1. Review of Linear Algebra

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Vector space and Basis

Definition 1

$u, v \in V$, $a \in \mathbb{C}$에 대해서 addition과 scalar multiplication이 정의될 수 있다면, set V를 vector space이라고 한다.

- (addition) $u + v \in V$
- (scalar multiplication) $au \in V$

Linear combination

Definition 2

A linear combination of a list $\{|v_1\rangle, ..., |v_m\rangle\}$ of vectors in V is a vector of the form

$$|v\rangle = \sum_i a_i |v_i\rangle = a_1 |v_1\rangle + \ + a_m |v_m\rangle \Rightarrow \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{pmatrix}$$

where $a_1, ..., a_m \in \mathbb{C}$. And, V is called the **span** of $\{|v_1\rangle, ..., |v_m\rangle\}$, denoted by $span(|v_1\rangle, ..., |v_m\rangle)$. In other words:

$$\text{span}(|v_1\rangle, ..., |v_m\rangle) = \{a_1 |v_1\rangle + \ + a_m |v_m\rangle : a_1, \ldots, a_m \in \mathbb{C}\}$$

Definition 3

A list $\{|v_1\rangle, ..., |v_m\rangle\}$ of vectors in V is called **linearly independent** if the *only* choice of $a_1, ..., a_m \in \mathbb{C}$ that makes

$$a_1 |v_1\rangle + \cdots + a_m |v_m\rangle = 0$$

is $a_1 = \cdots = a_m = 0$.

Definition 4

A list $\{|v_1\rangle, ..., |v_m\rangle\}$ of vectors in V is called **linearly dependent** if the *exists* a set of numbers $a_1, ..., a_m \in \mathbb{C}$ with $a_i \neq 0$ for at least one value of i, such that

$$a_1 |v_1\rangle + \cdots + a_m |v_m\rangle = 0.$$

Basis

Definition 5

A **basis** of V is a list of vectors in V that is linearly *independent* and *spans* V. The number of elements in the basis is defined to be the *dimension* of V.

Linear Operator

Definition 6

벡터를 입력으로 받아서 다른 벡터로 반환하는 함수 $A : V \to W$를 **operator**라고 한다. 입력 벡터와 출력 벡터의 벡터 공간은 동일할 수도 있고, 다를 수도 있다. 특히 Linear operator는 다음과 같이 동작한다.

$$A\left(\sum_i a_i \left|v_i\right\rangle\right) = \sum_i a_i A(\left|v_i\right\rangle)$$

만약 linear operator A가 출력 벡터를 입력 벡터와 동일한 벡터공간 V에 매핑한다면, "Linear operator on V" 라고 표현한다.

Theorem 7

Set of all linear operator $V \to W$는 다음과 같이 나타내며, 이 집합에서 addition과 scalar multiplication이 정의될 수 있기 때문에 vector space이다.

$$\mathcal{L}(V, W)$$

Matrix representation

Definition 8

$A : V \to W$이고 각 벡터공간의 basis가 $\{|v_i\rangle\}$, $\{|w_i\rangle\}$라고 하자. 그렇다면 V의 어떤 basis vector에 대해서 operator를 적용한 결과는 W 벡터공간안에 있는 벡터이다.

$$A\,|v_i\rangle \in W$$

따라서 이 벡터는 $\{|w_i\rangle\}$의 linear combination으로 나타낼 수 있다.

$$A\,|v_i\rangle = \sum_j A_{ij}\,|w_j\rangle$$

이때 coefficient A_{ij}를 operator A를 표현하는 matrix의 원소로 사용한다.

Inner product and Outer product

Inner product

Definition 9 (Inner product)

Inner product는 같은 벡터공간에 있는 두 벡터를 입력으로 받아 전달받아서 상수값을 반환한다.

$$(,) : V \times V \to \mathbb{C}$$

Any operation can be defined as *inner product* if it satisfies

- linearlity
$$\left(|v\rangle, \sum_i \lambda_i |w_i\rangle\right) = \sum_i \lambda_i (|v\rangle, |w_i\rangle)$$
- complex conjugation
$$(|v\rangle, |w\rangle) = (|w\rangle, |v\rangle)^*$$
- non-negative integer
$$(|v\rangle, |w\rangle) \geq 0$$
  with equality if and only if $|v\rangle = 0$.

Inner product

Definition 10 (Norm)

We define the *norm* of a vector $|v\rangle$ by

$$\sqrt{\langle v|v\rangle} = \| \, |v\rangle \, \|.$$

If norm of a vector $|v\rangle$ is $\| \, |v\rangle \, \| = 1$, we call it as *unit vector*.

Definition 11 (Orthogonality)

Vecotrs $|w\rangle$ and $|v\rangle$ are *orthogonal* if their inner product is zero.

$$\langle v|w\rangle = 0$$

Definition 12 (Orthonormal)

A set of vectors is *orthonormal* if each vector is a unit vector, and each vector pairs is orthogonal.

$$\langle v|w\rangle = 0, \quad \text{and} \quad \langle v|v\rangle = 1, \langle w|w\rangle = 1$$

Outer product

Definition 13

Outer product의 결과는 matrix이고, 따라서 $V \to V$를 수행하는 linear operator이다.

$$(|w\rangle \langle v|) |v'\rangle = |w\rangle \langle v|v'\rangle = \underbrace{\langle v|v'\rangle}_{\text{scalar}} \underbrace{|w\rangle}_{\text{vector } \in V}$$

Completeness

Theorem 14

특정 basis set $\{|i\rangle\}$*에 대해서 벡터를 linear combination으로 나타낼 때,*

$$|v\rangle = \sum_i v_i |i\rangle$$

각 basis에 대응되는 coefficient의 값은 벡터와 basis를 내적한 결과이다.

$$v_i = \langle i|v\rangle$$

Eigenvalue and Eigenvector

Eigenvalue and Eigenvector

Definition 15

Operator A에 대해서 다음 등식을 만족시키는 벡터 $|v\rangle$를 eigenvector, 실수 v를 eigenvalue라고 한다.

$$A|v\rangle = v|v\rangle$$

eigenvalue, vector를 구하는 방법은 $C(\lambda) = 0$을 만족시키는 λ가 eigenvalue가 된다. where $C(\lambda)$ is:

$$C(\lambda) = \text{det}(A - \lambda I)$$

Spectral decomposition

Definition 16

다음과 같이 Spectral decomposition을 할 수 있는 operator를 *diagonalizable*하다고 한다.

$$A = \sum_i \lambda_i |i\rangle \langle i|$$

Matrix properties: Hermitian and Unitary

Linear functional

Definition 17

Set of linear operator from $V \to \mathbb{C}$.

✓meaning: 벡터를 상수로 mapping시키는 operator를 Linear functional이라고 한다.

Riesz representation theorem

Theorem 18

Suppose:

- V*: finite-dimensional vector space*
- φ*: linear functional on* V

Then there is a unique vector $u \in V$ *s.t.*

$$\varphi |v\rangle = \langle u|v\rangle$$

Theorem 19

Suppose:

- V *is finite-dimensional vector space*
- A *is* $V \to V$ *linear operator*

Then there is a unique operator $B \equiv A^\dagger$ *s.t.*

$$(|v\rangle, A|w\rangle) = (B|v\rangle, |w\rangle)$$

$$\langle v|A|w\rangle = \langle v|B^\dagger|w\rangle = \langle v|(A^\dagger)^\dagger|w\rangle$$

Hermitian, normal and unitary

Definition 20 (Hermitian)

Operator A is the *Hermitian* operator such that

$$A^{\dagger} = A.$$

Corollary 21 (Projector)

Operator P is the projector onto the subspace W.Where W has orthonormal basis $\{|1\rangle, \cdots |k\rangle\}$ construct from orthonormal basis $\{|1\rangle, \cdots, |d\rangle\}$for V.

$$P \equiv \sum_{i=1}^{k} |i\rangle \langle i|$$

Hermitian, normal and unitary

Definition 22 (Normal)

Operator A is the *normal* operator such that

$$AA^\dagger = A^\dagger A.$$

- Hermitian operator is normal.
- Normal matrix is Hermitian if and only if it has real eigenvalues.
- Any normal operator A on a vector space V is *diagonal* with respect to some orthonormal basis for V.

Corollary 23 (Positive operator)

Operator B is positive operator such that

$$B = \sqrt{A^\dagger A}$$

Definition 24 (Unitary)

Operator U is *unitary* operator such that

$$UU^{\dagger} = U^{\dagger}U = I.$$

In other word,

$$U^{\dagger} = U^{-1}.$$

- Unitary operator is normal $\rightarrow$ has spectral decomposition
- Unitary operator preserve inner products.

$$(U|v\rangle, U|w\rangle) = \langle v|w\rangle$$

- U on orthonormal basis $\{|v_i\rangle\}$ write as

$$U = \sum_i |w_i\rangle |v_i\rangle,$$

where $|w_i\rangle = U|v_i\rangle$.

Tensor product

Tensor product

Definition 25

V와 W가 vector space이고 각각의 basis가 $\{i\}, \{j\}$일 때, 두 vector space의 tensor product는 다음과 같이 정의된다.

$$V \otimes W, \quad \{|i\rangle \otimes |j\rangle\} \in V \otimes W$$

Property of tensor product

By definition of tensor product, satisfies the following properties:
For an arbitrary scalar z and vectors $|v\rangle \in V$ and $|w\rangle \in W$

1) $z(|v\rangle \otimes |w\rangle) = (z\,|v\rangle) \otimes |w\rangle = |v\rangle \otimes (z\,|w\rangle)$,
2) $(|v_1\rangle + |v_2\rangle) \otimes |w\rangle = |v_1\rangle \otimes |w\rangle + |v_2\rangle \otimes w$,
3) $|v\rangle \otimes (|w_1\rangle + |w_2\rangle) = |v\rangle \otimes |w_1\rangle + |v\rangle \otimes |w_2\rangle$

Tensor product for Operators

Definition 26

V에 작용하는 operator A, W에 작용하는 operator B를 이용하면 tensor product vector space에대한 operator를 다음과 같이 나타낼 수 있다.

$$(A \otimes B)(|v\rangle \otimes |w\rangle) = A|v\rangle \otimes B|w\rangle$$

more general,

$$C = \sum_i c_i A_i \otimes B_i$$

Useful concepts: projector, trace and commutator

Operator functions

Definition 27

Normal operator A에 대해, operator function은 다음과 같이 적용된다.

$$f(A) = \sum_a f(a) |a\rangle \langle a|$$

✓ meaning: operator의 spectral decomposition에 대해 eigenvalue에 적용된다.

Trace

Definition 28

$$\text{tr}(A) = \sum_i A_{ii} = \sum_i \langle i|A|i\rangle$$

By definition of trace, satisfies the following properties:

- $\text{tr}(AB) = \text{tr}(BA)$
- $\text{tr}(A+B) = \text{tr}(A) + \text{tr}(B)$
- $\text{tr}(zA) = z \cdot \text{tr}(A)$
- (unitary invariant)
$$\text{tr}(UAU^\dagger) = \text{tr}(A)$$
- $\langle\psi|A|\psi\rangle = \text{tr}[A\,|\psi\rangle\,\langle\psi|]$

$*$ Proof:

Commutator

Definition 29

A, B is linear operator, commutator:

$$[A, B] = AB - BA$$

anti-commutator:

$$\{A, B\} = AB + BA$$

Theorem 30

If commutator is zero; $[A, B] = 0$ then if and only if there exists an orthogonal basis that diagonalizable A and B ***simultaneously****.*

Decompositions

Polar decomposition

Let A be a linear operator on a vector space V. Then there exists unitary U and positive operators J and K such that

$$A = UJ = KU$$

Singular value decomposition

Let A be a linear operator on a vector space V. Then there exists unitary U and V and a diagonal matrix D with nonnegative entries such that

$$A = UDV.$$

$*$ Proof:

Shmidt decomposition

Suppose we have a vector in a composite system $V \otimes W$. Then there exist orthonormal basis in V and W such that

$$|a\rangle = \sum_i \lambda_i |v_i\rangle \otimes |w_i\rangle$$

References

- Sheldon Axler, Linear Algebra Done Right, 3th
- Lecture notes for QU511: Quantum Computing (Fall 2024)

test block