Axiomatic path strength definition for fuzzy connectedness and the case of multiple seeds

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ABSTRACT

This paper presents an extension of the theory and algorithms for fuzzy connectedness. In this framework, a strength of connectedness is assigned to every pair of image elements by finding the strongest connecting path between them. The strength of a path is the weakest affinity between successive pairs of elements along the path. Affinity specifies the degree to which elements hang together locally in the image. A fuzzy connected object containing a particular seed element is computed via dynamic programming. In all reported works so far, the minimum of affinities has been considered for path strength and the maximum of path strengths for fuzzy connectedness. The question thus remained all along as to whether there are other valid formulations for fuzzy connectedness. One of the main contributions of this paper is a theoretical investigation under reasonable axioms to establish that maximum of path strengths of minimum of affinities along each path is indeed the one and only valid choice. The second contribution here is to generalize the theory and algorithms of fuzzy connectedness to the multi-seeded case. The importance of multi-seeded fuzzy connectedness is illustrated with examples taken from several real medical imaging applications.

Keywords: Fuzzy connectedness, image segmentation, digital topology, fuzzy affinity.

1. INTRODUCTION

The activity of image segmentation spans over three decades.[1] Most real objects often possess heterogeneous material compositions. Considerable progress has been made toward handling data inaccuracies and uncertainties in images using fuzzy subset theory as a mathematical vehicle.[2] However, attempts to handle the notion of hanging togetherness of image elements in the presence of graded composition also in the same fuzzy setting are sparse.[3–8] Except in some simple situations, considering each image element on its own or in conjunction with just its local neighbors is not sufficient for effective object definition. For effectively addressing hanging togetherness, we believe that the (spatial, topological) relationship among all image elements should be considered. In,[7] toward this goal, we developed a framework of fuzzy connected object definition theory and algorithms. In this framework, a local fuzzy relation called *affinity* is defined on the image domain which assigns to every pair of nearby image elements a strength of local hanging togetherness which has a value in [0,1]. Affinity between two elements depends on their spatial nearness as well as how similar their image intensities and intensity derived features are. A global fuzzy relation called *fuzzy connectedness* is defined on the image domain which assigns to every pair (c, d) of image elements a strength of global hanging togetherness that has a value in [0,1]. To determine this value, every possible path from c to d (a sequence of nearby elements starting from c and ending at d) is considered and the minimum affinity of pairwise elements along the path is determined. This affinity represents the strength of this path. The strength of hanging togetherness (connectedness) between c and d is the largest of the strengths of all paths between c and d. In defining a fuzzy connected object, the strength of connectedness between all possible pairs of image elements has to be determined. It has been shown[7] that this combinatorially horrendous problem can be solved via dynamic programming and that a fuzzy connected object containing a given image element o can be found by determining the strength of connectedness from o to all image elements. An investigation on how to define affinity in practical image segmentation including a scale-based formulation of fuzzy connectedness is reported in.[8] By allowing object regions to compete in

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terms of connectedness strengths to win membership of elements, a relative fuzzy connectedness framework is developed in[9] and its iterative extension is described in.[10] Improvement of the computational aspects of fuzzy connectedness are described in.[11,12] Investigations combining deformable boundary-based and Voronoi diagram-based methods with fuzzy connectedness are reported in.[13,14] The fuzzy connectedness framework and its extensions[7-10] have been effectively utilized in several medical applications.

There are two main motivations for the extensions considered in this paper. The first is theoretical and the second is practical. The theoretical motivation stems from the min-max construct used in defining fuzzy connectedness — to investigate if there are other reasonable constructs. We show in Section 2 that, under a set of reasonable axioms, min-max is the only valid construct which retains the essential theoretical structure, which in turn permits fuzzy connectedness to be computed via dynamic programming. The practical motivation stems from the fact that, in many real applications (such as multiple sclerosis lesion detection[15]), the number of objects to be delineated may be large (100s). In such cases, it may take a significant amount of an operator's time to specify the seeds — one per each object. Automatic selection of seeds by using a conservative segmentation strategy is therefore more attractive than manual selection. When seeds are selected automatically, in general, multiple points are identified in each fuzzy connected object, and it may be impossible to identify exactly one seed point per object. (The seeds identified in the same fuzzy connected object need not be connected in the hard sense). Further, there are applications (such as the definition of vessels in MR angiographic images) wherein multiple seed points per object are needed to ensure that thicker as well as subtle and fine aspects of the same object are captured in its delineation. Thus there is a practical need for the previous theoretical and algorithmic framework to be extended to the case of multiple seed points. We present these generalizations in Sections 2 and 3. In Section 4, results on several applications are presented, and our conclusions are stated in Section 5.

2. THEORY

To begin, we restate in Sections 2.1 and 2.2 some known concepts already discussed in[7] for completeness and establishing the terminology. Some new material is also covered in these sections.

2.1. Fuzzy Relations, Scenes, Binary Scenes

Let $\mathcal{A}$ and $\mathcal{B}$ be two fuzzy subsets[20] of any reference set X. A *2-ary fuzzy relation* ρ in X is a fuzzy subset of $X \times X$, $\rho = \{((x,y), \mu_\rho(x,y)) \mid x, y \in X\}$ where $\mu_\rho : X \times X \to [0,1]$. Since we are not interested in fuzzy m-ary relations for $m > 2$, we drop the qualifier "2-ary" for simplicity. We use μ subscripted by the fuzzy subset under consideration to denote the membership function of the fuzzy subset. For hard subsets, μ will denote their characteristic function. Let ρ be any fuzzy relation in X. ρ is said to be *reflexive*, if, $\forall x \in X, \mu_\rho(x,x) = 1$; *symmetric*, if, $\forall x, y \in X, \mu_\rho(x,y) = \mu_\rho(y,x)$; *transitive*, if, $\forall x, z \in X, \mu_\rho(x,z) = \max_{y \in X}[\min[\mu_\rho(x,y), \mu_\rho(y,z)]]$. A fuzzy relation ρ is called a *similitude relation* in X if it is reflexive, symmetric and transitive. The analogous concept for hard binary relations is an equivalence relation.

For any $n \geq 2$, asusual, we think of Z^n to be the result of digitization of R^n, and identify the element of Z^n with hypercubes in R^n, which will be called *spels* (an abbreviation for "space elements"). Z^n itself will be thought of as the set of all spels in R^n. A fuzzy relation α in Z^n is said to be a *fuzzy adjacency* if it is both reflexive and symmetric. It is desirable that α be such that $\mu_\alpha(c,d)$ is a non increasing function of the distance $\|c - d\|$ between c and d. It is not difficult to see that the hard adjacency relations commonly used in digital topology[21] are special cases of fuzzy adjacencies. We call the pair (Z^n, α), where α is a fuzzy adjacency, a *fuzzy digital space*. A *scene over* a fuzzy digital space (Z^n, α) is a pair $\mathcal{C} = (C, f)$ where $C = \{c \mid -b_j \leq c_j \leq b_j$ for some $b \in Z^n_+\}$; Z^n_+ is the set of n-tuples of positive integers; f, called *scene intensity*, is a function whose domain is C, called the *scene domain*, and whose range is a set of numbers (usually integers). $\mathcal{C}$ is a *binary scene over* (Z^n, α) if the range of f is $\{0, 1\}$. We assume in this paper that $n \geq 2$.

2.2. Fuzzy Affinity, Path

Let $\mathcal{C} = (C, f)$ be a scene over (Z^n, α). Any fuzzy relation κ in C is said to be a *fuzzy spel affinity* (or, *affinity* for short) *in* $\mathcal{C}$ if it is reflexive and symmetric. In practice, however, κ should be such that, for any $c, d \in C$, $\mu_\kappa(c,d)$ is a function of (i) the fuzzy adjacency between c and d; (ii) the homogeneity of the spel intensities at c and d; (iii) the closeness of the spel intensities and of the intensity-based features of c and d to some expected intensity and feature values for the object. Further, $\mu_\kappa(c,d)$ may depend on the actual location of c and d (i.e., μ_κ is shift variant). Some examples of μ_κ are given in,[7] and a detailed and an objective comparative study of a variety of functional forms for μ_κ is reported in[8] including a scale-based formulation. Throughout this paper, κ with appropriate subscripts and superscripts will be used to denote fuzzy spel affinities.

A nonempty *path* p_{cd} *in* $\mathcal{C}$ *from* c *to* d is a sequence $\langle c = c_1, c_2, \cdots, c_l = d\rangle$ of $l \geq 1$ spels in C; l is called *length* of the path. (Note that the successive spels in the sequence need not be "adjacent" in the sense adjacency is usually defined in digital topology.[21]) An empty path in $\mathcal{C}$, denoted $\langle\rangle$, is a sequence of no spels. Paths of length 2 will be referred to as *links*. The set of all paths in $\mathcal{C}$ from c to d is denoted by P_{cd}. (Note that c and d are not necessarily distinct.) The set of all paths in $\mathcal{C}$, defined as $\bigcup_{c,d\in C} P_{cd}$, is denoted by $P_\mathcal{C}$. We define a binary *join to* operation on $P_\mathcal{C}$, denoted "+" as follows. For every two nonempty paths $p_{cd} = \langle c_1, c_2, \cdots, c_{l_1}\rangle \in P_\mathcal{C}$ and $p_{de} = \langle d_1, d_2, \cdots, d_{l_2}\rangle \in P_\mathcal{C}$ [note that $c_{l_1} = d_1 = d$],

$$
\begin{aligned}
p_{cd} + p_{de} &= \langle c_1, c_2, \cdots, c_{l_1}, d_2, \cdots, d_{l_2}\rangle, && (1)\\
p_{cd} + \langle\rangle &= p_{cd}, && (2)\\
\langle\rangle + p_{de} &= p_{de}, && (3)\\
\langle\rangle + \langle\rangle &= \langle\rangle. && (4)
\end{aligned}
$$

Note that the join to operation between $p_{c_1c_2}$ and $p_{d_1d_2}$ is undefined if $c_2 \neq d_1$. It is shown in[7] (Proposition 2.1) that for any scene $\mathcal{C} = (C, f)$ over any fuzzy digital space (Z^n, α), the following relation holds for any two spels $c, e \in C$:

$$P_{ce} = \{p_{cd} + p_{de} \mid d \in C \text{ and } p_{cd} \in P_{cd} \text{ and } p_{de} \in P_{de}\}. \tag{5}$$

We define a binary relation *greater than* on $P_\mathcal{C}$, denoted ">", as follows. Let $p = \langle c_1, c_2, \cdots c_{l_p}\rangle$ and $q = \langle d_1, d_2, \cdots d_{l_q}\rangle$ be any paths in $\mathcal{C}$. We say that $p > q$ if and only if we can find a mapping g from the set of spels in q to the set of spels in p that satisfies all of the following conditions:

1. $g(d_i) = c_j$ only if $d_i = c_j$.
2. There exists some $1 \leq m \leq l_p$, for which $g(d_1) = c_m$.
3. For all $1 \leq j < l_q$, whenever $g(d_j) = c_i$, $g(d_{j+1}) = c_k$ for some $k \geq i$ and $c_i = c_{i+1} = \cdots = c_{k-1}$.

Some examples follow: $\langle c_1, c_2, c_3, c_3, c_4, c_5\rangle > \langle c_3, c_4, c_4, c_5\rangle$; $\langle c_1, c_2, c_3, c_4\rangle > \langle c_3, c_3, c_3, c_3, c_3, c_3\rangle$. It readily follows that every non empty path in $\mathcal{C}$ is greater than the empty path $\langle\rangle$ in $\mathcal{C}$.

2.3. Functional Form for Path Strength, Fuzzy Connectedness

Our aim is to assign a strength of connectedness to every pair (c, d) of spels in C. It makes sense to consider this strength of connectedness to be the largest of the strengths assigned to all paths between c and d. (The physical analogy one may consider is to think of c and d as being connected by many strings, each with its own strength. When c and d are pulled apart the strongest string will break at the end, which should be the determining factor for the strength of connectedness between c and d.) However, it is not so obvious as to how the strength of each path should be defined. Several measures based on the affinities along the path including their sum, product, and minimum all seem plausible. We establish in this section that minimum of affinities is the only valid choice for path strength under the assumptions stated in Axioms 1–4 below, which, we believe are all reasonable.

Let $\mathcal{C} = (C, f)$ be a scene over a fuzzy digital space (Z^n, α) and let κ be a fuzzy affinity in $\mathcal{C}$. A *fuzzy* κ-net $\mathcal{N}$ in $\mathcal{C}$ is a fuzzy subset of $P_\mathcal{C}$ with its membership function $\mu_\mathcal{N} : P_\mathcal{C} \to [0, 1]$. $\mu_\mathcal{N}$ assigns a strength to every path of $P_\mathcal{C}$. For any spels $c, d \in C$, $p_{cd} \in P_{cd}$ is called a *strongest path* from c to d if $\mu_\mathcal{N}(p_{cd}) = \max_{p\in P_{cd}}[\mu_\mathcal{N}(p)]$. One of the questions this paper addresses is how to assign strengths to paths, or equivalently, what the functional form of $\mu_\mathcal{N}$ should be. The idea of a κ-net is to set up a network of all possible paths between all possible pairs of spels in $\mathcal{C}$ with a strength assigned to every path. This is for facilitating the definition of fuzzy connectedness.

AXIOM 1. *For any scene $\mathcal{C}$ over (Z^n, α), for any affinity κ and κ-net $\mathcal{N}$ in $\mathcal{C}$, for any two spels $c, d \in C$, the strength of the link from c to d is the affinity between them; i.e., $\mu_\mathcal{N}(\langle c, d\rangle) = \mu_\kappa(c, d)$.*

AXIOM 2. *For any scene $\mathcal{C}$ over (Z^n, α), for any affinity κ and κ-net $\mathcal{N}$ in C, for any two paths $p_1, p_2 \in P_C$, $p_1 > p_2$ implies that $\mu_{\mathcal{N}}(p_1) \leq \mu_{\mathcal{N}}(p_2)$.*

AXIOM 3. *For any scene $\mathcal{C}$ over (Z^n, α), for any affinity κ and κ-net $\mathcal{N}$ in C, fuzzy κ-connectedness K in C is a fuzzy relation in C defined by the following membership function. For any $c, d \in C$,*

$$\mu_K(c,d) = \max_{p \in P_{cd}} [\mu_{\mathcal{N}}(p)]. \tag{6}$$

(For fuzzy connectedness, we shall always use the upper case form of the symbol used to represent the corresponding fuzzy affinity.)

AXIOM 4. *For any scene $\mathcal{C}$ over (Z^n, α), for any affinity κ and κ-net $\mathcal{N}$ in C, fuzzy κ-connectedness in C is a symmetric and transitive relation.*

Axiom 1 says that, a link is a basic unit in any path, and that the strength of a link (which will be utilized in defining path strength) should be simply the affinity between the two component spels of the link. This is the fundamental way in which affinity is brought into the definition of path strength. Note that, in a link $\langle c_i, c_{i+1} \rangle$ in a path, c_i and c_{i+1} may not always be adjacent (in the sense "adjacency" is usually considered in hard digital topology) — that is, c_i and c_{i+1} may be far apart differing in some of their coordinates by more than 1. In such cases, Axiom 1 guarantees that the strength of $\langle c_i, c_{i+1} \rangle$ is determined by $\mu_\kappa(c,d)$ and not by "tighter" paths of the form $\langle c_i = c_{i,1}, c_{i,2}, \cdots, c_{i,m} = c_{i+1} \rangle$ wherein the successive spels are indeed adjacent. Since κ is by definition reflexive and symmetric, this axiom guarantees that link strength is also a reflexive and symmetric relation in C. Axiom 2 guarantees that the strength of any path changes in a non increasing manner along the path. This property is sensible and becomes essential in casting fuzzy connected object tracking as a dynamic programming problem. Axiom 3 says essentially that the strength of connectedness between c and d should be the strength of the strongest path between them. Its reasonableness has already been discussed. Finally, Axiom 4 guarantees that fuzzy connectedness is a similitude relation in C. Its reflexivity is proved in Proposition 1 below. This property is essential to prove the main theorem (Theorem 2.6 in[7]) that permits devising a dynamic programming solution to the otherwise seemingly prohibitive combinatorial optimization problem of determining a fuzzy connected object. For all proofs and theoretical detailes, see.[22]

In the remainder of this section, we shall establish that, under these axioms, the only possible choice for $\mu_{\mathcal{N}}(p)$ is the minimum of the strengths of the links in p.

PROPOSITION 1. *For any scene $\mathcal{C} = (C, f)$ over (Z^n, α), for any affinity κ and κ-net $\mathcal{N}$ in C, fuzzy κ-connectedness K in C is a similitude relation.*

PROPOSITION 2. *For any scene $\mathcal{C} = (C, f)$ over (Z^n, α), for any affinity κ and κ-net $\mathcal{N}$ in C, and for any path p in C of length $l_p \leq 1$, $\mu_{\mathcal{N}}(p) = 1$.*

PROPOSITION 3. *For any scene $\mathcal{C} = (C, f)$ over (Z^n, α), for any affinity κ and κ-net $\mathcal{N}$ in C, and for any spels $c, d \in C$, similitudity of fuzzy κ-connectedness implies that for any path $p = \langle c = c_1, c_2, \cdots, c_l = d \rangle$ from c to d, $\mu_K(c,d) \geq \min_{1<i\leq l}[\mu_\kappa(c_{i-1}, c_i)]$.*

PROPOSITION 4. *For any scene $\mathcal{C} = (C, f)$ over (Z^n, α), for any affinity κ and κ-net $\mathcal{N}$ in C, for a strongest path $p_{cd} = \langle c = c_1, c_2, \cdots, c_l = d \rangle$ between any two spels $c, d \in C$, and for any other path $p = \langle c = c_1^{(p)}, c_2^{(p)}, \cdots c_{l_p}^{(p)} = d \rangle$ between $c, d \in C$, $min_{1<i\leq l}[\mu_\kappa(c_{i-1}, c_i)] \geq \min_{1<i\leq l_p}[\mu_\kappa(c_{i-1}^{(p)}, c_i^{(p)})]$.*

LEMMA 5. *For any scene $\mathcal{C} = (C, f)$ over (Z^n, α), for any affinity κ and κ-net $\mathcal{N}$ in C, and for any spels $c, d \in C$, similitudity of fuzzy κ-connectedness implies*

$$\mu_K(c,d) = \max_{p \in P_{cd}} \left[\min_{1<i\leq l_p} \left[\mu_\kappa(c_{i-1}^{(p)}, c_i^{(p)}) \right] \right], \tag{7}$$

where $p = \langle c_1^{(p)}, c_2^{(p)}, \cdots, c_{l_p}^{(p)} \rangle$.

LEMMA 6. *For any scene $\mathcal{C} = (C, f)$ over (Z^n, α), for any affinity κ and κ-net $\mathcal{N}$ in C, and for any two spels $c, d \in C$, the following functional form of μ_K*

$$\mu_K(c,d) = \max_{p \in P_{cd}} \left[\min_{1<i\leq l_p} \left[\mu_\kappa(c_{i-1}^{(p)}, c_i^{(p)}) \right] \right], \tag{8}$$

where p is the path $\langle c_1^{(p)}, c_2^{(p)}, \cdots, c_{l_p}^{(p)} \rangle$, implies similitudity of fuzzy κ-connectedness K.

THEOREM 7. *For any scene $\mathcal{C} = (C, f)$ over (Z^n, α), and for any affinity κ and κ-net $\mathcal{N}$ in $\mathcal{C}$, fuzzy κ-connectedness K in $\mathcal{C}$ is a similitude relation in C if and only if*

$$\mu_K(c, d) = \max_{p \in P_{cd}} \left[\min_{1 < i \leq l_p} \left[\mu_\kappa(c_{i-1}^{(p)}, c_i^{(p)}) \right] \right],$$

where p is the path $\langle c_1^{(p)}, c_2^{(p)}, \cdots, c_{l_p}^{(p)} \rangle$.

We have now established, based on Axioms 1 to 4, that the minimum of affinities along a path is the only plausible choice for path strength. Following the spirit of the above theorem, we define path strength by

$$\mu_{\mathcal{N}}(p) = \min_{1 < i \leq l_p} \left[\mu_\kappa(c_{i-1}^{(p)}, c_i^{(p)}) \right], \tag{9}$$

where p is the path $\langle c_1^{(p)}, c_2^{(p)}, \cdots, c_{l_p}^{(p)} \rangle$. For the remainder of this paper, we shall assume the above definition of path strength.

2.4. Fuzzy Connected Objects, Fuzzy Object Extraction

In this section, we shall develop the notion of fuzzy connected objects containing a set of specified seed spels, and study the properties of such objects.

Let $\mathcal{C} = (C, f)$ be any scene over (Z^n, α), let κ be any affinity in $\mathcal{C}$, and let θ be a fixed number in $[0, 1]$. Let S be any subset of C. We shall refer to S as the set of reference spels or seed spels and assume throughout that $S \neq \phi$. A *fuzzy $\kappa\theta$-object $\mathcal{O}_{K\theta}(s)$ of $\mathcal{C}$ containing a seed spel s of $\mathcal{C}$* is a fuzzy subset of C whose membership function is

$$\mu_{\mathcal{O}_{K\theta}(s)}(c) = \begin{cases} \eta(f(c)), & \text{if } c \in O_{K\theta}(s) \subset C, \\ 0, & \text{otherwise.} \end{cases} \tag{10}$$

In this expression, η is an *objectness function* whose domain is the range of f and whose range is [0,1]. It maps imaged scene intensity values into objectness values. For most segmentation purposes, η may be chosen to be a Gaussian whose mean and standard deviation correspond to the intensity value expected for the object region and its standard deviation (or some multiple thereof). The choice of η should depend on the particular imaging modality that generated $\mathcal{C}$ and the actual physical object under consideration. (When a hard segmentation is desired, $O_{K\theta}(s)$ (defined bellow) will constitute the (hard) set of spels that represents the extent of the physical object and η will simply be the characteristic function of $O_{K\theta}(s)$.) $O_{K\theta}(s)$ is a subset of C satisfying all of the following conditions:

$$\text{(i)} \quad s \in O_{K\theta}(s); \tag{11}$$

$$\text{(ii)} \quad \text{for any spels } c, d \in O_{K\theta}(s),\ \mu_K(c, d) \geq \theta; \tag{12}$$

$$\text{(iii)} \quad \text{for any spel } c \in O_{K\theta}(s) \text{ and any spel } d \notin O_{K\theta}(s),\ \mu_K(c, d) < \theta. \tag{13}$$

We shall refer to $O_{K\theta}(s)$ as the *support of* $\mathcal{O}_{K\theta}(s)$. In words, the support of $\mathcal{O}_{K\theta}(s)$ is a maximal subset of C such that it includes s and the strength of connectedness between any two of its spels is at least θ. (This definition is different from the one given in,[7] and we believe, is intuitively more sensible. Later on, we shall establish their theoretical equivalence.)

We now generalize the above concept of a fuzzy connected object from a single seed spel s to a set S of spels. A *fuzzy $\kappa\theta$-object $\mathcal{O}_{K\theta}(S)$ of $\mathcal{C}$ containing a set S* of *seed spels of $\mathcal{C}$* is a fuzzy subset of C whose membership function is

$$\mu_{\mathcal{O}_{K\theta}(S)}(c) = \begin{cases} \eta(f(c)), & \text{if } c \in O_{K\theta}(S) = \bigcup_{s \in S} O_{K\theta}(s), \\ 0, & \text{otherwise.} \end{cases} \tag{14}$$

We shall refer to $O_{K\theta}(S)$ as the *support of* $\mathcal{O}_{K\theta}(S)$. (Note that $O_{K\theta}(\{s\}) = O_{K\theta}(s)$.) In words, the support of $\mathcal{O}_{K\theta}(S)$ is simply the union of the support of the fuzzy connected objects containing the individual seed spels of S.

The following theorem gives us a guidance as to how a fuzzy $\kappa\theta$-object $O_{K\theta}(S)$ of $\mathcal{C}$ should be computed. It is not practical to use the definition directly for this purpose because of the combinatorial complexity. The following theorem provides a practical way of computing $O_{K\theta}(S)$.

THEOREM 8. *For any scene $\mathcal{C} = (C, f)$ over (Z^n, α), for any affinity κ in $\mathcal{C}$, for any $\theta \in [0, 1]$, for any objectness function η, and for any non empty set $S \subset C$, the support $O_{K\theta}(S)$ of the fuzzy $\kappa\theta$-object of $\mathcal{C}$ containing S equals*

$$O_{K\theta}(S) = \{c \mid c \in C \text{ and } \max_{s \in S}[\mu_K(s, c)] \geq \theta\}. \tag{15}$$

As a consequence of the above theorem, we shall show in the next section that $O_{K\theta}(S)$ can be computed via dynamic programming, given $\mathcal{C}$, κ, θ, η, and S. We shall now study some important properties of fuzzy $\kappa\theta$-objects of $\mathcal{C}$ containing S, eventually demonstrating the robustness of the definition of fuzzy $\kappa\theta$-objects with respect to seed specification (Theorem 11).

PROPOSITION 9. *For any scene $\mathcal{C} = (C, f)$ over (Z^n, α), for any affinity κ in $\mathcal{C}$, for any $\theta \in [0, 1]$, for any objectness function η, and for any spels $s_1, s_2 \in C$, $\mathcal{O}_{K\theta}(s_1) = \mathcal{O}_{K\theta}(s_2)$ if and only if $s_2 \in O_{K\theta}(s_1)$.*

It follows immediately from the above proposition that the fuzzy $\kappa\theta$-object $\mathcal{O}_{K\theta}(s)$ of $\mathcal{C}$ specified by s is unique for any given $\mathcal{C}$, s, θ, κ, and η, establishing the legitimacy of the definition of $\mathcal{O}_{K\theta}(s)$. The following is an immediate consequence of the above proposition.

COROLLARY 10. *For any scene $\mathcal{C} = (C, f)$ over (Z^n, α), for any affinity κ in $\mathcal{C}$, for any $\theta \in [0, 1]$, for any objectness function η, and for any seed spel $s \in C$, and a non empty set $S \subset C$, $\mathcal{O}_{K\theta}(s) = \mathcal{O}_{K\theta}(S)$ if and only if $S \subset O_{K\theta}(s)$.*

The above results lead us to the following theorem characterizing the robustness of specifying fuzzy $\kappa\theta$-objects through sets of seed spels.

THEOREM 11. *For any scene $\mathcal{C} = (C, f)$ over (Z^n, α), for any affinity κ in $\mathcal{C}$, for any $\theta \in [0, 1]$, for any objectness function η, and for any non empty sets $S_1, S_2 \subset C$, $\mathcal{O}_{K\theta}(S_1) = \mathcal{O}_{K\theta}(S_2)$ if and only if $S_1 \subset O_{K\theta}(S_2)$ and $S_2 \subset O_{K\theta}(S_1)$.*

The above theorem has important consequences in the practical utilization of the fuzzy connectedness algorithms presented in the next section. It states that the seeds must be selected from the same physical object and at least one seed must be selected from each physically connected region. High precision (reproducibility) of any segmentation algorithm with regard to subjective operator actions (and with regard to automatic operations minimizing these actions), such as specification of seeds, is essential for their practical utility. Generally, it is easy for human operators to specify spels within a region in the scene corresponding to the same physical object in a repeatable fashion. Theorem 11 guarantees that, even though the sets of spels specified in repeated trials may not be the same, as long as these sets are within the region of the same physical object in the scene, the resulting segmentations will be identical. Many region growing algorithms which adaptively change the spel inclusion criteria during the growth process cannot guarantee this robustness property. We note that Theorem 11 also establishes the uniqueness of the fuzzy $\kappa\theta$-object specified by S, asserting the legitimacy of the definition.

3. ALGORITHMS

In this section we present two algorithms for extracting a fuzzy $\kappa\theta$-object containing a set S of spels in a given scene $\mathcal{C}$ for a given affinity κ in $\mathcal{C}$, both based on dynamic programming.[23] The first algorithm, named $\kappa\theta FOEMS$ (*$\kappa\theta$-fuzzy object extraction for multiple seeds*), extracts a fuzzy $\kappa\theta$-object of $\mathcal{C}$ of strength θ generated by the set S of reference spels. In this algorithm, the value of θ is assumed to be given as input and the algorithm uses this knowledge to achieve an efficient extraction of the $\kappa\theta$-object. In the second algorithm, named $\kappa FOEMS$, we output what we call a *κ-connectivity scene $\mathcal{C}_{KS} = (C, f_{KS})$ of $\mathcal{C}$ generated by* the set S of reference spels defined by $f_{KS}(c) = \max_{s \in S}[\mu_K(s, c)]$. The correct behavious of these algorithms is established in.[22]

Algorithm $\kappa\theta FOEMS$ terminates faster than $\kappa FOEMS$ for two reasons. First, $\kappa\theta FOEMS$ produces the hard set based on (Theorem 8). Therefore, for any spel $c \in C$, once we find a path of strength θ or greater from any of the reference spels to c, we do not need to search for a better path upto c, and hence, can avoid further processing for c. This allows us to reduce computation. Second, certain computations are avoided for those spels $d \in C$ for which $\max_{s \in S}[\mu_K(s,d)] < \theta$.

Unlike $\kappa\theta FOEMS$, $\kappa FOEMS$ computes the best path from the reference spels of S to every spel c in C. Therefore, every time the algorithm finds a better path upto c, it modifies the connectivity value at c and subsequently processes other spels which are affected by this modification. The algorithm generates a connectivity scene $\mathcal{C}_{KS} = (C, f_{KS})$ of $\mathcal{C}$. Although, $\kappa FOEMS$ terminates slower, it has a practical advantage. After the algorithm terminates, one can interactively specify θ and thereby examine various $\kappa\theta$-objects and interactively select the best θ. The connectivity scene has interesting properties relevant to classification and in shell rendering and manipulation[24] of $\kappa\theta$-objects.

Algorithm $\kappa\theta FOEMS$
Input: $\mathcal{C}$, S, κ, η, and θ as defined in Section 2.
Output: $\mathcal{O}_{K\theta}(S)$ as defined in Section 2.
Auxiliary Data Structures: An n-D array representing a temporary scene $\mathcal{C}' = (C, f')$ such that f' corresponds to the characteristic function of $O_{K\theta}(S)$, and a queue Q of spels. We refer to the array itself by $\mathcal{C}'$ for the purpose of the algorithm.

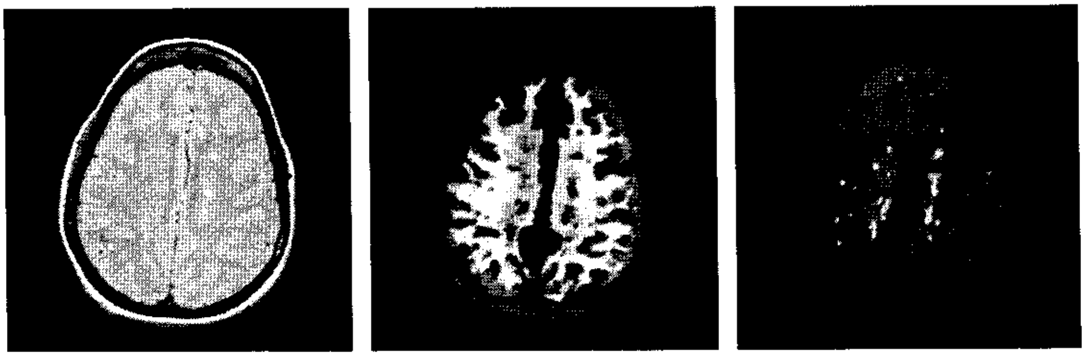
begin
1. set all elements of $\mathcal{C}'$ to 0 except those spels $s \in S$ which are set to 1;
2. push all spels $c \in C$ such that for some $s \in S$ $\mu_\kappa(s,c) \geq \theta$ to Q;
3. *while* Q is not empty *do*
4.   remove a spel c from Q;
5.   *if* $f'(c) \neq 1$ *then*
6.   set $f'(c) = 1$;
7.   push all spels d such that $\mu_\kappa(c,d) \geq \theta$ to Q;
    endif;
  endwhile;
8. create and output $\mathcal{O}_{K\theta}(S)$ by assigning the value $\eta(f(c))$ to all $c \in C$ for which $f'(c) > 0$ and 0 to the rest of the spels;

end

Algorithm $\kappa FOEMS$
Input: $\mathcal{C}$, S, and κ as defined Section 2.
Output: A scene $\mathcal{C}' = (C, f')$ representing the κ-connectivity scene $\mathcal{C}_{KS}$ of $\mathcal{C}$ generated by S.
Auxiliary Data Structures: An n-D array representing the connectivity scene $\mathcal{C}' = (C, f')$ and a queue Q of spels. We refer to the array itself by $\mathcal{C}'$ for the purpose of the algorithm.

begin
1. set all elements of $\mathcal{C}'$ to 0 except those spels $s \in S$ which are set to 1;
2. push all spels $c \in C$ such that, for some $s \in S$, $\mu_\kappa(s,c) > 0$ to Q;
3. *while* Q is not empty *do*
4.   remove a spel c from Q;
5.   find $f_{\max} = \max_{d \in C}[\min(f'(d), \mu_\kappa(c,d))]$;
6.   *if* $f_{\max} > f'(c)$ *then*
7.   set $f'(c) = f_{\max}$;
8.   push all spels e such that $\min[f_{max}, \mu_\kappa(c,e)] > f'(e)$ to Q;
    endif;
  endwhile;
9. output the connectivity scene $\mathcal{C}'$;

end



(a) (b) (c)

Figure 1. A 2D slice taken from a proton density weighted 3D MR scene of a multiple sclerosis patient's head. (b), (c) The κ-connectivity scenes for white matter regions and the lesions (somewhat hyper intense areas) in the slice shown in (a).

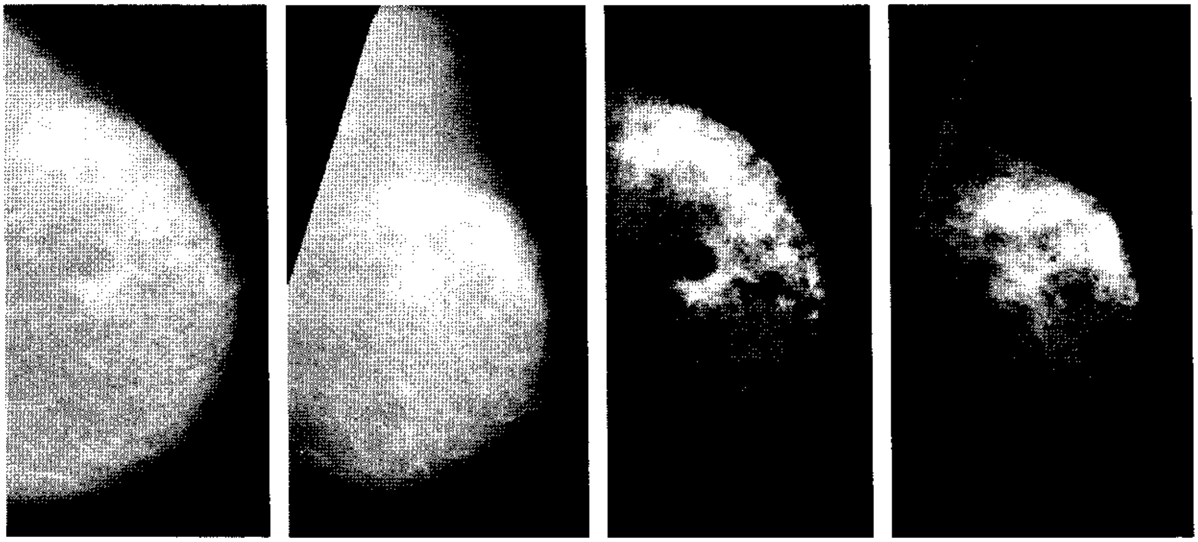
4. RESULTS AND DISCUSSION

We have been developing the fuzzy connectedness family of methods[7–10, 12] for the past seven years. They have been implemented in the 3DVIEWNIX software system[25] and evaluated on several thousands of patient data sets in several medical applications[15–19] as to their precision, accuracy and efficiency. In all these applications, we encounter multiple seeds. Although these applications utilized the algorithms described in this paper within the framework of the original fuzzy connectedness algorithm,[7] algorithms $\kappa FOEMS$ and $\kappa\theta FOEMS$, the proof of their correctness, the underlying theory of fuzzy $\kappa\theta$-objects containing a set of seed spels, and the axiomatic definition of fuzzy connectedness were not described previously. Since evaluation of these algorithms has already been done in specific medical applications, in this paper we shall only give examples without undertaking formal evaluation.

Our first example, illustrated in Figure 1, comes from a proton density (PD) weighted 3D MR scene of a multiple sclerosis (MS) patient's head. A 2D slice from the 3D scene of domain $256 \times 256 \times 53$ and of voxel size $0.86 \times 0.86 \times 3.0$ mm^3 is displayed in Figure 1(a). The fuzzy connectedness method was separately applied to the 3D scene ($n = 3$) to compute the white matter (WM) region and the lesions. Figures 1(b) and (c) demonstrate a slice (corresponding to the slice shown in Figure 1(a)) taken from the 3D κ-connectivity scenes for the two objects. In the two different executions of algorithm $\kappa FOEMS$, different affinities that were appropriate for the two different tissue regions (WM and lesions) were used. In this example, the seeds were specified manually, several seeds for the WM regions and a couple of them in each of the lesion blobs. An automatic seed selection method for this lesion detection task has been presented in[15] which consists of first delineating WM, gray matter, and cerebro-spinal fluid fuzzy objects and then detecting holes in the union of the support of these three fuzzy $\kappa\theta$-objects. For delineating the three objects, several seeds are specified on one slice manually in each of the three object regions.

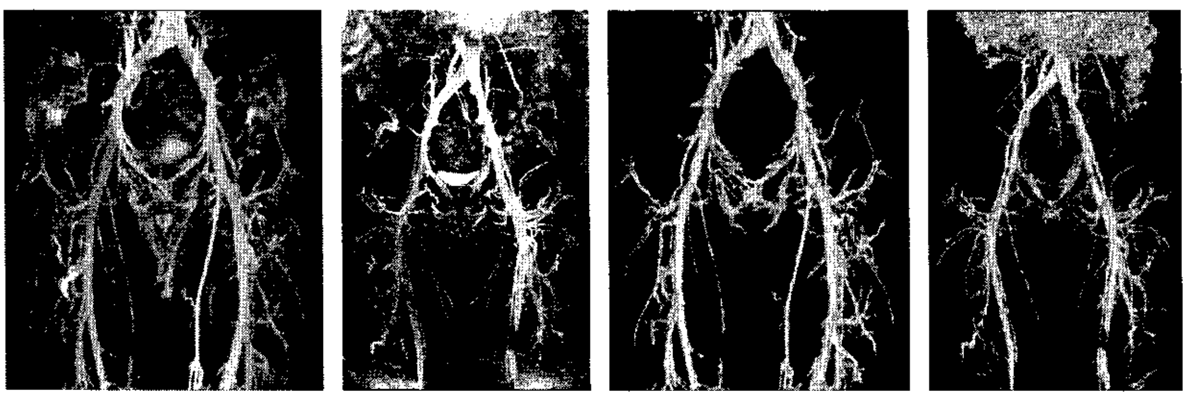
Figure 2 illustrates our second example — delineation of dense regions in digitized mammograms. The extent of the dense regions is known to indicate the risk of breast cancer. Therefore, the segmentation and quantification of these regions is potentially useful in the management and screening of patients. Figure 2(a) shows a digitized mammogram taken in the cranio-caudal (CC) projection, while (b) shows a digitized mammogram of the same breast taken in the medio-lateral-oblique (MLO) projection. In (b), the pectoral muscles projected in the scene were removed manually. The κ-connectivity scenes for Figures 2(a) and (b) are shown in Figures 2(c) and (d), respectively. Here, the seeds were automatically selected using an initial conservative thresholding.

Our third example, demonstrated in Figure 3, pertains to the application of separating veins and arteries in contrast enhanced MR scenes.[17] The separation is achieved by first segmenting the whole vessel tree in the scene and then separating arteries and veins within the vessel tree. Figure 3 demonstrates the results of the first step



(a) (b) (c) (d)

Figure 2. A digitized mammogram of a patient's breast (a) at CC projection and (b) at the MLO projection (after manually removing pectoral muscles). (c), (d) κ-connectivity scenes for dense regions in (a) and (b), respectively.



(a) (b) (c) (d)

Figure 3. (a), (b) Maximum intensity projection (MIP) renditions of contrast enhanced 3D MR scenes of two patients. The scene domains represent the body region from the belly to the knee. (c), (d) 3D renditions of segmented vessel trees obtained from the κ-connectivity scenes of the data sets in (a) and (b), respectively.

(the second step uses relative fuzzy connectedness and its variations[9,10] not discussed in this paper). Figures 3(a) and (b) show maximum intensity projection (MIP) renditions of the contrast enhanced 3D MR scenes of the body regions from the belly to the knee of two patients. The MIP renditions are created (without requiring any segmentation) by assigning to each pixel in the rendition a gray value that corresponds to the maximum scene intensity along the line of projection corresponding to the pixel. (c) and (d) show 3D renditions of the vessel trees segmented from their fuzzy connectedness scenes. The domains and voxel sizes for both these scenes are $512 \times 512 \times 60$ and $0.94 \times 0.94 \times 1.80$ mm^3, respectively.

5. CONCLUDING REMARKS

This paper makes two main contributions. First, continuing on a previously developed framework of fuzzy connectedness and object definition in multidimensional scenes,[7] it establishes that maximization of path strength of the minimum of affinities along each path is the one and the only valid choice for defining fuzzy connectedness. This uniqueness of the definition of fuzzy connectedness has been established starting from four axioms. (1) The strength of a link between two image elements is their affinity itself. (2) As a path grows longer, its strength does not increase. (3) The strength of connectedness between any two image elements is simply the strength of the strongest path between them. (4) Fuzzy connectedness is a symmetric and transitive relation. The first three axioms, we assert, are quite natural and inherent to the intuition and the idea of fuzzy connectedness. The fourth axiom is needed to devise practical algorithms for finding fuzzy connected objects.

The second contribution of this paper is the extension of the original single-seeded fuzzy connectedness theory and algorithms[7] to the case of multiple seeds. One of the main results of this extension is the demonstration that any subset of image elements in the support of the fuzzy connected object would generate the same fuzzy connectedness object provided that at least one seed is selected from each physically connected region, establishing the robustness of the delineation with respect to the choice of seed spels. Another main result is the demonstration that a dynamic programming solution suggested in the original framework[7] is applicable to the multiple seed case also with some modifications. The correctness of the new algorithms has been established in this paper. The importance of multiple-seeded fuzzy connectedness from both practicality and automation in image segmentation has been illustrated by several examples drawn from large ongoing medical applications where its utility has been established on hundreds of real data sets.

Among the four axioms which form the basis of the theory presented in this paper, the fourth has mainly an algorithmic motivation. It will be interesting to see how the theory and algorithms get modified (even at the sacrifice of algorithmic efficiency) if Axiom 4 is dropped.

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