Numerical Solutions for Ideal MHD

On the essential ingredients of the method

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MATH 521 Project Presentation

The Ideal MHD Equations

Can be written as conservations of mass, momentum, and energy

$$\partial_t \rho + \boldsymbol{\nabla} \cdot [\rho \boldsymbol{u}] = 0$$
$$\partial_t (\rho \boldsymbol{u}) + \boldsymbol{\nabla} \cdot [\rho \boldsymbol{u}\boldsymbol{u}^T + p_{\text{tot}} \mathbb{I} - \boldsymbol{B}\boldsymbol{B}^T] = 0$$
$$\partial_t E + \boldsymbol{\nabla} \cdot [(E + p_{\text{tot}})\boldsymbol{u} - (\boldsymbol{u} \cdot \boldsymbol{B})\boldsymbol{B}] = 0$$

and an evolution of the magnetic field

$$\partial_t \boldsymbol{B} + \boldsymbol{\nabla} \cdot [\boldsymbol{u}\boldsymbol{B}^T - \boldsymbol{B}\boldsymbol{u}^T] = 0$$

with $p_{\text{tot}} = p + \frac{1}{2}|\boldsymbol{B}|^2$ and $E = \frac{1}{2}\rho|\boldsymbol{u}|^2 + \frac{1}{2}|\boldsymbol{B}|^2 + \frac{p}{\gamma-1}$ where γ is the ratio of specific heats.

The Entire Interest

Consider writing a compact form of Ideal MHD with $\boldsymbol{U} = (\rho, \boldsymbol{u}, \boldsymbol{B}, E)$

$$\partial_t \boldsymbol{U} + \boldsymbol{\nabla} \cdot \boldsymbol{F}(\boldsymbol{U}) = 0$$

and $\boldsymbol{F}(\boldsymbol{U})$ the appropriate flux functions.

What are the essentials for simulating these equations?

Discretization & Constraints

Discretization

- Space $\longrightarrow$ Discontinuous Galerkin with Numerical Flux
- Time $\longrightarrow$ Runge-Kutta / Linear Multistep

Constraint

- 1D $\longrightarrow$ Constant B_x
- Multi-D $\longrightarrow$ $\nabla \cdot \boldsymbol{B} = 0$ $\longrightarrow$ Projection Method

1D 1st Order Discontinuous Galerkin Method

Consider 1-dimensional Ω so that $\Omega = \cup_j I_j$

Find $U \in V_h^k$ such that

$$\frac{\partial}{\partial t}\int_{\mathrm{I_j}} U v_h \, dx - \int_{I_j} f(U)\frac{\partial}{\partial x} v_h \, dx + f_{j+\frac{1}{2}} v^{-}_{h,j+\frac{1}{2}} - f_{j-\frac{1}{2}} v^{+}_{h,j-\frac{1}{2}} = 0$$

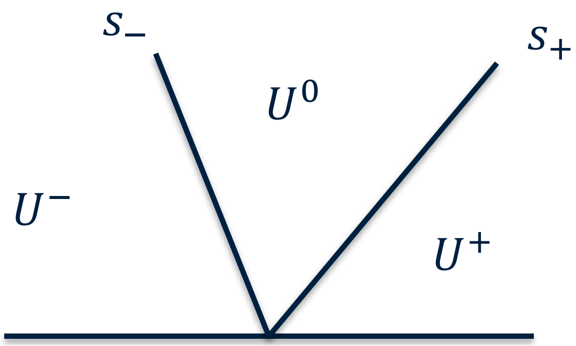
for all $v_h \in V_h^k$ and for all j.

$$\frac{\partial}{\partial t} U_j^h = \frac{1}{\Delta x_j}\left(f_{j-\frac{1}{2}} - f_{j+\frac{1}{2}}\right)$$

HLL Numerical Flux

- Single Intermediate State Riemann Solver
- $s_{\pm}$ from eigenvalues of flux Jacobian

$$A_p = \begin{bmatrix} u_1 & \rho & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & u_1 & 0 & 0 & \frac{-B_1}{\rho} & \frac{B_2}{\rho} & \frac{B_3}{\rho} & \frac{1}{\rho} \\ 0 & 0 & u_1 & 0 & \frac{-B_2}{\rho} & \frac{-B_1}{\rho} & 0 & 0 \\ 0 & 0 & 0 & u_1 & \frac{-B_3}{\rho} & 0 & \frac{-B_1}{\rho} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & B_2 & -B_1 & 0 & -u_2 & u_1 & 0 & 0 \\ 0 & B_3 & 0 & -B_1 & -u_3 & 0 & u_1 & 0 \\ 0 & \gamma p & 0 & 0 & \bar{\gamma}\mathbf{u}\cdot\mathbf{B} & 0 & 0 & u_1 \end{bmatrix}$$



$$\tilde{f}(U^-, U^+) = \frac{s_+ f_- - s_- f_+ + s_+ s_- (U^+ - U^-)}{s_+ - s_-}$$

2nd Order Adams-Bashforth

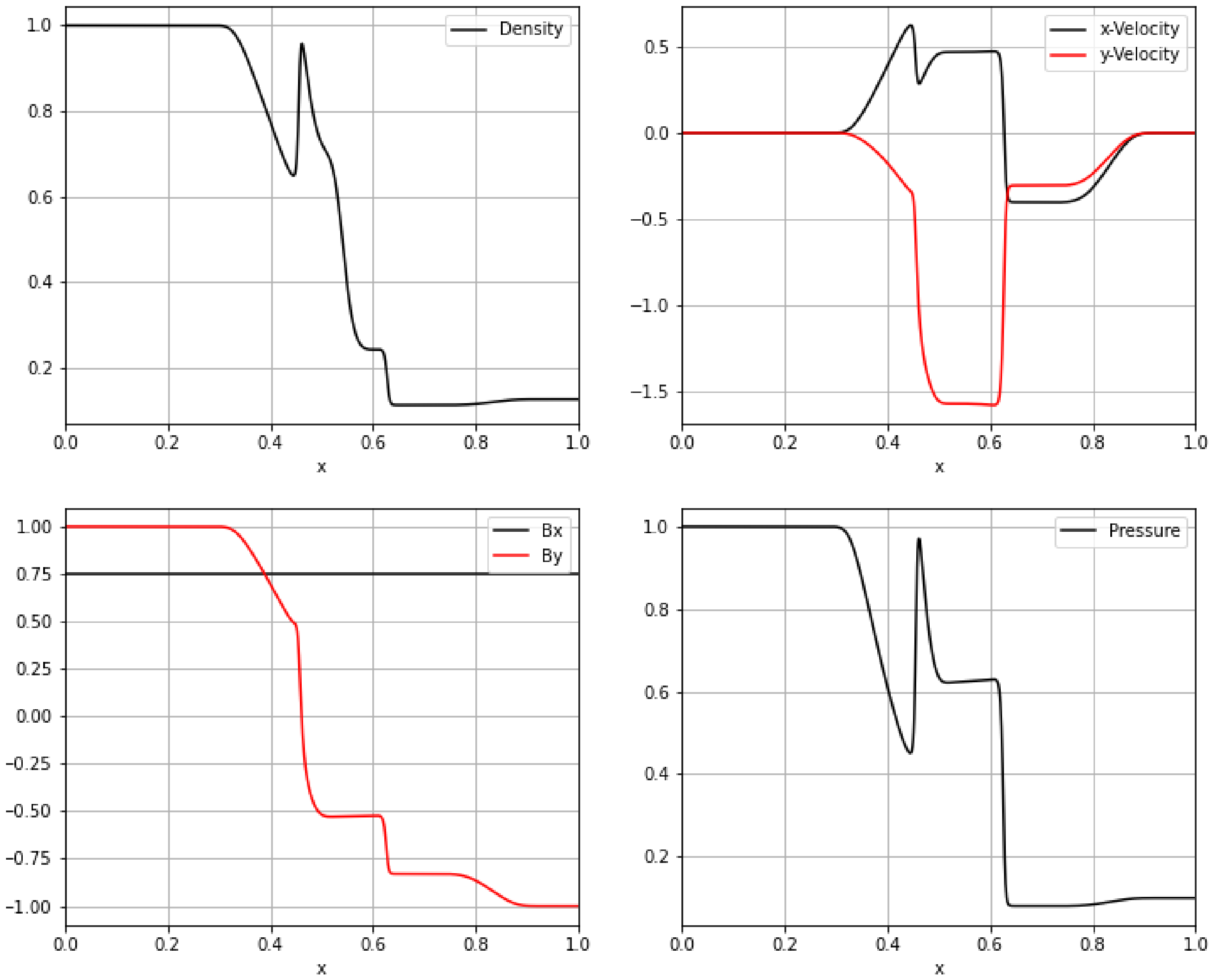
- Initialized with single FE step

$$U^{n+1} = U^n + \frac{\Delta t}{2}(F^n - F^{n-1})$$

$$U^{n+1} = U^n + \Delta t F^n$$

- CFL Condition

$$\Delta t \leq \frac{\Delta x}{s_{max}}, \qquad s_{max} = \max_j\left(\max(|\lambda_-(U_j)|, |\lambda_+(U_j)|) \right)$$


Density
x-Velocity
y-Velocity
Bx
By
Pressure
x

Questions?