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MTH 420

PS 8

① Trace - For square matrix $M \in F^{k \times k}$, the trace $(Tr(M))$ is sum of its diagonal elements. $Tr(M) = \sum_{i=1}^{k} M_{ii}$ 8/10

$(AB)_{ii} = \sum_{k=1}^{n} A_{ik} B_{ki}$, the trace is ✓

no second Σ here.

$Tr(AB) = \sum_{i=1}^{m} (AB)_{ii} = \sum_{i=1}^{m} \sum_{k=1}^{n} A_{ik} B_{ki} = \sum_{k=1}^{n} \sum_{i=1}^{m} B_{ki} A_{ik} = \sum_{k=1}^{n} \cancel{\sum_{i=1}} (BA)_{kk}$ ✓

$= Tr(BA)$

Therefore, $Tr(AB) = Tr(BA)$ for all $A \in \mathbb{F}^{m \times n}$, $B \in \mathbb{F}^{n \times m}$

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② $f_1(p) = \int_0^1 p(t)\,dt$, $f_2(p) = \int_1^2 p(t)\,dt$, $f_3(p) = \int_{-1}^{0} p(t)\,dt$

ⓐ Find three elements $p_1, p_2, p_3 \in P_2(\mathbb{R})$ such that $f_i(p_j) = \delta_{ij},\ i,j \in \{1,2,3\}$

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let each $p_j(t) = a_0 + a_1 t + a_2 t^2$, write integrals as linear functions:

$f_i(p) = \int_{a_i}^{b_i} (a_0 + a_1 t + a_2 t^2)\,dt = a_0 I_0^{(i)} + a_1 I_1^{(i)} + a_2 I_2^{(i)}$

$= I_k^{(i)} = \int_{a_i}^{b_i} t^k\,dt$

$$M = \begin{bmatrix} \int_0^1 1 & \int_0^1 t & \int_0^1 t^2 \\ \int_1^2 1 & \int_1^2 t & \int_1^2 t^2 \\ \int_{-1}^0 1 & \int_{-1}^0 t & \int_{-1}^0 t^2 \end{bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \\ 1 & 3/2 & \frac{7}{3} \\ 1 & -\frac{1}{2} & \frac{1}{3} \end{bmatrix}$$ - The matrix of functionals applied to the monomial basis $\{1, t, t^2\}$

solve system: $M \cdot \vec{c}_j = \vec{e}_j$

where $j = 1, 2, 3$, $\bar{e}_j$ is the standard basis vector

- use Gaussian elimination / matrix inversion to get $p_j(t) = c_{j0} + c_{j1}t + c_{j2}$

(b) Because $\dim(P_2(\mathbb{R})) = 3$, the dual space $(P_2(\mathbb{R}))$ also = 3.

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The three functionals f_1, f_2, f_3 are linearly independent and they form a basis.

Proof?

Also, the dual basis $\{p_1, p_2, p_3\}$ such that $f_i(p_j) = \delta_{ij}$, means that $\{f_1, f_2, f_3\}$ is linearly independent and hence a basis.

a bit more explanation?

Therefore, $\{f_1, f_2, f_3\}$ is basis of $(P_2(\mathbb{R}))$

(c) For any $a \in \mathbb{R}$, $e_a(p) = p(a)$

$e_0(p) = p(0)$
$e_1(p) = p(1)$
$e_{-1}(p) = p(-1)$

$$\begin{bmatrix} e_0(1) & e_0(t) & e_0(t^2) \\ e_1(1) & e_1(t) & e_1(t^2) \\ e_{-1}(1) & e_{-1}(t) & e_{-1}(t^2) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

$$\det \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} = 1 \cdot \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} - 0 + 0 = 1 \cdot (1 \cdot 1 - 1 \cdot (-1)) = 2$$

Therefore, the matrix is invertible b/c determinant $= 2$, which is nonzero, so the functionals $\{e_0, e_1, e_{-1}\}$ are linearly independent and spans the dual space.

$\{e_0, e_1, e_{-1}\}$ is basis of $(P_2(\mathbb{R}))$

want: $f_1 = \alpha e_0 + \beta e_1 + \gamma e_{-1}$, so $f_1(p) = \alpha p(0) + \beta p(1) + \gamma p(-1)$

$p(t)=1$: $f_1(1) = \int_0^1 1\,dt = 1 = \alpha \cdot 1 + \beta \cdot 1 + \gamma \cdot 1 \Rightarrow \alpha + \beta + \gamma = 1$

$p(t)=t$: $f_1(t) = \int_0^1 t\,dt = \frac{1}{2} = \alpha \cdot 0 + \beta \cdot 1 + \gamma \cdot (-1) \Rightarrow \beta - \gamma = \frac{1}{2}$

$p(t)=t^2$: $f_1(t^2) = \int_0^1 t^2\,dt = \frac{1}{3} = \alpha \cdot 0 + \beta \cdot 1 + \gamma \cdot 1 = \beta + \gamma = \frac{1}{3}$

$\alpha + \beta + \gamma = 1$

plug in

$\beta - \gamma = \frac{1}{2} \Rightarrow \beta = \frac{1}{2} + \gamma$

$\beta + \gamma = \frac{1}{3}$

plug in

$\frac{1}{2} + \gamma + \gamma = \frac{1}{3}$

$\frac{1}{2} + 2\gamma = \frac{1}{3}$

$2\gamma = \frac{1}{3} - \frac{1}{2} = 2\gamma = \frac{2}{6} - \frac{3}{6} = \frac{-1}{6}$

$2\gamma = \frac{-1}{6}$

$\boxed{\gamma = \frac{-1}{6} \cdot \frac{1}{2} = \frac{-1}{12}}$

$\alpha + \frac{1}{2} + \gamma + \gamma = 1$

$\alpha + \frac{1}{2} - \frac{1}{12} - \frac{1}{12} = 1$

$\alpha + \frac{6}{12} - \frac{1}{12} - \frac{1}{12} = 1$

$\alpha + \frac{4}{12} = 1$

$\alpha + \frac{1}{3} = 1$

$\boxed{\alpha = \frac{2}{3}}$

$\beta = \frac{1}{2} - \frac{1}{12}$

$\frac{6}{12} - \frac{1}{12} = \frac{5}{12}$

$\boxed{\beta = \frac{5}{12}}$

$$\boxed{f_1 = \frac{2}{3} e_0 + \frac{5}{12} e_1 - \frac{1}{12} e_{-1}}$$

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(e) A basis (p_1, p_2, p_3) is dual to (e_0, e_1, e_2) if:

$e_i(p_j) = \delta_{ij}$, $i,j \in \{0,1,2\}$, (q_1, q_2, q_3) is dual to (e_0, e_1, e_{-1})

This means that, p_1 is the polynomial such that $e_0(p_1) = 1$, $e_1(p_1) = 0$, $e_2(p_1) = 0$

q_1 is polynomial such that $e_0(q_1) = 1$, $e_1(q_1) = 0$, $e_{-1}(q_1) = 0$

So both p_1 and q_1 satisfy: $p(0)=1$
$p(1)=0$

However $p_1(2)=0$ and $q_1(-1)=0$

Unless p_1 and q_1 happen to satisfy both $p(2)$ and $p(-1)=0$, they're not the same.

- The first two conditions for p_1 and q_1 are the same.

Since the values 0 and 1 are fixed in both cases, the condition ($\cancel{\#}$ 2 and -1) is different, $p_1 \neq q_1$

Does $p_1 = q_1$? No because they differ on one evaluation point.

Therefore $p_1 \neq q_1$ and $p_2 \neq q_2$ a_6 5/5

(f) p_3 ; $e_0(p_3)=0$
$e_1(p_3)=0$
$e_2(p_3)=1$

q_3: $e_0(q_3)=0$
$e_1(q_3)=0$
$e_{-1}(q_3)=1$

Both polynomials vanish at 0 and 1 and are nonzero at third point (2 or -1)

Let $r(t)=(t)(t-1)$ that vanishes at 0 and 1:

$r(2)=1$ or $r(-1)=1$

$r(2)=2\cdot 1=2$, so $p_3(t)=\frac{1}{2}(t)(t-1)$

$r(-1)=(-1)(-2)=2$, so $q_3(t)=\frac{1}{2}(t)(t-1)$

Therefore, $p_3 = q_3 = \frac{1}{2}t(t-1)$

the third basis vector in both dual bases coincides b/c they both satisfy the same two vanishing conditions, and differ only in scaling at the third point (but get scaled by the same value), so they end up being the same

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix}$$

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we need to find all vectors $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} \in \mathbb{R}^5$ such that:

$v_1 \cdot x = 0, \quad v_2 \cdot x = 0, \quad v_3 \cdot x = 0$

System of equations:

$$1x_1 + 2x_2 + 0x_3 + 1x_4 - 1x_5 = 0$$
$$0x_1 + 1x_2 + 1x_3 + 3x_4 + 0x_5 = 0$$
$$1x_1 + 0x_2 + 1x_3 - 1x_4 + 1x_5 = 0$$

$x_1 + 2x_2 + x_4 - x_5 = 0$

$x_2 + x_3 + 3x_4 = 0$

$x_1 + x_3 - x_4 + x_5 = 0$

$$x_3 = -x_2 - 3x_4$$

$$x_1 + 2x_2 + x_4 - x_5 = 0 \Rightarrow x_1 = -2x_2 - x_4 + x_5$$

$$(-2x_2 - x_4 + x_5) + (-x_2 - 3x_4) - x_4 + x_5 = 0$$

$$-3x_2 - 5x_4 + 2x_5 = 0 \Rightarrow x_2 = -\frac{5}{3}x_4 + \frac{2}{3}x_5$$

$$x_3 = -\left(-\frac{5}{3}x_4 + \frac{2}{3}x_5\right) - 3x_4 = \frac{5}{3}x_4 - \frac{2}{3}x_5 - 3x_4 = -\frac{4}{3}x_4 - \frac{2}{3}x_5$$

$$x_1 = -2\left(-\frac{5}{3}x_4 + \frac{2}{3}x_5\right) - x_4 + x_5 = \frac{10}{3}x_4 - \frac{4}{3}x_5 - x_4 + x_5 = \frac{7}{3}x_4 - \frac{1}{3}x_5$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = x_4 \begin{bmatrix} 7/3 \\ -5/3 \\ -4/3 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1/3 \\ 2/3 \\ -2/3 \\ 0 \\ 1 \end{bmatrix}$$

Column vectors, not functionals? What are the corresponding functionals?

The basis for $S^\perp$ is $\left\{ \begin{bmatrix} 7/3 \\ -5/3 \\ -4/3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1/3 \\ 2/3 \\ -2/3 \\ 0 \\ 1 \end{bmatrix} \right\}$

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④ (a) Prove $(W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp$

- $(W_1 \cap W_2)^\perp$ consist of all functionals that vanish in $W_1 \cap W_2$

- $W_1^\perp + W_2^\perp$ consist of all functionals that can be written as the sum of a functional in $W_1^\perp$ and functional $W_2^\perp$

let $f \in (W_1 \cap W_2)^\perp$: $f(w)=0$ for all $w \in W_1 \cap W_2$.

since $W_1 \cap W_2$ and $W_1 \cap W_2 \subseteq W_2$, f must vanish on W_1 and on W_2. Therefore, $f \in W_1^\perp \cap W_2^\perp$ which implies No

$f \in W_1^\perp + W_2^\perp \Rightarrow (W_1 \cap W_2)^\perp \subseteq W_1^\perp + W_2^\perp$ 4/10

In reverse inclusion: let $f \in W_1^\perp + W_2^\perp$, $f = f_1 + f_2$ where $f_1 \in W_1^\perp$ and $f_2 \in W_2^\perp$. since f_1 vanishes on all vectors in W_1 and f_2 vanishes on all vectors in W_2, then sum f will vanish on all vectors in $W_1 \cap W_2$.

Thus, $f \in (W_1 \cap W_2)^\perp$ This part is ok

Therefore, $W_1^\perp + W_2^\perp \subseteq (W_1 \cap W_2)^\perp$ and we can conclude

$(W_1 \cap W_2)^\perp = W_1^\perp + W_2^\perp$

(b) Prove: $(W_1 + W_2)^\perp = W_1^\perp \cap W_2^\perp$

let $f \in (W_1 + W_2)^\perp$, so $f(w)=0$ for all $w \in W_1 + W_2$.

B/C $W_1 + W_2$ is the set of all sums of vectors from W_1 and W_2, f vanishes on all vectors in W_1 and W_2,

Therefore, $f \in W_1^\perp \cap W_2^\perp$

For reverse inclusion: let $f \in W_1^\perp \cap W_2^\perp$, f vanishes for all vectors in W_1 and W_2, f therefore vanishes in $W_1 + W_2$, $f \in (W_1 + W_2)^\perp$