MTH420 MIDTERM NAME: Ivan Wang

(1) Let $V = \{a_0 + a_1x + a_2x^2 : a_0, a_1, a_2 \in \mathbb{R}\} \leq \mathbb{R}[x]$. Let $\mathcal{B} = ((x-1)^2, x^2, x^2-1)$, which is an ordered basis of V, and let $\alpha = (x-1)^2 \in V$. Find $[\alpha]_{\mathcal{B}}$.

$(x-1)^2 = (x-1)(x-1) = x^2 - x - x + 1 = x^2 - 2x + 1$

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$a_2 = 1,\ a_1 = -2,\ a_0 = 1$

x^2 $\quad a_2 = 1,\ a_1 = 0,\ a_0 = 0$

$x^2 - 1$ $\quad a_2 = 1,\ a_1 = 0,\ a_0 = -1$

$$\begin{bmatrix} 1 & -2 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

$$[\alpha]_{\mathcal{B}} = \begin{bmatrix} 1 & -2 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

No.... $[\alpha]_B$ is a column vector.

(2) Let $V = \left\{ \begin{bmatrix} x & y \\ z & -x \end{bmatrix} : x, y, z \in \mathbb{R} \right\}$.

(a) Prove that V is a subspace of $\mathbb{R}^{2\times 2}$.

In order to be a subspace of $\mathbb{R}^{2\times 2}$ it must meet these conditions $(0, +, \cdot)$

Contains zero matrix

Closed under addition

Closed under scalar multiplication

Zero matrix: let $x, y, z = 0$

$0 \in \mathbb{R}$

so $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is contained

Condition satisfied

Closed under addition: let $A = \begin{bmatrix} x_1 & y_1 \\ z_1 & -x_1 \end{bmatrix}$, $B = \begin{bmatrix} x_2 & y_2 \\ z_2 & -x_2 \end{bmatrix}$

$A+B = \begin{bmatrix} x_1+x_2 & y_1+y_2 \\ z_1+z_2 & -(x_1+x_2) \end{bmatrix}$ where $x_1, x_2, \cancel{y_1}, y_2, z_1, z_2 \in \mathbb{R}$

Therefore, V is closed under addition.

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Closed under scalar multiplication: let c be a scalar.

$cx, cy, cz \in \mathbb{R}$

$c\left(\begin{bmatrix} x & y \\ z & -x \end{bmatrix}\right) = \begin{bmatrix} cx & cy \\ cz & -cx \end{bmatrix}$

Therefore, the set is closed under scalar multiplication

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Therefore, all the conditions have been meet, so $V = \left\{ \begin{bmatrix} x & y \\ z & -x \end{bmatrix} : x, y, z \in \mathbb{R} \right\}$ is a subspace of $\mathbb{R}^{2\times 2}$

(b) Let $\mathcal{B} = \left(\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\right)$, which is an ordered basis for V, and define

$$T\left(\begin{bmatrix}x & y\\ y & z\end{bmatrix}\right) = \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}x & y\\ z & -x\end{bmatrix} - \begin{bmatrix}x & y\\ z & -x\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix}.$$

Compute $M_{\mathcal{B},\mathcal{B}}(T)$.

$$T\left(\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\right) = \underbrace{\begin{bmatrix}a & b\\ c & d\end{bmatrix}}_{2\times 2}\underbrace{\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}}_{2\times 2} - \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

$$\begin{bmatrix}a & -b\\ c & -d\end{bmatrix} - \begin{bmatrix}a & b\\ -c & -d\end{bmatrix} = \begin{bmatrix}0 & -2b\\ 2c & 0\end{bmatrix}$$

$$T\left(\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\right) = \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} - \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

$$= \begin{bmatrix}0 & a\\ 0 & c\end{bmatrix} - \begin{bmatrix}c & d\\ 0 & 0\end{bmatrix} = \begin{bmatrix}-c & a-d\\ 0 & c\end{bmatrix}$$

$$T\left(\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\right) = \begin{bmatrix}a & b\\ c & d\end{bmatrix}\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix} - \begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}a & b\\ c & d\end{bmatrix}$$

$$\begin{bmatrix}b & 0\\ d & 0\end{bmatrix} - \begin{bmatrix}0 & 0\\ a & b\end{bmatrix} = \begin{bmatrix}b & 0\\ d-a & -b\end{bmatrix}$$

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$$M_{B,B}(T) = \begin{bmatrix}-c+b & -2b+a-d\\ 2c+d-a & c-b\end{bmatrix}$$

X

(3) Let 8/10

$$A = \begin{bmatrix} 1 & 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 & 0 \end{bmatrix}$$

6 (a) Find a basis for the row space of A. Show your work.

$$\xrightarrow[R_4-R_1]{R_3-R_1} \begin{bmatrix} 1 & 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & -1 & 1 & 0 & -1 \\ 0 & -1 & 1 & 1 & -2 \end{bmatrix} \xrightarrow[R_4+R_2]{R_3+R_2} \begin{bmatrix} 1 & 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$$\xrightarrow{R_1-R_2} \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \xrightarrow{R_1+R_4} \begin{bmatrix} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}$$

$$R_3 \leftrightarrow R_4 \quad \begin{bmatrix} 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} \right\}$$

row space consists of rows not columns. Otherwise OK.

$\boxed{\text{rank}(A) = 3}$ ↳ 3 columns

$$\left[\begin{matrix}1\\0\\1\\2\\0\end{matrix}\right], \left[\begin{matrix}0\\1\\-1\\0\\1\end{matrix}\right], \left[\begin{matrix}0\\0\\0\\1\\-1\end{matrix}\right]$$

(b) Find a basis for the null space of A. Show your work.

$Ax=0$

$x_1 + x_3 + 2x_4 = 0$

let $x_4 = x_5 = 0$

$x_2 - x_3 + x_5 = 0$

$x_4 - x_5 = 0$ $\boxed{x_4 = x_5}$

$x_1 + x_3 = 0$ $\boxed{x_1 = -x_3}$

$x_2 - x_3 = 0$

$\boxed{x_2 = x_3}$

$$\left\{ \left[\begin{matrix}-1\\1\\1\\0\\0\end{matrix}\right], \left[\begin{matrix}0\\0\\0\\1\\1\end{matrix}\right] \right\} = \text{null}(A)$$

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nullity = 2

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(4) Prove that, for any matrix A, the row space of A and the column space of A have the same dimension. (**Note:** The textbook calls these dimensions "row rank (A)" and the "column rank(A)," and only introduces "rank(A)" after proving that they are equal. So "they are equal because they are both the rank" is circular reasoning.)

rank-nullity Theorem: $\dim(\ker) + \dim(\text{im}) = V$

rank + nullity = m

relevant...

dimension of column

Ex) Consider an $m \times n$ matrix where $m < n$.

$$A = \begin{bmatrix} x & x & x & x \\ x & x & x & x \\ x & x & x & x \\ x & & & \end{bmatrix}$$

- let the last row be all 0's

?

Then the rank(A) = 2

and nullity (A) = 1

rank(A) + nullity (A) = 3

number of columns