MTH420 “PROBLEM SET” 13

This is not a problem set that’s going to actually be *due*. It’s just some problems on the last chunk of material for practice.

(1) Let $A = \begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 3 \end{bmatrix}$. Find the characteristic and minimal polynomials of A. Show your work and be sure to explain how you know the minimal polynomial is minimal.

(2) Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. Find the characteristic and minimal polynomials of A. Show your work and be sure to explain how you know the minimal polynomial is minimal.

(3) The characteristic polynomial of a matrix A is $(x-1)^3(x-2)^2(x+1)$.
  (a) List all possibilities for the minimal polynomial.
  (b) For each of the possibilities, find a Jordan matrix with that minimal polynomial (and the given characteristic polynomial).

(4) Find two Jordan matrices which have the same characteristic and minimal polynomial, but are not similar. Prove that your answer is correct.

(5) Let $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ be the operator defined by $T(X) = AX - XA$ where

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

Compute
  (a) The characteristic polynomial of T.
  (b) The minimal polynomial of T.
  (c) The Jordan form of T.

(6) Let D be the derivative operator on the space $C^\infty(\mathbb{R}, \mathbb{R})$ of infinitely differentiable functions $\mathbb{R} \to \mathbb{R}$, and let $f_1(x) = xe^{3x}\cos(2x)$.
  (a) Let W be the span of $\{D^k : k \in \mathbb{Z}_{\geq 0}\}$. Find the dimension of W.
  (b) Consider the operator D_W on W induced by D. Find its characteristic polynomial.
  (c) Find its minimal polynomial.
  (d) Briefly explain why D_W can’t be represented by a Jordan matrix with entries in $\mathbb{R}$.

(7) Let D be the derivative operator on the space $C^\infty(\mathbb{R}, \mathbb{C})$ of infinitely differentiable functions $\mathbb{R} \to \mathbb{C}$, and let $f_1(x) = xe^{3x}\cos(2x)$. Let U be the span of $\{D^k : k \in \mathbb{Z}_{\geq 0}\}$. Consider the operator D_U on U induced by D.
  (a) Find its Jordan form.
  (b) Find an ordered basis of U such that the Jordan form is the matrix of U with respect to that basis.
  (c) It is possible to do (a) without doing (b) first. Do you see why?

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