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A new definition of fractional derivative

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ABSTRACT

We give a new definition of fractional derivative and fractional integral. The form of the definition shows that it is the most natural definition, and the most fruitful one. The definition for $0 \leq \alpha < 1$ coincides with the classical definitions on polynomials (up to a constant). Further, if $\alpha = 1$, the definition coincides with the classical definition of first derivative. We give some applications to fractional differential equations.

1. Introduction

Fractional derivative is as old as calculus. L'Hospital in 1695 asked what does it mean $\frac{d^n f}{dx^n}$ if $n = \frac{1}{2}$. Since then, many researchers tried to put a definition of a fractional derivative. Most of them used an *integral form* for the fractional derivative. Two of which are the most popular ones.

(i) *Riemann–Liouville definition*. For $\alpha \in [n-1, n)$, the α derivative of f is

$$D_a^\alpha(f)(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d^n}{dt^n} \int_a^t \frac{f(x)}{(t-x)^{\alpha-n+1}}\, dx.$$

(ii) *Caputo definition*. For $\alpha \in [n-1, n)$, the α derivative of f is

$$D_a^\alpha(f)(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(x)}{(t-x)^{\alpha-n+1}}\, dx.$$

Now, all definitions including (i) and (ii) above satisfy the property that the fractional derivative is linear. This is the only property inherited from the first derivative by all of the definitions. However, the following are some of the setbacks of the other definitions:

(i) The Riemann–Liouville derivative *does not* satisfy $D_a^\alpha(1) = 0$ ($D_a^\alpha(1) = 0$ for the Caputo derivative), if α is not a natural number.

(ii) All fractional derivatives *do not* satisfy the known formula of the derivative of the product of two functions:

$$D_a^\alpha(fg) = fD_a^\alpha(g) + gD_a^\alpha(f).$$

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(iii) All fractional derivatives *do not* satisfy the known formula of the derivative of the quotient of two functions:

$$D_a^\alpha(f/g) = \frac{gD_a^\alpha(f) - fD_a^\alpha(g)}{g^2}.$$

(iv) All fractional derivatives *do not* satisfy the chain rule:

$$D_a^\alpha(f \circ g)(t) = f^{(\alpha)}\big(g(t)\big)\, g^{(\alpha)}(t).$$

(v) All fractional derivatives *do not satisfy*: $D^\alpha D^\beta f = D^{\alpha+\beta} f$ in general.

(vi) The Caputo definition *assumes that the function f is differentiable*.

Our interest in fractional derivatives started when Professor S. Momani showed us how to solve the differential equation

$$y^{\left(\frac{1}{2}\right)} + y = x^{\frac{1}{2}} + \frac{2}{\Gamma(2.5)} x^{\frac{3}{2}}; \qquad y(0) = 0$$

where $y^{(\frac{1}{2})}$ is the fractional derivative of y of order $\frac{1}{2}$. The known solution for this differential equation is not easy to be obtained, so we thought that a new definition of fractional derivative may facilitate some computations.

The object of this paper is to present a new, yet an easy definition of fractional derivative. The new definition seems to be a *natural extension* of the usual derivative, and it *satisfies the first four properties mentioned above*. Our definition coincides with the known fractional derivatives on polynomials (up to a constant multiple).

For the history and main results on fractional derivatives and fractional differential equations, we refer the reader to [1–4].

2. The definition

Let $f : [0, \infty) \to \mathbb{R}$ and $t > 0$. Then the definition of the derivative of f at t is $\frac{df}{dt} = \lim_{\varepsilon \to 0} \frac{f(t+\varepsilon) - f(t)}{\varepsilon}$. According to this, one has $\frac{dt^n}{dt} = nt^{n-1}$. So the question is: Can one put a similar definition for the fractional derivative of order α, where $0 < \alpha \le 1$? Or in general for $\alpha \in (n, n+1]$ where $n \in \mathbb{N}$.

Let us write T_α to denote the operator which is called the fractional derivative of order α. For $\alpha = 1$, T_1 satisfies the following properties:

(i) $T_1(af + bg) = aT_1(g) + bT_1(f)$, for all $a, b \in \mathbb{R}$ and f, g in the domain of T_1.
(ii) $T_1(t^p) = pt^{p-1}$ for all $p \in \mathbb{R}$.
(iii) $T_1(fg) = fT_1(g) + gT_1(f)$.
(iv) $T_1(\frac{f}{g}) = \frac{gT_1(f) - fT_1(g)}{g^2}$.
(v) $T_1(\lambda) = 0$, for all constant functions $f(t) = \lambda$.

Now, we present our new definition, which is the simplest and most natural and efficient definition of fractional derivative of order $\alpha \in (0, 1]$. We should remark that the definition can be generalized to include any α. However, the case $\alpha \in (0, 1]$ is the most important one, and once it is established, the other cases are simple.

Definition 2.1. Given a function $f : [0, \infty) \longrightarrow \mathbb{R}$. Then the "*conformable fractional derivative*" of f of order α is defined by

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}$$

for all $t > 0$, $\alpha \in (0, 1)$. If f is α-differentiable in some $(0, a)$, $a > 0$, and $\lim_{t \to 0^+} f^{(\alpha)}(t)$ exists, then define

$$f^{(\alpha)}(0) = \lim_{t \to 0^+} f^{(\alpha)}(t).$$

We will, sometimes, write $f^{(\alpha)}(t)$ for $T_\alpha(f)(t)$, to denote the conformable fractional derivatives of f of order α. In addition, if the conformable fractional derivative of f of order α exists, then we simply say f is α-differentiable.

We should remark that $T_\alpha(t^p) = p\,t^{p-\alpha}$. Further, our definition coincides with the classical definitions of R–L and of Caputo on polynomials (up to a constant multiple).

As a consequence of the above definition, we obtain the following useful theorem.

Theorem 2.1. *If a function $f : [0, \infty) \longrightarrow \mathbb{R}$ is α-differentiable at $t_0 > 0$, $\alpha \in (0, 1]$, then f is continuous at t_0.*

Proof. Since $f(t_0 + \varepsilon t_0^{1-\alpha}) - f(t_0) = \frac{f(t_0 + \varepsilon t_0^{1-\alpha}) - f(t_0)}{\varepsilon}\, \varepsilon$. Then,

$$\lim_{\varepsilon \to 0} [f(t_0 + \varepsilon t_0^{1-\alpha}) - f(t_0)] = \lim_{\varepsilon \to 0} \frac{f(t_0 + \varepsilon t_0^{1-\alpha}) - f(t_0)}{\varepsilon} \cdot \lim_{\varepsilon \to 0} \varepsilon.$$

Let $h = \varepsilon t_0^{1-\alpha}$. Then,

$$\lim_{h\to 0}[f(t_0+h)-f(t_0)] = f^{(\alpha)}(t_0)\ \cdot 0,$$

which implies that

$$\lim_{h\to 0} f(t_0+h) = f(t_0).$$

Hence, f is continuous at t_0. $\square$

One can easily show that T_α satisfies all the properties in the following theorem.

Theorem 2.2. *Let $\alpha \in (0, 1]$ and f, g be α-differentiable at a point $t > 0$. Then*

(1) $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$, *for all* $a, b \in \mathbb{R}$.
(2) $T_\alpha(t^p) = pt^{p-\alpha}$ *for all* $p \in \mathbb{R}$.
(3) $T_\alpha(\lambda) = 0$, *for all constant functions* $f(t) = \lambda$.
(4) $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.
(5) $T_\alpha(\frac{f}{g}) = \frac{gT_\alpha(f)-fT_\alpha(g)}{g^2}$.
(6) *If, in addition, f is differentiable, then* $T_\alpha(f)(t) = t^{1-\alpha}\frac{df}{dt}(t)$.

Proof. Parts (1) through (3) follow directly from the definition. We choose to prove (4) and (6) only since they are crucial. Now, for fixed $t > 0$,

$$\begin{aligned}
T_\alpha(fg)(t) &= \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})g(t+\varepsilon t^{1-\alpha}) - f(t)g(t)}{\varepsilon}\\
&= \lim_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha})g(t+\varepsilon t^{1-\alpha}) - f(t)g(t+\varepsilon t^{1-\alpha}) + f(t)g(t+\varepsilon t^{1-\alpha}) - f(t)g(t)}{\varepsilon}\\
&= \lim_{\varepsilon\to 0}\left(\frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}.g(t+\varepsilon t^{1-\alpha})\right) + f(t)\lim_{\varepsilon\to 0}\frac{g(t+\varepsilon t^{1-\alpha}) - g(t)}{\varepsilon}\\
&= T_\alpha(f)(t)\lim_{\varepsilon\to 0} g(t+\varepsilon t^{1-\alpha}) + f(t)T_\alpha(g)(t).
\end{aligned}$$

Since g is continuous at t, $\lim_{\varepsilon\to 0} g(t+\varepsilon t^{1-\alpha}) = g(t)$. This completes the proof of part (4). (5) can be proved in a similar way. To prove (6), let $h = \varepsilon t^{1-\alpha}$ in Definition 2.1, and then $\varepsilon = t^{\alpha-1}h$. Therefore,

$$\begin{aligned}
T_\alpha(f)(t) &= \lim\nolimits_{\varepsilon\to 0}\frac{f(t+\varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}\\
&= \lim_{h\to 0}\frac{f(t+h) - f(t)}{ht^{\alpha-1}}\\
&= t^{1-\alpha}\lim_{h\to 0}\frac{f(t+h) - f(t)}{h}\\
&= t^{1-\alpha}\frac{df}{dt}(t). \quad \square
\end{aligned}$$

Conformable fractional derivative of certain functions

(1) $T_\alpha(t^p) = pt^{p-\alpha}$ for all $p \in \mathbb{R}$.
(2) $T_\alpha(1) = 0$.
(3) $T_\alpha(e^{cx}) = cx^{1-\alpha}e^{cx},\ c \in \mathbb{R}$.
(4) $T_\alpha(\sin bx) = bx^{1-\alpha}\cos bx,\ b \in \mathbb{R}$.
(5) $T_\alpha(\cos bx) = -bx^{1-\alpha}\sin bx,\ b \in \mathbb{R}$.
(6) $T_\alpha(\frac{1}{\alpha}t^\alpha) = 1$.

However, it is worth noting the following conformable fractional derivatives of certain functions:

(i) $T_\alpha(\sin\frac{1}{\alpha}t^\alpha) = \cos\frac{1}{\alpha}t^\alpha$.
(ii) $T_\alpha(\cos\frac{1}{\alpha}t^\alpha) = -\sin\frac{1}{\alpha}t^\alpha$.
(iii) $T_\alpha(e^{\frac{1}{\alpha}t^\alpha}) = e^{\frac{1}{\alpha}t^\alpha}$.

One should notice that a function could be α-differentiable at a point but not differentiable, for example, take $f(t) = 2\sqrt{t}$. Then $T_{\frac{1}{2}}(f)(0) = \lim_{t\to 0^+} T_{\frac{1}{2}}(f)(t) = 1$, where $T_{\frac{1}{2}}(f)(t) = 1$, for $t > 0$. But $T_1(f)(0)$ does not exist. This is not the case for the known classical fractional derivatives.

Although the most important case for the range of α is $(0, 1)$, but, what if $\alpha \in (n, n+1]$ for some natural number n? What would be the definition?

Definition 2.2. Let $\alpha \in (n, n+1]$, and f be an n-differentiable at t, where $t > 0$. Then the *conformable fractional derivative* of f of order α is defined as

$$T_\alpha(f)(t) = \lim_{\varepsilon \to 0} \frac{f^{(\lceil \alpha \rceil - 1)}(t + \varepsilon t^{(\lceil \alpha \rceil - \alpha)}) - f^{(\lceil \alpha \rceil - 1)}(t)}{\varepsilon}$$

where $\lceil \alpha \rceil$ is the smallest integer greater than or equal to α.

Remark 2.1. As a consequence of Definition 2.2, one can easily show that

$$T_\alpha(f)(t) = t^{(\lceil \alpha \rceil - \alpha)} f^{\lceil \alpha \rceil}(t)$$

where $\alpha \in (n, n+1]$, and f is $(n+1)$-differentiable at $t > 0$.

Analysis of the definition

The previous definitions of fractional derivative do not enable us to study the analysis of α-differentiable functions. However, our definition makes it possible to prove basic analysis theorems like the Rolle's theorem and the mean value theorem.

Theorem 2.3 (*Rolle's Theorem for Conformable Fractional Differentiable Functions*)**.** *Let $a > 0$ and $f : [a, b] \to \mathbb{R}$ be a given function that satisfies*

(i) *f is continuous on $[a, b]$,*
(ii) *f is α-differentiable for some $\alpha \in (0, 1)$,*
(iii) *$f(a) = f(b)$.*

Then, there exists $c \in (a, b)$, such that $f^{(\alpha)}(c) = 0$.

Proof. Since f is continuous on $[a, b]$, and $f(a) = f(b)$, there is $c \in (a, b)$, which is a point of local extrema. With no loss of generality, assume c is a point of local minimum. So

$$f^{(\alpha)}(c) = \lim_{\varepsilon \to 0^+} \frac{f(c + \varepsilon c^{1-\alpha}) - f(c)}{\varepsilon} = \lim_{\varepsilon \to 0^-} \frac{f(c + \varepsilon c^{1-\alpha}) - f(c)}{\varepsilon}.$$

But, the first limit is non-negative, and the second limit is non-positive. Hence $f^{(\alpha)}(c) = 0$. $\square$

Theorem 2.4 (*Mean Value Theorem for Conformable Fractional Differentiable Functions*)**.** *Let $a > 0$ and $f : [a, b] \to \mathbb{R}$ be a given function that satisfies*

(i) *f is continuous on $[a, b]$.*
(ii) *f is α-differentiable for some $\alpha \in (0, 1)$.*

Then, there exists $c \in (a, b)$, such that $f^{(\alpha)}(c) = \frac{f(b) - f(a)}{\frac{1}{\alpha} b^\alpha - \frac{1}{\alpha} a^\alpha}$.

Proof. Consider the function

$$g(x) = f(x) - f(a) - \frac{f(b) - f(a)}{\frac{1}{\alpha} b^\alpha - \frac{1}{\alpha} a^\alpha} \left(\frac{1}{\alpha} x^\alpha - \frac{1}{\alpha} a^\alpha \right).$$

Then the function g satisfies the conditions of Rolle's theorem. Hence there exists $c \in (a, b)$, such that $g^{(\alpha)}(c) = 0$. Using the fact that $T_\alpha(\frac{1}{\alpha} t^\alpha) = 1$, the result follows. $\square$

Along the same lines in basic analysis, one can use the present mean value theorem to prove the following proposition.

Proposition 2.1. *Let $f : [a, b] \to \mathbb{R}$ be α-differentiable for some $\alpha \in (0, 1)$.*

(a) *If $f^{(\alpha)}$ is bounded on $[a, b]$ where $a > 0$. Then f is uniformly continuous on $[a, b]$, and hence f is bounded.*
(b) *If $f^{(\alpha)}$ is bounded on $[a, b]$ and continuous at a. Then f is uniformly continuous on $[a, b]$, and hence f is bounded.*

It is well-known that if $f'(t)$ is bounded on $I = [a, b]$, then f is uniformly continuous on I. However, the converse need not be true. To see this, consider $f(t) = 2\sqrt{t}$ on $I = [0, 1]$. Then f is uniformly continuous on $[0, 1]$ but $f'(t)$ is not bounded there. However, boundedness of $f^{(\alpha)}(t)$ for $0 < \alpha < 1$ and the continuity of f on I (continuity of f at 0 in the subspace topology is equivalent to right continuity of f at 0), which implies, by the above proposition, the uniform continuity of f on I.

3. Fractional integral

When it comes to integration, the most important class of functions to define the integral is the space of continuous functions.

So, using the Weierstrass theorem, it is enough to define the fractional integral on polynomials. This suggests the following.

Let $\alpha \in (0, \infty)$. *Define* $J_\alpha(t^p) = \frac{t^{p+\alpha}}{p+\alpha}$ for any $p \in \mathbb{R}$, and $\alpha \neq -p$.
If $f(t) = \sum_{k=0}^{n} b_k t^k$, then we define $J_\alpha(f) = \sum_{k=0}^{n} b_k J_\alpha(t^k) = \sum_{k=0}^{n} b_k \frac{t^{k+\alpha}}{k+\alpha}$.
If $f(t) = \sum_{k=0}^{\infty} b_k t^k$, where the series is uniformly convergent, then we define $J_\alpha(f) = \sum_{k=0}^{\infty} b_k \frac{t^{k+\alpha}}{k+\alpha}$.

Clearly, J_α is linear on its domain. Further, if $\alpha = 1$, then J_α is the usual integral.

Now, according to our definition, if $\alpha = \frac{1}{2}$, then

$$J_\alpha(\sin t) = \sum_{n=0}^{\infty} \frac{(-1)^n\, t^{2n+\frac{3}{2}}}{(2n+\frac{3}{2})(2n+1)!},$$

similarly for $\cos t$ and e^t, and for any $\alpha \in (0, 1)$.

These examples suggest the following *definition for the α-fractional integral of a function f starting from* $\mathbf{a} \geq 0$.

Definition 3.1. $I_\alpha^a(f)(t) = I_1^a(t^{\alpha-1}f) = \int_a^t \frac{f(x)}{x^{1-\alpha}}\,dx$, where the integral is the usual Riemann improper integral, and $\alpha \in (0, 1)$.

So, $I_{\frac{1}{2}}^0\left(\sqrt{t}\ \cos t\right) = \int_0^t \cos x\,dx = \sin\ t$, and $I_{\frac{1}{2}}^0\left(\cos 2\sqrt{t}\right) = \sin 2\sqrt{t}$.

One of the nice results is the following.

Theorem 3.1. *$T_\alpha\ I_\alpha^a(f)(t) = f(t)$, for $t \geq a$, where f is any continuous function in the domain of I_α.*

Proof. Since f is continuous, then $I_\alpha^a(f)(t)$ is clearly differentiable. Hence,

$$\begin{aligned} T_\alpha(I_\alpha^a(f))(t) &= t^{1-\alpha}\frac{d}{dt}I_\alpha^a(f)(t) \\ &= t^{1-\alpha}\frac{d}{dt}\int_a^t \frac{f(x)}{x^{1-\alpha}}\,dx \\ &= t^{1-\alpha}\frac{f(t)}{t^{1-\alpha}} \\ &= f(t). \quad \square \end{aligned}$$

4. Applications

Now we will solve fractional differential equations according to our definitions.

Example 4.1. $y^{(1/2)} + y = x^2 + 2x^{3/2},\ y(0) = 0$. This is the equation that we mentioned in the Introduction, approximating $\Gamma(2.5) = 1.33$ to 1.

Let us find a solution y_h of the homogeneous equation $y^{(1/2)} + y = 0$. We look for a solution of the form $y_h = e^{r\sqrt{x}}$. $y^{(1/2)} + y = 0 \implies \frac{r}{2}e^{r\sqrt{x}} + e^{r\sqrt{x}} = 0 \implies \frac{1}{2}r + 1 = 0$, the auxiliary equation. Hence, $y_h = e^{-2\sqrt{x}}$. It is easy to verify that $y_p(x) = x^2$ is a particular solution of the nonhomogeneous equation.

Now, the general solution is $y(x) = y_h(x) + y_p(x) = Ae^{-2\sqrt{x}} + x^2$, where A is constant. Finally, the initial condition $y(0) = 0$ implies that $A = 0$. Hence, $y(x) = x^2$.

We should remark here that such an equation is a known one that people solve using the R–L derivative, with $\frac{2}{\Gamma(2.5)}x^{\frac{3}{2}}$ replacing $2x^{3/2}$, and we get the same solution, but our method using our definition is much more easier.

Example 4.2. One can easily show that the auxiliary equation for $y^{(\alpha)} + y = 0,\ 0 < \alpha \leq 1$ is $\alpha r + 1 = 0$, so that the solution is given by $y(x) = e^{-\frac{1}{\alpha}x^\alpha}$. More details including the method of undetermined coefficients will be discussed in a forthcoming paper.

In the following example we will show the benefit of the fractional derivative product rule which allows us to use the idea of the integrating factor.

Example 4.3. $y^{(1/2)} + \sqrt{x}\,y = xe^{-x}$.

We solve this equation by multiplying it by $e^{I_{1/2}(\sqrt{x})} = e^x$. Then

$$e^x y^{(1/2)} + \sqrt{x}e^x y = x \implies (e^x y)^{(1/2)} = x \implies e^x y(x) = \frac{2}{3}x^{3/2} + C$$

$\implies y(x) = \frac{2}{3}x^{3/2}e^{-x} + Ce^{-x}$, C is constant, which can be easily verified to be a solution of the above equation.

Example 4.4. $y^{(\frac{1}{2})} = \frac{x^{\frac{3}{2}}+y\sqrt{x}}{2x+3y}$.

We assume that we are looking for a differentiable y. So, by (i) in Theorem 2.2, we have $y^{(\frac{1}{2})} = \sqrt{x}dy/dx$.

Hence, the fractional differential equation becomes $y'\sqrt{x} = \frac{x^{\frac{3}{2}}+y\sqrt{x}}{2x+3y}$. So, $y' = \frac{x+y}{2x+3y}$. This is a homogeneous differential equation of order one and can be solved easily.

Closing remarks

(i) We believe that the fractional derivative presented here, for $0 < \alpha < 1$, is local by nature. Here the following question is raised: Can its physical meaning be interpreted easier than those previously presented in history?

(ii) Note that the derivative presented here does not have any delay effect but the other fractional ones have (there is a kernel in the definition inside the integral)!.

(iii) Consider the very simple differential equation $y^{(1/2)}+y = 0$. If one has to solve it using the Caputo or Riemann–Liouville definition, then he/she must use either the Laplace transform or the fractional power series technique. However, using our definition and the fact $T_\alpha(e^{\frac{1}{\alpha}t^\alpha}) = e^{\frac{1}{\alpha}t^\alpha}$, one can easily see that $y = ce^{-2\sqrt{t}}$ is the general solution.

(iv) According to one of the classical definitions, $D^{\frac{1}{2}}(\sin ax) = a^{\frac{1}{2}}\sin(ax+\pi/4)$. So what if $a = -1$? Now according to our definition $T_\alpha(\sin\frac{1}{\alpha}t^\alpha) = \cos\frac{1}{\alpha}t^\alpha$. To evaluate $D^\alpha(\sin\frac{1}{\alpha}t^\alpha)$ using the classical definitions, it is not an easy job.

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