MTH420 PROBLEM SET 4

(1) Let F be a field and V a vector space over F. Let S be a subset of V. Prove that V has a basis $\mathcal{B}$ such that $S \subset \mathcal{B}$. (Note: the space V is not assumed to be finite dimensional. If V is finite dimensional, we proved the existence of $\mathcal{B}$ in class, so the job here is really to handle the infinite dimensional case. My recommendation would be to adapt the argument we used to prove that V has a basis.)

(2) Let F be a field and let X be a set with n elements. View $F^X = \{f : X \to F\}$ as an F-vector space in the usual way: so that, for each $f, g \in F^X$ and $c \in F$, the element $cf + g$ is, by definition, the function which sends each $x \in X$ to $cf(x) + g(x)$. What is the dimension of F^X? Prove that your answer is correct.

(3) Let us view $\mathbb{C}^{2\times 2}$ as a vector space over the field $\mathbb{R}$. (This means, we consider scalar multiplication only by elements of $\mathbb{R}$, even though the entries are in $\mathbb{C}$.) Find a basis, and the dimension.

(4) If A is a matrix with complex entries, $\overline{A}$ is defined to be the matrix obtained by conjugating each entry of A, and A^t is the transpose. For example,

$$\overline{\begin{bmatrix} 1+2i & 3-i \\ 2 & 2+3i \end{bmatrix}} = \begin{bmatrix} 1-2i & 3+i \\ 2 & 2-3i \end{bmatrix}, \qquad \text{and} \qquad \begin{bmatrix} 1+2i & 3-i \\ 2 & 2+3i \end{bmatrix}^t = \begin{bmatrix} 1+2i & 2 \\ 3-i & 2+3i \end{bmatrix}.$$

Continue viewing $\mathbb{C}^{2\times 2}$ as a vector space over $\mathbb{R}$. Prove that $\{A \in \mathbb{C}^{2\times 2} : \overline{A}^t = A\}$ is a subspace, and find a basis for it.

(5) Now let's switch back to viewing $\mathbb{C}^{2\times 2}$ as a 4-dimensional vector space over $\mathbb{C}$. Prove that $\{A \in \mathbb{C}^{2\times 2} : \overline{A}^t = A\}$ is **not** a subspace.

(6) Find three elements of $\mathbb{R}^3$ which are linearly dependent, even though any two of them are linearly independent.

(7) Prove that

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \end{bmatrix} \right\}$$

is a basis of $\mathbb{R}^{4\times 1}$.

(8) This problem is trivial once the next one is completed, but might be a useful warm-up if the next one looks too hard at first. Find c_1, c_2, c_3, c_4 such that

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}.$$

(9) Find c_1, c_2, c_3, c_4 such that

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} + c_4 \begin{bmatrix} 0 \\ 2 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}.$$

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