MTH420 PROBLEM SET 12

(1) Let
$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 8 \end{bmatrix}$$
   (a) Compute A^{adj}.
   (b) Check that your answer to the first part is plausible by computing $A \cdot A^{\mathrm{adj}}$.

(2) Let
$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$
   (a) Compute B^{adj}.
   (b) Check that your answer to the first part is plausible by computing $B \cdot B^{\mathrm{adj}}$.

(3) Let R be a commutative ring with unity and consider the space $T^2(R^{2\times 1})$ of bilinear functions $R^{2\times 1} \times R^{2\times 1} \to R$. (That is, we're looking at bilinear functions which take in a pair of column vectors and return a scalar.)
   (a) For any $A \in R^{2\times 2}, X, Y \in R^{2\times 1}$ let $f_A(X,Y) = X^t \cdot A \cdot Y$. (The ^{t} is a transpose.) Prove that $f_A \in T^2(R^{2\times 1})$.
   (b) Let $\underline{\mathrm{f}} : R^{2\times 2} \to T^2(R^{2\times 1})$ be the mapping which sends A to f_A for each A. Prove that it is an isomorphism.
   (c) Inside of $T^2(R^{2\times 1})$ we have the subspace of alternating elements. What subspace of $T^2(R^{2\times 1})$ does it correspond to?

(4) Let R be a commutative ring with unity, and M a free R-module with basis $\mathcal{B} = (\beta_1, \ldots, \beta_n)$. Let $\bigwedge^k(M)$ be the set of all k-linear alternating functions $M^k \to R$.
   (a) Prove that $\bigwedge^k(M)$ is a submodule of the $\{f : M^k \to R\}$.
   (b) Prove that it is free (i.e., has a basis) and describe a basis. How many elements does the basis have?

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