Ivan Wang
MTH 420
Problem set 4

① Let F be a field a V a vector space over F. Let S be a subset of V. Prove that V has a basis B such that $S \subset B$.

F is a field
V is vector space over F
S is subset of V

Let A be set of all linearly independent subsets of V that contain S. So $A_1 \leq A_2$ iff $A_1 \subseteq A_2$ for $A_1, A_2 \in A$

Show every chain in A has upper bound!

Let $\{A_i\}_{i \in I}$ be chain in A, where I is an index set.

Consider the set $A = \bigcup_{i \in I} A_i$

A is an upper bound because $A_i \subseteq A$ for all $i \in I$

A is also linearly independent because: $\sum_{k=1}^{n} c_k v_k = 0$ for some $v_k \in A$ and $c_k \in F$. Since A is union of A_i, each v_k belongs to A_{ik}. Since $\{A_i\}_{i \in I}$ is a chain, there exists A_j such that $A_{ik} \subseteq A_j$ for all k. Therefore, all v_k are in A_j and since A_j is linearly independent $c_k = 0$ for all k. And A is linearly independent, $S \subseteq A$ because $S \subseteq A_i$ for all i.

Zorn's lemma: A has maximal element B, i.e. B is a linearly independent subset of V containing S.

let set $B' = B \cup \{v\}$ and is linearly independent.

<u>Therefore:</u>

$S \subseteq B \subset B'$ $\Rightarrow$ $B' \in A$. However, $B \subseteq B'$ because it contradicts the maximality of B. Therefore, B must span V

Therefore, B is linearly independent and spans V, this means that B is a basis for V and $S \subseteq B$

This means that there exists a basis B of V such that $S \subseteq B$.

(2) let F be a field

X be a set with n elements

8/10.

<u>Basis</u>: Consider functions δ_x for $x \in X$

$$\delta_x(y) = \begin{cases} 1, & \text{if } y = x \\ 0, & \text{otherwise} \end{cases}$$

Good.

For any function $f \in F^X$, f can be express as a linear combination of $f = \sum_{x \in X} f(x)\delta_x$ } some detail here...

δ? what does a δ with no subscript mean? Do you mean δ_x?

<u>linear independence</u>: $\sum_{x \in X} c_x \delta_x = 0$, when at any $y \in X$

This? Now

$\sum_{x \in X} c_x \delta_x(y) = 0$ This means that $\delta(y)$ is 1 when $x = y$ and 0 otherwise, this holds for $y \in X$ all coefficients c_x must be zero, proving linear independence.

Therefore, $\{\delta_x \mid x \in X\}$ is a basis and has n elements, $\dim_F F^X = n$

as vector space over $\mathbb{R}$, Only allow scalar multiplication

$a+bi$ for $a,b \in \mathbb{R}$

$4 \times 2 = 8$ since there is total of 4 entries in 2×2 matrix with each entry contributing to two real parameters.

$\dim(\mathbb{C}^{2\times 2}) = 8$

Basis matrices corresponding to real parts:

$E_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $E_2 \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $E_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $E_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Basis matrices corresponding the imaginary parts:

$E_5 = \begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix}$, $E_6 = \begin{bmatrix} 0 & i \\ 0 & 0 \end{bmatrix}$, $E_7 = \begin{bmatrix} 0 & 0 \\ i & 0 \end{bmatrix}$, $E_8 = \begin{bmatrix} 0 & 0 \\ 0 & i \end{bmatrix}$

These matrices form a basis because spanning set: $A = \begin{bmatrix} a+bi & c+di \\ e+fi & g+hi \end{bmatrix}$ can be written as an linear combination of these basis matrices using real coefficients.

Therefore the set $\{E_1, E_2, \dots E_8\}$ forms basis $\mathbb{C}^{2\times 2}$ over $\mathbb{R}$, with $\dim(\mathbb{C}^{2\times 2}) = 8$

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④ Prove $\{A \in \mathbb{C}^{2\times 2} : A^{t} = A\}$ is subspace and find a basis.

S is a subspace if these conditions are met

$(0, +, \cdot)$

contains zero matrix

closed under addition

closed under scalar multiplication

Zero matrix is in S because $(0^t = 0)$, so $0 \in S$

Closed under addition is satisfied because if $A, B \in S$, this means that $A^t = A$, $B^t = B$. This means that $(A+B)^t = A^t + B^t = A + B$

Therefore, $A+B \in S$

Closed under scalar multiplication is satisfied because

if let $A \in S$ and $c \in \mathbb{R}$. $(cA)^t = cA^t = cA$

And cA is still symmetric, S is closed under real scalar multiplication

Basis: $A = \begin{bmatrix} a+bi & c+di \\ c-di & e+fi \end{bmatrix}$, $a,b,c,d,e,f \in \mathbb{R}$

The basis for this space consist of matrices to each independent real parameter:

- Real part of diagonal elements: $B_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $B_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$
- Imaginary part of diagonal elements: $B_3 = \begin{bmatrix} i & 0 \\ 0 & 0 \end{bmatrix}$, $B_4 = \begin{bmatrix} 0 & 0 \\ 0 & i \end{bmatrix}$
- Real part of symmetric off diagonal elements: $B_5 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$
- Imaginary part of symmetric $B_6 = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$

The 6 matrices are linearly independent and spans all possible symmetric complex matrices, hence forms a basis, $\dim(S) = 6$

order for S to be a subspace of $\mathbb{C}^{2\times 2}$ over $\mathbb{C}$, is closed under addition and complex scaler multiplication.

Closed under addition: let $A, B \in S$, $A^t = A$ and $B^t = B$

Therefore, $A+B \in S$ because $(A+B)^t = A^t + B^t = A+B$

Closed under scalar multiplication: let $A \in S$ and $x \in \mathbb{C}$ (scalar)

$(xA)^t = xA^t = xA$

condition is only true iff x is real $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \in S$

let $x = i$: $iA = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$

$\Rightarrow (iA)^t = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = iA$ — This is not symmetric in $\mathbb{C}^{2\times 2}$, because symmetry requires the off diagonal entries to be equal. Therefore, $iA \notin S$, violating closure under complex scalar multiplication.

Since S is not closed under complex scalar multiplication, it fails to be a subspace of $\mathbb{C}^{2\times 2}$ when viewed as a 4 dimensional vector space.

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6) let $V_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $V_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $V_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ 5/5

check pairwise independence: V_1 and V_2 are linearly independent b/c they are standard basis vectors.

- V_1 and V_3 are independent since no scalar multiple V_1 can equal V_3

- V_2 and V_3 are independent b/c no scalar multiple of V_2 can equal V_3.

$a\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ $\boxed{\begin{matrix} a+c=0 \\ b+c=0 \\ 0=0 \end{matrix}}$ $\Rightarrow 1V_1 + 1V_2 - 1V_3 = 0$

These are nonzero scalars, so the vectors are linearly dependent.

Therefore, $\{V_1, V_2, V_3\}$ satisfies the conditions stated.

7. Prove that $\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \end{bmatrix} \right\}$

check $c_1 V_1 + c_2 V_2 + c_3 V_3 + c_3 V_4 = 0$ where $c_1 = c_2 = c_3 = c_4 = 0$

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 2 & 3 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 1 & -2 \end{bmatrix} \xrightarrow[R_4 = R_4 + R_1]{R_2 = R_2 - 2R_1} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & -2 \end{bmatrix} \xrightarrow{R_4 = R_4 - 2R_2}$$

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & -6 \end{bmatrix} \xrightarrow{R_4 = R_4 + R_3} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -6 \end{bmatrix} \xrightarrow{R_4 = -\frac{1}{6} R_4} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_2 = R_2 - 2R_4} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 = R_2 - R_3} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 = R_1 - R_2}$$

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$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Thus, vectors are linearly independent, since we have 4 linearly vectors in $\mathbb{R}^{4\times1}$, these vectors form a basis of $\mathbb{R}^{4\times1}$

The set $\{V_1, V_2, V_3, V_4\}$ is a basis of $\mathbb{R}^{4\times1}$

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8. 10/10

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} c_1 + c_2 \\ 2c_1 + 3c_2 + c_3 + 2c_4 \\ c_3 \\ -c_1 + c_2 + c_3 - 2c_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$$

$c_1 + c_2 = 4$

$2c_1 + 3c_2 + c_3 + 2c_4 = 3$

$c_3 = 2$

$-c_1 + c_2 + c_3 - 2c_4 = 1$

plug in $c_3 = 2$

$-c_1 + c_2 + 2 - 2c_4 = 1$

$-c_1 + c_2 - 2c_4 = -1$

$-c_1 + c_2 = -1 + 2c_4$

$c_2 - c_1 = 2c_4 - 1$

$(4 - c_1) - c_1 = 2c_4 - 1$

$4 - 2c_1 = 2c_4 - 1$

$5 - 2c_1 = 2c_4$

$c_4 = \frac{5 - 2c_1}{2}$

Plug

continue next page

$c_4 = \frac{5-2c_1}{2}$

$c_1 + 3c_2 + 2 + 2 \times \frac{5-2c_1}{2} = 3$

$2c_1 + 3(4 - c_1) + 2 + (5 - 2c_1) = 3$

$2c_1 + 12 - 3c_1 + 2 + 5 - 2c_1 = 3$

$-3c_1 - 2c_1 + 2c_1 + 12 + 2 + 5 = 3$

$-3c_1 - 2c_1 + 2c_1 + 19 = 3$

$-3c_1 = -16$

$c_1 = \frac{16}{3}$

$c_2 = 4 - c_1 = 4 - \frac{16}{3} = \frac{-4}{3}$

$c_4 = \frac{5 - 2 \times \frac{16}{3}}{2} = \frac{-17}{6}$

$c_1 = \frac{16}{3}, \quad c_2 = \frac{-4}{3}, \quad c_3 = 2,$

$c_4 = \frac{-17}{6}.$

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$$c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$$\begin{bmatrix} c_1 + c_2 \\ 2c_1 + 3c_2 + c_3 + 2c_4 \\ c_3 \\ -c_1 + c_2 + c_3 - 2c_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

$c_1 + c_2 = b_1$

$2c_1 + 3c_2 + c_3 + 2c_4 = b_2$

$c_3 = b_3$

$-c_1 + c_2 + c_3 - 2c_4 = b_4$

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Great job

$-c_1 + c_2 + b_3 - 2c_4 = b_4$

$-c_1 + c_2 - 2c_4 = b_4 - b_3$

$\boxed{\text{Substitute } c_2 = b_1 - c_1}$

$-c_1 + (b_1 - c_1) - 2c_4 = b_4 - b_3$

$b_1 - 2c_1 - 2c_4 = b_4 - b_3$

$-2c_1 - 2c_4 = b_4 - b_3 - b_1$

$c_1 + c_4 = \frac{b_1 + b_3 - b_4}{2}$

$\boxed{\text{Substitute } c_2 = b_1 - c_1, \ c_3 = b_3}$

$2c_1 + 3(b_1 - c_1) + b_3 + 2c_4 = b_2$

$2c_1 + 3b_1 - 3c_1 + b_3 + 2c_4 = b_2$

$-c_1 + 3b_1 + b_3 + 2c_4 = b_2$

$-\left(\frac{b_1 + b_3 - b_4}{2} - c_4\right) + 3b_1 + b_3 + 2c_4 = b_2$

$\frac{-b_1 + b_3 - b_4}{2} + c_4 + 3b_1 + b_3 + 2c_4 = b_2$

$\frac{-b_1 + b_3 - b_4}{2} + 3b_1 + b_3 + 3c_4 = b_2$

(continue on back)

$$2\left(\frac{-b_1+b_3-b_4}{2}+3b_1+b_3+3c_4=b_2\right)2$$

$$-b_1+b_3-b_4+6b_1+2b_3+6c_4=2b_2$$

$$5b_1+b_3+b_4+6c_4=2b_2$$

$$6c_4=2b_2-5b_1-b_3-b_4$$

$$c_4=\frac{2b_2-5b_1-b_3-b_4}{6}$$

$$c_1=\frac{b_1+b_3-b_4}{2}-\frac{2b_2-5b_1-b_3-b_4}{6}\Rightarrow\frac{3(b_1+b_3-b_4)-(2b_2-5b_1-b_3-b_4)}{6}$$

$$\Rightarrow c_1=\frac{3b_1+3b_3-3b_4-2b_2+5b_1+b_3+b_4}{6}=\frac{8b_1+4b_3-2b_2-2b_4}{6}=\frac{4b_1+2b_3-b_2-b_4}{3}$$

$$c_2=b_1-c_1=b_1-\frac{4b_1+2b_3-b_2-b_4}{3}\Rightarrow\frac{3b_1-(4b_1+2b_3-b_2-b_4)}{3}$$

$$\Rightarrow\frac{3b_1-4b_1-2b_3+b_2+b_4}{3}=\frac{-b_1-2b_3+b_2+b_4}{3}$$

$$c_1=\frac{4b+2b_3-b_2-b_4}{3},\quad c_2=\frac{-b_1-2b_3+b_2+b_4}{3},\quad c_3=b_3,\quad c_4=\frac{2b_2-5b_1-b_3-b_4}{6}$$