

Cointegration in Forex Pairs Trading

1 Introduction

Forex pairs trading strategy that implements cointegration is a sort of convergence trading strategy based on statistical arbitrage using a mean-reversion logic. This strategy was first introduced by Morgan Stanley in the 1980s using stock pairs, but traders found that it could be used in FX trading as well.

If two pairs are cointegrated, it means that the *spread* between those pairs is about to converge over time based on an empirical study. It does not necessarily mean that the divergence is about to stop anytime soon, however some investors consider that it may be time to act, attempting to buy one pair and sell the other then wait for a convergence to happen. Unless the co-movements between the two currencies break over time, the ‘net’ trade should be zero in the bad scenario (excluding the trading costs).

The goal of this research is to review the cointegration in the FX market using three different approaches – Engle-Granger, Johansen test and the Hurst exponent – with some application in Eviews and Bloomberg. One fundamental property that pairs trading requires is that the instruments have to be cointegrated in order to ensure a connection between two FX currency pairs.

Definition (Cointegration): An n -dimensional time series y_t is cointegrated if the linear combination $B_1y_{1t} + \dots + B_nY_{nt}$ of the component variables is stationary.

If $n = 2$ for instance, the time series y_{1t} and y_{2t} are said to be cointegrated (co-moving as integrated series) with co-integrated (first order) parameter b if the same b holds for both equation in the VAR(1) process below:

Giving y_{1t} and y_{2t} two series (currency pairs), we have the following equation:

$$\begin{aligned}\Delta y_{1t} &= a_1(y_{1,t-1} - by_{2,t-1}) + e_{1,t} \\ \Delta y_{2t} &= a_2(y_{1,t-1} - by_{2,t-1}) + e_{2,t}\end{aligned}$$

1. If Δy_{1t} and Δy_{2t} were both $I(0)$, i.e. y_{1t} and y_{2t} were both $I(1)$; and

2. We could find a coefficient b such that:

$$y_{1,t-1} - by_{2,t-1} \sim I(0) \quad (1)$$

So the two side of the equation are balanced in the form $I(0) = I(0) + I(0)$.

If we look at two currency pairs y_{1t} and y_{2t} , with $y_{1t} - b y_{2t} = Z_t$ and Z_t a stationary time series with a zero means. Then we can set up a threshold T such as:

- If $y_{1t} - b y_{2t}$ is greater than T , take a short position in y_{1t} and long y_{2t}
- If $y_{1t} - b y_{2t}$ is lower than $-T$, go long y_{1t} and short y_{2t}

2 Engle-Granger approach

We can use a variety of tests for cointegration, but the easier one is the Engle-Granger test:

Step 1

1. We first verify that y_{1t} and y_{2t} are both $I(1)$
2. Compute the cointegration relationship $y_{1t} = b y_{2t} + e_t$ by using the ordinary least square (OLS) method, and save the residuals of the cointegrating regression e_t

Step 2

3. Verify that the cointegrated residuals e_t are stationary by using a ADF (Augmented Dickey-Fuller test) unit root test.

$$\Delta \hat{e}_t = \Pi \hat{e}_{t-1} + \sum_{i=1}^k (c_i \Delta \hat{e}_{t-i}) + \eta_t \quad (2)$$

Where $H_0 : \Pi = 0$ (unit root / no cointegration)

With:

$$\hat{t}_{\Pi=0} = \frac{\hat{\Pi}}{SE_{\hat{\Pi}}} \sim DF_{\Pi=0} \quad (3)$$

In case you fail to reject this test fail to reject the null hypothesis of a unit root against the autoregressive alternative, you will need to construct a model containing only first differences.

Cointegration means that the two time series co-move together in the long term, therefore cannot drift apart very much [and for too long] from each other (Granger, 1981). We are now going to apply this method to a few currency pairs using Eviews as software.

1. **Descriptive statistics:** The period under observation runs from 31 December 2002 to 21 July 2016 and covers the daily exchange rates of the US-Dollar expressed in foreign currencies (i.e. how many units of US dollars for 1 unit of Euro). Overall, the nine most important exchange rates in the world are used: the Euro (EUR), the Japanese Yen (JPY), the British Pound (GBP), the Swiss Franc (CHF), the Swedish Krona (SEK), the Norwegian Krona (NOK), the Australian Dollar (AUD), the New Zealand Dollar (NZD) and the Mexican Peso (MXN). All daily rates are converted by taking the natural logarithm. The data is extracted from Bloomberg using PX LAST as a field.
2. **Unit Root Tests:** Before we can carry out the cointegration analysis, we must apply unit root tests to all exchange rates (in natural logarithm). As precised earlier, we expect that the order of integration is one as many academic studies have shown (e.g. Meese and Singleton, 1982). We apply different unit root tests such as Phillips-Peron (PP) test, the KPSS-test by Kwiatkowski et al. (1992) and the DF-GLS test by Elliott et al. (1996) to the exchange rates (log) and in first difference.

2.1 Test on a few currency pairs

2.1.1 USD/EUR and USD/GBP

FX Literature has usually suggested to look at USD/EUR and USD/GBP as a Forex pairs. If we plot a chart of USD/EUR and USD/GBP over the past ten years (see Figure 1), we can see that both series are kind of co-moving together. The only period where the regime shifted was during the Great Financial crisis; the British pound plummeted by 36 % against the US Dollar whereas USD/EUR saw a 23 % drawdown during the same period.

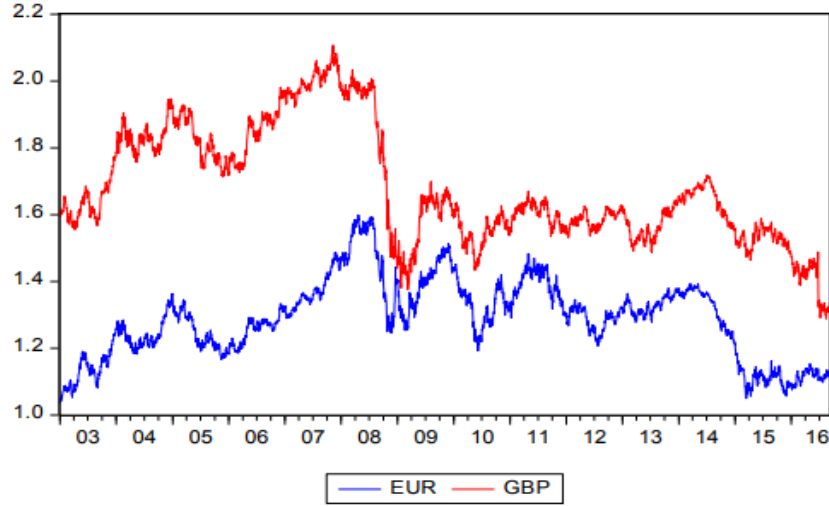


Figure 1: EURUSD and GBPUSD historical daily prices)

Eviews Application:

First step: Estimate the regression using a simple OLS regression (log terms):

$$USD/EUR_t = \alpha + \beta USD/GBP_t + e_t \quad (4)$$

Once we uploaded the data on Eviews, we can proceed to the regression by using Estimate Equation in the Quick Menu. As it is indicated in the equation specification, we enter the equation and chose the Least Square method (which should be the method set by default).

The results of our regression are shown in Table 1. We can see that the statistics are highly significant for both the constant and the explanatory variable. Therefore, we can *conclude* that the [log] EUR exchange rates can be explain by the [log] GBP exchange rates.

Table 1: Results of simple OLS regression (4)	
Variable	ln(EUR)
ln(GBP)	0.384*** (0.014)
Constant	0.053*** (0.007)
Obs.	3573
R ²	0.18
Standard errors in parentheses	
*** p<0.01, ** p<0.05, * p<0.1	

We are now going to find out if there is a possible cointegration between the two time series using the Engle-Granger approach, or in other words, test if the residuals are stationary.

Eviews help:

First, we have to create a new object in Eviews (Right click on the workfile, type of Object: Series), which we are going to name *Residuals*. Then, in command, all you need to do is type *Residuals = resid*, which will ‘copy paste’ the values in the object *resid* into the new object *Residuals*. The reason we do that is because *resid* is an object that Eviews creates by default, and therefore we cannot work directly with that object. Now that we have the values in a different object (Residuals), we are going to do a unit root ADF (Augmented Dickey-Fuller) test. To do so, double click on Residuals, click on View and select Unit Root Test.

With a t-statistic of -2.51 and a critical value (10% level) of -2.56, we cannot reject the Null hypothesis that the residuals have a unit root, hence there are not stationary (Results in Table 2).

Table 2: Results of ADF test

H_0 Hypothesis: <i>resid</i> has a unit root		t-Stat	Prob
ADF test statistic		-2.51	0.1120
Test critical values	1% level	-3.43	
	5% level	-2.86	
	10% level	-2.57	
Standard errors in parentheses			
*** p<0.01, ** p<0.05, * p<0.1			

However, we noticed from Figure 1 that a structural change occurred during the financial crisis, and a more interesting approach would be to use the Least Square with Breakpoints method that Eviews offers us in the estimation settings. When we select that method, we also have to change the maximum breaks to 1 in the options (as we can only notice 1 structural change). We can now rerun the regression, and the results are shown in Table 3.

Table 3: Results of simple OLS regression (4) with break

variable	ln(EUR)
1st Period : 12/31/2002 - 01/01/2008 (1306 obs.)	
ln(GBP)	0.966*** (0.021)
Constant	-0.360*** (0.012)
2nd Period : 01/02/2008 - 09/08/2016 (2267 obs.)	
ln(GBP)	0.888*** (0.014)
Constant	-0.144*** (0.007)
Obs.	3573
R ²	0.18
Standard errors in parentheses	
*** p<0.01, ** p<0.05, * p<0.1	

First, we can notice that the break happened in January 1st 2008 (in the beginning of the GFC), and that the coefficients are all statistically significant for the two samples. We are now going to find out if the residuals are stationary this time.

And this time, the residuals are stationary (Table 4) as the ADF statistic is significant at a 5-percent level, meaning that the two currency pairs are ‘comoving’. The cointegration beta (that we called β in regression 1) decreased from 0.965 in the first sample to 0.888 in the second sample.

Table 4: Results of ADF test

H_0 Hypothesis: <i>resid</i> has a unit root		t-Stat	Prob
ADF test statistic		-3.22**	0.02
Test critical values	1% level	-3.43	
	5% level	-2.86	
	10% level	-2.57	
Standard errors in parentheses			
*** p<0.01, ** p<0.05, * p<0.1			

We can now start working on a potential strategy, which consist in looking at the *spread* between EURUSD and GBPUSD. The statistic we are interested in is -3.22 (which is lower than the 5% critical value of -2.86), and indicates us a presence of mean reversion (i.e. deviations from ‘equilibrium value’ tend to mean revert, meaning that the EURUSD and GBPUSD cannot diverge too far away from each other). A way to trade this strategy would be to plot the residuals and define entry/exit levels on EURUSD vs. GBPUSD. We saw earlier that since the last regime change (Jan-08), we have the following parameters: $\beta = 0.888$ and $\alpha = -0.144$, therefore the time series of the residuals is define by the following formula:

$$e_t = USD/EUR_t - (\alpha + \beta USD/GBP_t) \quad (5)$$

Bloomberg Help:

We are now going to plot the residuals using the Bloomberg HS Spread Analysis function (Figure 2). On Bloomberg’s Terminal, **type EUR Curncy GBP Curncy HS** and press **GO**. As you can see below, you then need to change the two coefficients: Mult (or β) = 0.888 and Const (α) = -0.14. The chart on the bottom shows you the spread analysis (the residuals for us), and you can observe that there are usually within fluctuating within a range of [-0.15; 0.15].



Figure 2: EURUSD and GBPUSD spread analysis

How to implement the signal of the strategy?

Now that we showed that the two currency pairs are cointegrated (using Engle-Granger approach), we have to determine a Long/Short signal for our strategy. One good thing about Currency pairs' trading is that you are not that exposed to a potential bullish or bearish developing trend on a specific currency pair based on aggressive loose actions run by a central bank (Japan increasing QE size, or Fed starting a tightening cycle). By using this statistical 'arbitrage' strategy on EURUSD and GBPUSD, we are exposed to three different countries, which is better for diversification.

There are several signals we could use to build up a strategy based on those results. First of all, we need to decide if the signal is going to be static or 'stochastic'. I believe that using a time varying based on implied volatility leads to better results. You would have two moving bands:

- when the spreads (or residuals) exceed the Positive Band (i.e. 0.10), Short the spread, and
- when the spreads go below the Negative Band (i.e. -0.15), Long the spread.

When you long (resp. short) the spread, you are basically going Long (resp. short) 1 unit EURUSD and Short (resp. long) 0.88 unit of GBPUSD. Now, it is to the trader's choice to go for stochastic/time-varying or static bands.

2.1.2 USD/AUD and USD/NZD

In this subsection, we do the same exercise for the Aussie (AUD) and the Kiwi (NZD), two similar commodity-linked economies. If we first plot the two exchange rates (Figure 2), we can see that the two currencies have been also co-moving over the past decade, with a kiwi that has always been lower than the Aussie (vis-à-vis the US Dollar). We have the following equation:

$$USD/NZD_t = \alpha + \beta USD/AUD_t + e_t \quad (6)$$

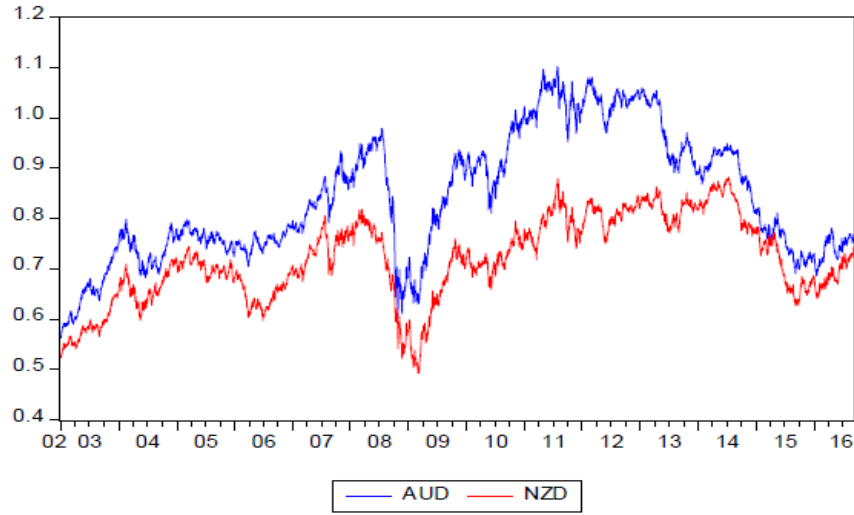


Figure 3: USD/AUD and USD/NZD historical daily prices

We go through the same steps, and we find that the residuals of the OLS method (without breaks) are also non-stationary; the ADF statistic that we get is -2.345, hence higher than the 10-percent critical value (-2.57). Despite the weakness of the Engle-Granger method with breaks, we will retest one more time to see the results on these two currencies.

If we use the LS method with Breaks (we include 1 Break as we did for *USD/EUR* and *Cable*), the statistic becomes significant at a 1-percent level (Results in Table 5 and 6)

Table 5: Results of simple OLS regression (6) with break

variable	ln(NZD)
1st Period : 12/31/2002 - 07/18/2013 (2753 obs.)	
ln(AUD)	0.695*** (0.005)
Constant	-0.229*** (0.001)
2nd Period : 07/19/2013 - 09/08/2016 (820 obs.)	
ln(AUD)	0.904*** (0.014)
Constant	-0.102*** (0.003)
Obs.	3573
R ²	0.87
Standard errors in parentheses	
*** p<0.01, ** p<0.05, * p<0.1	

Table 6: Results of ADF test

H_0 Hypothesis: <i>resid</i> has a unit root		t-Stat	Prob
ADF test statistic		-3.57***	0.006
Test critical values	1% level	-3.43	
	5% level	-2.86	
	10% level	-2.57	
Standard errors in parentheses			
*** p<0.01, ** p<0.05, * p<0.1			

This method is quite straightforward and depends on the choice of the dependent variable; its principal strength is that it is very easy to use based on observable data, however many academics have criticized this model as the results can come from spurious regression. The simulation results of Gregory et al. (1996) show the power of the Engle-Ganger test of the null hypothesis of no cointegration is dramatically reduced when there is a break in the cointegration relationship.

As we are dealing with an arbitrary amount of input series (10 different currency pairs), we are now going to introduce the Johansen Tests for Cointegration, which could be seen as the multivariate generalization of the augmented Dickey-Fuller test G. (Dwyer, 2015).

3 Johansen Tests for Cointegration

The Johansen test approaches the cointegration test by examining the number of independent linear combination for n time series. If there are n variables which all have unit roots, then there will be at most $n-1$ vectors that co-move together. In our cases, that would be at most 9 cointegrating vectors.

We first start by expressing the following vector autoregressive (VAR) model of order p :

$$y_t = C + A_1 y_{t-1} + \dots + A_p y_{t-p} + e_t \quad (7)$$

where y_t is a $n \times 1$ vector of variables that are integrated of order 1 (i.e. equivalent to $I(1)$) and e_t is an $n \times 1$ vector of white noises. By subtracting $A_1 y_{t-1} + \dots + A_p y_{t-p}$, this whole VAR can be written in the Error-Correction form as follows:

$$\Delta y_t = C + \Pi y_{t-1} + \sum_{i=1}^{p-1} (\gamma_i \Delta y_{t-i}) + e_t \quad (8)$$

where $\Pi = \sum_{i=1}^{p-1} (A_i - 1)$, and $\gamma = -\sum_{j=i+1}^p A_j$

This representation is also known as the Vector Error Correction Model (VECM) form of the VAR(p). The term Πy_{t-1} referred as the long-run part or the equilibrium correction of the model (is there any cointegration between the 10 currency pairs?). On the other hand, the other part of the equation defines the short-term adjustments and the γ_i are also called the short-term parameters.

Suppose $C=0$ (no constant or deterministic trend). Mathematically, we can say that the VAR is stable if:

$$\det(I_k - A_1 z - \dots - A_p z^p) \neq 0 \quad (9)$$

for $|z| \leq 1$.

In other words, if all the roots of the determinantal polynomial are outside the complex unit circle. Otherwise, if the polynomial has a unit root, which is to say $\det(I_k - A_1 z - \dots - A_p z^p) = 0$ for $z = 1$, the matrix Π is singular (no cointegration).

In Suppose now that the matrix Π has a rank of $rk(\Pi) = r$, where $0 < r < n$. said there are n vectors of variables, therefore we have at most $n - 1$ co-integrating relations. Then, as Helmut Lutkepohl specifies in Econometric Analysis with Vector Autoregressive Models, it is known (matrix theory) that there exist $(n \times r)$ matrices α and β with ranks $rk(\alpha) = rk(\beta) = r$, such that $rk(\Pi) = \alpha\beta'$.

Therefore, we have the following :

$$\Pi y_{t-1} = \alpha\beta' y_{t-1} \quad (10)$$

If we multiply by $(\alpha\alpha')^{-1}\alpha'$, we show that $\alpha\beta' y_{t-1}$ is $I(0)$, and there are r linearly independent cointegration relations among the components of y_{t-1} .

They are two tests conducted by the Johansen test, the trace test and the eigenvalue test, where the higher the rank (and/or the eigenvalues) the more independent cointegration relations. We are now going to use the Johansen test for our case study.

3.1 Application using commodity currencies

For this part, I decided to run a Johansen test for Cointegration on three commodity currencies – the Mexican Peso (MXN), the Norwegian Krone (NOK) and the Australian Dollar (AUD).

The results of the Johansen's maximum likelihood trace test are on Table 7. The Johansen cointegration test result shows there exists a cointegration relationship among the currencies (long term relationship). The value of Table 7 shows the first null hypothesis is refused under the significant 1-percent level (i.e. there exists a relationship between the three currencies). Indeed, those three countries are highly relying on commodity prices, therefore their currencies tend to co-move with the commodity business cycle. Over the past few years, the end of the super-cycle (shrinking demand in commodities) has dramatically impacted the three currencies.

Table 8 shows the estimated values of weight of corresponding cointegration vector.

Table 7: Johansen Cointegration Test results

Unrestricted Cointegration Rank Test
Trace

NB of CE(s)	Eigenvalue	Trade Stat.	Critical Value	Prob
None	0.013	40.80	29.80	0.002
At most 1	0.005	12.28	15.50	0.144
NB of CE(s)	0.001	2.43	3.84	0.119

Unrestricted Cointegration Rank Test
Maximum Eigenvalue

NB of CE(s)	Eigenvalue	Max Eigen Stat.	Critical Value	Prob
None	0.013	28.52	21.13	0.004
At most 1	0.005	9.85	14.26	0.222
NB of CE(s)	0.001	2.43	3.84	0.119

Standard errors in parentheses

*** p<0.01, ** p<0.05, * p<0.1

Table 8: Normalized Cointegrating Coefficient

LNOK	LMXN	LAUD
1.00	-0.89	-0.41
	(0.087)	(0.077)

4 Hurst exponent and half-life mean reversion

4.1 Hurst exponent

The Hurst exponent is a measure of persistence or long memory and could be used to define if a time series is trending or mean reverting. Its computation returns a value between 0 and 1, and the closer the value is to 0.5 the more random the time series has behaved historically (i.e. the underlying time series is a Geometric Brownian Motion).

If its value $H < 0.5$, then we tend to say that the series is mean reverting, whereas the series is trending if $H > 0.5$.

For instance, if we compute the Hurst Exponent on the EURUSD FX returns, we should get a value close to 0.5 as the FX returns should follow a random walk (Rogoff and Meese, 1983 RW model forecasts exchange rates better than

economic models).

The capital market theory is based on the assumption that security prices are martingales, which implies that the expected value of security price is the price in the previous period. Security prices follow random walks and returns from financial securities are unpredictable from past observations (weak-form Efficient Market Hypothesis).

We can measure the speed of diffusion as the variance of the series:

$$Var(\tau) = \langle |z_{t+\tau} - z_t|^2 \rangle \quad (11)$$

Where z are the log(prices), τ is time lag (arbitrary). We know by definition that the quadratic variation of the Brownian Motion is finite and equal to . In other words, we have the following:

$$\langle |z_{t+\tau} - z_t|^2 \rangle \sim \tau \quad (12)$$

However, if the time series is either mean reverting or trending, the equation above becomes:

$$\langle |z_{t+\tau} - z_t|^2 \rangle \sim \tau^{2H} \quad (13)$$

where H is the Hurst exponent and serves as an indicator of the degree to which a series trends.

If we calculate the Hurst exponent using the Hurst function developed in matlab (R. Weron, see file), we get values superior to 0.9 using the log prices, and we get values around 0.5 using the returns. This tells us that EURUSD (log) trends whereas EURUSD (returns) is a Geometric Brownian Motion.

Time series	H Exponent
EURUSD (log)	0.9374
EURUSD (returns)	0.5234

Note that trading literature is conflicted as to the usefulness of Hurst exponent, however it could be interesting to add it in your research when you are testing if the residuals are stationary. Once you discovered two cointegrated time series, include the H exponent to see if the spread usually returns to zero if it diverges too much.

For instance, popular pair trading stocks in the equity market would be:

- Royal Dutch Shell A vs. Royal Dutch Shell B (see Figure 4)
- JP Morgan vs. Goldman Sachs
- GDX vs. GLD ETF pairs



Figure 4: Spread analysis of Royal Dutch Shell Shares A versus B (Source: Bloomberg)

4.2 Half life of Mean Reversion

The Half Life of the mean reversion tells us how long it takes to prices (log) to revert to the series' mean and therefore measure the quality of stationarity. It is access using the stochastic differential equations known as Ornstein Uhlenbeck process (mean reverting process), which take the following formula:

$$dy(t) = (\lambda y(t - 1) + \mu)dt + d\epsilon \quad (14)$$

Where represents the noise (Wiener process in OU process).

Applying Ito's Lemma, we get the following analytical solution for the expected value of $y(t)$:

$$E[y(t)] = y_0 \exp(\lambda t) - \frac{\mu}{\lambda} (1 - \exp(\lambda t)) \quad (15)$$

We know that λ is negative for a mean-reverting process, and the expression above tells us that the expected value of the price decays exponentially to the value $-\frac{\mu}{\lambda}$ with the half-life of decay equals to $\frac{-\log(2)}{\lambda}$.

If we get a positive value for λ , prices do not mean revert and we should move on to a new strategies.

If we get $\lambda = 0$, then the half-life will be very long and the mean reverting (converging spreads) will be non-profitable over time. Note than Half life can also be used as a stop loss criterion; if a spread hasn't hit its exit target within the period given by the half life, we would then exit the trade. This can 'tell'

us if the relationship between two currency pairs or securities (stocks pair, yield spreads) has changed.

5 References

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