MTH420 PROBLEM SET 9

(1) Let V be a vector space, W a subspace of V, and f a linear functional on W. Prove there is a linear functional g on V such that $g(\alpha) = f(\alpha)$ for all $\alpha \in W$. (This is sometimes expressed as "f may be extended to a functional on V.") This is approximately, problem 12 from section 3.5, except I'm *not* telling you to assume V is finite dimensional.

(2) Let V be a vector space, W a subspace of V. Prove that W^* is isomorphic to V^*/W^0. (Proof should work even when V is not finite dimensional).

(3) Consider the space $\mathbb{R}[x]_{\le 2}$ of at-most-quadratic polynomials with real coefficients, and the differentiation operator $D : \mathbb{R}[x]_{\le 2} \to \mathbb{R}[x]_{\le 2}$. For $i = 0, 1, 2$, let f_i be the functional which maps $a_0 + a_1 x + a_2 x^2$ to a_i.
   (a) Compute $D^t(f_i)$ for each i. Express your answer as a linear combination of f_0, f_1, f_2.
   (b) For each $t \in \mathbb{R}$ let e_t be the "plug in t" functional $\mathbb{R}[x]_{\le 2} \to \mathbb{R}$. Express $D^t(e_1)$ as a linear combination of f_0, f_1, f_2.
   (c) Express $D^t(e_t)$ as a linear combination of f_0, f_1, f_2. (The coefficients in the combination will probably depend on the variable t, I would think.)

(4) Let $f = x^2 - 3x + 1$ and $g = x^3 - 2x$. Compute
   (a) $f(A)$ and $g(A)$ were $A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}$.
   (b) $f(g)$ and $g(f)$.
   (c) $f(T)(X)$ and $g(T)(X)$, where $T : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ is given by

$$T(X) = AX - XA, \qquad A = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix}.$$

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