

Chapter ML:VI

VI. Decision Trees

- ❑ Decision Trees Basics
- ❑ Impurity Functions
- ❑ Decision Tree Algorithms
- ❑ Decision Tree Pruning

Decision Tree Algorithms

ID3 Algorithm [Quinlan 1986] [CART Algorithm]

Setting:

- X is a multiset of feature vectors.
- C is a set of classes.
- $D = \{(\mathbf{x}_1, c_1), \dots, (\mathbf{x}_n, c_n)\} \subseteq X \times C$ is a multiset of examples.

Learning task:

- Fit D using a decision tree T .

Decision Tree Algorithms

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Characteristics of the ID3 algorithm:

1. Each splitting is based on one nominal feature and considers its complete domain. Splitting based on feature A with domain $dom(A) = \{a_1, \dots, a_m\}$:

$$X = \{\mathbf{x} \in X : \mathbf{x}|_A = a_1\} \cup \dots \cup \{\mathbf{x} \in X : \mathbf{x}|_A = a_m\}$$

2. Splitting criterion is information gain.

Decision Tree Algorithms

ID3 Algorithm (continued) [Mitchell 1997 version] [algorithm template]

ID3(D, Features)

1. Create a node t for the tree.
2. Label t with the most common class in D .
3. If all examples in D have the same class, return the single-node tree t .
4. If Features is empty, return the single-node tree t .

Decision Tree Algorithms

ID3 Algorithm (continued) [Mitchell 1997 version] [algorithm template]

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2. Label t with the most common class in D .
3. If all examples in D have the same class, return the single-node tree t .
4. If Features is empty, return the single-node tree t .

Otherwise:

5. Let A^* be the feature from Features that best classifies examples in D .

Assign t the decision feature A^* .

6. For each possible value “ a ” in $\text{dom}(A^*)$ do:

- ❑ Add a new tree branch below t , corresponding to the test $A^* = “a”$.
- ❑ Let D_a be the subset of D that has value “ a ” for A^* .
- ❑ If D_a is empty:

Then add a leaf node with the label of the most common class in D .

Else add the subtree **ID3**(D_a , Features $\setminus \{A^*\}$).

7. Return t .

Decision Tree Algorithms

ID3 Algorithm (continued) [\[algorithm template\]](#)

ID3(D, Features)

1. $t = \text{createNode}()$
2. $\text{label}(t) = \text{mostCommonClass}(D)$
3. IF $\forall (\mathbf{x}, c) \in D : c = \text{label}(t)$ THEN **return**(t) ENDIF // D is pure.
4. IF $\text{Features} = \emptyset$ THEN **return**(t) ENDIF // We are running out of features.
- 5.
- 6.
- 7.

Decision Tree Algorithms

ID3 Algorithm (continued) [\[algorithm template\]](#)

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5. $A^* = \text{argmax}_{A \in \text{Features}} (\text{informationGain}(D, A))$
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Decision Tree Algorithms

ID3 Algorithm (continued) [\[algorithm template\]](#)

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5. $A^* = \text{argmax}_{A \in \text{Features}} (\text{informationGain}(D, A))$
6. **FOREACH** $a \in \text{dom}(A^*)$ **DO**
 $D_a = \{(\mathbf{x}, c) \in D : \mathbf{x}|_{A^*} = a\}$
 IF $D_a = \emptyset$ THEN

 ELSE
 $\text{createEdge}(t, a, \text{ID3}(D_a, \text{Features} \setminus \{A^*\}))$
 ENDIF

 ENDDO
7. **return**(t)

Decision Tree Algorithms

ID3 Algorithm (continued) [\[algorithm template\]](#)

ID3(D, Features)

1. $t = \text{createNode}()$
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6. **FOREACH** $a \in \text{dom}(A^*)$ **DO**
 $D_a = \{(\mathbf{x}, c) \in D : \mathbf{x}|_{A^*} = a\}$
 IF $D_a = \emptyset$ THEN // We are running out of data.
 $t' = \text{createNode}()$
 $\text{label}(t') = \text{label}(t)$
 $\text{createEdge}(t, a, t')$
 ELSE
 $\text{createEdge}(t, a, \text{ID3}(D_a, \text{Features} \setminus \{A^*\}))$
 ENDIF
ENDDO
7. $\text{return}(t)$

Remarks:

- ❑ Step 3 of the [ID3 algorithm](#) checks the purity of D and, given this case, assigns the unique class to the respective node.
- ❑ The ID3 (Iterative Dichotomiser 3) was published by [Ross Quinlan](#) in 1986.

Decision Tree Algorithms

ID3 Algorithm: Example

Example set D for mushrooms, drawn from a set of feature vectors X over the three dimensions color, size, and points:

	Color	Size	Points	Edibility
1	red	small	yes	toxic
2	brown	small	no	edible
3	brown	large	yes	edible
4	green	small	no	edible
5	red	large	no	edible



Decision Tree Algorithms

ID3 Algorithm: Example (continued)

Top-level call of ID3. Analyze a splitting with regard to the feature “color” :

		toxic	edible		
$D _{\text{color}}$	=	red	1	1	$\leadsto \quad D_{\text{red}} = 2, \quad D_{\text{brown}} = 2, \quad D_{\text{green}} = 1$
		brown	0	2	
		green	0	1	

Decision Tree Algorithms

ID3 Algorithm: Example (continued)

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Estimated prior probabilities:

$$\hat{P}(\textit{Color}=\textit{red}) = \frac{2}{5} = 0.4, \quad \hat{P}(\textit{Color}=\textit{brown}) = \frac{2}{5} = 0.4, \quad \hat{P}(\textit{Color}=\textit{green}) = \frac{1}{5} = 0.2$$

Decision Tree Algorithms

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Conditional entropy:

$$\begin{aligned} H(\mathcal{A} \mid \mathcal{B}_1) &= H(\{A_1, A_2\} \mid \{B_{1,1}, B_{1,2}, B_{1,3}\}) \\ &= H(\{C=\text{toxic}, C=\text{edible}\} \mid \{\text{Color}=\text{red}, \text{Color}=\text{brown}, \text{Color}=\text{green}\}) \\ &= -(0.4 \cdot (\tfrac{1}{2} \cdot \log_2 \tfrac{1}{2} + \tfrac{1}{2} \cdot \log_2 \tfrac{1}{2}) + 0.4 \cdot (\tfrac{0}{2} \cdot \log_2 \tfrac{0}{2} + \tfrac{2}{2} \cdot \log_2 \tfrac{2}{2}) + 0.2 \cdot (\tfrac{0}{1} \cdot \log_2 \tfrac{0}{1} + \tfrac{1}{1} \cdot \log_2 \tfrac{1}{1})) = 0.4 \end{aligned}$$

Decision Tree Algorithms

ID3 Algorithm: Example (continued)

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$$H(\mathcal{A} \mid \mathcal{B}_2) = H(\{C=\text{toxic}, C=\text{edible}\} \mid \{\text{Size}=\text{small}, \text{Size}=\text{large}\}) = \dots \approx 0.55$$

$$H(\mathcal{A} \mid \mathcal{B}_3) = H(\{C=\text{toxic}, C=\text{edible}\} \mid \{\text{Points}=\text{yes}, \text{Points}=\text{no}\}) = \dots = 0.4$$

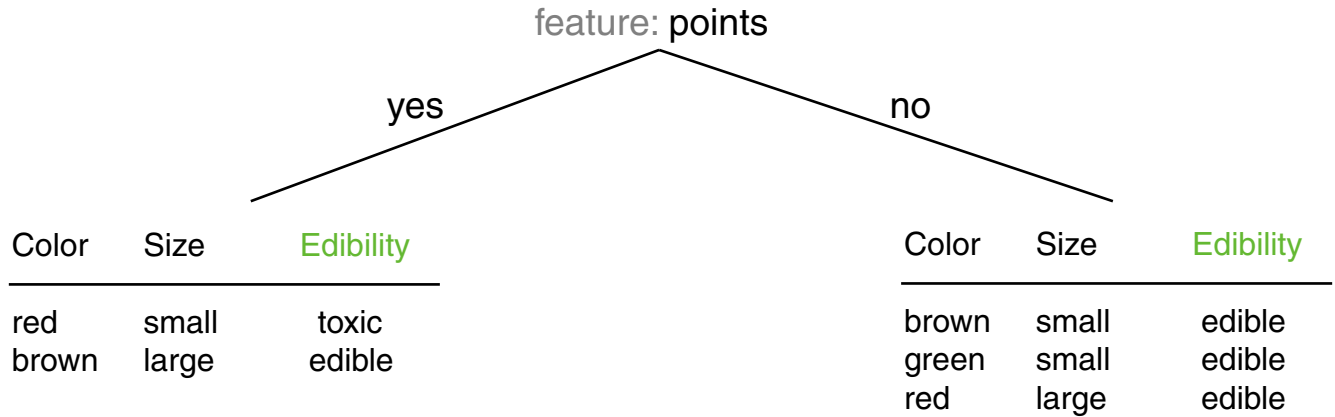
Remarks:

- ❑ The smaller $H(\mathcal{A} \mid \mathcal{B})$ is, the larger becomes the information gain. Hence, the difference $H(\mathcal{A}) - H(\mathcal{A} \mid \mathcal{B})$ needs not to be computed since $H(\mathcal{A})$ is constant within each recursion step.
- ❑ In the example, the information gain in the first recursion step becomes maximum for the features “color” and “points”.
- ❑ Notation. When used in the role of a random variable (here: in the argument of a probability P), features are written in italics and capitalized.
- ❑ Notation. The probabilities, denoted as $P(\cdot)$, are unknown and estimated by the relative frequencies, denoted as $\hat{P}(\cdot)$.

Decision Tree Algorithms

ID3 Algorithm: Example (continued)

Decision tree before the first recursion step:

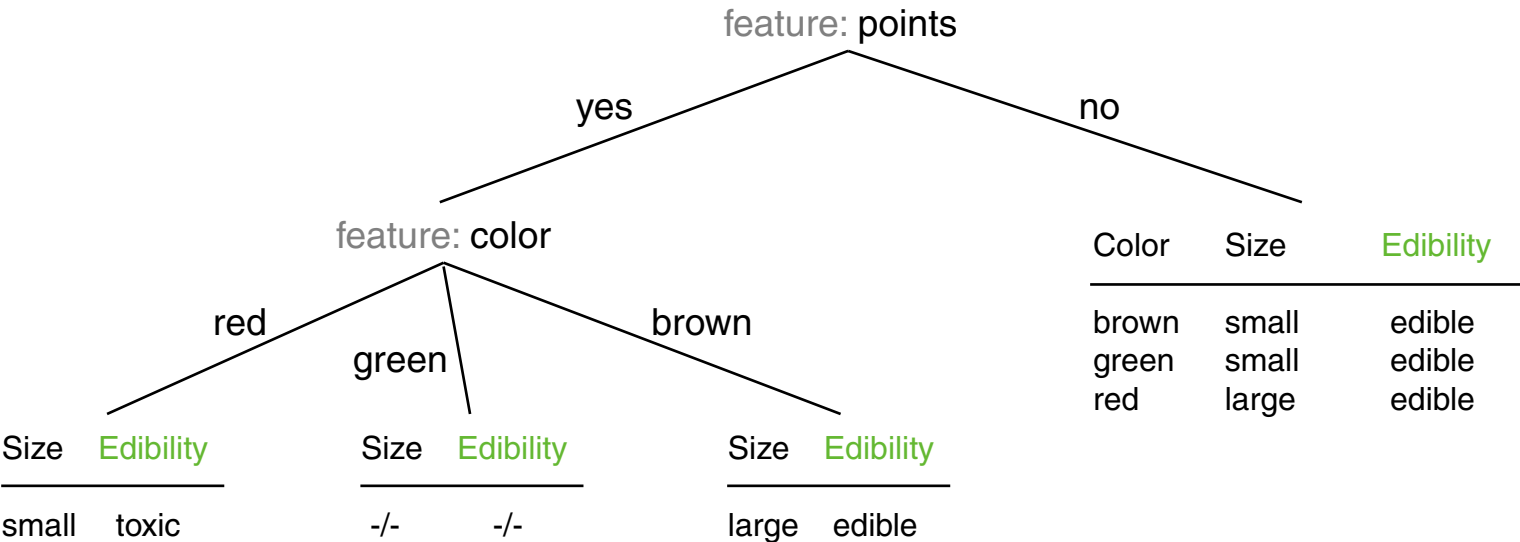


Choosing the feature “points” in Step 5 of the ID3 algorithm.

Decision Tree Algorithms

ID3 Algorithm: Example (continued)

Decision tree before the second recursion step:

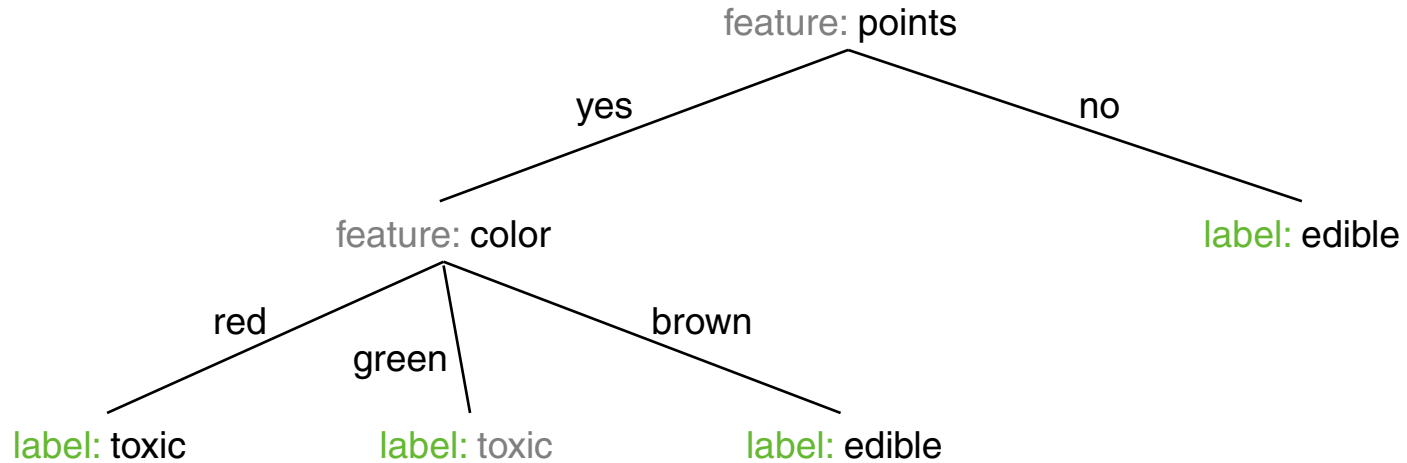


Choosing the feature “color” in Step 5 of the ID3 algorithm.

Decision Tree Algorithms

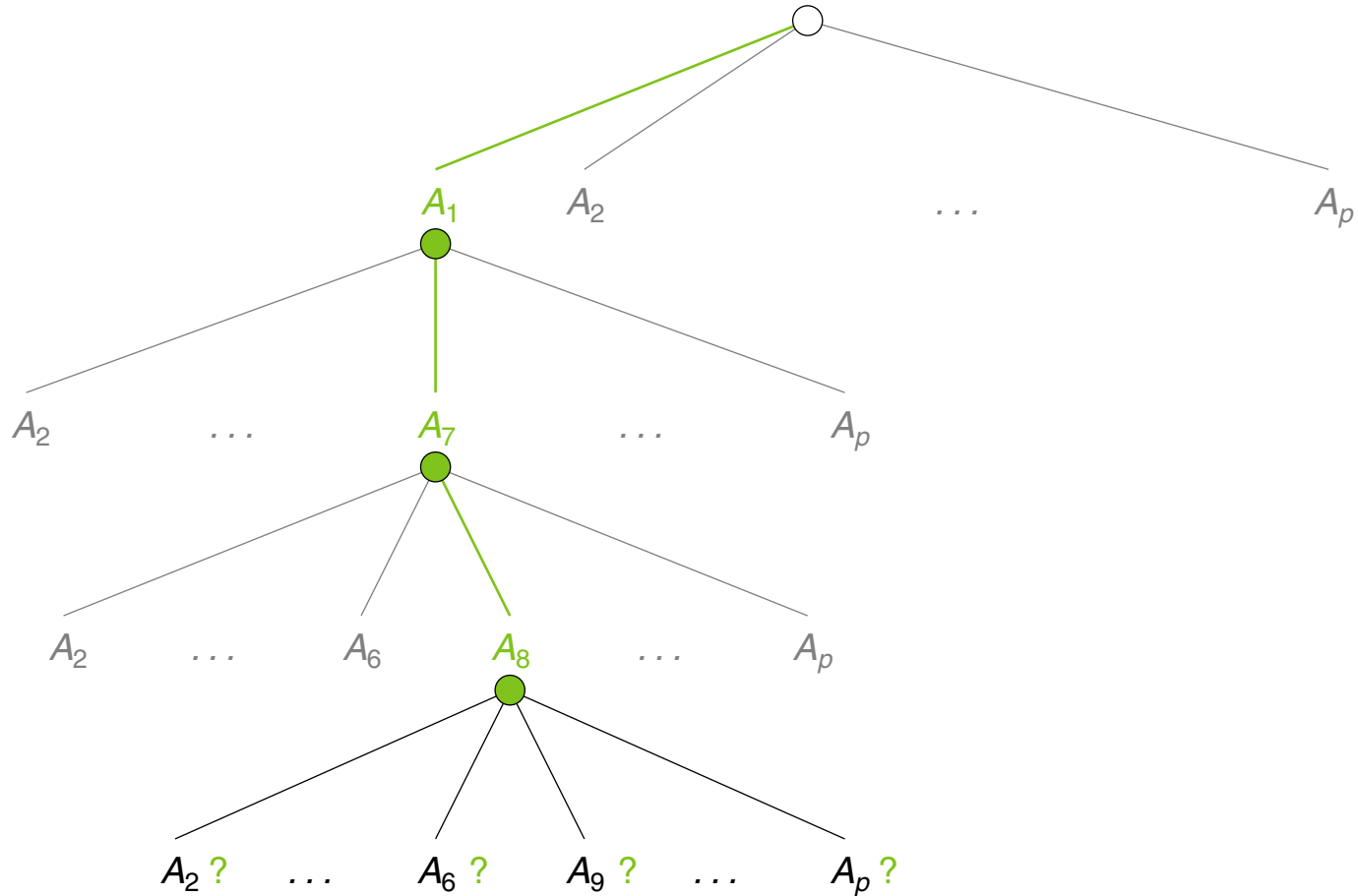
ID3 Algorithm: Example (continued)

Final decision tree after second recursion step:



Break of a tie: choosing the class “toxic” for D_{green} in Step 6 of the ID3 algorithm.

ID3 Algorithm: Search Space

$$\textit{Features} = \{A_1, A_2, \dots, A_p\}$$


Remarks (search space versus hypothesis space):

- ❑ The underlying [search space](#) of an algorithm that samples without replacement a single feature in each step (= monothetic splitting) consists of all permutations of the features in the feature set. In particular, if the number of features (= dimensionality of a feature vector \mathbf{x}) is p , then the search space contains $p!$ elements.
- ❑ The set of possible decision trees over D forms the hypothesis space H . The maximum size of H , i.e., the maximum number of decision trees for a data set D in a binary classification setting, is $2^{|D|}$: If the feature vectors are pairwise distinct, every subset of D can form a class while the complement of the subset will form the other class. The set of possible subsets of D is $\mathcal{P}(D)$, where $|\mathcal{P}(D)| = 2^{|D|}$.
- ❑ Observe that either $p! < 2^{|D|}$ or $p! > 2^{|D|}$ can hold. I.e., the search space due to feature ordering can be smaller or larger than its underlying hypothesis space. The former characterizes the typical situation; also note that both the search space and the hypothesis space grow exponentially in the number of features and examples respectively.
- ❑ The difference between search space size and hypothesis space size results from Step 6 of the [ID3 algorithm](#): the same feature selection order will lead to different decision trees when given different data sets. However, since the splitting operation in Step 6 is deterministic it has no effect on the search space.
- ❑ The runtime of the ID3 algorithm is in $O(p^2 \cdot n)$, i.e., significantly below $p!$ since only a small part of the search space is explored. At each split, the algorithm greedily (in fact, irrevocably) selects the most informative feature by applying information gain as a heuristic for feature selection.

Decision Tree Algorithms

ID3 Algorithm: Inductive Bias

Inductive bias is the rigidity in applying the (little bit of) knowledge learned from a training set for the classification of unseen feature vectors.

Observations:

- ❑ Decision tree search happens in the space of all hypotheses.
- ❑ To generate a decision tree, the ID3 algorithm needs per branch at most as many decisions as features are given.

Decision Tree Algorithms

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 - The target concept is a member of the hypothesis space.
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 - no backtracking takes place
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Decision Tree Algorithms

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Where the inductive bias of the ID3 algorithm becomes manifest:

1. Small decision trees are preferred.
2. Highly discriminative features tend to be closer to the root.

Is this justified?

Remarks (inductive bias) :

- ❑ The inductive bias of the ID3 algorithm is of a different kind than the inductive bias of the candidate elimination algorithm (or version space algorithm):
 1. The underlying hypothesis space H of the candidate elimination algorithm is incomplete. H corresponds to a coarsened view onto the space of all hypotheses since H contains only conjunctions of feature-value pairs as hypotheses.
However, this restricted hypothesis space is searched completely by the candidate elimination algorithm. Keyword: restriction bias
 2. The underlying hypothesis space H of the ID3 algorithm is complete since it contains all decision trees that can be constructed over D .
However, this complete hypothesis space is searched incompletely, but following a preference. Keyword: preference bias or search bias
- ❑ The inductive bias of the ID3 algorithm renders the algorithm robust wrt. noise.

Decision Tree Algorithms

CART Algorithm [Breiman 1984] [ID3 Algorithm]

Setting:

- X is a multiset of feature vectors. No restrictions are presumed for the features' measurement scales.
- C is a set of classes.
- $D = \{(\mathbf{x}_1, c_1), \dots, (\mathbf{x}_n, c_n)\} \subseteq X \times C$ is a multiset of examples.

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Decision Tree Algorithms

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Characteristics of the CART algorithm:

1. Each splitting is binary and considers one feature at a time.
2. Splitting criterion is the information gain or the Gini index.

Decision Tree Algorithms

CART Algorithm (continued)

Let A be a feature with domain $dom(A)$. Apply (probably multiple times) the respective rule to induce a finite number of binary splittings of X :

1. If A is nominal, choose $B \subset dom(A)$ such that $0 < |B| \leq |dom(A) \setminus B|$.
2. If A is ordinal, choose $a \in dom(A)$ such that $x_{\min} < a < x_{\max}$, where x_{\min} , x_{\max} are the minimum and maximum values of feature A in D .
3. If A is numeric, choose $a \in dom(A)$ such that $a = 0.5 \cdot (x_{l_1} + x_{l_2})$, where x_{l_1} , x_{l_2} are consecutive elements in the ordered value list of feature A in D .

Decision Tree Algorithms

CART Algorithm (continued)

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Adapt Step 5+6 to turn the ID3 into the CART algorithm:

- For all $A \in Features$ generate with the above rules all splittings of $D(t)$.
- Choose a splitting that maximizes the impurity reduction $\Delta \iota$:

$$\underline{\Delta \iota}(D(t), \{D(t_L), D(t_R)\}) = \iota(D(t)) - \frac{|D(t_L)|}{|D|} \cdot \iota(D(t_L)) - \frac{|D(t_R)|}{|D|} \cdot \iota(D(t_R)).$$

- Recursively call CART to process $D(t_L)$ and $D(t_R)$.

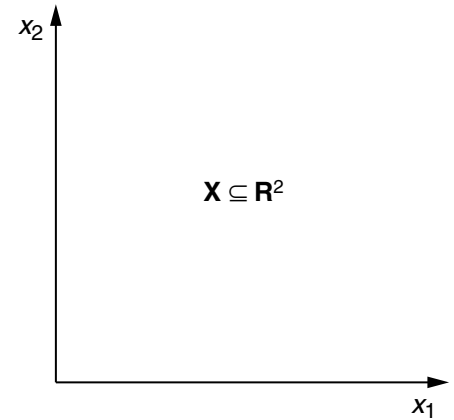
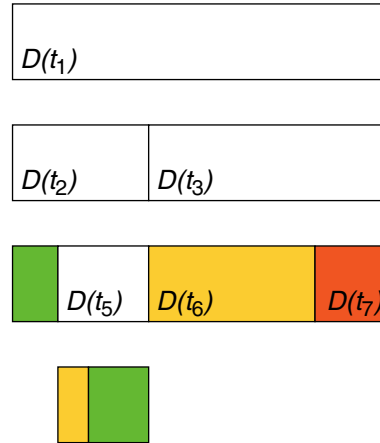
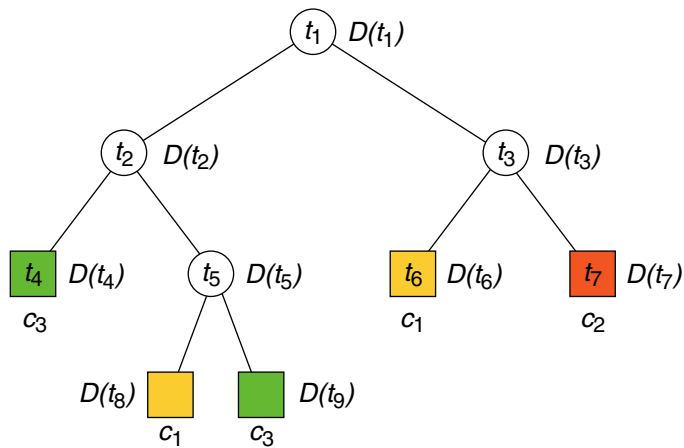
Remarks:

- ❑ t_L and t_R denote the left and right successor of t in the decision tree. These nodes are returned by the calls of the CART algorithm and connected to t via *createEdge()*.
- ❑ Since the CART algorithm creates binary splittings only, the feature A^* chosen in [Step 5](#) can be chosen again later on. Hence, a call of CART to process $D(t_L)$ (or $D(t_R)$) in [Step 6](#) passes the complete set of features as second parameter (and not: $Features \setminus \{A^*\}$).

Decision Tree Algorithms

CART Algorithm (continued)

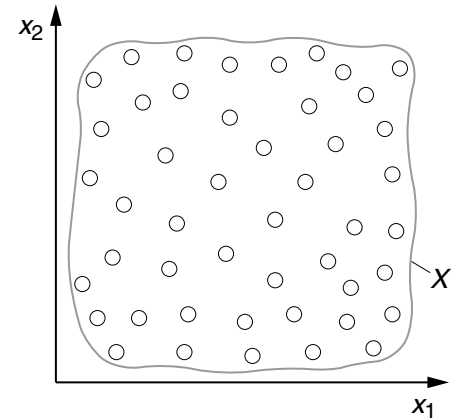
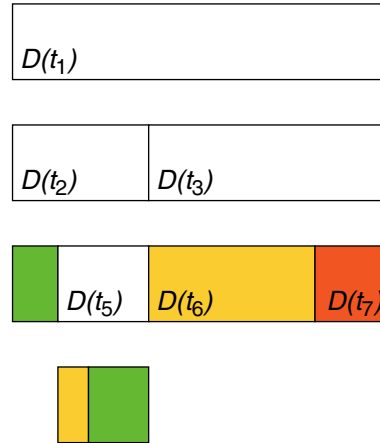
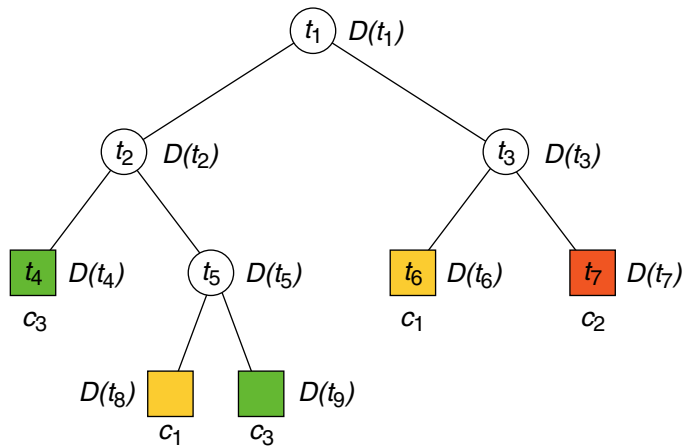
Illustration for two numeric features; i.e., the feature space X underlying X corresponds to a two-dimensional plane such as the \mathbb{R}^2 :



Decision Tree Algorithms

CART Algorithm (continued)

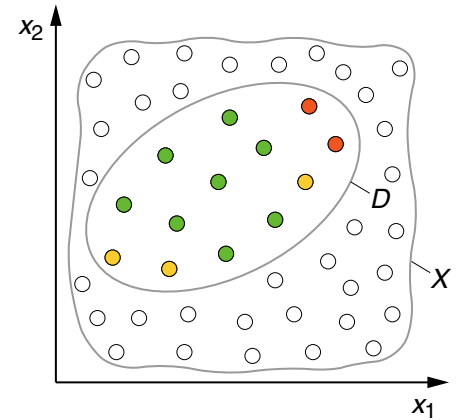
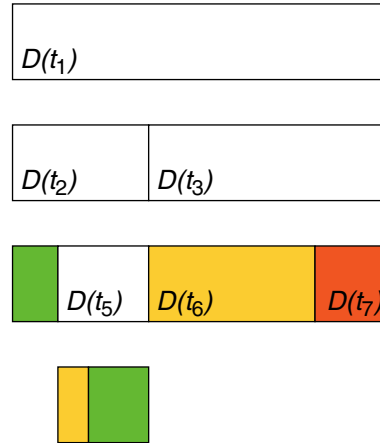
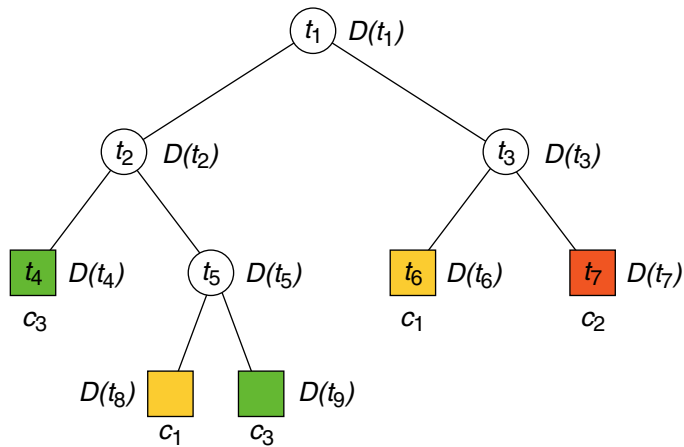
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Decision Tree Algorithms

CART Algorithm (continued)

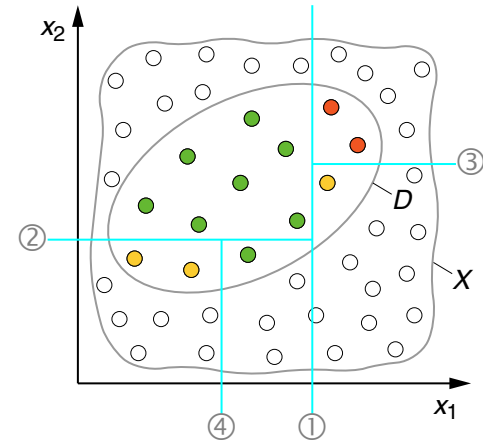
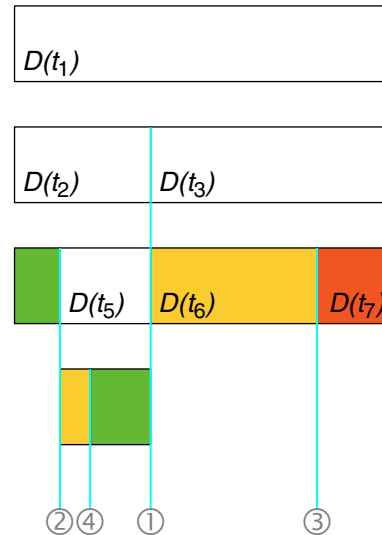
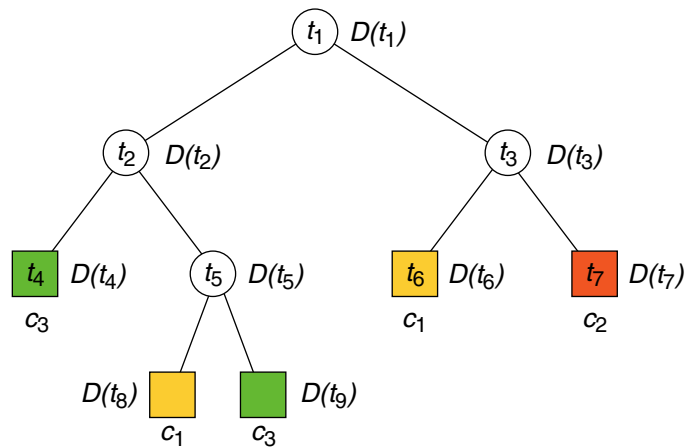
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Decision Tree Algorithms

CART Algorithm (continued)

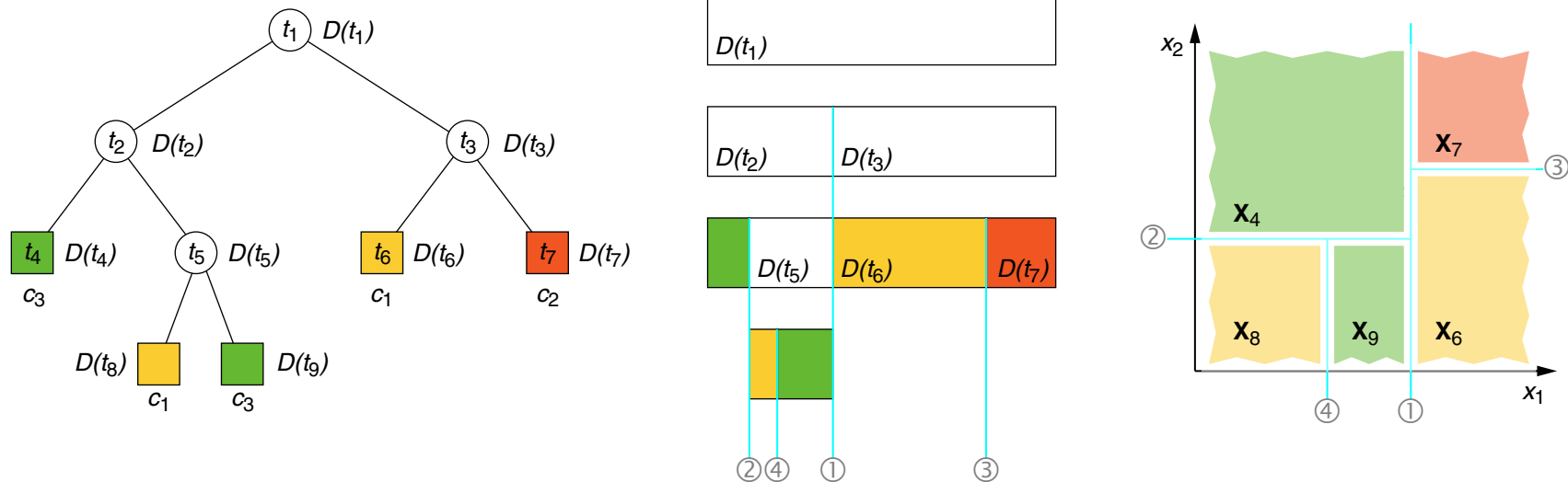
Illustration for two numeric features; i.e., the feature space X underlying X corresponds to a two-dimensional plane such as the \mathbb{R}^2 :



Decision Tree Algorithms

CART Algorithm (continued)

Illustration for two numeric features; i.e., the feature space \mathbf{X} underlying \mathbf{X} corresponds to a two-dimensional plane such as the \mathbb{R}^2 :



By the sequence of (here: four) splittings of D the feature space \mathbf{X} is cut into rectangular areas that are parallel to the two axes. Keyword: guillotine cuts