Exercises

Exercise 1: Propositional Logic

Consider the following formula in propositional logic:

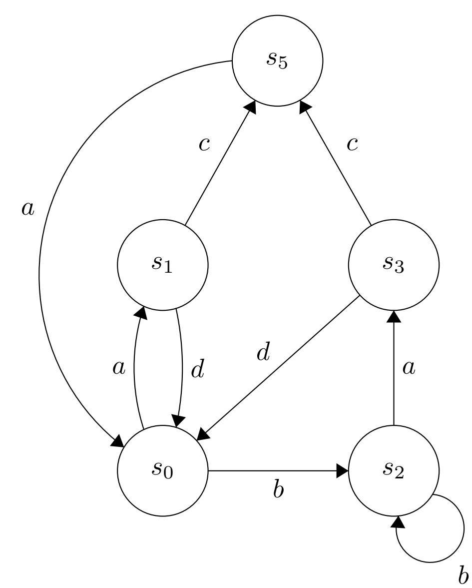
$$(A \vee \neg(B \wedge A \wedge C)) \Leftrightarrow (A \Rightarrow D) \wedge (D \Leftrightarrow (E \wedge (B \vee \neg(C \wedge \neg A)))) \wedge (B \Leftrightarrow ((C \vee E) \wedge (B \vee C))$$

Considering, that this formula contains all propositional symbols,

- how many models are there?
- is the formula valid? If not, in which models is it valid? Use both *model checking* and *propositional theorem proving* to justify your answer!

Exercise 2: Relations

Consider the following transition system:



Give a concrete definition (i.e. enumerate all relevant tuples) for the following Relations

- The intended relational semantics $[\![\alpha]\!] := R_\alpha$ for each action label $\alpha \in \{a, b, c, d\}$.
- $R_a; R_b$
- $R_a \cup R_d$
- $(R_a; R_c)^*$

Exercise 3: Propositional Dynamic Logic

Consider the following model labelled Kripke-structure representing a model m (the set of propositional symbols is $\Phi_0 := \{E, O\}$, and the set of action labels is $A_0 := \{a, b, c\}$):

$$K_m = (\{s_0, s_1, s_2\}, \{(s_0, a, s_1), (s_1, a, s_0), (s_1, b, s_1), (s_1, c, s_2)\}, \{s_0 \mapsto \{E\}, s_1 \mapsto \{O\}, s_2 \mapsto \emptyset\})$$

First, draw the transitions system for this model m.
Now, Verify whether the following sentences are satisfied not?

- $m, s_0 \models \langle (a; b)^* \rangle O$
- $m, s_1 \models \langle (a; b)^* \rangle O$
- $m, s_0 \models [(a; b)^*] O$
- $m, s_1 \models [(a; b)^*] O$
- $m, s_0 \models [a; a] E$
- $m, s_1 \models \langle b \rangle \top \Rightarrow O$
- $m, s_1 \models \langle a \cup b \rangle O$
- $m \models \langle \alpha \rangle \top \Rightarrow (E \Leftrightarrow \neg O)$

Exercise 4: Hoare Rule

Use propositional reasoning to prove the validity of the so-called *Hoare* rule.

$$\frac{\phi \wedge \psi \Rightarrow [\alpha]\phi}{\phi \wedge \psi \Rightarrow [\textbf{while } \psi \textbf{ do } \alpha]\phi} \equiv \frac{\phi \wedge \psi \Rightarrow [\alpha]\phi}{\phi \wedge \psi \Rightarrow [(\psi; \alpha)^*; \neg\psi]\phi}$$

Tip: Start by proving the that following (so-called *loop-invariant* rule) rule is valid:

$$\frac{\phi \Rightarrow [\alpha]\phi}{\phi \Rightarrow [\alpha^*]\phi}$$

To prove the loop invariance rule, you may have a look at *Modal Generalisation* and the *Induction* axiom. If this is too tricky right now, you can simply assume the loop invariance rule for now and proceed with the Hoare rule.

Using, the loop invariance rule, the axioms for test, and sequence, as well as considering the following propositional tautologies:

$$((\phi \wedge \psi) \Rightarrow \chi) \Leftrightarrow (\phi \Rightarrow (\psi \Rightarrow \chi))$$
$$\top \Leftrightarrow (\phi \Rightarrow (\neg\psi \Rightarrow (\neg\psi \wedge \phi)))$$

you should be able to conduct the proof.

Exercise 5: River-Crossing Puzzle

Consider the following well-known "river-crossing puzzle":

> Once upon a time a farmer went to a market and purchased a wolf, a goat, and a cabbage. On his way home, the farmer came to the bank of a river and rented a boat. But crossing the river by boat, the farmer could carry only himself and a single one of his purchases: the wolf, the goat, or the cabbage.
>
> If left unattended together, the wolf would eat the goat, or the goat would eat the cabbage.
>
> The farmer's challenge was to carry himself and his purchases to the far bank of the river, leaving each purchase intact. How did he do it?

Modeling this puzzle in Propositional Dynamic Logic, we assume the following propositional symbols:

F "The farmer is on river right".

G "The goat is on river right".

C "The cabbage is on river right".

W "The wolf is on river right".

Negating these states means that the respective object is on river left. Thus, the start space is $\neg F \neg G \neg C \neg W$ and the goal state is $FGCN$.

Answer the following questions:

- What are the states that should be avoided?
- Is there a composite action α (which is a sequence of atomic actions) such that the formula
  $$\neg F \neg G \neg C \neg W \Rightarrow [\alpha] FGCN$$
  becomes valid? Start by defining the set of atomic actions, then draw the respective Kripke-Structure of the problem. Finally, define α and show by reasoning of the model you have just draw that the above statement is valid!