MTH420 PROBLEM SET 8

(1) Let m, n be positive integers. Prove that, for all $A \in F^{m\times n}$ and $B \in F^{n\times m}$,
$$\mathrm{Tr}(AB) = \mathrm{Tr}(BA).$$
(Here, Tr means "trace".)

(2) Let $P_2(\mathbb{R})$ be the space of polynomial-functions $\mathbb{R} \to \mathbb{R}$ of the form $p(t) = a_0 + a_1 t + a_2 t^2$. (That is, $P_2(\mathbb{R})$ is the image of $\mathbb{R}[x]_{\le 2}$ under the mapping $\mathbb{R}[x] \to \mathbb{R}^{\mathbb{R}}$ which sends each polynomial to the corresponding polynomial-function.) Let
$$f_1(p) = \int_0^1 p(t)\,dt, \qquad f_2(p) = \int_1^2 p(t)\,dt, \qquad f_3(p) = \int_{-1}^0 p(t)\,dt.$$

(a) Find three elements $p_1, p_2, p_3 \in P_2(\mathbb{R})$ such that $f_i(p_j) = \delta_{i,j}$ for $i, j \in \{1, 2, 3\}$.

(b) Prove that $\{f_1, f_2, f_3\}$ is a basis of the dual space $(P_2(\mathbb{R}))^*$.

(c) For each $a \in \mathbb{R}$ we have the mapping $e_a : P_2(\mathbb{R}) \to \mathbb{R}$ given by $e_a(p) = p(a)$. Prove that $\{e_0, e_1, e_{-1}\}$ is a basis of $(P_2(\mathbb{R}))^*$.

(d) Express f_1 as a linear combination of e_0, e_1 and e_{-1}.

(e) The set $\{e_0, e_1, e_2\}$ is also a basis of $(P_2(\mathbb{R}))^*$. Suppose that (p_1, p_2, p_3) is the basis of $P_2(\mathbb{R})$ whose dual is (e_0, e_1, e_2) and (q_1, q_2, q_3) is the basis of $P_2(\mathbb{R})$ whose dual is (e_0, e_1, e_{-1}). Is $p_1 = q_1$ and $p_2 = q_2$? Justify your answer.

(f) (Optional) With the same set-up as in the previous problem, what do you observe about p_3 and q_3? Can you explain this observation?

(3) Let
$$S = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\} \subset \mathbb{R}^{5\times 1}.$$
Find a basis for $S^0 \le L(\mathbb{R}^{5\times 1}, \mathbb{R})$. (Notice that $L(\mathbb{R}^{5\times 1}, \mathbb{R})$ is a space of functions from $\mathbb{R}^{5\times 1}$ to $\mathbb{R}$, not a space of column vectors, or row vectors, etc. You can parametrize the elements of $L(\mathbb{R}^{5\times 1}, \mathbb{R})$ using column vectors, or row vectors, or tuples. It's a good idea. But be sure to explain your parametrization to me before trying to use it to communicate with me.)

(4) Let V be a vector space over a field F, and W_1, W_2 two subspaces of V.

(a) Prove that $(W_1 \cap W_2)^0 = W_1^0 + W_2^0$.

(b) Prove that $(W_1 + W_2)^0 = W_1^0 \cap W_2^0$.

(I suspect you will need to use the following fact: if S is a linearly independent set of a vector space, then the space has a basis which contains S.) (Note that we are not assuming V is finite dimensional.)

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