# Chapter ML:VII (continued)

## VII. Bayesian Learning

- Approaches to Probability
- Conditional Probability
- Bayes Classifier
- Exploitation of Data
- Frequentist versus Subjectivist

# Exploitation of Data

## Data Events

Data from a "predictor-response" setting:

$D = \{(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_n, y_n)\}$ (regression)

$D = \{(\mathbf{x}_1, c_1), \ldots, (\mathbf{x}_n, c_n)\}$ (classification)

- $D$ is the result of $n$ i.i.d. trials. I.e., $n$ objects are sampled independently and from the same probability distribution. All objects are characterized by a "response" variable that is either quantitative (a number $y$) or categorical (a class label $c$), and by $p$ "predictors" (a feature vector $\mathbf{x}$).

- $p(\mathbf{x}_i, c_i)$, $p(\mathbf{x}_i, c_i) := P(\mathbf{X}_i{=}\mathbf{x}_i, C_i{=}c_i)$, is the probability of the joint event $\{\mathbf{X}_i{=}\mathbf{x}_i, C_i{=}c_i\}$, i.e., (1) to get the vector $\mathbf{x}_i$, and, (2) that the respective object belongs to class $c_i$. The $p(\mathbf{x}_i, y_i)$ are defined analogously.

- The $Y_i$, $C_i$, and $\mathbf{X}_i$ are i.i.d. (multivariate) random variables. Typically, the $Y_i$ are of continuous type, the $C_i$ of discrete type, and the variables of the random vector $\mathbf{X}_i$, $\mathbf{X}_i := (X_{1,i}, \ldots, X_{p,i})^T$, of continuous type.

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# Exploitation of Data

## Data Events (continued)

Data from an "outcome-only" setting:

$D = \{y_1, \ldots, y_n\}$ (quantitative)

$D = \{c_1, \ldots, c_n\}$ (categorical)

- $D$ is the result of $n$ i.i.d. trials. I.e., $n$ outcomes are sampled independently and from the same probability distribution. All outcomes are characterized by either a number $y$ or a class label $c$.
- $p(y_i)$, $p(y_i) := P(Y_i{=}y_i)$, is the probability of the event $Y_i{=}y_i$.
  $p(c_i)$, $p(c_i) := P(C_i{=}c_i)$, is the probability of the event $C_i{=}c_i$.
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Remarks:

- The following remarks on the predictor-response setting are detailed for a categorical response variable $c$; they apply to a quantitative response variable $y$ as well.

- By experiment design, the $n$ joint events, $\{\mathbf{X}_1{=}\mathbf{x}_1, C_1{=}c_1\}, \ldots, \{\mathbf{X}_n{=}\mathbf{x}_n, C_n{=}c_n\}$, generating the data $D$ are mutually independent:

$$\begin{aligned} p(D) = p\left(\{(\mathbf{x}_1, c_1), \ldots, (\mathbf{x}_n, c_n)\}\right) &= \prod_{i=1,\ldots,n} p(\mathbf{x}_i, c_i) \\ &\stackrel{(1)}{=} \prod_{i=1,\ldots,n} \left(p(\mathbf{x}_i) \cdot p(c_i \mid \mathbf{x}_i)\right) \\ &\stackrel{(2)}{=} \prod_{i=1,\ldots,n} p(\mathbf{x}_i) \cdot \prod_{i=1,\ldots,n} p(c_i \mid \mathbf{x}_i) \end{aligned}$$

  (1) Usually *not* independent are any two events $\mathbf{X}_i{=}\mathbf{x}_i$ and $C_i{=}c_i$, $i = 1, \ldots, n$:

  $p(\mathbf{x}_i, c_i) \neq p(\mathbf{x}_i) \cdot p(c_i)$

  (2) See the maximum likelihood derivation of the logistic loss $L_\sigma(\mathbf{w})$.

- By experiment design, the probabilities, $p(\mathbf{x}_i)$, $i = 1, \ldots, n$, are independent, i.e., the probability of the joint event $\{\mathbf{X}_1{=}\mathbf{x}_1, \ldots, \mathbf{X}_n{=}\mathbf{x}_n\}$ is equal to the product of the singleton events: $p(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \prod_{i=1,\ldots,n} p(\mathbf{x}_i)$.

  A consistent and unbiased estimate for $p(\mathbf{x})$ is $\hat{p}(\mathbf{x}) = |\{(\mathbf{x}, \cdot\,) \in D\}| \cdot \frac{1}{|D|}$.

- By experiment design, the conditional probabilities, $p(c_i \mid \mathbf{x}_i)$, $i = 1, \ldots, n$, are *invariant under covariate shift*, i.e., invariant under a change of $p(\mathbf{x}_i)$. That is, the classification procedure, “determination of $c_i$ given some $\mathbf{x}_i$”, always runs the same way, regardless of how often $\mathbf{x}_i$ is encountered.

 

Remarks: (continued)

- The invariance of $p(c_i \mid \mathbf{x}_i)$ under a covariate shift can also be understood as the fact that any two events $\mathbf{X}_i{=}\mathbf{x}_i$ and $(C_i{=}c_i \mid \mathbf{X}_i{=}\mathbf{x}_i)$, $i = 1, \ldots, n$ are independent:

  $$\text{“}p(\mathbf{x}, (c \mid \mathbf{x}))\text{”} \;=\; p(\mathbf{x}) \cdot p(c \mid \mathbf{x}) = p(\mathbf{x}, c)$$

  However, this interpretation is problematic since standard probability theory does not allow a conditional event being combined with other events. See section Probability Basics of this part, conditional event algebra, and Lewis's triviality result for details.

- Within an outcome-only setting such as “flipping a coin”, the object features (coin diameter, coin age, etc.) are not used as predictors. I.e., one does not model the relationship between a response variable and predictors $\mathbf{x}$ but models (the probability of) a sequence of outcomes $D = \{y_1, \ldots, y_n\}$ or $D = \{c_1, \ldots, c_n\}$.

- The type of setting, be it predictor-response or outcome-only, is independent of data exploitation aspects such as
  - discriminative versus generative,
  - non-probabilistic versus probabilistic,
  - maximum likelihood versus Bayes, or
  - frequentist versus subjectivist.

# Exploitation of Data

## Typical Learning Settings

$D = \{(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_n, y_n)\},\ \ D = \{(\mathbf{x}_1, c_1), \ldots, (\mathbf{x}_n, c_n)\}$

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| (1) | RSS($\mathbf{w}$) : | $\sum_{(\mathbf{x},y)\in D} (y - \mathbf{w}^T\mathbf{x})^2$ | RSS for $D$ under a linear model, parameterized by $\mathbf{w}$. Least squares estimate: $\hat{\mathbf{w}} = \text{argmin}_{\mathbf{w}\in\mathbf{R}^{p+1}}\ \text{RSS}(\mathbf{w})$ |
| (2) | $p(D; \mathbf{w})$ : | $\prod_{(\mathbf{x},c)\in D} p(c \mid \mathbf{x}; \mathbf{w})$ | Probability of $D$ under a logistic model, parameterized by $\mathbf{w}$. Maximum likelihood estimate: $\mathbf{w}_{\text{ML}} = \text{argmax}_{\mathbf{w}\in\mathbf{R}^{p+1}}\ p(D; \mathbf{w})$ |
| (3) | $L(\mathbf{w})$ : | $\sum_{(\mathbf{x},c)\in D} l_\sigma(c, \sigma(\mathbf{w}^T\mathbf{x}))$ | Loss for $D$ under a logistic model, parameterized by $\mathbf{w}$. Minimum loss (= maximum likelihood) estimate: $\hat{\mathbf{w}} = \text{argmin}_{\mathbf{w}\in\mathbf{R}^{p+1}}\ L(\mathbf{w})$ |
| (4) | $p(c \mid \mathbf{x})$ : | $\frac{p(\mathbf{x} \mid c) \cdot p(c)}{p(\mathbf{x})}$ | Probability of $c$ given $\mathbf{x}$ via Bayes's rule. Maximum a posteriori class for $\mathbf{x}$ : $c_{\text{MAP}} = \text{argmax}_{c\in\{\oplus,\ominus\}}\ p(c \mid \mathbf{x})$ |

$D = \{y_1, \ldots, y_n\},\ \ D = \{c_1, \ldots, c_n\}$

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| (5) | $p(D; \theta)$ : | $\binom{n}{k} \cdot \theta^k \cdot (1-\theta)^{n-k}$ | Probability of $D$ under the binomial distribution, parameterized by $\theta$. Maximum likelihood estimate: $\theta_{\text{ML}} = \text{argmax}_{\theta\in[0;1]}\ p(D; \theta)$ |
| (6) | $p(\theta \mid D)$ : | $\frac{p(D \mid \theta) \cdot p(\theta)}{p(D)}$ | Probability of $\theta$ given $D$ via Bayes's rule. Maximum a posteriori hypothesis: $\theta_{\text{MAP}} = \text{argmax}_{\theta\in\{\theta_1,\theta_2\}}\ p(\theta \mid D)$ |

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(1) RSS($\mathbf{w}$) : $\displaystyle\sum_{(\mathbf{x},y)\in D} (y - \mathbf{w}^T\mathbf{x})^2$

RSS for $D$ under a linear model, parameterized by $\mathbf{w}$. Least squares estimate: $\hat{\mathbf{w}} = \text{argmin}_{\mathbf{w}\in\mathbf{R}^{p+1}}$ RSS($\mathbf{w}$)

(2) $p(D; \mathbf{w})$ : $\displaystyle\prod_{(\mathbf{x},c)\in D} p(c \mid \mathbf{x}; \mathbf{w})$

Probability of $D$ under a logistic model, parameterized by $\mathbf{w}$. Maximum likelihood estimate: $\mathbf{w}_{\text{ML}} = \text{argmax}_{\mathbf{w}\in\mathbf{R}^{p+1}} \, p(D; \mathbf{w})$

(3) $L(\mathbf{w})$ : $\displaystyle\sum_{(\mathbf{x},c)\in D} l_\sigma(c, \sigma(\mathbf{w}^T\mathbf{x}))$

Loss for $D$ under a logistic model, parameterized by $\mathbf{w}$. Minimum loss (= maximum likelihood) estimate: $\hat{\mathbf{w}} = \text{argmin}_{\mathbf{w}\in\mathbf{R}^{p+1}} \, L(\mathbf{w})$

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Probability of $c$ given $\mathbf{x}$ via Bayes's rule. Maximum a posteriori class for $\mathbf{x}$ : $c_{\text{MAP}} = \text{argmax}_{c\in\{\oplus,\ominus\}} \, p(c \mid \mathbf{x})$

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Probability of $D$ under the binomial distribution, parameterized by $\theta$. Maximum likelihood estimate: $\theta_{\text{ML}} = \text{argmax}_{\theta\in[0;1]} \, p(D; \theta)$

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Remarks (predictor-response vs. outcome-only setting) :

- (1),…, (4) Predictor-response setting, $\mathbf{x} \to y$ or $\mathbf{x} \to c$. The relation between $\mathbf{x}$ and $y$ or $c$ is captured by a model function $y(\mathbf{x})$. The data $D$ is exploited to fit $y(\mathbf{x})$, which in turn means to determine a parameter $w$ or parameter vector $\mathbf{w}$ for $y(\mathbf{x})$. Modeling and predicting a quantitative response variable $y$ is a regression task; modeling and predicting a categorical response variable $c$ is a classification task.

  An example for a categorical predictor-response setting is the classification of an email as spam ($c = \oplus$) or ham ($c = \ominus$), given a vector $\mathbf{x}$ of linguistic features for that email.

- (5), (6) Outcome-only setting, $y_1, \ldots, y_n$ or $c_1, \ldots, c_n$. Modeling a sole outcome variable means to fit the data $D$ using a suited distribution function, which in turn means to determine the distribution parameter $\theta$ or distribution parameters $\boldsymbol{\theta}$. Again, one can distinguish between different measurement scales, such as quantitative ($y$) or categorical ($c$).

  An example for a categorical outcome-only setting is a coin flip experiment where one has to fit the observations (number of heads and tails) under the binomial distribution, which in turn means to determine the distribution parameter $\theta$.

- (1),…, (6) Depending on the experiment setting, i.e., fitting of a model function vs. fitting of a distribution, either the symbol $w$ (or $\mathbf{w}$), or the symbol $\theta$ (or $\boldsymbol{\theta}$) may be used to denote the parameter (or parameter vector).

Remarks (discriminative vs. generative approach) :

(1), (2), (3) Discriminative approach to classification. Exploit the data to determine a decision boundary. Typically, "discriminative" implies "frequentist".

The optimization (argmin, argmax) considers $p(\mathbf{x})$, the distribution of the independent variables $\mathbf{x}$, implicitly via the multiplicity of $\mathbf{x}$ in the data $D$. Recall that $D$ is a multiset of examples.

(2), (3), (5) Maximum likelihood (ML) principle to parameter estimation.

(2) Recall the identities from the maximum likelihood derivation of the logistic loss $L_\sigma(\mathbf{w})$:

$$p(D;\mathbf{w}) = \prod_{(\mathbf{x},c)\in D} p(\mathbf{x},c;\mathbf{w}), \qquad \underset{\mathbf{w}\in\mathbf{R}^{p+1}}{\text{argmax}}\; p(D;\mathbf{w}) = \underset{\mathbf{w}\in\mathbf{R}^{p+1}}{\text{argmax}} \prod_{(\mathbf{x},c)\in D} p(c \mid \mathbf{x};\mathbf{w})$$

(1), (2) If the data comes from an exponential family and mild conditions are satisfied, least-squares estimates and maximum-likelihood estimates are identical.

(2), (3) Probabilistic model. The conditional class probability function (CCPF), $p(c \mid \mathbf{x})$, is estimated for all feature vectors (= at all quantiles). The model is not generative since the distribution of the independent variable, $p(\mathbf{x})$, is not modeled (but of course exploited implicitly via $D$).

Maximizing the probability under a logistic model is equivalent to minimizing the logistic loss $L_\sigma$. Hence, $\mathbf{w}_{\text{ML}} = \hat{\mathbf{w}}$.

Remarks (discriminative vs. generative approach) : (continued)

(4) Generative approach to classification. Exploit the data $D$ (here: estimate $p(\mathbf{x} \mid c)$ and $p(c)$ for all $\mathbf{x}$ and $c$) to provide a model for the joint probability distribution, $p(\mathbf{x}, c)$, from which $D$ is sampled.

(5) Generative approach. Assuming the conditions of the binomial data model, exploit the data $D$ (here: estimate the parameter $\theta$) to provide a model for the binomial probability distribution, $p(c)$, from which $D$ is sampled.

(6) Generative or discriminative approach. $p(\theta \mid D)$ can be estimated by either providing ($\rightarrow$ generative) or by *not* providing ($\rightarrow$ discriminative) a model for the probability distribution from which $D$ is sampled.

Remarks (ML principle vs. Bayes method) :

(1), (2), (3), (5) $\mathbf{w}$ (as well as $\theta$) is not the realization of a random variable—which would come along with a distribution—but an *exogenous parameter*, which is varied in order to find the maximum probability $p(D;\mathbf{w})$ (or $p(D;\theta)$ or the minimum loss $L(\mathbf{w})$.

The fact that $\mathbf{w}$ (or $\theta$) is an exogenous parameter and not a realization of a random variable is reflected by the notation, which uses a »;« instead of a »|« in the argument of $p()$.

(4) Application of Bayes's rule, presupposing that one can estimate the likelihoods $p(\mathbf{x} \mid \cdot\,)$ ( $p(x_j \mid \cdot\,)$ in case of Naive Bayes) at higher fidelity than the conditional class probabilities, $p(\,\cdot \mid \mathbf{x})$, from the data.

Under the Naive Bayes Assumption, $p(\mathbf{x} \mid c)$ is modeled as $\prod_{j=1}^{p} p(x_j \mid c)$.

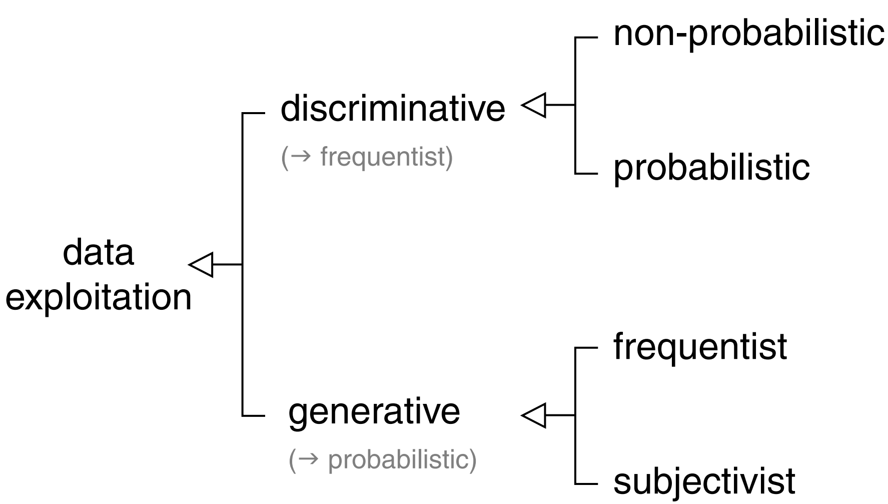
(4), (6) Likelihoods, $p(\mathbf{x} \mid \cdot\,)$, $p(D \mid \cdot\,)$, are computed for events under alternative classes $c$ or parameters $\theta$. The settings differ in that an event in (4) is about a feature vector $\mathbf{x}$, while an event in (6) is about a sequence $D$. (4) may (but not need to) apply the Naive Bayes assumption to compute the likelihood $p(\mathbf{x} \mid c)$, which is a common approximation for a nominal feature space and if data are sparse. For (6), if the data originate from a coin flip experiment, the likelihood $p(D \mid \theta)$ is computed via the binomial distribution.

If the prior probabilities, $p(c)$ or $p(\theta)$, are estimated also from $D$, we follow the frequentist paradigm; if the priors rely on subjective assessments we follow the subjectivist paradigm.

If we assume uniform priors, i.e., the $p(c)$ or the $p(\theta)$ are equally probable, MAP estimates and ML estimates are equal since $p(c \mid \mathbf{x}) \propto p(\mathbf{x} \mid c)$ or $p(\theta \mid D) \propto p(D \mid \theta)$, where »$\propto$« means "is proportional to".

# Exploitation of Data

## Learning Approaches Overview




Support vector machine

(1) Linear regression with least square estimates from $D$

(2) Logistic regression via $p()$ with ML estimates from $D$

(3) Logistic regression via $L()$ with ML estimates from $D$

(4) Bayes with ML estimates from $D$ as priors

(5) Probability model with ML estimate from $D$

(6) Bayes with subjective priors

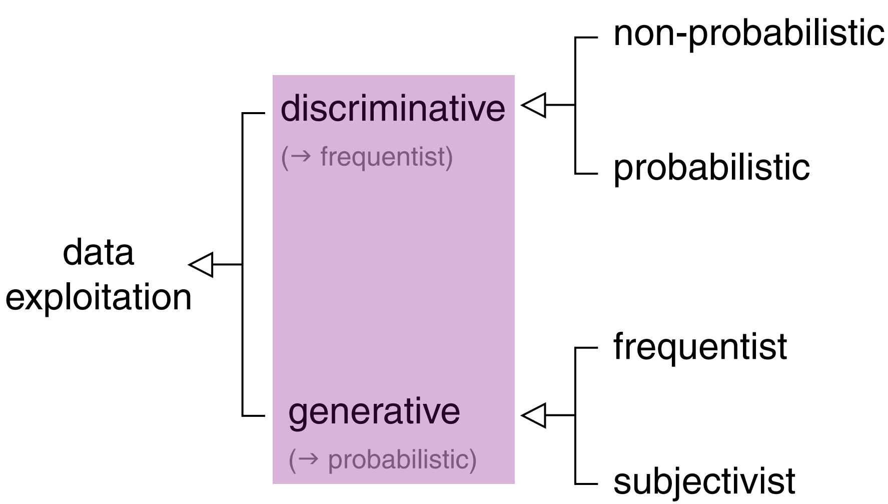
(4) Bayes with subjective priors

$D = \{(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_n, y_n)\}, \;\; D = \{(\mathbf{x}_1, c_1), \ldots, (\mathbf{x}_n, c_n)\}$

$D = \{y_1, \ldots, y_n\}, \;\; D = \{c_1, \ldots, c_n\}$

# Exploitation of Data

## Learning Approaches Overview (continued)

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$D = \{y_1, \ldots, y_n\}$, $D = \{c_1, \ldots, c_n\}$

discriminative : Determine a boundary to split $D$. $\rightarrow$ No model for the distribution of $D$.

generative : Provide a model for the probability distribution from which $D$ is sampled.

# Exploitation of Data

## Learning Approaches Overview (continued)

- data exploitation
  - discriminative (→ frequentist)
    - non-probabilistic
    - probabilistic
  - generative (→ probabilistic)
    - frequentist
    - subjectivist

Support vector machine

(1) Linear regression with least square estimates from $D$

(2) Logistic regression via $p()$ with ML estimates from $D$

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$D = \{y_1, \ldots, y_n\}$, $D = \{c_1, \ldots, c_n\}$

non-probabilistic : Threshold some model function (typically at zero). $\rightarrow$ Classification, Labeling

probabilistic : Estimate $p(c \mid \mathbf{x})$ at all quantiles. $\rightarrow$ Class probability estimation, CCPF

# Exploitation of Data

## Learning Approaches Overview (continued)

data exploitation
- discriminative (→ frequentist)
  - non-probabilistic
  - probabilistic
- generative (→ probabilistic)
  - frequentist
  - subjectivist

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(1) Linear regression with least square estimates from $D$
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(4) Bayes with subjective priors

$D = \{(\mathbf{x}_1, y_1), \ldots, (\mathbf{x}_n, y_n)\}$, $D = \{(\mathbf{x}_1, c_1), \ldots, (\mathbf{x}_n, c_n)\}$
$D = \{y_1, \ldots, y_n\}$, $D = \{c_1, \ldots, c_n\}$

frequentist : Consider a unique mechanism that generated the data $D$.
subjectivist : Specify beliefs for alternative mechanisms one of which generated $D$.

Remarks:

- We call a data exploitation approach “generative” if it provides us with a model for the probability distribution from which $D$ is sampled. With such a model we are able to generate arbitrary samples from the population where $D$ is sampled from.
- The overview does not show all but common combinations. In particular:
  - Typically, “discriminative” implies “frequentist”. The inverse does not apply: consider a Bayes classifier with priors estimated from the data.
  - Typically, “generative” implies “probabilistic”. The inverse does not apply: logistic regression provides a probabilistic model to classification.
- Discriminative approaches are further distinguished as “non-probabilistic” or “probabilistic”.
- Generative approaches are further distinguished as “frequentist” or “subjectivist”.

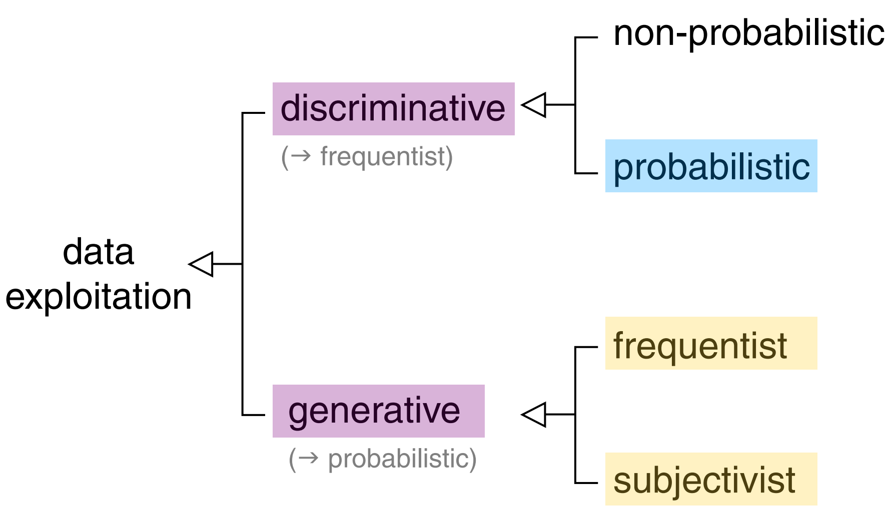
# Chapter ML:VII (continued)

## VII. Bayesian Learning

- Approaches to Probability
- Conditional Probability
- Bayes Classifier
- Exploitation of Data
- Frequentist versus Subjectivist

# Frequentist versus Subjectivist

## Logistic Regression versus Naive Bayes [data exploitation examples]

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$D = \{y_1, \ldots, y_n\},\ D = \{c_1, \ldots, c_n\}$

# Frequentist versus Subjectivist

## Logistic Regression versus Naive Bayes [data exploitation examples] (continued)

(2) $$\mathbf{w}_{\text{ML}} = \underset{\mathbf{w}\in\mathbf{R}^{p+1}}{\text{argmax}} \prod_{(\mathbf{x},c)\in D} p(c \mid \mathbf{x}; \mathbf{w})$$ (logistic regression)

(4) $$c_{\text{MAP}} = \underset{c\in\{\oplus,\ominus\}}{\text{argmax}} \ p(c \mid \mathbf{x})$$

Observation 1. Both approaches maximize $p(D)$ :

- (2), the "ML principle", determines the parameters $\mathbf{w}$ of the logistic model function such that $\prod_D p(c \mid \mathbf{x})$ becomes maximum. Note that a parameter vector $\mathbf{w}$ that maximizes $\prod_D p(c \mid \mathbf{x})$ will also maximize $\prod_D p(\mathbf{x}, c)$, and thus $p(D)$ (under the i.i.d. assumption).
- (4), the "Bayes method", determines for a given $\mathbf{x}$ its most probable class. By choosing $c_{\text{MAP}}$ for each $\mathbf{x}$, Bayes maximizes $p(D)$ by maximizing each factor of $\prod_D p(c \mid \mathbf{x})$. Note that $p(\mathbf{x})$ is constant per factor. Recall that Naive Bayes approximates $p(\mathbf{x} \mid c)$ with $\prod_{j=1}^{p} p(x_j \mid c)$.

# Frequentist versus Subjectivist

## Logistic Regression versus Naive Bayes [data exploitation examples] (continued)

(2) $\mathbf{w}_{\text{ML}} = \underset{\mathbf{w} \in \mathbf{R}^{p+1}}{\text{argmax}} \prod_{(\mathbf{x},c) \in D} p(c \mid \mathbf{x}; \mathbf{w})$ (logistic regression)

(4) $c_{\text{MAP}} = \underset{c \in \{\oplus, \ominus\}}{\text{argmax}} \dfrac{p(\mathbf{x} \mid c) \cdot p(c)}{p(\mathbf{x})}$ (Bayes)

Observation 1. Both approaches maximize $p(D)$ :

- (2), the "ML principle", determines the parameters $\mathbf{w}$ of the logistic model function such that $\prod_D p(c \mid \mathbf{x})$ becomes maximum. Note that a parameter vector $\mathbf{w}$ that maximizes $\prod_D p(c \mid \mathbf{x})$ will also maximize $\prod_D p(\mathbf{x}, c)$, and thus $p(D)$ (under the i.i.d. assumption).
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# Frequentist versus Subjectivist

## Logistic Regression versus Naive Bayes [data exploitation examples] (continued)

(2) $\mathbf{w}_{\text{ML}} = \underset{\mathbf{w}\in\mathbf{R}^{p+1}}{\text{argmax}} \prod_{(\mathbf{x},c)\in D} p(c \mid \mathbf{x}; \mathbf{w})$ (logistic regression)

(4) $c_{\text{MAP}} = \underset{c\in\{\oplus,\ominus\}}{\text{argmax}} \dfrac{\prod_{j=1}^{p} p(x_j \mid c) \cdot p(c)}{p(\mathbf{x})}$ (Naive Bayes)

Observation 1. Both approaches maximize $p(D)$ :

- (2), the "ML principle", determines the parameters $\mathbf{w}$ of the logistic model function such that $\prod_D p(c \mid \mathbf{x})$ becomes maximum. Note that a parameter vector $\mathbf{w}$ that maximizes $\prod_D p(c \mid \mathbf{x})$ will also maximize $\prod_D p(\mathbf{x}, c)$, and thus $p(D)$ (under the i.i.d. assumption).

- (4), the "Bayes method", determines for a given $\mathbf{x}$ its most probable class. By choosing $c_{\text{MAP}}$ for each $\mathbf{x}$, Bayes maximizes $p(D)$ by maximizing each factor of $\prod_D p(c \mid \mathbf{x})$. Note that $p(\mathbf{x})$ is constant per factor. Recall that Naive Bayes approximates $p(\mathbf{x} \mid c)$ with $\prod_{j=1}^{p} p(x_j \mid c)$.

# Frequentist versus Subjectivist

## Logistic Regression versus Naive Bayes [data exploitation examples] (continued)

(2) $\mathbf{w}_{\text{ML}} = \underset{\mathbf{w} \in \mathbf{R}^{p+1}}{\text{argmax}} \prod_{(\mathbf{x},c) \in D} p(c \mid \mathbf{x}; \mathbf{w})$ (logistic regression)

(4) $c_{\text{MAP}} = \underset{c \in \{\oplus,\ominus\}}{\text{argmax}} \prod_{j=1}^{p} p(x_j \mid c) \cdot p(c)$ (Naive Bayes)

Observation 1. Both approaches maximize $p(D)$ :

- (2), the "ML principle", determines the parameters $\mathbf{w}$ of the logistic model function such that $\prod_D p(c \mid \mathbf{x})$ becomes maximum. Note that a parameter vector $\mathbf{w}$ that maximizes $\prod_D p(c \mid \mathbf{x})$ will also maximize $\prod_D p(\mathbf{x}, c)$, and thus $p(D)$ (under the i.i.d. assumption).

- (4), the "Bayes method", determines for a given $\mathbf{x}$ its most probable class. By choosing $c_{\text{MAP}}$ for each $\mathbf{x}$, Bayes maximizes $p(D)$ by maximizing each factor of $\prod_D p(c \mid \mathbf{x})$. Note that $p(\mathbf{x})$ is constant per factor. Recall that Naive Bayes approximates $p(\mathbf{x} \mid c)$ with $\prod_{j=1}^{p} p(x_j \mid c)$.

# Frequentist versus Subjectivist

## Logistic Regression versus Naive Bayes [data exploitation examples] (continued)

(2) $\mathbf{w}_{\text{ML}} = \underset{\mathbf{w}\in\mathbf{R}^{p+1}}{\text{argmax}} \prod_{(\mathbf{x},c)\in D} p(c \mid \mathbf{x}; \mathbf{w})$ (logistic regression)

(4) $c_{\text{MAP}} = \underset{c\in\{\oplus,\ominus\}}{\text{argmax}} \prod_{j=1}^{p} p(x_j \mid c) \cdot p(c)$ (Naive Bayes)

Observation 1. Both approaches maximize $p(D)$ :

- (2), the "ML principle", determines the parameters $\mathbf{w}$ of the logistic model function such that $\prod_D p(c \mid \mathbf{x})$ becomes maximum. Note that a parameter vector $\mathbf{w}$ that maximizes $\prod_D p(c \mid \mathbf{x})$ will also maximize $\prod_D p(\mathbf{x}, c)$, and thus $p(D)$ (under the i.i.d. assumption).

- (4), the "Bayes method", determines for a given $\mathbf{x}$ its most probable class. By choosing $c_{\text{MAP}}$ for each $\mathbf{x}$, Bayes maximizes $p(D)$ by maximizing each factor of $\prod_D p(c \mid \mathbf{x})$. Note that $p(\mathbf{x})$ is constant per factor. Recall that Naive Bayes approximates $p(\mathbf{x} \mid c)$ with $\prod_{j=1}^{p} p(x_j \mid c)$.

# Frequentist versus Subjectivist

## Logistic Regression versus Naive Bayes [data exploitation examples] (continued)

(2) $\mathbf{w}_{\text{ML}} = \underset{\mathbf{w}\in\mathbf{R}^{p+1}}{\text{argmax}} \prod_{(\mathbf{x},c)\in D} p(c \mid \mathbf{x}; \mathbf{w})$ (logistic regression)

(4) $c_{\text{MAP}} = \underset{c\in\{\oplus,\ominus\}}{\text{argmax}} \prod_{j=1}^{p} p(x_j \mid c) \cdot p(c)$ (Naive Bayes)

Observation 2 (corollary). Both approaches model the covariate distribution:

- (2), the "ML principle", considers $p(\mathbf{x})$, the distribution of the independent variables $\mathbf{x}$, implicitly via the multiplicity of $\mathbf{x}$ in the data $D$. Recall that $D$ is a multiset of examples.

- (4), the "Bayes method", as a generative approach, models $p(\mathbf{x} \mid c)$ and $p(c)$, and hence also $p(\mathbf{x}, c)$, $p(\mathbf{x})$, and $p(c \mid \mathbf{x})$. The likelihoods, $p(\mathbf{x} \mid c)$ (or $p(x_j \mid c)$ under Naive Bayes), are estimated from $D$; the priors, $p(c)$, may be estimated by subjective assessments.

Remarks:

- ❑ Both approaches maximize $p(D)$ by maximizing $\prod_D p(c \mid \mathbf{x})$.

  Estimating $p(c \mid \mathbf{x})$ is usually significantly easier than estimating $p(\mathbf{x}, c)$.

- (4) Naive Bayes models $p(\mathbf{x} \mid c)$ as $\prod_{j=1}^{p} p(x_j \mid c)$, where $p(x_j \mid c)$ is estimated as $\hat{p}(x_j \mid c)$, $\hat{p}(x_j \mid c) = |\{(\mathbf{x}, c) \in D : \mathbf{x}|_j{=}x_j\}| \,/\, |\{(\,\cdot\,, c) \in D\}|$.

  Similarly, $p(c)$ can estimated as $\hat{p}(c)$, $\hat{p}(c) = |\{(\,\cdot\,, c) \in D\}|$; but, also a dedicated (and subjective) prior probability model can be stated.

  $p(\mathbf{x})$ can be computed with the Law of Total Probability, $p(\mathbf{x}) = \sum_{c \in \{\oplus, \ominus\}} p(\mathbf{x} \mid c) \cdot p(c)$. Note, however, that $p(\mathbf{x})$ is not required to compute $c_{\text{MAP}}$ for $\mathbf{x}$.

- (4) If for the Bayes method—aside from the likelihoods $p(x_j \mid c)$— also the class priors, $p(c)$, are computed from $D$, we follow the frequentist paradigm, similar to the ML principle. Only if the values for $p(c)$ (= the prior probability model) rely on subjective assessments, the Bayes method can be considered as subjectivist.

- ❑ Whether to apply the ML principle or the Bayes method is not a free choice; it depends on

  - the availability of data $D$,
  - the conditional strengths of the likelihoods, $p(\mathbf{x} \mid c)$,
  - the reliability of the assessments for the prior probabilities, $p(c)$, and,
  - whether or not subjective assessments shall be considered to estimate the priors $p(c)$.

- ❑ Synonymous: covariate, independent, predictor [variable / distribution].

Remarks: (continued)

- Observe the subtle distinction between “Bayes rule” and “Bayes method” made here. With the former we refer to the identity that connects the posterior probability, $P(A \mid B)$, and the likelihood, $P(B \mid A)$ (the “reversal of condition and consequence”). With the latter we refer to the *parameter estimation principle* where the maximum a posteriori probability is determined.

- Note that a class-conditional event “**X**$=\mathbf{x} \mid$ *C*$=c$” does not necessarily model a cause-effect relation: the event “*C*$=c$” may cause—but does not need to cause—the event “**X**$=\mathbf{x}$”.

  Examples:

  – A disease $c$ will cause the symptoms $\mathbf{x}$ (but not vice versa).

  – Weather conditions $\mathbf{x}$ will cause the decision “*EnjoySurfing*=yes” (but not vice versa).

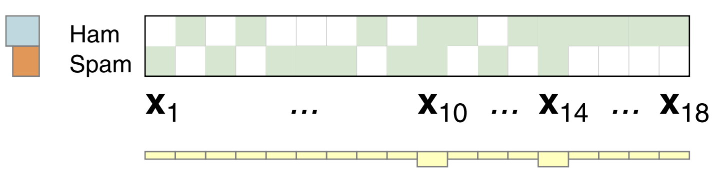
  Similarly, also if $\mathbf{x}$ is the independent variable of a function $y(\mathbf{x})$ that maps features to classes $c$, the cause-effect direction is not necessarily $\mathbf{x} \rightarrow c$, but can also be the other way around: Consider $y(\mathbf{x}) = c$ with “disease $c$” $\rightarrow$ ”symptoms $\mathbf{x}$”.

# Frequentist versus Subjectivist

## Logistic Regression versus Naive Bayes: Example

A multiset of examples $D$:

| | URLs | Spelling errors | Spam |
|---|---|---|---|
| 1 | 5 | 3 | yes |
| 2 | 4 | 1 | no |
| 3 | 4 | 3 | yes |
| ⋮ | ⋮ | ⋮ | ⋮ |
| 10 | 1 | 0 | no |
| 11 | 1 | 0 | yes |
| ⋮ | ⋮ | ⋮ | ⋮ |
| 15 | 1 | 4 | no |
| 16 | 1 | 4 | yes |
| ⋮ | ⋮ | ⋮ | ⋮ |
| 20 | 0 | 4 | no |




Learning task:

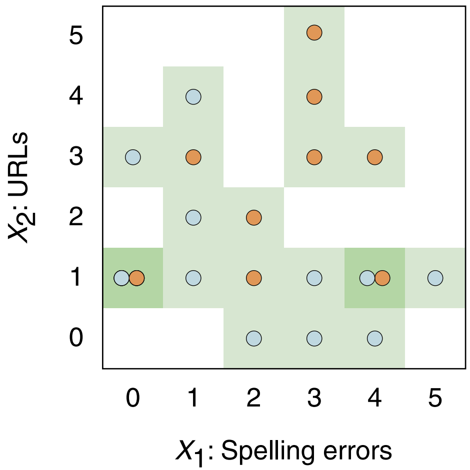
- Fit $D$ to compute a classifier for feature vectors $\mathbf{x}$, $\mathbf{x} \notin D$.

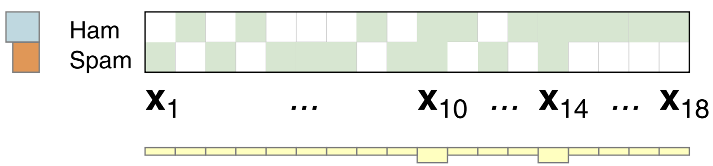
# Frequentist versus Subjectivist

## Logistic Regression versus Naive Bayes: Example (continued)

A multiset of examples $D$:

| | URLs | Spelling errors | Spam |
|---|---|---|---|
| 1 | 5 | 3 | yes |
| 2 | 4 | 1 | no |
| 3 | 4 | 3 | yes |
| ⋮ | ⋮ | ⋮ | ⋮ |
| 10 | 1 | 0 | no |
| 11 | 1 | 0 | yes |
| ⋮ | ⋮ | ⋮ | ⋮ |
| 15 | 1 | 4 | no |
| 16 | 1 | 4 | yes |
| ⋮ | ⋮ | ⋮ | ⋮ |
| 20 | 0 | 4 | no |




Learning task:

- Fit $D$ to compute a classifier for feature vectors $\mathbf{x}$, $\mathbf{x} \notin D$.

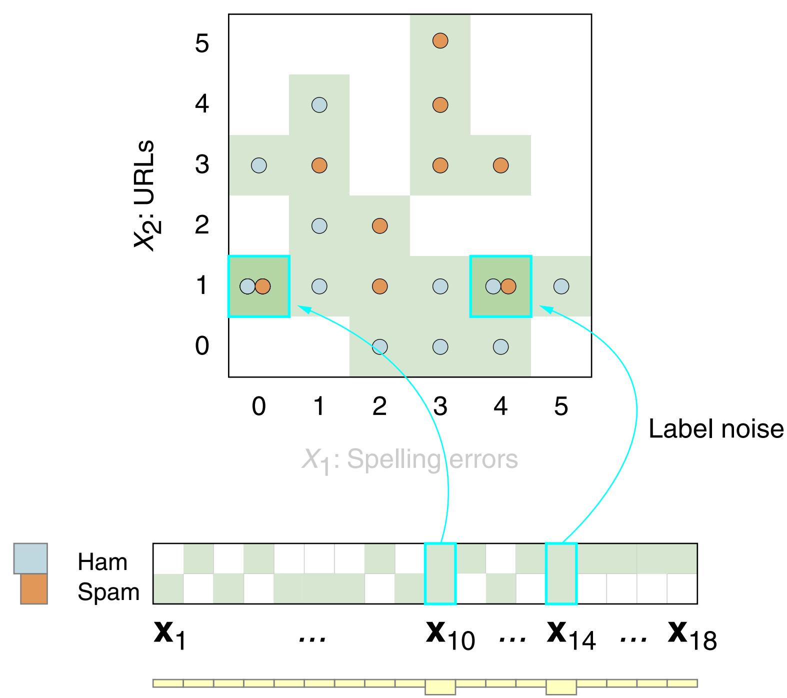
 

# Frequentist versus Subjectivist

## Logistic Regression versus Naive Bayes: Example (continued)

A multiset of examples $D$:

| | URLs | Spelling errors | Spam |
|---|---|---|---|
| 1 | 5 | 3 | yes |
| 2 | 4 | 1 | no |
| 3 | 4 | 3 | yes |
| ⋮ | ⋮ | ⋮ | ⋮ |
| **10** | **1** | **0** | **no** |
| **11** | **1** | **0** | **yes** |
| ⋮ | ⋮ | ⋮ | ⋮ |
| **15** | **1** | **4** | **no** |
| **16** | **1** | **4** | **yes** |
| ⋮ | ⋮ | ⋮ | ⋮ |
| 20 | 0 | 4 | no |



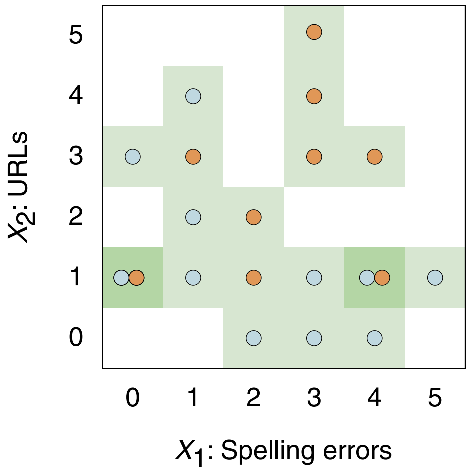

Learning task:

- Fit $D$ to compute a classifier for feature vectors $\mathbf{x}$, $\mathbf{x} \notin D$.

# Frequentist versus Subjectivist

## Logistic Regression versus Naive Bayes: Conditional Class Probabilities

Logistic regression:

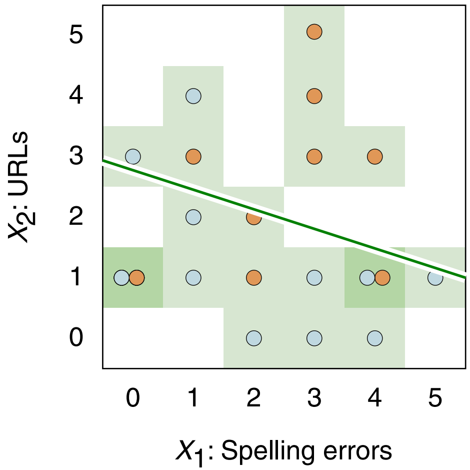



- Distribution of $D$.

# Frequentist versus Subjectivist

## Logistic Regression versus Naive Bayes: Conditional Class Probabilities (continued)

Logistic regression:

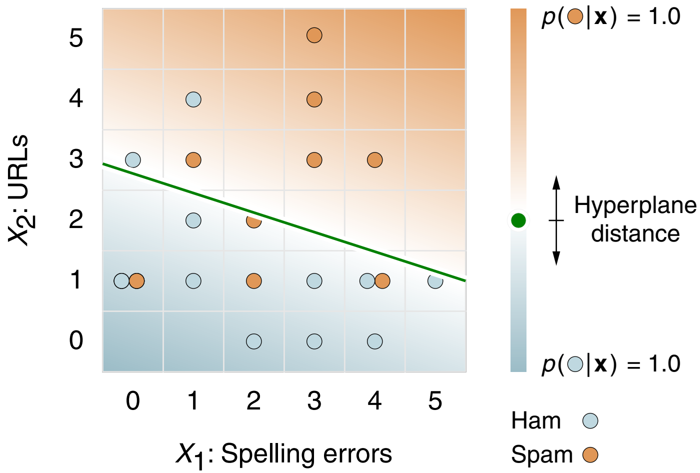



- Hyperplane $\langle \mathbf{w}_{\text{ML}}, \mathbf{x} \rangle = 0$.

# Frequentist versus Subjectivist

## Logistic Regression versus Naive Bayes: Conditional Class Probabilities (continued)

Logistic regression:

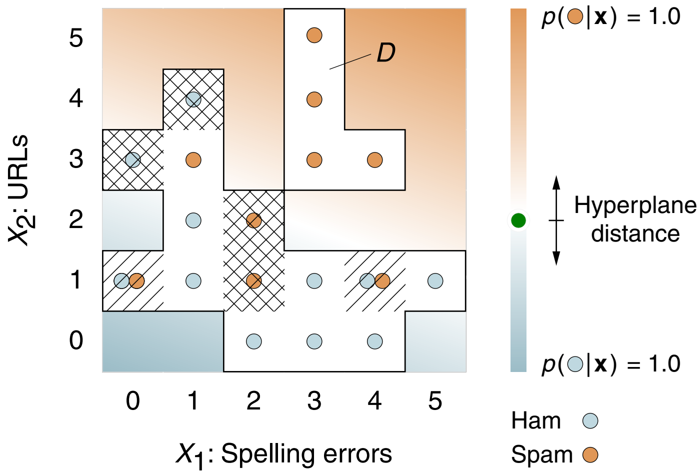



- Conditional class probabilities computed with $\mathbf{w}_{\text{ML}}$, the ML estimate for $\mathbf{w}$ given $D$.

# Frequentist versus Subjectivist

## Logistic Regression versus Naive Bayes: Conditional Class Probabilities (continued)

Logistic regression:

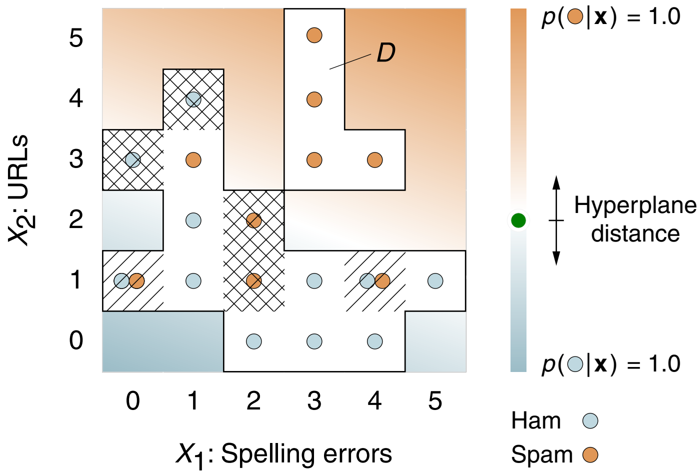



- Training error.

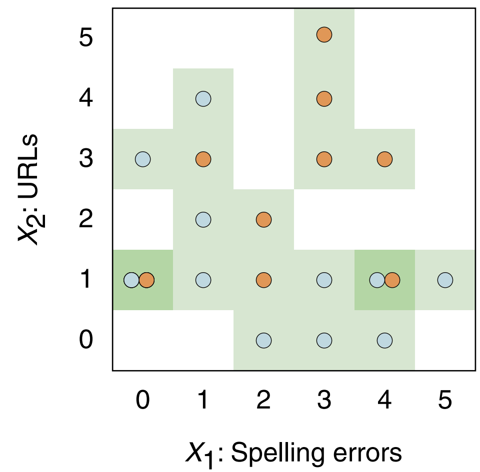
# Frequentist versus Subjectivist

## Logistic Regression versus Naive Bayes: Conditional Class Probabilities (continued)

Logistic regression:




Naive Bayes:

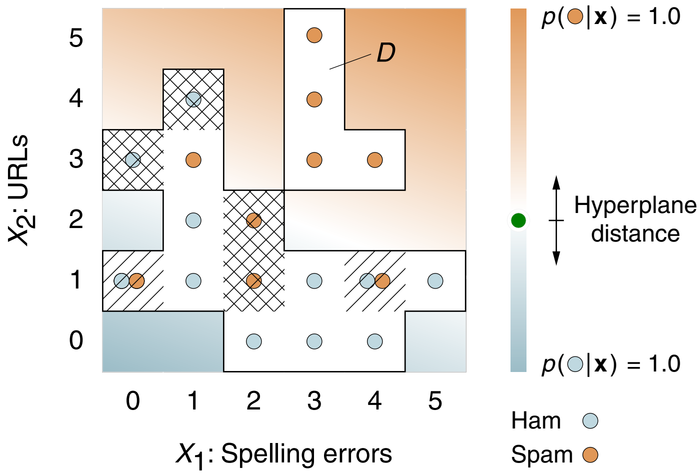



- Training error.
- Distribution of $D$.

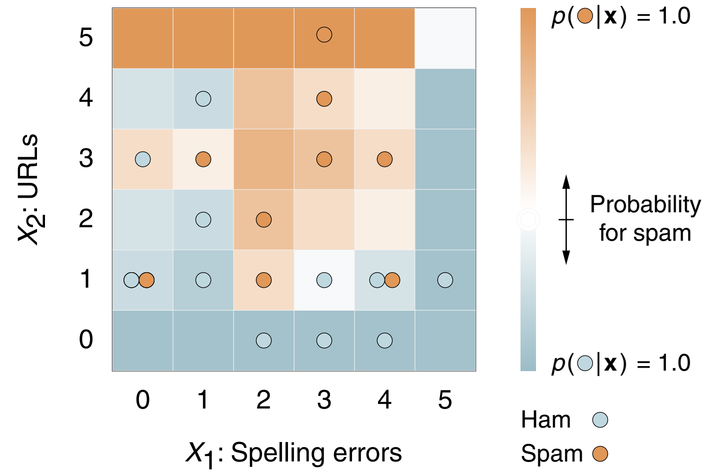
# Frequentist versus Subjectivist

## Logistic Regression versus Naive Bayes: Conditional Class Probabilities (continued)

Logistic regression:




Naive Bayes:



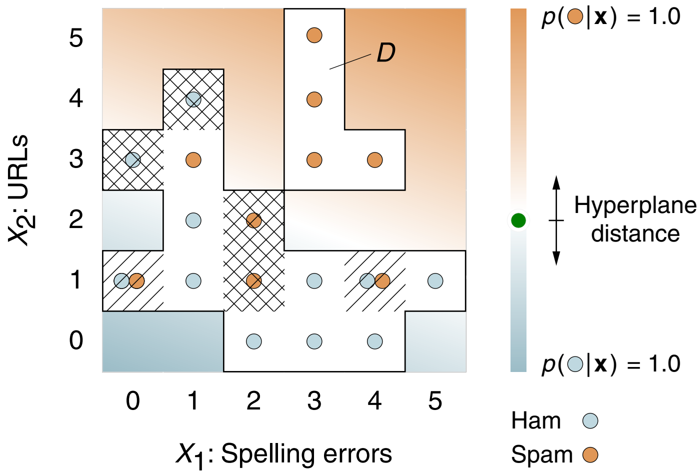

- Training error.

- Conditional class probabilities computed for the respective MAP class, using $p(c)$ estimates from $D$.

# Frequentist versus Subjectivist

## Logistic Regression versus Naive Bayes: Conditional Class Probabilities (continued)

Logistic regression:

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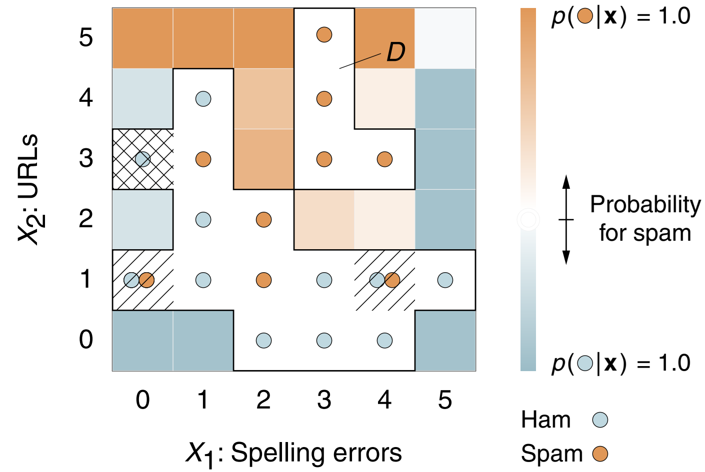


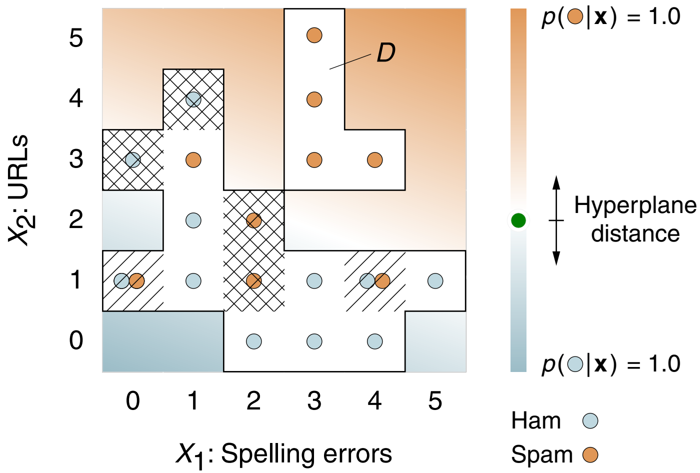

- Training error.

- Training error.

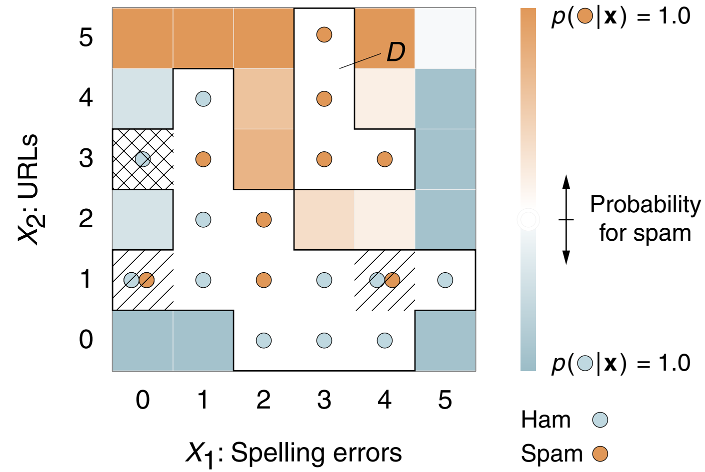
# Frequentist versus Subjectivist

## Logistic Regression versus Naive Bayes: Conditional Class Probabilities (continued)

Logistic regression:




Naive Bayes:




- Computation of a hyperplane.
- Approach: minimization of accumulated "misclassification distances" for examples in $D$.
- Discriminative and probabilistic.

- Computation of a probability distribution.
- Basis: class-conditional feature and class frequencies in $D$.
- Generative (implies probabilistic).

Remarks:

- Both approaches, logistic regression and Naive Bayes, estimate the conditional class probability function, $p(\text{Spam} \mid \mathbf{x})$ or $p(\text{Ham} \mid \mathbf{x}) = 1 - p(\text{Spam} \mid \mathbf{x})$. However, the two estimation approaches follow very different concepts.

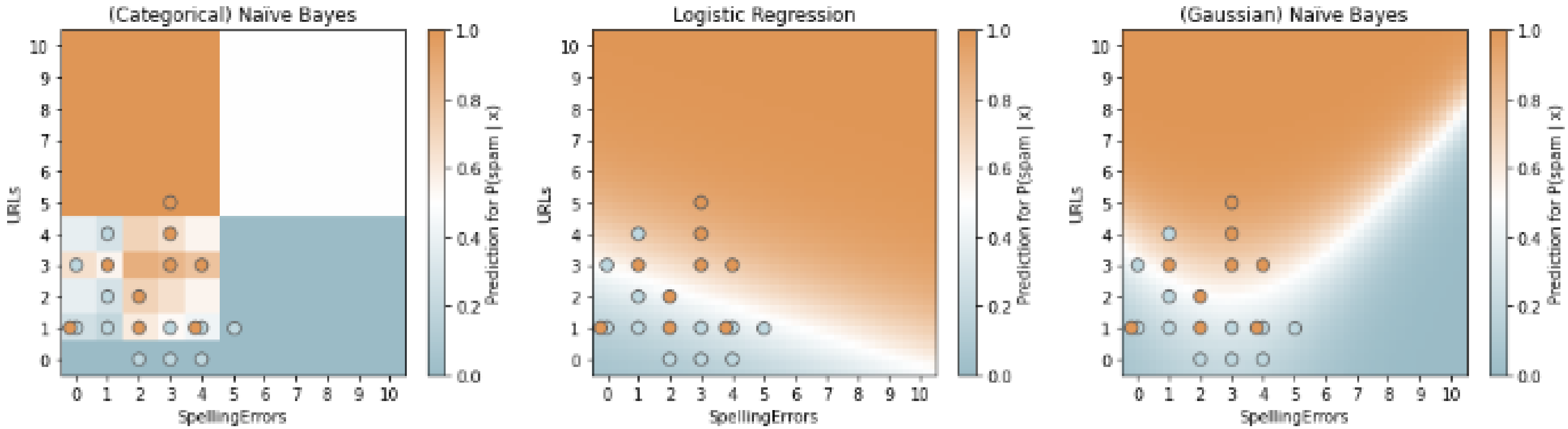
- Generalization characteristic:
  - The conditional class probability function as computed via logistic regression decides not only the feature space $\{0, 1, 2, 3, 4, 5\}^2$ but the entire $\mathbf{R}^2$. (Whether that makes sense is another matter.)
  - The conditional class probability function as computed via Naive Bayes provides class probability estimates for $\mathbf{x} \in \{0, 1, 2, 3, 4, 5\}^2$. The probabilities are estimated from the class-conditional feature frequencies (likelihood estimates) and class frequencies, $\hat{p}(x_1 \mid c)$, $\hat{p}(x_2 \mid c)$, and $\hat{p}(c)$, as found in $D$. Note that a vector $\mathbf{x} = (x_1, x_2)^T$ gets the probability of zero for class $c$, if $x_1$ or $x_2$ does not occur in some feature vector with class label $c$ in $D$.

- Handling of class imbalance and covariate distribution:
  - Logistic regression considers the $p(c)$ and the $p(\mathbf{x})$ implicitly via their multiplicity in $D$. I.e., the learned parameter vector $\mathbf{w}$ has the class imbalance as well as the covariate distribution “compiled in”.
  - Naive Bayes, again, estimates the $p(c)$ and the $p(\mathbf{x})$ from the frequencies in $D$. More specifically, $p(\mathbf{x})$ can be estimated from $\hat{p}(x_1 \mid c)$, $\hat{p}(x_2 \mid c)$, and $\hat{p}(c)$ with the Law of Total Probability. Note that the computation of $p(\mathbf{x})$ is not necessary for a ranking (= classification without class membership probability).

# Frequentist versus Subjectivist

## Naive Bayes: Smoothing and Continuous Likelihoods




$\rightsquigarrow$ *BOARD*

# Frequentist versus Subjectivist

## Naive Bayes: Prior Probability Models

Comparison of the conditional class probability function, $p(c \mid \mathbf{x})$, under Naive Bayes for three different prior probability models (= assessments of class priors), $p(c)$.


$p(c)$ estimates from $D$

$P_a(C{=}\text{Spam}) = \hat{p}(\text{Spam}) = 0.45$

$P_a(C{=}\text{Ham}) = \hat{p}(\text{Ham}) = 0.55$

Subjective assessments for $p(c)$

$P_b(C{=}\text{Spam}) = 0.6$

$P_b(C{=}\text{Ham}) = 0.4$

$P_c(C{=}\text{Spam}) = 0.8$

$P_c(C{=}\text{Ham}) = 0.2$

# Frequentist versus Subjectivist

## Classification: Bayes Optimum versus MAP versus Ensemble

$\rightsquigarrow$ *BOARD*

# Frequentist versus Subjectivist

## Advanced Bayesian Decision Making

Recall the Bayes rule,

$$P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)},$$

with $A$ and $B$ in the role of a "hypothesis event", $\boldsymbol{H}{=}h$, and a "data event", $\mathbf{D}{=}D$,

$$P(\boldsymbol{H}{=}h \mid \mathbf{D}{=}D) \;=\; \frac{P(\mathbf{D}{=}D \mid \boldsymbol{H}{=}h) \cdot P(\boldsymbol{H}{=}h)}{P(\mathbf{D}{=}D)}$$

# Frequentist versus Subjectivist

## Advanced Bayesian Decision Making (continued)

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rewritten using probability mass functions, pmf, (in case of discrete events) :

$$p(h \mid D) = \frac{p(D \mid h) \cdot p(h)}{p(D)}$$

- Likelihood: How well does $h$ explain (= entail, induce, evoke) the data $D$?
- Prior: How probable is the hypothesis $h$ a priori (= in principle)?
- Normalization: How probable is the observation of the data $D$?
- Posterior: How probable is the hypothesis $h$ when observing the data $D$?

# Frequentist versus Subjectivist

## Advanced Bayesian Decision Making (continued)

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# Frequentist versus Subjectivist

## Advanced Bayesian Decision Making (continued)

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# Frequentist versus Subjectivist

## Advanced Bayesian Decision Making (continued)

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- Posterior: How probable is the hypothesis $h$ when observing the data $D$?

Remarks:

- When using the Bayes method for a predictor-response setting, then $p(D)$, $p(D) := P(\mathbf{D}{=}D)$, is the probability of the data $D = \mathbf{x}$. I.e., **D** is a random vector whose domain is the feature space $\mathbf{X}$.
- When using the Bayes method for an outcome-only setting, then $p(D)$, $p(D) := P(\mathbf{D}{=}D)$, is the probability of the data $D = \{y_1, \ldots, y_n\}$ or $D = \{c_1, \ldots, c_n\}$. I.e., **D** is a random vector whose domain is $\mathbf{R}^n$ or $C^n$, where $C$ is the set of possible classes or class labels.
- $p(h) := P(\boldsymbol{H}{=}h)$ (also $p(\mathbf{w})$, $p(\theta)$, or similar) is the probability of choosing a certain $h$, a parameter vector $\mathbf{w}$, or some model function as hypothesis. I.e., *H* is a random variable whose domain is the set $H$ of possible hypotheses.
- Recall that $p()$ is defined via $P()$ and that the two notations can be used interchangeably, arguing about realizations of random variables and events respectively.

 