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NOTE

Fuzzy Connected Object Delineation: Axiomatic Path Strength Definition and the Case of Multiple Seeds

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This paper presents an extension of previously published theory and algorithms for object definition based on fuzzy connectedness. In this approach, a strength of connectedness is assigned between every pair of image elements. This is done by finding the strongest among all possible connecting paths between the two elements in each pair. The strength assigned to a particular path is defined as the weakest affinity between successive pairs of elements along the path. Affinity specifies the degree to which elements hang together locally in the image and is determined considering how close the elements are in space and in intensity features. A fuzzy connected object containing a particular seed element specified within the object is computed via dynamic programming. In all reported works so far, the minimum of affinities has been considered to define path strength and the maximum of path strengths has been used to define fuzzy connectedness. The question thus remained all along as to whether there are other valid formulations for fuzzy connectedness. One of the main contributions of this paper is a theoretical investigation under reasonable axioms to establish that maximum of path strengths of minimum of affinities along each path is indeed the one and only valid choice for defining fuzzy connectedness. In the past, a single fuzzy connected object was specified through a single seed element indicated interactively within the object region. When many objects are to be identified in a single image, interactive seed specification becomes cumbersome. Further, selecting exactly one element in each region automatically is more difficult than identifying a set of elements. Hence the theory and algorithms for the single-element case need to be generalized to multiple elements for more effective practical use. This is the second main contribution of this paper. The importance of multiple seeded fuzzy connected object definition from the considerations of both practicality

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and automation in image segmentation are described and illustrated with examples taken from several real medical applications.

1. INTRODUCTION

Multidimensional images are currently available through sensing devices that operate on a wide range of frequency in the electromagnetic spectrum—from ultrasound to visible light to X- and γ-rays [1]. Defining objects in these image data is fundamental to most computerized imaging applications [2, 3]. This activity, generally referred to as image segmentation, spans over three decades [4]. Most real objects often possess heterogeneous material compositions. Further, imaging devices have inherent limitations including spatial, parametric, and temporal resolutions, and the acquired images suffer from inevitable artifacts such as noise, blurring, and background variation. These effects cause the object regions to exhibit a gradation of intensity values. In spite of this graded composition, knowledgeable human observers usually do not have any difficulty in perceiving object regions as a gestalt. That is, image elements in these regions seem to hang together to form the object regions in spite of their gradation of values. These two notions—graded composition and hanging togetherness—must somehow be addressed by theories and strategies that aim at defining objects in acquired images.

The principle of attempting to retain data inaccuracies and object graded composition as realistically as possible in object representations derived from images, and subsequently in object visualization, manipulation, and analysis, is undoubtedly the right stand. Considerable progress has been made toward this goal using fuzzy subset theory as a mathematical vehicle [5]. However, attempts to handle the notion of hanging togetherness also in the same fuzzy setting are sparse [6–11]. Except in some simple situations, considering each image element on its own or in conjunction with just its local neighbors is not sufficient for effective object definition. For effectively addressing hanging togetherness, we believe that the (spatial, topological) relationship among all image elements should be considered. In [10], toward this goal, we developed a framework of fuzzy connected object definition theory and algorithms. In this framework, a local fuzzy relation called *affinity* is defined on the image domain which assigns to every pair of nearby image elements a strength of local hanging togetherness which has a value in [0, 1]. Affinity between two elements depends on their spatial nearness as well as how similar their image intensities and intensity-derived features are. A global fuzzy relation called *fuzzy connectedness* is defined on the image domain, which assigns to every pair (c, d) of image elements a strength of global hanging togetherness that has a value in [0, 1]. To determine this value, every possible path from c to d (a sequence of nearby elements starting from c and ending at d) is considered and the minimum affinity of pairwise elements along the path is determined. This affinity represents the strength of this path. The strength of hanging togetherness (connectedness) between c and d is the largest of the strengths of all paths between c and d. In defining a fuzzy connected object, the strength of connectedness between all possible pairs of image elements has to be determined. It has been shown [10] that this combinatorially horrendous problem can be solved via dynamic programming and that a fuzzy connected object containing a given image element o can be found by determining the strength of connectedness from o to all image elements. An investigation on how to define affinity in practical image segmentation including a scale-based formulation of fuzzy connectedness is reported in

[11]. By allowing object regions to compete in terms of connectedness strengths to win membership of elements, a relative fuzzy connectedness framework is developed in [12] and its iterative extension is described in [13]. Improvement of the computational aspects of fuzzy connectedness is described in [14, 15]. Investigations combining deformable boundary-based and Voronoi diagram-based methods with fuzzy connectedness are reported in [16, 17].

The fuzzy connectedness framework and its extensions [10–13] have been effectively utilized in several medical applications. These include multiple sclerosis lesion detection and quantification via magnetic resonance (MR) imaging [18–20] and blood vessel definition in MR angiography [21, 22], in the separation of bone and soft tissues from skin in computed tomography (CT) images for craniomaxillofacial 3D visualization [23], in automatic breast density estimation [24] via digitized mammograms for breast cancer risk assessment, and in brain tumor delineation via MR imaging [25]. Thousands of 3D images have been successfully processed so far, and, in some of these applications, the algorithms are in routine use in clinical trials. These experiences motivated us to further improve and extend this framework as proposed in this paper.

There are two main motivations for the extensions considered in this paper. The first is theoretical and the second is practical. The theoretical motivation stems from the min–max construct used in defining fuzzy connectedness—to investigate if there are other reasonable constructs. We show in Section 2 that, under a set of reasonable axioms, min–max is the only valid construct that retains the essential theoretical structure, which in turn permits fuzzy connectedness to be computed via dynamic programming. The practical motivation stems from the fact that, in many real applications (such as multiple sclerosis lesion detection [18]), the number of objects to be delineated may be large (100s). In such cases, it may take a significant amount of an operator's time to specify the seeds—one per each object. Automatic selection of seeds by using a conservative segmentation strategy is therefore more attractive than manual selection. When seeds are selected automatically, in general, multiple points are identified in each fuzzy connected object, and it may be impossible to identify exactly one seed point per object. (The seeds identified in the same fuzzy connected object need not be connected in the hard sense). Further, there are applications (such as the definition of vessels in MR angiographic images) wherein multiple seed points per object are needed to ensure that thicker as well as subtle and fine aspects of the same object are captured in its delineation. Thus there is a practical need for the previous theoretical and algorithmic framework to be extended to the case of multiple seed points. We present these generalizations in Sections 2 and 3. In Section 4, results on several applications are presented, and our conclusions are stated in Section 5.

2. THEORY

To begin, we restate in Sections 2.1 and 2.2 some known concepts already discussed in [10] for completeness and establishing the terminology. Some new material is also covered in these sections.

2.1. *Fuzzy Relations, Scenes, Binary Scenes*

Let X be any reference set. A *fuzzy subset* [26], $\mathcal{A}$, of X is a set of ordered pairs, $\mathcal{A} = \{(x, \mu_{\mathcal{A}}(x)) \mid x \in X\}$, where $\mu_{\mathcal{A}} : X \to [0, 1]$. $\mu_{\mathcal{A}}$ is called the *membership function* of $\mathcal{A}$ in X. The fuzzy subset $\mathcal{A}$ is called nonempty if there exists an $x \in X$ such that $\mu_{\mathcal{A}}(x) \neq 0$;

otherwise it is called the empty fuzzy subset of X. We use Φ to denote an empty fuzzy subset and ϕ to denote an empty hard set. The *fuzzy intersection* and *fuzzy union* between two fuzzy subsets, $\mathcal{A}$ and $\mathcal{B}$, of X are defined as follows: $\mathcal{A} \cap \mathcal{B} = \{(x, \mu_{\mathcal{A}\cap\mathcal{B}}(x)) \mid x \in X\}$, where $\mu_{\mathcal{A}\cap\mathcal{B}}(x) = \min[\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x)]$; $\mathcal{A} \cup \mathcal{B} = \{(x, \mu_{\mathcal{A}\cup\mathcal{B}}(x)) \mid x \in X\}$, where $\mu_{\mathcal{A}\cup\mathcal{B}}(x) = \max[\mu_{\mathcal{A}}(x), \mu_{\mathcal{B}}(x)]$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two fuzzy subsets of X. We say that $\mathcal{A}$ *includes* $\mathcal{B}$, denoted by $\mathcal{A} \subset \mathcal{B}$, if and only if $\forall x \in X$, $\mu_{\mathcal{A}}(x) \leq \mu_{\mathcal{B}}(x)$. A *2-ary fuzzy relation* ρ in X is a fuzzy subset of $X \times X$; $\rho = \{((x, y), \mu_\rho(x, y)) \mid x, y \in X\}$, where $\mu_\rho : X \times X \to [0, 1]$. Since we are not interested in fuzzy m-ary relations for $m > 2$, we drop the qualifier "2-ary" for simplicity. We use μ subscripted by the fuzzy subset under consideration to denote the membership function of the fuzzy subset. For hard subsets, μ will denote their characteristic function. Let ρ be any fuzzy relation in X. ρ is said to be *reflexive* if $\forall x \in X$, $\mu_\rho(x, x) = 1$, *symmetric* if $\forall x, y \in X$, $\mu_\rho(x, y) = \mu_\rho(y, x)$, and *transitive* if $\forall x, z \in X$, $\mu_\rho(x, z) = \max_{y \in X}[\min[\mu_\rho(x, y), \mu_\rho(y, z)]]$. A fuzzy relation ρ is called a *similitude relation* in X if it is reflexive, symmetric, and transitive. The analogous concept for hard binary relations is an equivalence relation.

For any $n \geq 2$, let n-dimensional Euclidean space R^n be subdivided into hypercuboids by n mutually orthogonal families of parallel hyperplanes. Assume, with no loss of generality, that the hyperplanes in each family have equal unit spacing so that the hypercuboids are unit hypercubes, and we shall choose coordinates so that the center of each hypercube has integer coordinates. The hypercubes will be called *spels* (an abbreviation for "space elements"). When $n = 2$, spels are called *pixels*, and when $n = 3$ they are called *voxels*. The coordinates of the center of a spel are represented by an n-tuple of integers, defining a point in Z^n. Z^n itself will be thought of as the set of all spels in R^n with the above interpretation of spels, and the concepts of spels and points in Z^n will be used interchangeably.

A fuzzy relation α in Z^n is said to be a *fuzzy adjacency* if it is both reflexive and symmetric. It is desirable that α be such that $\mu_\alpha(c, d)$ is a nonincreasing function of the distance $\|c - d\|$ between c and d. It is not difficult to see that the hard adjacency relations commonly used in digital topology [27] are special cases of fuzzy adjacencies. We call the pair (Z^n, α), where α is a fuzzy adjacency, a *fuzzy digital space*. Fuzzy digital space is a concept that characterizes the underlying digital grid system independent of any image-related concepts. We shall eventually tie this with image-related concepts to arrive at fuzzy object-related notions.

A *scene over* a fuzzy digital space (Z^n, α) is a pair $\mathcal{C} = (C, f)$ where $C = \{c \mid -b_j \leq c_j \leq b_j$ for some $b \in Z^n_+\}$; Z^n_+ is the set of n-tuples of positive integers; f, called *scene intensity*, is a function whose domain is C, called the *scene domain*, and whose range is a set of numbers (usually integers). $\mathcal{C}$ is a *binary scene over* (Z^n, α) if the range of f is $\{0, 1\}$. We assume in this paper that $n \geq 2$.

2.2. *Fuzzy Affinity, Path*

Let $\mathcal{C} = (C, f)$ be a scene over (Z^n, α). Any fuzzy relation κ in C is said to be a *fuzzy spel affinity* (or, *affinity* for short) *in* $\mathcal{C}$ if it is reflexive and symmetric. In practice, however, κ should be such that, for any $c, d \in C$, $\mu_\kappa(c, d)$ is a function of (i) the fuzzy adjacency between c and d; (ii) the homogeneity of the spel intensities at c and d; (iii) the closeness of the spel intensities and of the intensity-based features of c and d to some expected intensity and feature values for the object. Further, $\mu_\kappa(c, d)$ may depend on the actual location of c and d (i.e., μ_κ is shift variant). Some examples of μ_κ are given in [10], and a detailed and an objective comparative study of a variety of functional forms for μ_κ is reported in [11], including a scale-based formulation. Throughout this paper, κ with appropriate subscripts and superscripts will be used to denote fuzzy spel affinities.

A nonempty *path* p_{cd} *in* $\mathcal{C}$ *from* c *to* d is a sequence $\langle c = c_1, c_2, \ldots, c_l = d\rangle$ of $l \geq 1$ spels in C; l is called *length* of the path. (Note that the successive spels in the sequence need not be "adjacent" in the sense that adjacency is usually defined in digital topology [27].) An empty path in $\mathcal{C}$, denoted $\langle\ \rangle$, is a sequence of no spels. Paths of length 2 will be referred to as *link*s. The set of all paths in $\mathcal{C}$ from c to d is denoted by P_{cd}. (Note that c and d are not necessarily distinct.) The set of all paths in $\mathcal{C}$, defined as $\bigcup_{c,d\in C} P_{cd}$, is denoted by $P_{\mathcal{C}}$. We define a binary *join to* operation on $P_{\mathcal{C}}$, denoted "+" as follows. For every two nonempty paths, $p_{cd} = \langle c_1, c_2, \ldots, c_{l_1}\rangle \in P_{\mathcal{C}}$ and $p_{de} = \langle d_1, d_2, \ldots, d_{l_2}\rangle \in P_{\mathcal{C}}$ [note that $c_{l_1} = d_1 = d$],

$$p_{cd} + p_{de} = \left\langle c_1, c_2, \ldots, c_{l_1}, d_2, \ldots, d_{l_2}\right\rangle, \tag{2.1}$$

$$p_{cd} + \langle\ \rangle = p_{cd}, \tag{2.2}$$

$$\langle\ \rangle + p_{de} = p_{de}, \tag{2.3}$$

$$\langle\ \rangle + \langle\ \rangle = \langle\ \rangle. \tag{2.4}$$

Note that the join to operation between $p_{c_1c_2}$ and $p_{d_1d_2}$ is undefined if $c_2 \neq d_1$. It is shown in [10] (Proposition 2.1) that for any scene $\mathcal{C} = (C, f)$ over any fuzzy digital space (Z^n, α), the following relation holds for any two spels $c, e \in C$:

$$P_{ce} = \{p_{cd} + p_{de} \mid d \in \mathcal{C} \text{ and } p_{cd} \in P_{cd} \text{ and } p_{de} \in P_{de}\}. \tag{2.5}$$

We define a binary relation *greater than* on $P_{\mathcal{C}}$, denoted ">", as follows. Let $p = \langle c_1, c_2, \ldots c_{l_p}\rangle$ and $q = \langle d_1, d_2, \ldots d_{l_q}\rangle$ be any paths in $\mathcal{C}$. We say that $p > q$ if and only if we can find a mapping g from the set of spels in q to the set of spels in p that satisfies all of the following conditions:

1. $g(d_i) = c_j$ only if $d_i = c_j$;
2. there exists some $1 \leq m \leq l_p$, for which $g(d_1) = c_m$;
3. for all $1 \leq j < l_q$, whenever $g(d_j) = c_i$, $g(d_{j+1}) = c_k$ for some $k \geq i$ and $c_i = c_{i+1} = \cdots = c_{k-1}$.

Some examples follow: $\langle c_1, c_2, c_3, c_3, c_4, c_5\rangle > \langle c_3, c_4, c_4, c_5\rangle$; $\langle c_1, c_2, c_3, c_4\rangle > \langle c_3, c_3, c_3, c_3, c_3, c_3\rangle$. It readily follows that every nonempty path in $\mathcal{C}$ is greater than the empty path $\langle\ \rangle$ in $\mathcal{C}$.

2.3. *Functional Form for Path Strength, Fuzzy Connectedness*

Our aim is to assign a strength of connectedness to every pair (c, d) of spels in C. It makes sense to consider this strength of connectedness to be the largest of the strengths assigned to all paths between c and d. (The physical analogy one may consider is to think of c and d as being connected by many strings, each with its own strength. When c and d are pulled apart the strongest string will break at the end, which should be the determining factor for the strength of connectedness between c and d.) However, it is not so obvious as to how the strength of each path should be defined. Several measures based on the affinities along the path, including their sum, product, and minimum, all seem plausible. We prove in this section that minimum of affinities is the only valid choice for path strength under the assumptions stated in Axioms 1–4 below, which, we believe are all reasonable.

Let $\mathcal{C} = (C, f)$ be a scene over a fuzzy digital space (Z^n, α) and let κ be a fuzzy affinity in $\mathcal{C}$. A *fuzzy* κ-net $\mathcal{N}$ in $\mathcal{C}$ is a fuzzy subset of $P_{\mathcal{C}}$ with its membership function $\mu_{\mathcal{N}} : P_{\mathcal{C}} \rightarrow [0, 1]$. $\mu_{\mathcal{N}}$ assigns a strength to every path of $P_{\mathcal{C}}$. For any spels $c, d \in C$, $p_{cd} \in P_{cd}$ is called a *strongest path* from c to d if $\mu_{\mathcal{N}}(p_{cd}) = \max_{p \in P_{cd}}[\mu_{\mathcal{N}}(p)]$. One of the questions this paper addresses is how to assign strengths to paths or, equivalently, what the functional form of $\mu_{\mathcal{N}}$ should be. The idea of a κ-net is to set up a network of all possible paths between all possible pairs of spels in $\mathcal{C}$ with a strength assigned to every path. This is for facilitating the definition of fuzzy connectedness.

Axiom 1. For any scene $\mathcal{C}$ over (Z^n, α), for any affinity κ and κ-net $\mathcal{N}$ in $\mathcal{C}$, for any two spels $c, d \in C$, the strength of the link from c to d is the affinity between them; i.e., $\mu_{\mathcal{N}}(\langle c, d\rangle) = \mu_\kappa(c, d)$.

Axiom 2. For any scene $\mathcal{C}$ over (Z^n, α), for any affinity κ and κ-net $\mathcal{N}$ in $\mathcal{C}$, for any two paths $p_1, p_2 \in P_{\mathcal{C}}$, $p_1 > p_2$ implies that $\mu_{\mathcal{N}}(p_1) \leq \mu_{\mathcal{N}}(p_2)$.

Axiom 3. For any scene $\mathcal{C}$ over (Z^n, α), for any affinity κ and κ-net $\mathcal{N}$ in $\mathcal{C}$, fuzzy κ-connectedness K in $\mathcal{C}$ is a fuzzy relation in C defined by the following membership function. For any $c, d \in C$,

$$\mu_K(c, d) = \max_{p \in P_{cd}}[\mu_{\mathcal{N}}(p)]. \tag{2.6}$$

(For fuzzy connectedness, we shall always use the upper case form of the symbol used to represent the corresponding fuzzy affinity.)

Axiom 4. For any scene $\mathcal{C}$ over (Z^n, α), for any affinity κ and κ-net $\mathcal{N}$ in $\mathcal{C}$, fuzzy κ-connectedness in $\mathcal{C}$ is a symmetric and transitive relation.

Axiom 1 says that a link is a basic unit in any path and that the strength of a link (which will be utilized in defining path strength) should be simply the affinity between the two component spels of the link. This is the fundamental way in which affinity is brought into the definition of path strength. Note that, in a link $\langle c_i, c_{i+1}\rangle$ in a path, c_i and c_{i+1} may not always be adjacent (in the sense "adjacency" is usually considered in hard digital topology)—that is, c_i and c_{i+1} may be far apart, differing in some of their coordinates by more than 1. In such cases, Axiom 1 guarantees that the strength of $\langle c_i, c_{i+1}\rangle$ is determined by $\mu_\kappa(c, d)$ and not by "tighter" paths of the form $\langle c_i = c_{i,1}, c_{i,2}, \ldots, c_{i,m} = c_{i+1}\rangle$ wherein the successive spels are indeed adjacent. Since κ is by definition reflexive and symmetric, this axiom guarantees that link strength is also a reflexive and symmetric relation in C. Axiom 2 guarantees that the strength of any path changes in a nonincreasing manner along the path. This property is sensible and becomes essential in casting fuzzy connected object tracking as a dynamic programming problem. Axiom 3 says essentially that the strength of connectedness between c and d should be the strength of the strongest path between them. Its reasonableness has already been discussed. Finally, Axiom 4 guarantees that fuzzy connectedness is a similitude relation in C. Its reflexivity is proved in Proposition 0 below. This property is essential to prove the main theorem (Theorem 2.6 in [10]) that permits devising a dynamic programming solution to the otherwise seemingly prohibitive combinatorial optimization problem of determining a fuzzy connected object.

In the remainder of this section, we shall establish that, under these axioms, the only possible choice for $\mu_{\mathcal{N}}(p)$ is the minimum of the strengths of the links in p.

PROPOSITION 0. *For any scene $\mathcal{C} = (C, f)$ over (Z^n, α), for any affinity κ and κ-net $\mathcal{N}$ in $\mathcal{C}$, fuzzy κ-connectedness K in $\mathcal{C}$ is a similitude relation.*

Proof. Because of Axiom 4, we need to prove only the reflexivity of K.

We claim that $\langle c, c\rangle$ is a strongest path from c to c. This is because, by Axiom 1, $\mu_{\mathcal{N}}(\langle c, c\rangle) = \mu_\kappa(c, c) = 1$ since κ is reflexive. Therefore, by Axiom 3, $\mu_K(c, c) = 1$. ∎

PROPOSITION 1. *For any scene $\mathcal{C} = (C, f)$ over (Z^n, α), for any affinity κ and κ-net $\mathcal{N}$ in $\mathcal{C}$, and for any path p in $\mathcal{C}$ of length $l_p \leq 1$, $\mu_{\mathcal{N}}(p) = 1$.*

Proof. For any spel $c \in C$, and for any path p_c from c to c itself, p_c is always greater than both $\langle c\rangle$ and $\langle\ \rangle$. Thus, by Axiom 2, $\mu_K(c, c) \leq \min[\mu_{\mathcal{N}}(\langle c\rangle), \mu_{\mathcal{N}}(\langle\ \rangle)]$. By Proposition 0, $\mu_K(c, c)$ is always 1; and this is possible only when $\mu_{\mathcal{N}}(\langle c\rangle) = \mu_{\mathcal{N}}(\langle\ \rangle) = 1$. ∎

PROPOSITION 2. *For any scene $\mathcal{C} = (C, f)$ over (Z^n, α), for any affinity κ and κ-net $\mathcal{N}$ in $\mathcal{C}$, and for any spels $c, d \in C$, similitudity of fuzzy κ-connectedness implies that for any path $p = \langle c = c_1, c_2, \ldots, c_l = d\rangle$ from c to d, $\mu_K(c, d) \geq \min_{1<i\leq l}[\mu_\kappa(c_{i-1}, c_i)]$.*

Proof. The proof is by the method of induction. By hypothesis, p cannot be an empty path. When $l = 1$, c and d become identical, and the lemma trivially follows from the reflexivity property of fuzzy κ-connectedness (Proposition 0). By (2.6) in Axiom 3, $\mu_K(c_1, c_2) \geq \mu_{\mathcal{N}}(\langle c_1, c_2\rangle)$, and by Axiom 1, $\mu_{\mathcal{N}}(\langle c_1, c_2\rangle) = \mu_\kappa(c_1, c_2)$. Thus, $\mu_K(c_1, c_2) \geq \mu_\kappa(c_1, c_2)$ so that the proposition is proved for $l = 2$. Now we show that the lemma is true for $l = i$, assuming that it is true for $l = i - 1$. In other words, we have to prove that $\mu_K(c_1, c_i) \geq \min[\mu_\kappa(c_1, c_2), \mu_\kappa(c_2, c_3), \ldots, \mu_\kappa(c_{i-1}, c_i)]$, assuming that $\mu_K(c_1, c_{i-1}) \geq \min[\mu_\kappa(c_1, c_2), \mu_\kappa(c_2, c_3), \ldots, \mu_\kappa(c_{i-2}, c_{i-1})]$. From the arguments we have used above for c_1, c_2, it follows that $\mu_K(c_{i-1}, c_i) \geq \mu_\kappa(c_{i-1}, c_i)$. By Proposition 0, we obtain

$$\mu_K(c_1, c_i) \geq \min[\mu_K(c_1, c_{i-1}), \mu_K(c_{i-1}, c_i)]$$

or

$$\mu_K(c_1, c_i) \geq \min[\min[\mu_\kappa(c_1, c_2), \mu_\kappa(c_2, c_3), \ldots \mu_\kappa(c_{i-2}, c_{i-1})], \mu_\kappa(c_{i-1}, c_i)]$$

or

$$\mu_K(c_1, c_i) \geq \min[\mu_\kappa(c_1, c_2), \mu_\kappa(c_2, c_3), \ldots \mu_\kappa(c_{i-1}, c_i)].$$ ∎

PROPOSITION 3. *For any scene $\mathcal{C} = (C, f)$ over (Z^n, α), for any affinity κ and κ-net $\mathcal{N}$ in $\mathcal{C}$, for a strongest path $p_{cd} = \langle c = c_1, c_2, \ldots, c_l = d\rangle$ between any two spels $c, d \in C$, and for any other path $p = \langle c = c_1^{(p)}, c_2^{(p)}, \ldots c_{l_p}^{(p)} = d\rangle$ between $c, d \in C$, $min_{1<i\leq l}[\mu_\kappa(c_{i-1}, c_i)] \geq$ $\min_{1<i\leq l_p}[\mu_\kappa(c_{i-1}^{(p)}, c_i^{(p)})]$.*

Proof. Suppose that the proposition is not true; that is, $\min_{1<i\leq l}[\mu_\kappa(c_{i-1}, c_i)] <$ $\min_{1<i\leq l_p}[\mu_\kappa(c_{i-1}^{(p)}, c_i^{(p)})]$. From Proposition 2, $\mu_K(c, d) \geq \min_{1<i\leq l_p}[\mu_\kappa(c_{i-1}^{(p)}, c_i^{(p)})]$, which implies that $\mu_K(c, d) > \min_{1<i\leq l}[\mu_\kappa(c_{i-1}, c_i)]$. Clearly, for all $1 < i \leq l$, $p_{cd} >$ $\langle c_{i-1}, c_i\rangle$. Thus by Axioms 1 and 2, we have $\mu_{\mathcal{N}}(\langle c_{i-1}, c_i\rangle) = \mu_\kappa(c_{i-1}, c_i) \geq \mu_{\mathcal{N}}(p_{cd})$ for all $1 < i \leq l$. That is, $\mu_{\mathcal{N}}(p_{cd}) \leq \min_{1<i\leq l}[\mu_\kappa(c_{i-1}, c_i)]$, which implies that $\mu_K(c, d) >$

$\mu_{\mathcal{N}}(p_{cd})$. However, this contradicts (2.6) in Axiom 3. Hence, $min_{1<i\leq l}[\mu_\kappa(c_{i-1}, c_i)] \geq \min_{1<i\leq l_p}[\mu_\kappa(c_{i-1}^{(p)}, c_i^{(p)})]$. ■

LEMMA 4. *For any scene $\mathcal{C} = (C, f)$ over (Z^n, α), for any affinity κ and κ-net $\mathcal{N}$ in $\mathcal{C}$, and for any spels $c, d \in C$, similitudity of fuzzy κ-connectedness implies*

$$\mu_K(c, d) = \max_{p\in P_{cd}} \left[\min_{1<i\leq l_p} \left[\mu_\kappa\left(c_{i-1}^{(p)}, c_i^{(p)}\right)\right]\right], \tag{2.7}$$

where $p = \langle c_1^{(p)}, c_2^{(p)}, \ldots, c_{l_p}^{(p)}\rangle$.

Proof. Let $p_{cd} = \langle c = c_1, c_2, \ldots, c_l = d\rangle$ be a strongest path in $\mathcal{C}$ from c to d. By Propositions 0 and 2, we have $\mu_K(c, d) \geq \min_{1<i\leq l}[\mu_\kappa(c_{i-1}, c_i)]$. From Axioms 1 and 2, we have $\mu_{\mathcal{N}}(p_{cd}) \leq \min_{1<i\leq l}[\mu_\kappa(c_{i-1}, c_i)]$. Therefore, by (2.6) in Axiom 3, $\mu_K(c, d) = \mu_{\mathcal{N}}(p_{cd}) = \min_{1<i\leq l}[\mu_\kappa(c_{i-1}, c_i)]$. However, from Proposition 3, we have $min_{1<i\leq l}[\mu_\kappa(c_{i-1}, c_i)] \geq \min_{1<i\leq l_p}[\mu_\kappa(c_{i-1}^{(p)}, c_i^{(p)})]$ for any path $p = \langle c_1^{(p)}, c_2^{(p)}, \ldots c_{l_p}^{(p)}\rangle$ in P_{cd}. Hence,

$$\mu_K(c, d) = \max_{p\in P_{cd}} \left[\min_{1<i\leq l_p} \left[\mu_\kappa\left(c_{i-1}^{(p)}, c_i^{(p)}\right)\right]\right]. \tag{2.8}$$

The following corollary is a direct consequence of the above lemma and Axiom 3. ■

COROLLARY 5. *For any scene $\mathcal{C} = (C, f)$ over (Z^n, α), for any affinity κ and κ-net $\mathcal{N}$ in $\mathcal{C}$, and for any spels $c, d \in C$ and a strongest path $p_{cd} = \langle c = c_1, c_2, \ldots c_l = d\rangle$ in $\mathcal{C}$ from c to d, similitudity of fuzzy κ-connectedness implies $\mu_{\mathcal{N}}(p_{cd}) = \min_{1<i\leq l}[\mu_\kappa(c_{i-1}, c_i)]$.*

LEMMA 6. *For any scene $\mathcal{C} = (C, f)$ over (Z^n, α), for any affinity κ and κ-net $\mathcal{N}$ in $\mathcal{C}$, and for any two spels $c, d \in C$, the following functional form of μ_K,*

$$\mu_K(c, d) = \max_{p\in P_{cd}} \left[\min_{1<i\leq l_p} \left[\mu_\kappa\left(c_{i-1}^{(p)}, c_i^{(p)}\right)\right]\right], \tag{2.9}$$

where p is the path $\langle c_1^{(p)}, c_2^{(p)}, \ldots, c_{l_p}^{(p)}\rangle$, implies similitudity of fuzzy κ-connectedness K.

Proof. For any spel $c \in C$, the path $\langle c\rangle \in P_{cc}$ and $\mu_K(c, c) \geq \mu_{\mathcal{N}}(\langle c\rangle)$ by (2.6), and $\mu_{\mathcal{N}}(\langle c\rangle) = 1$ by Proposition 1. From (2.9), $\mu_K(c, c) \leq 1$ as $\mu_\kappa(c', d') \leq 1$ for any spels $c', d' \in C$. Hence, $\mu_K(c, c) = 1$ and K is reflexive.

For any spels $c, d \in C$, there is a one-to-one mapping between P_{cd} and P_{dc} as follows: every path $p_{cd} = \langle c = c_1^{(p)}, c_2^{(p)}, \ldots, c_l^{(p)} = d\rangle$ in P_{cd} is uniquely mapped to the path $p_{dc} = \langle d = c_l^{(p)}, c_{l-1}^{(p)}, \ldots, c_1^{(p)} = c\rangle$ in P_{dc}. From the symmetry of κ, $\min_{1<i\leq l}[\mu_\kappa(c_{i-1}, c_i)] = \min_{l\geq i>1}[\mu_\kappa(c_i, c_{i-1})]$. Therefore, by (2.9), $\mu_K(c, d) = \max_{p_{cd}\in P_{cd}}[\min_{1<i\leq l}[\mu_\kappa(c_{i-1}, c_i)]] = \max_{p_{dc}\in P_{dc}}[\min_{l\geq i>1}[\mu_\kappa(c_i, c_{i-1})]] = \mu_K(d, c)$. Hence, K is symmetric.

For any two spels $c, e \in C$, by (2.9),

$$\mu_K(c, e) = \max_{p\in P_{ce}} \left[\min_{1<i\leq l_p} \left[\mu_\kappa\left(c_{i-1}^{(p)}, c_i^{(p)}\right)\right]\right],$$

where $p = \langle c_1^{(p)}, c_2^{(p)}, \ldots, c_{l_p}^{(p)}\rangle$. By (2.5), $P_{ce} = \{p_1 + p_2 \mid d \in C,\ p_1 \in P_{cd} \text{ and } p_2 \in P_{de}\}$, where $p_1 = \langle c_1^{(p_1)}, c_2^{(p_1)}, \ldots, c_{l_{p_1}}^{(p_1)} \rangle$ and $p_2 = \langle c_1^{(p_2)}, c_2^{(p_2)}, \ldots, c_{l_{p_2}}^{(p_2)} \rangle$. Therefore,

$$\begin{aligned}
\mu_K(c, e) &= \max_{d \in C,\, p_1 \in P_{cd},\, p_2 \in P_{de}} \left[\min \left[\min_{1<i\le l_{p_1}} \left[\mu_\kappa \left(c_{i-1}^{(p_1)}, c_i^{(p_1)} \right) \right], \min_{1<j\le l_{p_2}} \left[\mu_\kappa \left(c_{j-1}^{(p_2)}, c_j^{(p_2)} \right) \right] \right] \right] \\
&= \max_{d \in C} \left[\min \left[\max_{p_1 \in P_{cd}} \left[\min_{1<i\le l_{p_1}} \left[\mu_\kappa \left(c_{i-1}^{(p_1)}, c_i^{(p_1)} \right) \right] \right], \max_{p_2 \in P_{de}} \left[\min_{1<j\le l_{p_2}} \left[\mu_\kappa \left(c_{j-1}^{(p_2)}, c_j^{(p_2)} \right) \right] \right] \right] \right] \\
&= \max_{d \in C} [\min[\mu_K(c, d), \mu_K(d, e)]] \text{ by (2.9).}
\end{aligned}$$

Hence K is transitive. ■

THEOREM 7. *For any scene $\mathcal{C} = (C, f)$ over (Z^n, α), and for any affinity κ and κ-net $\mathcal{N}$ in $\mathcal{C}$, fuzzy κ-connectedness K in $\mathcal{C}$ is a similitude relation in C if and only if*

$$\mu_K(c, d) = \max_{p \in P_{cd}} \left[\min_{1<i\le l_p} \left[\mu_\kappa \left(c_{i-1}^{(p)}, c_i^{(p)} \right) \right] \right],$$

where p is the path $\langle c_1^{(p)}, c_2^{(p)}, \ldots, c_{l_p}^{(p)} \rangle$.

Proof. Combine Lemmas 4 and 6. ■

We have now established, based on Axioms 1 to 4, that the minimum of affinities along a path is the only plausible choice for path strength. Following the spirit of the above theorem, we define path strength by

$$\mu_{\mathcal{N}}(p) = \min_{1<i\le l_p} \left[\mu_\kappa \left(c_{i-1}^{(p)}, c_i^{(p)} \right) \right], \tag{2.10}$$

where p is the path $\langle c_1^{(p)}, c_2^{(p)}, \ldots, c_{l_p}^{(p)} \rangle$. For the remainder of this paper, we shall assume the above definition of path strength.

2.4. *Fuzzy Connected Objects, Fuzzy Object Extraction*

In this section, we shall develop the notion of fuzzy connected objects containing a set of specified seed spels, and study the properties of such objects.

Let $\mathcal{C} = (C, f)$ be any scene over (Z^n, α), let κ be any affinity in $\mathcal{C}$, and let θ be a fixed number in [0, 1]. Let S be any subset of C. We shall refer to S as the set of reference spels or seed spels and assume throughout that $S \neq \phi$. A *fuzzy $\kappa\theta$-object $\mathcal{O}_{K\theta}(s)$ of $\mathcal{C}$ containing a seed spel s of C* is a fuzzy subset of C whose membership function is

$$\mu_{\mathcal{O}_{K\theta}(s)}(c) = \begin{cases} \eta(f(c)), & \text{if } c \in O_{K\theta}(s) \subset C, \\ 0, & \text{otherwise.} \end{cases} \tag{2.11}$$

In this expression, η is an *objectness function* whose domain is the range of f and whose range is [0,1]. It maps imaged scene intensity values into objectness values. For most segmentation purposes, η may be chosen to be a Gaussian whose mean and standard deviation correspond to the intensity value expected for the object region and its standard deviation (or some multiple thereof). The choice of η should depend on the particular imaging modality that generated $\mathcal{C}$ and the actual physical object under consideration. (When a hard segmentation is desired, $O_{K\theta}(s)$ (defined bellow) will constitute the (hard) set of spels that represents the extent of the physical object and η will simply be the characteristic function of $O_{K\theta}(s)$.) $O_{K\theta}(s)$ is a subset of C satisfying all of the following conditions:

$$\text{(i)}\quad s \in O_{K\theta}(s); \tag{2.12}$$

$$\text{(ii)}\quad \text{for any spels } c, d \in O_{K\theta}(s),\ \mu_K(c,d) \geq \theta; \tag{2.13}$$

$$\text{(iii)}\quad \text{for any spel } c \in O_{K\theta}(s) \text{ and any spel } d \notin O_{K\theta}(s),\ \mu_K(c,d) < \theta. \tag{2.14}$$

We shall refer to $O_{K\theta}(s)$ as the *support of* $\mathcal{O}_{K\theta}(s)$. In words, the support of $\mathcal{O}_{K\theta}(s)$ is a maximal subset of C such that it includes s and the strength of connectedness between any two of its spels is at least θ. (This definition is different from the one given in [10] and, we believe, is intuitively more sensible. Later on, we shall establish their theoretical equivalence.)

We now generalize the above concept of a fuzzy connected object from a single seed spel s to a set S of spels. A *fuzzy* $\kappa\theta$*-object* $\mathcal{O}_{K\theta}(S)$ *of* $\mathcal{C}$ *containing a set* S of *seed spels of* C is a fuzzy subset of C whose membership function is

$$\mu_{\mathcal{O}K\theta(S)}(c) = \begin{cases} \eta(f(c)), & \text{if } c \in O_{K\theta}(S) = \bigcup_{s\in S} O_{K\theta}(s), \\ 0, & \text{otherwise.} \end{cases} \tag{2.15}$$

We shall refer to $O_{K\theta}(S)$ as the *support of* $\mathcal{O}_{K\theta}(S)$. (Note that $O_{K\theta}(\{s\}) = O_{K\theta}(s)$.) In words, the support of $\mathcal{O}_{K\theta}(S)$ is simply the union of the support of the fuzzy connected objects containing the individual seed spels of S.

The following theorem gives us a guidance as to how a fuzzy $\kappa\theta$-object $\mathcal{O}_{K\theta}(S)$ of $\mathcal{C}$ should be computed. It is not practical to use the definition directly for this purpose because of the combinatorial complexity. The following theorem provides a practical way of computing $\mathcal{O}_{K\theta}(S)$.

THEOREM 8. *For any scene* $\mathcal{C} = (C, f)$ *over* (Z^n, α), *for any affinity* κ *in* $\mathcal{C}$, *for any* $\theta \in [0, 1]$, *for any objectness function* η, *and for any nonempty set* $S \subset C$, *the support* $O_{K\theta}(S)$ *of the fuzzy* $\kappa\theta$*-object of* $\mathcal{C}$ *containing* S *equals*

$$O_{K\theta}(S) = \left\{c \mid c \in C \text{ and } \max_{s\in S}[\mu_K(s,c)] \geq \theta\right\}. \tag{2.16}$$

Proof. Let Ω denote the set on the right-hand side of (2.16). By (2.12) and (2.15), each spel $s \in S$ is in the set $O_{K\theta}(S)$. Let $c \in O_{K\theta}(S)$. By (2.13) and (2.15), $\max_{s\in S}[\mu_K(s,c)] \geq \theta$. Hence, $c \in \Omega$ and $O_{K\theta}(S) \subset \Omega$. To prove $\Omega \subset O_{K\theta}(S)$, we take any spel $c \notin O_{K\theta}(S)$. By (2.14) and (2.15), $\max_{s\in S}[\mu_K(s,c)] < \theta$. Hence, $c \notin \Omega$ and $\Omega \subset O_{K\theta}(S)$. Therefore, $\Omega = O_{K\theta}(S)$. ■

As a consequence of the above theorem, we shall show in the next section that $\mathcal{O}_{K\theta}(S)$ can be computed via dynamic programming, given $\mathcal{C}$, κ, θ, η, and S. We shall now study some important properties of fuzzy $\kappa\theta$-objects of $\mathcal{C}$ containing S, eventually demonstrating the robustness of the definition of fuzzy $\kappa\theta$-objects with respect to seed specification (Theorem 11).

PROPOSITION 9. *For any scene* $\mathcal{C} = (C, f)$ *over* (Z^n, α), *for any affinity* κ *in* $\mathcal{C}$, *for any* $\theta \in [0, 1]$, *for any objectness function* η, *and for any spels* $s_1, s_2 \in C$, $\mathcal{O}_{K\theta}(s_1) = \mathcal{O}_{K\theta}(s_2)$ *if and only if* $s_2 \in O_{K\theta}(s_1)$.

Proof. Let $c \in O_{K\theta}(s_1)$; i.e., $\mu_K(s_1, c) \geq \theta$. Let $s_2 \in O_{K\theta}(s_1)$. Then $\mu_K(s_1, s_2) = \mu_K(s_2, s_1) \geq \theta$. By transitivity of K, $\mu_K(s_2, c) \geq \min[\mu_K(s_2, s_1), \mu_K(s_1, c)] \geq \theta$. Therefore, by Theorem 8, $c \in O_{K\theta}(s_2)$. Hence, $O_{K\theta}(s_1) \subset O_{K\theta}(s_2)$. As $\mu_K(s_2, s_1) \geq \theta$, by Theorem 8, $s_1 \in O_{K\theta}(s_2)$. Therefore, following the same argument, it can be shown that $O_{K\theta}(s_2) \subset O_{K\theta}(s_1)$. Hence, by (2.11), $\mathcal{O}_{K\theta}(s_1) = \mathcal{O}_{K\theta}(s_2)$.

To prove the "only if" part, let $s_2 \notin O_{K\theta}(s_1)$; i.e., $\mu_K(s_1, s_2) < \theta$. Therefore, by Theorem 8, $s_2 \notin O_{K\theta}(s_1)$. By reflexivity of K (see Proposition 0), $\mu_K(s_2, s_2) = 1 \geq \theta$. Therefore, by Theorem 8, $s_2 \in O_{K\theta}(s_2)$ so that $O_{K\theta}(s_1) \neq O_{K\theta}(s_2)$. Hence, by (2.11), $\mathcal{O}_{K\theta}(s_1) \neq \mathcal{O}_{K\theta}(s_2)$. ■

It follows immediately from the above proposition that the fuzzy $\kappa\theta$-object $\mathcal{O}_{K\theta}(s)$ of $\mathcal{C}$ specified by s is unique for any given $\mathcal{C}$, s, θ, κ, and η, establishing the legitimacy of the definition of $\mathcal{O}_{K\theta}(s)$. The following is an immediate consequence of the above proposition.

COROLLARY 10. *For any scene $\mathcal{C} = (C, f)$ over (Z^n, α), for any affinity κ in $\mathcal{C}$, for any $\theta \in [0, 1]$, for any objectness function η, for any seed spel $s \in C$, and a non empty set $S \subset C$, $\mathcal{O}_{K\theta}(s) = \mathcal{O}_{K\theta}(S)$ if and only if $S \subset O_{K\theta}(s)$.*

Proof. Combine Proposition 9 and (2.15). ■

The above results lead us to the following theorem characterizing the robustness of specifying fuzzy $\kappa\theta$-objects through sets of seed spels.

THEOREM 11. *For any scene $\mathcal{C} = (C, f)$ over (Z^n, α), for any affinity κ in $\mathcal{C}$, for any $\theta \in [0, 1]$, for any objectness function η, and for any nonempty sets $S_1, S_2 \subset C$, $\mathcal{O}_{K\theta}(S_1) = \mathcal{O}_{K\theta}(S_2)$ if and only if $S_1 \subset O_{K\theta}(S_2)$ and $S_2 \subset O_{K\theta}(S_1)$.*

Proof. Let $S_1 = \{s_1, s_2, \ldots, s_m\}$. Let $S_1 \subset O_{K\theta}(S_2) = \bigcup_{s' \in S_2} O_{K\theta}(s')$. This implies that, for each $s_i \in S_1$, there is some $s_i' \in S_2$ such that $s_i \in O_{K\theta}(s_i')$. (Note that s_i's are not necessarily distinct.) Therefore, by (2.15) and Proposition 9, $O_{K\theta}(S_1) = O_{K\theta}(s_1) \cup O_{K\theta}(s_2) \cup \cdots \cup O_{K\theta}(s_m) = O_{K\theta}(s_1') \cup O_{K\theta}(s_2') \cup \cdots \cup O_{K\theta}(s_m') \subset O_{K\theta}(S_2)$. Using $S_2 \subset O_{K\theta}(S_1)$ and similar arguments, it can be shown that $O_{K\theta}(S_2) \subset O_{K\theta}(S_1)$. Hence, by (2.15), $\mathcal{O}_{K\theta}(S_1) = \mathcal{O}_{K\theta}(S_2)$.

To prove the "only if" part, let us consider a spel $c \in S_2$ such that $c \notin O_{K\theta}(S_1)$; i.e., $\max_{s \in S_1}[\mu_K(s, c)] < \theta$. Therefore, by Theorem 8, $c \notin O_{K\theta}(S_1)$. By reflexivity of K (see Proposition 0), $\mu_K(c, c) = 1$, i.e., $\max_{s \in S_2}[\mu_K(s, c)] = 1 \geq \theta$. Therefore, by Theorem 8, $c \in O_{K\theta}(S_2)$ so that $O_{K\theta}(S_1) \neq O_{K\theta}(S_2)$. Hence, by (2.11), $\mathcal{O}_{K\theta}(S_1) \neq \mathcal{O}_{K\theta}(S_2)$. ■

The above theorem has important consequences in the practical utilization of the fuzzy connectedness algorithms presented in the next section. It states that the seeds must be selected from the same physical object and at least one seed must be selected from each physically connected region. High precision (reproducibility) of any segmentation algorithm with regard to subjective operator actions (and with regard to automatic operations minimizing these actions), such as specification of seeds, is essential for their practical utility. Generally, it is easy for human operators to specify spels within a region in the scene corresponding to the same physical object in a repeatable fashion. Theorem 11 guarantees that, even though the sets of spels specified in repeated trials may not be the same, as long as these sets are within the region of the same physical object in the scene, the resulting segmentations will be identical. Many region growing algorithms which adaptively change the spel inclusion criteria during the growth process cannot guarantee this robustness property. We note that Theorem 11 also establishes the uniqueness of the fuzzy $\kappa\theta$-object specified by S, asserting the legitimacy of the definition.

The following corollary is an immediate consequence of the above theorem.

COROLLARY 12. *For any scene* $\mathcal{C} = (C, f)$ *over* (Z^n, α), *for any affinity* κ *in* $\mathcal{C}$, *for any* $\theta \in [0, 1]$, *for any objectness function* η, *for any non empty set* $S \subset C$, *and for any* $s \in C$, $\mathcal{O}_{K\theta}(S) = \mathcal{O}_{K\theta}(S \cup \{s\})$ *if and only if* $\max_{s' \in S}[\mu_K(s', s)] \geq \theta$.

The following propositions establish the monotonic properties of the fuzzy $\kappa\theta$-objects with respect to the strength of connectedness θ and the set S of seed spels.

PROPOSITION 13. *For any scene* $\mathcal{C} = (C, f)$ *over* (Z^n, α), *for any affinity* κ *in* $\mathcal{C}$, *for any* $\theta_1, \theta_2 \in [0, 1]$, *for any objectness function* η, *and for any nonempty set* $S \subset C$, $\mathcal{O}_{K\theta_1}(S) \subset \mathcal{O}_{K\theta_2}(S)$ *if* $\theta_2 < \theta_1$.

Proof. For any spel $c \in O_{K\theta_1}(S)$, there exists a spel $s \in S$ for which $\mu_K(s, c) \geq \theta_1$. Since, $\theta_2 < \theta_1$, we have $\mu_K(s, c) > \theta_2$ and $c \in O_{K\theta_2}(S)$. Thus, $O_{K\theta_1}(S) \subset O_{K\theta_2}(S)$. Therefore, by (2.15), $\mathcal{O}_{K\theta_1}(S) \subset \mathcal{O}_{K\theta_2}(S)$. ■

PROPOSITION 14. *For any scene* $\mathcal{C} = (C, f)$ *over* (Z^n, α), *for any affinity* κ *in* $\mathcal{C}$, *for any* $\theta \in [0, 1]$, *for any objectness function* η, *and for any nonempty sets* $S_1, S_2 \subset C$, $\mathcal{O}_{K\theta}(S_1) \subset \mathcal{O}_{K\theta}(S_2)$ *if* $S_1 \subset S_2$.

Proof. For any spel $c \in O_{K\theta}(S_1)$, there exists a spel $s \in S_1$ for which $\mu_K(s, c) \geq \theta$; since, $S_1 \subset S_2$, we have $\max_{s' \in S_2}[\mu_K(s', c)] \geq \mu_K(s, c) \geq \theta$ and $c \in O_{K\theta}(S_2)$. Thus, $O_{K\theta}(S_1) \subset O_{K\theta}(S_2)$. Therefore following (2.15), $\mathcal{O}_{K\theta}(S_1) \subset \mathcal{O}_{K\theta}(S_2)$. ■

We close this section with the following property of $O_{K\theta}(S)$ which states the conditions under which $O_{K\theta}(S)$ is guaranteed to be (path) connected in a hard sense.

PROPOSITION 15. *For any scene* $\mathcal{C} = (C, f)$ *over* (Z^n, α), *for any affinity* κ *in* $\mathcal{C}$, *for any* $\theta \in [0, 1]$, *for any objectness function* η, *for any nonempty* $S \subset C$, *and for any spels* $c, d \in O_{K\theta}(S)$, *all spels in the strongest path from* c *to* d *in* $\mathcal{C}$ *are contained in* $O_{K\theta}(S)$ *if and only if* $\min_{s_1, s_2 \in S}[\mu_K(s_1, s_2)] \geq \theta$.

Proof. First, assume that $\min_{s_1, s_2 \in S}[\mu_K(s_1, s_2)] < \theta$; i.e., there exist some $s_1', s_2' \in S$ such that $\mu_K(s_1', s_2') < \theta$. By the reflexivity property of K, we have $S \subset O_{K\theta}(S)$; i.e., $s_1', s_2' \in O_{K\theta}(S)$ establishing that there exist spels $c(=s_1')$ and $d(=s_2')$ in $O_{K\theta}(S)$ such that $\mu_K(c, d) < \theta$. This proves the "only if" part.

Now we prove the "if" part. By the definition of $O_{K\theta}(S)$, and since $c, d \in O_{K\theta}(S)$, there exist $s_c, s_d \in S$ such that $\mu_K(c, s_c) \geq \theta$ and $\mu_K(s_d, d) \geq \theta$. From the hypothesis, $\min_{s_1, s_2 \in S}[\mu_K(s_1, s_2)] \geq \theta$, and therefore, $\mu_K(s_c, s_d) \geq \theta$. Hence, by the transitivity property of K, we get $\mu_K(c, d) \geq \theta$. From the fact that, whenever $\mu_K(c, d) \geq \theta$, all spels in the strongest path from c to d are in $O_{K\theta}(S)$, the proof is complete. ■

3. ALGORITHMS

In this section we present two algorithms for extracting a fuzzy $\kappa\theta$-object containing a set S of spels in a given scene $\mathcal{C}$ for a given affinity κ in $\mathcal{C}$, both based on dynamic programming [28]. The first algorithm, named $\kappa\theta FOEMS$ (*$\kappa\theta$-fuzzy object extraction for multiple seeds*), extracts a fuzzy $\kappa\theta$-object of $\mathcal{C}$ of strength θ generated by the set S of reference spels. In this algorithm, the value of θ is assumed to be given as input and the algorithm uses this knowledge to achieve an efficient extraction of the $\kappa\theta$-object. In the second algorithm, named

κ*FOEMS*, we output what we call a *κ-connectivity scene* $\mathcal{C}_{KS} = (C, f_{KS})$ *of* $\mathcal{C}$ *generated by* the set S of reference spels defined by $f_{KS}(c) = \max_{s \in S}[\mu_K(s, c)]$. In Propositions 16 and 17, we confirm the termination of both algorithms in a finite number of steps, while in Propositions 18 and 19, we prove their correctness.

Algorithm $\kappa\theta$*FOEMS* terminates faster than κ*FOEMS* for two reasons. First, $\kappa\theta$*FOEMS* produces the hard set based on (2.16). Therefore, for any spel $c \in C$, once we find a path of strength θ or greater from any of the reference spels to c, we do not need to search for a better path up to c, and hence can avoid further processing for c. This allows us to reduce computation. Second, certain computations are avoided for those spels $d \in C$ for which $\max_{s \in S}[\mu_K(s, d)] < \theta$.

Unlike $\kappa\theta$*FOEMS*, κ*FOEMS* computes the best path from the reference spels of S to every spel c in C. Therefore, every time the algorithm finds a better path up to c, it modifies the connectivity value at c and subsequently processes other spels which are affected by this modification. The algorithm generates a connectivity scene $\mathcal{C}_{KS} = (C, f_{KS})$ of $\mathcal{C}$. Although, κ*FOEMS* terminates slower, it has a practical advantage. After the algorithm terminates, one can interactively specify θ and thereby examine various $\kappa\theta$-objects and interactively select the best θ. The connectivity scene has interesting properties relevant to classification and in shell rendering and manipulation [29] of $\kappa\theta$-objects. However, we will not pursue these directions in this paper.

Algorithm $\kappa\theta$*FOEMS*
Input: $\mathcal{C}$, S, κ, η, and θ as defined in Section 2.
Output: $\mathcal{O}_{K\theta}(S)$ as defined in Section 2.
Auxiliary data structures: An n-D array representing a temporary scene $\mathcal{C}' = (C, f')$, such that f' corresponds to the characteristic function of $O_{K\theta}(S)$, and a queue Q of spels. We refer to the array itself by $\mathcal{C}'$ for the purpose of the algorithm.

begin

1. set all elements of $\mathcal{C}'$ to 0 except those spels $s \in S$ which are set to 1;
2. push all spels $c \in C$ such that for some $s \in S$ $\mu_\kappa(s, c) \geq \theta$ to Q;
3. *while* Q is not empty *do*
4.   remove a spel c from Q;
5.   *if* $f'(c) \neq 1$ *then*
6.   set $f'(c) = 1$;
7.   push all spels d such that $\mu_\kappa(c, d) \geq \theta$ to Q;

    endif;

endwhile;

8. create and output $\mathcal{O}_{K\theta}(S)$ by assigning the value $\eta(f(c))$ to all $c \in C$ for which $f'(c) > 0$ and 0 to the rest of the spels;

end

Algorithm κ*FOEMS*
Input: $\mathcal{C}$, S, and κ as defined Section 2.
Output: A scene $\mathcal{C}' = (C, f')$ representing the κ-connectivity scene $\mathcal{C}_{KS}$ of $\mathcal{C}$ generated by S.

Auxiliary data structures: An n-D array representing the connectivity scene $\mathcal{C}' = (C, f')$ and a queue Q of spels. We refer to the array itself by $\mathcal{C}'$ for the purpose of the algorithm.

begin

1. set all elements of $\mathcal{C}'$ to 0 except those spels $s \in S$ which are set to 1;
2. push all spels $c \in C$ such that, for some $s \in S$, $\mu_\kappa(s, c) > 0$ to Q;
3. *while* Q is not empty *do*
4. remove a spel c from Q;
5. find $f_{\max} = \max_{d \in C}[\min(f'(d), \mu_\kappa(c, d))]$;
6. *if* $f_{\max} > f'(c)$ *then*
7. set $f'(c) = f_{\max}$;
8. push all spels e such that $\min[f_{max}, \mu_\kappa(c, e)] > f'(e)$ to Q;

   endif;

   endwhile;
9. output the connectivity scene $\mathcal{C}'$;

end

PROPOSITION 16. *For any scene $\mathcal{C} = (C, f)$ over (Z^n, α), for any affinity κ in $\mathcal{C}$, for any $\theta \in [0, 1]$, for any nonempty set $S \subset C$, and for any objectness function η, algorithm $\kappa\theta$FOEMS terminates in a finite number of steps.*

Proof. The algorithm is iterative, and at each iteration in the *while-do* loop, it removes exactly one spel from queue Q. Since the number of elements in C is always finite, the algorithm fails to terminate in a finite number of steps only when some spel $c \in C$ visits Q infinitely many times. Following Step 7, c is pushed into Q only when some of its neighbors are modified. The number of neighbors of c is always finite (since C is finite); thus, c visits Q infinitely many times only when some spel $d \in C$ is modified infinitely many times. However, following Step 5, a spel is modified at most once. Hence $\kappa\theta$*FOEMS* terminates in a finite number of steps. ■

PROPOSITION 17. *For any scene $\mathcal{C} = (C, f)$ over (Z^n, α), for any affinity κ in $\mathcal{C}$, and for any nonempty set $S \subset C$, algorithm κFOEMS terminates in a finite number of steps.*

Proof. Following the same argument as in Proposition 16, the algorithm κ*FOEMS* fails to terminate in a finite number of Steps only when some spel $c \in C$ is modified infinitely many times. Following Step 6, the value at c strictly increases after every modification. Moreover, following Steps 5 and 7, during every modification, c is assigned a value from $\{\mu_\kappa(d, e) \mid d, e \in C\}$. However, since C is finite and hence $C \times C$ is finite, $\{\mu_\kappa(d, e) \mid d, e \in C\}$ contains only finitely many elements, so that c could be modified only finitely many times. Hence κ*FOEMS* terminates in a finite number of steps. ■

To prove the correctness of algorithms $\kappa\theta$*FOEMS* and κ*FOEMS* we need the following notations. Let $T_{c,S}(l)$ denote the set of all paths of length less than or equal to l from c to some $s \in S$. Let $L_{c,S}(\theta)$ denote the length of the shortest path (with minimum length) of strength greater than or equal to θ from c to some spel of S. If there is no such path from c to any spel in S, then we assume that $L_{c,S}(\theta)$ is a large number, essentially infinity.

PROPOSITION 18. *For any scene $\mathcal{C} = (C, f)$ over (Z^n, α), for any affinity κ in $\mathcal{C}$, for any $\theta \in [0, 1]$, for any nonempty set $S \subset C$, and for any objectness function η, when $\kappa\theta$FOEMS terminates, its output equals the fuzzy $\kappa\theta$-object $\mathcal{O}_{K\theta}(S)$ of $\mathcal{C}$ containing S.*

Proof. It is enough to prove that the set of spels with value 1 at the end of Step 7 is $O_{K\theta}(S)$. Initially, the algorithm starts with the spels of S (see Step 1). Thus, Step 7 guarantees that whenever a spel c is pushed into Q, there is always a path from some reference spel upto c such that the affinity values along the path are always greater than or equal to θ (so that the strength for the path is greater than or equal to θ). Therefore, the output must be a subset of $O_{K\theta}(S)$.

It may be observed from the algorithm that once a spel is set to "1", the algorithm never resets it to "0". We now prove that $O_{K\theta}(S)$ is a subset of the output of the algorithm by using the method of induction. Obviously, for any $c \in C$, $L_{c,S}(\theta) = 1$ implies that $c \in S$. Therefore, Step 1 guarantees that every $c \in C$ such that $L_{c,S}(\theta) = 1$ is set to "1". Now, assume that, for some $i > 1$, every $c \in C$ such that $L_{c,S}(\theta) < i$ is set to "1". Let $d \in C$ be a spel with $L_{d,S}(\theta) = i$ so that there is a path $\langle s = d_1, d_2, \ldots, d_i = d\rangle$ of strength greater than or equal to θ from $s \in S$ to d. Following (2.10), $\min_{1<j\leq i}[\mu_\kappa(d_{j-1}, d_j) \geq \theta$ and thus, $\mu_{\mathcal{N}}(\langle d_1, d_2, \ldots, d_{i-1}\rangle) \geq \theta$. Therefore, $L_{d_{i-1},S}(\theta) = i - 1$ and d_{i-1} is set to "1" and Step 7 guarantees that d (note that $\mu_\kappa(d_{i-1}, d_i) \geq \theta$) is pushed into Q which is set to "1" upon its removal from Q. Hence, $O_{K\theta}(S)$ is a subset of the output of algorithm *$\kappa\theta$FOEMS*. ■

PROPOSITION 19. *For any scene $\mathcal{C} = (C, f)$ over (Z^n, α), for any affinity κ in $\mathcal{C}$, and for any non empty set $S \subset C$, when κFOEMS terminates, its output equals the connectivity scene $\mathcal{C}_{KS} = (C, f_{KS})$ of $\mathcal{C}$ generated by S.*

Proof. Initially, the algorithm starts with the spels of S (see Step 1). Thus, Step 5 guarantees that whenever a spel c is set to f_{max} (in Step 7), there is always a path from some reference spel up to c with strength f_{max}. Therefore, for any $c \in C$, $f'(c) \leq f_{KS}(c)$.

It may be observed from the algorithm that it never reduces the value of any $c \in C$. We now prove that, for any $c \in C$, $f_{KS}(c) \leq f'(c)$ by using the method of induction. Obviously, for any $c \in C$, $T_{c,S}(1)$ being nonempty implies that $c \in S$, and following Proposition 1, the strength of any element of $T_{c,S}(1)$ is always "1". Thus, in Step 1, it is guaranteed that the algorithm sets every spel $c \in C$ to the strength of the best path of $T_{c,S}(1)$. Now let us assume that the algorithm sets every spel $c \in C$ to the strength of the best path of $T_{c,S}(i-1)$ for some $i > 1$. We show that the algorithm sets every spel $c \in C$ to the strength of the best path of $T_{c,S}(i)$ in Steps 5 to 8. Let $p_{sd} = \langle s = d_1, d_2, \ldots, d_i = d\rangle$ be the best among all paths in $T_{d,S}(i)$ from s to d for some $s \in S$ and any $d \in C$. Following (2.10), $\mu_{\mathcal{N}}(\langle d_1, d_2, \ldots, d_{i-1}\rangle)$, $\mu_\kappa(d_{i-1}, d_i) \geq \mu_{\mathcal{N}}(p_{sd})$. Following the hypothesis of the induction method, at certain iteration the algorithm sets d_{i-1} to $f'(d_{i-1}) = \max_{p\in T_{d_{i-1},S}(i-1)} \mu_{\mathcal{N}}(p) \geq \mu_{\mathcal{N}}(\langle d_1, d_2, \ldots, d_{i-1}\rangle) \geq \mu_{\mathcal{N}}(p_{sd})$ at Step 7, and Step 8 ensures that, in the same iteration, d is pushed into Q; and upon the removal of d from Q, it is assigned (see Steps 5 to 7) a value greater than or equal to $f'(d) = \min[f'(d_{i-1}), \mu_\kappa(d_i, d_{i-1})] \geq \mu_{\mathcal{N}}(p_{sd})$. Therefore, for any $c \in C$, $f_{KS}(c) \leq f'(c)$. Hence, by the results in the previous paragraph, for any $c \in C$, $f_{KS}(c) = f'(c)$. ■

4. RESULTS AND DISCUSSION

We have been developing the fuzzy connectedness family of methods [10–13, 15] for the past 7 years. They have been implemented in the 3DVIEWNIX software system [30] and evaluated on several thousands of patient data sets in several medical applications [18–25] as to their precision, accuracy, and efficiency. In all these applications, we encounter multiple seeds. Although these applications utilized the algorithms described in this paper within

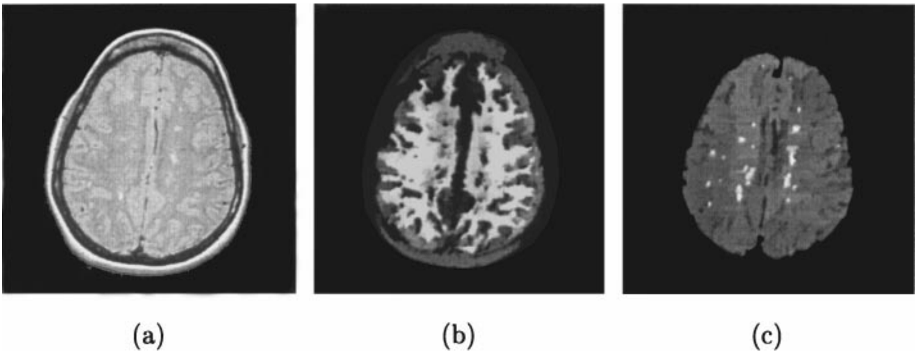

FIG. 1. A 2D slice taken from a proton density-weighted 3D MR scene of a multiple sclerosis patient's head. (b, c) The κ-connectivity scenes for white matter regions and the lesions (somewhat hyperintense areas) in the slice shown in (a).

the framework of the original fuzzy connectedness algorithm [10], algorithms $\kappa FOEMS$ and $\kappa\theta FOEMS$, the proof of their correctness, the underlying theory of fuzzy $\kappa\theta$-objects containing a set of seed spels, and the axiomatic definition of fuzzy connectedness were not described previously. Since evaluation of these algorithms has already been done in specific medical applications, in this paper we shall only give examples without undertaking formal evaluation.

Our first example, illustrated in Fig. 1, comes from a proton density (PD) weighted 3D MR scene of a multiple sclerosis (MS) patient's head. A 2D slice from the 3D scene of domain $256 \times 256 \times 53$ and of voxel size $0.86 \times 0.86 \times 3.0$ mm^3 is displayed in Fig. 1a. The fuzzy connectedness method was separately applied to the 3D scene ($n = 3$) to compute the white matter (WM) region and the lesions. Figures 1b and 1c demonstrate a slice (corresponding to the slice shown in Fig. 1a) taken from the 3D κ-connectivity scenes for the two objects. In the two different executions of algorithm $\kappa FOEMS$, different affinities that were appropriate for the two different tissue regions (WM and lesions) were used. In this example, the seeds were specified manually, several seeds for the WM regions and a couple of them in each of the lesion blobs. An automatic seed selection method for this lesion detection task has been presented in [18] which consists of first delineating WM, gray matter, and cerebro-spinal fluid fuzzy objects and then detecting holes in the union of the support of these three fuzzy $\kappa\theta$-objects. For delineating the three objects, several seeds are specified on one slice manually in each of the three object regions. This method is in routine clinical use in this application with 1500 3D scans processed to date.

Figure 2 illustrates our second example—delineation of dense regions in digitized mammograms. The extent of the dense regions is known to indicate the risk of breast cancer [31]. Therefore, the segmentation and quantification of these regions is potentially useful in the management and screening of patients. Figure 2a shows a digitized mammogram taken in the cranio-caudal (CC) projection, while (b) shows a digitized mammogram of the same breast taken in the medio-lateral-oblique (MLO) projection. In (b), the pectoral muscles projected in the scene were removed manually. The κ-connectivity scenes for Figs. 2a and 2b are shown in Figs. 2c and 2d, respectively. Here, the seeds were automatically selected using an initial conservative thresholding. In this application, the method has been evaluated on 60 pairs of mammograms [24]. The segmentations have been asserted to be visually acceptable in all cases. In addition, based on manual tracing on a subset of these data, the

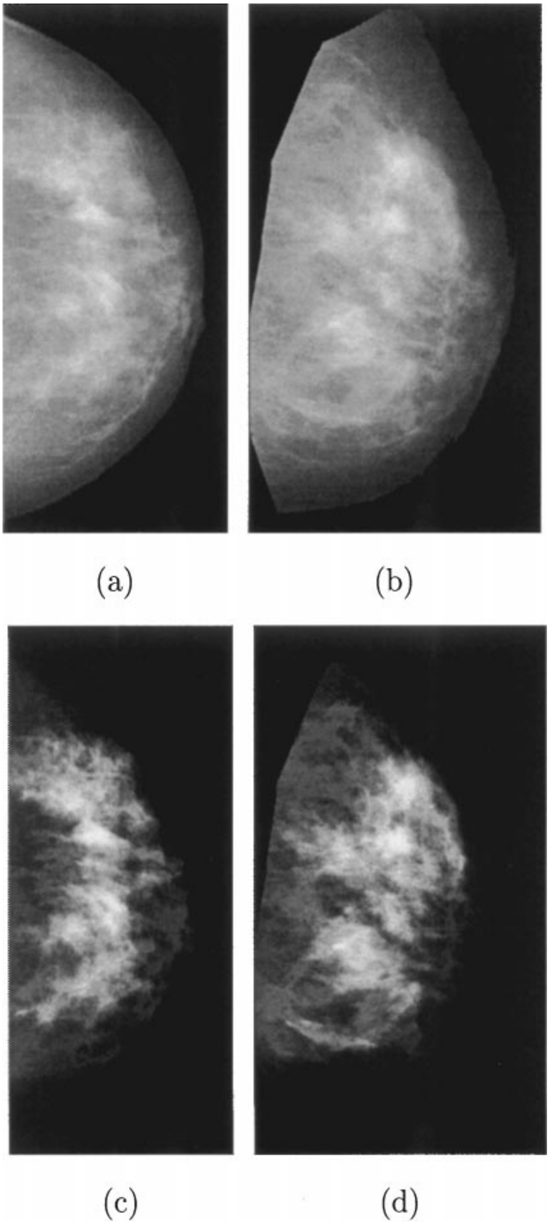

FIG. 2. A digitized mammogram of a patient's breast (a) at CC projection and (b) at the MLO projection (after manually removing pectoral muscles). (c, d) κ-Connectivity scenes for dense regions in (a) and (b), respectively.

disagreement of segmentations obtained by the fuzzy connectedness method and by manual outlining has been found to be within the range of variability of the latter method itself. Further, for the two projections of the same breast in the 60 pairs, a correlation of 0.96 has been achieved for the extent of the dense regions in the two projections.

Figures 3a and 3b show 3D renditions of the whole brain of a MS patient at two different angles. The brain parenchyma was segmented by thresholding the fuzzy connectedness scene computed from a 3D T1 weighted MR scene of the patient's head. The scene has

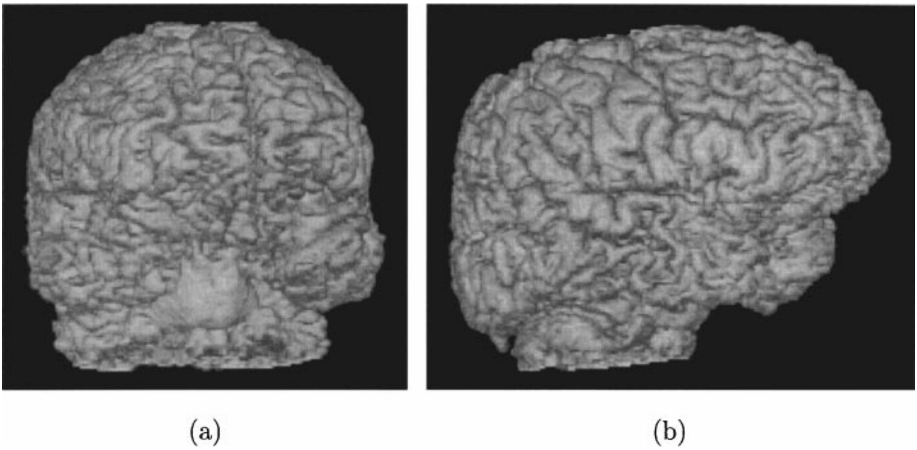

FIG. 3. (a, b) 3D renditions of the whole brain segmented by thresholding the κ-connectivity scene computed from a 3D T1-weighted MR scene of a MS patient's head.

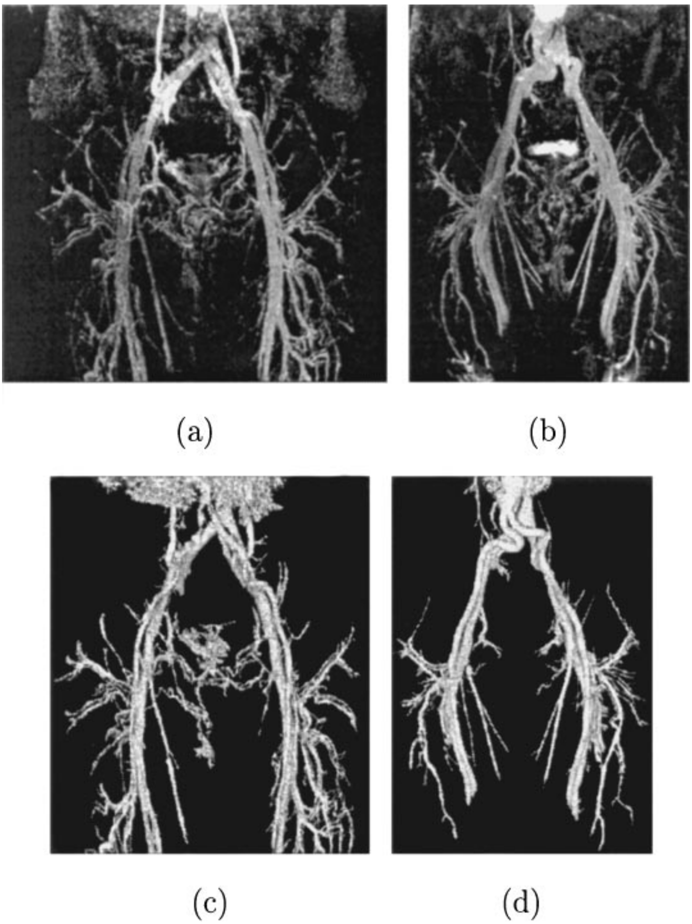

FIG. 4. (a, b) Maximum intensity projection (MIP) renditions of contrast enhanced 3D MR scenes of two patients. The scene domains represent the body region from the belly to the knee. (c, d) 3D renditions of segmented vessel trees obtained from the κ-connectivity scenes of the data sets in (a) and (b), respectively.

a domain of $256 \times 256 \times 124$ and voxel size of $0.94 \times 0.94 \times 1.0$ mm^3. Several seeds were manually specified within the brain tissue region for delineating the parenchymal region. Segmentation is useful in this application, not so much for visualizing the surface of the brain but for computing the parenchymal volume. It is known [32] that brain atrophy is an indicator of the severity of the MS disease. Segmentation is useful here for assessing disease and response to therapy.

Our fourth example, demonstrated in Fig. 4, pertains to the application of separating veins and arteries in contrast enhanced MR scenes [22]. The separation is achieved by first segmenting the whole vessel tree in the scene and then separating arteries and veins within the vessel tree. Figure 4 demonstrates the results of the first step (the second step uses iterative relative fuzzy connectedness [12, 13] not discussed in this paper). Figures 4a and 4b show maximum intensity projection (MIP) renditions of the contrast enhanced 3D MR scenes of the body regions from the belly to the knee of two patients. The MIP renditions are created (without requiring any segmentation) by assigning to each pixel in the rendition a gray value that corresponds to the maximum scene intensity along the line of projection corresponding to the pixel. Figures 4c and 4d show 3D renditions of the vessel trees segmented from their fuzzy connectedness scenes. The domains and voxel sizes for both these scenes are $512 \times 512 \times 60$ and $0.94 \times 0.94 \times 1.80$ mm^3, respectively. The application has been evaluated on about 150 3D scenes obtained from six different hospitals. Within the software all parameters as well as the threshold for vessel segmentation are computed automatically [22]. Several seeds are manually specified by operators on the slices of the scenes.

5. CONCLUDING REMARKS

This paper makes two main contributions. First, continuing on a previously developed framework of fuzzy connectedness and object definition in multidimensional scenes [10], it establishes that maximization of path strength of the minimum of affinities along each path is the one and the only valid choice for defining fuzzy connectedness. This uniqueness of the definition of fuzzy connectedness has been established starting from four axioms. (1) The strength of a link between two image elements is their affinity itself. (2) As a path grows longer, its strength does not increase. (3) The strength of connectedness between any two image elements is simply the strength of the strongest path between them. (4) Fuzzy connectedness is a symmetric and transitive relation. The first three axioms, we assert, are quite natural and inherent to the intuition and the idea of fuzzy connectedness. The fourth axiom is needed to devise practical algorithms for finding fuzzy connected objects.

The second contribution of this paper is the extension of the original single- seeded fuzzy connectedness theory and algorithms [10] to the case of multiple seeds. One of the main results of this extension is the demonstration that any subset of image elements in the support of the fuzzy connected object would generate the same fuzzy connectedness object provided that at least one seed is selected from each physically connected region, establishing the robustness of the delineation with respect to the choice of seed spels. Another main result is the demonstration that a dynamic programming solution suggested in the original framework [10] is applicable to the multiple seed case also with some modifications. The correctness of the new algorithms has been established in this paper. The importance of multiple-seeded fuzzy connectedness from both practicality and automation in image segmentation has been illustrated by several examples drawn from large ongoing medical applications where its utility has been established on hundreds of real data sets.

One of the drawbacks of the current fuzzy connectedness approach is having to select a threshold for the κ-connectivity scene. In applications involving a large number of studies based on a fixed image acquisition protocol, the threshold (as well as the affinity parameters) can be fixed without requiring per-case adjustment or they can be computed in an adaptive fashion based on the particular data set [24]. One alternative to determining a threshold for the κ-connectivity scene is to use one of the many automatic threshold selection methods that are available in the literature [33]. Another alternative, the one we are pursuing, is through the notion of relative fuzzy connectedness of objects [12]. The idea here is that every image element has a strength of connectedness with respect to each object in the image, background also being one object. An image element under question is grabbed by that object with respect to which it has the strongest connectedness. An iterative version [13] of this notion has also been developed to yield more accurate delineations.

Among the four axioms which form the basis of the theory presented in this paper, the fourth has mainly an algorithmic motivation. It will be interesting to see how the theory and algorithms get modified (even at the sacrifice of algorithmic efficiency) if Axiom 4 is dropped.

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