MTH420 PROBLEM SET 6

(1) Let V be the two dimensional subspace of $C^\infty(\mathbb{R})$ spanned by $e^{2x}\sin 3x$ and $e^{2x}\cos 3x$. Notice that $f' \in V$ for each $f \in V$, so that the formual $D(f) = f'$ defines an operator $D : V \to V$. Let $\mathcal{B} = (e^{2x}\sin 3x, e^{2x}\cos 3x.)$ and compute $M_{\mathcal{B},\mathcal{B}}(D)$.

(2) Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and for $X \in \mathbb{R}^{2\times 2}$ let $T(X) = AX - XA$.
   (a) Prove that T is linear.
   (b) Find the nullspace of T.
   (c) Find the image of T.
   (d) Find the matrix $M_{\mathcal{B},\mathcal{B}}(T)$ where

$$\mathcal{B} = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right).$$

(3) Take F an arbitrary field of characteristic 0 and let $D : F[x] \to F[x]$ be the unique linear transformation which maps x^n to nx^{n-1} for each positive integer n, and x^0 to 0. Find the nullspace, image, and nullity of D.

(4) Define $\mathcal{T} : \mathbb{R}^{2\times 2} \to L(\mathbb{R}^{2\times 2}, \mathbb{R}^{2\times 2})$ by $\mathcal{T}(A) = T_A$, where $T_A(X) = AX - XA$.
   (a) Prove that $\mathcal{T}$ is linear.
   (b) Find the nullspace of $\mathcal{T}$.
   (c) Find the rank of $\mathcal{T}$.

(5) Let $\mathbb{R}[x]_{\le 2} = \{a_0 + a_1x + a_2x^2 : a_0, a_1, a_2 \in \mathbb{R}\} \le \mathbb{R}[x]$. For $A \in \mathbb{R}^{2\times 2}$, let $q_A = \begin{bmatrix} x & 1 \end{bmatrix} A \begin{bmatrix} x \\ 1 \end{bmatrix} \in \mathbb{R}[x]_{\le 2}$. Then let $Q : \mathbb{R}^{2\times 2} \to \mathbb{R}[x]_{\le 2}$ be the function defined by $Q(A) = q_A$. It is linear and you may use this fact without including a proof in your solutions. (It's similar to (2)(a) and I don't think it's worthwhile to include two such similar problems in the set.) Take $\mathcal{B}$ the basis of $\mathbb{R}^{2\times 2}$ as in (2)(e) and $\mathcal{C} = (1, x, x^2)$ (an ordered basis of $\mathbb{R}[x]_{\le 2}$). Compute $M_{\mathcal{C},\mathcal{B}}(Q)$.

(6) Let $T : V \to W$ be a linear transformation. Let V' be the kernel of T and W' the image of T.
   (a) Prove that $T^{-1}(\{\beta\})$ (which is defined as $\{\alpha \in V : T(\alpha) = \beta\}$) is an element of V/V' for each $\beta \in W'$.
   (b) Define $U : W' \to V/V'$ by $U(\beta) = T^{-1}(\{\beta\})$. Prove that it is an isomorphism.

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