Johnson Noise in Resistors at Room Temperature

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The existence of noise consistent with Johnson noise is demonstrated in resistors at room temperature. Frequency dependent amplifier gain is discussed and calculated. Also, Johnson noise RMS voltage and experimental Boltzmann constant are calculated using Johnson and Nyquist's theory after separating amplifier noise from measurements. An experimental value of k_b is calculated to be $1.42 \cdot 10^{-23} \pm 7 \cdot 10^{-25}$, verifying the method's accuracy for measuring Johnson noise. We suggest constant resistance variable temperature experiments as an alternative to measuring Johnson noise.

Conductors will be ubiquitous in a modern society; they are at the heart of all electronic devices, enabling technological advances in transportation, computation, communication, and so forth. Hence, it is clear that a study of these materials in detail is certainly worth the time and effort. To that end, Johnson and Nyquist investigated, experimentally and theoretically respectively, the implications of thermal agitations in a conductor, specifically a real wire with resistance. They found there is a random induced voltage fluctuation which can be characterized by its power and its temperature. For brevity, Johnson and Nyquist derived equation (1) by calculating the noise power per standing wave mode for electromagnetic waves along a wire by means of the equipartition theorem [1, 2]. As the discovery is accredited to Johnson, this type of noise is dubbed Johnson Noise (JN), sometimes it is also called Johnson-Nyquist noise.

$$P/\Delta f = V_n^2/(R\Delta f) = 4k_bT \tag{1}$$

Where P is the dissipated power, V_n is the RMS voltage of the noise, R is the resistance of the conductor, k_b is Boltzmann's constant, T is the temperature in Kelvin, and Δf is the frequency bandwidth. Since $4k_bT$ is independent of Δf, choosing $\Delta f = 1[Hz]$ means the noise power spectral density for all resistances is a constant $B = 4k_bT$. This is a useful way to determine the accuracy of a measurement of Johnson noise.

Johnson noise and JN theory have numerous applications that are not obvious at first. Consider thermometry and secure key exchange systems; Qu et al. describe applications in measuring temperatures between $1mK$ and $800K$ with potential industrial uses which they suggest could reach up to $2000K$ measurements[3], and Kish presents a use of the Johnson Noise Scheme to enhanced secure key exchange systems [4]. Additionally, Johnson noise thermometry has been used to calibrate platinum resistance thermometers in nuclear power plants at 99% confidence over a temperature range of 273 to 1000 K [5]. Moreover, Johnson noise places a fundamental limit on the resolution of amplifiers as a signal will be indistinguishable from noise when the signal's maximum amplitude is less than that of the noise's.

Considering the order of magnitude of the noise power per hertz at room temperature, which is ~10^{-20}, an amplifier is essential to be able to make direct measurements. However, introducing an amplifier also introduces extra noise into the measurement that must be accounted for. The noise that an amplifier produces can be effectively represented by an ideal amplifier (with zero noise) and two noise sources that are amplified, those being a voltage noise and current noise. Additionally, the gain of such an amplifier does not necessarily have a constant frequency response and hence must be taken into consideration. Since noise amplitudes add by the squares of the RMS voltages, the total recorded voltage is given by equation (2).

$$V_{tot}^2 = G(f)^2 \left[V_{amp}^2 + (I_{amp}R)^2 + V_{JN}^2\right] \tag{2}$$

Where $G(f)$ is the frequency dependent gain of the amplifier, V_{tot} is the RMS voltage that is read by the sound card, V_{amp} is the noise RMS voltage from the amplifier, and $I_{amp}R$ is the product of the amplifier current noise and the total circuit resistance before the amplifier R. Hence, to calculate V_{JN}, which is the Johnson noise RMS voltage, the other three noise voltages are calculated and $G(f)$ is characterized. To accomplish this, a sound card connected to a computer was utilized, recording with a sampling frequency of 48kHz. However, to convert recorded amplitude on the computer back to a voltage, the sound card must be calibrated by feeding it signals of known voltages and analyzing them. The parameters that are determined in the calibration are the calibration factor C, and the input impedance L. By doing a plot and curve fit for these generated signals the parameters were determined to be $C = 9.412$ for the calibration factor, and $L = 1.9959\ k\Omega$ for the input impedance.

The frequency response of the amplifier was determined in a similar method to the calibration of the sound card. That is, sinusoidal signals with fixed amplitude were generated in audacity and sent to the amplifier. To prevent the sound card input from being saturated, the output signal was passed through a $1:1000$ voltage divider before being amplified and returned to the sound

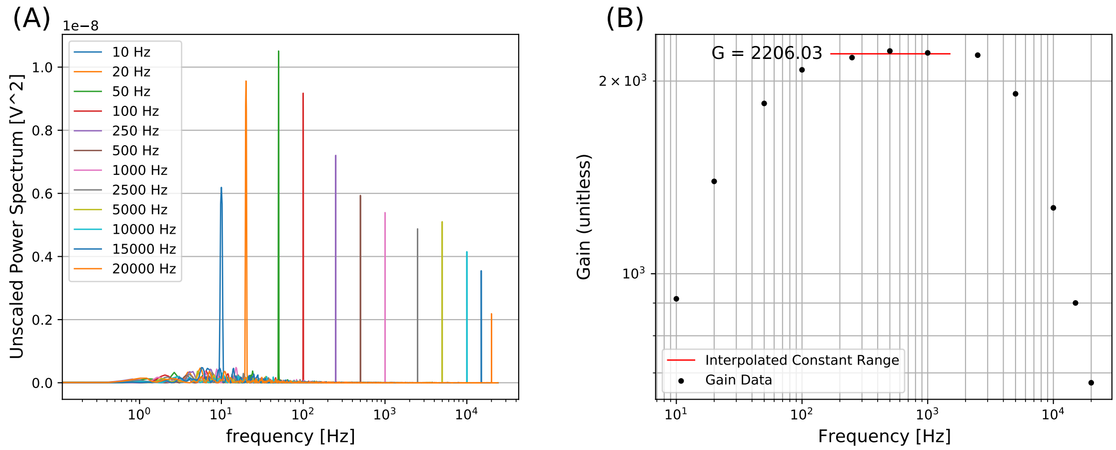

FIG. 1. **(A)** Plot of overlaid power spectra for the unamplified recorded signals. Note the total voltage of each spike is compared to the corresponding spike for the amplified signals to determine the gain at each frequency. **(B)** Amplifier frequency dependent gain for the 0Hz–20kHz range. Gain is approximated to be $G_0 = 2206.03$ in the 170Hz–1500Hz range by a least squares fit as shown by the red line.

card to be recorded. Exporting the recorded noise as wave files from audacity, the power spectra were taken and the total power of the frequency spike was taken. Figure 1(A) shows the overlayed power spectra for the unamplified files. To determine the voltage of each signal, the square root of the sum of the contributions to the signal spike was take. Then, dividing the amplified total voltage by the unamplified total voltage for each frequency, Figure 1(B) was obtained. To simplify the computations for Johnson noise, a region where the gain is approximately constant was chosen and the function $G(f)$ was interpolated by a constant least squares fit to be $G_0 = 2206.03$. It is worth noting that the $5kHz$, $10kHz$, and $15kHz$ signals have higher signal spikes than the trend of the other peaks may suggest they should have. However, as the gain is dependent on both the unamplified and amplified signals, it is inconsequential as long as the amplified $5kHz$, $10kHz$, and $15kHz$ signals are produced from corresponding output from the sound card that were used to plot the unamplified spectra.

With the sound card and amplifier characterized, at least in a frequency band, meaningful data can be extracted from noise measurements. Shielding a resistor from ambient noise, as there are plenty of electromagnetic fields that may influence the data otherwise, by placing it in a copper tube, one side is connected to a ground while the other to the input of the amplifier. The output of the amplifier is connected to the sound card which records the total noise into audacity, and is thence exported as a wave file to be analyzed in python. Repeating this process for resistances of 4.7Ω, 47Ω, 470Ω, $2.35k\Omega$, $4.7k\Omega$, $9.4k\Omega$, $23.5k\Omega$, $47k\Omega$, $94k\Omega$, $235k\Omega$, $470k\Omega$, $940k\Omega$, and $1.439M\Omega$, thirteen separate wave files were recorded. Taking the windowed spectral-density-scaled one-sided FFT for each of the wave files gives Figure 2(A). Scaling to get the noise power spectral density, following equation (3), for the pre-amplifier total signal in the constant gain region, Figure 2(B) is produced. It is key to notice that the four lowest resistance values have noise power spectral densities significantly larger than the other nine, as seen in Figure 2(B); indeed these noise power spectral densities are noticeably larger than the approximate order of magnitude 10^{-20} predicted by Johnson and Nyquist's theory. Therefore, these four data sets are not used in further calculation as they have significant noise power contributions from sources other than Johnson noise. The remaining collection of nine sets of data will be used to determine JN voltage and the experimental value of k_b.

$$PSD = \frac{ASD}{R \cdot (C \cdot G_0)^2} \tag{3}$$

Where PSD is the noise power spectral density, ASD is the amplitude spectral density as shown in Figure 2(A), R is the resistor value, C is the calibration factor, and G_0 is the constant gain.

Recall according to Johnson and Nyquist's theory, the Johnson noise power spectral density is $B = 4k_bT$. Hence, the time has come to address the noise from the amplifier. The amplifier voltage noise can be easily addressed by taking $R = 0$, as both Johnson noise and amplifier current noise should disappear. Ideally, a true $R = 0$ wire would be used, however superconductors were not readily available to use, so a solid core hookup wire must suffice. Calculating the RMS voltage per square root Hertz of this noise source in the constant gain region by taking the square root of the quotient of the sum of the squares of the ASD with the product of the number of points and the calibration factor times the constant gain squared, gives $V_{amp}/\sqrt{Hz} = 3.873 \cdot 10^{-9}$.

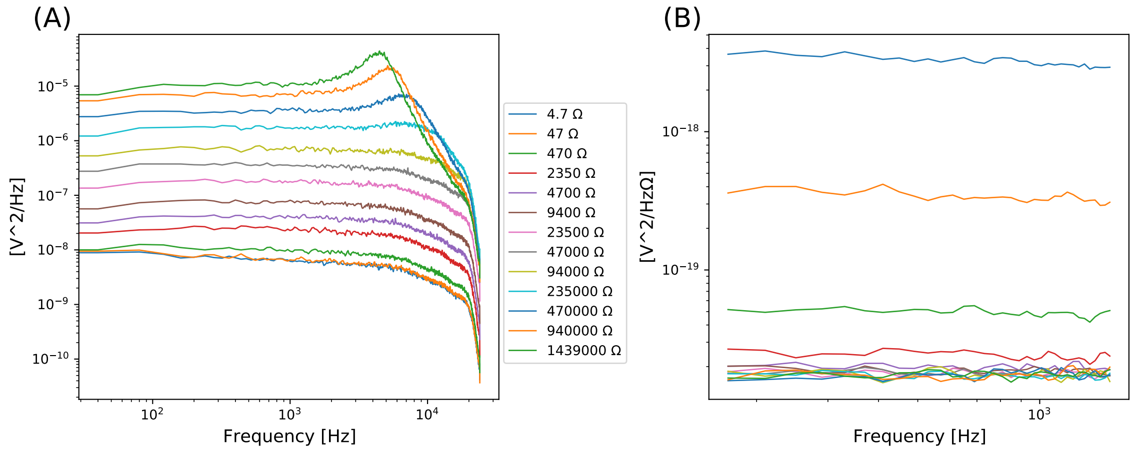

FIG. 2. **(A)** Plot of the windowed spectral-density-scaled one-sided FFT outputs for each total noise signal recorded – one for each resistance. Note that the amplitude generally increases as the resistance increases. **(B)** Plot of the total noise power per Hertz before the amplifier taken by scaling the outputs from (A) inside the constant gain region. The four smallest resistances are clearly separated from the rest, indicating a significant extraneous noise contribution to these signals.

The amplifier current noise was characterized by Dan Reed. Applying cross-correlation analysis to the Johnson noise wave files, it was determined that the current is of magnitude $1 \cdot 10^{-13}$. Furthermore, introducing a curve fit gave the value of $I_{amp} = 1.12 \cdot 10^{-13}$. Here, the aforementioned value is used to account for current noise. By rearranging equation (2), the Johnson noise per square root Hertz can be calculated. Hence, the external noise corrected noise power spectral density can be plotted to allow for a calculation of an experimental Boltzmann constant. Doing this plot, as shown in Figure (3), and applying a least squares constant fit to the nine reasonable data points, the constant noise power spectral density is determined to be $B = 1.714 \cdot 10^{-20}$. Then, as the measurements were taken in a warm room, with $T \approx 300K$ – it was an oversight to not record the temperature of the resistors nor the temperature of the room – k_b is calculated to have an experimental value of $1.42 \cdot 10^{-23}$. Comparing to the literature value of Boltzmann constant, $1.3806 \cdot 10^{-23}$, the experimental value of k_b is reasonably close.

Although the precise error calculations have escaped consideration, it is justifiable to place an uncertainty of 5% on the experimental k_b. This is reasonable as the contributions to the experimental k_b uncertainty are from temperature uncertainty, and B uncertainty. Clearly, estimating the temperature within 3 degrees is not difficult in a thermostat controlled house. Then, as uncertainties add in quadrature, the remaining uncertainty of 4.9% comes from B. This approximately means that each component in equation (3) can have up to a 2% uncertainty. In R the tolerances used were 1% or less. For C and G_0 the fits matched extremely well with the data, and the recorded data was taken accurately. Taking a 5% error means the literature value of k_b is within the uncertainty of $\pm\, 7 \cdot 10^{-25}$, suggesting the method is sufficiently accurate to conclude Johnson noise indeed exists in resistors at room temperature.

It would perhaps be worthwhile to conduct a similar experiment, although with a single resistance value, and vary the temperature instead to investigate Johnson noise. This is the premise of Johnson noise thermometry and may be easier to perform data analysis on.

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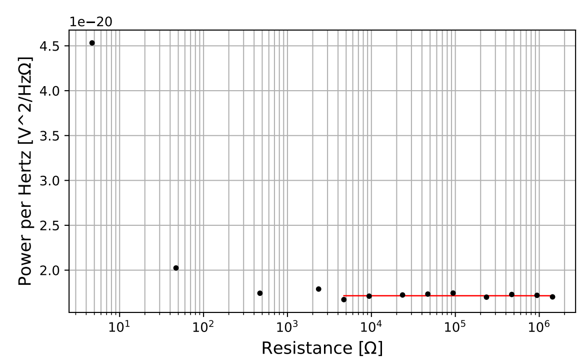

FIG. 3. Total Power per hertz plotted against resistance with a constant fit by least squares to the data, excluding the smallest 4 resistances. The constant, $B = 1.714 \cdot 10^{-20}$, should be equal to $4k_bT$ according to Johnson.

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