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MTH 420
PS 11

(1) Intersection: let $I = \bigcap_{I \in S} I$. This is the set of all elements $r \in R$ such that $r \in I$ for every ideal $I \in S$.

$$I \in \{r \in R \mid r \in I \text{ for all } I \in S\}$$

Show I is subgroup of $(R,+)$ (closed under addition and contains additive inverses)

- let $r_1, r_2 \in I$, this means that $r_1, r_2 \in I$ for all $I \in S$. Since $I \in S$ is an ideal and ideals are closed under addition, $r_1 + r_2 \in I$. Therefore $r_1 + r_2 \in \bigcap_{I \in S} I$ and I is closed under addition. ✓

- let $r \in I$, so $r \in I$ for every $I \in S$, $I \in S$ is an ideal and ideals are closed under additive inverses, so we can have $-r \in I$ for every $I \in S$. Therefore, $-r \in \bigcap_{I \in S} I$, which means I contains additive inverses. ✓

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Thus, I is a subgroup of $(R,+)$

Great job

Show I is closed under multiplication by elements of R

let $r \in I$ and $a \in R$. ~~show $a \cdot r \in I$ and $r \cdot a \in I$~~

Since $r \in I$ for all $I \in S$ and $I \in S$ is an ideal, $a \cdot r \in I$ for every $I \in S$. Therefore, $a \cdot r \in \bigcap_{I \in S} I$ (closed under left multiplication).

Since $r \in I$ for all $I \in S$, and each $I \in S$ is an ideal, we also have $r \cdot a \in I$ for every $I \in S$. Thus $r \cdot a \in \bigcap_{I \in S} I$ and I is closed under multiplication

Therefore, $\bigcap_{I \in S} I$ is an ideal of R

② For $a, b \in R$: $\varphi(a+b) = \varphi(a) + \varphi(b)$,
$\varphi(ab) = \varphi(a)\varphi(b)$

Prove $\ker \varphi := \{ a \in R ; \varphi(a) = 0_S \}$ is an ideal

Closed under addition: let $a, b \in \ker \varphi$, by def the kernel $\varphi(a) = 0_S$ and $\varphi(b) = 0_S$. Show $a+b \in \ker \varphi$!

$$\varphi(a+b) = \varphi(a) + \varphi(b) = 0_S + 0_S = 0_S$$

Thus, $a+b \in \ker \varphi$ is closed under addition.

Additive inverse: let $a \in \ker \varphi$, need to show that $-a \in \ker \varphi$. Since $\varphi(a) = 0_S$ and φ is a homomorphism,

~~not really needed.~~

$$\varphi(-a) = -\varphi(a) = -0_S = 0_S$$

Good

Therefore, $-a \in \ker \varphi$, so ~~ker~~ $\ker \varphi$ contains additive inverses.

Show that $\ker \varphi$ is closed under multiplication by elements of R

Closed under left multiplication: let $a \in \ker \varphi$ and $r \in R$. Since $a \in \ker \varphi$, we have $\varphi(a) = 0_S$. We need to show that $r \cdot a \in \ker \varphi$. Using homomorphism property of φ, we compute

$$\varphi(r \cdot a) = \varphi(r)\varphi(a) = \varphi(r) \cdot 0_S = 0_S$$

Good

Thus, $r \cdot a \in \ker \varphi$

Closed under right multiplication: let $a \in \ker \varphi$ and $r \in R$. Show $a \cdot r \in \ker \varphi$, using homomorphism property of φ, compute.

$$\varphi(a \cdot r) = \varphi(a)\varphi(r) = 0_S \cdot \varphi(r) = 0_S$$

Good

Thus, $a \cdot r \in \ker \varphi$

Therefore, $\ker \varphi$ is an ideal of R and we can conclude that the kernel of a ring homomorphism is always an ideal

$\boxed{\text{Q.E.D}}$

Excellent. Very thorough. You even caught some things I overlooked in checking my key. Well done.

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Prove I is subgroup of $(F[x], +)$

Closed under addition: let $f(x), g(x) \in I$, by def of I, $f(\alpha) = 0$ and $g(\alpha) = 0$. Show that $f(x) + g(x) \in I$, since $F[x]$ is a ring and the evaluation of polynomials is linear.

$$(f+g)(\alpha) = f(\alpha) + g(\alpha) = 0 + 0 = 0$$

Good

Thus, $f(x) + g(x) \in I$ and I is closed under addition.

Additive inverse: let $f(x) \in I$, by def $f(\alpha) = 0$ and we need to show that $-f(x) \in I$. Since evaluation is linear, we compute!

de

$$(-f)(\alpha) = -f(\alpha) = -0 = 0$$

Thus, $-f(x) \in I$ and so I contains additive inverse.

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Therefore, I is a subgroup of $(F[x], +)$

Prove I is closed under multiplication by elements of $F[x]$

Good

Closed under left multiplication: let $f(x) \in I$ and $h(x) \in F[x]$. Show that $h(x) f(x) \in I$, which since $f(x) \in I$, we have $f(\alpha) = 0$. Show that $(h(x) f(x))(\alpha) = 0$ by properties of polynomial evaluation:

$$(h(x) f(x))(\alpha) = h(\alpha) f(\alpha) = h(\alpha) \cdot 0 = 0$$

thus, $f(x)h(x) \in I$

Closed under right multiplication: For $f(x) \in I$ and $h(x) \in F[x]$, show $f(x) h(x) \in I$. Since $f(\alpha) = 0$:

this is not needed because $\cdot$ is commutative in $F[x]$

$$(f(x) h(x))(\alpha) = f(\alpha) h(\alpha) = 0 \cdot h(\alpha) = 0$$

thus, $f(x)h(x) \in I$

Thus, I is closed under multiplication by arbitrary elements of $F[x]$ on both sides, ideal of $F[x]$ and the set $\{f \in F[x] \mid f(\alpha) = 0\}$ is an ideal in $F[x]$

Therefore, I is ideal in $F[x]$ Q.E.D

④ (a) show zero vector, closed under addition, and closed under scalar multiplication.

zero vector: since M is an ideal, it contains additive identity of $F[x]$ which is the zero polynomial, thence $0 \in M$.

closed under addition: if $f(x), g(x) \in M$, then the properties of ideals, $f(x)+g(x) \in M$ (since ideals are closed under addition)

Closed under scalar multiplication: For any polynomial $f(x) \in M$ ~~[illegible]~~ and scalar $c \in F$, show $cF(x) \in M$. since M is an ideal and closed under multiplication by elements of $F[x]$, we have $cf(x) \in M$

therefore, M is closed under addition and scalar multiplication, so M is a subspace of $F[x]$.

(b) let M be a nonzero ideal in $F[x]$, show the quotient space $F[x]/M$ is finite-dimensional and describe the dimension of $F[x]/M$ depends on M.

Since M is a nonzero ideal in $F[x]$, by the structure of ideals in polynomial rings, M is generated by a single polynomial b/c $F[x]$ is a principal ideal domain.

There exists some nonzero polynomials $p(x) \in M$ such that $M = \langle p(x) \rangle$. This means every element of M is a multiple of $p(x)$ ($M = \{g(x)p(x) \mid g(x) \in F[x]\}$)

- The quotient space $F[x]/M$ consists of cosets of the form $f(x) + M$ where $f(x) \in F[x]$. Two polynomials $f(x)$ and $g(x)$ are in the same coset if their difference lies in M. Thus, the elements of $F[x]/M$ correspond to polynomials in $F[x]$ modulo the ideal M,

- since $M = \langle p(x) \rangle$, we can represent each coset uniquely by polynomials of degree strictly less than degree$(p(x))$. In other words, every element of $F[x]/M$ can be written as a polynomial $f(x) \in F[x]$ of degree less than $\deg(p(x))$, b/c any higher degree term would be equivalent to a multiple of $p(x)$ modulo M.

the set of polynomials of degree less than $deg(p(x))$

a ~~basis for $F[x]/M$~~ no, but it is bijective with $F[x]/M$

the set of polynomials of degree less than $deg(p(x))$

is given by $\{1, x, x^2, \dots, x^{deg(p(x))-1}\}$ 7/10

is linearly independent set of size $deg(p(x))$. Therefore,

the quotient space $F[x]/M$ is finite-dimensional with dimension equal

to $deg(p(x))$, which is the degree of the generator of ideal M.

(c) Show $(f+M) \oplus (g+M) = (f+g)+M$ is well-defined

prove if $f+M = f'+M$ and $g+M = g'+M$, then $(f+g)+M=$

$(f'+g')+M$

- $f+M = f'+M$ implies that $f-f' \in M$ 54/60
- $g+M = g'+M$ implies $g-g' \in M$

Consider: $(f+g)-(f'+g') = (f-f')+(g-g')$

since $f-f' \in M$ and $g-g' \in M$, we have $(f-f')-(g-g') \in M$,

means that $(f+g)-(f'+g') \in M$, Therefore, $(f+g)+M = (f'+g')+M$,

shows that the operation is well-defined.

(d) Show that the operation $(f+M) \otimes (g+M) = (f \cdot g)+M$ is well defined.

Show well-definedness: if $f+M = f'+M$ and $g+M = g'+M$, then

$(f \cdot g)+M = (f' \cdot g')+M$

- $f+M = f'+M$ implies that $f-f' \in M$
- $g+M = g'+M$ implies $g-g' \in M$

No. The RHS is $2fg - fg' - f'g$

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Consider: $(f \cdot g)-(f' \cdot g') = f \cdot (g-g') + (f-f') \cdot g$

since $f-f' \in M$ and $g-g' \in M$, both terms $f \cdot (g-g')$ and $(f-f') \cdot g$

lies in M, so $(f \cdot g)-(f' \cdot g') \in M$. Therefore, $(f \cdot g)+M = (f' g')+M$,

which shows ~~that~~ the operation is well-defined.

(e) Show commutativity of addition, commutativity of multiplication, Associativity of addition and multiplication, Existence of additive identity, existence of a multiplicative identity.

Commutativity of addition: Addition of cosets is defined by $(f+M) \oplus (g+M) = (f+g)+M$ and since addition $F[x]$ is commutative, $\oplus$ is commutative. ok.

Commutativity of multiplication: multiplication of cosets is defined by $(f+M) \otimes (g+M) = (f \cdot g)+M$ and since multiplication in $F[x]$ is commutative, $\otimes$ is commutative. ok.

Associativity of addition and multiplication: Both addition and multiplication in $F[x]$ are associative, so $\oplus$ and $\otimes$ are associative in $F[x]/M$. Need some details.

Existence of an additive identity: The coset $0+M$ is the additive identity as $(f+M) \oplus (0+M) = f+M$ for all $f+M \in F[x]/M$ ok.

Existence of a multiplicative identity: The coset $1+M$ is the multiplicative identity as $(f+M) \otimes (1+M) = (f \cdot 1)+M = f+M$ for all $f+M \in F[x]/M$ ok

therefore, $(F(x)/M, \oplus, \otimes)$ is commutative ring with unity.

additive inverses

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