# Minimum Spanning Trees

CS 2860: Algorithms and Complexity

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- Input: Graph $G = (V, E)$, edge weights/costs $w(e)$
- Spanning tree: A tree $T = (V, F)$ with edges from $E$ that connects every vertex of $G$
- Minimum spanning tree: A spanning tree $T$ of minimum total cost

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- Historical example: Electrifying Moravia
  - Region of present Czech republic
  - Build power lines to connect villages to electric grid
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  - Borůvka's algorithm, 1926
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# Properties of trees

## Definition

A tree is a connected graph with no cycles

We could also have chosen any of the following (equivalent statements):

- A tree is a connected graph with $n - 1$ edges
- A tree is a graph with $n - 1$ edges and no cycles
- A tree is a graph with a unique path between any two vertices
- A tree is a connected graph such that removing any edge leaves a disconnected graph

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# Properties of Spanning Trees

## Two properties of spanning trees

Let $T$ be a spanning tree in a graph $G = (V, E)$.

- Cycle property:
  1. Adding any edge $uv$ to $T$ creates a cycle $C$
  2. Removing any edge $u'v'$ from $C$ creates a new spanning tree
- Cut property:
  1. Removing any edge $uv$ from $T$ disconnects $T$ into two parts
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Proof (both cases): The new tree is a cycle-free graph with $n - 1$ edges

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# Min-cost spanning trees

## Min-cost spanning trees: Cycle rule

Let $G = (V, E)$ be a graph with edge weights, $T$ a spanning tree in $G$. Then $T$ is min-cost if and only if the following holds:

- Let $uv \in E$ be any edge not used in $T$
- Let $C$ be the cycle created by adding $uv$ to $T$
- Then $uv$ is the most expensive edge of $C$
  (or tied for most expensive, if there are ties)

Why?

- Suppose not. Let $uv$ be an edge not used in $T$, that is not the most expensive in its cycle $C$ in $T + uv$
- Create a tree $T'$ by adding $uv$, then deleting the most expensive edge $u'v'$ from $C$.
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[1] This was necessity. To show sufficiency is a little bit harder.

# Min-cost spanning trees: Option II

## Min-cost spanning trees: Cut rule

Let $G = (V, E)$ be a graph with edge weights, $T$ a spanning tree in $G$. Then $T$ is min-cost if and only if the following holds:

- For any split of $V$ into two parts $V = A \cup B$,
  $T$ contains a cheapest edge between $A$ and $B$.

Proof: Same as cycle rule (if the statement were false, you could create a cheaper tree from $T$ as $T - uv + u'v'$).

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- Prim's algorithm (the one you know):
  1. Begin with tree $T$ of single vertex $v$, no edges
  2. Repeat until $T$ is a spanning tree:
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  1. Begin with $T = \emptyset$
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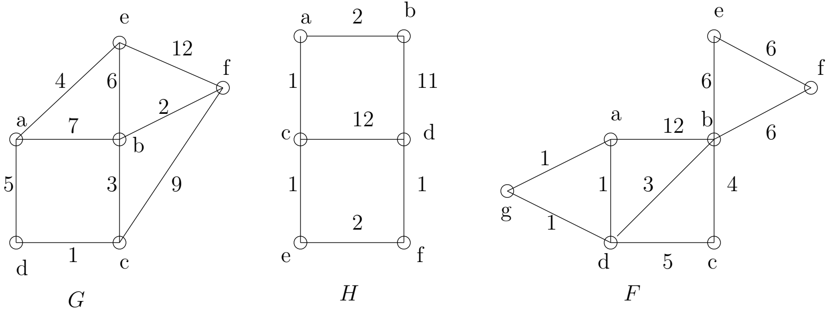
- Issues:
  - Algorithm workings
  - Correctness
  - Efficient implementation

# Using the algorithms

(1) Run Prim's algorithm (start from vertex *a*) and Kruskal's algorithm on graph *G* below. (2) Using Kruskal's algorithm, determine how many minimum spanning trees (MSTs) graph *F* has. (3) How many MSTs are in the graph $H'$ obtained from *H* by replacing cost 11 with 1 and 12 with 2?



# Using the algorithms: Solutions

1. Graph $G$ has a minimum spanning tree (MST) with the edges of costs 1, 2, 3, 4, 5.
2. Since in any MSP of $F$ only 2 edges of the 3 edges of cost 1 can be used and the same for edges of cost 6, $F$ has $3 \times 3 = 9$ MSTs (edges of cost 3 and 4 must be in each MST).
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# Spanning trees: Prim's algorithm

# Prim's algorithm, outline

Input: Graph $G = (V, E)$, edge weights $w$

- Start with a tree $T$ of a single vertex $v$, no edges
- Repeat until complete:
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- Discover/Extend central loop (Priority Queue)

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# First implementation

- Data structures:
  1. Vertices marked/unmarked, initially all unmarked
  2. Priority Queue pq of edges, initially empty
  3. The tree T
- Use a "visit" subroutine process(v):
  1. Mark $v$ as visited
  2. For every unmarked neighbour $u$ of $v$:
     2.1 Add edge $uv$ to pq with weight $w(uv)$
- Code, main loop:
  1. Select initial vertex $v$, process(v)
  2. Until pq is empty:
     2.1 e=pq.deleteMin() // pull cheapest edge $e$ from pq
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# First version, running time

- Our code performs:
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- Total time[2] $\mathcal{O}(|E| \log |E|)$
- Correctness proof (skipped):
  - Can show that the tree constructed follows cut rule

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# Advanced implementation

- The code adds edges to pq, but the algorithm needs to discover vertices
- This leads to a minor inefficiency (in space and time)
- Using an advanced priority queue (with decreaseKey method) we can write the code using a priority queue of vertices for space $\mathcal{O}(|V|)$, time $\mathcal{O}(|E| + |V| \log |V|)$ (theoretical)
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# Kruskal: Correctness

**Correctness**

Kruskal's algorithm creates a minimum spanning tree

- It clearly creates a spanning tree
- Min-cost via the cycle rule:
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# Implementation

1. Sort the edges
   - No problem
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Completes a cycle has two implementations:

- A slow one:
  - Search through the graph formed by $T$, starting from $u$, to see if we find $v$ (e.g., DFS from $u$ in $T$)
  - Up to $\mathcal{O}(n)$ time for a single check
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# Fast implementation

We need a data structure for connected components:

- Find(v) – return a name for the connected component of $T$ that contains $v$
- Union(u,v) – "merge" the components of $u$ and $v$ into one (by adding $uv$ to $T$)

Efficient implementation using this:

1. Sort the edges $E$ (e.g., merge sort) at $\mathcal{O}(|E| \log |E|)$ time
2. Initiate your Union-Find structure to "empty"
3. For every edge $uv$ in sorted order:
   3.1 If Find(u) == Find(v): ignore edge
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# $\star$ Prim: advanced implementation

# "Advanced PQ" interface

- In addition to the usual min-Priority Queue operations:
  - insert(item, weight): add item with value weight
  - minItem(), deleteMin(): Return min-weight item
- We will need a Priority Queue that also supports the following:
  1. contains(item): membership test
  2. currentValue(item): return current item priority/weight
  3. decreaseValue(item, newWeight): decrease weight of item to newWeight
- Implementations:
  1. Heap+Hash table with $\mathcal{O}(\log n)$ time per decreaseKey
  2. Fibonacci heaps: Advanced structure, not yet practical
     - Impractical: Pointer jumping, complex code
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# "Advanced PQ" interface

- In addition to the usual min-Priority Queue operations:
  - insert(item, weight): add item with value weight
  - minItem(), deleteMin(): Return min-weight item
- We will need a Priority Queue that also supports the following:
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  2. currentValue(item): return current item priority/weight
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- Implementations:
  1. Heap+Hash table with $\mathcal{O}(\log n)$ time per decreaseKey
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# Advanced Prim implementation: Outline

- Data structures:
  1. Min-Priority Queue pq containing vertices
  2. edgeTo[v] array codes edges, will encode tree
  3. Vertices marked/unmarked as before
- Principles:
  - pq contains candidate vertex to add to tree
  - Priority (weight) of vertex v in pq is cheapest known edge to v from tree
  - When we pull v from pq ("discover" step), we update neighbours u of v if a cheaper edge to u was found
  - If a cheaper edge was found, we call decreaseKey to *v* instead of a full insert

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# Full code

- Data structures:
  1. Min-Priority Queue pq containing vertices
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- Main loop:
  1. Select initial vertex $v$, call pq.insert(v,0)
  2. Until pq is empty:
     - 2.1 u=pq.deleteMin(); mark u
     - 2.2 For every neighbour $v$ of $u$:
       Call Update(v, w(uv))
- Support code: Update(Vertex v, Edge uv):
  1. If $v$ marked: return, do nothing
  2. If $v$ not in pq:
     - 2.1 edgeTo[v] = uv
     - 2.2 pq.insert(v, w(uv))
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# Advanced version: running time

- Our new implementation performs:
  1. $\mathcal{O}(|V|)$ insertions into pq
  2. $\mathcal{O}(|E|)$ update, decreaseKey calls
  3. $\mathcal{O}(|V|)$ deletions from pq
- With our familiar implementation:
  1. All operations take $\mathcal{O}(\log|V|)$ time
  2. Worst-case $\mathcal{O}(|E|\log|V|) = \mathcal{O}(|E|\log|E|)$ as before
  3. New best case: Zero decreaseKey calls, $\mathcal{O}(|E| + |V|\log|V|)$ time (maybe even just $\mathcal{O}(|E|)$?)
- With Fibonacci heaps (theoretical result):
  1. Insert/delete amounts to $\mathcal{O}(|V|\log|V|)$ time
  2. $\mathcal{O}(|E|)$ times decreaseKey takes $\mathcal{O}(|E|)$ total time
  3. In total worst-case time $\mathcal{O}(|E| + |V|\log|V|)$

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