Technical University of Denmark

02249 Computationally Hard problems

Project: Mirror Friendly Minimum Spanning Tree

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Not quite adequate	Adequate	Good

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Chapter 1

Introduction

This report will focus on a problem called MIRRORFRIENDLYMINIMUMSPAN-NINGTREE or $MFMST$ for short. We will present the problem in colloquial terms, show that the problem is in NP and that it is NP-complete. We have also present an algorithm for solving the optimization version of the problem, while providing a description of the algorithm and proving it's worst-case running time.

Problem: [MIRRORFRIENDLYMINIMUMSPANNINGTREE (MFMST)]
Input: An undirected, connected weighted graph $G = (V, E, w)$, where $V = \{1, \ldots, n\}$, $E = \{e_1, \ldots, e_m\}$ and $w : E \to \mathbb{N}_0$, and a number $B \in \mathbb{N}$.
Output: YES if there is a spanning tree $T \subseteq E$ for G such that

$$max \left\{ \sum_{e_i \in T} w(e_i), \sum_{e_i \in T} w(e_{m+1-i}) \right\} \leq B$$

and NO otherwise.

Chapter 2

NP Problems

2.1 Description of the problem in colloquial terms

A minimum spanning tree is a subgraph within an undirected, connected weighted graph that is a tree and connects all the vertices together with a weight less or equal to the weight of every other spanning tree. The main difference between a minimum spanning tree and a mirror friendly minimum spanning tree is the inequality described in Chapter 1. In a mirror friendly minimum spanning tree the inequality must be satisfied. It should be possible to mirror the spanning tree in such a way that the maximum of the spanning tree and the mirrored spanning tree is less than or equal to a fixed value, B. This also means that the mirror friendly minimum spanning tree may not be equal to the minimum spanning tree in the graph, i.e. it may have a larger weight than the minimum spanning tree.

2.1.1 Solve an example problem

Input: $V = \{1, 2, 3\}$, $E = \{e_1 = \{1, 2\}, e_2 = \{2, 3\}, e_3 = \{1, 3\}\}$, $w(e_i) = i$ for $i \in \{1, 2, 3\}$ and $B = 4$.

#	Spanning Tree	Mirrored Spanning Tree
1.	$w(e_1) + w(e_2) = 3$	$w(e_{3+1-1}) + w(e_{3+1-2}) = w(e_3) + w(e_2) = 5$
2.	$w(e_3) + w(e_1) = 4$	$w(e_{3+1-3}) + w(e_{3+1-1}) = w(e_3) + w(e_1) = 4$

Shown above are the sum of the weights for two pairs of spanning trees. The first pair has a spanning tree with a weight of 3 and a mirrored spanning tree

with a weight of 5. The maximum of these values would therefore be 5 which is more than B. Therefore the first pair does not satisfy the requirement to be a $MFMST$.

$$max\{3, 5\} \leq B \rightarrow 5 \nleq 4$$

The second pair has a spanning tree with a weight of 4 and a mirrored spanning tree with a weight of 4. The maximum of these values would therefore be 4 which is the same as B. The pair therefore satisfies the requirement to be a $MFMST$.

$$max\{4, 4\} \leq B \rightarrow 4 \leq 4$$

The algorithm would output YES.

2.1.2   Show that MFMST is in NP

MINDER DET HER IKKE MEGET OM SLIDES?

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1. Design a deterministic algorithm A which takes as input a problem instance X and random sequence R

Specify what the random sequence R consists of
Let the string R consist of bits: $R = r_1, r_2, \ldots, r_n$.

Specify how A interprets R as a guess
Consider the first m bits. If the ith bit is 1, mark the edge e_i.

Specify how A verifies the guess
If the marked edges create a mirror friendly minimum spanning tree with a weight less than or equal to B, answer YES, otherwise NO.

2. Show that the two conditions are met

↑ Skal have en anden titel - Måske det skal skrives om til at være med vores eget sprog? Carsten kan jo ikke lide når man er for korrekt.
If true the answer to X is YES, then there is a string R^* with positive probability such that $A(X, R^*) = YES$

Asssume that the answer is YES

Then there is a subset of the edges that creates a mirror friendly minimum spanning tree with a weight less than or equal to B.

Let $S \subseteq \{1, \ldots, m\}$ be the set that describe the edges' index

Construct the bit string $R^* = r_1, r_2, \ldots, r_m$ where $r_i = 1$ if and if only if $i \in S$

When A receives R^*, it will select the edges in S, verify that the edges form a spanning tree and that the weight of the edges are less than or equal to B and answer YES.

Altogether there is a string of length at most $p(n)$ that will give YES. The probability of randomly creating it is positive.

If true the answer to X is NO, then $A(X, R) = NO$ for all R

Assume that the answer is NO

Then no set of the edges create a mirror friendly minimum spanning tree with a weight less than or equal to B.

If R does not contain enough bits, the algorithm will correctly answer NO.

Otherwise the algorithm will mark some edges and compute their weight. This will be compared to B. But as no set of edges has a weight less than or equal to B, the answer is NO.

3. Show that A is p-bounded for some polynomial p

There are m edges.

It is checked that the string R consists of at least m bits. Time: $O(m)$.

Every edge is marked or not marked. Time: $O(m)$.

The weights of the marked edges are added. Time: $O(m)$.

The computed total weight is compared to B and the answer is returned. Time: $O(1)$.

Altogether the time is: $O(m)$.

2.2 Show that MFMST is NP-complete

2.2.1 Suitable problem P_c known to be NP-complete

Select a suitable problem P_c known to be NP-complete

We have chosen to use the problem 1-IN-3-SATISFIABILITY as our problem P_c which is known to be NP-complete.

Problem: [1-IN-3-SATISFIABILITY]
Input: A set of clauses $C = \{c_1, \ldots, c_k\}$ over n boolean variables $x_1, \ldots, x_n$, where every clause contains exactly three literals.
Output: YES if there is a satisfying assignment such that every clause has exactly one true literal, i.e., if there is an assignment

$$a : \{x_1, \ldots, x_n\} \to \{0, 1\}$$

such that every clause c_j is satisfied and no clause has two or three satisfied literals, and NO otherwise.

Prove that MFMST is reducable from problem P_c

We wish to prove

1-IN-3-SATISFIABILITY $\leq_p$ MIRRORFRIENDLYMINIMUMSPANNINGTREE

In order to prove this we use component design. When given an instance (X, C) of 1-IN-3-SATISFIABILITY we construct an instance $(G = (V, E, w), B)$ of MFMST. This is done by building small "component" graphs that are later connected to form the desired graph G. The components will have different "responsibilities". Some ensure a correct setting of the truth values, others which test satisfiability and components to connect them.

Outline of the transformation

Let $X = \{x_1, \ldots, x_n\}$ be the boolean variables and $C = \{c_1, \ldots, c_m\}$ the clauses over X. We set $B = n + 2m$ for the MFMST instance. We set $w(e_i) = i$. The vertex set of G Maaske skal $a()$ defineres lidt foer, jeg sad og

fattede ingenting, samme for $v()$:

$$V = \{x_1, \ldots, x_n\} \cup \{\overline{x}_1, \ldots, \overline{x}_n\} \cup \bigcup_{j=1}^{m} \{a_1(j), a_2(j), a_3(j)\}$$

The truth setting components are defined to be the edges.

$$\forall x_1 \in X \text{ let } T_1 \text{ be the vertex with incoming edges } x_1 \text{ and } \overline{x}_1$$

These components ensure that every spanning tree has to contain at least one of the edges x_1 or $\overline{x}_1$.

The components for clause satisfiability are defined by the following.

$$\forall c_j \in C: \text{ let } S_j \text{ be the set of vertices}$$
$$S_j = \{v_1(j), v_2(j), v_3(j)\}$$

Two edges per triangle are needed to make a spanning tree. The edge not chosen is the *true* literal which satisfies c_j.

The connecting components are defined by the following. Let $c_j = l_1 \vee l_2 \vee l_3$ where l_k are literals. More specifically l_k is a boolean variable x_i or negated boolean variable $\overline{x}_i$. For every clause c_j the following three vertices are added.

$$K_j = \{(v_1(j) \vee l_1), (v_2(j) \vee l_2), (v_3(j) \vee l_3)\}$$

The vertex set V is therefore:

$$V = \bigcup_{i=1}^{n} T_i \cup \bigcup_{j=1}^{m} S_j \cup \bigcup_{j=1}^{m} K_j$$

The transformation T can be performed in time polynomial in n and m.

Answer to X is YES then answer to $T(X)$ is YES

Assume that $a : X \rightarrow \{0, 1\}$ is an assignment which satisfies all clauses c_j. Consider the graph $G = (V, E, w)$ constructed above. We begin to construct $T \subseteq E$ by selecting n edges as follows:

$$x_i \in T \Leftrightarrow a(x_i) = 1$$

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$$\overline{x}_i \in T \Leftrightarrow a(x_i) = 0$$

Then all T_i are covered. For every clause c_j at least one connecting vertex is covered by an l_k (by virtue of some literal in c_j is set to 1). Assume that for clause c_j this is l_1. We add $a_2(j)$ and $a_3(j)$ to T. These two edges cover the other two connecting vertices and the three vertices of the triangle S_j. The set T is a spanning tree and has weight $B = n + 2m$.

Answer to $T(X)$ is YES then answer to X is YES

Lets assume that $T \subseteq E$ is a spanning tree for G with a mirrored spanning tree where both have a weight less than or equal to B. In order to cover the vertices at least one of the edges x_i or $\overline{x}_i$ has to be in T. In order to cover the vertices of S_j, the spanning tree T has to contain at least two of the edges. Therefore $|T| = B = n + 2m$, and we have that T contains exactly one edge from every T_i and exactly two edge from every S_j.
We define an assignment: $a(x_i) = 1$ if $x_i \in T$ and 0 if $\overline{x}_i \in T$
As T contains exactly one of x_i or $\overline{x}_i$ the assignment a is well defined. We still need to show that the assignment satisfies all clauses c_j. Let $c_j = l_1 \vee l_2 \vee l_3$ be a clause where l_k are literals. For every component S_j exactly two edges belong to T, say $\{v_1(j), v_2(j)\}$ and $\{v_2(j), v_3(j)\}$. They cover also the connecting vertices $(v_1(j) \vee l_1)$ and $(v_2(j) \vee l_2)$. The third connecting vertex $(v_3(j) \vee l_3)$ attached to S_j has to be covered by l_3. By the construction of G l_3 corresponds to a literal in c_j and by the construction of the assignment a, l_3 is set to true and clause c_j is then satisfied because only one literal in a clause must be true.

Chapter 3

Algorithm

3.1 Find an algorithm which solves the optimizing version of the problem

There are several algorithms for solving the Minimum Spanning Tree problem. Kruskals, Prims and Chazelles to name a few. We've chosen to use Prims method that uses an *Indexed Minimum Priority Queue*, to keep track of which edge to check next. The algorithm also makes sure not to visit edges twice.

The problem we have to solve isn't solved solely by using Prims algorithm. Prims algorithm only finds the actual MST, however we need to find a mirror friendly MST, therefore we had to make modifications to the algorithm.

Most noticable changes to the algorithm:

Mirror Weight can now be calculated for a given MST.

Search using exclusions of edges to use. We have to exclude some edges to not always find the same MST.

Mark edges as required before trying to find the MST.

3.1.1 Life of the algorithm

Here we will describe the lifecycle of the algorithm.

Birth The algorithm is initialized with G the undirected, connected weighted graph. All required fields are set to be of the minimum required size, which is equal to the amount of vertices in G ($|V|$).

Preschool The algorithm starts with no initial knowledge, so to start off it finds the MST. With the knowledge of the MST at hand, the mirrored weight of the MST can be calculated. Whichever weight is highest is used as the current *threshold* value. The threshold value will never increase. The algorithm uses a priority queue to prioritize which edges to look at, lowest weight first.

Highschool The algorithm now has enough knowledge to start exploring the graph in different ways. To make sure the same MST isn't found again, a set is created with the edges in the MST, but edges that are required to form a spanning tree are removed from the set. A powerset of the remaining edges are created. Each set in the powerset is then used to indicate which edges to exclude when performing a new search. Now try to find a MST using each and every exclusion included in the powerset.

University The algorithm was able to bruteforce a solution, but we want to make it smarter. To enhance the algorithm we have to limit the amount of exclusion sets to check. This is done by using some simple heuristics. If the current exclusion results in a MST not being found then the set of edges in the exclusion should never be tried again, we therefore add the set to a set of sets never to check again. From now on we skip every exclusion set found that is a superset of or equal to any of the sets never to check again. Earlier we specified that the threshold would never increase, but it will in fact decrease. While finding new MSTs using the exclusion sets, the threshold value will be updated if the maximum value of the weight and mirror weight is below the current threshold. The threshold value would be set to the maximum of the weight and the mirror weight, also the MST will be remembered.

Working 9-5 The algorithm still could run better, so instead of overdoing it's workload, it could minimize it even further. When the threshold value is not to be updated, we could check if it is so high that whatever set is the current set to be excluded could be used as a permanent exclusion. In the case that the minimum value of the weight and the

mirror weight is equal to or larger than the current threshold value, we
then choose to add the exclusion to the set of sets never to check again.

Retirement At last the algorithm solves the problem. Though it has aged
quite a bit and with age comes various defects, one of which could be
the ability to sort things properly. Therefore it could be an advantage
to avoid looking through the powerset of exclusions based on weight in
a sorted fashion, but simply pick them in a random order. Even with a
life worth of knowledge it is uncertain which way would be the optimal,
a sort based search or random picking based search.

Termination One may be lucky enough to end their days when their time is
due, others unfortunately terminate prematurely due to various factors,
in this case one factor could be the allowed running time. Either way,
the algorithm will return the spanning tree currently marked as the
best MFMST along with the threshold value.

3.2 Prove the worst-case running time of the algorithm

We've outlined the aspects of our algorithm. Now on to the running time.
The core part of the algorithm is finding the MST within the current graph
excluding the edges specified as the current exclusion.
For every MST exploration the algorithm must:

1. Create three arrays of size $|V|$. One of the arrays must have specified
   all it's values, which is a $O(|V|)$ operation.

2. Empty the priority queue. For each item removed from the queue a
   call to the $Visit$ method is made. Each vertex can be added at most
   once, therefore there will be $|V|$ calls to $Visit$.

3. Make one deletion and one insert/change operation on the priority
   queue per call to $Visit$. The priority queue has $O(log_2|V|)$ insertion,
   change and deletion.

4. Calculate the weight and mirror weight, both of these action take
   $O(|E|)$.

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5. Check the threshold value and maybe update the threshold value or add an exclusion to the never check again set, we claim these actions over the long run executes in constant time $O(1)$.

So to sum up, for every MST exploration the running time will be

$$O(|V| + |V| * (log_2|V|) + |E|)$$

After the initial MST is found, the required edges can be calculated, this takes $O(|E|)$.

Now on to the powerset of exclusions. For any set of size n, the powerset of that set will contain 2^n subsets (including the empty set). So in our worst-case the set of exclusions to try would be of size $2^{|E|}$. For every exclusion in the powerset we have to run the MST exploration. The generation of each subset of the powerset is assumed to be done in constant time $O(1)$.

Putting all this together gives us the following running time:

$$O(2^{|E|} * (|V| + |V| * (log_2|V|) + |E|) + |E|) \Rightarrow O(2^{|E|})$$

Our algorithm has a polynomial bound worst-case running time of $O(2^{|E|})$.

Chapter 4

Implementation

4.1 Implement the algorithm developed in 3.1

We have chosen to make our implementation in C#. We've included our
.NET Solution in the zip.

4.1.1 Running the program

The program can be run in two ways.

1. MFMSTProject.exe [max_execution_time] < input.file

2. MFMSTProject.exe input.file [max_executation_time]

For both, the [max_execution_time] is an optional parameter stating how
many milliseconds the program may use in trying to find the MFMST. If
none is set, the default is 20000 milliseconds as per requirement for the
contest. If no input.file is specified it will look for a file with the following
relative path: $TestFiles/test03.uwg$.

4.1.2   Example run

MFMSTProject.exe < TestFiles/test03.uwg

Time taken: 187 ms
Solution:
Edge 1: {1, 2}; weight: 50, mirrored weight: 51
Edge 2: {2, 3}; weight: 88, mirrored weight: 13
Edge 3: {3, 4}; weight: 66, mirrored weight: 10
Edge 5: {5, 6}; weight: 77, mirrored weight: 14
Edge 6: {6, 7}; weight: 3, mirrored weight: 52
Edge 7: {7, 8}; weight: 7, mirrored weight: 61
Edge 8: {8, 9}; weight: 18, mirrored weight: 38
Edge 9: {9, 10}; weight: 84, mirrored weight: 69
Edge 10: {10, 11}; weight: 45, mirrored weight: 61
Edge 13: {13, 14}; weight: 53, mirrored weight: 75
Edge 14: {14, 15}; weight: 5, mirrored weight: 4
Edge 15: {15, 16}; weight: 17, mirrored weight: 20
Edge 16: {16, 17}; weight: 73, mirrored weight: 57
Edge 17: {17, 18}; weight: 17, mirrored weight: 33
Edge 18: {18, 19}; weight: 57, mirrored weight: 7
Edge 19: {19, 20}; weight: 7, mirrored weight: 49
Edge 20: {20, 21}; weight: 49, mirrored weight: 7
Edge 21: {21, 22}; weight: 7, mirrored weight: 57
Edge 22: {22, 23}; weight: 33, mirrored weight: 17
Edge 24: {24, 25}; weight: 20, mirrored weight: 17
Edge 25: {25, 26}; weight: 4, mirrored weight: 5
Edge 29: {29, 30}; weight: 61, mirrored weight: 45
Edge 31: {2, 10}; weight: 38, mirrored weight: 18
Edge 32: {4, 12}; weight: 61, mirrored weight: 7
Edge 33: {5, 13}; weight: 52, mirrored weight: 3
Edge 34: {18, 25}; weight: 14, mirrored weight: 77
Edge 36: {20, 27}; weight: 10, mirrored weight: 66
Edge 37: {21, 28}; weight: 13, mirrored weight: 88
Edge 38: {22, 29}; weight: 51, mirrored weight: 50

Solution value: 1080 / 1071

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