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Developing a formalism for four-body interactions within star clusters using a core particle spray code

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Abstract

In this paper, we develop a four-body formalism for the core particle spray code, `corespray`, using a theoretical framework developed by Leigh et al. (2016) and an analysis of extra-tidal simulations created for the M3 globular cluster. We create this formalism by studying the $3 + 1$ final state for four-body interactions, which involve a triple and an escaper star. With the assumption that the lowest mass star in each encounter is the escaper, graphical representations of spatial and kinematic parameter spaces are created using the modified `corespray` code. These parameter spaces simulate the ejection of extra-tidal stars along with their recoil triples from the cores of globular clusters. We find that the spatial and proper motion parameter spaces are consistent with the results found by Grondin et al. (2022) for three-body interactions. However, the escape velocity distributions reveal that kicked triples in four-body interactions have the lowest average escape velocities, followed by kicked binaries in three-body interactions. The resulting average binary and triple mass fractions imply that about 10% of simulated binaries escape M3, as opposed to $< 5\%$ of triples.

1. INTRODUCTION

There is often interest among astronomers concerning the birth and evolution of gravitationally bound stars. Since stars do not form in isolation, but rather in clustered environments, they tend to exist in systems of up to N bodies. For researchers working with astrodynamics, understanding this N-body problem is essential in developing a theory to explain the phenomena of stellar and binary interactions in globular clusters (GCs) (Leigh et al. 2016). Given the increase in complexity as more bodies are added, the N-body problem is often considered in its simplest form as the three-body problem, and simplifying assumptions are made to solve it through approximation.

The study of galactic dynamics is often reliant on data collected concerning how GCs evolve over time. These findings can provide insight into star formation processes from initial GC size, and due to their high densities, are also ideal for observing how stars are affected by various processes (i.e tidal stripping, two-body relaxation) over time (Grondin et al. 2022). While these stellar processes can explain a GC's mass loss, they cannot quite explain the interactions within the core of a GC. As a result, understanding three-body interactions is vital for observing how the cores of GCs lose mass over time (Grondin et al. 2022).

Stars within the cores of GCs get ejected due to three-body interactions, and enter the Galactic halo, becoming extra-tidal stars (Grondin et al. 2022). These extra-tidal stars can provide a wealth of information about the cluster's evolution (i.e mass loss, formation, core dynamics) (Grondin et al. 2022). However, tracing these isolated stars back to their birth cluster is a challenge, as a result of large displacements or differing chemical signatures. Through the use of high-dimensional data analysis and a particle spray code called `corespray` developed by Grondin et al. (2022), the birth cluster of any isolated extra-tidal star can be identified. The `corespray` code is a Python-based particle spray code that samples three-body interactions of extra-tidal stars for any Galactic GC. Thus far, this code has been used by Grondin et al. (2022) to identify extra-tidal stars in one GC, M3, as a case study.

While approximations of the three-body problem can provide meaningful insights into the mass loss and evolution of a GC, they are not exhaustive in explaining all of the possible outcomes within the cores of star clusters. In particular, three-body interactions achieve final states that are limited to just the $2 + 1$ and $1 + 1 + 1$ configurations. As a result, it is important to consider the four-body problem as well, which can have multiple different outcomes (Leigh et al. 2016). Therefore, studying the four-body interaction is imperative to developing a comprehensive understanding of the spatial and kinematic properties of stars kicked from the cores of their GCs.

In this paper, we discuss modifications made to the current `corespray` algorithm that extend its methodology towards the four-body problem. The introduction of an additional star creates the opportunity for various final states to occur, such as two binaries $(2+2)$, a triple and a single star $(3+1)$, and a binary and two stars $(2+1+1)$ Leigh et al. (2016). This paper focuses on the $3+1$ case, and seeks to derive conclusions that explain differences between single stars kicked via three-body encounters and ones kicked via four-body encounters. In Section 2.a, we outline the original `corespray` methodology, and briefly discuss the mathematical theory involved. Section 2.b presents the four-body modifications made to `corespray`, sourced from Leigh et al. (2016). Features that add new functionalities for simulating four-body interactions will also be explained. Spatial and kinematic results from `corespray`'s modification are detailed in Section 3, along with a discussion on their relevance in Section 4. The paper will then conclude with a summary of key findings and possibilities for future study in Section 5.

2. METHODOLOGY

In this section, we give a brief overview of the `corespray` methodology developed by Grondin et al. (2022). We also present a four-body formalism that allows `corespray` to sample $3+1$ interactions, inspired by Leigh et al. (2016).

2.a. *Three-Body `Corespray` Particle Spray Code*

The `corespray` particle spray code is a Python-based algorithm that identifies extra-tidal kicked stars belonging to any galactic GC. To simulate these kicked stars, `corespray` initially defines three-dimensional position and velocity vectors within a cluster, which are created for the single and binary components of three-body systems. These vectors are defined at a random time along the orbit of the GC around the Milky Way galaxy. Using the `galpy` Python package, `corespray` then generates kicked stars with known orbital parameters, which allows orbits to be integrated forward in time (Bovy 2015).

To create simulations of three-body interactions, `corespray` samples masses of single stars (m_s) and recoil binaries (m_a, m_b) from a power law distribution found in Salpeter (1955). Then, the probability P_s that a single star escapes the system is given by Equation 1 (Valtonen & Karttunen 2006).

$$P_s = \frac{m_s^{-3}}{m_s^{-3} + m_a^{-3} + m_b^{-3}} \tag{1}$$

Randomly sampling Equation 1 lets `corespray` determine if the single star escapes the system. If so, the total energy E_0 of the system is computed using Equation 2 (Valtonen & Karttunen 2006),

$$E_0 = \frac{1}{2}m\dot{\mathbf{r}}_s^2 - G\frac{m_s m_B}{r_s} + E_B \tag{2}$$

where m_B is the mass of the recoil binary, $\mathbf{r}_s$ is the position vector between the single star and the centre of mass of the binary, $m = (m_s m_B)/(m_s + m_B)$ is the reduced mass of the system, and E_B is the binary's binding energy. The distribution of possible kick velocities that the single star could have is then given by $f(v_s)$, which is computed using E_0 and Equation 3 (Valtonen & Karttunen 2006),

$$f(v_s)dv_s = \frac{(3.5|E_0|^{7/2}(m_s M/m_B))v_s dv_s}{(|E_0| + \frac{1}{2}(m_s M/m_B)v_s^2)^{9/2}} \tag{3}$$

where $M = m_s + m_B$ and v_s is the escape velocity. The peak escape velocity $v_{s,peak}$ can then be obtained by maximizing $df/dv_s = 0$, giving Equation 4 (Valtonen & Karttunen 2006).

$$v_{s,peak} = \sqrt{\frac{(M - m_s)}{m_s M}}\sqrt{|E_0|} \tag{4}$$

`Corespray` then randomly samples Equation 3 between bounds set by Equation 4 to determine if the kicked single star is extra-tidal in nature. To conclude whether a star is extra-tidal, `corespray` compares its escape velocity to that of the GC's, which is found in Baumgardt & Hilker (2018), to verify if it is larger. This process is repeated until N extra tidal stars have been simulated, after which the escape velocities are projected onto the initial three-dimensional velocity vectors. Doing so determines the directions of motion of the extra-tidal stars (Grondin et al. 2022).

Over the course of one azimuthal period of the GC around the Milky Way, simulated escaper stars are integrated from their escape time to the present day. The integration of these stars is done in a combined Milky Way and King potential that explains influences from both the GC and the Galaxy (Grondin et al. 2022). With the `corespray` computed escape times and velocities, `galpy` can be used to determine multiple orbital parameters, such as spatial positions, proper motions, radial velocities, and distances.

2.b. *Modified Four-Body Version of `Corespray`*

The three-body `corespray` method is well developed and has produced interesting results (see Grondin et al. (2022)). However, its focus is on one type of encounter in the cores of GCs, which follows from $2 + 1$ interactions. In order to sample and simulate additional interactions within the cores of GCs, we introduce a four-body formalism for `corespray` (Leigh et al. 2016). This formalism largely follows the same procedure as what is described in Section 2.a, with a few key differences.

The first of these differences involves how the masses are sampled and assigned. Each four-body simulation samples masses of single stars (m_s), along with recoil triples (m_a, m_b, m_c). The total mass of the recoil triple (m_t) is also stored by `corespray`. Then, `corespray` assigns the lowest mass star of the four to be the escaper, while assigning the remaining stars to form the triple in the order they were sampled. This assumption follows from the results found by Leigh et al. (2016).

The next key difference is the calculation of the initial energy E_0. After `corespray` determines that a single star escapes a system, the initial energy $E_{0,four}$ is computed using Equation 5,

$$E_{0,four} = \frac{1}{2}(m_s + m_a)\dot{\mathbf{r}}_s^2 - G\frac{m_b m_c}{r_s} + E_{B,1} + E_{B,2} \quad (5)$$

where $E_{B,1}, E_{B,2}$ are the binding energies of the first and second binaries respectively. We denote the first binary to be the one containing the lowest mass star m_s.

The remaining key differences concern the calculation of the escape velocity distribution $f(v_e)dv_e$ and its peak $v_{e,peak}$. While the calculation of $f(v_e)dv_e$ does not differ significantly from Equation 3, we now consider the addition of a power law index constant n, given by Equation 6 (Valtonen & Karttunen 2006),

$$n - 3 = 18L^2 \quad (6)$$

where L is the total angular momentum of the system. In our procedure, we assume a total angular momentum of $L = 0$, resulting in a power law index of $n = 3$. Consequently, Equation 7 is used to create a new distribution of escape velocities for single stars (Leigh et al. 2016),

$$f(v_e)dv_e = \frac{((n-1)|E_0|^{n-1}(m_s M/m_B))v_e dv_e}{(|E_0| + \frac{1}{2}(m_s M/m_B)v_e^2)^n} \quad (7)$$

where $M = m_s + m_b + m_c$ and v_e is the escape velocity of the escaper. The total mass of the final system depends on these three singular masses due to an important result found by Leigh et al. (2016). In their study, they find that the four-body problem reduces to a three-body problem since the escaper star is mostly affected by the central binary in the triple, while not being considerably influenced by the third body (m_a).

The peak of Equation 7 is computed using Equation 8 (Leigh et al. 2016).

$$v_{\mathrm{e,peak}} = \left(n - \frac{1}{2}\right)^{-1/2} \sqrt{\frac{(M - m_s)}{m_s M}} \sqrt{|E_{0,four}|} \quad (8)$$

Following this calculation, `corespray` computes escape times and velocities in order to determine the same orbital parameters as in the three-body case. Subsequent analysis of spatial and kinematic parameter spaces created using the modified version of `corespray` lead to graphical representations of four-body systems, which depict extra-tidal stars along with their recoil triples.

3. RESULTS

We present here several parameter spaces simulated for M3 using the modified version of `corespray`, along with comparisons for the results of its three-body and four-body formalisms. The first important result from our tests is the similarity we find between the parameter spaces generated by the modified `corespray` and the original version, seen in Figure 1.

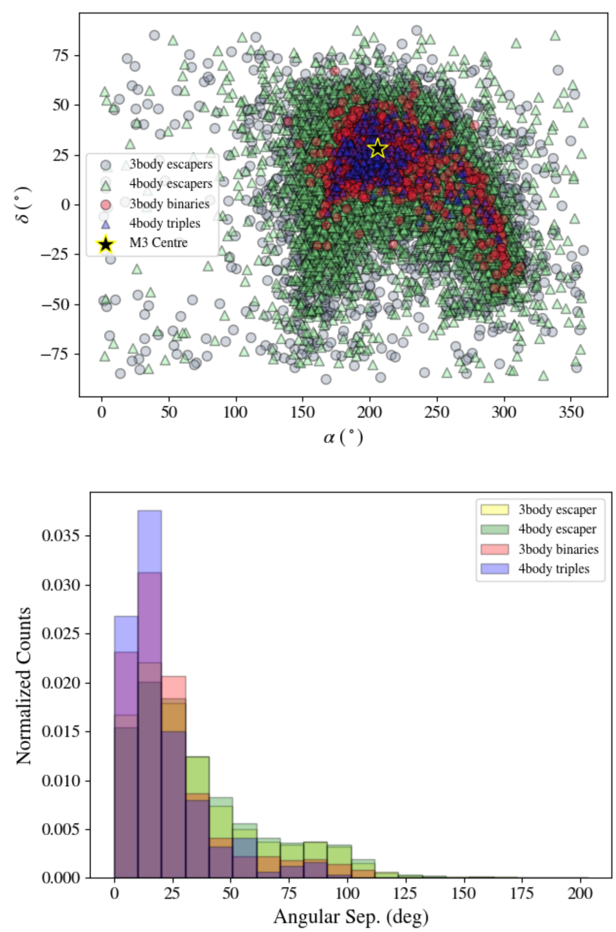

Figure 1: (Top panel) A spatial distribution of 10,000 stars of M3 simulated with `corespray`. The centre of M3 is shown by a yellow outlined star in the top panel. Three-body extra-tidal stars and binaries are shown by grey and red circles, while four-body extra-tidal stars and triples are indicated by green and blue triangles, respectively. (Bottom panel) Angular separation of escaper stars, binaries, and triples from three and four-body interactions.

After simulating spatial and angular separation parameter spaces for four-body interactions, we find that they are consistent with the three-body interactions simulated by Grondin et al. (2022). In particular, we see that most extra-tidal triples are clustered around the core of M3, which is in line with the original `corespray`'s result of extra-tidal binaries behaving this way as well. It also appears that three and four-body escaper stars alike are capable of receiving large kicks that push them away from the centre of M3. Repeated simulations of 10,000 stars for M3 show that the resulting average binary and triple mass fractions imply about 10% of simulated binaries escape M3, as opposed to $< 5\%$ of triples.

We also find that the escape velocity distribution of three and four-body escapers, binaries, and triples varies considerably. Figure 2 shows an escape velocity distribution resulting from the 10,000 star sample of M3 depicted in Figure 1.

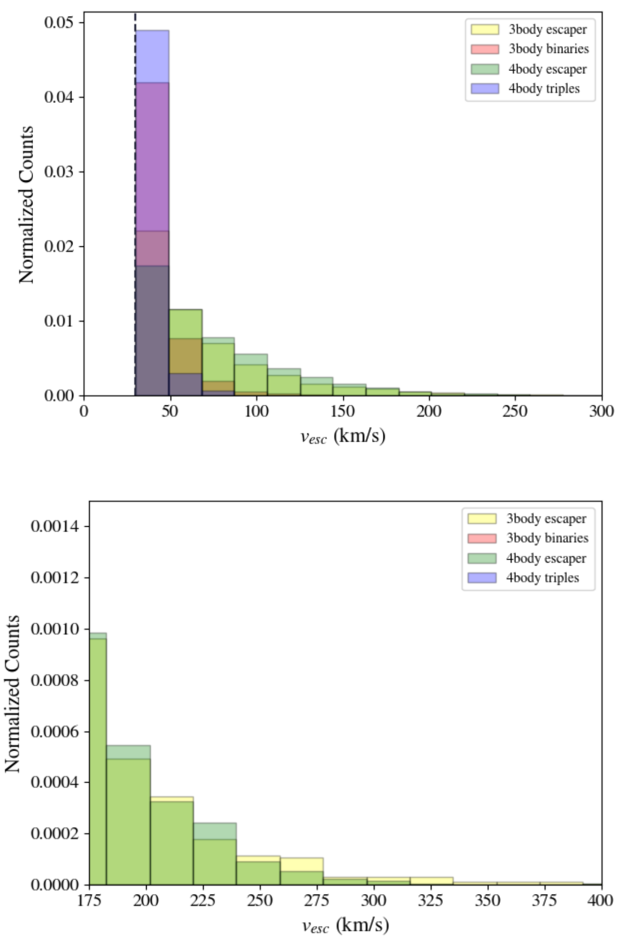

Figure 2: (Top panel) Escape velocity distribution of all escape velocities for simulated extra-tidal stars, binaries, and triples of 10,000 simulated stars of M3. (Bottom panel) A zoomed-in plot is also shown for high velocity escapers. The dotted vertical line represents the central escape velocity of M3.

From Figure 2, it is apparent that four-body triples have the lowest escape velocities on average, implying that triples are more centrally concentrated around the core of M3 than three-body binaries. This result is also supported by the bottom panel of Figure 1. Possible reasoning for this is discussed in Section 4.

Similar comparisons to what are seen in Figures 1 and 2 were also created for M15 and M92. The parameters for these GCs were obtained from Baumgardt & Hilker (2018), and analysis with the modified `corespray` yields trends similar to what is seen for M3. In the interest of maintaining brevity, their simulations are not included in this paper.

4. DISCUSSION

In this section, we discuss the significance and meaning of the results from Section 3.

An immediate trend we notice from Figure 1 is that extra-tidal binaries and triples are clustered around the core of M3. This can be explained using the distribution from Figure 2. According to the escape velocities for the various stellar bodies, we see that four-body triples and three-body binaries have relatively low escape velocities that are close to the central escape velocity of M3. Since binaries and triples have higher total masses than single stars, it follows that they are more difficult to accelerate. As a result, kick velocities for binaries and triples are much lower than those of the escaper stars due to the dynamics involved in N-body interactions.

In addition, Figure 2 shows that on average, there are more four-body escapers at higher velocities than three-body ones. We believe this result stems from our assumption that four-body escapers are always the lowest mass stars from their encounters, and lower mass objects are easier to accelerate.

Contrary to this, we notice that the highest velocity escapers ($v_{esc} \geq 250\,\mathrm{km\,s^{-1}}$) come from the three-body interaction. We believe this result is conditional depending on the $3+1$ interaction however; future tests involving $2+2$ or $2+1+1$ interactions have the potential to yield different results.

Our preliminary testing of the four-body version of `corespray` concludes with comparisons of mass and escape velocity distributions for three and four-body interactions. These results are shown in Figure 3.

The histograms representing various mass distributions in Figure 3 display the comparisons between three and four-body encounters. In the top panel of the figure, it is evident that there is a large number of escaper stars from both types of interactions with relatively low masses. This conclusion is within reason, as lower mass objects are generally easier to put into motion. Since the escape velocity needed to escape M3 is a set value

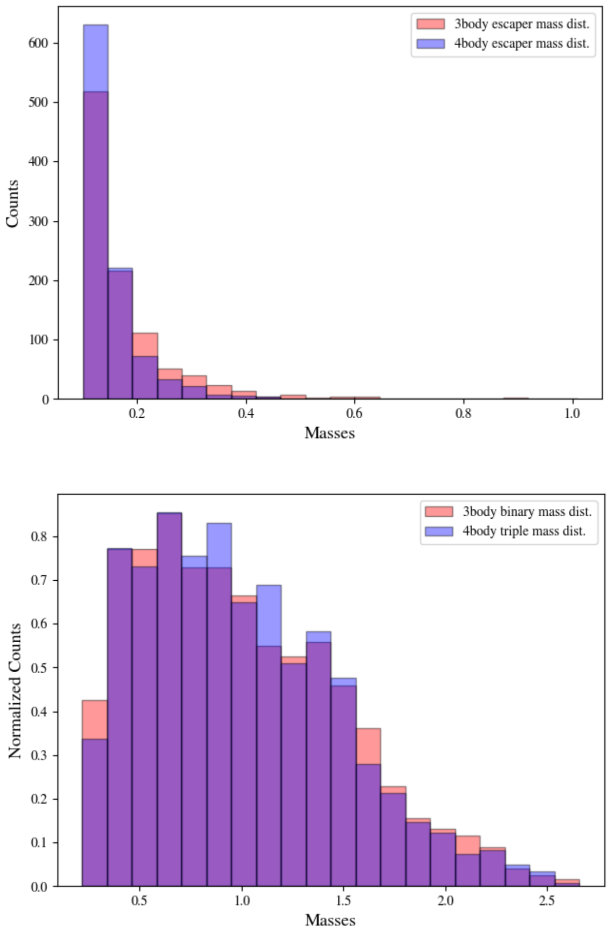

Figure 3: (Top panel) A mass distribution of escaper stars from three and four-body interactions for 1000 simulated stars in M3's core. (Bottom panel) A distribution of binary and triple masses from three-body and four-body interactions respectively, also for 1000 stars.

less than $50\,\mathrm{km\,s^{-1}}$, we can expect most escaper stars to be able to escape the GC's core. A more notable result is seen near the tail end of the distribution. We observe there that several small bins corresponding to three-body escapers exist in the higher mass end of the scale, which suggests that three-body interactions are more capable of producing high-mass escapers. This is a consequence from our assumption that four-body interactions eject the lowest mass star involved. Based on our assumption, high mass escapers from four-body interactions are reliant on four high mass stars being brought together. This fact implies that our simulations do not contain four-body systems consisting of four high mass stars, which is reasonable given that this is very unlikely to occur in nature.

On the other hand, the bottom panel of Figure 3 shows similar trends in the binary and triple masses, preventing us from making a distinct conclusion. Future study could provide insights towards this trend.

5. CONCLUSIONS AND FUTURE WORK

In this study, we develop a new four-body formalism for the particle spray code `corespray` that allows for the simulation of $3+1$ stellar interactions within the cores of GCs. Referencing the insights for the chaotic four-body problem from Leigh et al. (2016), we modify the methodology of the original `corespray` to implement our new formalism.

The modification of `corespray` is achieved by first analyzing literature, such as Leigh et al. (2016) and Valtonen & Karttunen (2006), to identify where changes to the original code can be made. Through this modification, we find that sampling escape velocities for $3+1$ interactions requires a slightly different distribution than that of Equation 3. From Leigh et al. (2016), we find a general escape velocity distribution formula (Equation 7), which supports the four-body modification with the inclusion of a fixed power law index constant n obtained from Valtonen & Karttunen (2006). `Corespray`'s original method of sampling masses is also altered in the four-body formalism; it now always takes the lowest mass star in the interaction to be the escaper (Leigh et al. 2016). In addition, the calculation of the initial energy of the system differs in the four-body `corespray` due to the inclusion of an additional star. Therefore, the binding energy of another binary system (Equation 5) must be considered to determine the total energy of the four-body system.

Testing the four-body formalism by creating simulations of the GC M3 lead to the results found in Figures 1, 2, and 3. As we expect from the three-body simulations, four-body triples tend to cluster around the core of the GC due to slower recoil kick velocities compared to single escaper stars. The figures also show that four-body escaper stars have faster escape velocities on average, yet three-body escapers are capable of reaching higher velocities overall.

Comparisons between binary and triple mass distributions show that similar trends exist between the masses, which could be explained through future study.

Additional modifications that could be implemented into `corespray` include developing formalisms that cover $2+2$ and $2+1+1$ stellar interactions, as well as optimizing the particle spray code process to create a variety of parameter spaces. With these improvements, along with the testing of other Milky Way GCs, we could potentially uncover deeper insights into galactic formation and evolution on the galactic scale.

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