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MTH 420
PS 9

① Proof using Zorn's lemma:

what's $\mathbb{F}$?
?

Let $\mathcal{F} = \{(U, h) \mid W \subseteq U \subseteq V,$ U is a subspace, $h: U \to \mathbb{F}$ linear, $h|_W = f\}$

This set that contains all linear functions h to f for some subspace U of V.

f? or F? or $\mathbb{F}$?

Partial ordering: $(U_1, h_1) \leq (U_2, h_2)$ if: $U_1 \subseteq U_2$

and $h_2|_{U_2} = h_1$

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Thus, (U_2, h_2) extends to (U_1, h_1)

Apply Zorn's lemma: Take any chain $\{(U_i, h_i)\}_{i \in I} \subseteq \mathcal{F}$

$\bar{U} = \bigcup_{i \in I} U_i$, for each $\alpha \in \bar{U}$ define $h(\alpha) = h_i(\alpha)$ where $\alpha \in U_i$

The chain is totally ordered, the values assigned to α are consistent. So h is well-defined and linear on $\bar{U}$, and
explain
$h|_W = f$. Therefore $(\bar{U}, h) \in \mathcal{F}$ and is an upper bound of the chain.

By Zorn's lemma, there exists a maximal element $(U_{max}, h_{max}) \in \mathcal{F}$

Show $U_{max} = V$: Suppose not, then there exists $v_0 \in V \setminus U_{max}$

$U' = U_{max} + \mathbb{F}v_0$

We want to extend h_{max} to U', so every element in U' can be written uniquely as $u + \lambda v_0$, where $u \in U_{max}$, $\lambda \in \mathbb{F}$

Define: $h'(u + \lambda v_0) = h_{max}(u) + \lambda a$

Thus, $(U', h') \in \mathcal{F}$ and strictly extends (U_{max}, h_{max}) contradicts maximality

Therefore, there exist a linear functional $g: V \to \mathbb{F}$ such that $g|_W = f$.

② Prove $W^* \cong V^*/W^0$:

Define map $\phi: V^* \to W^*$ by restriction: $\phi(\varphi) = \varphi|_W$

ϕ is surjective linear map, b/c any $f \in W^*$ can be extended to V using Zorn's lemma.

The kernel of ϕ is precisely W^0 and all $\varphi \in V^*$ will vanish on W.

- By the First isomorphism Theorem for vector spaces:

$$V^*/W^0 \cong \mathrm{Im}(\phi) = W^*$$

- QED

③ ⓐ let $p(x) = a_0 + a_1 x + a_2 x^2 \Rightarrow D(p) = a_1 + 2a_2 x$

compute

$D^t(f_0)(p) = f_0(D(p)) =$ Constant term $D(p) = a_1 \Rightarrow D^t(f_0) = f_1$

$D^t(f_1)(p) = f_1(D(p)) = D(p) = 2a_2 \Rightarrow D^t(f_1) = 2f_2$

$D^t(f_2)(p) = f_2(D(p)) = D(p) = 0 \Rightarrow D^t(f_2) = 0$

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ⓑ let and $e_1(p) = p(1) = a_0 + a_1 + a_2$:

$D^t(e_1)(p) = e_1(D(p)) = D(p)(1)$

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$D(p) = a_1 + 2a_2 x$: $D(p)(1) = a_1 + 2a_2 \Rightarrow D^t(e_1) = f_1 + 2f_2$

ⓒ $e_t(p) = p(t)$, $D(p)(t) = a_1 + 2a_2 t \Rightarrow D^t(e_t)(p) = D(p)(t) =$
$a_1 + 2a_2 t$ Therefore, $D^t(e_t) = f_1 + 2tf_2$

$f(x) = x^2 - 3x + 1$ $\quad$ $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$

$g(x) = x^3 - 2x$

$f(A) = A^2 - 3A + 1$

$A^2 = A \cdot A$ $\quad$ $A^2 = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}$

$= \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} - 3 \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \cancel{\begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix}} \begin{bmatrix} 6 & 9 \\ 3 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = A$

$f(A) = A$

$g(A) = A^3 - 2A$

$A^3 = A \cdot A^2$ $\quad$ $A^3 = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 7 & 12 \\ 4 & 7 \end{bmatrix} = \begin{bmatrix} 26 & 39 \\ 15 & 22 \end{bmatrix}$

$g(A) = \begin{bmatrix} 26 & 39 \\ 15 & 22 \end{bmatrix} - 2 \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 26 & 39 \\ 15 & 22 \end{bmatrix} - \begin{bmatrix} 4 & 6 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 22 & 33 \\ 13 & 18 \end{bmatrix}$

$g(A) = \begin{bmatrix} 22 & 33 \\ 13 & 18 \end{bmatrix}$ -4 $\quad$ 6/10

b) $f(g(x)) = g(x)^2 - 3g(x) + 1$

$g(x) = x^3 - 2x$

$g(x)^2 = (x^3 - 2x)^2 = (x^3 - 2x)(x^3 - 2x) = x^6 - 2x^4 - 2x^4 + 4x^2$

$x^6 - 4x^4 + 4x^2$

$f(g(x)) = x^6 - 4x^4 + 4x^2 - 3(x^3 - 2x) + 1 = x^6 - 4x^4 + 4x^2 - 3x^3 + 6x + 1$

$f(g(x)) = x^6 - 4x^4 - 3x^3 + 4x^2 + 6x + 1$ $\quad$ 40/50 Good

$g(x) = x^3 - 2x$

$g(f(x)) =$

$f = x^2 - 3x + 1 \qquad f^3 = (x^2-3x+1)^3 = x^6 - 9x^5 + 30x^4 - 45x^3 + 30x^2 - 9x + 1$

$g(f(x)) = x^6 - 9x^5 + 30x^4 - 45x^3 + 30x^2 - 9x + 1 \quad - 2(x^2 - 3x + 1)$

$x^6 - 9x^5 + 30x^4 - 45x^3 + 30x^2 - 9x + 1 - 2x^2 + 6x - 2$

6/10 $\qquad -4$

(c) ~~Let $T(X) = AX - XA$, $f(T) = T^2 - 3T + I$, $g(T) = T^3 - 2T$~~

~~Apply to general matrix $X \in \mathbb{R}^{2\times 2}$~~

~~define $T(X) = AX - XA$~~

~~$T(X) = AX - XA$~~

~~$f(T) = T^2 - 3T + I$~~

~~$g(T) = T^3 - 2T$~~

$f(x) = x^2 - 3x + 1$
$g(x) = x^3 - 2x$

$T(X) = AX - XA$

compute: $f(T)(X)$: $f(T)(X) = T(X)^2 - 3T(X) + I$

compute: $g(T)(X)$: $g(T)(X) = T(X)^3 - 2T(X)$