-circuit, we mean a 3-terminal circuit which has three 2-terminal circuits [$T_1\ T_2, T_2\ T_3\ ,\ T_1\ T_3$] combined s.t. These three circuits have a common point

(other than a terminal).



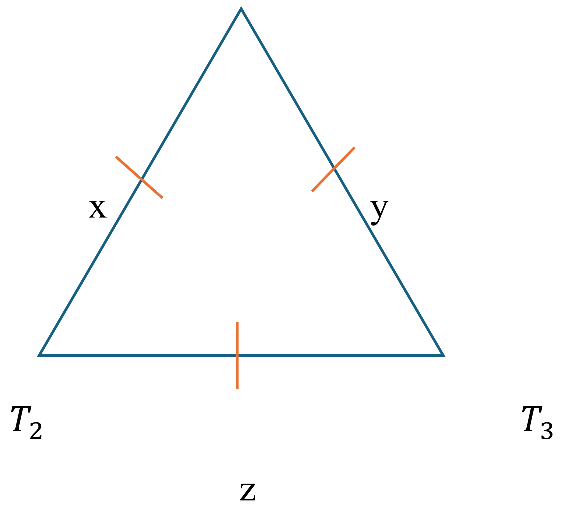
T_3



T_3

It is also called a T-circuit.

A delta circuit is also a 3-terminal circuit involving three 2-terminal circuits $[\, T_1\ T_2, T_2\ T_3\,, T_1\ T_3\,]$ Here the common points between any two 2- terminal circuits are the three terminals $[\, T_1\ T_2, T_3\,]$ each being common to exactly two of the 2-terminal circuits.



Conclusion

Since posets are the building blocks of lattice, I began this representation by discussing posets (Partially Ordered Sets), which is the subject before lattice. I have covered lattices up to this point in terms of concept, example, and several important theorems. The ideals of lattices have also been addressed, including their fundamental ideas, variants, and theorems.

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