MTH420 PROBLEM SET 3

(1) Let F be a field, and let m, n be positive integers with $m < n$. Show that, for every $A \in F^{n\times m}$ and every $B \in F^{m\times n}$ the matrix AB is non-invertible.

(2) Is the matrix
$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 2 & 3 & 4 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 4 \end{bmatrix} \in \mathbb{R}^{4\times 4}$$
invertible? If so, find the inverse. If not, prove it.

(3) Which of the following are subspaces of $\mathbb{R}^{2\times 2}$? (Prove that your answers are correct.)
   (a) $\left\{A \in \mathbb{R}^{2\times 2} : A^2 = A\right\}$
   (b) $\left\{A \in \mathbb{R}^{2\times 2} : A\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} A\right\}$
   (c) $\left\{A \in \mathbb{R}^{2\times 2} : A\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = 0\right\}$ (Here, 0 means the 2×2 matrix, all of whose entries are 0.)

(4) Find the intersection of all the subspaces from problem (3).

(5) Find the sum of all the subspaces from problem (3).

(6) Find a basis for the space from problem (5), and prove that it is a basis.

(7) Consider the space $\mathbb{R}^{\mathbb{R}}$ of all functions from $\mathbb{R}$ to $\mathbb{R}$. Which of the following are subspaces? (Prove that your answers are correct.)
   (a) The set of all continuous elements.
   (b) The set of all odd elements.
   (c) The set of all discontinuous elements.
   (d) The set of all solutions to the ODE $y'' - 2y' + y = x^2$.

Date: February 7, 2025.