MTH420 PROBLEM SET 7

(1) Let $T : \mathbb{R}^3 \to \mathbb{R}^2$ and $U : \mathbb{R}^2 \to \mathbb{R}^3$ be linear transformations.
   (a) Prove that $U \circ T$ is not invertible.
   (b) Give an example where $T \circ U$ is invertible.

(2) Let T be a linear operator on a vector space V, and U a second linear operator on V such that $T \circ U = I$ (the identity operator which maps evrey element of V to itself. Does it follow that U is the inverse of T?
   (a) Prove that if V is finite dimensional, it does.
   (b) Show that if V is not finite dimensional, it does not.

(3) Fix three integers m, n and p and a field F. For each $B \in F^{p\times m}$ we have the transformation $L_B : F^{m\times n} \to F^{p\times n}$ given by $L_B(X) = BX$. Prove that L_B is invertible if and only if $p = m$ and B is an invertible matrix.

(4) Suppose that V and W are vector spaces and $U : V \to W$ is an isomorphism. If $\varphi : L(V, V) \to L(W, W)$ is given by $\varphi(T) = U \circ T \circ U^{-1}$, prove that φ is also an isomorphism.

(5) Let V and W be two vector spaces of the same finite dimension over the same field F. Let T be an operator on V and S an operator on W. Consider

$$\{A \in F^{n\times n} : A = [T]_{\mathcal{B}} \text{ for some ordered basis } \mathcal{B} \text{ of } V\} \text{ and}$$

$$\{A \in F^{n\times n} : A = [S]_{\mathcal{C}} \text{ for some ordered basis } \mathcal{C} \text{ of } W\}.$$

   (a) Show that they are equal, whenever there is an isomorphism $U : V \to W$ such that $S = U \circ T \circ U^{-1}$.
   (b) Show that there is an isomorphism $U : V \to W$ such that $S = U \circ T \circ U^{-1}$, whenever they have an element in common.

(This is basically problem 9 from section 3.4 of Hoffman and Kunze. They suggest an approach.)

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