PAPER

Efficient Algorithms for Finding a Tree 3-Spanner on Permutation Graphs

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SUMMARY A tree 3-spanner T of a graph G is a spanning tree of G such that the distance between any two vertices in T is at most 3 times of their distance in G. Madanlal et al. have presented an $O(n+m)$ time algorithm for finding a tree 3-spanner of a permutation graph. However, the complexity of their algorithm is not optimal, and their algorithm can not be easily parallelized. In this paper, we will propose an improved algorithm to solve the same problem in $O(n)$ time. Moreover, our algorithm can be easily parallelized so that a tree 3-spanner of a permutation graph can be found in $O(\log n)$ time with $O(\frac{n}{\log n})$ processors on the EREW PRAM computational model.

key words: *tree, spanner, permutation graph, algorithm*

1. Introduction

Let $G=(V,E)$ be a connected graph with vertex set V and edge set E. A graph $H=(V',E')$ is a *spanning subgraph* of G if $V'=V$ and $E' \subseteq E$. If H is an acyclic connected spanning subgraph, then H is a *spanning tree* of G. A spanning tree T is a *tree t-spanner* of G if the distance between any two vertices in T is at most t times of their distance in G, where t is a positive integer. The concept of the tree spanner has applications in distributed environments [8].

Cai and Corneil [2] showed that the problem to determine whether a graph has a tree t-spanner on unweighted graphs can be solved in polynomial time when $t \leq 2$, and it is NP-complete when $t \geq 4$. The case where $t=3$ is an open problem; however, it is conjectured by Cai to be NP-complete [1]. Many researchers have studied the existence of a tree spanner on special types of graphs. For example, Madanlal et al. presented algorithms for finding tree 3-spanners on interval, permutation, and regular bipartite graphs [7]. Besides, the spanner problems on the hypercube and the bounded degree graphs were discussed in [4] and [3], respectively.

In the algorithm of Madanlal et al. [7], a tree 3-spanner of a permutation graph can be found in $O(n+m)$ time, where n is the number of vertices and m is the number of edges. However, their algorithm is not optimal, and it can not be easily parallelized. In this paper, we will present an improved algorithm to solve the same problem in $O(n)$ time, assuming the input permutation graph is connected. Our algorithm can be easily parallelized so that finding a tree 3-spanner of a permutation graph can be done in $O(\log n)$ time with $O(\frac{n}{\log n})$ processors on the EREW PRAM computational model. The rest part of this paper is organized as follows. In Sect. 2, we introduce the permutation graph and the $O(n+m)$ algorithm presented by Madanlal et al. In Sect. 3, we propose an $O(n)$ algorithm for finding a tree 3-spanner on permutation graphs and then show its correctness. The parallel algorithm for the same problem is described in Sect. 4. Finally, we give concluding remarks in Sect. 5.

2. Review

A permutation graph is defined as follows [5]. Let π be a permutation sequence $\pi(1), \pi(2), \cdots, \pi(n)$ of the numbers $1, 2, \cdots, n$, and let $\pi^{-1}(i)$ be the position in the π sequence where the number i can be found. Then, a permutation graph $G=(V,E)$ is with vertex set $V=\{1, 2, \cdots, n\}$ and edge set E, where $(i,j) \in E$ if and only if $(i-j)(\pi^{-1}(i)-\pi^{-1}(j)) < 0$. A permutation graph can be represented by a permutation diagram. In the diagram, there are two sequences $1, 2, \cdots, n$ and $\pi(1), \pi(2), \cdots, \pi(n)$. Each vertex i in G corresponds to a line between i and $\pi^{-1}(i)$ in the diagram. Two vertices are joined by an edge in G if and only if the corresponding two lines intersect in the diagram. For example, in Fig. 1, $\pi(1)=3, \pi^{-1}(1)=4$, and $(1,6) \in E$. From the definition, we know that for three vertices i, j, and k, $i<j<k$, if i is adjacent to k, then j is adjacent to i or k.

Permutation graphs were proposed by Pnueli et al. [9], who also described an $O(n^3)$ algorithm for testing if a given undirected graph is a permutation graph. An

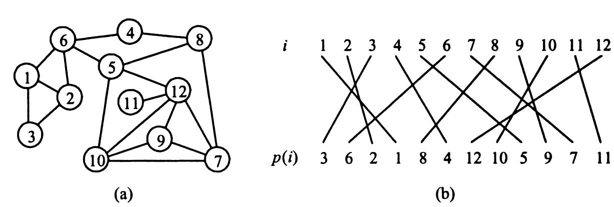

Fig. 1 An example of a permutation graph. (a) A permutation graph. (b) The corresponding diagram.

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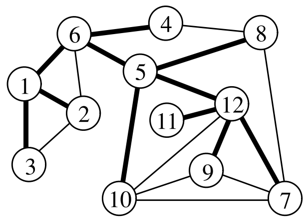

Fig. 2 The resulting tree 3-spanner.

improved $O(n^2)$ algorithm for the recognition of a permutation graph was presented by Spinrad [10].

The algorithm of Madanlal et al. constructs a tree 3-spanner T of a permutation graph by several stages. Let $A_i = \{j | j > i \text{ and } j \text{ is adjacent to } i\}$ for $1 \le i \le n$. Define

$$P_{\max}(i) = \max \begin{cases} \max A_i & \text{if } A_i \neq \phi; \\ i & \text{otherwise.} \end{cases}$$

Initially, let $i = 1$, $r_1 = 1$, $\alpha_0 = 1$, and $\alpha_1 = P_{\max}(1)$. In stage i, add edge (r_i, α_i) to T and find vertex sets S_i and R_i, where $S_i = \{v | v \text{ is in interval } (\alpha_{i-1}, \alpha_i) \text{ and } v \text{ is adjacent to } r_i\}$ and $R_i = \{v | v \text{ is in interval } (\alpha_{i-1}, \alpha_i) \text{ and } v \text{ is not adjacent to } r_i\}$. For all vertices $v \in S_i$, add edge (v, r_i) to T; for all vertices $v \in R_i$, add edge (v, α_i) to T. If $\alpha_i = n$, then the resulting graph T is a tree 3-spanner of the input permutation graph and stop this algorithm. Otherwise, determine r_{i+1} and α_{i+1} as follows for the next stage. Let $\alpha_{i+1} = \max_{v \in R_i} P_{\max}(v)$. Moreover, let $T_i = \{v | v \in R_i \text{ and } v \text{ is adjacent to } \alpha_{i+1}\}$. Then, choose r_{i+1} from T_i such that if v is not adjacent to r_{i+1} for some v in interval (α_i, α_{i+1}), then q is not adjacent to v for all $q \in T_i$. Let $i = i + 1$ and do the next stage.

We use Fig. 1 to illustrate the algorithm of Madanlal et al. In stage 1, $r_1 = 1$, $\alpha_1 = P_{\max}(1) = 6$, $S_1 = \{2, 3\}$, $R_1 = \{4, 5\}$, and $(1, 6)$ is an edge of T. We add edges $(2, 1)$, $(3, 1)$, $(4, 6)$, and $(5, 6)$ to T. In addition, $\alpha_2 = \max\{P_{\max}(4), P_{\max}(5)\} = \max\{8, 12\} = 12$, $T_1 = \{5\}$, and $r_2 = 5$. In stage 2, $S_2 = \{8, 10\}$, $R_2 = \{7, 9, 11\}$, and $(5, 12)$ is an edge of T. We add edges $(8, 5)$, $(10, 5)$, $(7, 12)$, $(9, 12)$, and $(11, 12)$ to T. Since $\alpha_2 = 12$, we stop the algorithm. The resulting tree 3-spanner is shown in Fig. 2.

In the algorithm of Madanlal et al., it needs to determine $\alpha_{i+1} = \max_{v \in R_i} P_{\max}(v)$ in each stage i. Since finding $P_{\max}$ for all vertices takes $O(m)$ time [7], the complexity of their algorithm is $O(n + m)$ even when the permutation diagram is given.

3. An $O(n)$ Algorithm for Finding a Tree 3-Spanner

In this section, we shall present an improved $O(n)$ algorithm for finding a tree 3-spanner of a permutation graph, which bases on the idea of the algorithm of Madanlal et al. In their algorithm, the most consuming step is to determine r_{i+1} and α_{i+1}. Thus, we design a new skill to reduce the complexity of this step. Let $L(i) = \max\{\pi(1), \pi(2), \cdots, \pi(i)\}$ and $U(i) = \max\{\pi^{-1}(1), \pi^{-1}(2), \cdots, \pi^{-1}(i)\}$, $i = 1, 2, \cdots, n$. Denote the set of vertices adjacent to i and i itself by $N(i)$. We can observe that the vertex of the maximal label in $N(i)$ is $L(\pi^{-1}(i))$. Moreover, in $N(i)$, the vertex of the most right position on the π sequence is $\pi(U(i))$. With the above concept, we can easily find a tree 3-spanner. Our algorithm is described below, in which notation is the same as that in Sect. 2. Moreover, the execution from Step 3.1 to Step 3.5 is called a *stage*.

Table 1 The values of $\pi(i)$, $\pi^{-1}(i)$, $L(i)$, and $U(i)$.

i	1	2	3	4	5	6	7	8	9	10	11	12
$\pi(i)$	3	6	2	1	8	4	12	10	5	9	7	11
$\pi^{-1}(i)$	4	3	1	6	9	2	11	5	10	8	12	7
$L(i)$	3	6	6	6	8	8	12	12	12	12	12	12
$U(i)$	4	4	4	6	9	9	11	11	11	11	12	12

Algorithm A

Input: A connected permutation graph G with the π sequence

Output: A tree 3-spanner T of G

Step 1. Let $L(i) = \max\{\pi(1), \pi(2), \cdots, \pi(i)\}$ and $U(i) = \max\{\pi^{-1}(1), \pi^{-1}(2), \cdots, \pi^{-1}(i)\}$ for $i = 1, 2, \cdots, n$.

Step 2. Let $i = 1$, $r_i = 1$, $\alpha_0 = 1$, and $\alpha_1 = L(\pi^{-1}(1))$.

Step 3. While $\alpha_i \le n$, do the following substeps.
/* Begin a new stage.*/

Step 3.1. Add edge (r_i, α_i) to T.

Step 3.2. For each vertex v in interval (α_{i-1}, α_i), if v is adjacent to r_i, then add edge (v, r_i) to T; otherwise, add edge (v, α_i) to T.

Step 3.3. If $\alpha_i = n$, then stop Algorithm A.

Step 3.4. Let $\alpha_{i+1} = L(U(\alpha_i))$ and $r_{i+1} = \pi(U(\alpha_i))$.

Step 3.5. Let $i = i + 1$.

We illustrate Algorithm A by the example of Fig. 1. The values of $\pi(i)$, $\pi^{-1}(i)$, $L(i)$, and $U(i)$, $i = 1, 2, \cdots, 12$, are shown in Table 1. In stage 1, $r_1 = 1$ and $\alpha_1 = L(\pi^{-1}(1)) = L(4) = 6$. We add edge (1, 6) to T. Since vertices 2 and 3 are adjacent to r_1, we add edges (2, 1) and (3, 1) to T. Vertices 4 and 5 are not adjacent to r_1, and we add edges (4, 6) and (5, 6) to T. Moreover, $\alpha_2 = L(U(\alpha_1)) = L(U(6)) = L(9) = 12$ and $r_2 = \pi(U(\alpha_1)) = \pi(9) = 5$. In stage 2, we add edge (5, 12) to T. Since vertices 8 and 10 are adjacent to r_2 while vertices 7, 9, and 11 are not, we add edges (8, 5), (10, 5), (7, 12), (9, 12), and (11, 12) to T. Because of $\alpha_2 = 12$, we stop this algorithm. The resulting tree 3-spanner is the same as the tree shown in Fig. 2.

Suppose the input connected permutation graph G needs k stages in executing Algorithm A. To prove the correctness of our algorithm, we have to show that r_{i+1}

and α_{i+1} defined in the algorithm of Madanlal et al. can be substituted by our r_{i+1} and α_{i+1} in each stage i of Algorithm A, $1 \le i \le k-1$.

Lemma 1: For any vertex i, $1 \le i \le n$, $P_{\max}(i) = L(\pi^{-1}(i))$.

Proof: By the definition of $P_{\max}(i)$, we have $\pi^{-1}(P_{\max}(i)) \le \pi^{-1}(i)$. Since $L(\pi^{-1}(i)) = \max\{\pi(1), \pi(2), \cdots, \pi(\pi^{-1}(i))\}$, we obtain $L(\pi^{-1}(i)) = P_{\max}(i)$. Assume to the contrary that $L(\pi^{-1}(i)) = j > P_{\max}(i) \ge i$ for some j. Then, $\pi^{-1}(j) < \pi^{-1}(i)$ and j is adjacent to i. It contradicts the definition of $P_{\max}(i)$ that $P_{\max}(i)$ is the vertex of the maximal label adjacent to i. Therefore, $P_{\max}(i) = L(\pi^{-1}(i))$. □

Lemma 2: Each $\alpha_{i+1}, 1 \le i \le k-1$, defined in the algorithm of Madanlal et al. is equal to $L(U(\alpha_i))$.

Proof: In the algorithm of Madanlal et al., $\alpha_{i+1} = \max_{v \in R_i} P_{\max}(v)$, where $R_i = \{j | j$ is in interval (α_{i-1}, α_i) and j is adjacent to $\alpha_i\}$. Since $P_{\max}(v) = L(\pi^{-1}(v))$, if $\pi^{-1}(u) > \pi^{-1}(v)$ for any two vertices u and v, then $P_{\max}(u) \ge P_{\max}(v)$. Thus, $\alpha_{i+1} = P_{\max}(u)$ for some $u \in R_i$ with $\pi^{-1}(u) = \max\{\pi^{-1}(j) | j \in R_i\}$. Since any vertex in R_i is of smaller label than α_i and is adjacent to α_i, we obtain $\pi^{-1}(u) = \max\{\pi^{-1}(1), \pi^{-1}(2), \cdots, \pi^{-1}(\alpha_i)\} = U(\alpha_i)$ and $\alpha_{i+1} = P_{\max}(u) = L(\pi^{-1}(u)) = L(U(\alpha_i))$. □

Lemma 3: Each r_{i+1}, $1 \le i \le k-1$, defined in the algorithm of Madanlal et al. can be substituted by $\pi(U(\alpha_i))$.

Proof: In their algorithm, they choose r_{i+1} from T_i, where $T_i = \{j | j \in R_i$ and j is adjacent to $\alpha_{i+1}\}$, such that if v is not adjacent to r_{i+1} for some v in interval (α_i, α_{i+1}), then q is not adjacent to v for all $q \in T_i$. In fact, the simplest way to choose r_{i+1} is to choose the vertex u from T_i with $\pi^{-1}(u) = \max\{\pi^{-1}(q) | q \in T_i\}$. The reason is that if u is not adjacent to some vertex v in interval (α_i, α_{i+1}), then for any vertex $q \in T_i$, $q \ne u$, we have $q < v$ and $\pi^{-1}(q) < \pi^{-1}(u) < \pi^{-1}(v)$ and then q is not adjacent to v. Since any vertex of T_i is of smaller label than α_i and is adjacent to α_i, $\pi^{-1}(u) = \max\{\pi^{-1}(1), \pi^{-1}(2), \cdots, \pi^{-1}(\alpha_i)\} = U(\alpha_i)$. Therefore, $u = \pi(\pi^{-1}(u)) = \pi(U(\alpha_i))$. In the algorithm of Madanlal et al., there may be more than one vertex which can be r_{i+1}. However, $u = \pi(U(\alpha_i))$ must be able to be r_{i+1}. We claim that r_{i+1} defined in the algorithm of Madanlal et al. can be substituted by $\pi(U(\alpha_i))$. □

Theorem 4: Algorithm A finds a tree 3-spanner of a permutation graph in $O(n)$ time.

Proof: By the above lemmas, we know that r_{i+1} and α_{i+1} defined in the algorithm of Madanlal et al. can be substituted by our r_{i+1} and α_{i+1} in each stage i of Algorithm A, $1 \le i \le k-1$. For the vertices v in interval (α_{i-1}, α_i), our algorithm uses the same manner of Madanlal et al. to add edge (v, r_i) or (v, α_i) to T. Therefore, the resulting tree found by Algorithm A is a tree 3-spanner of a permutation graph. To compute $L(i)$ and $U(i)$, $i = 1, 2, \cdots, n$, we can scan the π and π^{-1} sequences from left to right and keep each latest maximal label during the scanning. Thus, $L(i)$ and $U(i)$ can be computed in $O(n)$ time. Step 2 can be done in $O(1)$ time, and Step 3 takes $O(n)$ time totally after k stages. The complexity of Algorithm A is therefore $O(n)$. □

4. A Parallel Algorithm for Finding a Tree 3-Spanner

In the algorithm of Madanlal et al., it is not easy to parallelize the computation of r_{i+1} and α_{i+1}, especially on computing r_{i+1} which needs to test vertices in T_i and vertices in interval (α_i, α_{i+1}). Based on Algorithm A, we can easily derive a parallel algorithm for finding a tree 3-spanner of a permutation graph in $O(\log n)$ time with $O(\frac{n}{\log n})$ processors on the EREW PRAM computational model. Instead of executing stages, we find all r_i and α_i at first, then add edges to construct a tree 3-spanner.

Let $s(i) = \pi(U(i))$ and $t(i) = L(U(i))$ for all i, $i = 1, 2, \cdots, n$. We use $t(i)$ as the parent of vertex i, $i = 1, 2, \cdots, n-1$, to construct a tree T_n. Since the input permutation graph is connected, T_n is rooted at vertex n. For instance, all $t(i)$ of Fig. 1, $i = 1, 2, \cdots, n-1$, are 6, 6, 6, 8, 12, 12, 12, 12, 12, 12, 12, respectively. We can use edges $(i, t(i))$ to represent a rooted tree as shown in Fig. 3. In T_n, there is a path from vertex 1 to the root. We call such a path the $(1, n)$-*path*, denoted by 1-v_1-v_2-…-n. In Fig. 3, the $(1, n)$-path is 1-6-12.

Suppose the input permutation graph G needs k stages in executing Algorithm A. Then, we have the following lemmas.

Lemma 5: In the $(1, n)$-path 1-v_1-v_2-…-n, vertices v_1, v_2, $\cdots$, n are equal to $\alpha_1, \alpha_2, \cdots, \alpha_k$ of Algorithm A, respectively.

Proof: Since v_1 is the parent of vertex 1, $v_1 = t(1) = L(U(1)) = \alpha_1$. Similarly, v_2 is the parent of v_1, and $v_2 = t(v_1) = L(U(v_1)) = L(U(\alpha_1)) = \alpha_2$. With the argument, we can conclude that $v_1 = \alpha_1$, $v_2 = \alpha_2, \cdots, n = \alpha_k$. □

Corollary 6: The length of the $(1, n)$-path is equal to

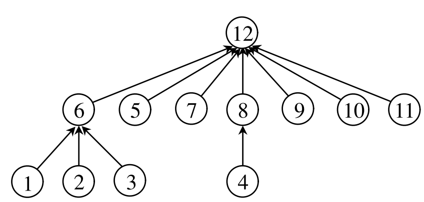

Fig. 3 The rooted tree constructed by edges $(i, t(i))$, $i = 1, 2, \cdots, 11$.

the number of stages in executing Algorithm A.

Lemma 7: In the $(1, n)$-path 1-v_1-v_2-…-v_{k-1}-n, vertices $s(1), s(v_1), s(v_2), \cdots, s(v_{k-1})$ are equal to $r_1, r_2, \cdots, r_k$ of Algorithm A, respectively.

Proof: By Lemma 5, we have $v_1 = \alpha_1$, $v_2 = \alpha_2$, $\cdots$, $v_{k-1} = \alpha_{k-1}$. For vertex 1, $s(1) = \pi(U(1)) = 1 = r_1$. For vertex v_1, $s(v_1) = \pi(U(v_1)) = \pi(U(\alpha_1)) = r_2$. Therefore, we can deduce that $s(1) = r_1$, $s(v_1) = r_2$, $\cdots$, $s(v_{k-1}) = r_k$. □

When all r_i and α_i, $i = 1, 2, \cdots, k$, are obtained, we shall construct a tree 3-spanner by adding edges of G. In Algorithm A, each vertex u in interval (a_{i-1}, a_i) in stage i is adjacent to either r_i or α_i. For parallel processing, we will find the *adjacency pair* $[a_u, b_u]$ of each vertex u, $u = 1, 2, \cdots, n$, such that u is adjacent to either $s(a_u)$ or b_u. The following is our parallel algorithm.

Algorithm B
Input: A connected permutation graph with the π sequence
Output: A tree 3-spanner T
Step 1. Let $L(i) = \max\{\pi(1), \pi(2), \cdots, \pi(i)\}$ and $U(i) = \max\{\pi^{-1}(1), \pi^{-1}(2), \cdots, \pi^{-1}(i)\}$ for $i = 1, 2, \cdots, n$.
Step 2. Let $s(i) = \pi(U(i))$ and $t(i) = L(U(i))$ for $i = 1, 2, \cdots, n$.
Step 3. For $i = 1, 2, \cdots, n-1$, use $t(i)$ as the parent of vertex i to construct a tree T_n rooted at vertex n.
Step 4. Find the $(1, n)$-path 1-v_1-v_2-…-n on T_n. Suppose the length of the $(1, n)$-path is k.
Step 5. Transfer the $(1, n)$-path into intervals $P_1 = [1, v_1]$, $P_2 = [v_1, v_2], \cdots, P_k = [v_{k-1}, n]$.
Step 6. For each vertex $u \le v_1$, let the adjacency pair $[a_u, b_u]$ of u be P_1. For each vertex $u > v_1$, let the adjacency pair $[a_u, b_u]$ of u be some $P_i = [v_{i-1}, v_i]$, $2 \le i \le k$, such that $v_{i-1} < u \le v_i$.
Step 7. For $u = 1, 2, \cdots, n$, if u is adjacent to $s(a_u)$, then add edge $(u, s(a_u))$ to T; otherwise, add edge (u, b_u) to T.

In the example of Fig. 1, the $(1, n)$-path is 1-6-12, and the intervals are [1, 6] and [6, 12]. For vertices 1, 2, 3, 4, 5, and 6, their adjacency pairs are all [1, 6], and these vertices are adjacent to vertex $s(1) = \pi(U(1)) = 1$ or vertex 6. For other vertices, their adjacency pairs are [6, 12], and these vertices are adjacent to vertex 5 or 12, where $s(6) = \pi(U(6)) = 5$. The resulting tree is the same as the tree shown in Fig. 2.

Now, we consider the complexity of Algorithm B. It is trivial that Steps 3, 5, and 7 can be done in $O(\log n)$ time with $O(\frac{n}{\log n})$ processors on the EREW PRAM model. By the prefix maxima computation [6], Step 1 can be done in $O(\log n)$ time with $O(\frac{n}{\log n})$ processors on the EREW PRAM model.

Step 2 needs additional computation to avoid concurrent read and write. Let $g_1 = \pi(U(1))$. For $i = 2, 3, \cdots, n$, let $g_i = 0$ if $U(i) = U(i-1)$ and let $g_i = \pi(U(i))$ otherwise. Then, use the prefix maxima computation to obtain all $s(i) = \max\{g_1, g_2, \cdots, g_i\}$, $i = 1, 2, \cdots, n$. Similarly, we can let $h_1 = L(U(1))$ and obtain all $t(i) = \max\{h_1, h_2, \cdots, h_i\}$ by the same technique, where $h_i = 0$ if $U(i) = U(i-1)$ and $h_i = L(U(i))$ otherwise, $i = 2, 3, \cdots, n$.

In Step 4, since the successor (i.e. the parent) of each vertex is known, we can apply the list ranking technique [6] to find the $(1, n)$-path in $O(\log n)$ time with $O(\frac{n}{\log n})$ processors on the EREW PRAM model.

Step 6 also needs additional computation to avoid concurrent read and write. Let $[x_u, y_u]$ be the *temporary pair* of vertex u. For each vertex u of G, if u is a vertex v_i of the $(1, n)$-path 1-v_1-v_2-…-v_k for some $i, 1 \le i \le k$, then let $[x_u, y_u] = P_i = [v_{i-1}, v_i]$; otherwise, let $[x_u, y_u] = P_k$. We say $[x_i, y_i] < [x_j, y_j]$ if $x_i < x_j$ and $y_i < y_j$. By the suffix minima computation [6], we can obtain all adjacency pairs $[a_u, b_u]$, where $[a_u, b_u] = \min\{[x_u, y_u], [x_{u+1}, y_{u+1}], \cdots, [x_n, y_n]\}$, $u = 1, 2, \cdots, n$. Thus, Step 6 can be done in $O(\log n)$ time with $O(\frac{n}{\log n})$ processors on the EREW PRAM computational model. Table 2 shows how to find $s(i), t(i)$, and the adjacency pair $[a_i, b_i]$ for each vertex i of Fig. 1.

Theorem 8: Algorithm B finds a tree 3-spanner of a permutation graph in $O(\log n)$ time with $O(\frac{n}{\log n})$ processors on the EREW PRAM computational model.

Proof: The complexity of Algorithm B is shown in the above paragraphs. We shall prove that the graph constructed by Algorithm B is the same as the tree constructed by Algorithm A. For each vertex u of G, the adjacency pair of u is $[a_u, b_u] = [v_{i-1}, v_i] = [\alpha_{i-1}, \alpha_i]$ for some i, $1 \le i \le k$ such that $\alpha_{i-1} < u \le \alpha_i$. Consider the following two cases.

Case 1. $u = \alpha_i$. Since α_i is adjacent to r_i and $r_i = \pi(U(\alpha_{i-1})) = \pi(U(v_{i-1})) = s(v_{i-1}) = s(a_u)$, u is adjacent to $s(a_u)$.
Case 2. $\alpha_{i-1} < u < \alpha_i$. Since $r_i = s(a_u)$ and $\alpha_i = b_u$, u is adjacent to $s(a_u)$ or b_u.

Therefore, if u is adjacent to $s(a_u)$, then add edge $(u, s(a_u))$ to T; otherwise, add edge (u, b_u)

Table 2 The process of finding $s(i), t(i)$, and $[a_i, b_i]$ for each vertex i.

i	1	2	3	4	5	6	7	8	9	10	11	12
g_i	1	0	0	4	5	0	7	0	0	0	11	0
$s(i)$	1	1	1	4	5	5	7	7	7	7	11	11
h_i	6	0	0	8	12	0	12	0	0	0	12	0
$t(i)$	6	6	6	8	12	12	12	12	12	12	12	12
x_i	6	6	6	6	6	1	6	6	6	6	6	6
y_i	12	12	12	12	12	6	12	12	12	12	12	12
a_i	1	1	1	1	1	1	6	6	6	6	6	6
b_i	6	6	6	6	6	6	12	12	12	12	12	12

to T. Since all vertices of G are in intervals $[1, v_1], [v_1, v_2], \cdots, [v_{k-1}, n]$ and we add edges for all vertices, the resulting tree T found by Algorithm B is the same as the tree found by Algorithm A. $\square$

5. Concluding Remarks

In this paper, we propose an $O(n)$ algorithm for finding a tree 3-spanner of a permutation graph. Our algorithm can be easily parallelized so that the same problem can be done in $O(\log n)$ time with $O(\frac{n}{\log n})$ processors on the EREW PRAM computational model. Comparing with our algorithm, the algorithm of Madanlal et al. is not efficient and can not be parallelized.

Madanlal et al. have presented an algorithm for finding a tree 3-spanner on interval graphs [7]. Since the classes of interval graphs and permutation graphs are subclasses of trapezoid graphs, there may exist a polynomial time algorithm for finding a tree 3-spanner on trapezoid graphs. It is interesting to study the tree 3-spanner problem on such a type of graphs.

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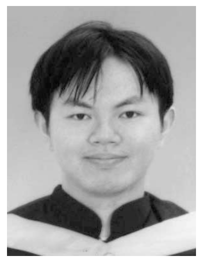

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