

# Chapter ML:II

## II. Machine Learning Basics

- ❑ Regression
- ❑ Concept Learning: Search in Hypothesis Space
- ❑ Concept Learning: Search in Version Space
- ❑ Measuring Performance

# Regression

## Classification versus Regression

$X$  is a  $p$ -dimensional feature space or input space. Example:

Customer 1	
house owner	yes
income (p.a.)	51 000 EUR
repayment (p.m.)	1 000 EUR
credit period	7 years
SCHUFA entry	no
age	37
married	yes
...	

...

Customer n	
house owner	no
income (p.a.)	55 000 EUR
repayment (p.m.)	1 200 EUR
credit period	8 years
SCHUFA entry	no
age	?
married	yes
...	

## Classification:

- $C = \{-1, 1\}$  is a set of classes. Similarly:  $C = \{0, 1\}$ ,  $C = \{\text{no}, \text{yes}\}$
- $D = \{(\mathbf{x}_1, c(\mathbf{x}_1)), \dots, (\mathbf{x}_n, c(\mathbf{x}_n))\} \subseteq X \times C$  is a set of examples.
- $c(\mathbf{x}_i)$  is the ground truth for the creditworthiness class,  $\mathbf{x}_i \in X$ .

# Regression

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### Regression:

- $Y \subseteq \mathbb{R}$  is the output space.
- $D = \{(\mathbf{x}_1, y_1), \dots, (\mathbf{x}_n, y_n)\} \subseteq X \times Y$  is a set of examples.
- $y_i$  is the **ground truth** for the credit line **value**,  $\mathbf{x}_i \in X$ .

# Regression

## The Linear Regression Model

- Given  $\mathbf{x}$ , predict a real-valued output under a linear model function:

$$y(\mathbf{x}) = w_0 + \sum_{j=1}^p w_j \cdot x_j$$

- Vector notation with  $x_0 = 1$  and  $\mathbf{w} = (w_0, w_1, \dots, w_p)^T$ :

$$y(\mathbf{x}) = \mathbf{w}^T \mathbf{x}$$

- Given  $\mathbf{x}_1, \dots, \mathbf{x}_n$ , assess goodness of fit as residual sum of squares:

$$\text{RSS}(\mathbf{w}) = \sum_{i=1}^n (y_i - y(\mathbf{x}_i))^2 = \sum_{i=1}^n (y_i - \mathbf{w}^T \mathbf{x}_i)^2 \quad (1)$$

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- Estimate  $\mathbf{w}$  by minimizing the residual sum of squares:

$$\hat{\mathbf{w}} = \underset{\mathbf{w} \in \mathbb{R}^{p+1}}{\text{argmin}} \text{RSS}(\mathbf{w}) \quad (2)$$

## Remarks:

- ❑ A *residual* is the difference between an observed value  $y_i$  and the estimated value  $y(\mathbf{x}_i)$  of the model function.
- ❑ The residual sum of squares, RSS, is the sum of squares of the residuals. It is also known as the sum of squared residuals, SSR, or the sum of squared errors of prediction, SSE.
- ❑ The RSS term quantifies the regression error—or similarly, the goodness of fit—in the form of a single value.
- ❑ RSS provides several numerical and [theoretical advantages](#), but it is not the only possibility to assess the goodness of fit (= error) between observed values and the model function. Alternative approaches for quantifying the error include absolute residual values or a polynomial in the residual values.
- ❑ The error computation is also called loss computation, cost computation, or generally, performance computation. Similarly, for the right-hand side of [Equation \(1\)](#) the following names are used: error function, loss function, cost function, or generally, performance term. Measures that quantify this kind of performance are called *effectiveness* measures. This term must not be confused with *efficiency* measures, which quantify the computational effort or runtime performance of a method.

## Remarks (continued) :

- ❑ From a statistical viewpoint, both  $\mathbf{x} = x_1, \dots, x_p$  and  $y$  represent *random variables* (vectorial and scalar respectively). Each feature vector,  $\mathbf{x}_i$ , and outcome,  $y_i$ , is the result of a random experiment and hence is governed by a—usually unknown—probability distribution.
- ❑ The distributions of the random variables  $y_i$  and  $(y_i - y(\mathbf{x}_i))$  are identical.
- ❑ [Equation \(2\)](#): Estimating  $\mathbf{w}$  by RSS minimization is based on the following assumptions:
  1. The random variables  $y_i$  are statistically independent. Actually, the conditional independence of the  $y_i$  under  $\mathbf{x}_i$  is sufficient.
  2. The means  $E(y_i)$  lie on a straight line, known as the true (population) regression line:  $E(y_i) = \mathbf{w}^{*T} \mathbf{x}_i$ . I.e., the relation between the observed  $(\mathbf{x}, y) \in X \times Y$  can be completely explained by a linear model function.
  3. The probability distributions  $P(y_i | \mathbf{x}_i)$  have the same variance.

The three assumptions are called the *weak set* (of assumptions). Along with a fourth assumption about the distribution shape of  $y_i$  they become the *strong set* of assumptions.

- ❑ In order to avoid cluttered notation, we won't use different symbols to distinguish random variables from ordinary variables. I.e., if  $\mathbf{x}, x, y$  denote a (vectorial or scalar) random variable this fact will become clear from the context.

# Regression

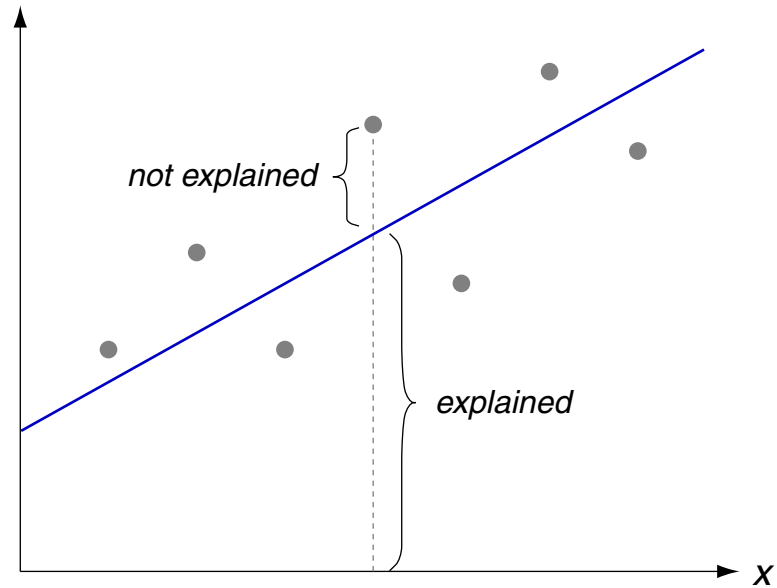
## One-Dimensional Feature Space





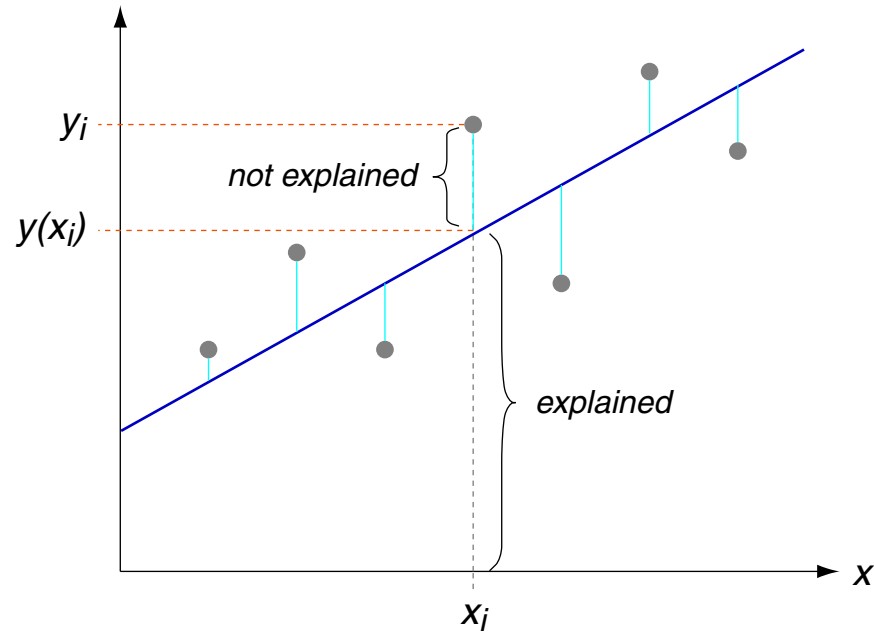
# Regression

## One-Dimensional Feature Space (continued)



# Regression

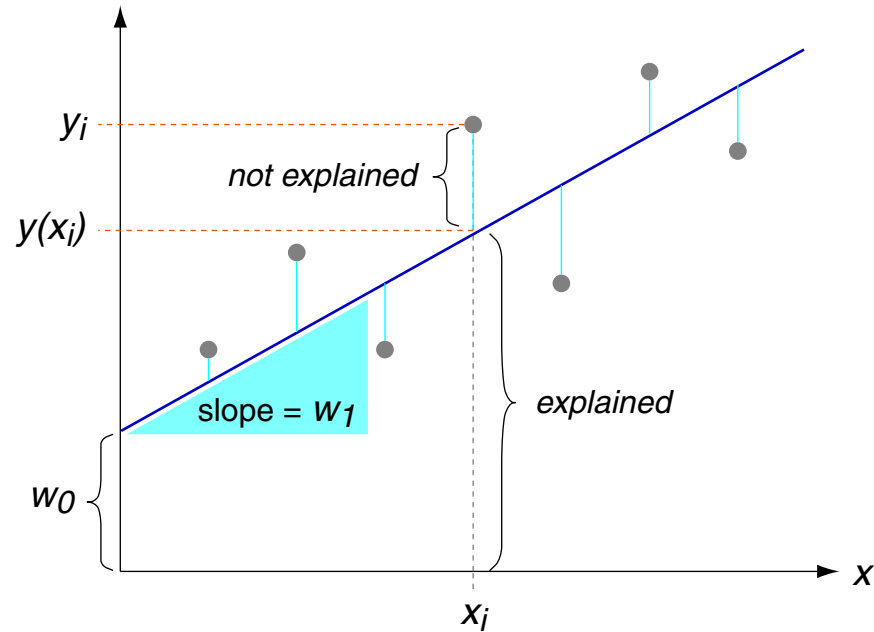
## One-Dimensional Feature Space (continued)



$$\text{RSS} = \sum_{i=1}^n (y_i - y(x_i))^2$$

# Regression

## One-Dimensional Feature Space (continued)



$$y(x) = w_0 + w_1 \cdot x, \quad \text{RSS}(w_0, w_1) = \sum_{i=1}^n (y_i - w_0 - w_1 \cdot x_i)^2$$

# Regression

## One-Dimensional Feature Space (continued) [higher-dimensional]

Minimize  $\text{RSS}(w_0, w_1)$  by a direct method:

$$1. \quad \frac{\partial}{\partial w_0} \sum_{i=1}^n (y_i - w_0 - w_1 \cdot x_i)^2 = 0$$

$$\leadsto \dots \leadsto w_0 = \frac{1}{n} \sum_{i=1}^n y_i - \frac{w_1}{n} \sum_{i=1}^n x_i = \bar{y} - w_1 \cdot \bar{x}$$

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$$2. \quad \frac{\partial}{\partial w_1} \sum_{i=1}^n (y_i - w_0 - w_1 \cdot x_i)^2 = 0$$

$$\rightsquigarrow \dots \rightsquigarrow w_1 = \frac{\sum_{i=1}^n (x_i - \bar{x}) \cdot (y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2}$$

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$$2. \quad \frac{\partial}{\partial w_1} \sum_{i=1}^n (y_i - w_0 - w_1 \cdot x_i)^2 = 0$$

$$\leadsto \dots \leadsto \hat{w}_1 \equiv w_1 = \frac{\sum_{i=1}^n (x_i - \bar{x}) \cdot (y_i - \bar{y})}{\sum_{i=1}^n (x_i - \bar{x})^2}$$

# Regression

## Higher-Dimensional Feature Space

- Recall Equation (1):

$$\text{RSS}(\mathbf{w}) = \sum_{\mathbf{x}_i \in D} (y_i - \mathbf{w}^T \mathbf{x}_i)^2$$

- Let  $\mathbf{X}$  denote the  $n \times (p + 1)$  matrix,  
where row  $i$  is the extended input vector  $(1 \ \mathbf{x}_i^T)$ ,  $\mathbf{x}_i \in D$ .

Let  $\mathbf{y}$  denote the  $n$ -vector of outputs in the training set  $D$ .

# Regression

## Higher-Dimensional Feature Space

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$$\leadsto \text{RSS}(\mathbf{w}) = (\mathbf{y} - \mathbf{X}\mathbf{w})^T(\mathbf{y} - \mathbf{X}\mathbf{w})$$

$\text{RSS}(\mathbf{w})$  is a quadratic function in  $p + 1$  parameters.



# Regression

## Higher-Dimensional Feature Space (continued) [one-dimensional]

Minimize  $\text{RSS}(\mathbf{w})$  by a direct method:

$$\frac{\partial \text{RSS}}{\partial \mathbf{w}} = -2\mathbf{X}^T(\mathbf{y} - \mathbf{X}\mathbf{w}) = 0, \quad \frac{\partial^2 \text{RSS}}{\partial \mathbf{w} \partial \mathbf{w}^T} = -2\mathbf{X}^T\mathbf{X} \quad [\text{Wikipedia } \underline{1}, \underline{2}, \underline{3}]$$

$$\mathbf{X}^T(\mathbf{y} - \mathbf{X}\mathbf{w}) = 0$$

$$\Leftrightarrow \mathbf{X}^T\mathbf{X}\mathbf{w} = \mathbf{X}^T\mathbf{y}$$

$$\leadsto \mathbf{w} = (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y}$$

# Regression

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$$\mathbf{X}^T(\mathbf{y} - \mathbf{X}\mathbf{w}) = 0$$

$$\Leftrightarrow \mathbf{X}^T \mathbf{X} \mathbf{w} = \mathbf{X}^T \mathbf{y}$$

$$\leadsto \mathbf{w} = \underbrace{(\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T}_{\text{Pseudoinverse of } \mathbf{X}} \mathbf{y}$$

Pseudoinverse of  $\mathbf{X}$  [[Wikipedia](#)]

Normal equations.

If  $\mathbf{X}$  has full column rank  $p + 1$ .

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If  $\mathbf{X}$  has full column rank  $p + 1$ .

Pseudoinverse of  $\mathbf{X}$  [[Wikipedia](#)]

$$\hat{y}(\mathbf{x}_i) = \mathbf{x}_i^T \hat{\mathbf{w}} \quad \text{Regression function with least squares estimator } \hat{\mathbf{w}}.$$

$$\begin{aligned} \hat{\mathbf{y}} &= \mathbf{X} \hat{\mathbf{w}} && \text{The } n\text{-vector of fitted values at the training input.} \\ &= \mathbf{X} (\mathbf{X}^T\mathbf{X})^{-1}\mathbf{X}^T\mathbf{y} \end{aligned}$$

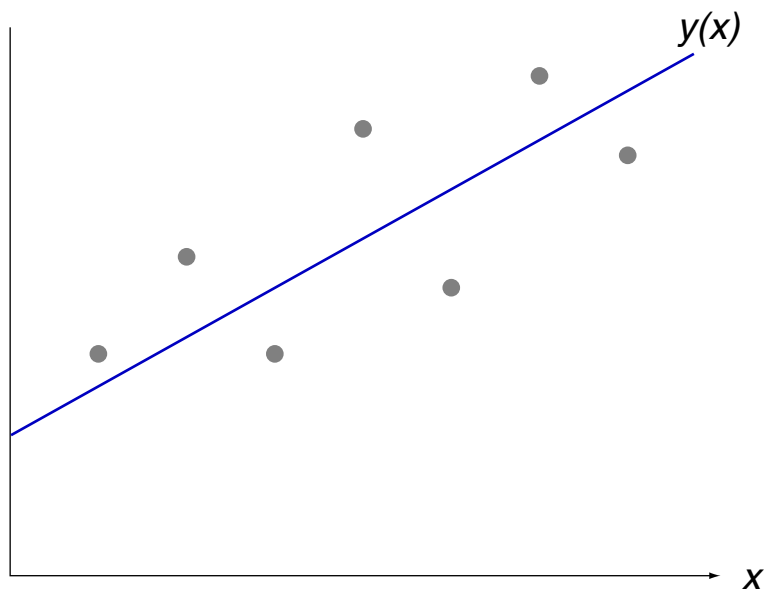
## Remarks:

- ❑ A curve fitting (or regression) method that is based on the minimization of squared residuals is called a *method of least squares*.
- ❑ Various approaches for operationalizing the method of least squares have been devised, in particular for the case of linear model functions. From a numerical viewpoint one can distinguish iterative methods, such as the LMS algorithm, and direct methods, such as solving the normal equations via computing the pseudoinverse.
- ❑ More on direct methods. While solving the normal equations is usually fast, it suffers from several deficits: it is numerically unstable and requires singularity handling. Numerically more stable and more accurate methods are based on the QR decomposition and the singular value decomposition, SVD.
- ❑ QR decomposition can deal with problems of up to  $10^4$  variables, provided a dense problem structure. For significantly larger problems (additional 1-2 orders of magnitudes) as well as for sparse matrices iterative solvers are the choice. Even larger, dense problems may be tackled with Artificial Neural Networks.

# Regression

## Linear Regression for Classification (illustrated for $p = 1$ )

Regression learns a real-valued function given as  $D = \{(\mathbf{x}_1, y_1), \dots, (\mathbf{x}_n, y_n)\}$ .



$$y(x) = (w_0 \ w_1) \begin{pmatrix} 1 \\ x \end{pmatrix}$$

# Regression

## Linear Regression for Classification (illustrated for $p = 1$ )

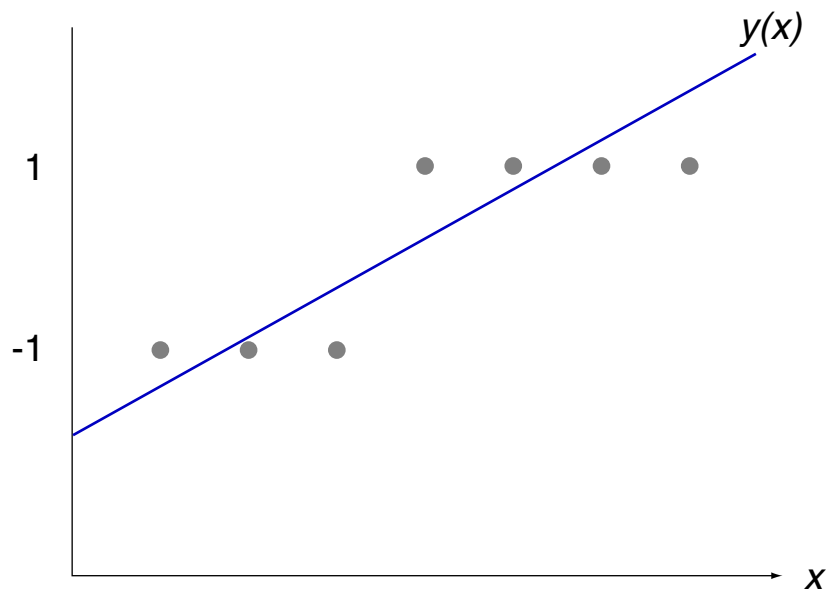
Binary-valued ( $\pm 1$ ) functions are also real-valued.



# Regression

## Linear Regression for Classification (illustrated for $p = 1$ )

Use linear regression to learn  $\mathbf{w}$  from  $D$ , where  $y_i = \pm 1 \approx y(\mathbf{x}_i) = \mathbf{w}^T \mathbf{x}_i$ .

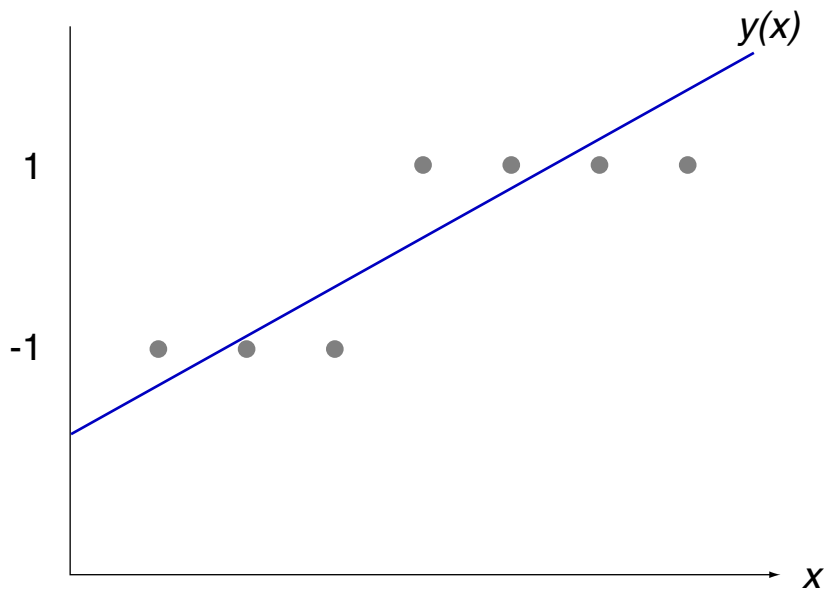


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The function “ $\text{sign}(\mathbf{w}^T \mathbf{x}_i)$ ” is likely to agree with  $y_i = \pm 1$ .

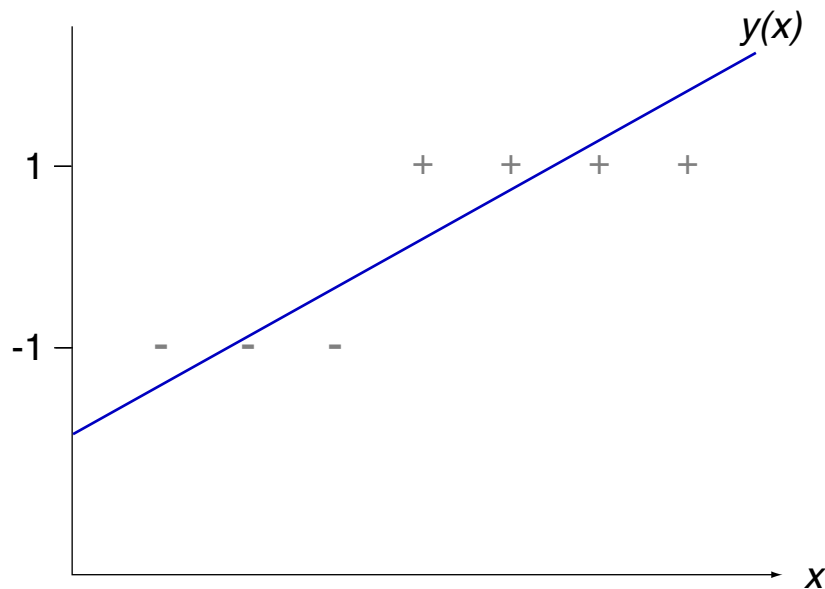
- Regression:  $y(\mathbf{x}) = \mathbf{w}^T \mathbf{x}$
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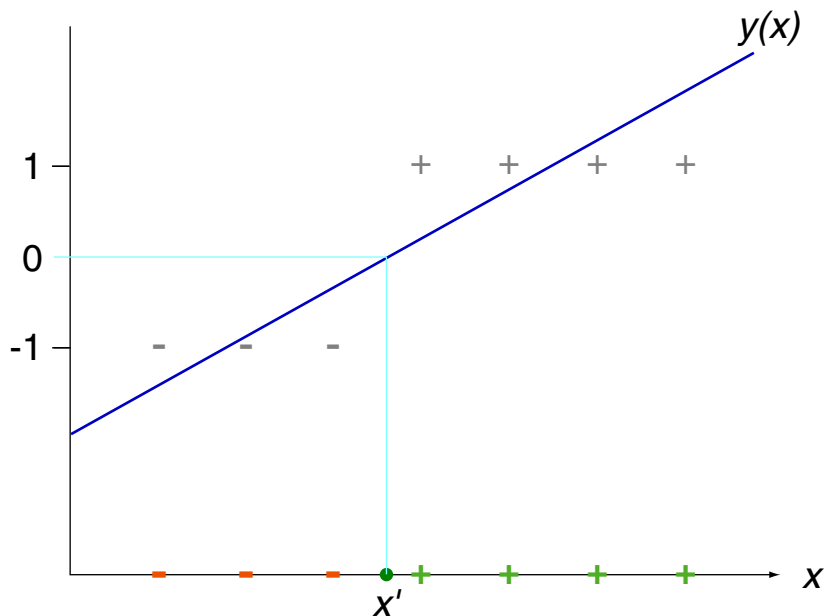
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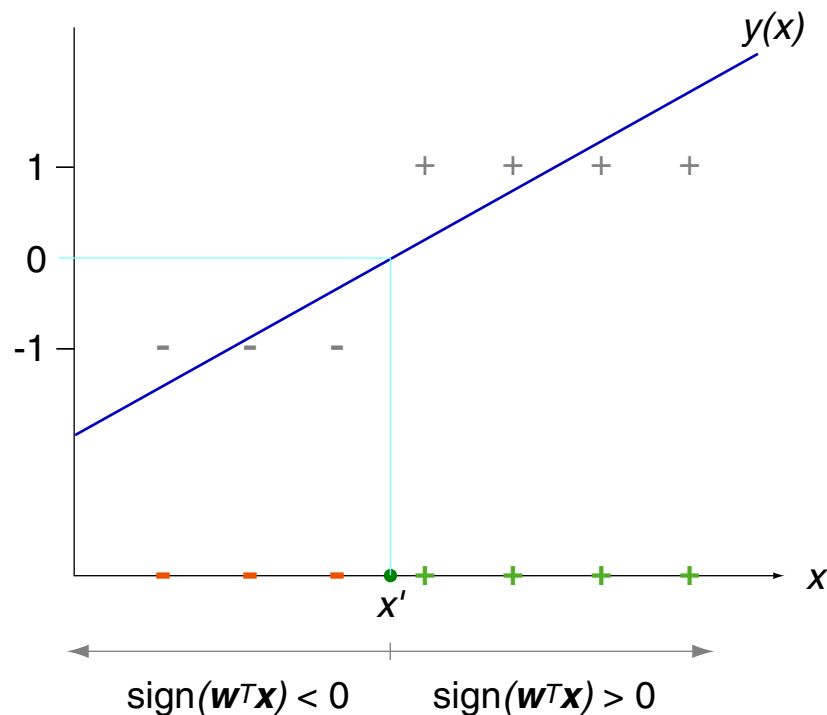
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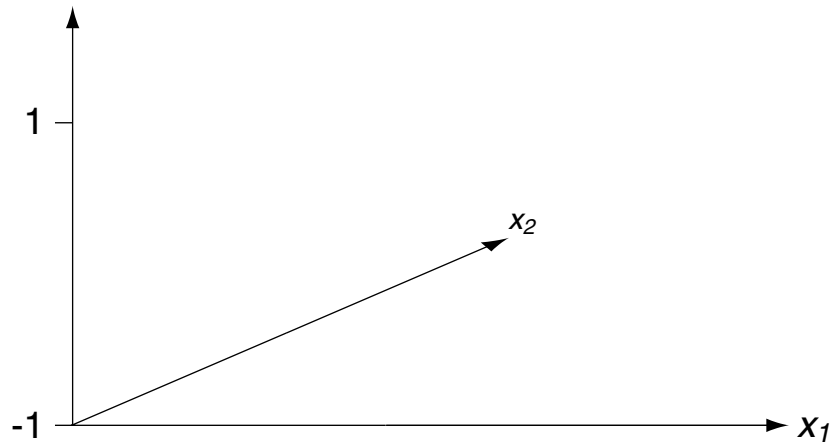
Use linear regression to learn  $\mathbf{w}$  from  $D$ , where  $y_i = \pm 1 \approx y(\mathbf{x}_i) = \mathbf{w}^T \mathbf{x}_i$ .



- The discrimination point, •, is defined by  $w_0 + w_1 \cdot x' = 0$ .
- For  $p = 2$  we are given a discrimination *line*.

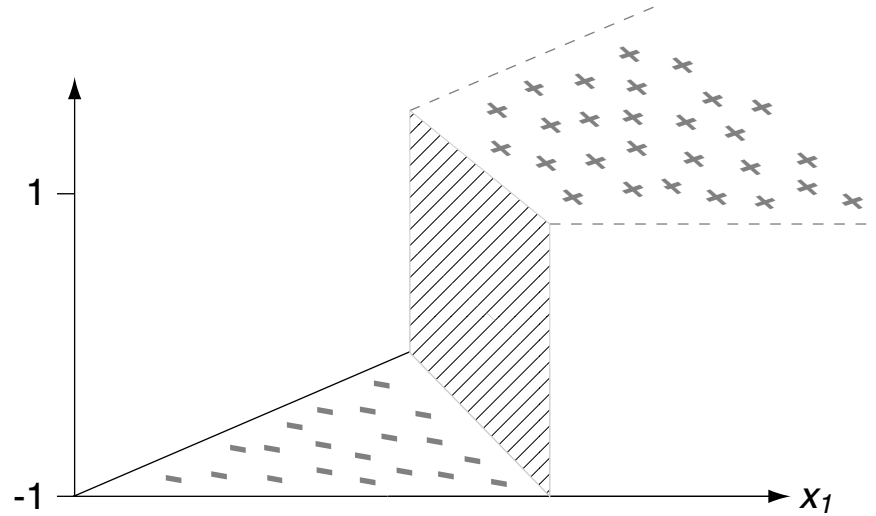
# Regression

Linear Regression for Classification (illustrated for  $p = 2$ )



# Regression

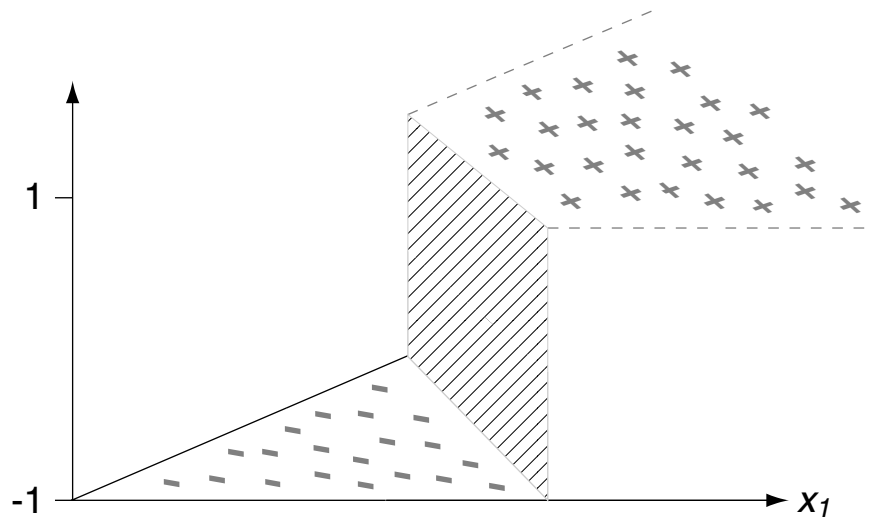
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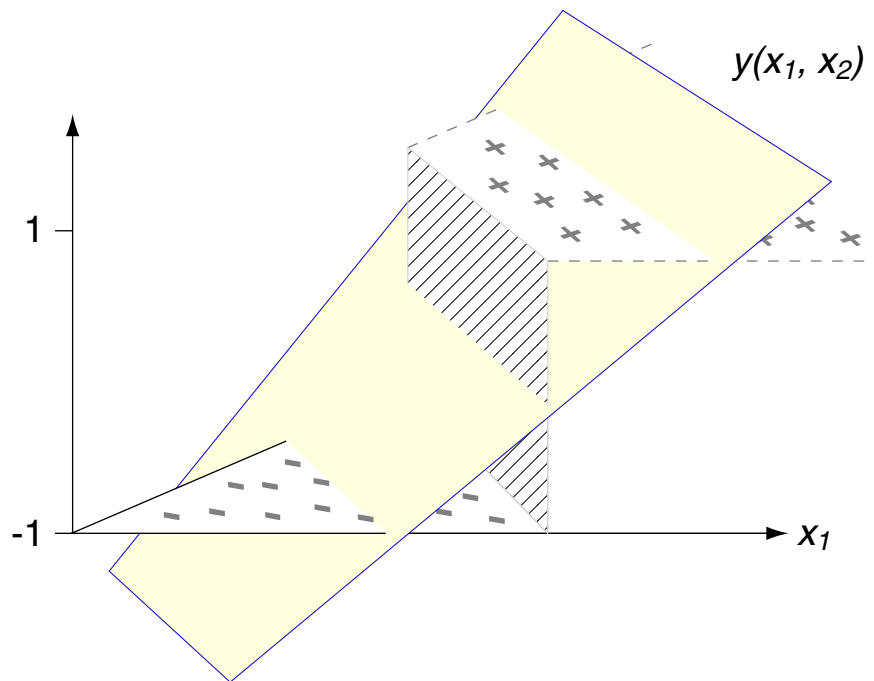


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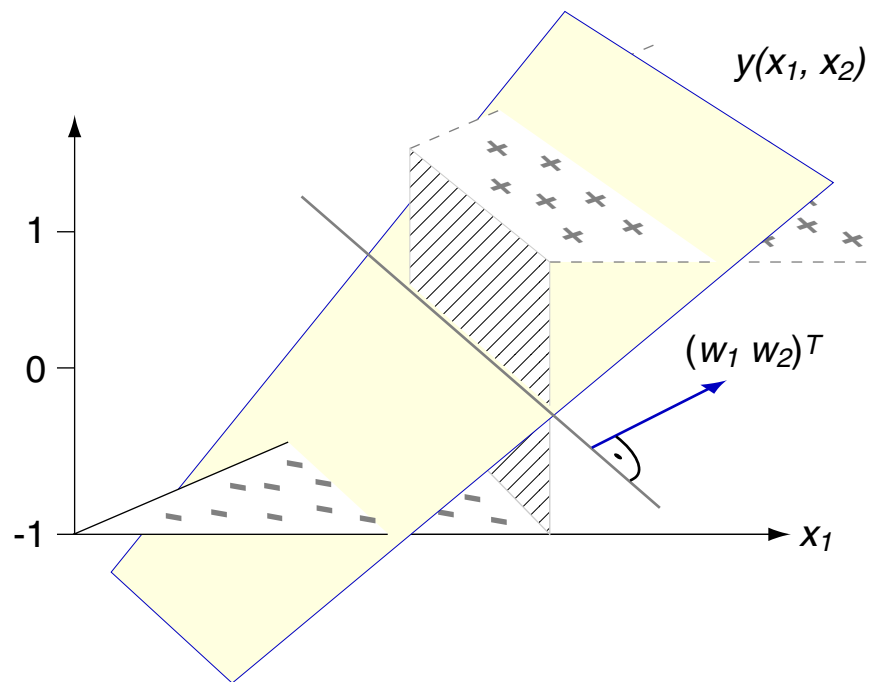
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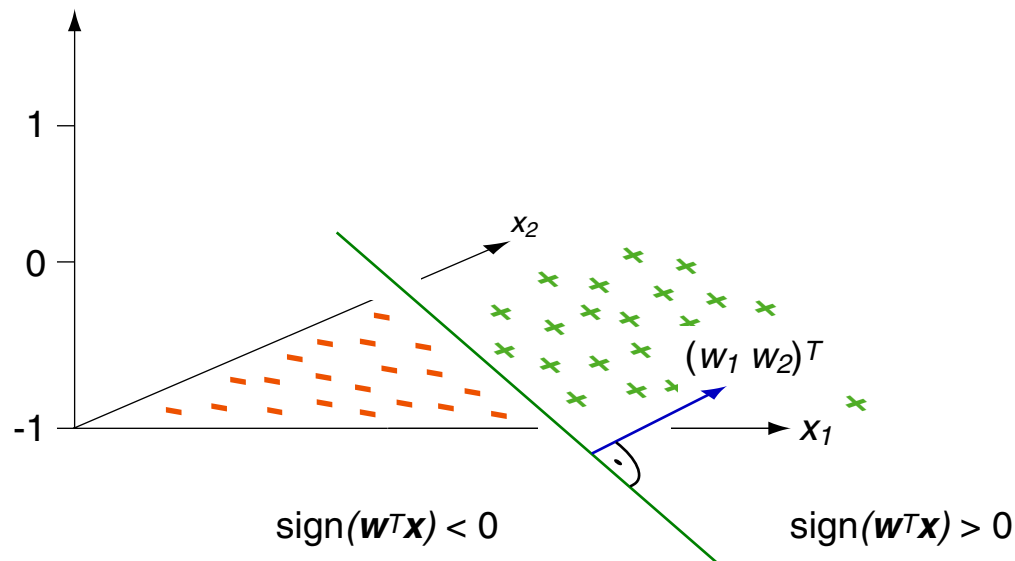
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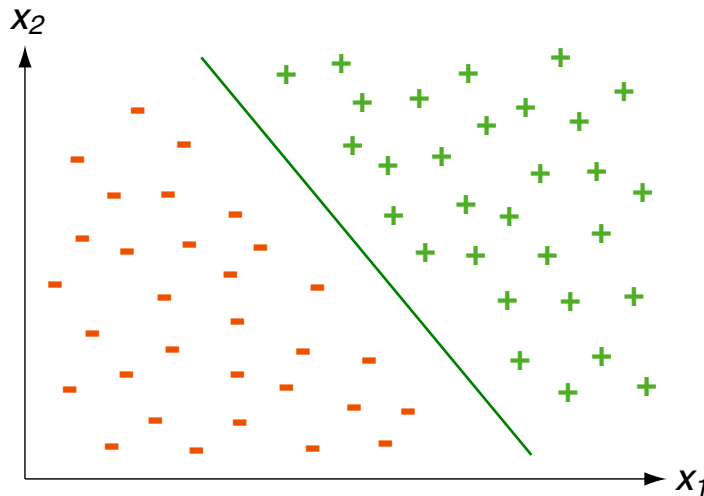


- ❑ The discrimination line, —, is defined by  $w_0 + w_1 \cdot x_1 + w_2 \cdot x_2 = 0$ .
- ❑ For  $p = 3$  ( $p > 3$ ) we are given a discriminating (hyper)plane.

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# Regression

## The Linear Model Function: Variants

The components (variables, random variables) of the input vector  $\mathbf{x} = (x_1, \dots, x_p)$  may stem from different sources [Hastie et al. 2001] :

1. quantitative inputs
2. transformations of quantitative inputs, such as  $\log x_j$ ,  $\sqrt{x_j}$
3. basis expansions, such as  $x_j = (x_1)^j$
4. encoding of a qualitative variable  $g$ ,  $g \in \{1, \dots, p\}$ , as  $x_j = I(g = j)$
5. interactions between variables, such as  $x_3 = x_1 \cdot x_2$

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No matter the source of the  $x_j$ , the model is still linear in the parameters  $\mathbf{w}$  :

$$y(\mathbf{x}) = w_0 + \sum_{j=1}^p w_j \cdot \phi_j(x_j)$$

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3. basis expansions, such as  $x_j = (x_1)^j$
4. encoding of a qualitative variable  $g, g \in \{1, \dots, p\}$ , as  $x_j = I(g = j)$
5. interactions between variables, such as  $x_3 = x_1 \cdot x_2$

No matter the source of the  $x_j$ , the model is still linear in the parameters  $\mathbf{w}$  :

$$y(\mathbf{x}) = w_0 + \sum_{j=1}^p w_j \cdot \phi_j(x_j)$$

- linear in the parameters: constant  $w_j$  and additive combination

# Regression

## The Linear Model Function: Variants

The components (variables, random variables) of the input vector  $\mathbf{x} = (x_1, \dots, x_p)$  may stem from different sources [Hastie et al. 2001] :

1. quantitative inputs
2. transformations of quantitative inputs, such as  $\log x_j$ ,  $\sqrt{x_j}$
3. basis expansions, such as  $x_j = (x_1)^j$
4. encoding of a qualitative variable  $g$ ,  $g \in \{1, \dots, p\}$ , as  $x_j = I(g = j)$
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No matter the source of the  $x_j$ , the model is still linear in the parameters  $\mathbf{w}$  :

$$y(\mathbf{x}) = w_0 + \sum_{j=1}^p w_j \cdot \phi_j(x_j)$$

- linear in the parameters: constant  $w_j$  and additive combination
- basis functions: input variables (space) become(s) feature variables (space)

# Regression

## The Linear Model Function: Properties of the Solution

### Theorem 1 (Gauss-Markov)

Let  $D = \{(\mathbf{x}_1, y_1), \dots, (\mathbf{x}_n, y_n)\}$  be a set of examples to be fitted with a linear model function as  $y(\mathbf{x}) = \mathbf{x}^T \mathbf{w}$ . Within the class of linear unbiased estimators for  $\mathbf{w}$ , the least squares estimator  $\hat{\mathbf{w}}$  has minimum variance, i.e., is most efficient.

# Regression

## The Linear Model Function: Properties of the Solution

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Related followup issues:

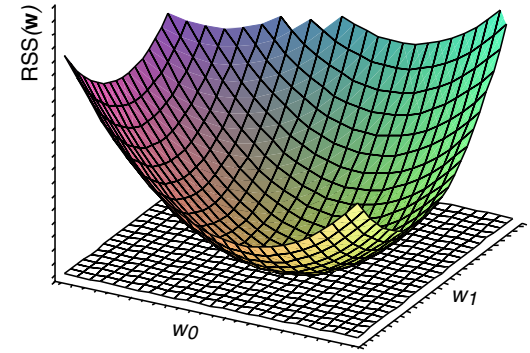
- ❑ mean and variance of  $\hat{\mathbf{w}}$
- ❑ proof of the Gauss-Markov theorem
- ❑ weak set and strong set of assumptions
- ❑ efficiency and consistency of unbiased estimators
- ❑ rank deficiencies, where the feature number  $p$  exceeds  $|D| = n$
- ❑ relation of mean least squares and the maximum likelihood principle



# Regression

## Methods of Least Squares: Iterative versus Direct Methods

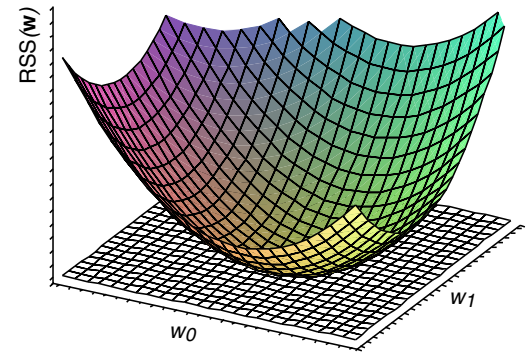
$$\underset{\mathbf{w}}{\operatorname{argmin}} \operatorname{RSS}(\mathbf{w}), \quad \text{with } \operatorname{RSS}(\mathbf{w}) = \sum_{i=1}^n (y_i - \mathbf{w}^T \mathbf{x}_i)^2$$



# Regression

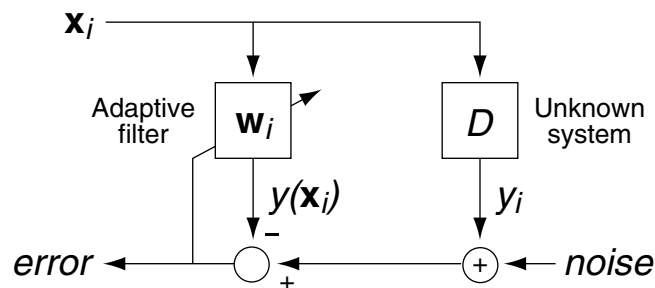
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## LMS algorithm:

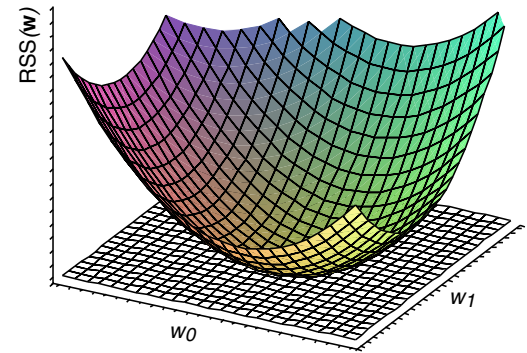
- ❑ applicable as online algorithm
- ❑ robust algorithm structure
- ❑ unsatisfactory convergence
- ❑ allows stochastic sampling



# Regression

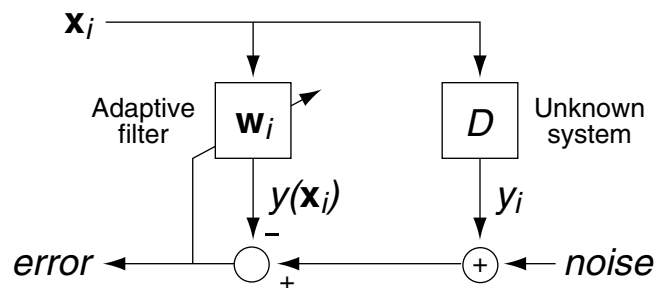
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### LMS algorithm:

- ❑ applicable as online algorithm
- ❑ robust algorithm structure
- ❑ unsatisfactory convergence
- ❑ allows stochastic sampling



### Normal equations:

- ❑ needs complete data
- ❑ numerically unstable
- ❑ requires singularity handling
- ❑ hardly applicable to big data

$$\hat{\mathbf{w}} = (\mathbf{X}^T \mathbf{X})^{-1} \mathbf{X}^T \mathbf{X} \mathbf{y}$$

## Remarks:

- ❑ Recall the definitions for [residual](#) and [method of least squares](#).
- ❑ The principle of RSS minimization is orthogonal to (= independent of) the type of the model function  $y$ , i.e., independent of its dimensionality as well as its linearity or nonlinearity.
- ❑ To fit the parameters  $\mathbf{w}$  of a (one-dimensional, multi-dimensional, linear, nonlinear) model function  $y$ , both the LMS algorithm and direct methods exploit information about the derivative of the RSS term with respect to  $\mathbf{w}$ . I.e., even if *classification* and not regression is the goal, the distance to the decision boundary (and not the zero-one-loss) is computed, since the latter is not differentiable.
- ❑ For a linear model function  $y$ ,  $\text{RSS}(\mathbf{w})$  is a convex function and hence a single, global optimum exists.
- ❑ Forthcoming: A main goal of machine learning approaches is to avoid overfitting. Overfitting in turn is caused by an inadequate (too high) model function complexity—or, similarly, by insufficient data. A means to reduce the model function complexity is *regularization*.
- ❑ Regularization will introduce additional constraints for the model function  $y$  or the parameter vector  $\mathbf{w}$ , respectively. With regularization the [minimization expression \(2\)](#) will consist of two summands: a performance term such as the RSS term, and a penalizing term such as a norm. As before, the first term captures the model function's goodness depending on  $\mathbf{w}$ , whereas the second term restricts the absolute values of the model function's parameters  $\mathbf{w}$ .