MTH420 PROBLEM SET 9

(1) Take $f, g \in F[[X]]$ (the algebra of formal power series). Prove that $fg = 0$ implies either f or g is 0.

(2) Take $a, b \in F$ with $a \neq 0$. Prove that $\{(ax+b)^n : n \text{ a non-negative integer}\}$. Is a basis of $F[x]$.

(3) Let $A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$.

   (a) Find the polynomials p_1, p_2, p_3 such that $p_i(j) = \delta_{i,j}$.

   (b) Compute $p_i(A)$ for each i.

(4) Suppose that $L : F[x] \to F$ is an F-algebra homomorphism. (That is, it's not only linear, but also satisfies $L(fg) = L(f)L(g)$ for all $f, g \in F[x]$. Prove that L is either 0, or the evaluation functions e_t given by $e_t(f) = f(t)$, for some $t \in F$.

(5) Which of the following subsets of $\mathbb{Q}[x]$ are ideals? Justify your answers.

   (a) The set of all elements of even degree.

   (b) The set of all elements of degree > 5.

   (c) The set of all elements of degree < 5.

   (d) The set of all elements f such that $f(1) = 0$.

   (e) The set of all elements f such that $f(0) = 1$.

   (f) The set of all elements f such that $f(0) = f(1) = 0$.

   (g) The set of all elements f such that $f(0) = f(1)$.

   (h) The image of the linear operator T defined by

$$T\left(\sum_{i=0}^{n} c_i x^i\right) = \sum_{i=0}^{n} \frac{c_i}{i+1} x^{i+1}.$$

   (i) The set of all elements f such that $f\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Date: April 11, 2025.