Chapter S:III

III. Informed Search

- Best-First Search
- Best-First Search for State-Space Graphs
- Cost Functions for State-Space Graphs
- Evaluation of State-Space Graphs
- Algorithm A*
- BF* Variants
- Hybrid Strategies

Cost Functions

Overview

BF defines a schema for the design of search strategies for state-space graphs. Up to this point, the evaluation functions f remained unspecified.

Questions:

- How to compute f?
- How to evaluate a solution path?
- How to evaluate a search space graph?
- How to identify a most promising solution base?

Answering these question gives rise to a taxonomy of Best-First algorithms.

Cost Functions

Overview (continued)

The answers are developed in several steps by the following concepts:

1. Recursive cost functions (for paths)
2. Solution cost (for a given solution path)
3. Optimum solution cost (for a complete search space graph)
4. Estimated solution cost (for a given solution base)
5. Estimated optimum solution cost (for a partial search space graph)

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Names of the respective cost functions:

		Solution	
		given	optimum searched
Exploration	complete	C_P	C^*
	partial	$\widehat{C}_P$	$\widehat{C}$

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Names of the respective cost functions:

		Solution	
		given	optimum searched
Exploration	complete	$C_P(s)$	$C^*(s)$
	partial	$\widehat{C}_P(s)$	$\widehat{C}(s) \rightsquigarrow n$

n represents a most promising solution base.

Cost Functions

If solution graphs are known, the solution cost for a solution graph can be determined.

Definition 1 (Cost Function C_P)

For an OR-graph G and let M be an ordered set. A function C_P, which assigns each solution path P in G and each node n in P a cost value $C_P(n)$ in M is denoted by C_P.

Usage and notation of C_P :

- No provisions are made how to compute $C_P(n)$ for a solution path P. $C_P(s)$ specifies the cost of a solution path P for s :

$$f(\gamma) = C_P(s) \text{ with } P \text{ backpointer path of } \gamma.$$

Remarks:

- As ordered set M usually $\mathbf{R} \cup \{\infty\}$ is chosen.
- $C_P(n)$ should be seen as binary function with arguments P and n.
- The cost value $C_P(n)$ is meaningful only if n is a node in P.
- Solution cost does not measure efforts for finding a solution. Solution cost aggregate properties of operations (and decompositions) in a solution graph to form a cost value.
- Instead of cost functions we may employ merit functions or, even more general, weight functions. The respective notations are Q_P for merits, and W_P for weights.
  E.g., hill-climbing algorithms often employ merit functions.
- A cost function can be a complex accounting rule, considering the properties of a solution path:
  1. node costs, such as the processing effort of a manufacturing machine,
  2. edge costs, such as the cost for transportation or transmission, and
  3. terminal payoffs, which specify a lump value for the remaining solution effort at leaf nodes.
- At places where the semantics was intuitively clear, we have already used the notation $C_P(n)$ to denote the solution cost of a problem associated with node n. Definition 1 catches up for the missing notation and semantics.

Cost Functions

If the entire search space graph rooted at a node s is known, the optimum solution cost for the root node s can be determined.

Definition 2 (Optimum Solution Cost C^*, Optimum Solution)

Let G be an OR-graph with root node s and let $C_P(n)$ denote a cost function for G.

The optimum solution cost for a node n in G, $C^*(n)$, is defined as

$$C^*(n) = \inf\{C_P(n) \mid P \text{ is solution path for } n \text{ in } G\}$$

A solution path with solution cost $C^*(n)$ is called optimum solution path for n. The optimum solution cost for s, $C^*(s)$, is abbreviated as C^*.

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Remarks:

- If G contains no solution path for n, let $C^*(n) = \infty$.

Cost Functions

If the entire search space graph rooted at a node s is known, the optimum solution cost extending a solution base for s can be determined.

Definition 3 (Optimum Solution Cost C_P^* for a Solution Base)

Let G be an explored subgraph of an OR graph $\mathcal{G}$ with root node s and cost function $C_P(n)$ for $\mathcal{G}$.

The optimum solution cost for node n in G based on a solution base P, $C_P^*(n)$, is defined as

$$C_P^*(n) = \inf\{C_{P'}(n) \mid P' \text{ is solution path in } \mathcal{G} \text{ extending } P\}$$

Cost Functions

If the search space graph rooted at a node s is known partially, the optimum solution cost extending a solution base for s can be estimated.

Definition 4 (Estimated Optimum Solution Cost $\widehat{C}_P$ for a Solution Base)

Let G be an explored subgraph of an OR graph $\mathcal{G}$ with root node s and cost function $C_P(n)$ for $\mathcal{G}$.

The estimated optimum solution cost for a node n in G based on a solution base P in G, $\widehat{C}_P(n)$, returns an estimate of $C_P^*(n)$.

$\widehat{C}_P(n)$ is optimistic, if and only if $\widehat{C}_P(n) \leq C_P^*(n)$.

Usage of $\widehat{C}_{P'}$:

- In BF we use $f(n) = \widehat{C}_P(s)$ with P backpointer path of n.
- In order to emphasize the dualism of estimated and real values, we often write $f^*(n)$ instead of $C_P^*(n)$, i.e., $f(n)$ is an estimate of $f^*(n)$, the optimum solution path cost for s when extending P.

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Remarks:

- Estimated optimum solution cost values are computed on basis of the explored subgraph G of the underlying search space graph $\mathcal{G}$. So, values may change over time with G.
- In the setting of Definition 3 we assume

  $$C^*(s) = \inf\{C_P^*(s) \mid P \text{ is solution base in } G\}$$

  Therefore, it is essential for search algorithms to keep available solution bases that are important for this result. Optimistically estimating $C_P^*(n)$ in BF will direct the search into promising directions.

Cost Functions

If the search space graph rooted at a node s is known partially, the optimum solution cost for s can be estimated.

Definition 5 (Estimated Optimum Solution Cost $\widehat{C}$ [Overview]**)**

Let G be an explored subgraph of an OR graph $\mathcal{G}$ with root node s and cost function $C_P(n)$ for $\mathcal{G}$.

The estimated optimum solution cost for a node n in G, $\widehat{C}(n)$, is defined as follows:

$$\widehat{C}(n) = \inf\{\widehat{C}_P(n) \mid P \text{ is solution base in } G\}$$

A solution base P for s with $\widehat{C}_P(s) = \widehat{C}(s)$ is called most promising solution base (for s).

Cost Functions

If the search space graph rooted at a node s is known partially, the optimum solution cost for s can be estimated.

Definition 6 (Estimated Optimum Solution Cost $\widehat{C}$ [Overview]**)**

Let G be an explored subgraph of an OR graph $\mathcal{G}$ with root node s and cost function $C_P(n)$ for $\mathcal{G}$.

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A solution base P for s with $\widehat{C}_P(s) = \widehat{C}(s)$ is called most promising solution base (for s).

Cost Functions

Cost Concept in Uniform-Cost Search

- Edge weight.
  Encode either cost values or merit values, which are accounted if the respective edges become part of the solution.
  $c(n, n')$ denotes the cost value of an edge from n to n'.

- Path cost.
  The cost of a path, C_P, results from applying a *cost measure* F, which specifies how cost of a continuing edge is combined with the cost of the rest of the path.

  Examples:
  Sum cost $:=$ the sum of all edge costs of a path P from s to n:

$$C_P(s) = \sum_{i=0}^{k-1} c(n_i, n_{i+1}), \text{ with } n_0 = s \text{ and } n_k = n$$

  Maximum cost $:=$ the maximum of all edge costs of a path:

$$C_P(s) = \max_{i \in \{0,\ldots,k-1\}} c(n_i, n_{i+1}), \text{ with } n_0 = s \text{ and } n_k = n$$

- Estimated optimum solution cost.
  The cost value for the solution base is taken as the estimate of optimum solution cost.

$$\widehat{C}_P(n) := C_P(n)$$

Cost Functions

Recursive Cost Functions

The computation of the evaluation functions f would be nearly impracticable if the cost of paths were based on complex global properties of the path and had to be computed from scratch for each additionally explored node.

Definition 7 (Recursive Cost Function, Cost Measure)

A cost function C_P for a solution path P is called recursive, if for each node n in P holds:

$$C_P(n) = \begin{cases} F[E(n)] & n \text{ is leaf in } P \text{ (and, hence, } n \text{ is goal node)} \\ F[E(n), C_P(n')] & n \text{ is inner node in } P \text{ and } n' \text{ direct successor of } n \text{ in } P \end{cases}$$

- n' denotes the direct successor of n in P,
- $E(n) \in \mathbf{E}$ denotes a set of *local* properties of n with respect to P,
- F is a function that prescribes how local properties of n are accounted (better: combined) with properties of the direct successor of n:

$$F : \mathbf{E} \times M \to M, \quad \text{where } M \text{ is an ordered set.}$$

F is called cost measure.

Remarks:

- If for a solution base its merits, quality, or other positive aspects are measured, F is called merit measure.

Cost Functions

Recursive Cost Functions (continued)

If the search space graph rooted at a node s is known partially and a recursive cost function is used, cost estimates for a solution base

1. can be built upon estimates for optimum solution cost of non-goal leaf nodes in this solution base and,
2. can be computed by taking the estimations of h for granted and propagating the cost values bottom-up. Keyword: *Face-Value Principle*

Definition 8 (Heuristic Function h)

Let G be an OR graph. A function h, which assigns each node n in G an estimate $h(n)$ of the optimum solution cost value $C^*(n)$, the optimum cost of a solution path for n, is called heuristic function (for G).

In order to emphasize the dualism of estimated and real values, we often write $h^*(n)$ instead of $C^*(n)$, i.e., $h(n)$ is an estimate of $h^*(n)$.

Remarks:

- If algorithm BF were equipped with a dead end recognition function $\perp(n)$, no unsolvable node would be stored. A dead end recognition could also be incorporated in h in such a way that h returns ∞ for unsolvable nodes.

Corollary 9 (Estimated Solution Cost $\widehat{C}_P$ for a Solution Base)

Let G, $\mathcal{G}$, s, and $C_P(n)$ be defined as for $\widehat{C}_P$. Let the cost function be recursive based on F and E, and let h be a heuristic function.

Using the face-value principle, the estimated solution cost for solution bases P in G is computed as follows:

$$\widehat{C}_P(n) = \begin{cases} c(n) & n \text{ is leaf in } P \text{ and } P \text{ is solution path} \\ h(n) & n \text{ is leaf in } P \text{ but } P \text{ is no solution path} \\ F[E(n), \widehat{C}_P(n')] & n \text{ is inner node in } P \text{ and } n' \text{ direct successor of } n \text{ in } P \end{cases}$$

Evaluation of State-Space Graphs

Recursive Cost Functions and Efficiency

If the search space graph is an OR graph rooted at a node s and is known partially and a recursive cost function is used that is defined via a

1. monotone cost measure F, i.e., for e, e', c, c' with $e \leq e', c \leq c'$ we have

$$F[e, c] \leq F[e', c']$$

the (estimated) optimum solution cost can be computed bottom-up.

If additionally the recursive cost function is based on an

2. underestimating heuristic function h, i.e., $h(n) \leq C^*(n)$

then the estimated solution cost $\widehat{C}_P(s)$ is underestimating optimum solution cost $C^*_P(s)$ for a solution base P.

Corollary 10 (Optimum Solution Cost C^* [GBF, Overview])

Let G be an OR graph rooted at s. Let $C_P(n)$ be a recursive cost function for G based on E and a monotone cost measure F.

The optimum solution cost $C^*(n)$ for a node n in G can be computed as follows:

$$C^*(n) = \begin{cases} c(n) & n \text{ is goal node and leaf in } G \\ \infty & n \text{ is unsolvable leaf node in } G \\ \min_i\{F[E(n), C^*(n_i)]\} & n \text{ is inner OR node in } G, \\ & n_i \text{ direct successors of } n \text{ in } G \end{cases}$$

Compare to Bellman's equations.

Corollary 11 (Estimated Optimum Solution Cost $\widehat{C}$ [GBF, Overview])

Let G be an explored subgraph of a state-space graph $\mathcal{G}$ rooted at s. Let h be a heuristic function and let $C_P(n)$ be a recursive cost function for $\mathcal{G}$ based on E and a monotone cost measure F.

Using the face-value principle, the estimated optimum solution cost $\widehat{C}(n)$ for a node n in G can be computed as follows:

$$\widehat{C}(n) = \begin{cases} c(n) & n \text{ is goal node and leaf in } G \\ h(n) & n \text{ is leaf in } G \text{ but no goal node} \\ \min_i\{F[E(n), \widehat{C}(n_i)]\} & n \text{ is inner OR node in } G \\ & n_i \text{ direct successors of } n \text{ in } G \end{cases}$$

Remarks:

- Computing $\widehat{C}(n)$ is based on an explored subgraph G of a state-space graph. That means, all solution bases in G are considered and not only those, maintained by BF in its traversal tree.

  BF algorithms will use only the solution bases in the traversal tree to compute $\widehat{C}(n)$.

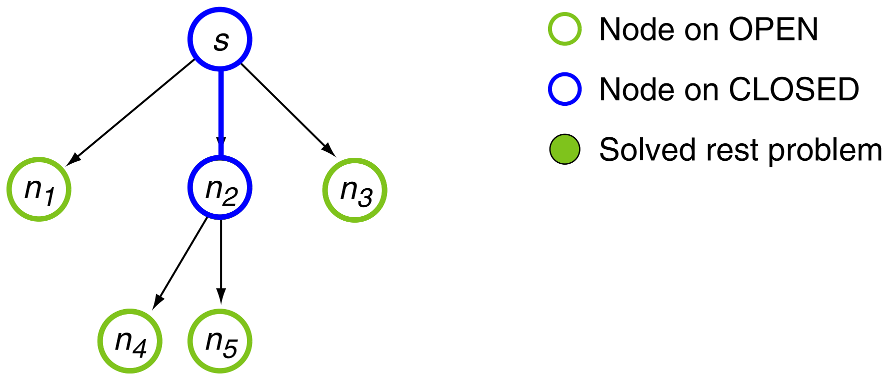
  Q. Will this lead to different results? A. Different results can be obtained only from discarded paths. For the computation of $\widehat{C}(n)$ an algorithm can use only the direct successor nodes of n that still have a backpointer reference to n, i.e., $\widehat{C}(n)$ may be to large. Since we are only interested in $\widehat{C}(s)$, the correct value will be computed if we are using an order-preserving cost function.

- $\widehat{C}(n)$ computes for a node n the minimum of the estimated costs among all solution bases rooted at n (paths from n to leaf node in G). In particular, $\widehat{C}(s)$ computes the estimated optimum solution cost for the entire problem, and it hence defines a most promising solution base.

- $\widehat{C}_P$, the estimated solution cost for a solution base P is a recursive cost function. Hence, $\widehat{C}_{P_{s-n}}(s)$ can be computed bottom-up, from n to s along path P_{s-n}.

Evaluation of State-Space Graphs

Illustration of $\widehat{C}(n)$, $\widehat{C}_P(n)$



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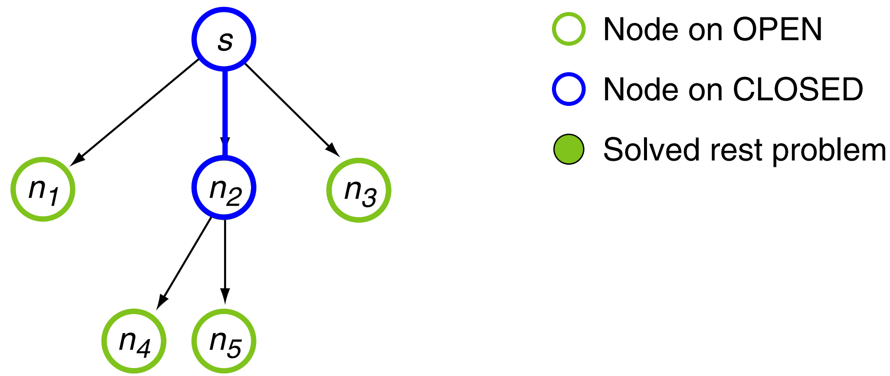
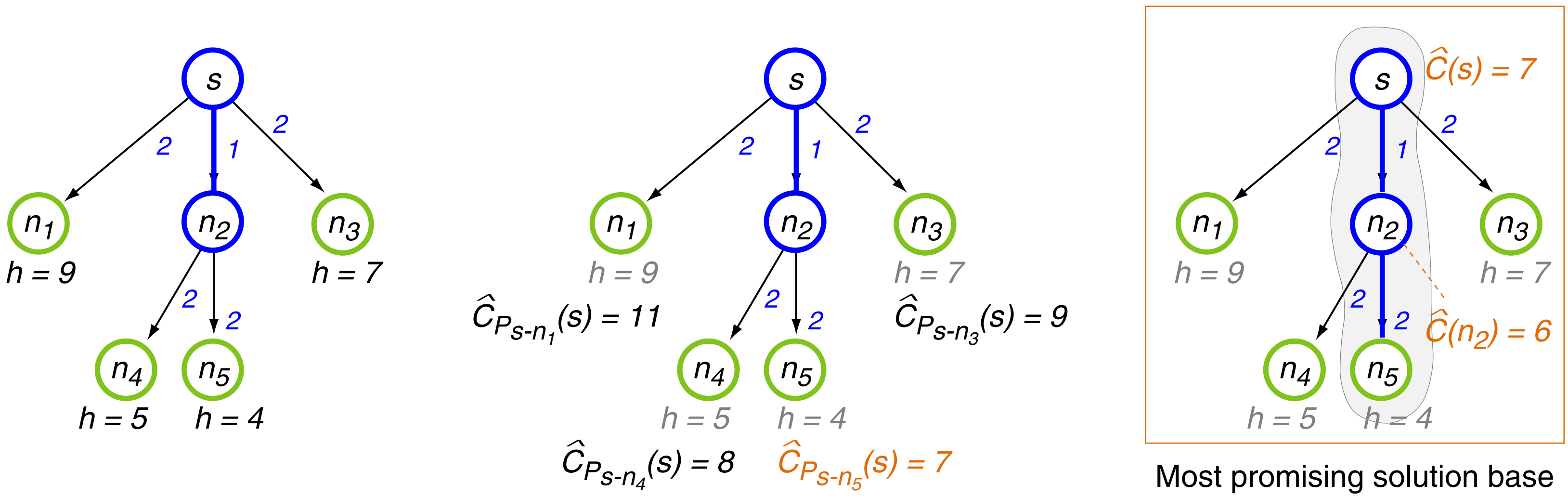


Computation of $\widehat{C}_{P_{s-n}}(s)$ for each node n on OPEN:



Most promising solution base

Evaluation of State-Space Graphs

Additive Cost Measures

To compute $\widehat{C}_{P_{s-n}}(s)$, a bottom-up propagation from n to s may not be necessary. Dependent on the cost measure F, it can be sufficient to pass a single (several) parameter(s) *top-down*, from a node to its direct successors.

Illustration for F = "$+$" and a path $P_{s-n} = (s, n_1, \ldots, n_k, n)$ from s to n:

$$\begin{aligned}
\widehat{C}_{P_{s-n}}(s) &= F[E(s), \widehat{C}_{P_{s-n}}(n_1)] \\
&= F[E(s),\ F[E(n_1),\ F[E(n_2),\ \ldots,\ F[E(n_k), h(n)] \ldots\ ]\ ]\ ] \\
&= c(s, n_1) + c(n_1, n_2) + \ \ldots\ + c(n_k, n) + h(n) \\
&= g_{P_{s-n}}(n) + h(n)
\end{aligned}$$

Evaluation of State-Space Graphs

Additive Cost Measures

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$$\begin{aligned}
\widehat{C}_{P_{s-n}}(s) &= F[E(s), \widehat{C}_{P_{s-n}}(n_1)] \\
&= F[E(s),\ F[E(n_1),\ F[E(n_2),\ \ldots,\ F[E(n_k), h(n)] \ldots]\,]\,] \\
&= c(s, n_1) + c(n_1, n_2) + \ \ldots \ + c(n_k, n) + h(n) \\
&= g_{P_{s-n}}(n) + h(n)
\end{aligned}$$

Definition 12 (Additive Cost Measure)

Let G be an OR graph, n a node in G, n' a direct successor of n, and F a cost measure. F is called additive cost measure iff ($\leftrightarrow$) it is of the following form:

$$F[E(n), \widehat{C}(n')] = E(n) + \widehat{C}(n')$$

Remarks:

- $g_{P_{s-n}}(n)$ is the sum of the edge costs of a path $P_{s-n} = (s, n_1, \ldots, n_k, n)$ from s to n.
- $h(n)$ estimates the rest problem cost at node n.
- Here, we use the computation of estimated optimum solution cost extending a solution base for recursive cost functions.

Evaluation of State-Space Graphs

Relation to the Algorithm BF [GBF]

$\mathrm{BF}^*(s, \textit{successors}, \star, f)$

```
    ...
    LOOP
      IF (OPEN = ∅) THEN RETURN(Fail);
      n = min(OPEN, f);    // Find most promising solution base.
      IF ⋆(n) THEN RETURN(n);    // Delayed termination.
```

Define $f(n)$ as $\widehat{C}_P(s)$ with P backpointer path of n:

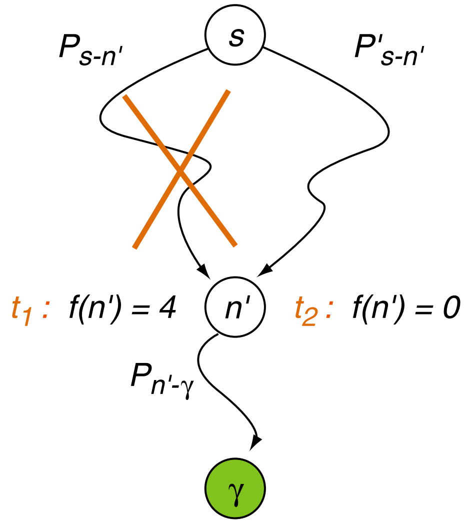
- ➜ $f(n)$ is a recursive evaluation function.
- ➜ Algorithm BF becomes Algorithm Z.

Delayed termination:

- ➜ Algorithm BF becomes Algorithm BF*.
- ➜ Algorithm Z becomes Algorithm Z*.

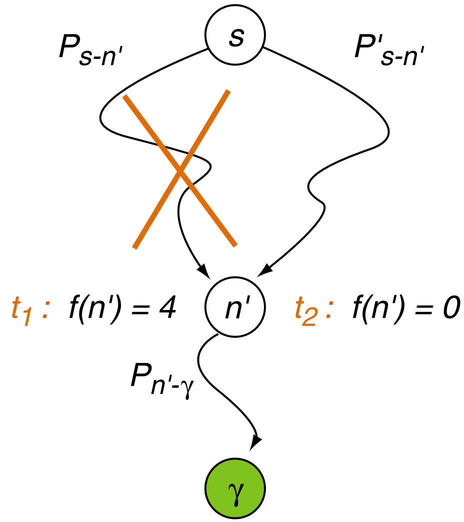
Recall that BF discards the inferior of two paths leading to the same node:

5. **FOREACH** n' IN *successors*(n) **DO**
   ...
   IF $(n' \notin$ OPEN AND $n' \notin$ CLOSED$)$
   THEN ...
   ELSE
     $n'_{old} =$ *retrieve*$(n',$ OPEN $\cup$ CLOSED$)$;
     IF $(f(n') < f(n'_{old}))$
     THEN
       update_backpointer(n'_{old}, n);
       IF $n'_{old} \in$ CLOSED THEN ...ENDIF
     ENDIF



Recall that BF discards the inferior of two paths leading to the same node:

```
5.    FOREACH n' IN successors(n) DO
        ...
        IF (n' ∉ OPEN AND n' ∉ CLOSED)
        THEN ...
        ELSE
          n'_old = retrieve(n', OPEN ∪ CLOSED);
          IF (f(n') < f(n'_old))
          THEN
            update_backpointer(n'_old, n);
            IF n'_old ∈ CLOSED THEN ...ENDIF
          ENDIF
```



→ An optimistic evaluation function f is not sufficient for Z* to be optimum.

→ Necessary: cost estimations for alternative solution bases must be independent of their shared continuation.

Formally: The cost function $\widehat{C}_P(s)$ must be *order-preserving*.

Definition 13 (Order-Preserving)

A cost function $\widehat{C}_P(s)$ is called order-preserving if for all nodes n' and paths $P_{s-n'}$, $P'_{s-n'}$ from s to n', and for all nodes n and paths $P_{n'-n}$ from n' to n holds:

$$\widehat{C}_{P_{s-n'}}(s) \leq \widehat{C}_{P'_{s-n'}}(s) \quad \Rightarrow \quad \widehat{C}_{P_{s-n}}(s) \leq \widehat{C}_{P'_{s-n}}(s)$$

P_{s-n} and P'_{s-n} denote the paths from s to n that result from concatenating the paths $P_{s-n'}$ and $P'_{s-n'}$ with $P_{n'-n}$.

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P_{s-n} and P'_{s-n} denote the paths from s to n that result from concatenating the paths $P_{s-n'}$ and $P'_{s-n'}$ with $P_{n'-n}$.

Corollary 14 (Order-Preserving)

If a cost function $\widehat{C}_P(s)$ is order-preserving, then for all nodes n' and paths $P_{s-n'}$, $P'_{s-n'}$ from s to n', and for all nodes n and paths $P_{n'-n}$ from n' to n holds:

$$\widehat{C}_{P_{s-n}}(s) > \widehat{C}_{P'_{s-n}}(s) \quad \Rightarrow \quad \widehat{C}_{P_{s-n'}}(s) > \widehat{C}_{P'_{s-n'}}(s)$$

and

$$\widehat{C}_{P_{s-n'}}(s) = \widehat{C}_{P'_{s-n'}}(s) \quad \Rightarrow \quad \widehat{C}_{P_{s-n}}(s) = \widehat{C}_{P'_{s-n}}(s)$$

Definition 15 (Order-Preserving for Solution Paths)

A cost function $\widehat{C}_P(s)$ is called order-preserving for solution paths if for all nodes n' and paths $P_{s-n'}$, $P'_{s-n'}$ from s to n', and for all nodes γ and paths $P_{n'-\gamma}$ from n' to γ holds:

$$\widehat{C}_{P_{s-n'}}(s) \leq \widehat{C}_{P'_{s-n'}}(s) \quad \Rightarrow \quad \widehat{C}_{P_{s-\gamma}}(s) \leq \widehat{C}_{P'_{s-\gamma}}(s)$$

$P_{s-\gamma}$ and $P'_{s-\gamma}$ denote the paths from s to γ that result from concatenating the paths $P_{s-n'}$ and $P'_{s-n'}$ with $P_{n'-\gamma}$.

Evaluation of State-Space Graphs

Optimum Solution Cost and Order Preservation (continued)

Definition 15 (Order-Preserving for Solution Paths)

A cost function $\widehat{C}_P(s)$ is called order-preserving for solution paths if for all nodes n' and paths $P_{s-n'}$, $P'_{s-n'}$ from s to n', and for all nodes γ and paths $P_{n'-\gamma}$ from n' to γ holds:

$$\widehat{C}_{P_{s-n'}}(s) \;\leq\; \widehat{C}_{P'_{s-n'}}(s) \quad \Rightarrow \quad \widehat{C}_{P_{s-\gamma}}(s) \;\leq\; \widehat{C}_{P'_{s-\gamma}}(s)$$

$P_{s-\gamma}$ and $P'_{s-\gamma}$ denote the paths from s to γ that result from concatenating the paths $P_{s-n'}$ and $P'_{s-n'}$ with $P_{n'-\gamma}$.

Corollary 16 (Order-Preserving for Solution Paths)

An order-preserving cost function $\widehat{C}_P(s)$ is order-preserving for solution paths.

Evaluation of State-Space Graphs

Optimum Solution Cost and Order Preservation (continued)

Lemma 17 (Order-Preserving)

Evaluation functions f that rely on additive cost measures $F[e, c] = e + c$ are order-preserving.

Define $f(n)$ as $\widehat{C}_{P_{s-n}}(s) = g_{P_{s-n}}(n) + h(n)$ ($f = g + h$ for short):

➜ Algorithm Z* becomes Algorithm A*.

Lemma 17 (Order-Preserving)

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Define $f(n)$ as $\widehat{C}_{P_{s-n}}(s) = g_{P_{s-n}}(n) + h(n)$ ($f = g + h$ for short):

→ Algorithm Z* becomes Algorithm A*.

Proof (of Lemma)

Let $P_{s-n'} = (s, n_{1,1}, \ldots, n_{1,k}, n')$, $P'_{s-n'} = (s, n_{2,1}, \ldots, n_{2,l}, n')$ be paths from s to n', where

$$\widehat{C}_{P_{s-n'}}(s) = c(s, n_{1,1}) + \ldots + c(n_{1,k}, n') + h(n') \ \leq \ c(s, n_{2,1}) + \ldots + c(n_{2,l}, n') + h(n') = \widehat{C}_{P'_{s-n'}}(s)$$

Let $P_{n'-n} = (n', n_1, \ldots, n_r, n)$ be a path from n' to n. Then follows:

$$\begin{array}{rrcl} & c(s, n_{1,1}) + \ldots + c(n_{1,k}, n') + & & c(s, n_{2,1}) + \ldots + c(n_{2,l}, n') + \\ & c(n', n_1) + \ldots + c(n_r, n) + h(n) & \leq & c(n', n_1) + \ldots + c(n_r, n) + h(n) \\ \Leftrightarrow & \widehat{C}_{P_{s-n}}(s) & \leq & \widehat{C}_{P'_{s-n}}(s) \end{array}$$

Evaluation of State-Space Graphs

Optimum Solution Cost and Order Preservation (continued)

Lemma 17 (Order-Preserving)

Evaluation functions f that rely on additive cost measures $F[e,c] = e + c$ are order-preserving.

Define $f(n)$ as $\widehat{C}_{P_{s-n}}(s) = g_{P_{s-n}}(n) + h(n)$ ($f = g + h$ for short):

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Proof (of Lemma)

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$$\widehat{C}_{P_{s-n'}}(s) = c(s, n_{1,1}) + \ldots + c(n_{1,k}, n') + h(n') \ \leq\ c(s, n_{2,1}) + \ldots + c(n_{2,l}, n') + h(n') = \widehat{C}_{P'_{s-n'}}(s)$$

Let $P_{n'-n} = (n', n_1, \ldots, n_r, n)$ be a path from n' to n. Then follows:

$$\begin{array}{lrcl}
 & c(s, n_{1,1}) + \ldots + c(n_{1,k}, n') + & & c(s, n_{2,1}) + \ldots + c(n_{2,l}, n') + \\
 & c(n', n_1) + \ldots + c(n_r, n) + h(n) & \leq & c(n', n_1) + \ldots + c(n_r, n) + h(n) \\
\Leftrightarrow & \widehat{C}_{P_{s-n}}(s) & \leq & \widehat{C}_{P'_{s-n}}(s)
\end{array}$$

Remarks:

- $g(n)$ denotes the sum of the edge cost values along the backpointer path from s to n. Since A* as BF* variant maintains for each node generated at each point in time a unique backpointer, there is only one solution base for each terminal node in the explored subgraph G of the search space graph for which a cost value has to be computed. Therefore, $g_{P_{s-n}}(n)$ can be seen as a function $g(n)$ that is only dependent from n.

Evaluation of State-Space Graphs

Optimum Solution Cost and Order Preservation (continued)

Example for a cost function that is recursive but *not* order-preserving:

$$\widehat{C}_P(n) = \begin{cases} c(n) & n \text{ is goal node and leaf in } P \\ h(n) & n \text{ is leaf in } P \text{ but no goal node} \\ F[E(n), \widehat{C}_P(n')] & n \text{ is inner node in } P \text{ and} \\ \qquad = |c(n,n') + \widehat{C}_P(n') - 5| & n' \text{ is direct successor of } n \text{ in } P \end{cases}$$

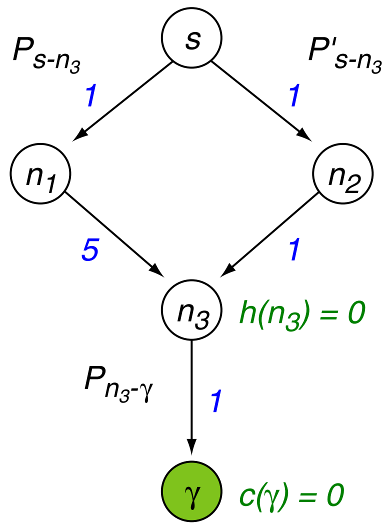
Example for a cost function that is recursive but *not* order-preserving:

$$\widehat{C}_P(n) = \begin{cases} c(n) & n \text{ is goal node and leaf in } P \\ h(n) & n \text{ is leaf in } P \text{ but no goal node} \\ F[E(n), \widehat{C}_P(n')] & n \text{ is inner node in } P \text{ and} \\ \quad = |c(n,n') + \widehat{C}_P(n') - 5| & n' \text{ is direct successor of } n \text{ in } P \end{cases}$$

$$P_{s-n_3} = (s, n_1, n_3)$$

$$P'_{s-n_3} = (s, n_2, n_3)$$

$$P_{n_3-\gamma} = (n_3, \gamma)$$



Evaluation of State-Space Graphs

Optimum Solution Cost and Order Preservation (continued)

Example for a cost function that is recursive but *not* order-preserving:

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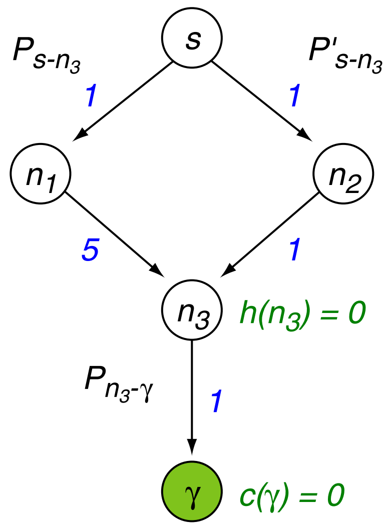
$$P_{s-n_3} = (s, n_1, n_3)$$

$$P'_{s-n_3} = (s, n_2, n_3)$$

$$P_{n_3-\gamma} = (n_3, \gamma)$$

$$\widehat{C}_{P_{s-n_3}}(s) = |1 + |5 + 0 - 5| - 5| = 4$$



Evaluation of State-Space Graphs

Optimum Solution Cost and Order Preservation (continued)

Example for a cost function that is recursive but *not* order-preserving:

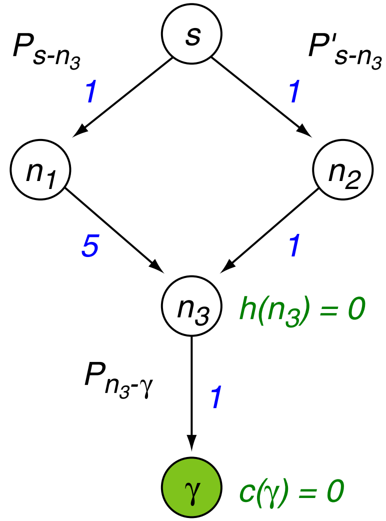
$$\widehat{C}_P(n) = \begin{cases} c(n) & n \text{ is goal node and leaf in } P \\ h(n) & n \text{ is leaf in } P \text{ but no goal node} \\ F[E(n), \widehat{C}_P(n')] & n \text{ is inner node in } P \text{ and} \\ \quad = |c(n,n') + \widehat{C}_P(n') - 5| & n' \text{ is direct successor of } n \text{ in } P \end{cases}$$

$$P_{s-n_3} = (s, n_1, n_3)$$

$$P'_{s-n_3} = (s, n_2, n_3)$$

$$P_{n_3-\gamma} = (n_3, \gamma)$$

$$\widehat{C}_{P_{s-n_3}}(s) = |1 + |5 + 0 - 5| - 5| = 4$$



Optimum Solution Cost and Order Preservation (continued)

Example for a cost function that is recursive but *not* order-preserving:

$$\widehat{C}_P(n) = \begin{cases} c(n) & n \text{ is goal node and leaf in } P \\ h(n) & n \text{ is leaf in } P \text{ but no goal node} \\ F[E(n), \widehat{C}_P(n')] & n \text{ is inner node in } P \text{ and} \\ \quad = |c(n,n') + \widehat{C}_P(n') - 5| & n' \text{ is direct successor of } n \text{ in } P \end{cases}$$

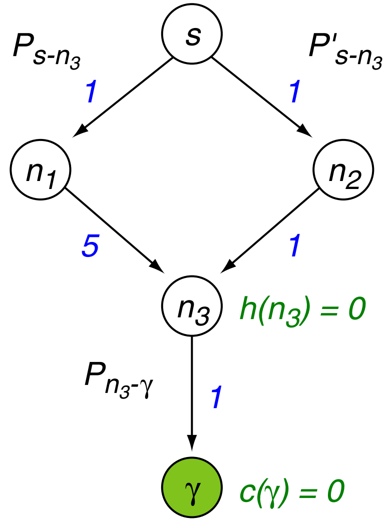
$$P_{s-n_3} = (s, n_1, n_3)$$

$$P'_{s-n_3} = (s, n_2, n_3)$$

$$P_{n_3-\gamma} = (n_3, \gamma)$$

$$\widehat{C}_{P_{s-n_3}}(s) = |1+|5+0-5|-5| = 4$$

$$\widehat{C}_{P'_{s-n_3}}(s) = |1+|1+0-5|-5| = 0$$



Evaluation of State-Space Graphs

Optimum Solution Cost and Order Preservation (continued)

Example for a cost function that is recursive but *not* order-preserving:

$$\widehat{C}_P(n) = \begin{cases} c(n) & n \text{ is goal node and leaf in } P \\ h(n) & n \text{ is leaf in } P \text{ but no goal node} \\ F[E(n), \widehat{C}_P(n')] & n \text{ is inner node in } P \text{ and} \\ \quad = |c(n,n') + \widehat{C}_P(n') - 5| & n' \text{ is direct successor of } n \text{ in } P \end{cases}$$

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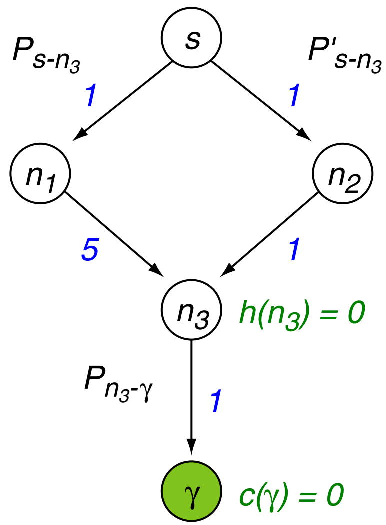
$$\widehat{C}_{P_{s-n_3}}(s) = |1 + |5 + 0 - 5| - 5| = 4$$

$$\widehat{C}_{P'_{s-n_3}}(s) = |1 + |1 + 0 - 5| - 5| = 0$$

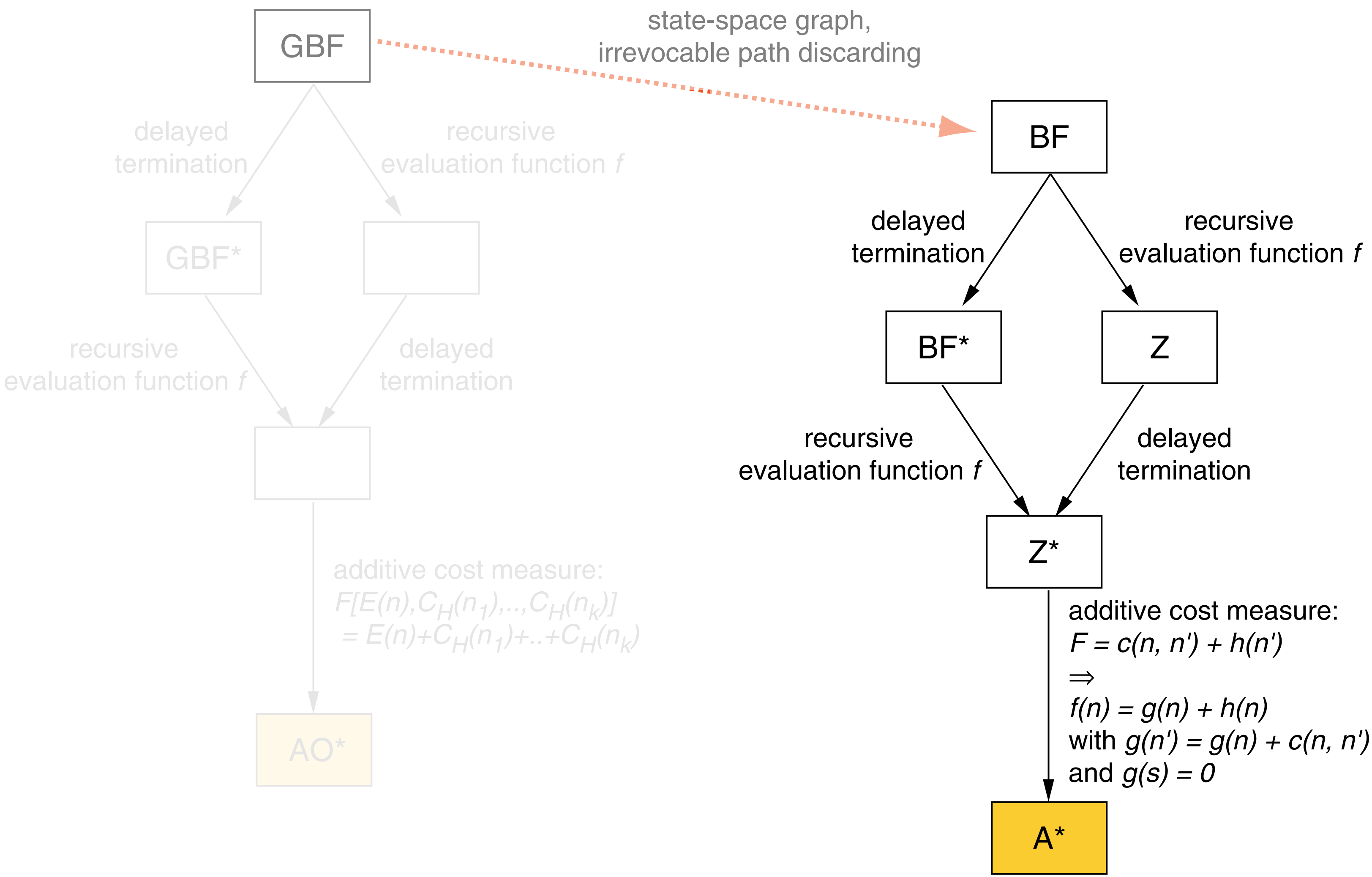
$$\widehat{C}_{P_{s-\gamma}}(s) = |1 + |5 + |1 + 0 - 5| - 5| - 5| = 0$$

$$\widehat{C}_{P'_{s-\gamma}}(s) = |1 + |1 + |1 + 0 - 5| - 5| - 5| = 4$$



Evaluation of State-Space Graphs

Taxonomy of Best-First Algorithms



Algorithm A*

Algorithm: A*

Input: s. Start node representing the initial problem.
successors(n). Returns the successors of node n.
$\star(n)$. Predicate that is *True* if n is a goal node.
$h(n)$. Heuristic cost estimation for node n, where $f(n) = \widehat{C}_{P_{s-n}}(s) = g(n) + h(n)$

Output: An optimum goal node or the symbol *Fail*.

Algorithm A* [BF, BF$\star$]

$A^*(s,$ *successors*$, \star, h)$

1. *insert*$(s,$ OPEN$)$;
   $g(s) = 0$;
2. **LOOP**
3. IF (OPEN $= \emptyset$) THEN RETURN(*Fail*);
4. $n =$ *min*(OPEN, $g + h$); // Most promising solution base minimizes $f(n)$.
   IF $\star(n)$ THEN RETURN(n); // Delayed termination.
   remove(n, OPEN); *push*(n, CLOSED);
5. **FOREACH** n' IN *successors*(n) **DO** // Expand n.
   add_backpointer(n', n);
   $g(n') = g(n) + c(n, n')$;
   IF ($n' \notin$ OPEN AND $n' \notin$ CLOSED)
   THEN // n' encodes a new state.
   insert(n', OPEN);
   ELSE // n' encodes an already visited state.
   $n'_{old} =$ *retrieve*(n', OPEN $\cup$ CLOSED);
   IF $(g(n') < g(n'_{old}))$
   THEN // The state of n' is reached via a cheaper path.
   update_backpointer(n'_{old}, n); $g(n'_{old}) = g(n')$;
   IF $n'_{old} \in$ CLOSED THEN *remove*(n'_{old}, CLOSED); *insert*(n'_{old}, OPEN); ENDIF
   ENDIF
   ENDIF
   ENDDO
6. **ENDLOOP**

Remarks:

- $g(n)$ is the sum of the edge costs of the current backpointer path $P_{s-n} = (s, \ldots, n)$ from s to n.
- $h(n)$ estimates the optimum rest problem cost at node n.
- $h(n) = c(n)$ is assumed for all goal nodes n. Often, we even have $h(n) = c(n) = 0$ for all goal nodes n.
- Although only the order-preserving property of $f = g + h$ was mentioned, we still assume that the following equivalence holds:

  “Solution base $P_{s-n'}$ can be completed by $P_{n'-\gamma}$ to a solution path.”

  $\Leftrightarrow$

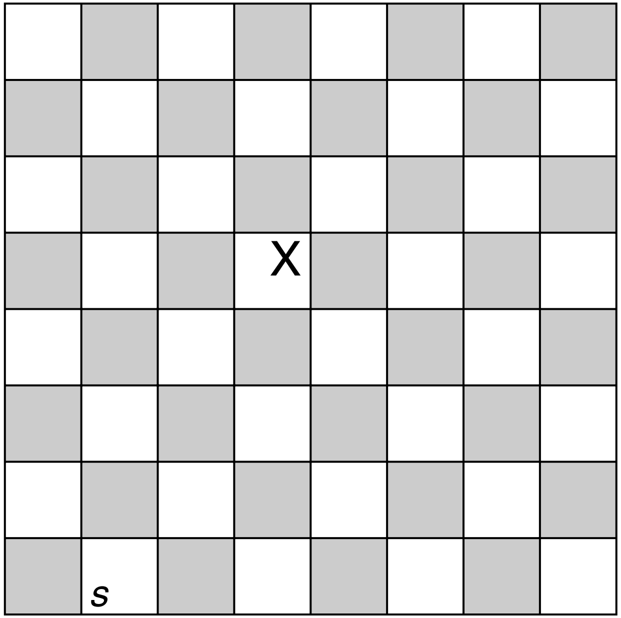
  “Solution base $P'_{s-n'}$ can be completed by $P_{n'-\gamma}$ to a solution path.”

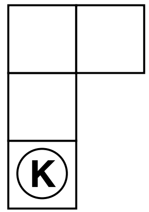
  This equivalence is trivially satisfied, if $\star(\gamma)$ checks only local properties of γ but not properties of the backpointer path of γ.

Algorithm A*

Example: Knight Moves

Search a shortest sequence of knight moves leading from s to X.





Knight move

Let n' be a direct successor of n.

- $f(n') = g(n') + h(n')$
- $g(n') = g(n) + c(n, n')$
- $g(s) = 0$
- $c(n, n') = 1$

Algorithm A*

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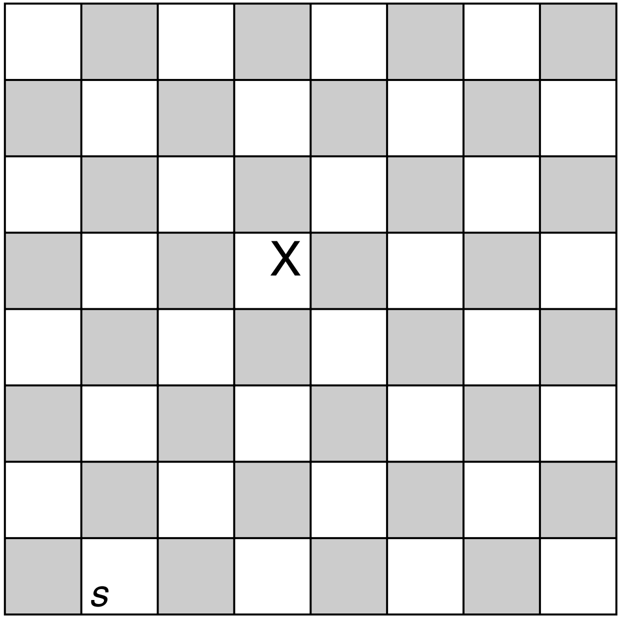

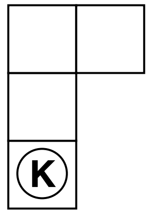

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Let n' be a direct successor of n.

- $f(n') = g(n') + h(n')$
- $g(n') = g(n) + c(n, n')$
- $g(s) = 0$
- $c(n, n') = 1$

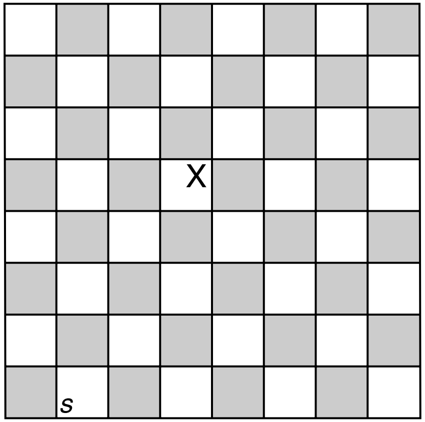
$$h_1 = \lceil \frac{\#rows}{2} \rceil$$

$$h_2 = \lceil \frac{\max\{\#rows,\ \#columns\}}{2} \rceil$$

$$h_3 = \lceil \frac{\#rows\ +\ \#columns}{3} \rceil$$

Algorithm A*

Example: Knight Moves (continued)



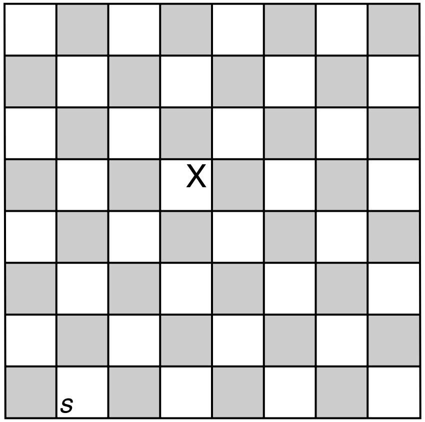
OPEN	CLOSED
$\{s\}$	$\{\}$

n	$g(n)$	$h_1(n)$	$f(n)$
s	0	2	2

$$h = h_1 = \lceil \frac{\#rows}{2} \rceil$$

Algorithm A*

Example: Knight Moves (continued)



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$\{s\}$	$\{\}$

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s	0	2	2

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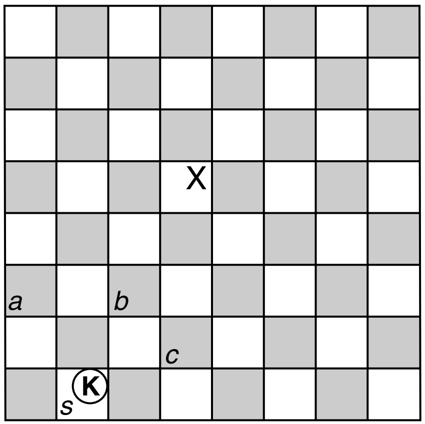
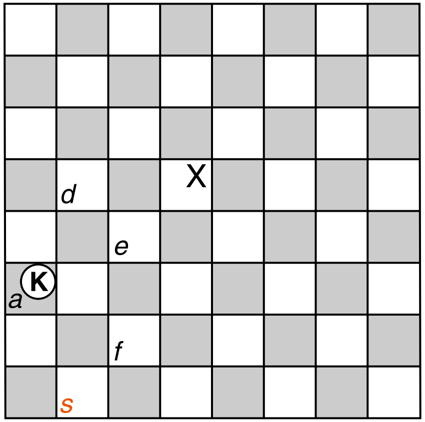


OPEN	CLOSED
$\{a, b, c\}$	$\{s\}$

n	$g(n)$	$h_1(n)$	$f(n)$
s	0	2	2
a	1	1	2
b	1	1	2
c	1	2	3

Algorithm A*

Example: Knight Moves (continued)

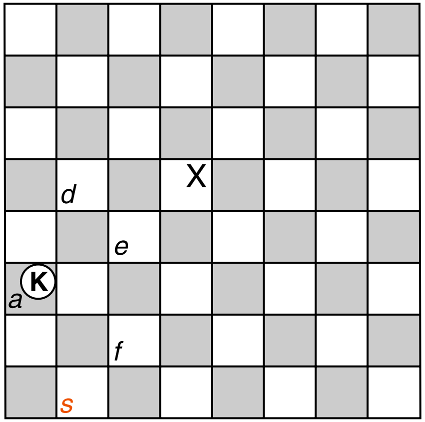


OPEN	CLOSED
$\{d, b, c, e, f\}$	$\{a, s\}$

n	$g(n)$	$h_1(n)$	$f(n)$
s	0	2	2
a	1	1	2
b	1	1	2
c	1	2	3
d	2	0	2
e	2	1	3
f	2	2	4

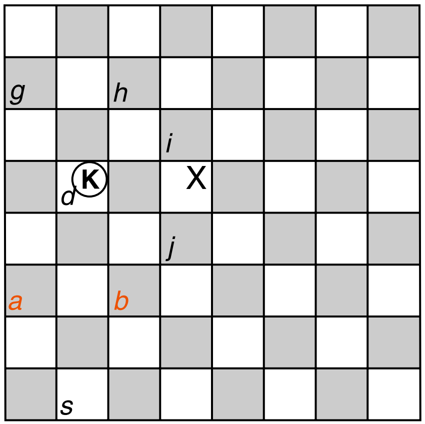
Algorithm A*

Example: Knight Moves (continued)



OPEN	CLOSED
$\{d, b, c, e, f\}$	$\{a, s\}$

n	$g(n)$	$h_1(n)$	$f(n)$
s	0	2	2
a	1	1	2
b	1	1	2
c	1	2	3
d	2	0	2
e	2	1	3
f	2	2	4

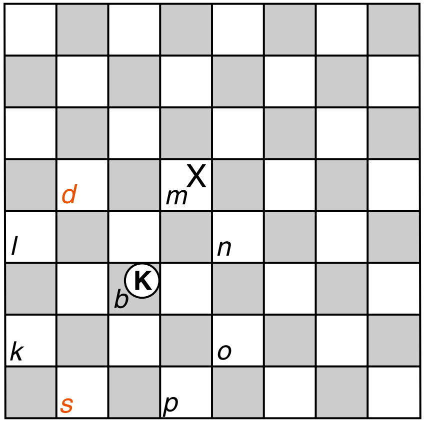


OPEN	CLOSED
$\{b, c, e, f, g, h, i, j\}$	$\{d, a, s\}$

n	$g(n)$	$h_1(n)$	$f(n)$
s	0	2	2
a	1	1	2
b	1	1	2
c	1	2	3
d	2	0	2
e	2	1	3
f	2	2	4
g	3	1	4
h	3	1	4
i	3	1	4
j	3	1	4

Algorithm A*

Example: Knight Moves (continued)

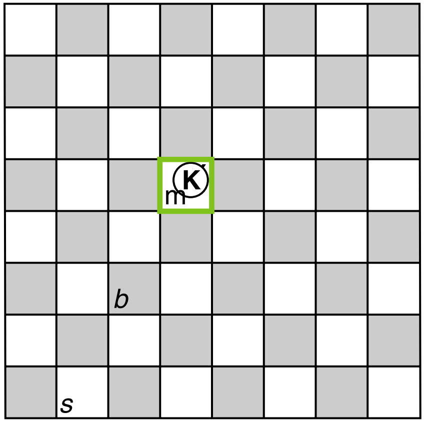


OPEN	CLOSED
$\{m, c, e, l, n, f, g, h, i, j, k, o, p\}$	$\{b, d, a, s\}$

n	$g(n)$	$h_1(n)$	$f(n)$
s	0	2	2
a	1	1	2
b	1	1	2
c	1	2	3
d	2	0	2
e	2	1	3
f	2	2	4
g	3	1	4
h	3	1	4
i	3	1	4
j	3	1	4
m	2	0	2
l	2	1	3
n	2	1	3
k	2	2	4
o	2	2	4
p	2	2	4

Algorithm A*

Example: Knight Moves (continued)



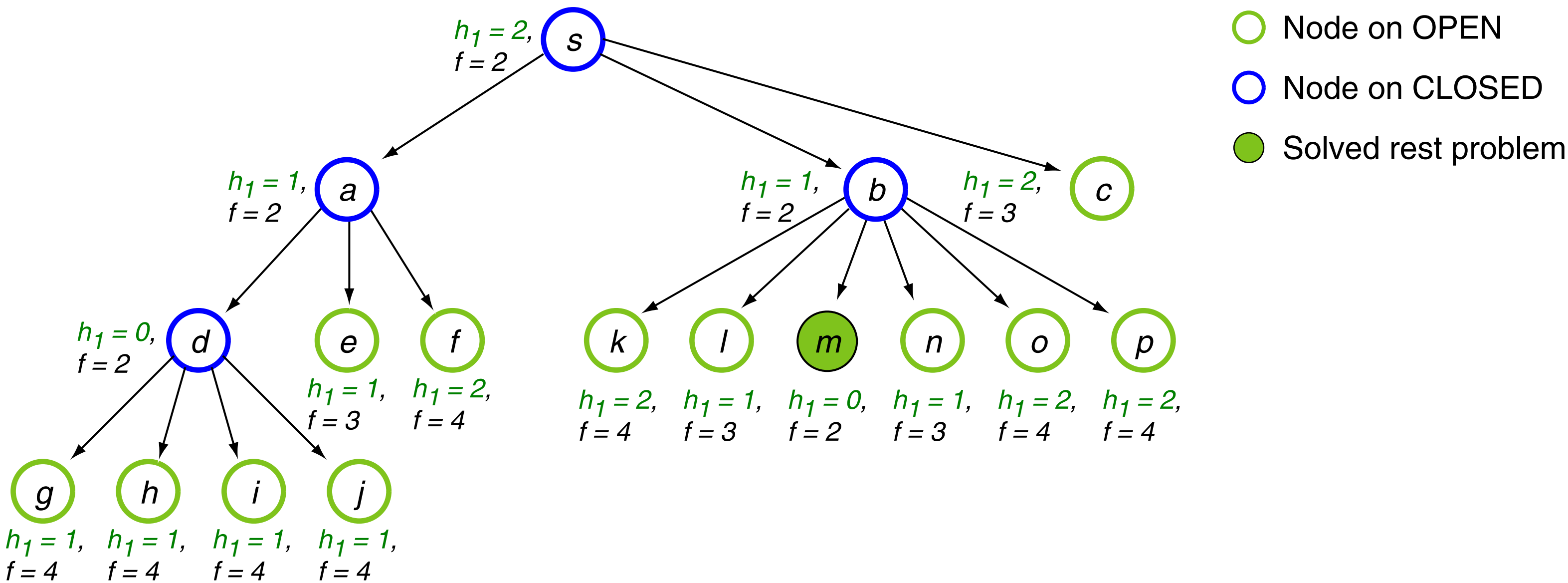
OPEN	CLOSED
$\{c, e, l, n,$ $f, g, h, i, j, k, o, p\}$	$\{m, b, d, a, s\}$

n	$g(n)$	$h_1(n)$	$f(n)$
s	0	2	2
a	1	1	2
b	1	1	2
c	1	2	3
d	2	0	2
e	2	1	3
f	2	2	4
g	3	1	4
h	3	1	4
i	3	1	4
j	3	1	4
m	2	0	2
l	2	1	3
n	2	1	3
k	2	2	4
o	2	2	4
p	2	2	4

Algorithm A*

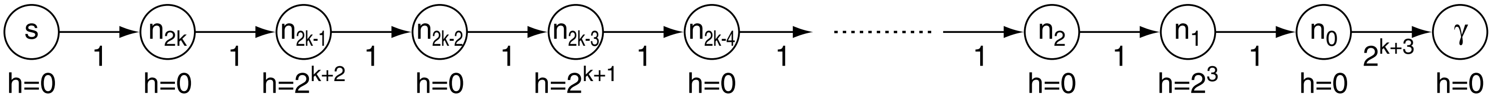
Example: Knight Moves (continued)

Analyzed part of the search space graph:



Algorithm A*

Exponential Runtime Example

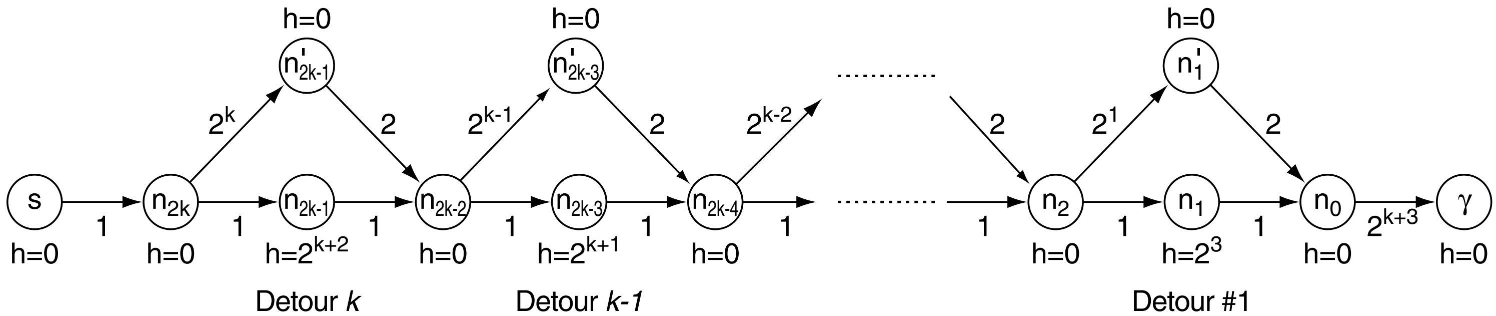


- ❑ Optimum cost path with path cost $2^{k+3} + 2k + 1$.
- ❑ Additional cost for using detours starting from $n_{2j}, 1 \leq j \leq k$ less than 2^{j+1}.
- ❑ At each point in time before A* terminates,
  - at most one node $n_{2j}, 1 \leq j \leq k$ is on OPEN,
  - any two nodes on OPEN share the initial part of their backpointer paths (starting from s to the predecessor of the leftmost of the two),
  - for any two non-goal nodes on OPEN with different position from left to right the leftmost node has a higher f-value,
  - for two nodes n_{2j+1} and $n'_{2j+1}, 1 \leq j < k$ on OPEN n_{2j+1} has a higher f-value,
  - for γ on OPEN the f-value is maximal wrt. OPEN.

➜ A* requires more than 2^k node expansions.

Algorithm A*

Exponential Runtime Example (continued)

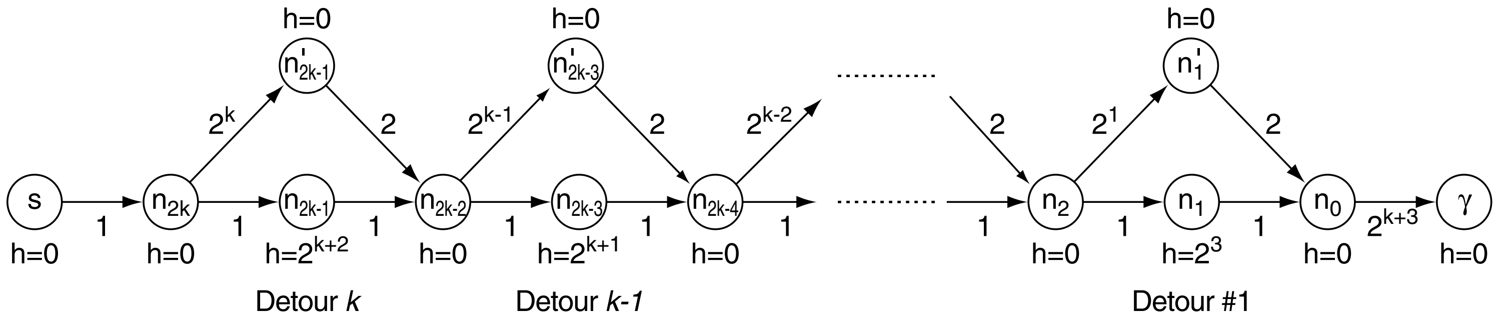


- Optimum cost path with path cost $2^{k+3} + 2k + 1$.
- Additional cost for using detours starting from $n_{2j}, 1 \leq j \leq k$ less than 2^{j+1}.
- At each point in time before A* terminates,
  - at most one node $n_{2j}, 1 \leq j \leq k$ is on OPEN,
  - any two nodes on OPEN share the initial part of their backpointer paths (starting from s to the predecessor of the leftmost of the two),
  - for any two non-goal nodes on OPEN with different position from left to right the leftmost node has a higher f-value,
  - for two nodes n_{2j+1} and $n'_{2j+1}, 1 \leq j < k$ on OPEN n_{2j+1} has a higher f-value,
  - for γ on OPEN the f-value is maximal wrt. OPEN.

➜ A* requires more than 2^k node expansions.

Algorithm A*

Exponential Runtime Example (continued)



- Optimum cost path with path cost $2^{k+3} + 2k + 1$.
- Additional cost for using detours starting from $n_{2j}, 1 \leq j \leq k$ less than 2^{j+1}.
- At each point in time before A* terminates,
  - at most one node $n_{2j}, 1 \leq j \leq k$ is on OPEN,
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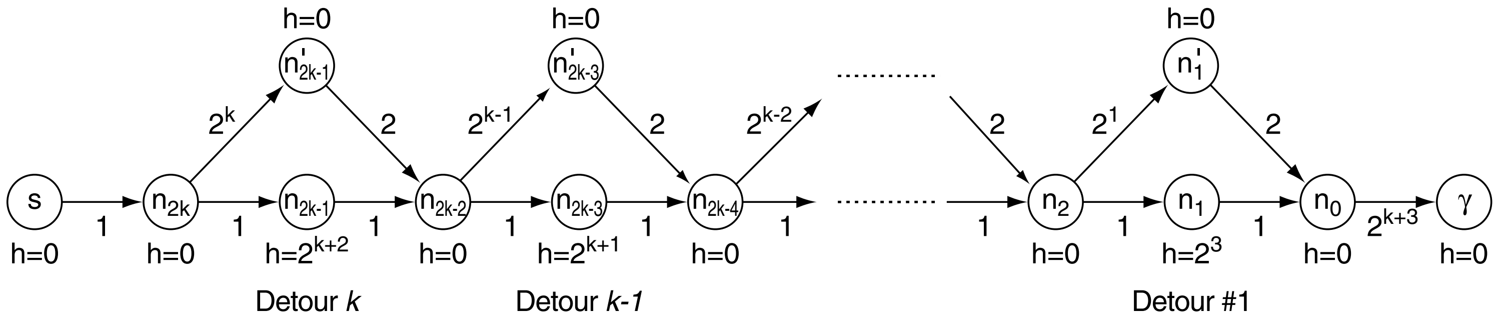

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