Shortest paths; Dijkstra's algorithm

CS 2860: Algorithms and Complexity

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Shortest path problems

Shortest path problems

- Ubiquitous question – What is the best/fastest/cheapest way to get from A to B?
- Natural graph interpretation
- Variants:
  - Directed or undirected?
  - Weights: No weights, positive weights, positive and negative weights?
  - Questions: From A to B? From A to everywhere? All-to-all?

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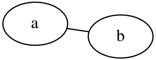

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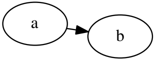

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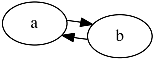

Bidirectional

- Directed edge (arc): $u \to v$
  - Can be followed in the given direction only
- Undirected edge: uv, $\{u, v\}$
  - Can be followed in either direction
- Common reduction:
  - Replace edge uv by two arcs $u \to v$, $v \to u$ – bidirectional arcs
  - No shortest path will want to use both arcs uv and vu
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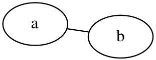

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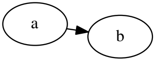

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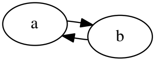

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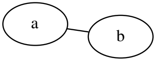

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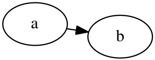

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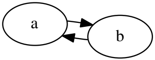

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Distinction 2: Weights type

- Weights – abstract cost notion
  - Roads: Distance, time, cost, . . .
  - Abstract graphs: e.g., time offset (see later)
- Unit weights – all edges are equal
  - We care only about the number of hops in our paths
- Non-negative weights – natural notion
  - Example: Roads long or short, but not (e.g.) minus five meters
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- One-to-one – From London to Novosibirsk
  - Usually no faster solution than single-source
  - (But see A* heuristic)
- Single-source: From London to anywhere
  - Focus today
- All-pairs shortest paths
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Single-source shortest paths

Structure of shortest paths

Shortest paths tree

For any graph G, with a source vertex s and non-negative edge lengths, shortest paths from s to all other vertices can be captured via a rooted spanning out-tree (branching).

Observations:

1. There can be several shortest paths from s to v – but we only need one.
2. Any prefix of a shortest path is a shortest paths
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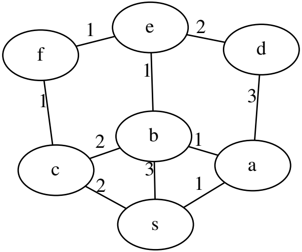

Example graph – We'll find the out-tree and distances from s afterwards, but the result is shown now

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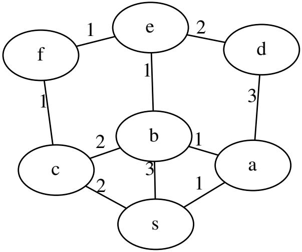

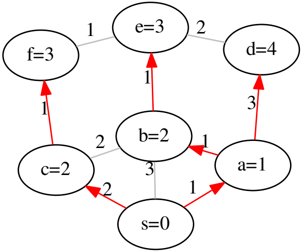

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Shortest path trees

- Representing the shortest paths efficiently (small space, quick information extraction)
- Logically we think of the representation as a tree
- Physically we can use the following:
  - Array distanceTo[n] storing the distance from s to every vertex
  - Array parentOf[n] storing the last vertex u in the shortest path from s to the vertex v
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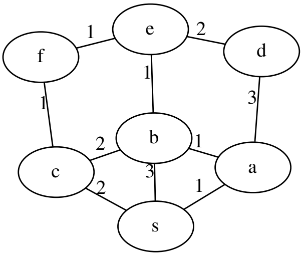

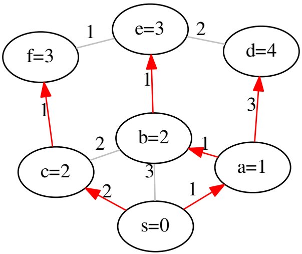

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parentOf							
distanceTo							

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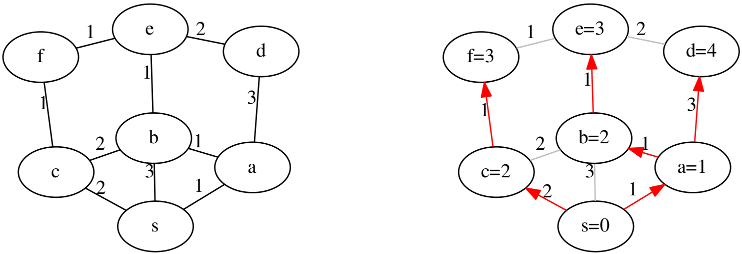

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parentOf	-	s	a	s	a	b	c
distanceTo	0	1	2	2	4	3	3

Correctness criterion for shortest paths

Correctness criterion: Non-improvement

The array distanceTo[v] encodes correct shortest single-source paths if and only if

$$\text{distanceTo[v]} \leq \text{distanceTo[u]} + \text{weight}(uv)$$

holds for every edge uv in the graph.

- Condition is clearly necessary (it must hold)
- Can show sufficiency – see course book ★

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Dijkstra's algorithm

Warm-up: No weights

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  3. Unit-weight case (no weights)
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Reminder: Breadth-first search

- Easier algorithm with unit weights only: BFS
- Initialise:
  1. All vertices unmarked
  2. Array distanceTo[n] initialised to ∞
  3. Array parentOf[n] initialised with null
- Start BFS from vertex s:
  1. Create empty Queue q
  2. Mark s, add to queue
  3. distanceTo[s]=0
  4. Until q is empty:

     4.1 Dequeue vertex u from queue

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Dijkstra: Idea and pseudocode

- Idea:
  - Modify BFS to visit closest vertices first
  - Paths with more hops can still be shorter – vertices may overtake each other in the queue
- Implementation: Three vertex types:
  1. Marked vertices (finished – path locked)
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Update step

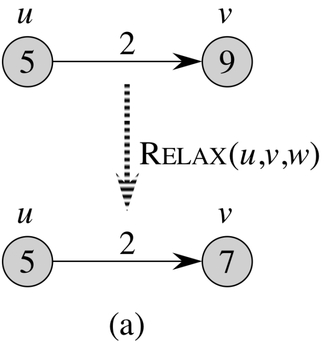

Illustration of update step (called "Relax" above)

Dijkstra: code

Very close to Advanced Prim:

- Data structures:
    1. Min-Priority Queue pq containing queued vertices
    2. parentOf[v] array encoding shortest-path tree
    3. distanceTo[v] array
- Main loop:
    1. Select initial vertex v, call pq.insert(v,0)
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            Call Update(v, uv)

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- Support code: Update(Vertex v, Edge uv):
  1. Let newDistance = distanceTo[u] + weight(uv)
  2. If *v* marked: return, do nothing
  3. If *v* not in pq:
     - 3.1 parentOf[v] = u
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  4. If newDistance < pq.currentValue(v):
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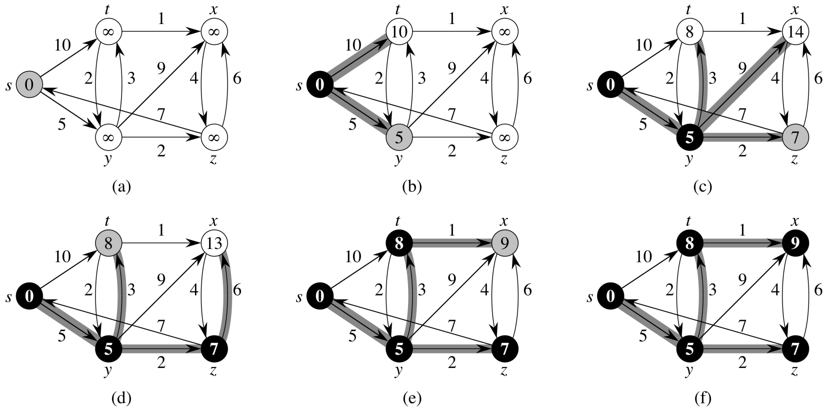
Illustration



Dijkstra in action: black=marked; gray=current; ∞=unseen
Thick edges=Best known path at the time

Dijkstra: Correctness

Suppose we added arc uv in the last iteration.
We know that the array distanceTo[v] encodes correct shortest single-source paths if and only if

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Running time of Dijkstra

Dijkstra's algorithm (as described here) has a running time of

- $\mathcal{O}(|E| \log |V|)$ if a "normal" heap+hash implementation of Min-Priority Queue+decreaseValue is used
- $\mathcal{O}(|E| + |V| \log |V|)$ if "theoretical" Fibonacci heaps are used

- The algorithm performs:
  1. $|V|$ insert operations
  2. $|V|$ deleteMin operations
  3. $|E|$ decreaseValue operations (at most)
- There are at most $|V| = n$ entries in pq
- Data structure profiles:
  - Binary heaps: $\mathcal{O}(\log n)$ for insert, deleteMin, decreaseValue
  - Fibonacci: $\mathcal{O}(\log n)$ for insert, deleteMin;
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Dijkstra: Complexity

Running time of Dijkstra

Dijkstra's algorithm (as described here) has a running time of

- $\mathcal{O}(|E| \log |V|)$ if a "normal" heap+hash implementation of Min-Priority Queue+decreaseValue is used
- $\mathcal{O}(|E| + |V| \log |V|)$ if "theoretical" Fibonacci heaps are used

- The algorithm performs:
  1. $|V|$ insert operations
  2. $|V|$ deleteMin operations
  3. $|E|$ decreaseValue operations (at most)
- There are at most $|V| = n$ entries in pq
- Data structure profiles:
  - Binary heaps: $\mathcal{O}(\log n)$ for insert, deleteMin, decreaseValue
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Dijkstra's algorithm (as described here) has a running time of

- $\mathcal{O}(|V|^2 + |E|) = \mathcal{O}(|V|^2)$ if we did not use priority queues (original version)

- Original version (1956)[1]
  - insert is mark vertex as queued, $\mathcal{O}(1)$ time
  - deleteMin is scan through all vertices, $\mathcal{O}(|V|)$ time
  - decreaseValue is reassign distanceTo array, $\mathcal{O}(1)$ time
- $(|V| \cdot \mathcal{O}(1)) + (|V| \cdot \mathcal{O}(|V|)) + (|E| \cdot \mathcal{O}(1)) = \mathcal{O}(|V|^2 + |E|)$
- Dijkstra was a strong motivation for developing fast decreaseValue operations

[1] Heapsort was published 1964, AVL trees 1962

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An example of Dijkstra's use

Go to Slide 9.

★Other applications

- Paths in directed acyclic graphs
  - Can compute in $\mathcal{O}(|E| + |V|)$ time
  - Can also find longest paths (normally difficult)
- Application: job scheduling
  - Collection of jobs with processing times and precedence
  - Nodes: Job i starts and Job i ends
  - Arcs Start(i) $\to$ End(i), weight = processing time of job
  - Arcs End(i) $\to$ Start(j), weight 0: Precedence
  - Longest path determines critical path – minimum processing time
- Can even use negative weight arc Start(i) $\to$ End(j) to say "Job j must end at the earliest X minutes before job i starts"
  - I.e., "no cooldown" – job i starts not-too-long after job j ends

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