MTH420 PROBLEM SET 11

(1) Let R be a ring, and S a set of ideals in R. (Don't assume S is finite.) Prove that

$$\bigcap_{I \in S} I$$

is also an ideal in R.

(2) Let R and S be rings, and $\varphi : R \to S$ a ring homomorphism, i.e., a function such that for any $a, b \in R$

$$\varphi(a+b) = \varphi(a) + \varphi(b), \qquad \varphi(ab) = \varphi(a)\varphi(b).$$

Prove that the kernel $\ker \varphi := \{a \in R; \varphi(a) = 0_S\}$ is an ideal.

(3) Let F be a field, $\mathcal{A}$ an F-algebra, and α an element of $\mathcal{A}$. Prove that

$$\{f \in F[x] : f(\alpha) = 0\}$$

is an ideal in $F[x]$.

(4) Take $M \neq \{0\}$ an ideal in $F[x]$.
  (a) Prove that M is a subspace of $F[x]$.
  (b) Consider the quotient space $F[x]/M = \{f + M : f \in F[x]\} = \{\{f + m : m \in M\} : f \in F[x]\}$. Prove that it is finite-dimensional, and describe how the dimension depends on M. (You'll need to use what we've learned about the set of ideals in $F[x]$.)
  (c) Prove that $(f + M) \oplus (g + M) = (f + g) + M$ gives a well defined binary operation on $F[x]/M$.
  (d) Prove that $(f+M) \otimes (g+M) = (f \cdot g) + M$ give a well defined binary operation on $F[x]$.
  (e) Prove that $(F[x]/M, \oplus, \otimes)$ is a commutative ring with unity.

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