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MTH420
PS 4

(1) formal power series over a field $\mathbb{F}$ is series of form:

$$f(x) = \sum_{n=0}^{\infty} a_n x^n, \quad a_n \in \mathbb{F}$$

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So $f(x)g(x) = \left(\sum_{n=0}^{\infty} a_n x^n\right)\left(\sum_{m=0}^{\infty} b_m x^m\right) = \sum_{k=0}^{\infty}\left(\sum_{i=0}^{k} a_i b_{k-i}\right)x^k$

here, you use notation you haven't set up yet

let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$ and $f(x)g(x)=0$

here you set up notation, It's more clear if the set up is before the use.

This means that $\forall k \geq 0, \sum_{i=0}^{k} a_i b_{k-i} = 0$

and $a_n = 0$ $(f \equiv 0)$ or $b_n = 0$ $(g \equiv 0)$

unclear. Knowing the sum is zero doesn't tell us all, or any, of the terms is/are 0.

$1+2+(-5)+(-4)+6 = 0$

all nonzero

- Suppose $f \neq 0$ and $g \neq 0$, this means that $a_n \neq 0$ and the smallest index n such that $b_n \neq 0$. Let $k = m+n$, then

$$\sum_{i=0}^{k} a_i b_{k-i} = 0$$

← Defs of m & n are not made clearly.

the terms $a_m b_n$ appears when $i=m$, for all other i, either $a_i = 0$ for $i<m$ or $b_{k-i}=0$ for $k-i<n$ because $a_i=0$ for $i<m$ and $b_j=0$ for $j<n$ by minimality.

So the only non zero term in the sum is $a_m b_n \neq 0$

Thus $\sum_{i=0}^{k} a_i b_{k-i} = a_m b_n \neq 0$ contradiction

Therefore, if $fg=0$, then either $f \equiv 0$ or $g \equiv 0$, which proves $\mathbb{F}[[x]]$ is an integral domain.

(2) Given field $\mathbb{F}$ and $a, b \in \mathbb{F}$ with $a \neq 0$, let $S = \{(ax+b)^n$

show S is a basis of $\mathbb{F}[x]$, the ring (the $\mathbb{F}$-vector space) of all polynomials over $\mathbb{F}$.

Show S is linear independent over $\mathbb{F}$

Suppose that
$$\sum_{k=0}^{n} c_k (ax+b)^k = 0,$$

we must prove $c_k = 0$ for all k.

Proof of linear independence seems to trail off before it's finished.

let $y = ax+b \Rightarrow x = \frac{y-b}{a}$

$(ax+b)^k = y^k$

the linear combination becomes: $\sum_{k=0}^{n} c_k y^k = 0$ ← Ok, ...

??

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Show spanning $\mathbb{F}[x]$

let $f(x) \in \mathbb{F}[x]$, make substitution $y = ax+b$ or $x = \frac{y-b}{a}$

$f(x) = f\left(\frac{y-b}{a}\right) = g(y)$ ← I guess this is intended to define g!

$g(x) = \sum_{k=0}^{n} c_k y^k = \sum_{k=0}^{n} c_k (ax+b)^k$ thus $f(x) = \sum_{k=0}^{n} c_k (ax+b)^k$

The polynomial $f(x)$ can be written as a linear combination of the the elements in S, so they span $\mathbb{F}[X]$

$\boxed{\text{Therefore, } \{(ax+b)^n \mid n \in \mathbb{Z}_{\geq 0}\} \text{ is a basis of } \mathbb{F}[X]}$

(3) Given matrix
$$A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$$

(a) Find p_1, p_2, p_3 such that $p_i(j) = \delta_{ij}$

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matrix is upper triangular (special case of Jordan matrix),
the eigenvalues of a matrix comes from the roots of

$\det(A-\lambda I)=0$

$diag(A) = [2,2,1,3,3]$ and the eigenvalues of A is
$\lambda=2, \lambda=1, \lambda=3$

<u>construct interpolation polynomials:</u> $p_i(\lambda_j)=\delta_{ij}$

$p_1(x)$: 1 at $x=1$, 0 at $x=2, x=3$

$p_2(x)$: 1 at $x=2$, 0 at $x=1, x=3$

$p_3(x)$: 1 at $x=3$; 0 at $x=1, x=2$

$$p_1(x) = \frac{(x-2)(x-3)}{(1-2)(1-3)} = \frac{(x-2)(x-3)}{(-1)(-2)} = \frac{(x-2)(x-3)}{2}$$

$$p_2(x) = \frac{(x-1)(x-3)}{(2-1)(2-3)} = \frac{(x-1)(x-3)}{(1)(-1)} = -(x-1)(x-3)$$

$$p_3(x) = \frac{(x-1)(x-2)}{(3-1)(3-2)} = \frac{(x-1)(x-2)}{(2)(1)} = \frac{(x-1)(x-2)}{2}$$

(b) <u>compute $p_i(A)$ for each i</u>

let $I = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$A-2I$ plug into P_1
$A-3I$ P_2
$A-I$ P_3

$$P_1(A) = \frac{(A-2I)(A-3I)}{2}$$
$$P_2(A) = -(A-I)(A-3I)$$
$$P_3(A) = \frac{(A-I)(A-2I)}{2}$$

$$A-2I = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad A-I = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$A-3I = \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P_1(A) = \frac{(A-2I)(A-3I)}{2}, \quad \text{let } M_1 = A-2I,\ M_2 = A-3I$$

$$M_1 M_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Rightarrow \frac{1}{2} \begin{bmatrix} 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P_2(A) = -(A-I)(A-3I),$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$P_3(A) = \frac{(A-I)(A-2I)}{2} \qquad \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

$$\Rightarrow \frac{1}{2} \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

show if $L \neq 0$ that $\exists t \in F$ such $L(f) = f(t)$ for all $f \in F[x]$

case: $L = 0$ this is one of the outcomes

case: $L \neq 0$ let $x \in F[x]$ be the variable, then define

$t = L(x) \in F \Rightarrow L(f) = f(t)$ for all $f \in F[x]$

Prove $L(f) = f(t)$ for all monomials

let $f(x) = x^n$ (Proof by induction)

$L(x^n) = t^n$ for all $n \geq 0$

Base case: $n = 0$, $x^0 = 1$, therefore $L(1) = t^0$

Inductive step: let $L(x^k) = t^k$, then

$$L(x^{k+1}) = L(x \cdot x^k) = L(x) \cdot L(x^k) = t \cdot t^k = t^{k+1}$$

Therefore $L(x^n) = t^n$ for all $n \in \mathbb{N}$

let $f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \in F[x]$

L is linear so: $L(f) = a_0 L(1) + a_1 L(x) + \ldots + a_n L(x^n) = a_0 + a_1 t + \ldots + a_n t^n = f(t)$

$\Rightarrow L(f) = f(t) = e_t(f)$

Therefore, every nonzero F-algebra homomorphism $L: F[x] \to F$ is of form: $L(f) = f(t)$ for some $t \in F$

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Good

(5) The subset $I \subseteq \mathbb{Q}[x]$ is an ideal if $0 \in I$, if $f, g \in I$ then $f+g \in I$ (closed under addition) and $f \in I$, and $h \in \mathbb{Q}[x]$, then $hf \in I$

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(a) Not ideal because if we let $f(x) = x^2 \in I$, however $x \cdot f(x) = x^3$ has odd degree (so it's not in the set), it's not closed under multiplication.

+5

(b) Not ideal because it's not closed under addition. let $f = x^6$ and $g = -x^{6} + x^4 \in \mathbb{Q}[x]$, then $f+g = x^4$ and that has degree 4 (less than 5).

+5

(c) Not ideal because it's not closed under multiplication. let $f(x) = x^3 \in I$, but $x^3 \cdot x^3 = x^6 \notin I$.

+5

(d) Ideal because this is the kernel of the evaluation homomorphism $\varepsilon_1 : \mathbb{Q}[x] \to \mathbb{Q}$, $f \mapsto f(1)$ and kernels of ring homomorphisms are always ideals.

+5

(e) Not ideal because $f(0) = 1$ (this means that it doesn't contain 0), therefore violates $0 \in I$ condition.

+5

(f) $f(0) = f(1) = 0$

Ideal, this is intersection of ideals (x) and $(x-1)$ $(x) \cap (x-1)$

The set of polynomials divisible by both x and $x-1$.

+5

(g) Not ideal because let $f(x) = 1 \in I$, if we multiply $x \cdot f(x) = x$ and $x(0) = 0 \neq x(1) = 1$ (Not in the set).

+5

(h) $+5$ $\mathbb{Q}[x] \to \mathbb{Q}[x]$, $f(x) \mapsto \int_0^x f(t)\,dt$ the image is set of polynomials with zero constant term,

The ideal (x) in $\mathbb{Q}[x]$, which is ideal

(i) Ideal b/c it's the kernel of ring homomorphism, $\phi : \mathbb{Q}[x] \to \mathbb{Q}^{2\times 2}$, $f \mapsto f($

The kernels of ring homomorphisms are always ideals.