Ivan Wang

MTH 420

PS 6

① $B = (e^{2x}\sin 3x,\ e^{2x}\cos 3x)$

$f_1 = e^{2x}\sin 3x,\quad f_2 = e^{2x}\cos 3x$

$D(f_1) = \frac{d}{dx}(e^{2x}\sin 3x) \Rightarrow e^{2x}(2\sin 3x + 3\cos 3x) = 2e^{2x}\sin 3x + 3e^{2x}\cos 3x$

Take derivatives

$D(f_2) = \frac{d}{dx}(e^{2x}\cos 3x) \Rightarrow e^{2x}(2\cos 3x - 3\sin 3x) = 2e^{2x}\cos 3x - 3e^{2x}\sin 3x$

Express $D(f_1), D(f_2)$:

$D(f_1) = 2f_1 + 3f_2$ ✓ $M_{BB}(D) \cancel{=} \begin{bmatrix} 2 & 3 \\ -3 & 2 \end{bmatrix}$ 6/10

$D(f_2) = -3f_1 + 2f_2$

From this we deduce $M_{BB}(D)\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}2\\3\end{bmatrix}$, which tells us the first column not the first row

② let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and for $X \in \mathbb{R}^{2\times 2}$, let $T(X) = AX - XA$

ⓐ Prove that T is linear: Show $X, Y \in \mathbb{R}^{2\times 2}$ and any scalar c

$T(X+Y) = T(X) + T(Y)$, $T(cX) = cT(X)$ Additivey

$T(X+Y) = A(X+Y) - (X+Y)A$ good.

$T(X+Y) = AX + AY - XA - YA$ ✓ 10/10

$T(X+Y) = (AX - XA) + (AY - YA) = T(X) + T(Y)$

Homogeneity: $T(cX) = A(cX) - (cX)A$

$T(cX) = c(AX) - c(XA) = c(AX - XA) = cT(X)$

therefore, since the additivey and homogeneity property holds T is linear transformation

(b) Find null space of T. $T(X) = AX - XA = 0$

$$AX = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{bmatrix}$$

$$XA = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} a+3b & 2a+4b \\ c+3d & 2c+4d \end{bmatrix}$$

$AX - XA = 0$:
$$\begin{bmatrix} a+2c & b+2d \\ 3+4c & 3b+4d \end{bmatrix} - \begin{bmatrix} a+3b & 2a+4b \\ c+3d & 2c+4d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$a+2c-(a+3b)=0 \Rightarrow a+2c-a-3b=0$
$2c-3b=0$, $2c=3b$ ✓ ok

$(b+2d)-(2a+4b)=0 \Rightarrow b+2d-2a-4b=0$
$-2a-3b+2d=0$ ok.

$(3a+4c)-(c+3d)=0 \Rightarrow 3a+4c-c-3d=0$
$3a+3c=3d$
$a+c=d$ OK.

These equations don't lead to that conclusion. Show work

4/10

$(3b+4d)-(2c+4d)=0$
$3b-2c=0$
$3b=2c$

Null space $=$ ~~span~~ $\left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$

No. It's clear that A itself is in the nullspace of T, so I know there's a mistake somewhere.

(c) Find image of T: rank-nullity theorem gives: $\dim(\text{Im } T) = 4-1 = 3$

To find basis, we compute standard basis matrices:

$E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$

$B = \left\{\begin{bmatrix} 0 & -2 \\ 3 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\right\}$

...d the matrix $M_{B,B}(T)$ where $B = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right)$

$$T(E_{11}) = AE_{11} - E_{11}A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ -2 \\ 3 \\ 0 \end{bmatrix}$$

$$T(E_{12}) = AE_{12} - E_{12}A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} -3 & -3 \\ 0 & 3 \end{bmatrix}$$

$$\begin{bmatrix} -3 \\ -3 \\ 0 \\ 3 \end{bmatrix}$$

$$T(E_{21}) = AE_{21} - E_{21}A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & -4 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix} \times$$

7/10

$$\begin{bmatrix} 2 \\ 0 \\ 1 \\ -4 \end{bmatrix}$$

$$T(E_{22}) = AE_{22} - E_{22}A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -3 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 2 \\ -3 \\ 0 \end{bmatrix}$$

$$M_{BB}(T) = \begin{bmatrix} 0 & -3 & 2 & 0 \\ -2 & -3 & 0 & 2 \\ 3 & 0 & 1 & -3 \\ 0 & 3 & -4 & 0 \end{bmatrix}$$

✓ ✓ ✗ ✓

③ Given linear transformation

$D: F[x] \to F[x]$:

$D(x^n) = nx^{n-1}$ for all $n \geq 1$ and $D(1) = 0$

Null space: $\ker(D)$ consists of all polynomials $f(x) \in F[x]$ such that:

$D(f) = 0$

let $f(x) = a_0 + a_1x + a_2x^2 + \dots + a_nx^n$

$D(f) = a_1 + 2a_2x + 3a_3x^2 + \dots + na_nx^{n-1}$

continue on back

When $D(f)=0$: $D(f)=a_1+2a_2x+3a_3x^2+\dots+na_nx^{n-1}=0$

$$\boxed{a_1=0,\ 2a_2=0,\ 3a_3=0,\ \dots,\ na_n=0}$$

Therefore, the field F has characteristic 0, we have $n\neq 0$ for all n. Each equation forces $a_n=0$, which means that the only polynomial satisfying $D(f)=0$ is the constant polynomial $f(x)=a_0$

So the nullspace D is the space of constant polynomials:

$$\ker(D)=\{a_0 \mid a_0\in F\}$$ Good

Thus, $\text{nullity}(D)=1$ b/c space has dim of 1 Good

Image of D: $D(f)=a_1+2a_2x+3a_3x^2+\dots+na_nx^{n-1}$

because every polynomial of form:

$$b_0+b_1x+b_2x^2+\dots+b_{n-1}x^{n-1}$$

is obtained by differentiating the polynomial, the image of D consists of all polynomials that do not contain a constant term.

Therefore: $\text{im}(D)=\{g(x)\in F[x] \mid g(x)$ - with no constant term$\}$

The dimension of this space is infinite, so D has infinite rank.

6/10

Define $T: \mathbb{R}^{2\times 2} \to L(\mathbb{R}^{2\times 2}, \mathbb{R}^{2\times 2})$ by $T(A) = T_A$ where $T_A(X) = AX - XA$

ⓐ Check that all $A, B \in \mathbb{R}^{2\times 2}$ and c, d are scalars,

$$T(cA + dB) = cT(A) + dT(B)$$

$$T(cA + dB) = T_{cA+dB}$$

$$T_{cA+dB}(X) = (cA + dB)X - X(cA + dB)$$

$$= cAX + dBX - XcA - XdB = c(AX - XA) + d(BX - XB)$$

Therefore, $T_{cA+dB}(X) = cT_A(X) + dT_B(X)$, so T is a linear transformation ok. 10/10

ⓑ Find $\ker(T)$ ok 5/5

$$T_A(X) = AX - XA = 0 \quad \text{for all } X \in \mathbb{R}^{2\times 2}$$

This means A must commute with every $X \in \mathbb{R}^{2\times 2}$. And the only matrices that commute with all matrices in $\mathbb{R}^{2\times 2}$ are scalar multiples of identity matrix,

$A = \lambda I$, $\uparrow \lambda \in \mathbb{R}$ which means the nullspace: $\ker(T) = \{\lambda I \mid \lambda \in \mathbb{R}\}$

True, but show your work.

Since the space is one-dimensional, $\dim(\ker(T)) = 1$

ⓒ Rank of T: up ~~dim rank~~ rank-nullity theorem:

$$\dim(\ker(T)) + \text{rank}(T) = 4$$

$$= 1 + \text{rank}(T) = 4$$

$$\boxed{\text{rank}(T) = 3}$$

5/5

5) The basis for $\mathbb{R}^{2\times 2}$ is $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$

$-E_{ij}$ is the matrix with 1 in (i,j) position and 0 else

The basis C for $\mathbb{R}[x]_{\leq 2}$ is $C = \{1, x, x^2\}$

$E_{11}: q_{E_{11}} = [x \ 1] E_{11} [x \ 1] = [x \ 1] \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} [x \ 1] = [x \ 1] \begin{bmatrix} x \\ 1 \end{bmatrix} = x^2$

$E_{12}: q_{E12} = [x \ 1] E_{12} [x \ 1] = [x \ 1] \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} [x \ 1] = [x \ 1] \begin{bmatrix} 1 \\ 0 \end{bmatrix} = x$

$E_{21}: q_{E_{21}} = [x \ 1] E_{21} [x \ 1] = [x \ 1] \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} [x \ 1] = [x \ 1] \begin{bmatrix} 0 \\ x \end{bmatrix} = \cancel{x}$

$E_{22}: q_{E_{22}} = [x \ 1] E_{22} [x \ 1] = [x \ 1] \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} [x \ 1] = [x \ 1] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1$

$Q(E_{11}) = x^2$ is $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ in C

$Q(E_{12}) = x$ is $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ in C

$Q(E_{21}) = 0$ is $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ in C

$Q(E_{22}) = 1$ is $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ in C

$M_{C,\beta}(Q) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$

8/10

---

⑥ (a) The preimage of β is $T^{-1}(\{\beta\}) = \{\alpha \in V \mid T(\alpha) = \beta\}$ consists of all solutions to $T(\alpha) = \beta$.

Let $\alpha_1, \alpha_2 \in T^{-1}(\{\beta\})$: $T(\alpha_1) = \beta$, $T(\alpha_2) = \beta$

8/10

then $T(\alpha_1 - \alpha_2) = \beta - \beta = 0$

so $\alpha_1 - \alpha_2 \in V'$ and all elements in $T^{-1}(\{\beta\})$ differ by an element of V' forms a coset in V/V'.

$T^{-1}(\{\beta\}) = \alpha_1 + V'$

Needs more detail. Seems pretty close to a proof that $T^{-1}(\{\beta\})$ is contained in $\alpha_2 + V'$. But not $=$..

Therefore, every $\beta \in W'$ corresponds to exactly one coset in V/V' $T^{-1}(\{\beta\}) \in V/V'$

Defie $U: W' \to V/V'$ by $U(\beta) = T^{-1}(\{\beta\})$. Prove that it is an isomorphism.

$$U(\beta) = T^{-1}(\{\beta\}) = \alpha + V'$$

not really needed

Show U is well defined: For each $\beta \in W'$, there exists at least one $\alpha \in V$ such that $T(\alpha) = \beta$ and all choices of α differ by an element of V'. So the coset $\alpha + V'$ is uniquely determined

Show U is linear: let $\beta_1, \beta_2 \in W'$

so $U(\beta_1) = \alpha_1 + V'$, $U(\beta_2) = \alpha_2 + V'$, for some $\alpha_1, \alpha_2 \in V$. (introduce each new variable clearly.)

T is linear

$$T(\alpha_1 + \alpha_2) = T(\alpha_1) + T(\alpha_2) = \beta_1 + \beta_2$$

Therefore, $U(\beta_1 + \beta_2) = (\alpha_1 + \alpha_2) + V' = U(\beta_1) + U(\beta_2)$

check scalars: introduce β, α, c properly & explain your logic.

$$U(c\beta) = (c\alpha) + V' = cU(\beta)$$

7/10

Therefore, U is linear

Show U is bijective

- Injectivity - if $U(\beta) = V'$, then $\beta = 0$, means $\ker(U) = \{0\}$
- Surjectivity - Every coset in V/V' is of form $\alpha + V'$, and each such α maps to some $\beta \in W'$

Therefore, U ~~satis~~ satisfies bijective linear transformation conditions, so it is isomorphism.

76/100 Good work