Course 02249 Computationally Hard Problems
Fall 2013, DTU Compute

Solution to assignment PROJECT

Students' names: Andreas Hallberg Kjeldsen and Morten Chabert Eskesen

Study nr. s092638 and s133304

Date. 04.11.2013

Not quite adequate	Adequate	Good
☹	😐	☺

Problem: [MIRRORFRIENDLYMINIMUMSPANNINGTREE (MFMST)]
Input: An undirected, connected weighted graph $G = (V, E, w)$, where $V = \{1, \ldots, n\}$, $E = \{e_1, \ldots, e_m\}$ and $w : E \to \mathbb{N}_0$, and a number $B \in \mathbb{N}$.
Output: YES if there is a spanning tree $T \subseteq E$ for G such that

$$max\left\{\sum_{e_i \in T} w(e_i), \sum_{e_i \in T} w(e_{m+1-i})\right\} \leq B$$

and NO otherwise.

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a) Description of the problem in colloquial terms

A minimum spanning tree is a subgraph within a undirected, connected weighted graph that is a tree and connects all the vertices together with a weight less or equal to the weight of every other spanning tree. The main difference between a minimum spanning tree and a mirror friendly minimum spanning tree is the inequality described above. In a mirror friendly minimum spanning tree the inequality must be satisfied. It should be possible to mirror the spanning tree in such a way that the maximum of the spanning tree and the mirrored spanning tree is less than or equal to a fixed value, B. This also means that the mirror friendly minimum spanning tree may not be equal to the minimum spanning tree in the graph, i.e. it may have a larger weight than the minimum spanning tree.

Solve an example problem

Input: $V = \{1, 2, 3\}$, $E = \{e_1 = \{1, 2\}, e_2 = \{2, 3\}, e_3 = \{1, 3\}\}$, $w(e_i) = i$ for $i \in \{1, 2, 3\}$ and $B = 4$.

Spanning Tree	Mirrored Spanning Tree
$e_1 + e_2 = 3$	$e_{3+1-1} + e_{3+1-2} = 5$
$e_3 + e_1 = 4$	$e_{3+1-3} + e_{3+1-1} = 4$

$$max\,\{e_3 + e_1, e_{3+1-3} + e_{3+1-1}\} \leq 4$$

$$max\,\{4, 4\} \leq 4$$

Output would be a spanning tree consisting of the edges: e_3 and e_1.

b) Show that MFMST is in NP

1. Design a deterministic algorithm A which takes as input a problem instance X and random sequence R

Specify what the random sequence R consists of
Let the string R consist of bits: $R = r_1, r_2, \ldots, r_n$.

Specify how A interprets R as a guess
Consider the first m bits. If the i-th bit is 1, mark the edge e_i.

Specify how A verifies the guess
If the marked edges create a mirror friendly minimum spanning tree with a weight less than or equal to B, answer YES, otherwise NO.

2. Show that the two conditions are met

If the answer to X is YES, then there is a string R^* with positive probability such that $A(X, R^*) = YES$

> Asssume that the answer is YES
>
> Then there is a subset of the edges that creates a mirror friendly minimum spanning tree with a weight less than or equal to B.
>
> Let $S \subseteq \{1, \ldots, m\}$ be the set that describe the edges' index
>
> Construct the bit string $R^* = r_1, r_2, \ldots, r_m$ where $r_i = 1$ if and if only if $i \in S$
>
> When A receives R^*, it will select the edges in S, check that the weight of the edges are less than or equal to B and answer YES.
>
> Altogether there is a string of length at most $p(n)$ that will give YES. The probability of randomly creating it is positive.

If the answer to X is NO, then $A(X, R) = NO$ for all R

> Assume that the answer is NO
>
> Then no set of the edges create a mirror friendly minimum spanning tree with a weight less than or equal to B.
>
> If R does not contain enough bits, the algorithm will correctly answer NO.

Otherwise the algorithm will mark some edges and compute their weight. This will be compared to B. But as no set of edges has a weight less than or equal to B, the answer is NO.

3. Show that A is p-bounded for some polynomial p

There are m edges.

It is checked that the string R consists of at least m bits. Time: $O(m)$.

Every edge is marked or not marked. Time: $O(m)$.

The weights of the marked edges are added. Time: $O(m)$.

The computed total weight is compared to B and the answer is returned. Time: $O(1)$.

Altogether the time is: $O(m)$.

c) Show that MFMST is NP-complete

Suitable problem P_c known to be NP-complete

Prove $P_c \leq_p$ MirrorFriendlyMinimumSpanningTree.

Outline of the transformation

Answer to X is YES then answer to $T(X)$ is YES

Answer to $T(X)$ is YES then answer to X is YES

Time Analysis

d) Find an algorithm which solves the optimizing version of the problem

e) Prove the worst-case running time of the algorithm

f) Implement the algorithm developed in d)