Homework 2 solution

Max marks: 110

Due on September 17, 2021, 9 AM, before class.

Row	x_1	x_2	x_3	f
0	0	0	0	0
1	0	0	1	1
2	0	1	0	1
3	0	1	1	0
4	1	0	0	1
5	1	0	1	0
6	1	1	0	0
7	1	1	1	1

Table 1: Truth table for a 3-way light switch

1 Sept 10th Lecture

Problem 1 *If the SOP form for $\bar{f} = A\bar{B}\bar{C} + \bar{A}\bar{B}$, then give the POS form for f. [10 marks]*

Solution

Take inversion on both sides

$$\begin{aligned} \bar{\bar{f}} &= \overline{A\bar{B}\bar{C} + \bar{A}\bar{B}} \\ f &= \overline{A\bar{B}\bar{C}} \cdot \overline{\bar{A}\bar{B}} && \text{by DeMorgan's} \\ &= (\bar{A} + B + C)(A + B) && \text{by DeMorgan's} \end{aligned}$$

Problem 2 *Use DeMorgan's Theorem to find f if $\bar{f} = (A + BC)D + EF$. [10 marks]*

Solution

Take inversion on both sides

$$\begin{aligned} \bar{\bar{f}} &= \overline{(A + BC)D + EF} \\ f &= \overline{((A + BC)D)} \cdot \overline{EF} && \text{by DeMorgan's} \\ &= (\overline{(A + BC)} + \bar{D})(\bar{E} + \bar{F}) && \text{by DeMorgan's} \\ &= (\bar{A}\overline{(BC)} + \bar{D})(\bar{E} + \bar{F}) && \text{by DeMorgan's} \\ &= (\bar{A}(\bar{B} + \bar{C}) + \bar{D})(\bar{E} + \bar{F}) && \text{by DeMorgan's} \end{aligned}$$

Problem 3 *Implement the function in Table 1 using only NAND gates. [10 marks]*

Solution

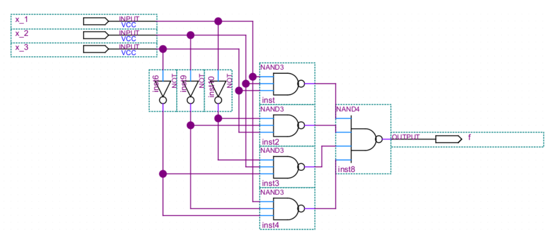
To implement the function using NAND gates, we seek the SOP form of the function,

	$\bar{x}_1$		x_1	
	$\bar{x}_2$	x_2	x_2	$\bar{x}_2$
$\bar{x}_3$	0	1	0	1
x_3	1	0	1	0

.

The function cannot be simplified beyond minterms.

$$\begin{aligned} f &= \bar{x}_1\bar{x}_2x_3 + \bar{x}_1x_2\bar{x}_3 + x_1\bar{x}_2\bar{x}_3 + x_1x_2x_3 \\ &= \overline{\overline{\bar{x}_1\bar{x}_2x_3}} + \overline{\overline{\bar{x}_1x_2\bar{x}_3}} + \overline{\overline{x_1\bar{x}_2\bar{x}_3}} + \overline{\overline{x_1x_2x_3}} \\ &= \overline{\overline{\bar{x}_1\bar{x}_2x_3} \cdot \overline{\bar{x}_1x_2\bar{x}_3} \cdot \overline{x_1\bar{x}_2\bar{x}_3} \cdot \overline{x_1x_2x_3}} \end{aligned}$$



Problem 4 *Implement the function in Table 1 using only NOR gates. [10 marks]*

Solution

To implement the function using NAND gates, we seek the POS form of the function. We plot the K-map for $\bar{f}$,

	$\bar{x}_1$		x_1	
	$\bar{x}_2$	x_2	x_2	$\bar{x}_2$
$\bar{x}_3$	1	0	1	0
x_3	0	1	0	1

.

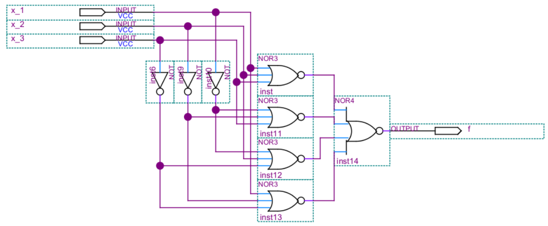
The function $\bar{f}$ cannot be simplified further,

$$\bar{f} = \bar{x}_1\bar{x}_2\bar{x}_3 + \bar{x}_1\bar{x}_2x_3 + \bar{x}_1x_2\bar{x}_3 + x_1\bar{x}_2x_3$$

Taking inverse of both sides and observing $\bar{\bar{f}} = f$.

$$\begin{aligned} f &= (x_1 + x_2 + x_3)(x_1 + \bar{x}_2 + \bar{x}_3)(\bar{x}_1 + x_2 + \bar{x}_3)(\bar{x}_1 + \bar{x}_2 + x_3) \\ &= \overline{\overline{(x_1 + x_2 + x_3)(x_1 + \bar{x}_2 + \bar{x}_3)(\bar{x}_1 + x_2 + \bar{x}_3)(\bar{x}_1 + \bar{x}_2 + x_3)}} \end{aligned}$$

$$= \overline{\overline{(x_1 + x_2 + x_3)} + \overline{(x_1 + \bar{x}_2 + \bar{x}_3)} + \overline{(\bar{x}_1 + x_2 + \bar{x}_3)} + \overline{(\bar{x}_1 + \bar{x}_2 + x_3)}}$$



2 Sept 13th Lecture

Problem 5 *Find the minimum-cost SOP and POS forms for the function* $f(x_1, x_2, x_3) = m(1, 2, 3, 5)$. *[1, Prob 2.37] [10 marks]*

Solution

Minimum cost SOP

	$\bar{x}_1$		x_1	
	$\bar{x}_2$	x_2		$\bar{x}_2$
$\bar{x}_3$	0	1	0	0
x_3	1	1	0	1

$$f = \bar{x}_1 x_2 + \bar{x}_2 x_3 \tag{1}$$

Cost = 2 AND + 1 OR + (2 * (2 input per AND gates) + 2 input per OR gate) inputs = 9

To find Minimum cost POS, we draw K-map for $\bar{f}$.

	$\bar{x}_1$		x_1	
	$\bar{x}_2$	x_2		$\bar{x}_2$
$\bar{x}_3$	0	1	0	0
x_3	1	1	0	1

$$\bar{f} = \bar{x}_2 \bar{x}_3 + x_1 x_2 \tag{2}$$
$$\implies f = (x_2 + x_3)(\bar{x}_1 + \bar{x}_2) \tag{3}$$

Cost = 2 OR + 1 AND + (2 * (2 inputs per OR gate) + 2 input AND gate) inputs = 9

Problem 6 *Find the minimum-cost SOP and POS forms for the function* $f(x_1, x_2, x_3) = \sum m(1, 4, 7) + D(2, 5)$. *[1, Prob 2.38] [10 marks]*

Solution

Minimum cost SOP

	$\bar{x}_1$			x_1
	$\bar{x}_2$	x_2		$\bar{x}_2$
$\bar{x}_3$	0	d	0	1
x_3	1	0	1	d + d + d

$$f = x_1 \bar{x}_2 + x_1 x_3 + \bar{x}_2 x_3 \tag{4}$$

Cost = 3 AND + 1 OR + (3 * (2 input per AND gate) + 3 inputs per OR gate) inputs = 13

To find minimum cost POS, we draw K-map for $\bar{f}$,

	$\bar{x}_1$		x_1	
	$\bar{x}_2$	x_2		$\bar{x}_2$
$\bar{x}_3$	1	d + d + d	1	0
x_3	0	1	0	d

$$\bar{f} = \bar{x}_1 \bar{x}_3 + \bar{x}_1 x_2 + x_2 \bar{x}_3 \tag{5}$$
$$\implies f = (x_1 + x_3)(x_1 + \bar{x}_2)(\bar{x}_2 + x_3) \tag{6}$$

Cost = 3 OR + 1 AND + (3*(2 inputs per OR gate)+3 inputs per AND gate) inputs = 13

Problem 7 *Find the minimum-cost SOP and POS forms for the function* $f(x_1, x_2, x_3, x_4) = \prod M(0, 1, 2, 4, 5, 7, 8, 9, 10, 12, 14, 15)$. *[1, Prob 2.39] [10 marks]*

Solution

The function f is zero at the maxterms. We draw the following K-map,

		$\bar{x}_1$		x_1	
		$\bar{x}_2$	x_2		$\bar{x}_2$
$\bar{x}_3$	$\bar{x}_4$	0	0	0	0
	x_4	0	0	1	0
x_3	x_4	1	0	0	1
	$\bar{x}_4$	0	1	0	0

$$f = \bar{x}_2 x_3 x_4 + \bar{x}_1 x_2 x_3 \bar{x}_4 + x_1 x_2 \bar{x}_3 x_4 \tag{7}$$

Cost = 3 AND gates + 1 OR gate + (3+4+4 inputs to the AND gates + 3 inputs to the OR gate) = 18.

To find the POS form, we draw K-map for $\bar{f}$,

		$\bar{x}_1$		x_1	
		$\bar{x}_2$	x_2		$\bar{x}_2$
$\bar{x}_3$	$\bar{x}_4$	1 + 1	1 + 1	1	1 + 1
	x_4	1	1	0	1
x_3	x_4	0	1	1	0
	$\bar{x}_4$	1	0	1	1 + 1

$$\bar{f} = \bar{x}_1\bar{x}_3 + \bar{x}_3\bar{x}_4 + x_2x_3x_4 + x_1\bar{x}_2\bar{x}_3 + x_1x_3\bar{x}_4 + \bar{x}_2x_3\bar{x}_4$$

$$\implies f = (x_1 + x_3)(x_3 + x_4)(x_2 + x_3 + x_4)(\bar{x}_1 + x_2 + x_3)$$

Problem 8 *Find the minimum-cost SOP and POS forms for the function* $f(x_1, x_2, x_3, x_4) = \sum m(0, 2, 8, 9, 12, 15) + D(1, 3, 6, 7)$. *[1, Prob 2.40] [10 marks]*

Solution

The K-map for f is

		$\bar{x}_1$		x_1	
		$\bar{x}_2$	x_2		$\bar{x}_2$
$\bar{x}_3$	$\bar{x}_4$	1	0	1	1 + 1
	x_4	d	0	0	1
x_3	x_4	d	d	1	0
	$\bar{x}_4$	1	d	0	0

$$f = \bar{x}_1\bar{x}_2 + x_1\bar{x}_3\bar{x}_4 + x_1\bar{x}_2\bar{x}_3 + x_2x_3x_4$$

Cost = 4 AND gates + 1 OR gate + (2 + 3 + 3 +3 inputs to the AND gates + 4 inputs to the OR gate) = 20
The K-map for $\bar{f}$ is

		$\bar{x}_1$		x_1	
		$\bar{x}_2$	x_2		$\bar{x}_2$
$\bar{x}_3$	$\bar{x}_4$	0	1	0	0
	x_4	d	1 + 1	1	0
x_3	x_4	d	d	0	1
	$\bar{x}_4$	0	d	1	1 + 1

$$\bar{f} = \bar{x}_1x_2 + x_2\bar{x}_3x_4 + x_1x_3\bar{x}_4 + x_1\bar{x}_2x_3. \quad (8)$$

$$\implies f = (x_1 + \bar{x}_2)(\bar{x}_2 + x_3 + \bar{x}_4)(\bar{x}_1 + \bar{x}_3 + x_4)(\bar{x}_1 + x_2 + \bar{x}_3). \quad (9)$$

Cost = 4 OR gates + 1 AND gate + (2 + 3 + 3 + 3 inputs to OR gates and 4 inputs to the AND gate) = 20

Problem 9 *Derive a minimum-cost realization of the four-variable function that is equal to 1 if exactly two or exactly three of its variables are equal to 1; otherwise it is equal to 0. [1, Prob 2.46] [10 marks]*

Solution

Row	x_1	x_2	x_3	x_4	f	Reason
0	0	0	0	0	0	
1	0	0	0	1	0	
2	0	0	1	0	0	
3	0	0	1	1	1	2-var are one
4	0	1	0	0	0	
5	0	1	0	1	1	2-var
6	0	1	1	0	1	2-var
7	0	1	1	1	1	3-var
8	1	0	0	0	0	
9	1	0	0	1	1	2-var
10	1	0	1	0	1	2-var
11	1	0	1	1	1	3-var
12	1	1	0	0	1	2-var
13	1	1	0	1	1	3-var
14	1	1	1	0	1	3-var
15	1	1	1	1	0	

K-map for the function f is

		$\bar{x}_1$		x_1	
		$\bar{x}_2$	x_2		$\bar{x}_2$
$\bar{x}_3$	$\bar{x}_4$	0	0	1	0
	x_4	0	1	1 + 1	1
x_3	x_4	1	1	0	1
	$\bar{x}_4$	0	1	1 + 1	1

$$f = x_2\bar{x}_3x_4 + x_2x_3\bar{x}_4 + x_1\bar{x}_2x_4 + \bar{x}_1x_3x_4 + x_1x_3\bar{x}_4 + x_1x_2\bar{x}_3$$

Cost = 5 AND gates + 1 OR gate + (5*3 inputs per AND gate + 5 inputs to the OR gate) = 26
K-map for the inverted function $\bar{f}$ is

		$\bar{x}_1$		x_1	
		$\bar{x}_2$	x_2		$\bar{x}_2$
$\bar{x}_3$	$\bar{x}_4$	1 + 1 + 1 + 1	1	0	1
	x_4	1	0	0	0
x_3	x_4	0	0	1	0
	$\bar{x}_4$	1	0	0	0

$$\bar{f} = \bar{x}_1\bar{x}_3\bar{x}_4 + \bar{x}_2\bar{x}_3\bar{x}_4 + \bar{x}_1\bar{x}_2\bar{x}_3 + \bar{x}_1\bar{x}_2\bar{x}_4 + x_1x_2x_3x_4$$

$$f = (x_1 + x_3 + x_4)(x_2 + x_3 + x_4)(x_1 + x_2 + x_3)(x_1 + x_2 + x_4)(\bar{x}_1 + \bar{x}_2 + \bar{x}_3 + \bar{x}_4)$$

		$\bar{x}_1 \bar{x}_2 \bar{x}_3$	$\bar{x}_1 \bar{x}_2 x_3$	$\bar{x}_1 x_2 x_3$	$\bar{x}_1 x_2 \bar{x}_3$	$x_1 \bar{x}_2 \bar{x}_3$	$x_1 \bar{x}_2 x_3$	$x_1 x_2 x_3$	$x_1 x_2 \bar{x}_3$
$\bar{x}_4$	$\bar{x}_5$	1	1	d	1	1	1	0	1
	x_5	1	d	1	1	d	1	0	1
x_4	x_5	1	d	d	1	1	d	0	0
	$\bar{x}_5$	0	1	1	0	0	1	0	0

		$x_1 = 0/1$: $\bar{x}_2 \bar{x}_3$	$\bar{x}_2 x_3$	$x_2 x_3$	$x_2 \bar{x}_3$
$\bar{x}_4$	$\bar{x}_5$	1 1	1 1	d 0	1 1
	x_5	1 d	d 1	1 0	1 1
x_4	x_5	1 1	d d	d 0	1 0
	$\bar{x}_5$	0 0	1 1	1 0	0 0

Table 2: Two ways to represent K-map for f in problem 10. Either way is correct. The essential minterm for the Essential Prime implicant is indicated with the same color.

		$\bar{x}_1 \bar{x}_2 \bar{x}_3$	$\bar{x}_1 \bar{x}_2 x_3$	$\bar{x}_1 x_2 x_3$	$\bar{x}_1 x_2 \bar{x}_3$	$x_1 \bar{x}_2 \bar{x}_3$	$x_1 \bar{x}_2 x_3$	$x_1 x_2 x_3$	$x_1 x_2 \bar{x}_3$
$\bar{x}_4$	$\bar{x}_5$	0	0	d	0	0	0	1	0
	x_5	0	d	0	0	d	0	1	0
x_4	x_5	0	d	d	0	0	d	1	1
	$\bar{x}_5$	1	0	0	1	1	0	1	1

		$x_1 = 0/1$: $\bar{x}_2 \bar{x}_3$	$\bar{x}_2 x_3$	$x_2 x_3$	$x_2 \bar{x}_3$
$\bar{x}_4$	$\bar{x}_5$	0/0	0/0	d/1	0/0
	x_5	0/d	d/0	0/1	0/0
x_4	x_5	0/0	d/d	d/1	0/1
	$\bar{x}_5$	1/1	0/0	0/1	1/1

Table 3: Two ways to represent 5-var K-map for $\bar{f}$ in problem 10. The essential minterms for Essential Prime Implicants (EPI) is shown in the same color.

Cost = 5 OR gates + 1 AND gate + (4 * 3 inputs per OR gate + 4 inputs to one OR gate + 5 inputs to 1 AND gate = 27

The minimal cost representation is the SOP representation:

$$f = x_2\bar{x}_3x_4 + x_2x_3\bar{x}_4 + x_1\bar{x}_2x_4 + \bar{x}_1x_3x_4 \\ + x_1x_3\bar{x}_4 + x_1x_2\bar{x}_3$$

Problem 10 *Find the minimum-cost SOP and POS forms for the function* $f(x_1, \ldots, x_5) = \sum m(0, 1, 3, 4, 6, 8, 9, 11, 13, 14, 16, 19, 20, 21, 22, 24, 25) + D(5, 7, 12, 15, 17, 23)$. *[1, Prob 2.42] [10 marks]*

Solution

The K-map for the function is in Table 2.

$$f = \bar{x}_1x_5 + \bar{x}_1x_3 + x_2x_3 + \bar{x}_3\bar{x}_4 + \bar{x}_2x_5$$

Cost = 5 AND gates + 1 OR gate + (5*2 inputs per AND gate + 5 inputs to one OR gate) = 21

The K-map for the function inverse is given in Table 3

$$\bar{f} = x_1x_2x_3 + \bar{x}_3x_4\bar{x}_5 + x_1x_2x_4$$
$$\implies f = (\bar{x}_1 + \bar{x}_2 + \bar{x}_3)(x_3 + \bar{x}_4 + x_5) \\ (\bar{x}_1 + \bar{x}_2 + \bar{x}_4)$$

Cost = 3 OR gate + 1 AND gate + (3*3 inputs to the OR gates and 3 inputs to the AND gate)=16.

References

[1] S. Brown and Z. Vranesic. *Fundamentals of Digital Logic with Verilog Design: Third Edition.* McGraw-Hill Higher Education, 2013.

		$\bar{x}_1$				x_1			
		$\bar{x}_2$		x_2		$\bar{x}_2$		x_2	
		$\bar{x}_3$	x_3		$\bar{x}_3$	$\bar{x}_3$	x_3		$\bar{x}_3$
$\bar{x}_4$	$\bar{x}_5$	0	4	12	8	16	20	28	24
	x_5	1	5	13	9	17	21	29	25
x_4	x_5	3	7	15	11	19	23	31	27
	$\bar{x}_5$	2	6	14	10	18	22	30	26

		$x_1 = 0/1$			
		$\bar{x}_2$		x_2	
		$\bar{x}_3$	x_3		$\bar{x}_3$
$\bar{x}_4$	$\bar{x}_5$	0/16	4/20	12/28	8/24
	x_5	1/17	5/21	13/29	9/25
x_4	x_5	3/19	7/23	15/31	11/27
	$\bar{x}_5$	2/18	6/22	14/30	10/26

Table 4: K-map for 5-variables with numbered minterms