Ivan Wang

MTH 420

PS 7

① (a) Function is invertible iff it's bijective (injective, surjective)

T is a function that goes from $\mathbb{R}^3$ to $\mathbb{R}^2$, meaning higher dimension to lower dimension.

Rank-nullity Thm: $\dim(\ker(T)) + \dim(\operatorname{im}(T)) = 3$

T maps to $\mathbb{R}^2$ and $\dim(\operatorname{im}(T)) \leq 2$, therefore, $\dim(\ker(T)) \geq 1$, so T is not injective

10/10

U is a function that goes from $\mathbb{R}^2$ to $\mathbb{R}^3$, lower dimension to higher dimension.

This means that image of U is ≤ 2.
So U is not surjective.

$U \circ T : \mathbb{R}^3 \to \mathbb{R}^2$ Since T is not injective there exists a nonzero vector such that $T(v)=0$.

$U(T(v)) = U(0) = 0$

Therefore $U \circ T$ is not injective, and so it cannot be invertible ~~or~~ bijective? But ok

4/5

ⓑ let $T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$

$T \circ U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ — identity transformation of $\mathbb{R}^2$ and is invertible.

Not bad but "linear transformation" does not mean "matrix". These are classes of objects which are related in various ways but not the same thing

② (a) Because $T \circ U = I$, this means that U is injective

For $U(v) = U(w)$ for $v, w \in V$, so apply T

$$T(U(v)) = T(U(w))$$

because $T \circ U = I$: $I(v) = I(w) \Rightarrow v = w$, therefore U is injective.

- In an finite-dimensional vector space, an injective transformation is also surjective b/c of rank-nullity thm (the image must have same dimension as the domain). This means that U is surjective and U is bijective.

Therefore, U is bijective and has an inverse.

So $T \circ U = I$, it follows that $T = U^{-1}$

and this means $U \circ T = I$. 10/10

U is a inverse of T.

(b) consider $V = \ell^2(\mathbb{N})$, the space of square-summable sequences of real numbers

$$\ell^2(\mathbb{N}) = \{(x_1, x_2, x_3, \ldots) \mid \sum_{i=1}^{\infty} x_i^2 < \infty\}$$

$$U: \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N}) : U(x_1, x_2, x_3, x_4, \ldots) = (0, x_1, x_2, x_3, \ldots)$$

shifts all coordinates to the right and puts 0 in 1st position

ok

$T: \ell^2(\mathbb{N}) \to \ell^2(\mathbb{N})$; $T(x_1, x_2, x_3, x_4, \ldots) = (x_2, x_3, x_4, \ldots)$

Check $(T \circ U)(x_1, x_2, x_3, \ldots) = T(0, x_1, x_2, x_3, \ldots) = (x_1, x_2, x_3, \ldots)$
$= I(x_1, x_2, x_3 \ldots)$

So $T \circ U = I$, however U is not surjective b/c no sequence gets mapped to a sequence whose first coordinate is nonzero. There is no $y \in \ell^2(\mathbb{N})$ such that $U(y) = (1, 0, 0 \ldots)$ which proves that U is not onto 5/5 Good

therefore U is not surjective $U \circ T \neq I$, so U is not the inverse of T.

---

③ For L_B to be invertible, it must be injective and surjective.

what's X here? $X \in F^{m \times n}$?

- For L_B to be injective, the only solution to $L_B(x) = BX = 0$ should be $X = 0$. This is true iff B is injective as a linear transformation on column vectors. B has full column

← But X wasn't a col vec?

rank, so $\ker(B) = \{0\}$

why?

Because B is $p \times m$, it only injective if $\mathrm{rank}(B) = m$, so $p \geq m$. If $p < m$, then B has a nontrivial kernel, means there exist nonzero matrices X such that $BX = 0$, which makes L_B is not injective.

For L_B to be surjective, every matrix $Y \in F^{p \times n}$ must have preimage $X \in F^{m \times n}$ such that $BX = Y$. This is only possible iff B spans all of F^p, so B has full row rank.

Spanning without sense

F^p or $F^{p \times n}$

$\text{rank}(B) = p$ and this condition can only hold if $p \leq m$. Since B has at most $\min(p, m)$ linearly independent rows,

4/10 only tackle one part of "if and only if"

- For L_B to be both injective and surjective, we need, $\text{rank}(B) = m$ (to satisfy injectivity, which requires $p \geq m$), and $\text{rank}(B) = p$ (to satisfy surjectivity, which requires $p \leq m$)

The only way this can be held true is when $p = m$ (B is a square matrix of size $m \times m$ and it must have full rank.

φ is isomorphism if it's bijective and linear.

Prove linearity: For $T_1, T_2 \in T(V,V)$ and α, β be scalars,

$\varphi(\alpha T_1 + \beta T_2) = U \circ (\alpha T_1 + \beta T_2) \circ U^{-1}$

apply linearity of composition: $U \circ (\alpha T_1 + \beta T_2) \circ U^{-1} = \alpha (U \circ T_1 \circ U^{-1}) + \beta (U \circ T_2 \circ U^{-1})$

$\circ$ is not linear in general e.g. $f, g, h: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2 + 1$, $g(x) = \sin x$, $h(x) = e^x$, then $f \circ (2g + 3h) \neq 2 f \circ g + 3 f \circ h$ ...

Therefore $\varphi(T_1) = U \circ T_1 \circ U^{-1}$ and $\varphi(T_2) = U \circ T_2 \circ U^{-1}$

we get $\varphi(\alpha T_1 + \beta T_2) = \alpha \varphi(T_1) + \beta \varphi(T_2)$

so φ is linear.

Prove φ is injective: let $\varphi(T)=0$

$U \circ T \circ U^{-1} = 0$

41/50

Good!

$T = U^{-1} \circ 0 \circ U = 0$

8/10

Therefore, $\ker(\varphi) = \{0\}$, This means that φ is injective.

Prove φ is surjective: For $S \in L(W,W)$, define $T = U^{-1} \circ S \circ U$

Then: $\varphi(T) = U \circ (U^{-1} \circ S \circ U) \circ U^{-1} = S$

Therefore, every $S \in L(W,W)$ has a preimage in $L(V,V)$, φ is surjective.

Therefore, φ satisfies all conditions, so it is a bijective linear map and is isomorphism.

5 (a) Since $U: V \to W$ is isomorphism, if $B = \{v_1, \dots v_n)$ is a basis for V, then $C = \{U(v_1), \dots, U(v_n)\}$ is a basis for W

- The matrix of T relative to B is $[T]_B = A$

Because $S = U T U^{-1}$, the matrix of S relative to C is

$[S]_C = P [T]_B P^{-1}$

where P is the change of the basis matrix from B to C, given the column vectors of $U(v_i)$.

Therefore, the conjugation preserves similarity classes, every matrix in A_V corresponds to one in A_W, this proves

$A_V = A_W$

(b) Suppose there is a common matrix A for both A_V and A_W, this means that there exist bases B for V and C for W such that,

$$A = [T]_\beta = [S]_C$$

$U: V \to W$ has unique isomorphism that maps B to C:

$U(v_i) = c_i$ for each basis vector $v_i \in B$, $c_i \in C$

Because U is an isomorphism

$$UT(v_i) = U\left(\sum_j A_{ji} v_j\right) = \sum_j A_{ji} U(v_j) = \sum_j A_{ji} c_j = S(U(v_i))$$

Therefore, $U \circ S = S \circ U$, $S = UTU^{-1}$