Chapter S:VII

VII. Game Playing

- Game Playing Introduction
- Evaluation Functions for Game Trees
- Propagation Algorithms for Game Trees

Game Playing Introduction

The game tree search here focuses on two-player, perfect-information games.

- Examples: chess, checkers, Go
- The rules of the game define legal moves; there is no room for chance.
- There is definite initial state s.
- Three different goal states are distinguished:
  1. win (W)
  2. loss (L)
  3. draw (D)

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  1. win (W)
  2. loss (L)
  3. draw (D)

A game tree is a representation of all possible plays of a game:

- The root node represents the initial state s.
- Leaf nodes represent goal states.
- Each path from the root s to a leaf nodes represents a complete game.

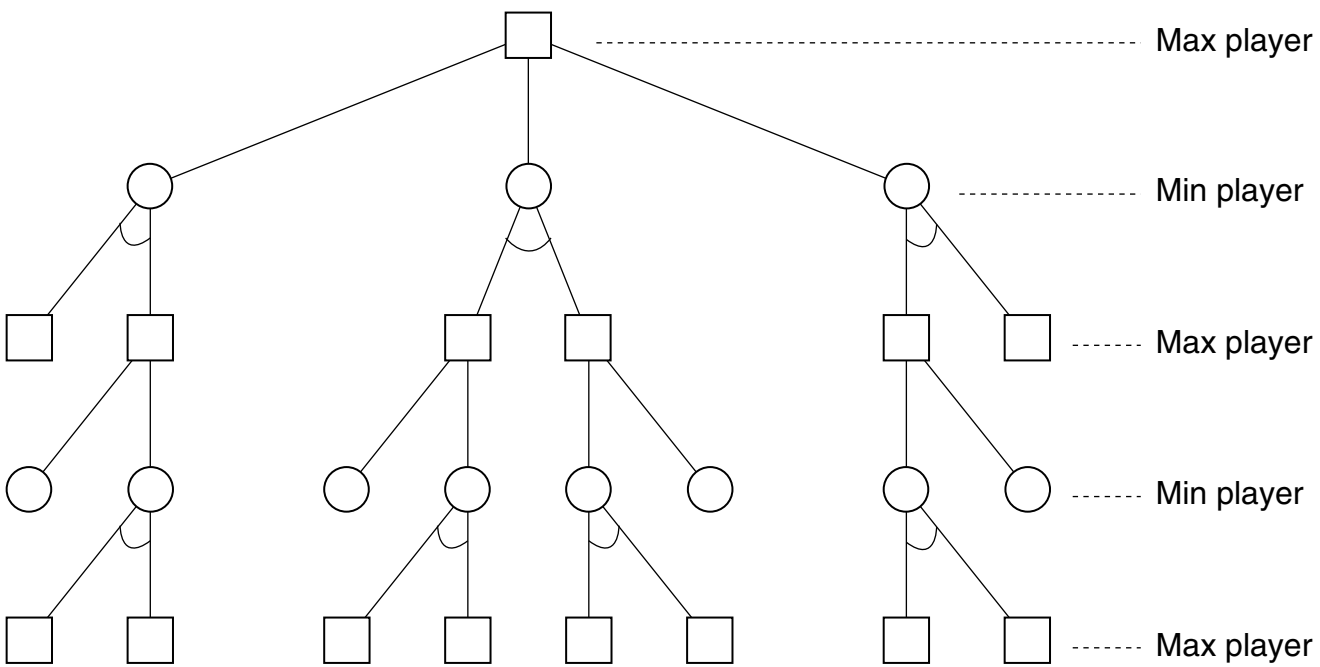
Game Playing Introduction

Definition 80 (Game Tree)

A game tree is a tree whose nodes (both inner nodes and leaf nodes) are either of type “Max” or “Min”. In particular holds:

1. All nodes at the same level of a game tree are of the same type.
2. Max nodes and Min nodes alternate between any two consecutive levels.

Illustration:



Remarks:

- Game trees are AND-OR trees.
- In general, the OR nodes of player 2 are the AND nodes of player 1—and vice versa.
- Player 1 is called “Max player” or Max, and player 1 is called “Min player” or Min.
- Without loss of generality we agree on the following: the interpretation of game trees is always done from the viewpoint of the Max player. Hence, the nodes of the Min player are considered as AND nodes, and all labelings reflect Max’s view.
- Graphical notation and summary:

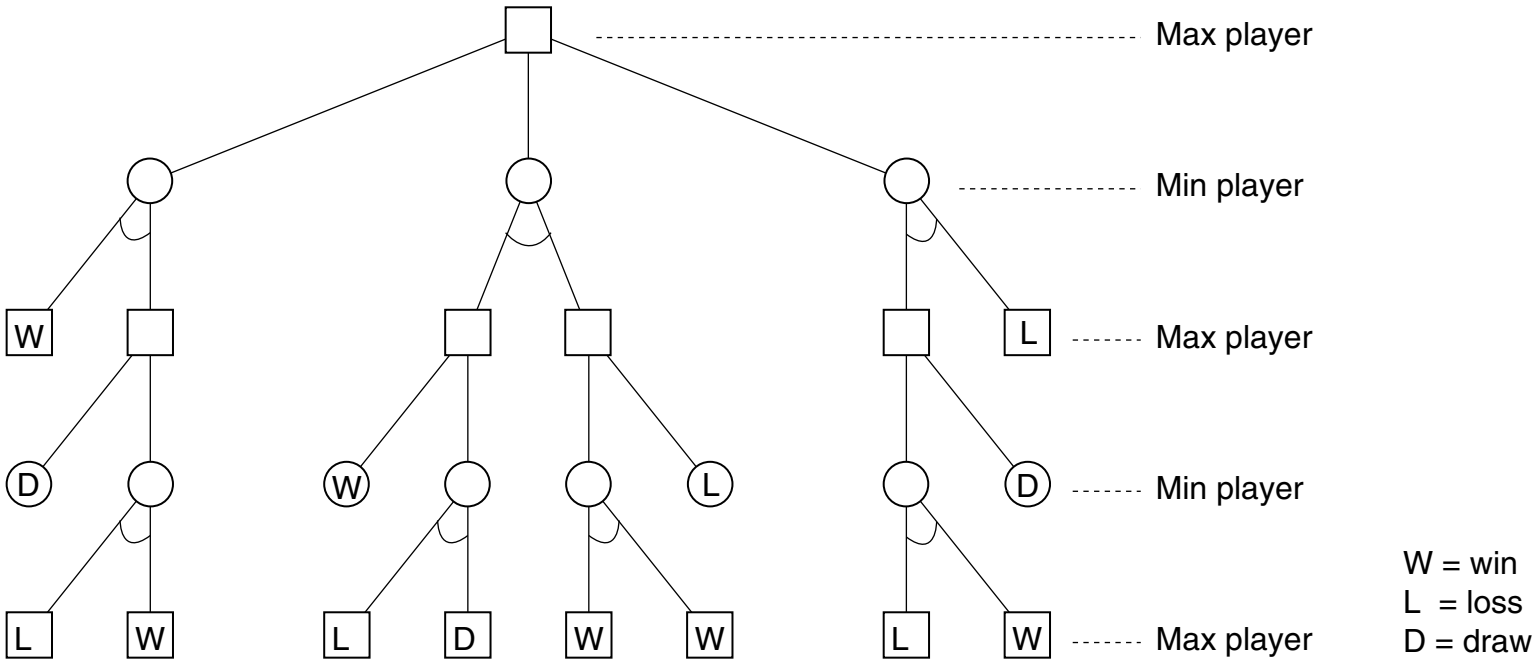
  □ = player 1 = Max player = own view = Max node = OR node

  ◯ = player 2 = Min player = adversarial view = Min node = AND node

- Finally, we assume that both players play optimum.

Game Playing Introduction

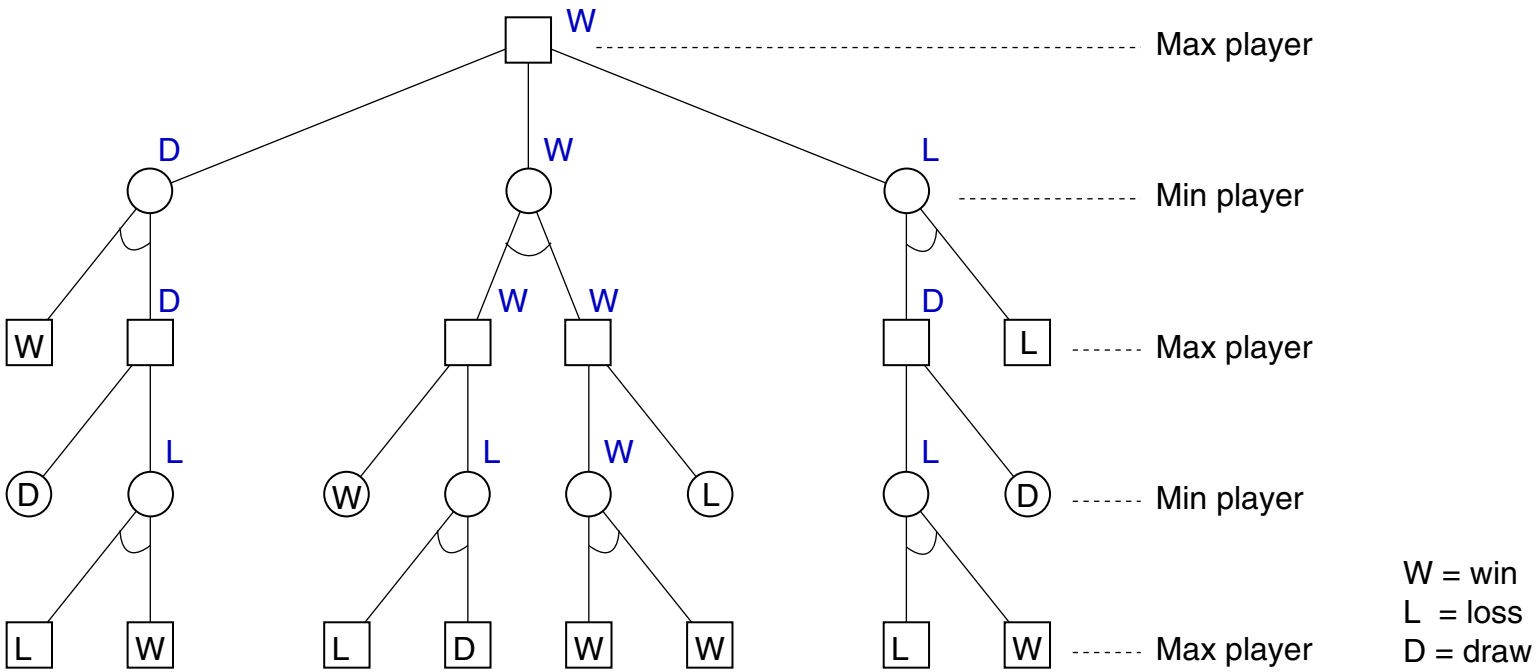
Illustration (continued):



Question: Is there a strategy such that Max will (start and) inevitably win?

Game Playing Introduction

Illustration (continued):



Question: Is there a strategy such that Max will (start and) inevitably win?

Game Playing Introduction

Definition 81 (Status Labeling Procedure)

Let T be a complete game tree whose leaf nodes are labeled with “win”, “loss”, and “draw” respectively. Then the label or *status* of an inner node $n \in T$ with *successors*(n) is defined as follows.

If n is of type “Max” then

$$\mathit{status}(n) = \begin{cases} \text{win} & \leftrightarrow \exists n' \in \mathit{successors}(n) : \mathit{status}(n') = \text{win} \\ \text{loss} & \leftrightarrow \forall n' \in \mathit{successors}(n) : \mathit{status}(n') = \text{loss} \\ \text{draw} & \leftrightarrow \exists n' \in \mathit{successors}(n) : \mathit{status}(n') = \text{draw} \wedge \\ & \quad\ \nexists n' \in \mathit{successors}(n) : \mathit{status}(n') = \text{win} \end{cases}$$

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If n is of type "Min" then

$$\mathit{status}(n) = \begin{cases} \text{win} & \leftrightarrow \forall n' \in \mathit{successors}(n) : \mathit{status}(n') = \text{win} \\ \text{loss} & \leftrightarrow \exists n' \in \mathit{successors}(n) : \mathit{status}(n') = \text{loss} \\ \text{draw} & \leftrightarrow \exists n' \in \mathit{successors}(n) : \mathit{status}(n') = \text{draw} \wedge \\ & \quad\ \nexists n' \in \mathit{successors}(n) : \mathit{status}(n') = \text{loss} \end{cases}$$

Remarks:

- Obviously node labeling in a game tree happens bottom-up, which usually implies that the game tree must be completely known.
- Compare the Definition “Status Labeling Procedure” to the Definition “Solved-Labeling Procedure”, which defines whether or not an AND-OR graph contains a solution graph.
  [S:II Search Space Representation]

Game Playing Introduction

Definition 82 (Game Strategy, Solution Tree, Winning Strategy)

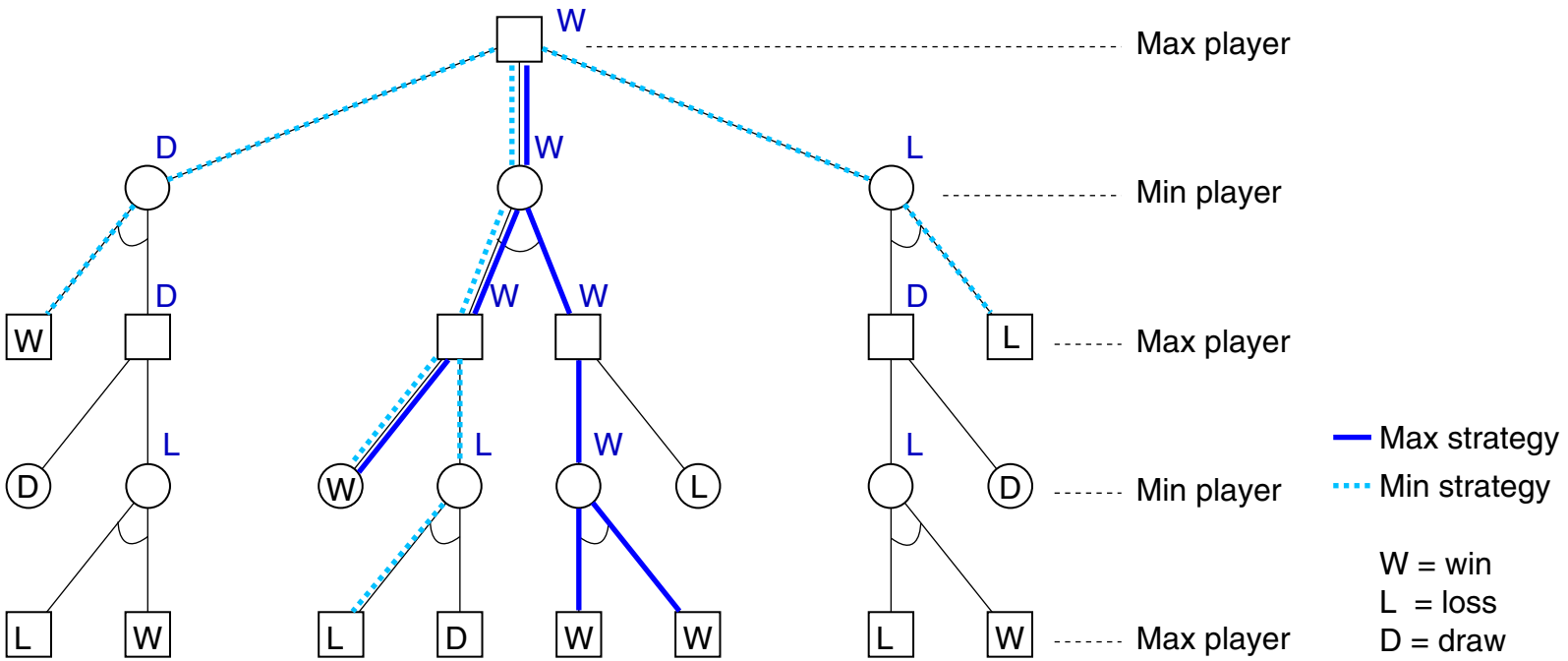
A game strategy for player Max is a subtree T^+ of T. T^+ has the root s, it contains for each inner Max node exactly one successor and for each inner Min node all successors.

A game strategy for player Min is a subtree T^- of T. T^- has the root s, it contains for each inner Min node exactly one successor and for each inner Max node all successors.

A winning strategy (for player Max) specifies how s is labeled with “win”, irrespective of the moves of player Min.

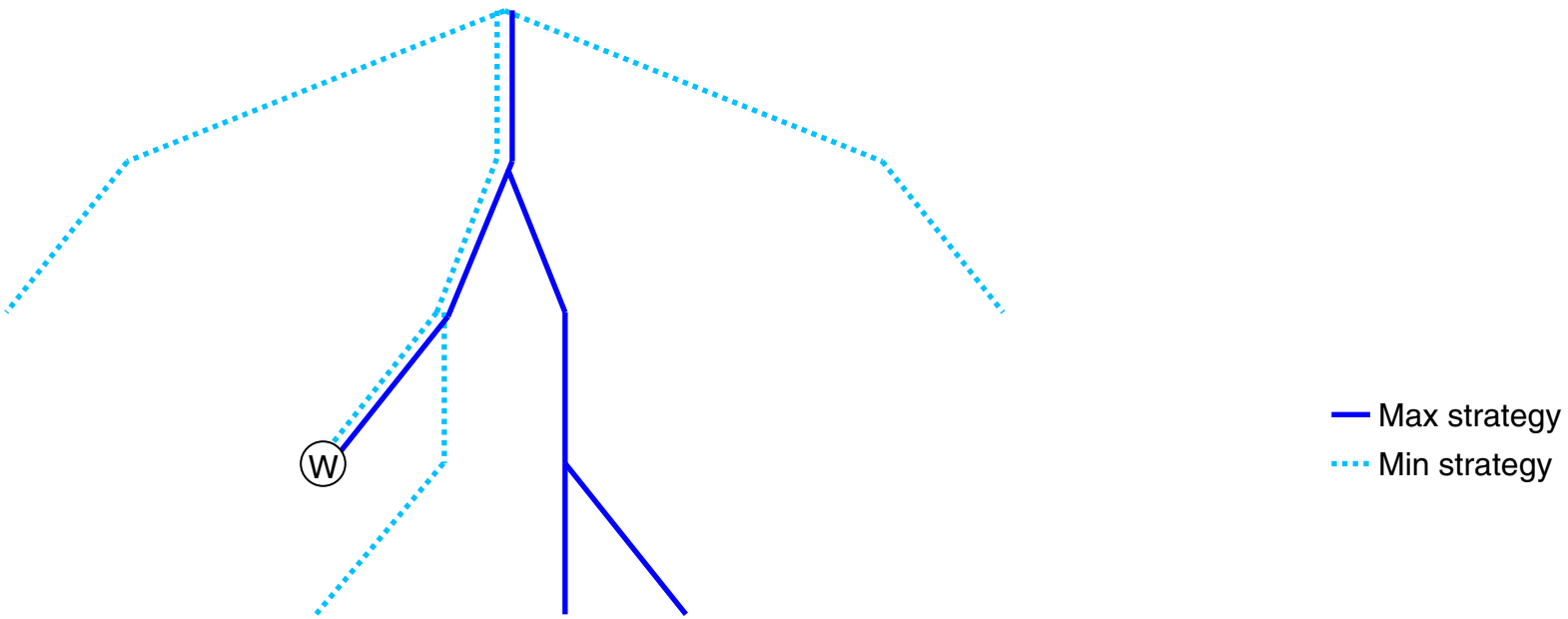
Game Playing Introduction

Illustration (continued):



Game Playing Introduction

Two strategies T^+, T^-, of Max and Min respectively, share at each level either one or no edge:



- The intersection of two strategies T^+, T^-, defines a path that corresponds to the play of the players Max and Min if both stick to their strategy.
- The intersection of two strategies T^+, T^-, defines the leaf $(T^+ \sqcap T^-)$, which corresponds to the final position of the game.

Game Playing Introduction

Strategy considerations for two-player, perfect-information games:

1. Let player Max choose a strategy T^+ and disclose it to player Min.
2. Min chooses T^- such that a leaf is reached that is as unfavorable as possible for Max. The label of this leaf computes as follows:

$$\min_{T^-} \mathit{status}(T^+ \sqcap T^-)$$

3. (With foresight) Max chooses T^+ such that the most unfavorable leaf is as good as possible for him. The label of this leaf computes as follows:

$$\max_{T^+} \min_{T^-} \mathit{status}(T^+ \sqcap T^-)$$

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By changing the roles we obtain: $\min_{T^-} \max_{T^+} \mathit{status}(T^+ \sqcap T^-)$

It holds (here without proof):

$$\min_{T^-} \max_{T^+} \mathit{status}(T^+ \sqcap T^-) = \underline{\mathit{status}(s)} = \max_{T^+} \min_{T^-} \mathit{status}(T^+ \sqcap T^-)$$

Remarks:

- The strategy considerations formalize the fact that both players play optimum.
- The following priority between status values is supposed: “loss” $<$ “draw” $<$ “win”
- The shown connections regarding *status*$(T^+ \sqcap T^-)$ can be proven inductively, considering a generic game tree, and starting with the leaf nodes.
- Observation: It is irrelevant whether a strategy is chosen up-front and disclosed, or whether decisions are made during the play.
- Other consequences: In order to proof whether a root node can be labeled with “win” and “loss” respectively, only one strategy T^+ or T^- is required. However, in order to proof whether a root node can be labeled with “draw”, two strategies T^+ and T^- are required.

Game Playing Introduction

The labeling of game trees is possible without distinguishing between Max and Min players. Instead, each node can be labeled from the viewpoint of that player who is currently moving.

Definition 83 (Mstatus Labeling Procedure)

Let T be a complete game tree whose leaf nodes are labeled with “win”, “loss”, and “draw” respectively. Then, under the mover-oriented viewpoint, the label or *mstatus*(n) of an inner node $n \in T$ with *successors*(n) is defined as follows.

$$\mathit{mstatus}(n) = \begin{cases} \text{win} & \leftrightarrow \exists n' \in \mathit{successors}(n) : \mathit{mstatus}(n') = \text{loss} \\ \text{loss} & \leftrightarrow \forall n' \in \mathit{successors}(n) : \mathit{mstatus}(n') = \text{win} \\ \text{draw} & \text{otherwise} \end{cases}$$

Remarks:

- By encoding the status “win” as 1, “loss” as -1, and “draw” as 0, the definition of *mstatus*(n) can be reformulated as follows:

$$\textit{mstatus}(n) = \max_{n' \in \textit{successors}(n)} \{-\textit{mstatus}(n')\}$$

- The mover-oriented labeling is also called Neg-Max labeling.

Evaluation Functions for Game Trees

Status labeling requires the generation of nearly the complete game tree. The following order of magnitudes illustrate the infeasibility of this prerequisite.

- A complete game tree for checker contains about 10^{40} inner nodes. If we processed 3 billion nodes per second, tree generation would last about 10^{21} centuries. [Samuel 1959]
- Chess: about 10^{120} inner nodes, generated within about 10^{101} centuries.
- Even if time were not the problem, storage space would be: There are about 10^{80} atoms in the observable universe.

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- Even if time were not the problem, storage space would be: There are about 10^{80} atoms in the observable universe.

➜ We need heuristics to evaluate game positions.

➜ Evaluation basis are features that characterize game positions.

➜ Distinguish between an immediate and a look-ahead evaluation of game positions.

Remarks:

- Immediate (also called “static”) evaluations of game positions are usually not used for a decision. Instead, for a certain search horizon (= search depth, bounded look-ahead) the possible game position are generated, and then the nodes at the search horizon are evaluated.
- The evaluations at the search horizon are considered as “true” values (recall: *face-value principle*). The values are propagated back up to that point where a decision is to be met, i.e., the starting point of the search.

  > *“The backed-up evaluations give a more accurate estimate of the true values of Max’s possible moves than would be obtained by applying the static evaluation function directly to those moves and not looking ahead to their consequences.”*
  >
  > [Barr/Feigenbaum 1981]

Evaluation Functions for Game Trees

Definition 84 (Minimax Rule)

Let T be an incomplete (= partially explored) game tree. The leafs of T form the current search horizon and can be evaluated with a function e. Then, the value $v(n)$ of a node $n \in T$ is defined as follows.

$$v(n) = \begin{cases} e(n) \in \mathbf{R} & n \text{ is leaf node} \\ \max_{n' \in \mathit{successors}(n)} \{v(n')\} & n \text{ is of type “Max”} \\ \min_{n' \in \mathit{successors}(n)} \{v(n')\} & n \text{ is of type “Min”} \end{cases}$$

Remarks:

- The minimax rule is a natural extension of the status labeling procedure for partially explored game trees.
- Most game playing algorithms are based on variants of the minimax rule.
- When operationalizing (= implementing) the minimax rule, the available computing resources are spent for two aspects:
  1. Generation of a portion of the game tree.
  2. Evaluation of the game positions at the leafs of the generated game tree portion.

  Consider the tradeoff between the quality (complexity) of an evaluation function e and the attainable search depth (amount of search effort) for look-ahead: Where to invest the available computing resources?
- For a fixed evaluation function e the search efforts is proportional to the number of generated leaf nodes. Hence, in game theory the leaf node number is the standard measure to asses the complexity of game playing algorithms.

Propagation Algorithms for Game Trees [MINIMAX, SOLVE, ALPHA-BETA]

Algorithm: MINIMAX-DFS

Input: n. A node in a game tree T.
successors(n). Returns the successors of node n.
$e(n) \in \mathbf{R}$. Evaluation function for a leaf node $n \in T$.

Output: The value $v(n)$ of the node n.

MINIMAX-DFS$(n, \textit{successors}, e)$

```
1. IF successors(n) = ∅
   THEN RETURN(e(n))
   ELSE
     FOREACH n' IN successors(n) DO
       v(n') = MINIMAX-DFS(n', successors, e);
     ENDDO
   ENDIF

2. IF nodeType(n) = 'Max'
   THEN RETURN(max{v(n') | n' ∈ successors(n)})
   ELSE RETURN(min{v(n') | n' ∈ successors(n)})
```

Remarks:

- A backtracking variant of algorithm MINIMAX-DFS, the algorithm MINIMAX-BT, would not generate all successors of a node at once, but generate and evaluate only one successor node at a time.
- The algorithm MINIMAX-DFS (as well as the algorithm MINIMAX-BT) generates more nodes then necessary. The algorithm SOLVE introduced below illustrates this fact. Note, however, that the algorithm SOLVE employs a two-valued evaluation function e.

Propagation Algorithms for Game Trees [MINIMAX, SOLVE, ALPHA-BETA]

Algorithm: SOLVE

Input: n. A node in a game tree T.
successors(n). Returns the successors of node n.
$e(n) \in$ {win, loss}. Evaluation function for a leaf node $n \in T$.

Output: The value $v(n)$ of the node n.

SOLVE$(n,$ *successors*$, e)$

```
1. IF successors(n) = ∅
   THEN RETURN(e(n))
   ELSE
     FOREACH n' IN successors(n) DO
       v(n') = SOLVE(n', successors, e);
       IF nodeType(n) = 'Max'
       THEN IF v(n') = 'win' THEN RETURN(win)
       ELSE IF v(n') = 'loss' THEN RETURN(loss)
     ENDDO
   ENDIF

2. IF nodeType(n) = 'Max'
   THEN RETURN(loss)
   ELSE RETURN(win)
```

Propagation Algorithms for Game Trees

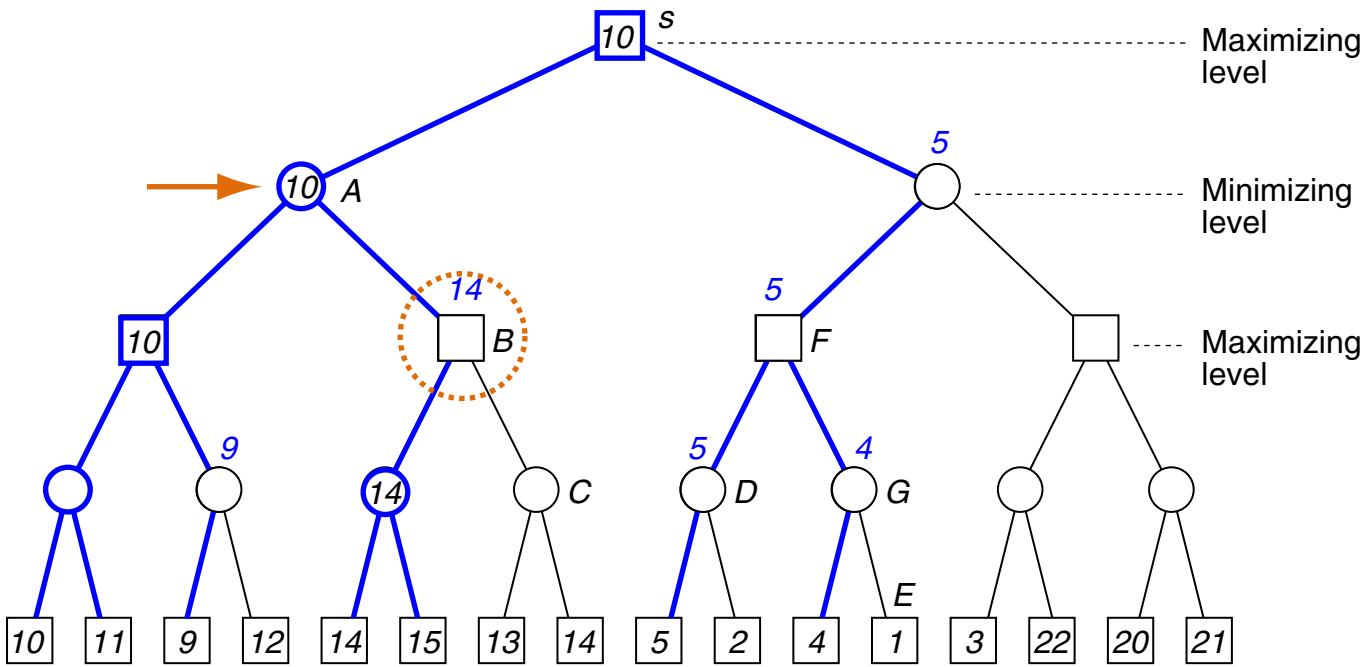
The pruning rationale used by the algorithm SOLVE is not restricted to two-valued evaluation functions but can be applied to multivalued (continuous) evaluation functions as well.

Overview of propagation algorithms:

	two-valued evaluation function	multivalued evaluation function
without pruning	MINIMAX	
with pruning	SOLVE	ALPHA-BETA

Propagation Algorithms for Game Trees

Game tree with propagated minimax values:

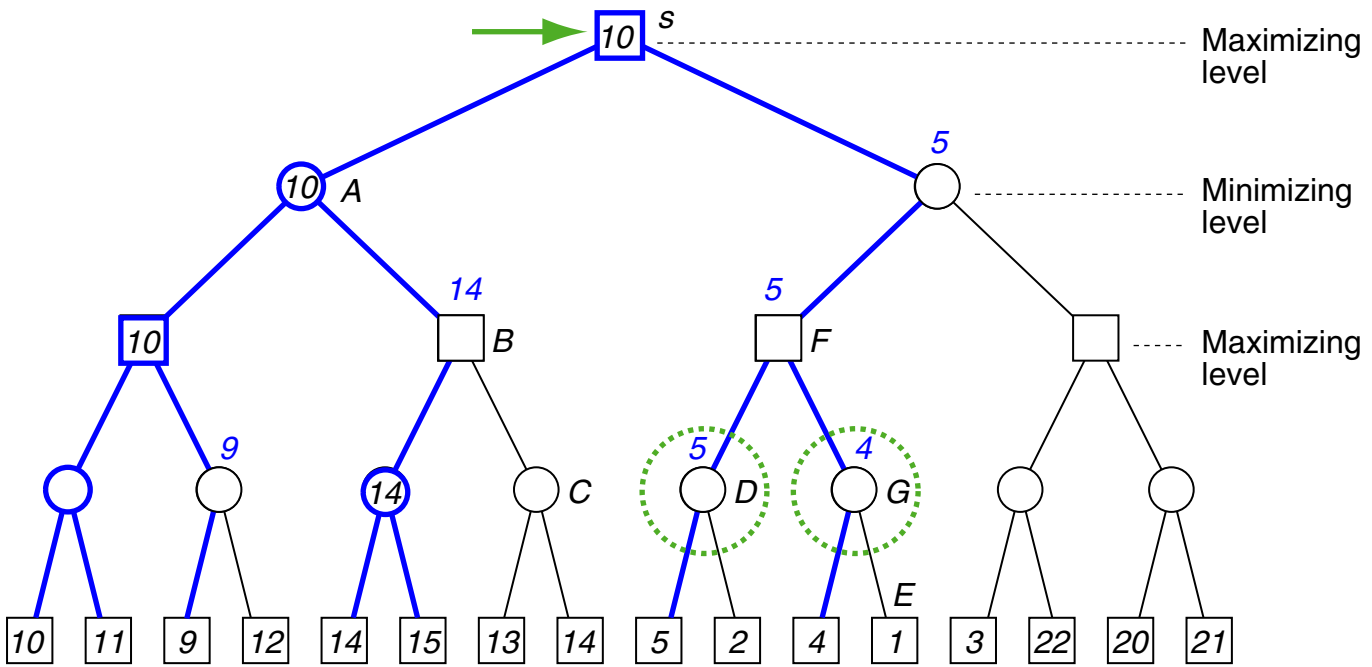


Argumentation:

1. The inspection of node B (its successors) yields information whether $v(A)$ must be decreased. What must hold for B to decrease $v(A)$?
   If we learn that this condition cannot be fulfilled anymore, the inspection of B can be aborted.

Propagation Algorithms for Game Trees

Game tree with propagated minimax values:



Argumentation:

2. The inspection of the nodes D or G (their successors) yield information whether $v(s)$ can be increased. What must hold for D or G to increase $v(s)$?
   If we learn that this condition cannot be fulfilled anymore, the inspection of D (G) can be aborted.

Remarks:

- The generalization of the previous argumentation will lead to the concept of α-bounds and β-bounds.
- For the node D, the value 10, which is here obtained from s, forms an α-bound.
- For the node B, the value 10, which is here obtained from A, forms a β-bound.
- Without loss of generality, the inspection of the nodes goes strictly from left to right.

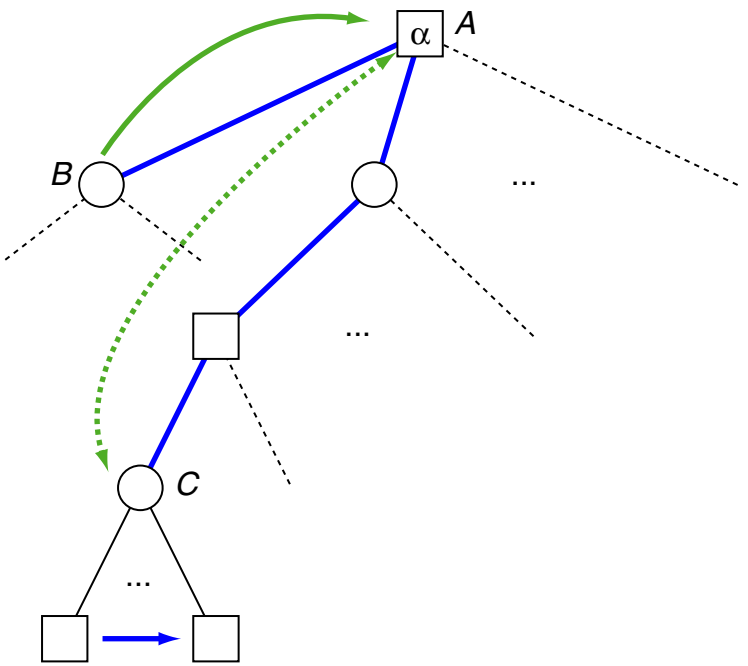
Propagation Algorithms for Game Trees

Definition 85 (α-Bound)

The α-bound is a lower bound and is used to prune (= to abort the inspection) of a Min node n. The value of α is defined as the currently maximum value of all predecessors of n that are of type “Max”.

The inspection of the Min node n can be aborted if $v(n) \leq \alpha$.

Illustration:



Remarks:

- The current α-bound can be found at the Max node A; it was propagated from the Min node B.
  Q. What must hold for the Min node C to increase $v(A) = \alpha$?
  A. C can increase α as long as $v(C) > \alpha$ holds.
- Observe that the successors of the Min node C can never increase $v(C)$. Hence, as soon as C got a value $v(C) \leq \alpha$, the investigation of the remaining successors of C can be aborted.
- At each point in time holds: The current value of α defines for the highest (closest to the root) Max node $\hat{n}$ with $v(\hat{n}) = \alpha$ a lower bound of the final value that would be computed for $\hat{n}$ by the algorithm MINIMAX-DFS.

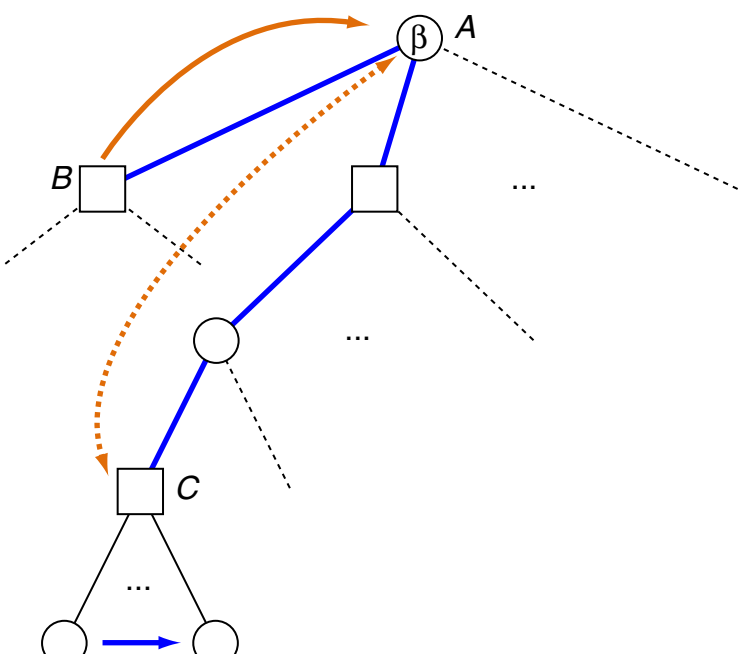
Propagation Algorithms for Game Trees

Definition 86 (β-Bound)

The β-bound is an upper bound and is used to prune (= to abort the inspection) of a Max node n. The value of β is defined as the currently minimum value of all predecessors of n that are of type “Min”.

The inspection of the Max node n can be aborted if $v(n) \geq \beta$.

Illustration:



Remarks:

- The current β-bound can be found at the Min node A; it was propagated from the Max node B.
  Q. What must hold for the Max node C to decrease $v(A) = \beta$?
  A. C can decrease β as long as $v(C) < \beta$ holds.
- Observe that the successors of the Max node C can never decrease $v(C)$. Hence, as soon as C got a value $v(C) \geq \beta$, the investigation of the remaining successors of C can be aborted.
- At each point in time holds: The current value of β defines for the highest (closest to the root) Min node $\hat{n}$ with $v(\hat{n}) = \beta$ an upper bound of the final value that would be computed for $\hat{n}$ by the algorithm MINIMAX-DFS.

Propagation Algorithms for Game Trees

The α-β-pruning scheme:

> *"Perform the backtracking version of minimax search with one exception; if in the course of updating the minimax value of a given node n crosses a certain bound, then no further exploration is needed beneath that node; its current-value $v(n)$ can be transmitted to its father as if all of its sons have been evaluated."*

[Pearl 1981, p. 233]

Remarks:

- The effectiveness of α-β pruning depends on the evaluation order of the terminal nodes.
- Given a fixed evaluation effort, it can be shown that—if the terminal nodes (terminal values) are randomly ordered—the attainable search depth is extended by 33% with α-β pruning.
  [Pearl 1981]

Propagation Algorithms for Game Trees

Algorithm: ALPHA-BETA

Input: n. A node in a game tree T.
successors(n). Returns the successors of node n.
$e(n) \in \mathbf{R}$. Evaluation function for a node $n \in T$.
α, β. Two numbers in $\mathbf{R}$ with $\alpha < \beta$. Initially, $\alpha = -\infty$, $\beta = +\infty$.

Output: The minimax value $v(n)$ of n if $v(n) \in (\alpha, \beta)$, or
α if $v(n) \leq \alpha$, or
β if $v(n) \geq \beta$.

Propagation Algorithms for Game Trees [MINIMAX, SOLVE, ALPHA-BETA]

```
ALPHA-BETA(n, successors, e, α, β)

1.  IF successors(n) = ∅
    THEN RETURN(e(n))
    ELSE
      IF nodeType(n) = 'max'

2.    THEN
        FOREACH n' IN successors(n) DO
          α = max(α, ALPHA-BETA(n', successors, e, α, β));
          IF α ≥ β THEN RETURN(β);
        ENDDO
        RETURN(α);

3.    ELSE
        FOREACH n' IN successors(n) DO
          β = min(β, ALPHA-BETA(n', successors, e, α, β));
          IF β ≤ α THEN RETURN(α);
        ENDDO
        RETURN(β);

      ENDIF
    ENDIF
```