An Introduction to Numerically Solving the Ideal MHD Equations

I. ABSTRACT

The Ideal MHD equations are used to predict the behaviour of astrophysical and laboratory plasmas by modeling them as magnetically coupled fluids; with applications such as in astrophysics, fusion based reactors, and plasma propulsion, the creation of accurate and robust numerical tools for MHD is topical. The aim of the project is to provide an accessible introduction to the Discontinuous Galerkin methods used for numerical MHD. This is accomplished by a walk-through of the considerations required to implement a first order DG scheme for one-dimensional MHD, with numerical verification, and the treatment of the divergence-free condition on the magnetic field for multi-dimensional MHD with a projection method.

II. INTRODUCTION

The MHD equations govern the evolution of electrically conducting fluids, modeling some plasmas, liquid metals, and electrolytes. In particular, the Ideal MHD equations can be applied to model non-relativistic plasmas with negligible resistivity. Written in conservative form, the equations are

$$\begin{aligned}
&\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \\
&\frac{\partial (\rho \mathbf{u})}{\partial t} + \nabla \cdot \left(\rho \mathbf{u}\mathbf{u}^T + \left(p + \frac{1}{2}\mathbf{B}\cdot\mathbf{B} \right) I - \mathbf{B}\mathbf{B}^T \right) = 0 \\
&\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot \left(\mathbf{u}\mathbf{B}^T - \mathbf{B}\mathbf{u}^T \right) = 0 \\
&\frac{\partial E}{\partial t} + \nabla \cdot \left(\left(E + p + \frac{1}{2}\mathbf{B}\cdot\mathbf{B} \right) \mathbf{u} - (\mathbf{u}\cdot\mathbf{B})\mathbf{B} \right) = 0
\end{aligned}$$

where ρ is the density, $\mathbf{u}$ is the velocity field, $\mathbf{B}$ is the magnetic field, and E is the energy. p is the pressure, and is computed by the equation of state, which closes the system with γ, the adiabatic index.

$$p = (\gamma - 1)\left(E - \rho \frac{\mathbf{u}\cdot\mathbf{u}}{2} - \frac{\mathbf{B}\cdot\mathbf{B}}{2} \right)$$

For physical solutions, the magnetic field must be divergence free. That is, $\nabla \cdot \mathbf{B} = 0$. In one-dimensional MHD, this constraint imposes constant B_1; however, in multiple dimensions, this constraint is more complicated. The above formulation is written for the conserved variables $\mathbf{U} = (\rho, \rho\mathbf{u}, \mathbf{B}, E)^T$, but using the primitive variables $\mathbf{W} = (\rho, \mathbf{u}, \mathbf{B}, p)^T$. Typically, model problems are given in terms of the primitive quantities, and evolved using the conserved quantities.

For applications of Ideal MHD, numerical techniques are heavily used. Many numerical MHD codes, such as the Athena or Ramses open source codes, are available and are actively used in research and industry. To that end, it is important to understand the requirements and motivation for the numerical simulations, and the resulting schemes that are implemented.

Problems of greater physical scale and complexity are being considered, increasing the physical computational resources required. The modern response to these demands has been to rely on increasingly parallel setups [1]. In response, numerical schemes that parallelize well – those which do not require global communication – have become of great interest and frequent topic in literature.

Discontinuous Galerkin (DG) methods are favourable for tackling these types of problems as they offer a scalable, inherently local formulation for arbitrary spatial-order schemes. As the numerical solution is computed in each cell on a chosen set of basis functions, typically polynomials, the spatial dependence of the solution is independent of the scheme's desired convergence order, which allows the control of numerical dissipation [2]. Here, the DG formulation is applied to a model problem to demonstrate and explain the components required for a successful numerical solution to the Ideal MHD equations.

III. DISCONTINUOUS GALERKIN SCHEME

The discontinuous Galerkin scheme that was implemented is now described. Generally, the approach here follows the RKDG method presented for one-dimensional systems by Cockburn, Lin, and Shu, as presented in [3]. The method considers equations of the form (1), where d is the number of dimensions, equipped with suitable initial and boundary conditions

$$\frac{\partial}{\partial t}\mathbf{U} + \sum_{i=1}^{d} \frac{\partial}{\partial x_i}\left(\mathbf{f}_i(\mathbf{U})\right) = 0 \tag{1}$$

The Ideal MHD equations are written in this way by taking $\mathbf{U}$ to be the column vector of the variables in the time derivatives, and $\mathbf{f}_i$ to be the column vector of the i$^{\text{th}}$ rows of each quantity inside the divergence operator. To focus on the framework, only the $d = 1$ case is considered for now.

An approximate solution of (1) with a purely initial condition, meaning with compactly supported or periodic boundaries, is desired. First discretizing spatially, the partition $I_j = (x_{j-1/2}, x_{j+1/2})$ so that the computational domain $I = \cup_j I_j$. Denote $\Delta x_j = x_{j+1/2} - x_{j-1/2}$ and $h = \sup_j \Delta x_j$ for later. The approximate solution is in the finite dimensional space

$$V_h^k = \left\{ v : v \in P^k(I_j)\ \forall j \right\}$$

where $P^k(I_j)$ is the space of degree k polynomials for each of its components in I_j. Multiplying (1) by a test function $v_h \in V_h^k$ for each component, integrating over I_j, and integrating by parts, the weak formulation for each component of $\mathbf{U}$ reads find $U \in V_h^k$ such that

$$\begin{aligned} \frac{\partial}{\partial t}\int_{I_j} U v_h \, dx - \int_{I_j} f(\mathbf{U}) \frac{\partial}{\partial x} v_h \, dx + \\ \mathbf{f}_{j+1/2} v^-_{h,j+1/2} - \mathbf{f}_{j-1/2} v^+_{h,j-1/2} \ = \ 0 \end{aligned}$$

for all $v_h \in V_h^k$ and holds all cells I_j. Now, choosing $k = 0$ restricts the approximation $\mathbf{U}^h$ to be constant in each cell and causes the derivative of the test functions to vanish. This choice is to simplify implementation and will result in a first-order accurate method. Replacing $\mathbf{U}$ by its approximate counterpart in V_h^0, and evaluating the remaining integral, the weak form can be written as

$$\frac{\partial}{\partial t} U_j^h = \frac{1}{\Delta x_j}\left(\mathbf{f}_{j-1/2} - \mathbf{f}_{j+1/2}\right) \tag{2}$$

with the remaining steps to fully discretize being time-discretization and determining the fluxes evaluated at the cell interfaces, the latter of which is considered next.

IV. NUMERICAL FLUX

To evaluate $\mathbf{f}$ at the cell boundaries, a numerical flux function, dependent on the solution values on either side, $\tilde{\mathbf{f}}(\mathbf{U}^-, \mathbf{U}^+)$ must be chosen subject to some constraints. As the DG method determines the solution for each I_j, the numerical flux is the only inter-cell communication that occurs and hence has conditions that must be met for convergence to a solution. The approach to determine a numerical flux which will give convergence is to consider a method that solves Riemann problems exactly or approximately.

A Riemann problem is a conservation equation equipped with two-state piece-wise constant initial conditions, and its time-evolution is considered. The solution is given by the characteristics of the governing equation, dividing the domain into states separated by waves; these are determined by the eigensystem of Jacobian of the fluxes [4], $\mathbf{f}$. For one-dimensional MHD, there are seven waves which are given by Powel [5]: one entropy, two Alfven, and four magneto-acoustic – two fast and two slow – which have wave speeds

$$u_1,\ u_1 \pm c_a,\ u_1 \pm c_s,\ u_1 \pm c_f$$

with the Alfven, slow, and fast speeds given by

$$\begin{aligned} c_a &= \frac{B_1}{\sqrt{\rho}} \\ c_{f,s}^2 &= \frac{1}{2}\left(\frac{\gamma p + \mathbf{B}\cdot\mathbf{B}}{\rho} \pm \sqrt{\left(\frac{\gamma p + \mathbf{B}\cdot\mathbf{B}}{\rho}\right)^2 - \frac{4\gamma p B_1^2}{\rho^2}} \right) \end{aligned}$$

and are nested in the sense that $c_s \leq c_a \leq c_f$.

An approach to simplify the system is to consider only the maximum and minimum wave speeds, with a single averaged intermediate state. Harten et al. [6] proposed the HLL approximate Riemann solver which has this structure. In the paper, it is shown that an explicit scheme of the form

$$v_j^{n+1} = v_j^n - \frac{\Delta t}{\Delta x}\left(\tilde{f}_{j+1/2}^n - \tilde{f}_{f-1/2}^n\right)$$

satisfies the weak form of the associated conservation law and entropy condition in the limit as $\Delta x \to 0$ provided that the numerical flux is consistent in the sense that $\tilde{f}(U,U) = U$, and consistent with the entropy condition in the sense that the numerical entropy flux is $\tilde{F}(U,U) = F(u)$ and an inequality of the kind $Q_j^{n+1} \leq Q_j^n - \frac{\Delta t}{\Delta x}\left(\tilde{F}_{j+1/2}^n - \tilde{F}_{j-1/2}^n\right)$ is satisfied. The numerical HLL flux that satisfies the above conditions for left and right wave speeds a_- and a_+ is given as

$$\tilde{f}(U^-, U^+) = \frac{s_+ f_- - s_- f_+ + s_+ s_- (U^+ - U^-)}{s_+ - s_-} \tag{3}$$

where $s_+ = \max(a_+, 0)$, $s_- = \min(a_-, 0)$. In its application to Ideal MHD here, the a_- and a_+ are chosen by the algorithm in Davis [7], with

$$a_- = \min(\lambda_-(U^-), \lambda_-(U^+))$$
$$a_+ = \max(\lambda_+(U^-), \lambda_+(U^+)$$

where $\lambda_\pm(\mathbf{U}) = u_1 \pm c_f|_{\mathbf{U}}$. Miyoshi et al. [8] present a comprehensive list of HLL type Riemann solvers for Ideal MHD, the one used here is simple and has low computational cost. Wu and Shu [9] showed that this approach can be made to preserve the positivity of pressure and density for Ideal MHD when applied with an appropriate limiter, but is not implemented here. Using a Riemann solving technique to determine the flux at each interface introduces a stability condition

V. TEMPORAL DISCRETIZATION

With the semi-discrete in space formulation complete for a single cell, the remaining work to compute a numerical solution to the Ideal MHD equations lies in the implementation. To advance the computation forward in time, an explicit 2nd order Adams-Bashforth time-stepping method is used. That is, for an evolutionary ODE $\frac{d}{dt}U = F(U)$, discretized temporally gives

$$U^{n+1} = U^n + \frac{\Delta t}{2}\left(F^n - F^{n-1}\right)$$

where superscripts denote the value at time $n\Delta t$. This two-step method is initiated by a single step of Forward Euler, $U^{n+1} = U^n + \Delta t F^n$, which maintains the second order accuracy for a single use.

Some consideration must be taken when choosing the time-step size Δt. The requirement is that the CFL condition must be met for all cells. Since each cell is advanced with the same time-step, Δt must be chosen such that the speediest waves in each Riemann problem do not interact with adjacent ones [10]. This is accomplished by

$$\Delta t \leq \frac{\Delta x}{c_{max}} \tag{4}$$

where $c_{max} = \max_j(\max(|\lambda_-(U_j)|, |\lambda_+(U_j)|))$. Now, all the ingredients for a numerical solution have been assembled.

VI. NUMERICAL TEST

To assemble the introduced components, the algorithm for numerically solving the one-dimensional Ideal MHD equations is:

At time t^n, for each cell in the computation domain,

1. Compute the following for each boundary

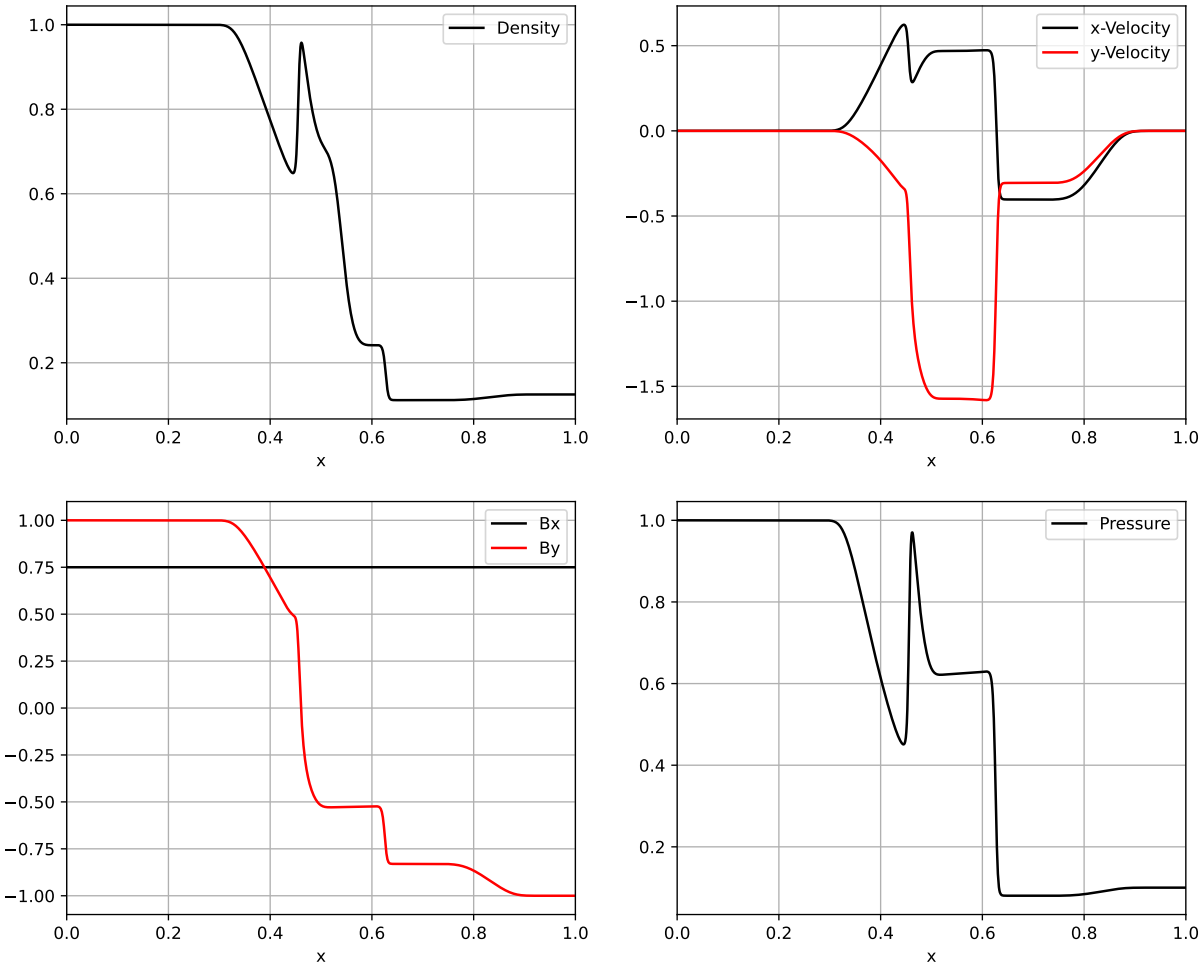
   i. a_- and a_+
   ii. $\tilde{\mathbf{f}}$ from (2) with s_+ and s_- computed from values in i.
2. Store the right-hand-side of (1) as F^n Wait for 2. to be completed for all cells then, for each cell again
3. Compute U^{n+1} with 2nd order Adams-Bashforth using F^n and F^{n-1}
4. Update $U^n := U^{n+1}$
5. Compute maximum Δt by (3)
6. Repeat from step 1 if $t < T$ with t increased by Δt

A common model problem for the one-dimensional Ideal MHD equations is the Brio-Wu shock tube problem, and is chosen to demonstrate the shock capturing capabilities and convergence of the implemented algorithm. The problem reads on the interval $[0, 1]$ with zero-flux boundary conditions, a left and right state are present at $t = 0$ given by

$$\mathbf{W}_L = (1, 0, 0, 0, 0.75, 1, 0, 1)^T$$
$$\mathbf{W}_R = (0.125, 0, 0, 0, 0.75, -1, 0.1)^T$$

and is evolved until $t = 0.1$. Fig. 1 presents the solutions components along the x-axis that were computed using $n_x = 1000$ grid points. The results show the characteristic behaviour of shocks, but lacks the sharpness of a higher-order method. Comparing to the Brio-Wu results of [11], the method here does not provide the same resolution, as expected of the first order method, but provides a good approximation of the correct behaviour.

FIG. 1. Brio–Wu shock tube problem solution at time $t = 0.1$, displaying non-zero components.



VII. CONCLUSION

The components critical to the successful implementation of a numerical scheme for the Ideal MHD equations have been explained. To demonstrate the process, the one-dimensional Ideal MHD equations are discretized with a first order discontinuous Galerkin spatial discretization fitted with the HLL flux, and second order Adams-Bashforth time-stepping is implemented and applied to the Brio-Wu shock tube problem as an example.

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