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MTH 420
PS 5

1) a) W is subspace of $\mathbb{R}[x]$: if it contains zero vector, closed under addition, closed under scalar multiplication.

- $W := \{ a_0 + a_1 x + a_2 x^2 : a_0, a_1, a_2 \in \mathbb{R} \} \leq \mathbb{R}[x]$

$0 + 0x + 0x^2 = 0 \in W$, therefore zero vector is satisfied.

these are the same set.

- let $v := \{ b_0 + b_1 x + b_2 x^2 : b_0, b_1, b_2 \in \mathbb{R} \} \leq \mathbb{R}[x]$

$b_0, b_1, b_2 \in \mathbb{R}$

we can now do this: $(a_0 + a_1 x + a_2 x^2) + (b_0 + b_1 x + b_2 x^2)$

$\Rightarrow (a_0 + b_0) + (a_1 + b_1)x + (a_2 + b_2)x^2$, this results in polynomial $c_0 + c_1 x + c_2 x^2$, with $c_0, c_1, c_2 \in \mathbb{R}$, therefore W is closed under addition.

Now let c be a scalar, $c \in \mathbb{R}$ and $(a_0 + a_1 x + a_2 x^2) \in W$

$c(a_0 + a_1 x + a_2 x^2) = ca_0 + (c a_1)x + (ca_2)x^2$

this results in polynomial ~~___~~ with real coefficients.

Therefore, W is closed under scalar multiplication.

therefore, W contains the zero vector, it's closed under addition, and is closed under scalar multiplication

7/10

(b) Prove $\{x^2, (x-1)^2, (x-2)^2\}$ is basis of W

x^2

$(x-1)^2 = (x-1)(x-1) = x^2 - x - x + 1 = x^2 - 2x + 1$

$(x-2)^2 = (x-2)(x-2) = x^2 - 2x - 2x + 4 = x^2 - 4x + 4$

Check if W can be written as linear combination using these functions.
$a_0 + a_1 x + a_2 x^2$

$c_1 x^2 + c_2 (x-1)^2 + c_3 (x-2)^2 = c_1 x^2 + c_2 (x^2 - 2x + 1) + c_3 (x^2 - 4x + 4)$

$\Rightarrow c_1 x^2 + c_2 x^2 - 2c_2 x + c_2 + c_3 x^2 - 4c_3 x + 4c_3$

$(c_1 + c_2 + c_3) x^2 + (-2c_2 - 4c_3) x + (c_2 + 4c_3)$

$c_1 + c_2 + c_3 = a_2$ $\quad -2c_2 - 4c_3 = a_1$ $\quad c_2 + 4c_3 = a_0$

system has unique solution (a_0, a_1, a_2), the given set spans W

? Needs more justification

check linear independence:

$(c_1 + c_2 + c_3) x^2 + (-2c_2 - 4c_3) x + (c_2 + 4c_3) = 0$

not a clear logical explanation of why something is true. Looks like scratch work

$c_1 + c_2 + c_3 = 0$

$-2c_2 - 4c_3 = 0$

$c_2 + 4c_3 = 0$

$-2c_2 + 4c_3 = 0$
$+2c_2 + 8c_3 = 0$

$12c_3 = 0$

$c_3 = 0$

$c_2 + 4(0) = 0$

$c_2 = 0$

$c_1 = 0$

Since $c_1 = c_2 = c_3 = 0$, it shows linear independence

Since set is linearly independent and span W, it basis of W.

6/10

Let $B = (x^2, (x-1)^2, (x-2)^2)$, and compute $[(x+1)^2]_B$

$(x+1)(x+1) = x^2+x+x+1 = x^2+2x+1$

$(x+1)^2 = \cancel{x^2 (x+1)(x+1) = x^2+x+x+1} = x^2+2x+1$

$c_1x^2 + c_2(x-1)^2 + c_3(x-2)^2$

$(x-1)^2 = (x-1)(x-1) = x^2-x-x+1 = x^2-2x+1$

$(x-2)^2 = (x-2)(x-2) = x^2-2x-2x+4 = x^2-4x+4$

$x^2+2x+1 = c_1x^2 + c_2(x^2-2x+1) + c_3(x^2-4x+4)$

$x^2+2x+1 = c_1x^2 + c_2x^2 - 2c_2x + c_2 + c_3x^2 - 4c_3x + 4c_3$

$= (c_1+c_2+c_3)x^2 + (-2c_2 - 4c_3)x + (c_2+4c_3)$

$c_1+c_2+c_3 = 1$

$-2c_2-4c_3=2$

$c_2+4c_3=1$

$-2c_2-4c_3=2$

$2c_2+8c_3=2$

$4c_3=4$

$\boxed{c_3=1}$

$c_2+4=1$

$\boxed{c_2=-3}$

$c_1-3+1=1$

$\boxed{c_1=3}$

Therefore, $[(x+1)^2]_B = \begin{bmatrix} 3 \\ -3 \\ 1 \end{bmatrix}$ 10/10

d) let $C=(1,x,x^2)$, which is usually the most convenient basis for W. Compute the change of basis matrix $P_{B,C}$

$B=(x^2,(x-1)^2,(x-2)^2)$

$a_0+a_1x+a_2x^2$

$x^2=0+0x+1x^2$ coordinate vector: $\begin{bmatrix}0\\0\\1\end{bmatrix}$

$(x-1)^2=(x-1)(x-1)=x^2-x-x+1=x^2-2x+1$

coordinate vector: $\begin{bmatrix}1\\-2\\1\end{bmatrix}$

$(x-2)^2=(x-2)(x-2)=x^2-2x-2x+4=x^2-4x+4$ coordinate vector: $\begin{bmatrix}4\\-4\\1\end{bmatrix}$

$P_{B,C}=\begin{bmatrix}0&1&4\\0&-2&-4\\1&1&1\end{bmatrix}$? 6/10

(e) $[(x+1)^2]_C$ $(x+1)^2=x^2+2x+1$

$[(x+1)^2]_C=\begin{bmatrix}1\\2\\1\end{bmatrix}$

$\underset{3\times3}{\begin{bmatrix}0&1&4\\0&-2&-4\\1&1&1\end{bmatrix}}\underset{3\times1}{\begin{bmatrix}3\\-3\\1\end{bmatrix}}=\begin{bmatrix}0(3)+1(-3)+4(1)\\0(3)-2(-3)-4(1)\\3-3+1\end{bmatrix}$

$P_{B,C}=\begin{bmatrix}0&1&4\\0&-2&-4\\1&1&1\end{bmatrix}$ $=\begin{bmatrix}1\\2\\1\end{bmatrix}$

identity holds ✓

57/68
Good

8/8

$A, X \in F^{n \times n}$ let $B_A(X) = AXA$. Is B_A linear?

Check for:

$B_A(X+Y) = B_A(X) + B_A(Y)$ let $X, Y \in \mathbb{F}^{n \times n}$

$B_A(cX) = cB_A(X)$ c is scalar $\in \mathbb{F}$

$B_A(X+Y) = A(X+Y)A = AXA + AYA = B_A(X) + B_A(Y)$

additivity is satisfied.

10/10

$B_A(cX) = A(cX)A = cAXA = cB_A(X)$

homogeneity is satisfies

since, additivity and homogeneity is satisfies, B_A is linear in X

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(3) For $A, X \in F^{n \times n}$ let $f_X(A) = AXA$. Is f_X linear?

Check for: $f_X(A+B) = f_X(A) + f_X(B)$

$f_X(cA) = cf_X(A)$

$f_X(A+B) = (A+B)X(A+B) = AXA + AXB + BXA + BXB$

$\neq$ not equal to $f_X(A) + f_X(B)$ No additivity

$f_X(cA) = (cA)X(cA) = cAX(cA) = c^2AXA = c^2 f_X(A)$

$c^2 f_X(A) \neq c f_X(A)$ No homogeneity

Therefore, f_X is not linear.

ok. 10/10