

or alternatively

$$\frac{P_2}{P_1} = 1 + \frac{2\gamma}{\gamma+1} (M_1^2 - 1) \quad (\text{A.10})$$

$$= \frac{2\gamma}{\gamma+1} M_1^2 - \frac{\gamma-1}{\gamma+1} \quad (\text{A.11})$$

$$\frac{\rho_2}{\rho_1} = \frac{\gamma+1}{\gamma-1 + \frac{2}{M_1^2}} \quad (\text{A.12})$$

$$M_2^2 = \frac{M_1^2 + \frac{2}{\gamma-1}}{\frac{2\gamma}{\gamma-1} M_1^2 - 1} \quad (\text{A.13})$$

The ratio of stagnation (total) pressure across the wave is

$$\frac{P_{t2}}{P_{t1}} = \frac{1}{\left(\frac{2\gamma}{\gamma+1} M_1^2 - \frac{\gamma-1}{\gamma+1} \right)^{\frac{\gamma}{\gamma-1}}} \frac{1}{\left(\frac{\frac{\gamma+1}{2} M_1^2}{1 + \frac{\gamma-1}{2} M_1^2} \right)^{\frac{\gamma}{\gamma-1}}} \quad (\text{A.14})$$

Using the transformation from wave-fixed to laboratory frame, we have

$$[w] = -[u] \quad (\text{A.15})$$

so that

$$\frac{[u]}{a_1} = \frac{2}{\gamma+1} \left(M_1 - \frac{1}{M_1} \right) \quad (\text{A.16})$$

$$(\text{A.17})$$

We can analytically express the shock adiabat or Hugoniot as pressure-volume relationship

$$\frac{P_2}{P_1} = \frac{\frac{\gamma+1}{\gamma-1} \frac{v_2}{v_1}}{\frac{\gamma+1}{\gamma-1} \frac{v_2}{v_1} - 1}, \quad (\text{A.18})$$

or equivalently

$$\frac{v_2}{v_1} = \frac{\frac{P_2}{P_1} + \frac{\gamma+1}{\gamma-1}}{\frac{\gamma+1}{\gamma-1} \frac{P_2}{P_1} + 1}. \quad (\text{A.19})$$

Alternatively, as a pressure-temperature relationship

$$\left(\frac{P_1}{P_2} \right)^2 + \left[\frac{\gamma+1}{\gamma-1} \left(1 - \frac{T_1}{T_2} \right) \right] \left(\frac{P_1}{P_2} \right) - \frac{T_1}{T_2} = 0, \quad (\text{A.20})$$

which has the explicit solutions

$$\frac{P_1}{P_2} = \sqrt{\left[\frac{1}{2} \frac{\gamma+1}{\gamma-1} \left(1 - \frac{T_1}{T_2} \right) \right]^2 + \frac{T_1}{T_2}} - \left[\frac{1}{2} \frac{\gamma+1}{\gamma-1} \left(1 - \frac{T_1}{T_2} \right) \right], \quad (\text{A.21})$$

or equivalently

$$\frac{T_1}{T_2} = \frac{\left(\frac{P_1}{P_2}\right)^2 + \frac{\gamma+1}{\gamma-1} \frac{P_1}{P_2}}{\frac{\gamma+1}{\gamma-1} \frac{P_1}{P_2} + 1}. \quad (\text{A.22})$$

Another useful equation is *Prandtl's relation*,

$$w_1 w_2 = a^{*2}, \quad (\text{A.23})$$

where a^* is the sound speed at a sonic point obtained in a fictitious isentropic process in the upstream flow.

$$a^* = \sqrt{2 \frac{\gamma-1}{\gamma+1} h_t}, \quad h_t = h + \frac{w^2}{2} \quad (\text{A.24})$$

A.2 Reflected Shock Waves

Several relationships for reflected waves can be derived by based on the fact that fluid adjacent to a stationary surface must be stationary. Figure 6.8 (Section 6.6) illustrates a possible geometry for wave reflection. The above condition requires that

$$u_1 = u_3 = 0. \quad (\text{A.25})$$

Therefore, the jump in velocity across the reflected wave,

$$[u]_R = u_3 - u_2 = -u_2 \quad (\text{A.26})$$

is the exact opposite of the jump in velocity across the incident wave,

$$[u]_I = u_2 - u_1 = u_2, \quad (\text{A.27})$$

or

$$[u]_I = -[u]_R \quad (\text{A.28})$$

The Rayleigh line equation (6.16) can be expressed in terms of jumps in properties, i.e.

$$[u]^2 = -[P][v] \quad (\text{A.29})$$

Now we relate the Rayleigh line of each wave

$$[P]_R[v]_R = [P]_I[v]_I. \quad (\text{A.30})$$

Pressure Jump

Using the perfect gas Hugoniot relationship (A.18) for both the incident and reflected waves, we can eliminate the volume jumps and find a relationship between the pressure ratios across the incident and reflected waves. Using the notation in Section 6.6,

$$\frac{P_3}{P_2} = \frac{(3\gamma-1)\frac{P_2}{P_1} - (\gamma-1)}{(\gamma-1)\frac{P_2}{P_1} + (\gamma+1)} \quad (\text{A.31})$$

The pressure ratio across the reflected shock is always less than across the incident shock and has a limiting value for large incident shock speeds of

$$\frac{P_3}{P_2} \rightarrow \frac{3\gamma - 1}{\gamma - 1} \quad \text{as} \quad \frac{P_2}{P_1} \rightarrow \infty \quad (\text{A.32})$$

On the other hand, for small incident shock speeds, the pressure ratio across the reflected and incident shock waves approaches 1. In this limit, if we expand about the initial state,

$$\frac{P_3}{P_2} - 1 = \frac{P_2}{P_1} - 1 - \frac{\gamma - 1}{2\gamma} \left(\frac{P_2}{P_1} - 1 \right)^2 + \left(\frac{\gamma - 1}{2\gamma} \right)^2 \left(\frac{P_2}{P_1} - 1 \right)^3 + \dots, \quad (\text{A.33})$$

and retain only the first order terms of the series, we obtain the acoustic result, i.e. the pressure rise across the reflected shock is equal to the rise across the incident shock. In other words, the total pressure rise ($P_3 - P_1$) is twice the pressure rise due to the incident wave ($P_2 - P_1$).

$$P_3 - P_1 \approx 2(P_2 - P_1) \quad \text{for acoustic waves} \quad (\text{A.34})$$

Mach Number

Similarly, we can determine an expression for the reflected shock Mach number. First, we define the incident and reflected shock Mach numbers.

$$M_I = \frac{U_I}{a_1} \quad (\text{A.35})$$

$$M_R = \frac{U_R + u_2}{a_2}. \quad (\text{A.36})$$

Then, using the velocity jump relation (A.16) and recalling (A.25), we relate the two Mach numbers

$$M_R - \frac{1}{M_R} = \frac{a_1}{a_2} \left(M_I - \frac{1}{M_I} \right). \quad (\text{A.37})$$

The left-hand side is a function α of the incident shock speed

$$\alpha = \frac{a_1}{a_2} \left(M_I - \frac{1}{M_I} \right). \quad (\text{A.38})$$

For a specified incident shock Mach number, we can compute α and find the reflected shock Mach number by solving the resulting quadratic equation

$$M_R = \frac{\alpha + \sqrt{\alpha^2 + 4}}{2}. \quad (\text{A.39})$$

From the incident shock jump conditions, α ranges between zero and a maximum value which is only a function of γ . Taking the limit as $M_I \rightarrow \infty$, we find that

$$\alpha_{max} = \frac{\gamma + 1}{\sqrt{2\gamma(\gamma - 1)}} \quad (\text{A.40})$$

which means that the reflected shock Mach number ranges between one and a maximum value of

$$M_{R,max} = \sqrt{\frac{2\gamma}{\gamma - 1}}. \quad (\text{A.41})$$

Enthalpy

For strong incident shock waves, we can derive from the reflected shock relationships (6.46)-(6.48), the approximate results

$$h_2 \approx h_1 + \frac{1}{2}U_I^2 \quad (\text{A.42})$$

$$h_3 \approx h_2 + \frac{1}{2}U_I^2 \quad (\text{A.43})$$

so that the enthalpy behind a strong reflected shock wave is

$$h_3 \approx h_1 + U_I^2 \quad (\text{A.44})$$

which is very useful in estimations of the reservoir enthalpy in the reflected shock tunnels.

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A.3 Detonation Waves in Perfect Gases

The jump conditions given in Section 6.1 are

$$\rho_1 w_1 = \rho_2 w_2 \quad (\text{A.45})$$

$$P_1 + \rho_1 w_1^2 = P_2 + \rho_2 w_2^2 \quad (\text{A.46})$$

$$h_1 + \frac{w_1^2}{2} = h_2 + \frac{w_2^2}{2} \quad (\text{A.47})$$

$$s_2 \geq s_1 \quad (\text{A.48})$$

Perfect-Gas, 2- γ Model

For a detonation, we assume two perfect gases, reactant (1) and product (2), with different specific heats and molecular weights. In this case, there will be two gas constants. We also assume an energy release, q , due to exothermic chemistry. Now our thermodynamic relations are

$$h_1 = c_{P1}T + h_{0,1} \quad (\text{A.49})$$

$$h_2 = c_{P2}T + h_{0,2} \quad (\text{A.50})$$

$$q = h_{0,1} - h_{0,2} \quad (\text{A.51})$$

$$P_1 = \rho_1 R_1 T_1 \quad (\text{A.52})$$

$$P_2 = \rho_2 R_2 T_2 \quad (\text{A.53})$$

$$c_{P1} = \frac{\gamma_1 R_1}{\gamma_1 - 1} \quad (\text{A.54})$$

$$c_{P2} = \frac{\gamma_2 R_2}{\gamma_2 - 1} \quad (\text{A.55})$$

We substitute these into the jump conditions to yield:

$$\frac{P_2}{P_1} = \frac{1 + \gamma_1 M_1^2}{1 + \gamma_2 M_2^2} \quad (\text{A.56})$$

$$\frac{v_2}{v_1} = \frac{\gamma_2 M_2^2}{\gamma_1 M_1^2} \frac{1 + \gamma_1 M_1^2}{1 + \gamma_2 M_2^2} \quad (\text{A.57})$$

$$\frac{T_2}{T_1} = \frac{\gamma_1 R_1}{\gamma_2 R_2} \frac{\frac{1}{\gamma_1 - 1} + \frac{1}{2} M_1^2 + \frac{q}{a_1^2}}{\frac{1}{\gamma_2 - 1} + \frac{1}{2} M_2^2} \quad (\text{A.58})$$

Additionally, the entropy variation along adiabat is

$$ds = \frac{1}{2T} (v_1 - v)^2 d(\rho w)^2 \quad (\text{A.59})$$

A.4 Chapman-Jouguet Conditions

At the CJ point, the isentrope, and Hugoniot and Rayleigh line are all tangent.

$$\frac{P_{\text{CJ}} - P_1}{v_{\text{CJ}} - v_1} = \left(\frac{\partial P}{\partial v} \right)_{\mathcal{H}} = \left(\frac{\partial P}{\partial v} \right)_s \quad (\text{A.60})$$

which implies that the product velocity is *sonic relative to the wave*

$$w_{2,\text{CJ}} = a_2 \quad (\text{A.61})$$

Jouguet's Rule

$$\frac{w^2 - a^2}{v^2} = \left[1 - \frac{\mathcal{G}}{2v}(v_1 - v) \right] \left[\left(\frac{\partial P}{\partial v} \right)_{Hug} - \frac{\Delta P}{\Delta v} \right] \quad (\text{A.62})$$

where \mathcal{G} is the Grúnesen parameter.

The flow downstream of a detonation is subsonic relative to the wave for points above the CJ state and supersonic for states below.

A.5 Two- γ CJ Conditions

Using the CJ conditions and the perfect-gas, 2- γ model, and the Mach number for the upper CJ (detonation) point

$$M_{CJ} = \sqrt{\mathcal{H} + \frac{(\gamma_1 + \gamma_2)(\gamma_2 - 1)}{2\gamma_1(\gamma_1 - 1)}} + \sqrt{\mathcal{H} + \frac{(\gamma_2 - \gamma_1)(\gamma_2 + 1)}{2\gamma_1(\gamma_1 - 1)}} \quad (\text{A.63})$$

where the parameter \mathcal{H} is the nondimensional energy release

$$\mathcal{H} = \frac{(\gamma_2 - 1)(\gamma_2 + 1)q}{2\gamma_1 R_1 T_1} = \frac{\gamma_2^2 - 1}{2} \frac{q}{a_1^2}, \quad (\text{A.64})$$

$$q = h_{01} - h_{02}. \quad (\text{A.65})$$

The inverse relationship is

$$\frac{q}{\gamma_1 R_1 T_1} = \frac{1}{2} \left(\frac{\gamma_2}{\gamma_1} \right)^2 \frac{(1 + \gamma_1 M_{CJ}^2)^2}{(\gamma_2^2 - 1) M_{CJ}^2} - \frac{1}{\gamma_1 - 1} - \frac{M_{CJ}^2}{2} \quad (\text{A.66})$$

Other quantities of interest include

- CJ pressure

$$\frac{P_{CJ}}{P_1} = \frac{1 + \gamma_1 M_{CJ}^2}{\gamma_2 + 1} \quad (\text{A.67})$$

- CJ density

$$\frac{\rho_{CJ}}{\rho_1} = \frac{\gamma_1(\gamma_2 + 1)M_{CJ}^2}{\gamma_2(1 + \gamma_1 M_{CJ}^2)} \quad (\text{A.68})$$

- CJ temperature

$$\frac{T_{CJ}}{T_1} = \frac{P_{CJ}}{P_1} \frac{\rho_1}{\rho_{CJ}} \frac{R_1}{R_2} \quad (\text{A.69})$$

- CJ sound speed

$$\frac{a_{CJ}}{a_1} = \frac{\gamma_2}{\gamma_1} \frac{1 + \gamma_1 M_{CJ}^2}{(1 + \gamma_2) M_{CJ}} \quad (\text{A.70})$$

- Effective energy release parameter

$$\frac{q}{a_1^2} = \frac{\gamma_2 + 1}{2(\gamma_2 - 1)} \left(\frac{a_{CJ}}{a_1} \right)^2 - \left(\frac{1}{\gamma_1 - 1} + \frac{M_{CJ}^2}{2} \right) \quad (\text{A.71})$$

A.6 One- γ CJ Conditions

A common approximation used in analytical and some numerical studies is to assume that ratio of specific heats γ has a common value and is constant in both reactants and products; usually, the additional assumptions are made that the molar masses in reactants and products are equal. The analytical expressions for the CJ state are particularly simple for this case:

$$M_{\text{CJ}} = \sqrt{\frac{\gamma+1}{2} \frac{q}{C_p T_1}} + 1 + \sqrt{\frac{\gamma+1}{2} \frac{q}{C_p T_1}} \quad (\text{A.72})$$

$$\frac{P_{\text{CJ}}}{P_1} = \frac{\gamma M_{\text{CJ}}^2 + 1}{\gamma + 1} \quad (\text{A.73})$$

$$\frac{\rho_{\text{CJ}}}{\rho_1} = \frac{(\gamma + 1) M_{\text{CJ}}^2}{1 + \gamma M_{\text{CJ}}^2} \quad (\text{A.74})$$

$$\frac{T_{\text{CJ}}}{T_1} = \frac{(\gamma M_{\text{CJ}}^2 + 1)^2}{M_{\text{CJ}}^2 (\gamma + 1)^2} \quad (\text{A.75})$$

A useful alternative relationship between energy release and CJ Mach number is

$$M_{\text{CJ}} - \frac{1}{M_{\text{CJ}}} = \sqrt{\frac{2q(\gamma^2 - 1)}{\gamma R T_1}} \quad (\text{A.76})$$

which yields the following relation for the effective energy release parameter

$$\frac{q}{a_1^2} = \frac{\gamma^2 - 1}{2} \left(M_{\text{CJ}} - \frac{1}{M_{\text{CJ}}} \right)^2 \quad (\text{A.77})$$

A.7 Strong detonation approximation

A useful limit for approximate analysis is $M_{\text{CJ}} \gg 1$. This simplifies the expressions for the CJ properties (A.67)-(A.69).

$$U_{\text{CJ}} \approx \sqrt{2(\gamma_2^2 - 1)q} \quad (\text{A.78})$$

$$\rho_{\text{CJ}} \approx \frac{\gamma_2 + 1}{\gamma_2} \rho_1 \quad (\text{A.79})$$

$$P_{\text{CJ}} \approx \frac{1}{\gamma_2 + 1} \rho_1 U_{\text{CJ}}^2 \quad (\text{A.80})$$

$$a_{\text{CJ}} \approx \frac{\gamma_2 U_{\text{CJ}}}{\gamma_2 + 1} \quad (\text{A.81})$$

$$u_{\text{CJ}} \approx \frac{U_{\text{CJ}}}{\gamma_2 + 1} \quad (\text{A.82})$$

A.8 Reflection of Detonation

A detonation wave incident on a rigid surface will reflect as a shock wave which propagates into the detonation products. The computation of the properties behind the reflected wave proceed in the same fashion as with the previous discussion for shock waves (Appendix A.2). When we compare the reflection of a detonation wave (traveling at CJ velocity) with a nonreactive shock wave of the same speed, we find that the pressure behind the resulting reflected shock wave is much higher in the case of the incident shock than the detonation. This is because the momentum flux behind the shock wave is higher than that behind the detonation. The chemical energy release in the detonation increases the equilibrium post-incident-wave temperature and

lowers the post-incident-wave pressure, density, and particle velocity compared to a shock wave of the same speed.

`demo_CJ_and_shock_state.m` demonstrates how to compute the following four states.

- CJ state: equilibrium behind a CJ detonation
- Frozen post-reflected-shock state resulting from reflection of a CJ detonation
- Frozen post-incident-shock state behind a shock traveling at the CJ speed
- Frozen post-reflected-shock state resulting from reflection of a frozen shock wave

for stoichiometric hydrogen-air mixtures.

CJ state

CJ speed 1968. (m/s)
 CJ pressure 1.54 (MPa)
 CJ temperature 2940. (K)
 CJ density 1.511 (kg/m³)
 w2 (wave frame) 1092. (m/s)
 u2 (lab frame) 875.7 (m/s)
 a2 (frozen) 1127. (m/s)
 a2 (equilibrium) 1091. (m/s)
 gamma2 (frozen) 1.242 (m/s)
 gamma2 (equilibrium) 1.163 (m/s)

Reflected shock (equilibrium) from CJ detonation

Reflected wave speed 782.9 (m/s)
 Reflected shock pressure 3.74 (MPa)
 Reflected shock temperature 3297. (K)
 Reflected shock density 3.200 (kg/m³)

Incident Shock (frozen) at CJ speed

shock speed 1968. (m/s)
 shock pressure 2.74 (MPa)
 shock temperature 1530. (K)
 shock density 4.506 (kg/m³)
 w2 (wave frame) 366.1 (m/s)
 u2 (lab frame) 1602. (m/s)
 a2 (frozen) 895.8 (m/s)
 gamma2 (frozen) 1.319 (m/s)

Reflected shock (frozen)

Shock speed 1968. (m/s)
 Reflected wave speed 599.4 (m/s)
 Reflected shock pressure 18.63 (MPa)
 Reflected shock temperature 2832. (K)
 Reflected shock density 16.55 (kg/m³)

In this example, the pressure behind the reflection of a frozen shock wave is 18.6 MPa as compared with 3.74 MPa behind the reflection of the detonation. The ratio of reflected to incident post-wave pressure is 6.97 for the frozen incident shock wave and only 2.46 for the detonation case. The ratio computed for the

detonation is found to be very insensitive to the mixture composition with both computed and measured values being close to 2.5 (see [Shepherd et al., 1991](#)).

Following the derivation of Stanyukovich and Zel'dovich ([Stanyukovich, 1960](#), p. 372-372), the ratio of post-reflected to post-incident pressure can be approximately computed for detonations using ideas similar to those for incident shock waves together with the strong detonation approximation (Appendix [A.7](#)). Using the notation of Section [6.6](#), the Rayleigh line for the detonation can be written:

$$u_2^2 = (P_2 - P_1)(v_1 - v_2) \quad (\text{A.83})$$

and applying the strong detonation approximation ([A.82](#)), this is

$$u_2^2 \approx \frac{P_2 v_1}{\gamma + 1} \quad (\text{A.84})$$

where we have dropped the subscripts on γ and assumed it has the same value for state 2 and 3. For the reflected shock wave, the Rayleigh line is

$$u_2^2 = (P_3 - P_2)(v_2 - v_3) . \quad (\text{A.85})$$

We can eliminate v_3 by using the following form of the Hugoniot relation

$$\frac{v_3}{v_2} = \frac{(\gamma + 1)P_3 + (\gamma - 1)P_2}{(\gamma - 1)P_3 + (\gamma + 1)P_2} . \quad (\text{A.86})$$

The volumes v_2 and v_1 can be eliminated by using the strong detonation relation ([A.79](#))

$$v_2 \approx \frac{\gamma v_1}{\gamma + 1} \quad (\text{A.87})$$

which results in a quadratic for the pressure ratio P_3/P_2 . The solution to the quadratic is

$$\frac{P_3}{P_2} = \frac{5\gamma + 1 + \sqrt{17\gamma^2 + 2\gamma + 1}}{4\gamma} . \quad (\text{A.88})$$

For values of γ between 1 and 5/3 (realistic for gases), this approximate formula give values of the pressure ratio between 2.4 and 2.5. Despite the very rough nature of the strong detonation approximation, the resulting values are in reasonable agreement with detailed computations and experimental data as discussed in [Shepherd et al. \(1991\)](#).

A.9 Oblique Shocks in Perfect Gases

The jump conditions for oblique shocks in perfect gases can be expressed analytically in terms of the wave angle β and upstream flow state by applying the transformation from laboratory coordinates to wave-fixed coordinates and applying the usual shock jump conditions with a normal shock Mach number M_{1n} specified as

$$M_{1n} = M_1 \sin \beta \quad (\text{A.89})$$

where the freestream Mach number $M_1 = u_1/a_1$. From Appendix A.1 the ratios of properties across the shock are found to be

$$\frac{P_2}{P_1} = \frac{2\gamma M_1^2 \sin^2 \beta - (\gamma - 1)}{\gamma + 1} \quad (\text{A.90})$$

$$\frac{\rho_2}{\rho_1} = \frac{(\gamma + 1)M_1^2 \sin^2 \beta}{(\gamma - 1)M_1^2 \sin^2 \beta + 2} \quad (\text{A.91})$$

$$\frac{w_1}{u_1} = \cos \beta \quad (\text{A.92})$$

$$\frac{v}{u_1} = \frac{(\gamma - 1)M_1^2 \sin^2 \beta + 2}{(\gamma + 1)M_1^2 \sin \beta} \quad (\text{A.93})$$

The flow deflection angle is given by

$$\tan \theta = \frac{2 \cot \beta (M_1^2 \sin^2 \beta - 1)}{(\gamma + 1)M_1^2 - 2(M_1^2 \sin^2 \beta - 1)} \quad (\text{A.94})$$

Prandtl's relationship is a tidy way to express the relationship between upstream and downstream normal velocities in terms of a reference sonic state a_*^2

$$w_1 w_2 = a_*^2 - \frac{\gamma - 1}{\gamma + 1} v^2 \quad (\text{A.95})$$

$$a_*^2 = \frac{2}{\gamma + 1} a_1^2 + \frac{\gamma - 1}{\gamma + 1} u_1^2 \quad (\text{A.96})$$

A.10 Prandtl-Meyer Expansion in Perfect Gas

The definition of the Prandtl-Meyer function is

$$d\omega = \sqrt{M^2 - 1} \frac{du}{u} \quad (\text{A.97})$$

The key idea for further developments is that along a streamline in steady, supersonic flow is that for isentropic flow, the variation in all properties can be related to the stagnation state and the local flow velocity or Mach number. In the case of a perfect gas, these can be expressed as analytic relationships. In the adiabatic flow of a perfect gas, the conservation of energy can be expressed as

$$h_t = h + \frac{u^2}{2} \quad (\text{A.98})$$

where $()_t$ is the stagnation state. This can be rewritten using the perfect gas relationship $h = C_p T$ with $C_p = \gamma/(\gamma - 1)$ where $\gamma = C_p/C_v$ is constant for a perfect gas.

$$\frac{T_t}{T} = 1 + \frac{\gamma - 1}{2} M^2 \quad (\text{A.99})$$

Using the perfect gas entropy relationships, the gas thermodynamic state can be obtained analytically

$$\frac{P_t}{P} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{\gamma}{\gamma - 1}}, \quad (\text{A.100})$$

$$\frac{\rho_t}{\rho} = \left(1 + \frac{\gamma - 1}{2} M^2\right)^{\frac{1}{\gamma - 1}}. \quad (\text{A.101})$$

$$(\text{A.102})$$

To simplify the Prandtl-Meyer function we start with the perfect gas expression for sound speed $a = \sqrt{\gamma R T}$ as a function of Mach number

$$\frac{a_t^2}{a^2} = 1 + \frac{\gamma - 1}{2} M^2. \quad (\text{A.103})$$

Defining $u = aM$, this leads to the expression for velocity changes as

$$\frac{du}{u} = \frac{da}{a} + \frac{dM}{M}, \quad (\text{A.104})$$

substitute and simplifying, we find that

$$\frac{du}{u} = \frac{1}{1 + \frac{\gamma + 1}{2} M^2} \frac{dM}{M}, \quad (\text{A.105})$$

and the Prandtl-Meyer function is defined by

$$d\omega = \frac{\sqrt{M^2 - 1}}{1 + \frac{\gamma + 1}{2} M^2} \frac{dM}{M}. \quad (\text{A.106})$$

The integration is tedious but can be performed analytically (the convention is to set $\omega(M = 1) = 0$) to yield

$$\omega(M) = \sqrt{\frac{\gamma + 1}{\gamma - 1}} \tan^{-1} \sqrt{\frac{\gamma - 1}{\gamma + 1} (M^2 - 1)} - \tan^{-1} \sqrt{M^2 - 1} \quad (\text{A.107})$$

The use of this relationship is discussed in detail in books on compressible flow [Liepmann and Roshko \(1957\)](#), [Thompson \(1972\)](#). The key result is that the Mach numbers upstream M_1 and downstream M_2 of an expansion wave that turns the flow through an angle $\Delta\theta$ are determined implicitly by the relation

$$\omega(M_2) = \omega(M_1) \pm |\Delta\theta| \quad (\text{A.108})$$

where the choice of sign depends on the direction of the turn and we always have $M_2 > M_1 > 1$.

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A.11 Shock Tubes

The perfect gas shock jump conditions can be combined to obtain the following relationship between the absolute change in velocity across the shock $\Delta u = |[u]|$ and the jump in pressure across the shock $\Delta P = |[P]|$

$$\frac{\Delta P}{P_1} = \gamma \frac{\Delta u}{a_1} \left(\frac{\gamma+1}{4} \frac{\Delta u}{a_1} + \sqrt{\left(\frac{\gamma+1}{4} \frac{\Delta u}{a_1} \right)^2 + 1} \right), \quad (\text{A.109})$$

or equivalently

$$\frac{\Delta u}{a_1} = \frac{\frac{1}{\gamma} \frac{\Delta P}{P_1}}{\sqrt{1 + \frac{\gamma+1}{2\gamma} \frac{\Delta P}{P_1}}}. \quad (\text{A.110})$$

The relationship between pressure and velocity change across an expansion wave can be computed from the Riemann invariants and the perfect gas isentropic relationships

$$\frac{\Delta P}{P_1} = \left[1 - \frac{\gamma-1}{2} \frac{\Delta u}{a_1} \right]^{2\gamma/(\gamma-1)} - 1. \quad (\text{A.111})$$

The maximum value of the velocity change is achieved with expansion to $P = 0$

$$\Delta u_{\max} = \frac{2}{\gamma-1} a_1. \quad (\text{A.112})$$

These relationships can be used together to match pressure and velocity at states 2 and 3 to obtain the following relationship between driver (state 4) and driven section (state 1) conditions and the shock Mach number $M_s = U_s/a_1$.

$$\frac{P_4}{P_1} = \left[1 - \frac{a_1}{a_4} \frac{\gamma_4-1}{\gamma+1} \left(M_s - \frac{1}{M_s} \right) \right]^{\frac{-2\gamma_4}{\gamma_4-1}} \left[1 + \frac{2\gamma_1}{\gamma_1+1} (M_s^2 - 1) \right] \quad (\text{A.113})$$

The limiting shock Mach number for $P_4/P_1 \rightarrow \infty$ is

$$M_s \rightarrow \frac{a_4}{a_1} \frac{\gamma_1+1}{\gamma_4-1} \quad (\text{A.114})$$

These formulas are useful guides for shock tube performance at modest pressures and shock Mach numbers but have to be used cautiously at high pressures and for strong shocks, see Section 7.11 for a discussion of some of these issues.

Appendix B

Differentials on the Hugoniot

In the numerical solution of the jump conditions for CJ velocity described in Chapter 8, we take advantage of some special properties of solutions near the CJ point. The key result is that the deviation in detonation speed $\delta w = w_1 - W_{\text{CJ}}$ is a quadratic function of the deviation of downstream specific volume $\delta v = v_2 - v_{\text{CJ}}$. This appendix derives this result for a generic equation of state, expresses the results for ideal gases in terms of standard thermodynamic derivatives and verifies the results using the perfect gas analytical expressions.

B.1 Differential Relationships on the Hugoniot

We will use the following notation for partial derivatives evaluated at a specific state.

$$\left(\frac{\partial f}{\partial x}\right)_1 = f_{1,x} \quad (\text{B.1})$$

$$\left(\frac{\partial^2 f}{\partial x^2}\right)_1 = f_{1,xx} \quad (\text{B.2})$$

where f is a function of x and 1 is the state where we evaluate the derivative. Also if we are holding y constant, where f is also a function of y , we will express that as

$$\left(\frac{\partial f}{\partial x}\right)_y = (f_{,x})_y \quad (\text{B.3})$$

Recall

- Rayleigh Line

$$P - P_1 = -\left(\frac{w_1}{v_1}\right)^2 (v - v_1) \quad (\text{B.4})$$

- Hugoniot

$$h - h_1 = (P - P_1) \frac{(v + v_1)}{2} \quad (\text{B.5})$$

Now we find the Taylor expansion of P about P_2 along the Hugoniot.

$$P = P_2 + P_{2\mathcal{H},v} \delta v + \frac{1}{2} P_{2\mathcal{H},vv} (\delta v)^2 + \dots \quad (\text{B.6})$$

We want an equation of the form

$$\delta w_1 = C(\delta v)^n \quad (\text{B.7})$$

that explains how the curve $w_1(v_2/v_1)$ behaves near the post-shock state, so we perturb the system from this state, state 2, where w_1^* is the specific value of w_1 that produces the chosen state 2.

$$v = v_2 + \delta v \quad (\text{B.8})$$

$$w_1 = w_1^* + \delta w_1 \quad (\text{B.9})$$

The equation of the Rayleigh line (B.4) becomes

$$P - P_1 = -\frac{1}{v_1^2}(w_1 + \delta w_1)^2((v_2 + \delta v) - v_1) \quad (\text{B.10})$$

and combined with the Hugoniot pressure expansion (B.6)

$$\begin{aligned} (P_2 - P_1) + P_{2\mathcal{H},v}\delta v + \frac{1}{2}P_{2\mathcal{H},vv}(\delta v)^2 + \dots \\ = -\frac{1}{v_1^2}(w_1^{*2} + 2w_1^*\delta w_1 + (\delta w_1)^2)(v_2 + \delta v - v_1) \end{aligned} \quad (\text{B.11})$$

Now we can group terms in powers of δv

- Zero Order

$$(P_2 - P_1) = -\frac{w_1^{*2}}{v_1^2}(v_2 - v_1) \quad (\text{B.12})$$

Because this is (B.4) evaluated at state 2, these terms cancel.

- Higher Order

$$P_{2\mathcal{H},v} + \frac{1}{2}P_{2\mathcal{H},vv}(\delta v)^2 + \dots = -\frac{w_1^{*2}}{v_1^2} \frac{1}{v_1^2}(2w_1^*\delta w_1 + (\delta w_1)^2)(v_2 + \delta v - v_1) \quad (\text{B.13})$$

It is important to remember that the derivatives of pressure are evaluated along the Hugoniot so $P_{2\mathcal{H},v}$ is the slope of the equilibrium Hugoniot.

B.2 CJ Point Analysis

From (B.4), the slope of the Rayleigh line is

$$-\frac{w_1^2}{v_1^2} \quad (\text{B.14})$$

which is

$$-\frac{U_{\text{CJ}}^2}{v_1^2} \quad (\text{B.15})$$

in the CJ case. At the CJ point, the Rayleigh line and the Hugoniot are tangent (i.e. have the same slope) so the slope of the Hugoniot, $P_{2\mathcal{H},v}$, is the equal to the slope of the Rayleigh line. Now it is clear that the first order terms of (B.13)

$$P_{2\mathcal{H},v} = -\frac{w_1^{*2}}{v_1^2} = -\frac{U_{\text{CJ}}^2}{v_1^2} \quad (\text{B.16})$$

also cancel. The remaining higher order terms equation is

$$\frac{1}{2}P_{2\mathcal{H},vv}(\delta v)^2 + \dots = -\frac{1}{v_1^2}(2w_1^*\delta w_1 + (\delta w_1)^2)(v_2 + \delta v - v_1) \quad (\text{B.17})$$

and if we only retain the lowest order term on each side, the equation simplifies to

$$\frac{v_1^2}{2} P_{2\mathcal{H},vv} (\delta v)^2 = -\delta w_1 [2 w_1^* (v_2 - v_1)] \quad (\text{B.18})$$

Therefore in the CJ case, (B.11) reduces to the form (B.7) we want

$$\delta w_1 = \frac{w_1^*}{4(P_2 - P_1)} P_{2\mathcal{H},vv} (\delta v)^2 \quad (\text{B.19})$$

B.3 Derivatives of Pressure

It would be convenient if we could express $P_{2\mathcal{H},vv}$ in terms of quantities that we can measure. We can use (B.5) to accomplish this. We would like to express enthalpy as a function of pressure and specific volume: $h(P, v)$

$$h(P, v) = h_2 + [h_{2,P}dP + h_{2,v}dv] + \frac{1}{2} [h_{2,PP}dP^2 + 2h_{2,Pv}dPdv + h_{2,vv}dv^2] \quad (\text{B.20})$$

To evaluate $P_{2\mathcal{H},vv}$, we will look again at states near state 2. Equation B.20 close to state 2 is

$$\begin{aligned} h(P, v) = h_2 + [h_{2,P}(P - P_2) + h_{2,v}(v - v_2)] + \\ \frac{1}{2} [h_{2,PP}(P - P_2)^2 + 2h_{2,Pv}(P - P_2)(v - v_2) + h_{2,vv}(v - v_2)^2] \end{aligned} \quad (\text{B.21})$$

To simplify this equation, substitute $v - v_2 = \delta v$.

$$\begin{aligned} h(P, v) = h_2 + [h_{2,P}(P - P_2) + h_{2,v}\delta v] + \\ \frac{1}{2} [h_{2,PP}(P - P_2)^2 + 2h_{2,Pv}(P - P_2)\delta v + h_{2,vv}(\delta v)^2] \end{aligned} \quad (\text{B.22})$$

Now we can group terms

$$h(P, v) = [h_2] + \delta v [h_{2,v}] + (\delta v)^2 \left[\frac{h_{2,vv}}{2} \right] + (P - P_2) [h_{2,P} + \delta v(h_{2,Pv})] + (P - P_2)^2 \left[\frac{h_{2,PP}}{2} \right] \quad (\text{B.23})$$

Substituting (B.23) the Hugoniot equation B.5 gives

$$\begin{aligned} [h_2 - h_1] + \delta v [h_{2,v}] + (\delta v)^2 \left[\frac{h_{2,vv}}{2} \right] + (P - P_2) [h_{2,P} + \delta v(h_{2,Pv})] + (P - P_2)^2 \left[\frac{h_{2,PP}}{2} \right] \\ = [(P - P_2) + (P_2 - P_1)] \left(\frac{v_1 + v_2}{2} + \frac{\delta v}{2} \right) \end{aligned} \quad (\text{B.24})$$

If we substitute the Taylor expansion for Hugoniot pressure (B.6 this equation becomes

$$\begin{aligned} \left[(h_2 - h_1) - (P_2 - P_1) \frac{v_2 + v_1}{2} \right] + \delta v \left[h_{2,v} - \frac{P_2 - P_1}{2} \right] + (\delta v)^2 \left[\frac{h_{2,vv}}{2} \right] \\ = \left(P_{2\mathcal{H},v} \delta v + \frac{1}{2} P_{2\mathcal{H},vv} (\delta v)^2 \right) \left[\frac{v_2 + v_1}{2} - h_{2,P} + \delta v \left(\frac{1}{2} - h_{2,Pv} \right) \right] - \\ \left(P_{2\mathcal{H},v}^2 (\delta v)^2 + P_{2\mathcal{H},v} P_{2\mathcal{H},vv} (\delta v)^3 + \frac{1}{4} P_{2\mathcal{H},vv}^2 (\delta v)^4 \right) \left[\frac{h_{2,PP}}{2} \right] \end{aligned} \quad (\text{B.25})$$

As before we can group powers of δv

- Zero Order

$$(h_2 - h_1) - (P_2 - P_1) \frac{v_2 + v_1}{2} = 0 \quad (\text{B.26})$$

This is exactly the Hugoniot curve expression (B.5) evaluated at state 2. Therefore, these terms cancel.

- First Order

$$h_{2,v} - \frac{P_2 - P_1}{2} = P_{2\mathcal{H},v} \left(\frac{v_2 + v_1}{2} - h_{2,P} \right) \quad (\text{B.27})$$

so

$$P_{2\mathcal{H},v} = \left[h_{2,v} - \frac{P_2 - P_1}{2} \right] \left[\frac{2}{v_2 + v_1 - 2h_{2,P}} \right] \quad (\text{B.28})$$

- Higher Order

$$\begin{aligned} (\delta v)^2 \frac{h_{2,vv}}{2} = & (\delta v)^2 P_{2\mathcal{H},v} \left(\frac{1}{2} - h_{2,Pv} \right) + \frac{(\delta v)^2}{2} P_{2\mathcal{H},vv} \left[\frac{v_2 + v_1}{2} - h_{2,P} + \delta v \left(\frac{1}{2} - h_{2,Pv} \right) \right] \\ & - \left(P_{2\mathcal{H},v}^2 (\delta v)^2 + P_{2\mathcal{H},v} P_{2\mathcal{H},vv} (\delta v)^3 + \frac{1}{4} P_{2\mathcal{H},vv}^2 (\delta v)^4 \right) \left[\frac{h_{2,PP}}{2} \right] \end{aligned} \quad (\text{B.29})$$

If we only keep the lowest order term on each side, the equation simplifies to

$$\frac{h_{2,vv}}{2} = P_{2\mathcal{H},v} \left(\frac{1}{2} - h_{2,Pv} \right) + \frac{1}{2} P_{2\mathcal{H},vv} \left[\frac{v_2 + v_1}{2} - h_{2,P} \right] - P_{2\mathcal{H},v}^2 \left[\frac{h_{2,PP}}{2} \right] \quad (\text{B.30})$$

Solving for $P_{2\mathcal{H},vv}$, we get

$$P_{2\mathcal{H},vv} = [h_{2,vv} + P_{2\mathcal{H},v}^2 (h_{2,PP}) + P_{2\mathcal{H},v} (2h_{2,Pv} - 1)] \left[\frac{2}{v_2 + v_1 - 2h_{2,P}} \right] \quad (\text{B.31})$$

B.4 Thermodynamic Analysis

We would like to express the derivatives of enthalpy as functions of quantities that we can determine so that we can evaluate the derivatives of pressure. To determine these derivatives of enthalpy we need two fundamental equations as well as the definitions of the Grüneisen Coefficient, \mathcal{G} , and the equilibrium sound speed, a_{eq} .

$$dh = Tds + vdP \quad (\text{B.32})$$

$$Tds = de + Pdv \quad (\text{B.33})$$

$$\mathcal{G} = v (P_e)_v \quad (\text{B.34})$$

$$a_{eq}^2 = -v^2 (P_v)_s \quad (\text{B.35})$$

First we will evaluate the first order partial derivatives of enthalpy

$$(h,P)_v = (e,P + (Pv),P)_v \quad (\text{B.36})$$

$$= (e,P)_v + v \quad (\text{B.37})$$

$$= \frac{v}{\mathcal{G}} + v \quad (\text{B.38})$$

$$(h,P)_v = v \frac{\mathcal{G} + 1}{\mathcal{G}} \quad (\text{B.39})$$

Similarly

$$(h, v)_P = (Ts, v + vP, v)_P \quad (\text{B.40})$$

$$= (Ts, v)_P \quad (\text{B.41})$$

$$= -(Ts, P)_v (P, v)_s \quad (\text{B.42})$$

$$= -(e, P)_v (P, v)_s \quad (\text{B.43})$$

$$= -\left(\frac{v}{\mathcal{G}}\right) \left(\frac{-a_{eq}^2}{v^2}\right) \quad (\text{B.44})$$

$$(h, v)_P = \frac{a_{eq}^2}{v} \frac{1}{\mathcal{G}} \quad (\text{B.45})$$

To evaluate the second order partial derivatives of enthalpy, we need to take derivatives of the above expressions

$$(h, PP)_v = (v, P)_v \left(1 + \frac{1}{\mathcal{G}}\right) - \frac{v}{\mathcal{G}^2} (\mathcal{G}, P)_v \quad (\text{B.46})$$

$$(h, vv)_P = \frac{a_{eq}}{v\mathcal{G}} \left(2(a_{eq, v})_P - \frac{a_{eq}}{v\mathcal{G}} (\mathcal{G} + v(\mathcal{G}, v)_P)\right) \quad (\text{B.47})$$

$$([(h, P)_v], v)_P = (v, v)_P \left(1 + \frac{1}{\mathcal{G}}\right) - \frac{v}{\mathcal{G}^2} (\mathcal{G}, v)_P \quad (\text{B.48})$$

$$([(h, v)_P], P)_v = \frac{a_{eq}}{v\mathcal{G}} \left(2(a_{eq, P})_v - \frac{a_{eq}}{v\mathcal{G}} (\mathcal{G}(v, P)_v + v(\mathcal{G}, P)_v)\right) \quad (\text{B.49})$$

The mixed partials should be equivalent so

$$\mathcal{G} + 1 + \left(\frac{a_{eq}^2}{v\mathcal{G}} - \frac{v}{\mathcal{G}}\right) (\mathcal{G}, P)_v = -\frac{a_{eq}^2}{v^2} (v, P)_v + \frac{2a_{eq}}{v} (a_{eq, P})_v \quad (\text{B.50})$$

Plugging in these expressions for derivatives of enthalpy into the derivatives of pressure gives

$$P_{2\mathcal{H}, v} = \left[\frac{a_{2eq}^2}{v_2} \frac{1}{\mathcal{G}_2} - \frac{P_2 - P_1}{2}\right] \left[\frac{2\mathcal{G}_2}{\mathcal{G}_2(v_1 - v_2) - 2v_2}\right] \quad (\text{B.51})$$

$$\begin{aligned} P_{2\mathcal{H}, vv} &= \frac{a_{2eq}}{v_2\mathcal{G}_2} \left[\frac{2\mathcal{G}_2}{\mathcal{G}_2(v_1 - v_2) - 2v_2}\right] \left[2(a_{2eq, v})_P - \frac{a_{2eq}}{v_2\mathcal{G}_2} (\mathcal{G}_2(v_{2, P})_v + v_2(\mathcal{G}_{2, v})_P)\right] \\ &+ P_{2\mathcal{H}, v}^2 \left[\frac{2\mathcal{G}_2}{\mathcal{G}_2(v_1 - v_2) - 2v_2}\right] \left[(v_{2, P})_v \left(1 + \frac{1}{\mathcal{G}_2}\right) - \frac{v_2}{\mathcal{G}_2^2} (\mathcal{G}_{2, P})_v\right] \\ &+ P_{2\mathcal{H}, v} \left[\frac{2\mathcal{G}_2}{\mathcal{G}_2(v_1 - v_2) - 2v_2}\right] [2h_{2, Pv} - 1] \end{aligned} \quad (\text{B.52})$$

B.5 Perfect Gas Analysis

For a perfect gas, the specific heat is constant and the equation of state is given by

$$P = \rho RT \quad (\text{B.53})$$

$$h = c_P T \quad (\text{B.54})$$

The derivatives of enthalpy are

$$h_{,P} = \frac{\gamma}{\gamma-1}v \quad (\text{B.55})$$

$$h_{,v} = \frac{\gamma}{\gamma-1}P \quad (\text{B.56})$$

$$h_{,PP} = 0 \quad (\text{B.57})$$

$$h_{,vv} = 0 \quad (\text{B.58})$$

$$h_{,Pv} = \frac{\gamma}{\gamma-1} \quad (\text{B.59})$$

Plugging these derivatives of enthalpy into the derivatives of pressure (B.51)-(B.52), we get

$$P_{2\mathcal{H},v} = \left[\frac{\gamma}{\gamma-1}P_2 - \frac{P_2 - P_1}{2} \right] \left[\frac{2}{v_1 + v_2 - 2\frac{\gamma}{\gamma-1}v_2} \right] \quad (\text{B.60})$$

$$\begin{aligned} &= \left[P_1 + P_2 \frac{\gamma+1}{\gamma-1} \right] \left[\frac{\gamma-1}{v_1(\gamma-1) + v_2(\gamma+1)} \right] \\ P_{2\mathcal{H},vv} &= \left[0 + 0 + P_{2\mathcal{H},v} \left(2\frac{\gamma}{\gamma-1} - 1 \right) \right] \left[\frac{2}{v_1 + v_2 - 2\frac{\gamma}{\gamma-1}v_2} \right] \\ &= 2P_{2\mathcal{H},v} \left[\frac{\gamma+1}{\gamma-1} \right] \left[\frac{\gamma-1}{v_1(\gamma-1) + v_2(\gamma+1)} \right]^2 \end{aligned} \quad (\text{B.61})$$

We can check the validity of (B.60)-(B.61) if we use the perfect gas expressions in the Hugoniot equation (B.5) directly, i.e.

$$\frac{\gamma}{\gamma-1}(Pv - P_1v_1) = (P - P_1)\frac{(v + v_1)}{2} \quad (\text{B.62})$$

which simplifies as follows

$$P_1 \left[\frac{(v + v_1)}{2} - v_1 \frac{\gamma}{\gamma-1} \right] = P \left[\frac{(v + v_1)}{2} - v \frac{\gamma}{\gamma-1} \right] \quad (\text{B.63})$$

$$(P_1v - P_1v_1) = (Pv - P_1v_1) \left(\frac{\gamma+1}{\gamma-1} \right) \quad (\text{B.64})$$

Now, we perturb the system as before and look at terms of the same order

$$P_1 \left[\frac{(v_2 + \delta v + v_1)}{2} - v_1 \frac{\gamma}{\gamma-1} \right] = \left(P_2 + P_{2\mathcal{H},v}\delta v + \frac{1}{2}P_{2\mathcal{H},vv}(\delta v)^2 \right) \left[\frac{(v_2 + \delta v + v_1)}{2} - (v_2 + \delta v) \frac{\gamma}{\gamma-1} \right] \quad (\text{B.65})$$

$$\begin{aligned} \left[P_1 \left(v_2 - v_1 \frac{\gamma+1}{\gamma-1} \right) \right] + \delta v &= \left(P_2 + P_{2\mathcal{H},v}\delta v + \frac{1}{2}P_{2\mathcal{H},vv}(\delta v)^2 \right) \\ &\quad \left[\left(v_1 - v_2 \frac{\gamma+1}{\gamma-1} \right) - \delta v \left(\frac{\gamma+1}{\gamma-1} \right) \right] \end{aligned} \quad (\text{B.66})$$

• Zero Order

$$P_1 \left(v_2 - v_1 \frac{\gamma+1}{\gamma-1} \right) = P_2 \left(v_1 - v_2 \frac{\gamma+1}{\gamma-1} \right) \quad (\text{B.67})$$

- First Order

$$1 = -P_2 \left(\frac{\gamma+1}{\gamma-1} \right) + P_{2\mathcal{H},v} \left(v_1 - v_2 \frac{\gamma+1}{\gamma-1} \right) \quad (\text{B.68})$$

so

$$P_{2\mathcal{H},v} = \left[1 + P_2 \left(\frac{\gamma+1}{\gamma-1} \right) \right] \left[\frac{\gamma-1}{v_1(\gamma-1) - v_2(\gamma+1)} \right] \quad (\text{B.69})$$

This is exactly (B.60).

- Higher Order

$$\begin{aligned} 0 = & -(\delta v)^2 P_{2\mathcal{H},v} \left(\frac{\gamma+1}{\gamma-1} \right) + \\ & \frac{(\delta v)^2}{2} P_{2\mathcal{H},vv} \left[\left(v_1 - v_2 \frac{\gamma+1}{\gamma-1} \right) - \delta v \left(\frac{\gamma+1}{\gamma-1} \right) \right] \end{aligned} \quad (\text{B.70})$$

If we only keep the lowest order term on each side, the equation simplifies to

$$0 = -P_{2\mathcal{H},v} \left(\frac{\gamma+1}{\gamma-1} \right) + \frac{P_{2\mathcal{H},vv}}{2} \left(v_1 - v_2 \frac{\gamma+1}{\gamma-1} \right) \quad (\text{B.71})$$

Solving for $P_{2\mathcal{H},vv}$

$$P_{2\mathcal{H},vv} = 2P_{2\mathcal{H},v} \left[\frac{\gamma+1}{\gamma-1} \right] \left[\frac{\gamma-1}{v_1(\gamma-1) - v_2(\gamma+1)} \right] \quad (\text{B.72})$$

We see that this expression for $P_{2\mathcal{H},vv}$ is identical to (B.61) which verifies that the general solution reduces correctly to the perfect gas model solution.