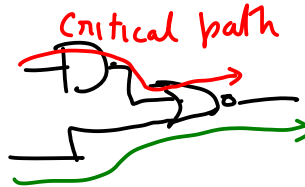


# Logic minimization: Minimum-cost circuits

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## 1 Logic minimization

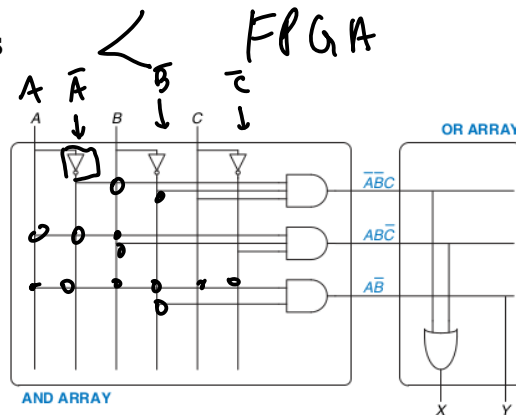
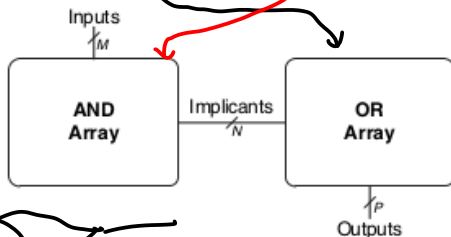
A general optimization criteria for multi-level logic are to Minimize some combination of:

1. Area occupied by the logic gates and interconnect;
2. the Critical Path Delay of the longest path through the logic;
3. the Degree of Testability of the circuit, measured in terms of the percentage of faults covered by a specified set of test vectors, for an appropriate fault model (Eg., single stuck faults, multiple stuck faults, etc.);
4. Power consumed by the logic gates.

In this course, we will start with two-level multi-input circuits and a criteria based on the number of gates/transistors/diodes.

## 2 Programmable Logic Arrays

Sum of Product form



## 3 Two-level circuits

The cost that we are going to consider in this class depend upon:

1. Number of gates.
2. Number of input to the gates.

More gates need more transistors, more area on the chip. More-inputs the gate need more transistors within each gate. Number of gate inputs can be considered secondary criterion to the number of gates.

**Example 1.** Find the cost of the following Boolean expression  $X = \bar{A}\bar{B}C + \bar{A}B\bar{C} + A\bar{B}\bar{C}$ .

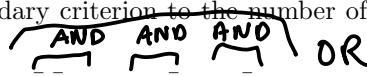
**Problem 1.** Find the cost of the following Boolean expression  $X = \bar{A}\bar{B}C + \bar{A}B\bar{C} + \bar{B}C$ .

3 AND gates + 1 OR gate + 3 input + 3 input + 2 input + 3 inputs to the OR gate = 15



$x + yz$  has the cost  $-2 + 2 + 2 = 6$

AND 2 gates  $z + y \cdot z$  OR gate has 2 inputs AND gate has 2 inputs

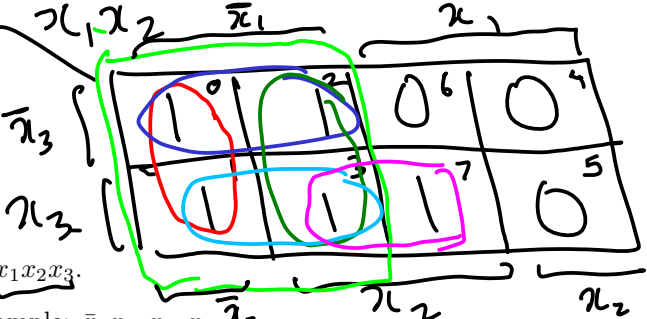


Problem 1

$$A \bar{B} C + \bar{A} B \bar{C} + \bar{B} C$$

$$\begin{aligned} & 3 \text{ AND gates} + 1 \text{ OR gate} + 3 \text{ input} + 3 \text{ input} + 2 \text{ input} \\ & + 3 \text{ inputs to the OR gate} \\ & = 15 \end{aligned}$$

Implicants =  $\bar{x}_1, \bar{x}_2, \bar{x}_3, x_1, x_2, x_3, \bar{x}_1 \cdot \bar{x}_2, \bar{x}_1 \cdot \bar{x}_3, \bar{x}_2 \cdot \bar{x}_3, \bar{x}_1 \cdot x_2, \bar{x}_1 \cdot x_3, \bar{x}_2 \cdot x_3, \bar{x}_1 \cdot x_2 \cdot x_3$



#### 4 Terminology for K-maps

Running Example:  $f = \sum m(0, 1, 2, 3, 7) = \bar{x}_1 + x_1 x_2 x_3$

**Literal** A single variable or its complement. Example:  $\bar{x}_1, x_1, x_2, x_3$

**Implicant** A product term which is true for a function. All minterms are implicants. Example:  $x_1 x_2 x_3, \bar{x}_1, m_0 = \bar{x}_1 \bar{x}_2 \bar{x}_3, \bar{x}_1 x_3, \bar{x}_1 \bar{x}_3$

**Prime Implicant** An implicant that cannot be combined into fewer literals. Example:  $\bar{x}_1, x_2 x_3$

**Essential Prime Implicant** An implicant that cannot be combined into fewer literals. Example:  $x_2 x_3$

**Cover** : List of Prime Implicants that account for all  $f = 1$ .

**Cost** : Number of gates (excluding not gate on literals) and number of inputs to each gate.

**Example 2.** Find minimum cost expression for the function  $f(x_1, x_2, x_3) = \prod M(4, 5, 6)$

truth table

$x_1$	$x_2$	$x_3$	$f$
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	0

Combining: in the k-map if it's can be grouped in the boolean expression

$$f = \sum m(0, 1, 2, 3, 7) = m_0 + m_1 = \bar{x}_1 \cdot \bar{x}_2 \cdot \bar{x}_3 + \bar{x}_1 \cdot \bar{x}_2 \cdot x_3 = \bar{x}_1 \cdot \bar{x}_2 (\bar{x}_3 + x_3) = \bar{x}_1 \cdot \bar{x}_2$$

**Problem 2.** Find minimum cost expression for the function  $f(x_1, x_2, x_3) = \prod M(2, 5, 6)$

#### 4.1 Incompletely specified functions or Don't cares

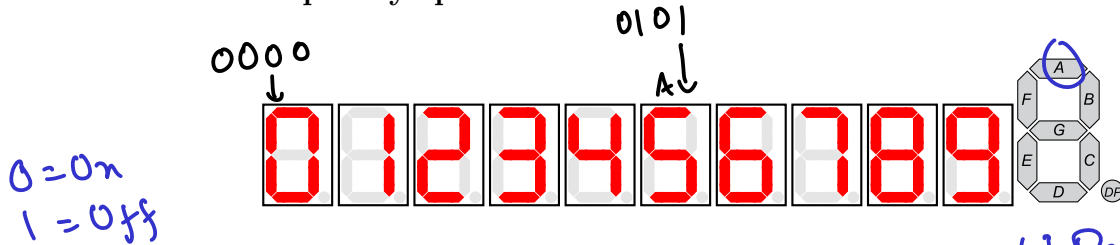
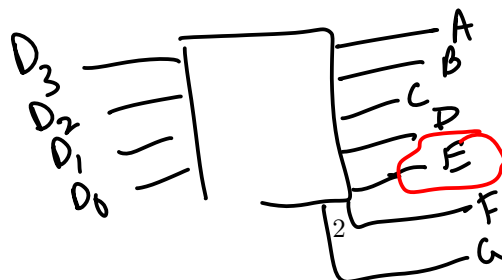


Figure 1: 7 Segment Representations of Each Integer



#	$D_3$	$D_2$	$D_1$	$D_0$	$A$
0	0	0	0	0	0
1	0	0	0	1	1
2	0	0	1	0	2
3	0	0	1	1	3
4	0	1	0	0	4
5	0	1	0	1	5
6	0	1	1	0	6
7	0	1	1	1	7
8	1	0	0	0	8
9	1	0	0	1	9

$$f(x_1, x_2, x_3) = \prod M(4, 5, 6)$$

① Find all  $PI = \{\bar{x}_1, x_2 x_3\}$

② Find all  $EPI = \{\bar{x}_1, x_2 x_3\}$

③ Remove the ones that are covered by the EPI

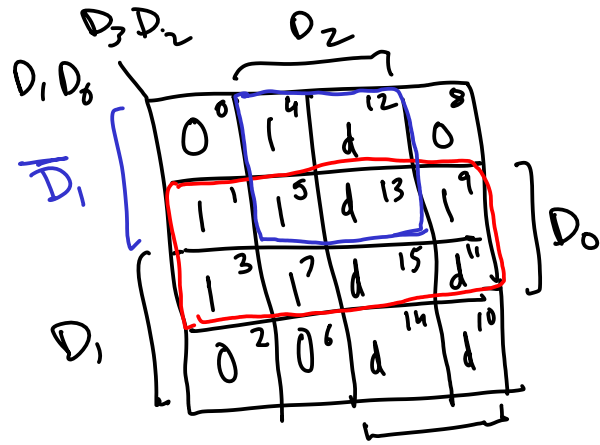
④ If some ones are still remaining then choose a subset of PIs that form a cover

	$x_1$	0	1
$x_2$	0	0	1
1	0	1	0

Take note of these EPI

The list of these EPIs is the minimum cost expression/circuit

	BCD Value				LED Segment
	$D_3$	$D_2$	$D_1$	$D_0$	E
0	0	0	0	0	0
1	0	0	0	1	1
2	0	0	1	0	0
3	0	0	1	1	1
4	0	1	0	0	1
5	0	1	0	1	1
6	0	1	1	0	0
7	0	1	1	1	1
8	1	0	0	0	0
9	1	0	0	1	1
10	1	0	1	0	d
	1	0	1	1	d
	1	1	0	0	d
	1	1	0	1	d
	1	1	1	0	d
	1	1	1	1	d



①  $PI = \{ D_0, D_2 \bar{D}_1 \}$   
 ②  $EPI = \{ D_0, D_2 \bar{D}_1 \}$   
 $f = D_0 + D_2 \bar{D}_1$

Example 3. Find minimum cost expression for the function

Sum of products (SOP)  $f(x_1, \dots, x_4) = \sum m(2, 4, 5, 6, 10) + D(12, 13, 14, 15)$  don't care  
 product of sums (POS)

$x_1, x_2, x_3, x_4$

$m_n$  will mean something different

$x_1$	$x_2$	$x_3$	$x_4$	$f$
0	0	0	0	0

Problem 3. Find minimum cost expression for the function

$f(x_1, \dots, x_4) = \sum m(0, 2, 4, 6, 7, 8, 9, 13) + D(1, 12, 15)$

order of the  $m_2$

$$\textcircled{1} \text{ PI} = \{x_2 \bar{x}_3, x_3 \bar{x}_4, x_2 \bar{x}_4\}$$

$$\textcircled{2} \text{ EPI} = \{x_2 \bar{x}_3, x_3 \bar{x}_4\}$$

$$f = x_2 \bar{x}_3 + x_3 \bar{x}_4$$

$$\text{Cost} = 3 + 6 = 9 \quad \underline{\underline{\text{SOP}}}$$

To find POS

$$\bar{f} =$$

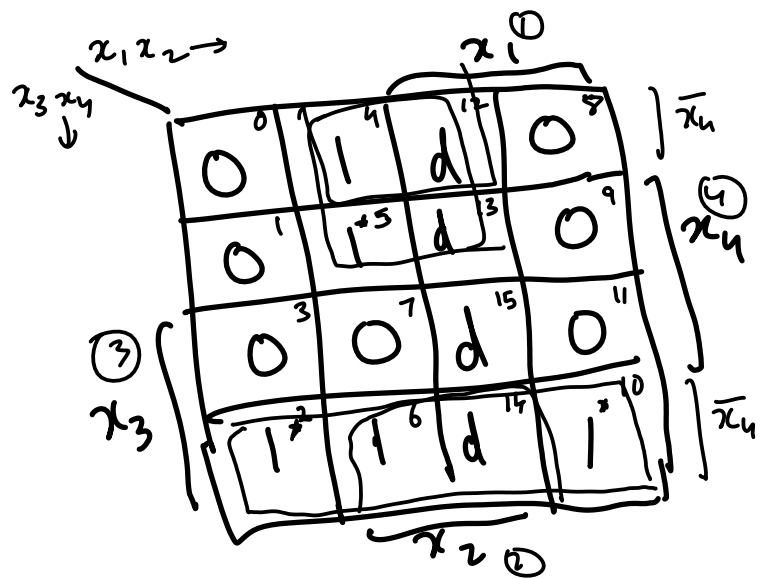
$$\textcircled{1} \text{ PI}$$

$$\textcircled{2} \text{ EPI} = \{x_3 x_4, \bar{x}_2 \bar{x}_3\}$$

$$\bar{f} = x_3 x_4 + \bar{x}_2 \bar{x}_3$$

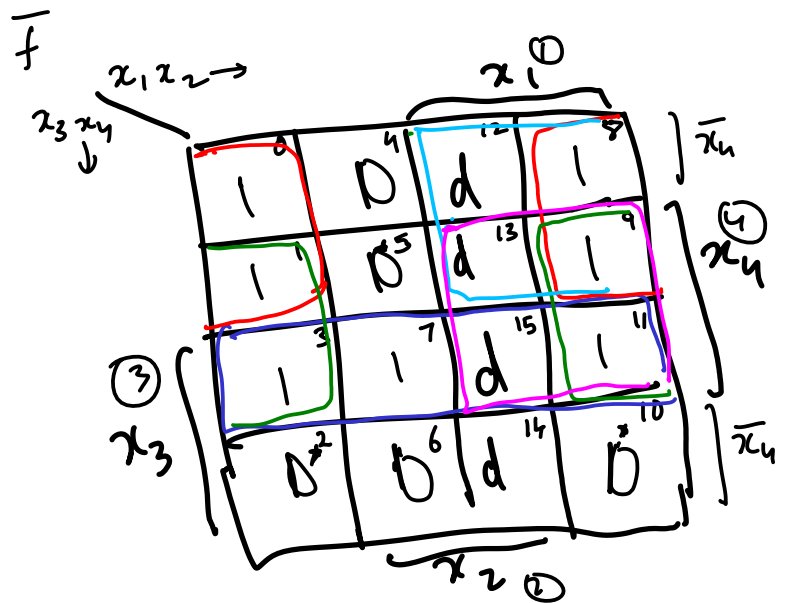
$$f = (\bar{x}_3 + \bar{x}_4) \cdot (x_2 + x_3)$$

$$\text{Cost} = 9$$



4-var Kmap

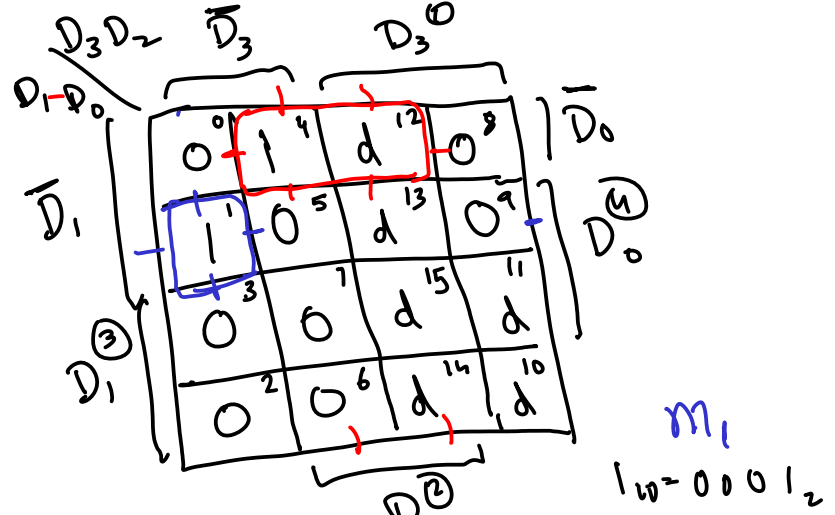
Box size 1 : 4 literals  
 2 : 3 literals  
 4 : 2 literals  
 8 : 1 literal  
 16 : 0 literals



#	$D_3$	$D_2$	$D_1$	$D_0$	A
0	0	0	0	0	0
1	0	0	0	1	0
2	0	0	1	0	0
3	0	0	1	1	0
4	0	1	0	0	0
5	0	1	0	1	0
6	0	1	1	0	0
7	0	1	1	1	0
8	1	0	0	0	0
9	1	0	0	1	0
10	1	0	1	0	0
11	1	0	1	1	0
12	1	1	0	0	0
13	1	1	0	1	0
14	1	1	1	0	0
15	1	1	1	1	0

4-var K map

Grouping of  
 1 = 4 literals  
 2 = 3 literals  
 4 = 2 literals  
 8 = 1 literal  
 16 = 0 literals



$$\textcircled{1} \text{ PI} = \{ \overline{D_2} \overline{D_1} \overline{D_0}, \overline{D_3} \overline{D_2} \overline{D_1} D_0 \} = \text{EPI}$$

$$\textcircled{2} \text{ EPI}$$

$$\textcircled{3} f = D_2 \overline{D_1} \overline{D_0} + \overline{D_3} \overline{D_2} \overline{D_1} D_0$$

Cost = 2 AND gates

+ 1 OR gate

+ 3 inputs to 1<sup>st</sup> and gate

+ 4 " " 2<sup>nd</sup> " "

+ 2 inputs to OR gate

$$= 12$$

What about grouping zeros?

Biggest grouping of zeros

$$PI = \left\{ \bar{D}_1, (\bar{D}_2 \cdot \bar{D}_0), \bar{D}_3, \bar{D}_2 \bar{D}_0 \right\}$$

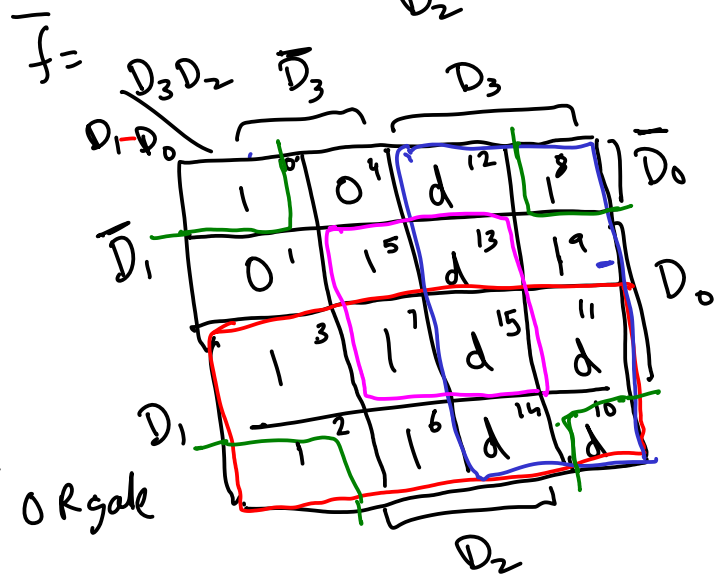
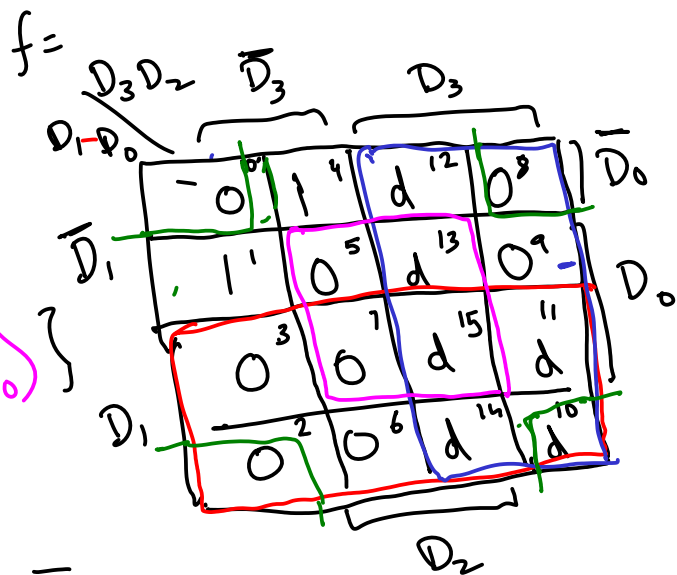
$\downarrow$                        $\downarrow$   
 $(D_2 + D_0)$            $(\bar{D}_2 + \bar{D}_0)$

$$EPI = \left\{ \bar{D}_1, (D_2 + D_0), \bar{D}_3, D_2 + D_0 \right\}$$

$$f = \bar{D}_1 \cdot (D_2 + D_0) \cdot \bar{D}_3 \cdot (D_2 + \bar{D}_0)$$

Cost = 2 OR gates  
 + 1 AND gate  
 + 2 inputs to the 1st OR gate  
 + 2        "  
 + 4

= 11

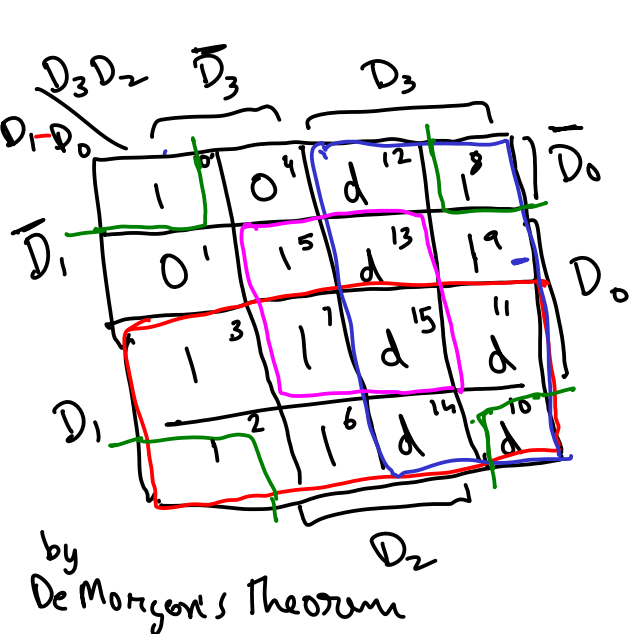


$$PI = \{ D_1, D_3, \bar{D}_2 \bar{D}_0, D_2 D_0 \} = EPI$$

$$\bar{f} = D_1 + D_3 + \bar{D}_2 \bar{D}_0 + D_2 D_0$$

$$f = \overline{D_1 + D_3 + \bar{D}_2 \bar{D}_0 + D_2 D_0}$$

$$= \bar{D}_1 \cdot \bar{D}_3 \cdot (D_2 + D_0) \cdot (\bar{D}_2 + \bar{D}_0)$$



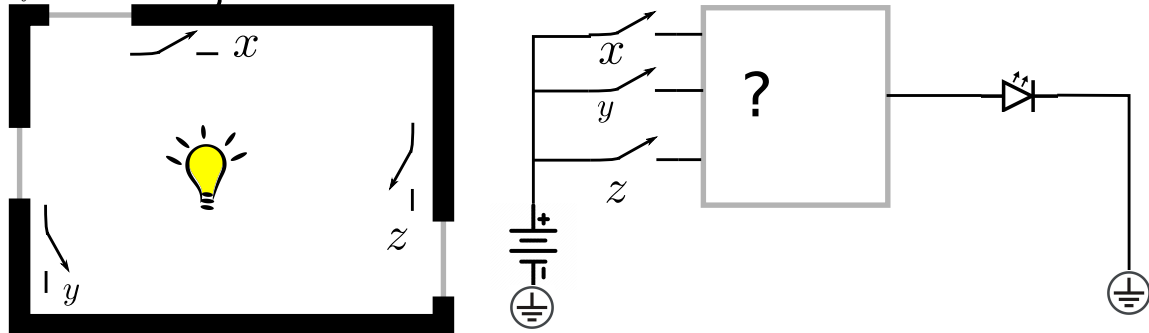


## 5 A few more Boolean problems

**Example 4.** Simplify the following Boolean expression:

$$f = x_1\bar{x}_3\bar{x}_4 + x_2\bar{x}_3\bar{x}_4 + x_1\bar{x}_2\bar{x}_3$$

**Example 5.** (Assume that a large room has three doors and that a switch near each door controls a light in the room. It has to be possible to turn the light on or off by changing the state of any one of the switches.)



**Problem 4.** A simple security system for two doors consists of a card reader and a keypad.

A person may open a particular door if he or she has a card containing the corresponding code and enters an authorized keypad code for that card. Note that card-code and keypad-code are different. The outputs from the card reader are given in the table below.

To unlock a door, a person must hold down the proper keys on the keypad and, then, insert the card in the reader. The authorized keypad code for door 1 is 10, and the authorized keypad code for door 2 is 11. If the card has an invalid code or if the wrong keypad code is entered, the alarm will ring (when the card is inserted). If the correct keypad code is entered, the corresponding door will be unlocked when the card is inserted.

Design the logic circuit for this simple security system. Your circuit's inputs will consist of a card code AB, and a keypad code CD. The circuit will have three outputs XYZ (if X is 1, door 1 will be opened; if Y is 1, door 2 will be opened; if Z is 1, the alarm will sound).

Find the minimal cost two-level circuit using K-maps for X, Y, Z. Provide the minimal cost. (It can be either of SOP/POS forms)

	A	B
No card inserted	0	0
Valid card-code for door 1	0	1
Valid card-code for door 2	1	1
Invalid card code	1	0

#	A	B	C	D	O <sub>1</sub>	O <sub>2</sub>
0	0	0	0	0	0	0
1	0	0	1	0	0	0
2	0	0	0	1	0	0
3	0	0	1	1	0	0
4	0	1	0	0	1	0
5	0	1	0	1	1	0
6	0	1	1	0	1	1
7	0	1	1	1	1	1
8	1	0	0	0	0	0
9	1	0	0	1	0	0
10	1	0	1	0	0	0
11	1	0	1	1	0	0
12	1	1	0	0	1	0
13	1	1	0	1	1	0
14	1	1	1	0	1	1
15	1	1	1	1	1	1

$$O_1 = m_4 = \bar{A} B C \bar{D}$$

$$O_{10} = 0110$$

$$O_2 = m_{15} = A B C D$$

A	B	C	D	Z
0	0	0	0	0
0	0	0	1	0
0	0	1	0	0
0	0	1	1	0
0	1	0	0	1
0	1	0	1	1
0	1	1	0	0
0	1	1	1	1
1	0	0	0	1
1	0	0	1	0
1	0	1	0	1
1	0	1	1	0
1	1	0	0	1
1	1	1	0	0
1	1	1	1	1
1	1	1	1	0
1	1	1	1	1
1	1	1	1	0
1	1	1	1	1

No card inserted

← 6<sup>th</sup> row

← 15<sup>th</sup> row

Z	B	A
0	4	12
1	5	13
3	7	15
2	6	14

8

9

11

10

Inputs:  $x, y, z$

Output: light on on iff  $f = 1$

$$f(x, y, z) = ?$$

3-var K-map

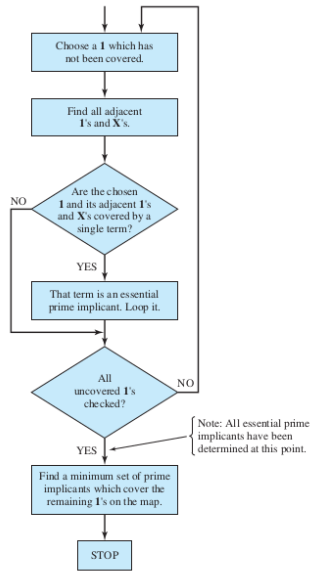
		$x$			
		0	1	0	1
$z$	0	0	1	0	1
	1	1	0	1	0
		$y$			

$x$	$y$	$z$	$f$
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	0
1	1	1	1

$$f = \sum m(1, 2, 4, 7)$$

$$f = \bar{x}\bar{y}z + \bar{x}y\bar{z} + x\bar{y}\bar{z}$$

$$\text{Cost} = 5 \text{ gates} + (12 + 4) \text{ inputs} = 21$$



**Example 6.**

		<i>AB</i>			
		<i>00</i>	<i>01</i>	<i>11</i>	<i>10</i>
<i>CD</i>	<i>00</i>	<i>X</i>	<i>1</i>		<i>1</i>
	<i>01</i>		<i>1</i>	<i>1</i>	<i>1</i>
	<i>11</i>		<i>X</i>	<i>X</i>	
	<i>10</i>		<i>1</i>		<i>1</i>

**Problem 5.** Find the minimum SOP (sum of products) and POS (product of sum) expression for the function  $f(a, b, c, d) = \prod M(5, 7, 13, 14, 15) \cdot \prod D(1, 2, 3, 9)$