Vertex colouring of graphs

CS 2860: Algorithms and Complexity

Magnus Wahlström and Gregory Gutin

October 16, 2017

Bipartite graphs

Bipartite Graph

A graph G is bipartite if its vertices can be partitioned into two sets, partite sets V_1 and V_2, such that every edge has one end-vertex in V_1 and the other in V_2.

In other words, a graph is bipartite if and only if its vertices can be coloured into two colours, say red and blue or 1 and 2, such that every edge has one red vertex and one blue vertex.

Bipartite Graph Theorem

A graph G is bipartite if and only if it has no cycles of odd length (3, 5, 7, etc.).

Q: Are trees bipartite graphs?
A: The above theorem implies that every tree is a bipartite graph (it has no cycles and so no odd length cycles).

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Bipartite and non-bipartite graphs

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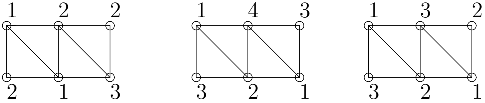

Non-bipartite graph

Q: What is the minimum number of edges that we need to delete from the graph H above to make it bipartite?
A: Deleting the two 'diagonal' edges makes it odd-cycle free. Deletion of only one edge in H will leave H with at least one cycle of length 3.

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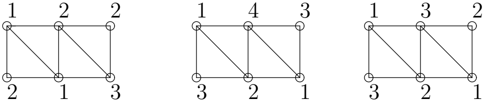

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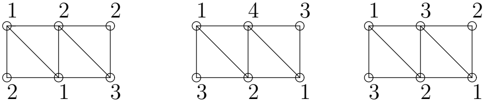

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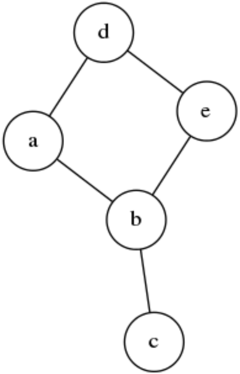
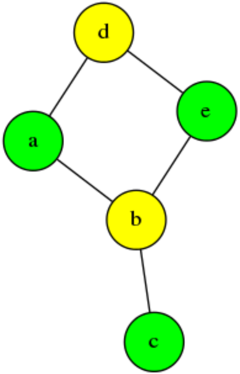
Bipartite graph recognition

Problem: Bipartite graphs

Find an algorithm that tries to colour the vertices of $G = (V, E)$ in two colours such that every edge goes between vertices of different colours. Give the time complexity of the algorithm.



Input



Output

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Find an algorithm that tries to colour the vertices of $G = (V, E)$ in two colours such that every edge goes between vertices of different colours. Give the time complexity of the algorithm.

1. Pick an uncoloured vertex v
2. Colour v with colour 1
3. For every neighbour u of v:
   3.1 If u is uncoloured, colour u opposite of v, recursively colour neighbours of u
   3.2 If u is coloured, and colour(u)=colour(v): Fail

Strategy: Propagation

Time complexity: $\mathcal{O}(|V| + |E|)$ by the sum-of-degrees theorem.

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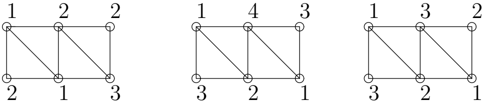
Vertex Colouring

- A vertex colouring of a graph $G = (V, E)$ is an assignment that assigns a colour to every vertex of G.
- A proper vertex colouring assigns different colours to end-vertices of all edges.



- A graph is said to be k-colourable if it has a proper vertex colouring with at most k colours.
- The chromatic number of a graph G, $\chi(G)$, is the minimum k such that G is k-colourable.

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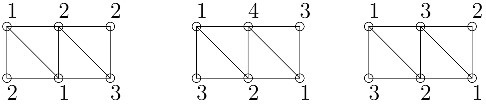

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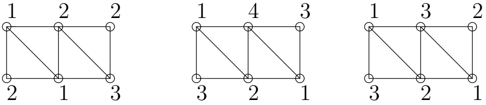

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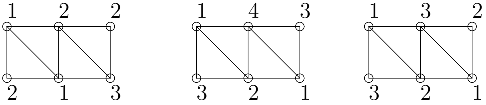

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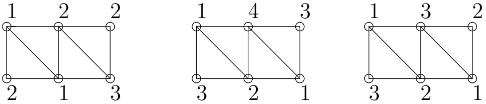

To see that the chromatic number of the above graph is 3, it is enough to observe that H is vertex 3-colourable and H has no proper vertex 2-colouring as H contains (as a subgraph) K_3.

A complete graph K_n on n vertices has vertices adjacent to each other. So $\chi(K_n) = n$.

Qs: Is there an efficient algorithm to decide whether a graph is 3-colourable? Is there an efficient algorithm to compute the chromatic number of a graph?

A: Both are highly unlikely. See the remaining lectures.

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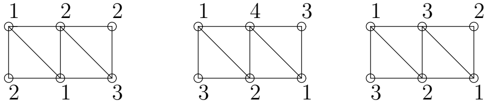

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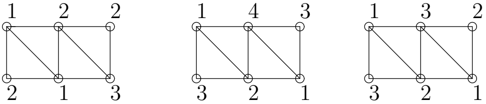

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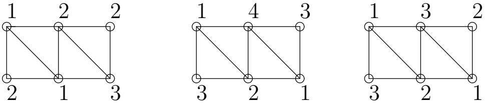

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