IIITV/TDA/EC429

Topological Data Analysis

End Sem Examination*

Duration: 180 Minutes (Maximum - 35 Points)

December 28, 2023

1. For a simplicial complex, the Betti numbers b_i are the dimensions of the homology groups, $H_i = ker(\partial_i)/img(\partial_{i+1})$ of the chain complex

$$\cdots \longrightarrow C_{i+1} \xrightarrow{\partial_{i+1}} C_i \xrightarrow{\partial_i} C_{i-1} \longrightarrow \cdots \longrightarrow C_{-1} = 0 \qquad (1)$$

where, C_i is the space of formal $\mathbb{R}$-linear sums (free vector space over $\mathbb{R}$) of oriented i-dimensional faces, i.e. oriented subsets of the abstract simplicial complex of size $i+1$, ∂_i is the boundary map. Endowing each C_i with an inner-product, we get maps $\partial_i^* : C_{i-1} \longrightarrow C_i$ (i.e. the transpose of ∂_i), and thus a Laplacian, $\Delta_i : C_i \longrightarrow C_i$, for each i, defined by $\Delta_i = \partial_{i+1}\partial_{i+1}^* + \partial_i^*\partial_i$.

For each i we define the set of harmonic i-forms to be

$$\mathcal{H}_i := \{c \in C_i | \Delta_i c = 0\}. \qquad (2)$$

[C1] Claim: (Hodge theory) For each i, $\mathcal{H}_i \cong H_i$. The Betti numbers are the dimensions of the $\mathcal{H}_i$.

(a) For the simplicial complex shown in Figure 1, find out b_1 using standard algorithm discussed in class.

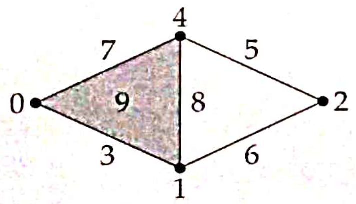

Figure 1: A simplicial complex

(b) Using [C1] calculate b_1.

(c) Argue about correctness of claim [C1]. [Hint: Elementary linear algebra]

[10 Points]

*This is an open book/ notes examination.

~~1-0~~ + 2-1 + 4-2 ~~+0-4~~ + ~~1-0~~

2. A filtered simplicial complex consists of a nested sequence of simplicial complexes: $K_0 \subset K_1 \subset \cdots \subset K_n$. Applying homology, we obtain a sequence of vector spaces and linear maps called a *persistence module*:

$$H_k(K_0) \to H_k(K_1) \to \cdots \to H_k(K_n). \tag{3}$$

The images of these maps and compositions of these maps are called persistent homology vector spaces. Given a persistence module

$$M_0 \to M_1 \to \cdots \to M_n, \tag{4}$$

the *rank function* assigns each pair (i, j) with $1 \leq i < j \leq n$ the rank of the linear map $M_i \to M_j$ given by comoposition of maps. If $i = j$ then this is the rank of the identity map on M_i which equals the dimension of M_i. Assume that the filteration on the simplicial complex of Figure 1 is defined with respect to the lexicographic ordering on the numbers assigned to the simplices, i.e.

$$X_0 \hookrightarrow X_1 \hookrightarrow X_2 \hookrightarrow \cdots \hookrightarrow X_9 \tag{5}$$

where $X_k = \{i^{th} simplex | i \leq k\}$.

(a) Write down boundary matrices for ∂_1 and ∂_2, ordering the simplices using the order in which they appear in the filtration (5).

(b) Apply the persistence algorithm to reduce these matrices using column operations.

(c) Give the birth-death pairs for H_0 and H_1.

(d) Plot the corresponding barcodes and persistence diagrams.

(e) Determine the rank functions for H_0 and H_1 .

[20 Points]

3. Landspaces For $b < d$, let $f_{(b,d)} : \mathbb{R} \to \mathbb{R}$ be the piecewise-linear "tent" function given by

$$f_{(b,d)}(t) = \begin{cases} t - b & if\; b \leq t \leq \frac{b+d}{2} \\ d - t & if\; \frac{b+d}{2} \leq t \leq d \\ 0 & otherwise. \end{cases} \tag{6}$$

Considere a persistence module with persistence diagram $D = \{(b_1, d_1), \ldots, (b_m, d_m)\}$. For $k = 1, 2, \ldots$ the k-th *persistence landscap function*, $\lambda_k : \mathbb{R} \to \mathbb{R}$, is given by letting $\lambda_k(t)$ be equal to the k-th largest value of $\{f_{(b_1,d_1)}(t), \ldots, f_{(b_m,d_m)}(t)\}$. The sequence of these functions is called the *persistence landscape*. The persistence landscape may also be viewed as the function $\lambda : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ given by $\lambda(k, t) = \lambda_k(t)$. Graph the persistence landscapes for H_0 and H_1 corresponding to the persistence diagrams found in previous exercise. [5 Points]

4. Let F denote a finite collection of closed, convex sets in $\mathbb{R}^d$. Prove that if the intersection of any $d + 1$ sets of $\mathbb{F}$ is non-empty, then the whole collection has non-empty intersection. [5 Points]