Ques 13. The sum of three positive numbers is constant. Prove that their product is maximum when they are equal.

$$x + y + z = k$$

$$f = xyz \quad , \quad \phi = x + y + z - k$$

$$F(x,y,z) = \cancel{x+y+} \; xyz + \lambda[x + y + z - k] \quad \text{--- (1)}$$

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \Rightarrow yz + \lambda = 0 \Rightarrow -\lambda = yz \quad \text{--- (2)}$$

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \Rightarrow xz + \lambda = 0 \Rightarrow -\lambda = +xz \quad \text{--- (3)}$$

$$\frac{\partial f}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0 \Rightarrow xy + \lambda = 0 \Rightarrow -\lambda = +xy \quad \text{--- (4)}$$

From eqⁿ — (2), (3) & (4) we get

$$-\lambda = yz = xz = xy$$

$$\Rightarrow xz = yz \Rightarrow x = y \quad | \quad \Rightarrow xy = yz \Rightarrow x = z$$

i.e, $x = y = z$

then $x + y + z = k$

$$x + x + x = k$$

$$3x = k$$

$$x = \frac{k}{3}$$

$$\Rightarrow x = y = z = \frac{k}{3}$$

$$f(x,y,z) = x \cdot y \cdot z = \frac{k}{3} \cdot \frac{k}{3} \cdot \frac{k}{3}$$

$$\boxed{f(x,y,z) = \frac{k^3}{27}}$$

Ques 14. Using Beta & Gamma functions evaluate the followings :-

(a) $\int_0^\infty \frac{x^c}{c^x}\, dx = \frac{\Gamma(c+1)}{(\log c)^{c+1}}$; $c>1$

$c^x = e^{\log c^x}$
$c^x = e^{x \log c}$

$= \int_0^\infty \frac{x^c}{c^x}\, dx$

$= \int_0^\infty x^c \, e^{-x\log c} . dx$

let $x \log c = t$
$\log c \, dx = dt$
lim : 0 to ∞

$= \int_0^\infty \left(\frac{t}{\log c}\right)^c e^{-t} \frac{dt}{\log c}$

$= \frac{1}{(\log c)^{c+1}} \int_0^\infty t^{c+1-1} . e^{-t} . dt$

$= \frac{1}{(\log c)^{c+1}} \int_0^\infty t^{(c+1)-1} e^{-t} . dt$

$= \boxed{\frac{\Gamma c+1}{(\log c)^{c+1}}}$

Ans.

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Maths Assignment - 1

Ques 1. If $y = \sqrt{x+2}$, find y_n

$y = (x+2)^{1/2}$

We know the formula for n^{th} derivative $(ax+b)^m$

$y = (ax+b)^m$

$y_n = m(m-1)(m-2) \ldots m-(n-1)\,(ax+b)^{m-n} \cdot a^n$

So, when $y = (x+2)^{1/2}$

$y_n = \frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right) \ldots \left(\frac{1}{2}-(n-1)\right)(x+2)^{\frac{1}{2}-n} \cdot 1^n$ Ans

Ques 3. If $u = \log\sqrt{x^2+y^2+z^2}$ show that $(x^2+y^2+z^2)\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = 1$

Solution: $u = \log\sqrt{x^2+y^2+z^2}$

$u = \log(x^2+y^2+z^2)^{1/2}$

$u = \frac{1}{2}\log(x^2+y^2+z^2)$

$$\frac{\partial u}{\partial x} = \frac{1}{2} \cdot \frac{1}{x^2+y^2+z^2} \cdot 2x$$

$$\frac{\partial u}{\partial x} = \frac{x}{x^2+y^2+z^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{-x^2+y^2+z^2}{(x^2+y^2+z^2)^2} \quad \text{---(1)}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{-y^2+x^2+z^2}{(x^2+y^2+z^2)^2} \quad \text{---(2)}$$

$$\frac{\partial^2 u}{\partial z^2} = \frac{-z^2+x^2+y^2}{(x^2+y^2+z^2)^2} \quad \text{---(3)}$$

Adding eq ①, ② and ③ and multiplying it by $x^2+y^2+z^2$ we get,

$$= (x^2+y^2+z^2)\left(\frac{-x^2+y^2+z^2}{(x^2+y^2+z^2)^2} + \frac{-y^2+x^2+z^2}{(x^2+y^2+z^2)^2} + \frac{-z^2+x^2+y^2}{(x^2+y^2+z^2)^2}\right)$$

$$= (x^2+y^2+z^2)\left(\frac{-\cancel{x^2}+\cancel{y^2}+\cancel{z^2}-\cancel{y^2}+\cancel{x^2}+z^2-\cancel{z^2}+x^2+y^2}{(x^2+y^2+z^2)^2}\right)$$

$$= \cancel{(x^2+y^2+z^2)}\left(\frac{\cancel{x^2+y^2+z^2}}{(x^2+y^2+z^2)^{\cancel{2}}}\right)$$

$$= 1$$

LHS = RHS

Hence proved

Ques 4: If $x = e^r\cos\theta$, $y = e^r\sin\theta$ show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-2r}\left(\frac{\partial^2 u}{\partial r^2} + \frac{\partial^2 u}{\partial \theta^2}\right)$

~~We don't have u here~~
~~how can I differenciate it??~~

Ques 5 → Verify euler's theorem for the functions:

(i) $u = (x^{1/2}+y^{1/2})(x^n+y^n)$

(ii) $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$

Solution: (i) $u = (x^{1/2}+y^{1/2})(x^n+y^n)$

putting $x = xt$ and $y = yt$

$$u = (x^{1/2}\cdot t^{1/2} + y^{1/2}\cdot t^{1/2})\cdot(x^n t^n + y^n t^n)$$

$$u = t^{1/2}(x^{1/2}+y^{1/2})\cdot t^n(x^n+y^n)$$

$$u = t^{1/2+n}(x^{1/2}+y^{1/2})\cdot(x^n+y^n)$$

Degree of u is $\left(\frac{1}{2}+n\right)$.

$$u = (x^{1/2} + y^{1/2})(x^n + y^n) \quad \text{--- } ①$$

partially diff. eq ① w.r.t x

$$\frac{\partial u}{\partial x} = (x^{1/2} + y^{1/2}) \cdot n x^{n-1} + (x^n + y^n) \cdot \frac{1}{2} x^{\frac{1}{2}-1}$$

$$\frac{\partial u}{\partial x} = (x^{1/2} + y^{1/2}) \cdot n x^{n-1} + (x^n + y^n) \cdot \frac{1}{2} x^{-1/2}$$

Multiply both sides by 'x';

$$x \frac{\partial u}{\partial x} = (x^{1/2} + y^{1/2}) \cdot n x^n + (x^n + y^n) \cdot \frac{1}{2} x^{1/2} \quad \text{--- } ②$$

partially diff. eq ① w.r.t y

$$\frac{\partial u}{\partial y} = (x^{1/2} + y^{1/2}) \cdot n y^{n-1} + (x^n + y^n) \cdot \frac{1}{2} y^{-1/2}$$

Multiply both sides with y

$$y \frac{\partial u}{\partial y} = (x^{1/2} + y^{1/2}) n y^n + (x^n + y^n) \cdot \frac{1}{2} y^{1/2} \quad \text{--- } ③$$

Adding eq^n ② and ③ we get;

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2} x^{1/2}(x^n + y^n) + (x^{1/2} + y^{1/2}) \cdot n x^n + \frac{1}{2} y^{1/2}(x^n + y^n) + (x^{1/2} + y^{1/2}) n y^n$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2} x^{1/2}(x^n + y^n) + \frac{1}{2} y^{1/2}(x^n + y^n) + (x^{1/2} + y^{1/2}) n x^n + (x^{1/2} + y^{1/2}) n y^n$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2}(x^n + y^n)(x^{1/2} + y^{1/2}) + n(x^{1/2} + y^{1/2})(x^n + y^n)$$

$$x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \frac{1}{2} u + n u$$

$$\boxed{x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = \left(\frac{1}{2} + n\right) u}$$

$$\therefore n = \frac{1}{2} + n$$

Hence verified

(ii) $u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$

$$u = x^0 \sin^{-1}\left(\frac{x}{y}\right) + x^0 \tan^{-1}\left(\frac{y}{x}\right)$$

$$u = x^0 \sin^{-1}\left(\frac{1}{\frac{y}{x}}\right) + x^0 \tan^{-1}\left(\frac{y}{x}\right)$$

$$u = x^0 \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right)$$

$$u = x^0 f\left(\frac{y}{x}\right) \quad \text{---} (1)$$

Since eq^n (1) is in the form of $u = x^n f(y/x)$ so it is a homogeneous function at $\boxed{n = 0}$

Now, we know that, by Euler's theorem

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Here $n = 0$

So,

$$\boxed{x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0}$$

Let's verify Euler's theorem

$$u = \sin^{-1}\left(\frac{x}{y}\right) + \tan^{-1}\left(\frac{y}{x}\right) \quad \text{---} (1)$$

partially diff eq^n (1) w.r.t x

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \frac{1}{y} + \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{-y}{x^2}$$

Multiply x on both sides

$$x\frac{\partial u}{\partial x} = \cancel{\frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \frac{x}{y}} + \cancel{\frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{-y}{x}}$$

partially diff eq^n (1) w.r.t y

$$\frac{\partial u}{\partial y} =$$

$$\frac{\partial u}{\partial x} = \frac{\cancel{y}}{\sqrt{y^2-x^2}} \cdot \frac{1}{\cancel{y}} + \frac{\cancel{x^2}}{x^2+y^2} \cdot \left(\frac{-y}{\cancel{x^2}}\right)$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{y^2-x^2}} - \frac{y}{x^2+y^2}$$

Multiply by x on both sides

$$x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2-x^2}} - \frac{xy}{x^2+y^2} \quad \text{---} \ (2)$$

partially diff. eq (1) w.r.t y, we get

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\left(\frac{x}{y}\right)^2}} \cdot \frac{-x}{y^2} + \frac{1}{1+\left(\frac{y}{x}\right)^2} \cdot \frac{1}{x}$$

$$\frac{\partial u}{\partial y} = \frac{\cancel{y}}{\sqrt{y^2-x^2}} \cdot \frac{-x}{y^{\cancel{2}}} + \frac{x^{\cancel{2}}}{x^2+y^2} \cdot \frac{1}{\cancel{x}}$$

$$\frac{\partial u}{\partial y} = \frac{-x}{y\sqrt{y^2-x^2}} + \frac{x}{x^2+y^2}$$

Multiply by y on both sides

$$y\frac{\partial u}{\partial y} = \frac{-x\cancel{y}}{\cancel{y}\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2}$$

$$y\frac{\partial u}{\partial y} = \frac{-x}{\sqrt{y^2-x^2}} + \frac{xy}{x^2+y^2} \quad \text{---} \ (3)$$

Adding eq^n (2) and (3) we get,

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \cancel{\frac{x}{\sqrt{y^2-x^2}}} - \cancel{\frac{xy}{x^2+y^2}} - \cancel{\frac{x}{\sqrt{y^2-x^2}}} + \cancel{\frac{xy}{x^2+y^2}}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

$$= nu$$

Hence verified.

Ques 6 → Show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2\tan(u)$

where $u = \sin^{-1}\left(\frac{x^3+y^3+z^3}{ax+by+cz}\right)$

Solution: $u = \sin^{-1}\left(\frac{x^3+y^3+z^3}{ax+by+cz}\right)$

putting $x = xt$ and $y = yt$ $z = zt$ in above eq^n

$$\sin u = \frac{x^3t^3 + y^3t^3 + z^3t^3}{axt + byt + czt}$$

$$\sin u = \frac{t^3(x^3+y^3+z^3)}{t(ax+by+cz)}$$

$$\sin u = t^2\left(\frac{x^3+y^3+z^3}{ax+by+cz}\right)$$

$\boxed{\text{Degree} = 2}$

Now we can solve it with deduction method

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{nf(u)}{f'(u)}$$

$$x\frac{\partial u}{\partial x} + y\cdot\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \frac{n\,f(\sin u)}{f'(\sin u)}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2\frac{\sin u}{\cos u}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 2\tan u$$

Hence proved

Ques 7 → If $u = x \log(xy)$, where $x^3 + y^3 + 3xy = 1$, find $\frac{du}{dx}$

Solution: $u = x \log(xy)$

$x^3 + y^3 + 3xy = 1$

By using total differenciation;

$$du = \left(\frac{\partial u}{\partial x}\right)dx + \left(\frac{\partial u}{\partial y}\right)dy$$

$$\frac{\partial u}{\partial x} = \left(\frac{\partial u}{\partial x}\right) + \left(\frac{\partial u}{\partial y}\right)\left(\frac{dy}{dx}\right)$$

$u = x \log xy$ --- ①

Partially diff above eq^n ① w.r.t x

$$\frac{\partial u}{\partial x} = \log xy + \frac{xy}{xy}$$

$$\frac{\partial u}{\partial x} = 1 + \log xy$$

Partially diff eq ① w.r.t y

$$\frac{\partial u}{\partial y} = (x)\left(\frac{1}{xy}\right)(x) = \frac{x}{y}$$

$x^3 + y^3 + 3xy = 1$

By diff. w.r.t x we get,

$$3x^2 + 3y^2 \frac{dy}{dx} + 3y + 3x \frac{dy}{dx} = 0$$

$$\frac{dy}{dx}[3x + 3y^2] = -[3x^2 + 3y]$$

$$\frac{dy}{dx} = -\left[\frac{x^2 + y}{x + y^2}\right]$$

Now, $\frac{du}{dx} = \frac{\partial u}{\partial x} + \left(\frac{\partial u}{\partial y}\right)\left(\frac{dy}{dx}\right)$

$$= 1 + \log xy + \frac{x}{y}\left[\frac{-(x^2+y)}{(x + y^2)}\right]$$

$$= 1 + \log xy - \frac{x(x^2+y)}{y(x+y^2)}$$

$$\boxed{\frac{du}{dx} = 1 + \log xy - \frac{x(x^2+y)}{y(x+y^2)}}$$ Ans

Ques 10 → If $u = x \sin y$ and $v = y \sin x$, then find $\frac{\partial(u,v)}{\partial(x,y)}$

Solution: $$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$$

$$\frac{\partial u}{\partial x} = \sin y$$

$$\frac{\partial u}{\partial y} = x \cos y$$

$$\frac{\partial v}{\partial x} = y \cos x$$

$$\frac{\partial v}{\partial y} = \sin x$$

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \sin y & x \cos y \\ y \cos x & \sin x \end{vmatrix}$$

$$= \boxed{(\sin x \cdot \sin y) - (y \cos x \cdot x \cos y)} \text{ Ans}$$

$$\cancel{\therefore \sin x \cdot \sin y = \frac{\cos(x-y) - \cos(x+y)}{2}}$$

$$= \cancel{\frac{\cos(x-y) - \cos(x+y)}{2}}$$

Ques 11 → If $u = xyz$, $v = x^2+y^2+z^2$, $w = x+y+z$

then compute the Jacobian $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$

Solution: $f_1 = u - xyz \neq 0$

$f_2 = v-(x^2+y^2+z^2) = 0 \Rightarrow f_2 = v - x^2 - y^2 - z^2 = 0$

$f_3 = w - x - y - z$

Solution: First we will find $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\[2mm] \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\[2mm] \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{vmatrix}$$

$\dfrac{\partial u}{\partial x} = yz$; $\dfrac{\partial u}{\partial y} = xz$; $\dfrac{\partial u}{\partial z} = xy$

$\dfrac{\partial v}{\partial x} = 2x$; $\dfrac{\partial v}{\partial y} = 2y$; $\dfrac{\partial v}{\partial z} = 2z$

$\dfrac{\partial w}{\partial x} = 1$; $\dfrac{\partial w}{\partial y} = 1$; $\dfrac{\partial w}{\partial z} = 1$

$$\frac{\partial(u,w,w)}{\partial(x,y,z)} = \begin{vmatrix} yz & xz & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix}$$

$$= 1(2xz^2 - 2xy^2) - 1(2yz^2 - 2yx^2) + 1(2zy^2 - 2zx^2)$$

$$= 2xz^2 - 2xy^2 - 2yz^2 + 2yx^2 + 2zy^2 - 2zx^2$$

$$= 2(xz^2 - \cancel{2}xy^2 - \cancel{2}yz^2 + \cancel{2}yx^2 + zy^2 - zx^2)$$

We know that, $\dfrac{\partial(u,v,w)}{\partial(x,y,z)} \times \dfrac{\partial(x,y,z)}{\partial(u,v,w)} = 1$

So, $\boxed{\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \dfrac{1}{2(xz^2 - xy^2 - yz^2 + yx^2 + zy^2 - zx^2)}}$ Ans

Ques 2: $y = e^{x} \sin^3 x$

$$\left\{ \begin{array}{l} \sin 3x = 3 \sin x - 4 \sin^3 x \\ \sin^3 x = \dfrac{3 \sin x - \sin 3x}{4} \end{array} \right.$$

$$y = \frac{e^x}{4} [3 \sin x - \sin 3x]$$

$$y = \frac{3}{4} \underbrace{e^x \sin x} - \frac{1}{4} \underbrace{e^x \sin 3x}$$

$a_1 = 1, \ b_1 = 1$ $\qquad$ $a_2 = 1, \ b_2 = 3$

$$r_1 = \sqrt{a_1^2 + b_1^2} \qquad r_2 = \sqrt{a_2^2 + b_2^2}$$

$$= \sqrt{2} = (2)^{1/2} \qquad = \sqrt{10} = (10)^{1/2}$$

$$\theta_1 = \tan^{-1} \frac{b_1}{a_1} = \tan^{-1} \frac{1}{1} \qquad \theta_2 = \tan^{-1} \frac{b_2}{a_2} = \tan^{-1} \frac{3}{3}$$

$$\tan^{-1} 1 = \frac{\pi}{4} \qquad = \tan^{-1} 3$$

We know,

$$\left\{ \begin{array}{l} y = e^{ax} \sin(bx + c) \\ y_n = r^n e^{ax} \sin(bx + c + n\theta) \end{array} \right\}$$

Then,

$$y_n = \frac{3}{4} \times (2)^{n/2} e^x \sin\left(x + n\frac{\pi}{4}\right) - \frac{1}{4} (10)^{n/2} e^x \sin(3x + n \tan^{-1} 3) \text{ Ans}$$

Ques 12 → Let x, y, z be the length, breadth, height of the rectangular box respectively.

$V = xyz = \phi$ $\quad\quad$ $S = 2yz + 2zx + xy$

$\phi = V - xyz$

$F = f + \lambda\phi$

$$\frac{\partial S}{\partial x} + \lambda\frac{\partial \phi}{\partial x} = 0 \Rightarrow 2z + y + \lambda(-yz) = 0 \quad \text{--- (1)}$$

$$\frac{\partial S}{\partial y} + \lambda\frac{\partial \phi}{\partial y} = 0 \Rightarrow 2z + x + \lambda(-xz) = 0 \quad \text{--- (2)}$$

$$\frac{\partial S}{\partial z} + \lambda\frac{\partial \phi}{\partial z} = 0 \Rightarrow 2y + 2x + \lambda(-xy) = 0 \quad \text{--- (3)}$$

Comparing (1) & (2)

$$\lambda = \frac{y + 2z}{yz}, \quad \lambda = \frac{x + 2z}{xz}; \quad \boxed{x = y}$$

putting this in V;

$$xyz = x^2 z = V \qquad z = \frac{V}{x^2}$$

putting this in S,

$$x^2 + 2x\left(\frac{V}{x^2}\right) + 2\left(\frac{V}{x^2}\right)x = x^2 + \frac{2V}{x} + \frac{2V}{x} \Rightarrow x^2 + \frac{4V}{x}$$

$$\frac{\partial S}{\partial x} = 2x - \frac{4V}{x^2} = 0$$

$$\frac{2x^3 - 4V}{x^2} = 0 \Rightarrow 2x^3 - 4V = x^3 = \frac{\cancel{2}4V}{\cancel{2}}$$

$$\boxed{x = 3\sqrt{2V}}$$

Same for y, $\boxed{y = 3\sqrt{2V}}$

$$z = \frac{V}{x^2} = \frac{V}{(3\sqrt{2V})^2} = \frac{V}{9 \times 2V} = \frac{V}{18V} = \boxed{\frac{1}{18} = z}$$

Ques 13

$f = xyz$ $\qquad\qquad$ $x+y+z = k$

$\phi = x+y+z-K$

$$F(x,y,z) = xyz + \lambda[x+y+z-K] \quad \text{---} ①$$

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \Rightarrow yz + \lambda = 0 \Rightarrow -\lambda = yz \quad \text{---} ②$$

$$\frac{\partial f}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \Rightarrow xz + \lambda = 0 \Rightarrow -\lambda = +xz \quad \text{---} ③$$

$$\frac{\partial f}{\partial x} + \lambda \frac{\partial \phi}{\partial z} = 0 \Rightarrow xy + \lambda = 0 \Rightarrow -\lambda = +xy \quad \text{---} ④$$

From eq ②, ③ and ④, we get

$$-\lambda = yz = xz = xy$$

$$x\cancel{z} = y\cancel{z} \Rightarrow x = y \quad \Big| \quad \Rightarrow x\cancel{y} = \cancel{y}z \Rightarrow x = z$$

i.e. $x = y = z$

then, $x+y+z = k$

$x+x+x = k$

$3x = k$

$x = \frac{k}{3}$

$$x = y = z = \frac{k}{3}$$

$$f(x,y,z) = x \cdot y \cdot z = \frac{K}{3} \cdot \frac{K}{3} \cdot \frac{K}{3}$$

$$f(x,y,z) = \frac{K^3}{27} \quad \text{Ans}$$

Ques 14 → Using Beta & Gamma functions evaluate the following -

(a) $\int_0^\infty \frac{x^c}{c^x}\,dx = \frac{\Gamma(c+1)}{(\log c)^{c+1}}$; $c>1$

$= \int_0^\infty \frac{x^c}{c^x}\,dx$

$= \int_0^\infty x^c e^{-x\log c}\,dx$

let $x\log c = t$

$\log c\,dx = dt$

lim: 0 to ∞

$= \int_0^\infty \left(\frac{t}{\log c}\right)^c e^{-t}\,\frac{dt}{\log c}$

$= \frac{1}{(\log c)^{c+1}} \int_0^\infty t^{c+1-1} e^{-t}\,dt$

$= \frac{1}{(\log c)^{c+1}} \int_0^\infty t^{(c+1)-1} e^{-t}\cdot dt$

$= \frac{\Gamma c+1.}{(\log c)^{c+1}}$

Hence proved.