Brief Summary of Kerr Spacetimes

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Motivated by the extensive number of papers that consider the Kerr spacetime at length, and lack thereof concise descriptions, a brief summary of the properties of the spacetime and its causal structure is presented. Differences to the Schwarzschild solution, such as the creation of an ergosphere, the ring-like singularity, and regions with CTCs, are given. Some applications of the solution are discussed, and topics of ongoing research over fifty years after Kerr's paper are put forth.

I. INTRODUCTION

The Schwarzschild solution to the Einstein equation has provided students of general relativity with useful insight into the structure of curved spacetime due to a spherically symmetric mass. Previous works have discussed in length the causal structure that is revealed as the size of a mass, suppose a collapsing star, decreases below $r = 2M$ and down to the $r = 0$ limit; the singularities for the Schwarzschild metric in (t, r, θ, ϕ) coordinates then must be considered. The nature of these singularities have been resolved, namely through the use of Kruskal-Szekeres coordinates and their extension. Although the Schwarzschild spacetime gives a deeper understanding of reality, it is indeed limited to the spherically symmetric case, limiting the practical use of the Schwarzschild metric in studying astrophysical black holes.

Realistically, it cannot be expected that stellar collapse is spherically symmetric. Rather, the non-spherical collapse is often attacked by means of numerically simulating the Einstein equation, often resulting in an axisymmetric spacetimes [1]. In fact, as orbiting matter falls into the black hole, its angular momentum is transferred to the black hole, resulting in near maximally rotating black holes in many astrophysics situations. For such a charge-neutral rotating black hole, Kerr spacetime and the Kerr metric fit the bill for characterizing the situation. The behavior of geodesics in this spacetime have been considered by numerous works previously, take [1], [3], [4], [5], for example. This paper relies on the work of Hartle and Hawking and Ellis to present a brief summary of Kerr spacetime's properties, causal structure, and interesting features. Some questions open to research are given in the conclusion whose answer's have evaded physicists so far.

II. THE KERR SPACETIME

The spacetime for a rotating black hole is described by the Kerr Metric. Such a black hole has mass M and an angular momentum J, note that the coordinate axis for ϕ is aligned with the angular momentum. Using geometric units, that is $c = 1$, $G = 1$, and *Boyer-Lindquist coordinates* – the equivalent to Schwarzschild coordinates for Schwarzschild spacetime – the metric can be written as

$$\begin{aligned} \mathrm{d}s^2 \; = \; & -\left(1 - \tfrac{2Mr}{\rho^2}\right)\mathrm{d}t^2 - \tfrac{4Mar\sin^2\theta}{\rho^2}\mathrm{d}\phi\mathrm{d}t + \tfrac{\rho^2}{\Delta}\mathrm{d}r^2 \\ & + \rho^2\mathrm{d}\theta^2 + \left(r^2 + a^2 + \tfrac{2Mra^2\sin^2\theta}{\rho^2}\right)\sin^2\theta\mathrm{d}\phi^2 \end{aligned} \tag{1}$$

where the quantities a, ρ, and Δ are defined as

$$a \equiv J/M \quad \rho^2 \equiv r^2 + a^2\cos^2\theta \quad \Delta \equiv r^2 - 2Mr + a^2 \tag{2}$$

Now that there is a formalism of the geometry, it is worth presenting the interesting properties of Kerr spacetime in general.

Firstly, note that the metric in the limit as r tends to infinity recovers the Minkowski metric in polar form, explicitly $\mathrm{d}s^2 = -\mathrm{d}t^2 + \mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2)$. That is to say, the Kerr metric is *asymptotically flat*.

Next, consider the Killing vectors of the metric. Since the metric is independent of t, and ϕ, the killing vectors are $\xi^\alpha = (1, 0, 0, 0)$, and $\eta^\alpha = (0, 0, 0, 1)$. It is also noteworthy that the metric is invariant under a reflection across the $\theta = \pi/2$ plane. These symmetries are labeled *Stationary* for t independence, and *Axisymmetric* for ϕ independence. Also, these properties of the Kerr spacetime are to be expected of a rotating object, although it is interesting to consider that it is in fact spacetime that is rotating rather than a rotating object situated in a static spacetime.

The parameter a is named the *Kerr Parameter* and ranges between 0 and M, and effectively describes the rate the black hole is rotating at. As such, it is expected that the Schwarzschild metric should be recovered when there is no rotation. Indeed, the $a = 0$ case reduces the Kerr metric to the Schwarzschild metric as expected.

Taking another look at the Kerr metric, specifically on the hunt for singularities, when $\rho = 0$ or $\Delta = 0$, the metric is indeed singular. For the $\Delta = 0$ singularities, solving the given definition in (2) – assuming $a \leq M$ – gives the radii

$$r_\pm = M \pm \sqrt{M^2 - a^2} \tag{3}$$

These radii are coordinate singularities just as the singularity in the Schwarzschild metric at $r = 2M$. This can be shown by transforming to generalized Eddington-Finkelstein coordinates, where the metric is no longer singular at $r = r_\pm$. Plotting the lightcones on a v, r graph, it is apparent timelike world lines travel in decreasing r or at best at constant $r = r_\pm$. Hence, the outer radius, r_+ is the event horizon, analogous to $r = 2M$ for the Schwarzschild solution. Moreover, the causal structure of the regions separated by the $r_\pm$ radii is discussed in more detail later. Nonetheless, the (perhaps more interesting) singularity at $\rho = 0$ occurs when $r = 0$ and $\theta = \pi/2$. It is indeed a real singularity in spacetime and deserves a more detailed investigation.

Perhaps the easiest way to understand the nature of the $r = 0$ singularity is to make proper use of the Kerr-Schild coordinates. To make the transformation, the following equations are applied.

$$x + iy = (r + ia)\sin\theta \exp i \int (\mathrm{d}\phi + a\Delta^{-1}\mathrm{d}r),$$

$$z = r\cos\theta, \quad \bar{t} = \int (\mathrm{d}t + (r^2 + a^2)\Delta^{-1}\mathrm{d}r) - r \quad (4)$$

In these coordinates, the Kerr metric is written as

$$\begin{aligned} \mathrm{d}s^2 \; = \; & -\mathrm{d}\bar{t}^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \\ & + \tfrac{2Mr^3}{r^4 + a^2 z^2}\left[\mathrm{d}\bar{t} + \tfrac{r(x\mathrm{d}x + y\mathrm{d}y)}{a^2 + r^2} + \tfrac{a(y\mathrm{d}x - x\mathrm{d}y)}{a^2 + r^2} + \tfrac{z\mathrm{d}z}{r}\right]^2 \end{aligned} \quad (5)$$

which is subject to the constraining equation $r(x, y, z)$, where r is simply a value rather than a coordinate now, given implicitly is

$$x^2 + y^2 + z^2 = r^2 + a^2\left[1 - \frac{z^2}{r^2}\right] \quad (6)$$

Applying the boundary of the real singularity, that is $r = 0$, $z = 0$ to (6), in Kerr-Schild coordinates, the singularity is at $x^2 + y^2 = a^2$, which is a ring rather than a point. For the maximal analytic extension of the Kerr solution, a combination of Kerr coordinates (r, θ, ϕ_+, u_+) and their extension $(r, \theta, -\phi_-, u_-)$ is used. That is, using Kerr coordinates, (1) can be extended to a space with $r < 0$ that is accessed by 'passing' through $r = 0$ at some $\theta \neq \pi/2$. Unlike the the maximally extended Schwarzschild solution where the extra region cannot be traveled to, the region inside the singularity ring *can* be traversed.

Another such difference from the Schwarzschild spacetime is apparent in the region $r > r_+$. For a static observer – that is constant holding r, ϕ, and θ, components of their world line constant with time – in Schwarzschild spacetime, they are free to be anywhere outside of the black hole horizon. However, investigating the condition $u_{obs} \cdot u_{obs} = -1$ for all values of r, it is found that for observers sufficiently close to the horizon, the condition can not be satisfied at all. Specifically, the boundary is

$$r = r_e(\theta) = M + \sqrt{M^2 - a^2 cos^2\theta} \quad (7)$$

The interior of this surface is called the *Ergosphere*; it is the region for which it is impossible for an observer to remain stationary with respect to infinity. Evidently $r_e \geq r_+$ for all θ given that $a \neq 0$, so effects from the ergosphere are indeed visible and not hidden by the event horizon and will be discussed in the following section.

III. CAUSAL STRUCTURE

In this section, the discussion of the causal structure is limited to the $\theta = \pi/2$ plane. Generally, this plane of Kerr spacetime can be divided into four regions according to the value of r. That is, **I** $(r > 2M)$, **II** $(r_+ < r < 2M)$, **III** $(r_- < r < r_+)$, and **IV** $(r < r_-)$. Note that $r = 2M$ is indeed r_e evaluated for $\theta = \pi/2$, the boundary of the ergosphere.

Consider region **I**; the properties to note are that timelike trajectories can travel in both increasing and decreasing r, and can orbit the Kerr black hole in both prograde and retrograde orbits – i.e. in the direction of rotation of the black hole and opposite it. The inner boundary of region **I** is called the *Stationary limit*, and in the plane is the ring at which timelike trajectories can at best remain apparently stationary to an observer at infinity. Informally, at and inside the boundary $r = 2M$, all timelike trajectories orbit the black hole in the same direction of rotation.

Region **II** presents an interesting class of trajectories that have surprising applications, for example these trajectories can be used to extract energy from black holes. However, this topic deserves its own paper. Characteristically, timelike and null trajectories can still increase and decrease in r, but must now appear to have prograde orbits about the black hole. Consequentially, phenomena such as gravitational lensing, which for the non-rotating

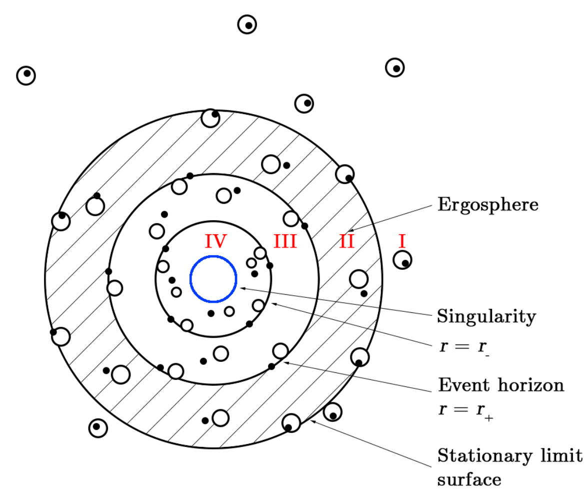

FIG. 1. The $\theta = \pi/2$ plane of the Kerr spacetime for a non-maximally spinning black hole $(a < M)$ rotating counter-clockwise as viewed from $\theta = 0$. The dots are flashes of light, while the closest circle to each dot represents the position of the flash a short time later. This is shown as Fig. 30 in [3].

case gave a deflection angle as a function of the impact parameter b, now has a ϕ dependence, meaning radial symmetry of deflection is lost. Also, inside region **II**, all null trajectories have prograde orbits; as a note on world lines outside of the $\theta = \pi/2$ plane, timelike trajectories without precession are impossible inside this region.

The boundary between regions **II** and **III**, that is the radius $r = r_+ = M + \sqrt{M^2 - a^2}$, is the event horizon. On this ring, null trajectories can at most remain at $r = r_+$ or decrease in radius, as timelike trajectories do. It is worth noting that this is indeed the 'point of no return', if any object, including light, crosses r_+, it cannot return. Once inside region **III**, null and timelike trajectories must follow paths of decreasing r until r_-. At this boundary, null trajectories can again remain at a fixed radius. r_- has the same properties as r_+, however, as once a particle has crossed r_+, it cannot cross back, r_- is not considered an event horizon, rather just a point of interest or inner horizon.

Finally, region **IV** contains the ring singularity at $r = 0$ and the extension to $r < 0$. Note that the singularity is only for $\theta = \pi/2$ and traveling towards $r = 0$ with any other θ coordinate will not pass through the singularity, rather the trajectory passes to the $r < 0$ extension of the Kerr solution. It is critical to observe that this $r < 0$ region only exists in the pure Kerr spacetime; that is when the metric is perturbed from the Kerr metric, the inner horizon becomes a curvature singularity, preventing the ring singularity from being reached [8]. In this sense, the causality of Kerr spacetime in region **IV** is silly to consider, although it can be intriguing. Indeed the causality up to $r = 0$ is similar to region **II**, however, upon crossing into the $r < 0$ domain, closed-timelike-curves (CTCs) become possible. Such trajectories prevent the establishment of a proper chronology as an observer could complete cyclic loops indefinitely. These peculiarities in the solutions of the Einstein field equation are the subject of many papers and are interesting to consider properly.

IV. CONCLUSION

Summarily, the Kerr family of solutions provides a general spacetime for neutrally charged rotating black holes. As expected, the Schwarzschild spacetime in a subset of the Kerr solution, for which $a = 0$. In Kerr spacetime, black holes create an ergosphere, inside which null and timelike curves must following the direction the black hole is rotating. These black holes have two horizons for the non-maximal rotation case, which shield the ring-like singularity from observation. There is indeed an analytic extension of Kerr spacetime into the $r < 0$ domain, where strange phenomena, like the naked singularity and CTCs, exist.

The power of these spacetimes is perhaps best demonstrated when considered with the cosmic censorship conjecture – the postulate that any singularity is concealed by an event horizon. As Hartle suggests, suppose there is a binary neutron star merger event. Before merging to form a black hole, it is a necessity to employ the entire range of classical and quantum physics of matter to understand the system in detail. However, after the black hole is created, the two parameters J and M describe the system in its entirety.

Although Roy Kerr's paper giving the solution to Einstein's field equation was published over fifty years ago, there are still many topics of ongoing research such as Achronal regions and chronology horizons, physically reasonable sources and 'interior solutions' for the Kerr spacetime, energy extraction from black holes, and black hole formation limits.

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