MTH 309 MIDTERM 2 APRIL 12TH

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> Please show **all** your work! Answers without supporting work will not be given credit.
> Clearly label your problems on separate paper.

62/80

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1. (28 points) Given the matrices A, B, and C:

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ -4 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 4 & 1 & 2 \\ 3 & 2 & -2 & 3 \\ -1 & 0 & 3 & 3 \\ 2 & 1 & 2 & -2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -1 & 1 & 2 & 7 \\ 0 & 3 & 1 & -3 & 6 \\ 0 & 0 & 4 & 2 & 1 \\ 0 & 0 & 0 & -2 & 11 \\ 0 & 0 & 0 & 0 & 5 \end{bmatrix}$$

Compute the following determinants.
a) $\det(A)$ b) $\det(B)$ c) $\det(C)$ d) $\det(A^2)$ e) $\det(2B)$ f) $\det(A) + \det(C)$

det(A) =

$1\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} - 2\begin{bmatrix} 2 & 2 \\ -4 & 2 \end{bmatrix} + 1\begin{bmatrix} 2 & 3 \\ -4 & 1 \end{bmatrix}$

$(6-2) - 2(4+8) + (2+12)$

$4 - \cancel{28} 8 - 16 + 2 + 12$

det(A) = -6

12

2. (16 points) Consider the following vectors in $\mathbb{R}^3$:

$$\vec{b}_1 = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}, \quad \vec{b}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \quad \vec{b}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

The set $\beta = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\}$ is a basis for $\mathbb{R}^3$.

(a) Compute $[\vec{u}]_\beta$, the coordinate vector of $\vec{u}$ relative to the basis β.

(b) Let $\vec{w} \in \mathbb{R}^3$ be a vector such that $[\vec{w}]_\beta = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$. Compute the vector $\vec{w}$.

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3. (16 points) Decide which of the following sets of vectors are subspaces of $\mathbb{R}^3$. Justify your answers.

(a) The set V_2 consisting of all vectors $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ such that $a_1 \geq a_2$.

(b) The set V_3 consisting of all vectors $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ such that $a_1^2 + a_2^2 + a_3^2 = 9$.

(c) The set V_1 consisting of all vectors $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ such that $a_1 = a_3 = 0$.

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4. (20 points) For each of the statements given below decide if it is true or false. If it is true explain why. If it is false explain why or give a counterexample.

(a) If a square matrix A has the property that $A^T A = I$, then the only possibility is $\det(A)=1$.

(b) No set of 4 vectors spans $\mathbb{R}^5$.

(c) If $\beta = \{\vec{b}_1, \vec{b}_2, \cdots \vec{b}_n\}$ is a basis for a vector space V, then the equation $c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + \vec{b}_n = \vec{0}$ has an infinite number of solutions.

(d) The set of vectors $\{x + 1, x^2 + 2, -1 - x\}$ from $\mathbb{P}_2$ is a linearly dependent set.

Wang

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 3 & 2 \\ -4 & 1 & 2 \end{bmatrix}$$

$$\det(A) = 1\begin{bmatrix} 3 & 2 \\ 1 & 2 \end{bmatrix} - 2\begin{bmatrix} 2 & 2 \\ -4 & 2 \end{bmatrix} + 1\begin{bmatrix} 2 & 3 \\ -4 & 1 \end{bmatrix}$$

$$6-2-2(4+8)+(2+12)$$

$$6-2-8-16+2+12$$

$$6-2-8-16+14$$

a) $\det(A) = -6$

$$6-2-8-2$$

$$4-8-2 = -4+2 = -6$$

$$B = \begin{bmatrix} 1 & 4 & 1 & 2 \\ 3 & 2 & -2 & 3 \\ -1 & 0 & 3 & 3 \\ 2 & 1 & 2 & -2 \end{bmatrix}$$

$$\det(B) = -1 \cdot C_{31} + \cancel{0 \cdot C_{32}} + 3 \cdot C_{33} + 3 \cdot C_{34}$$

$$-1(-1)^4\begin{bmatrix} 4 & 1 & 2 \\ 2 & -2 & 3 \\ 1 & 2 & -2 \end{bmatrix} + 3(-1)^6\begin{bmatrix} 1 & 4 & 2 \\ 3 & 2 & 3 \\ 2 & 1 & -2 \end{bmatrix} + 3(-1)^7\begin{bmatrix} 1 & 4 & 1 \\ 3 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix}$$

$$4\begin{bmatrix} -2 & 3 \\ 2 & -2 \end{bmatrix} - 1\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix} + 2\begin{bmatrix} 2 & -2 \\ 1 & 2 \end{bmatrix}$$

Continue on back

$$4\begin{bmatrix} 2 & 3 \\ 2 & -2 \end{bmatrix} - 1\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix} + 2\begin{bmatrix} 2 & -2 \\ 1 & 2 \end{bmatrix}$$

$$4(4-6) - 1(-4-3) + 2(4-2)$$

$$16 - 24 \cancel{+4} + 3 + 8 \cancel{-4}$$

$$= \cancel{-8} + 3 + \cancel{8} = 3$$

$$-1 \cdot 3 = -3$$

$$3(-1)^6 \begin{bmatrix} 1 & 4 & 2 \\ 3 & 2 & 3 \\ 2 & 1 & -2 \end{bmatrix}$$

$$1\begin{bmatrix} 2 & 3 \\ 1 & -2 \end{bmatrix} - 4\begin{bmatrix} 3 & 3 \\ 2 & -2 \end{bmatrix} + 2\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$$

$$-4 - 3 - 4(-6 - 6) + 2(3 - 4)$$

$$-4 - 3 + 24 + 24 + 6 - 8$$

$$(1)$$

$$-4 - 3 + 54 - 8$$

$$-4 + 51 - 8 = 47 - 8 = 39$$

$$3 \cdot 39 = 117$$

$$3(-1)^7 \begin{bmatrix} 1 & 4 & 1 \\ 3 & 2 & -2 \\ 2 & 1 & 2 \end{bmatrix}$$

$$1\begin{bmatrix} 2 & -2 \\ 1 & 2 \end{bmatrix} - 4\begin{bmatrix} 3 & -2 \\ 2 & 2 \end{bmatrix} + 1\begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}$$

$$4 + 2 - 4(6 + 4) + 1(3 - 4)$$

$$6 - 24 - 16 + 3 - 4$$

$$-3 \cdot -35 = 105$$

$$\det(B) = -3 + 117 + 105 = 219$$

-1

$\det(C) = 1 \cdot 3 \cdot 4 \cdot -2 \cdot 5 = -120$

d) $\det(A^2) = \det(A)^2$

$\det(A) = -6$

$-6^2 = 36$

e) $\det(2B) = 219 \cdot 2 = 438$

-2

f) $\det(A) + \det(C) = -6 + -120 = -126$

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2) $[\vec{u}]_\beta = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}$, $\vec{u} = c_1 \vec{b}_1 + c_2 \vec{b}_2 + c_3 \vec{b}_3$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 4 & -1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - 4R_1} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & -5 & 4 & -4 \\ 0 & -1 & 2 & -1 \end{array}\right]$$

$-1-4(1)$
$0-4(-1)$
$0-4(1)$

$\frac{10}{5} - \frac{4}{5} = \frac{6}{5}$

$-\frac{1}{5} + \frac{4}{5}$

$$\xrightarrow{-\frac{1}{5}R_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & -4/5 & 4/5 \\ 0 & -1 & 2 & -1 \end{array}\right] \xrightarrow{R_3 + R_2} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & -4/5 & 4/5 \\ 0 & 0 & 6/5 & -1/5 \end{array}\right] \xrightarrow{\frac{5}{6}R_3} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & -4/5 & 4/5 \\ 0 & 0 & 1 & \frac{-5}{30} \end{array}\right]$$

$\frac{4}{5} + \frac{4}{5}\left(\frac{-5}{30}\right)$

$\frac{4}{5} + \frac{-20}{150} \Rightarrow \frac{120}{150} - \frac{20}{150} = \frac{100}{150}$

$$\xrightarrow{R_2 + \cancel{8}\,4/5 R_3} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & \frac{10}{15} \\ 0 & 0 & 1 & -5/30 \end{array}\right]$$

$\boxed{\text{Continue on Back}}$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 1 \\ 0 & 1 & 0 & 10/15 \\ 0 & 0 & 1 & -5/30 \end{array}\right] \xrightarrow[\cancel{R_1 - R_2}]{R_1 + R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 25/30 \\ 0 & 1 & 0 & 10/15 \\ 0 & 0 & 1 & -5/30 \end{array}\right]$$

$$\frac{25}{30} - \frac{20}{30} = \frac{5}{30}$$

$$\xrightarrow{R_1 - R_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 5/30 \\ 0 & 1 & 0 & 10/15 \\ 0 & 0 & 1 & -5/30 \end{array}\right]$$

$$\frac{5}{30}\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{5}{30} \\ \frac{20}{30} \\ \frac{5}{30} \end{bmatrix} \qquad \frac{10}{15}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{10}{15} \\ \frac{-10}{15} \\ 0 \end{bmatrix}$$

$$-\frac{5}{30}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/30 \\ 0 \\ -5/30 \end{bmatrix}$$

Right idea but something goes wrong!

-4

a) $$[\vec{u}]_\beta = \begin{bmatrix} 5/30 \\ 20/30 \\ 5/30 \end{bmatrix} \begin{bmatrix} 10/15 \\ -10/15 \\ 0 \end{bmatrix} \begin{bmatrix} 5/30 \\ 0 \\ -5/30 \end{bmatrix} = \begin{bmatrix} \frac{30}{30} \\ 0 \\ 0 \end{bmatrix}$$

b) $$[\vec{w}]_\beta = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$$

+

$$2\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 2 \end{bmatrix}$$

$$1\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$-3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix}$$

$$\vec{w} = \begin{bmatrix} 6 \\ 7 \\ -1 \end{bmatrix}$$

- $\vec{0} \in V_2$
- Closed by addition
  $\vec{u}, \vec{v}, \vec{n} \in V_2$, $\vec{u}+\vec{v}+\vec{n} \in V_2$
- Closed by scalar multiplication
  $\vec{u} \in V_2$, $c\vec{u} \in V_2$

a)

$0 \geq 0$ ✓ $\quad -1\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$

$0+0+0 \in V_2$

$3+2+2 \in V_2$ ✓

$V_2 \not\subseteq \mathbb{R}^3$ (not in) because if we let $a_1 = 3$ and $a_2 = -2$ and multiply it by the scalar -1. It will make $a_2 > a_1$.

b) $\{a_1^2 + a_2^2 + a_3^2 = 9\}$ only if ~~there's~~ there's two ~~the~~ a_n ~~=~~ $= 3$ or -3

~~OR if one $a = 9$~~

$V_3 \not\subseteq \mathbb{R}^3$ (not in) because we can't use $\vec{0}$ in V_3 ~~$a_1^2 + a_2^2 + a_3^2$ is linearly independent~~ has to equal 0.

$a_1^2 + a_2^2 + a_3^2 = 0$

-6

c) $V_1 \not\subseteq \mathbb{R}^3$ (not in) ~~because~~ unless $a_2 = 0$, if $a_2 = \{$Any $\mathbb{R}$ that isn't $0\}$ then it won't forfill $\vec{0} \in V_2$

all vectors in $\begin{bmatrix} 0 \\ a_2 \\ 0 \end{bmatrix}$, so can have $\vec{0}$.

4) a) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \to \det = 1-0 = 1$ Test cases

$1 \cdot 1 = 1$

-2

True, because $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\det(A) = 1$

and $A^T \cdot A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \det = 1$

try $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$

$\det = -1$.

$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \det = 1$

b) $\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \in \mathbb{R}^5$

True because $\mathbb{R}^5$ can be span by 5 vectors.

c) If $\beta = \{\vec{b_1}, \vec{b_2} \dots \vec{b_n}\}$, then equation $c_1\vec{b_1} + c_2\vec{b_2} + \dots c\vec{b_n} = \vec{0}$ has infinite number of solutions

False because the equation equals $\vec{0}$ which means it's linearly independent (No infinite number of solutions)

d) $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & 2 & -1 \end{bmatrix} \xrightarrow{\text{swap } R_1, R_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 2 & -1 \end{bmatrix} \xrightarrow{R_3 - R_1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & -2 & 0 \end{bmatrix}$

no pivot free variable

-3

No! $-1+1$

~~True~~ False, because there's no solution (free variable in column 3). It cannot be linearly dependent.