Digital circuit design notes

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[1]The notes are from the following books [1, 2]. Not intended for distribution outside the class.

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Chapter 1

Boolean Algebra

1.1 Learning objectives

1. Representing digital circuits

2. Converting between different notations: Boolean expression, logic networks and switching circuits

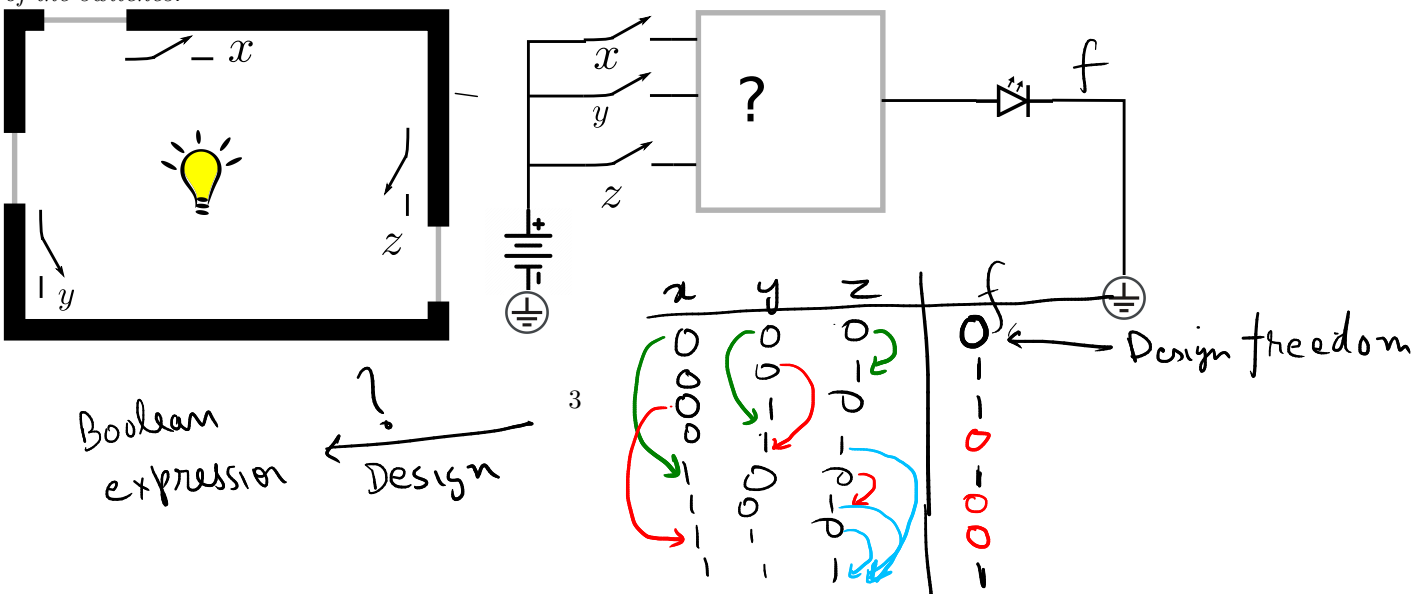
3. Converting between different logic network specifications: truth table, minterm, maxterms, product of sums canonical form and sum of product canonical form.

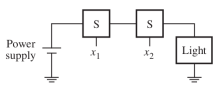
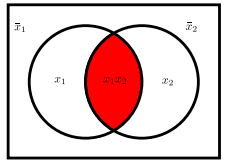
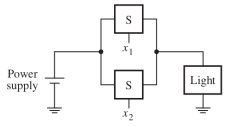
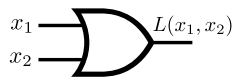
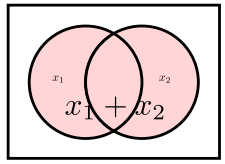
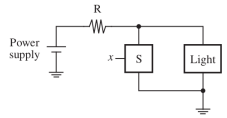
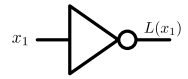
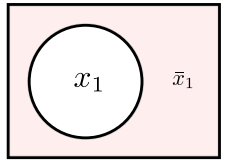
4. Introduce truth tables as Behavioral Verilog

5. This handout has 11 homework problems totaling to 140 marks

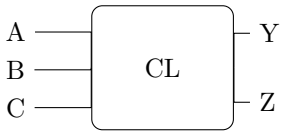
1.2 Motivating Problem

Example 1. *Assume that a large room has three doors and that a switch near each door controls a light in the room. It has to be possible to turn the light on or off by changing the state of any one of the switches.*



Name	C/Verilog	Boolean expr.	Truth Table	Switching circuit	(ANSI) symbol	Venn diagram
AND Gate	`L = x1 & x2`	$L = x_1 \cdot x_2 = x_1 x_2$	x_1 x_2 | $x_1 \cdot x_2$ 0 0 | 0 0 1 | 0 1 0 | 0 1 1 | 1			
OR Gate	`L = x1 | x2`	$L = x_1 + x_2$	x_1 x_2 | $x_1 + x_2$ 0 0 | 0 0 1 | 1 1 0 | 1 1 1 | 1			
NOT Gate	`L = ~ x1`	$L = \bar{x}_1 = x_1'$	x_1 | $\bar{x}_1$ 0 | 1 1 | 0			

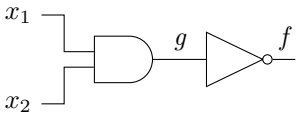
1.3 Digital circuits or networks



$$Y = F(A, B, C) \qquad Z = G(A, B, C)$$

1.4 Two input networks

Example 2. *Convert the following (ANSI) network into a Boolean expression, a truth table and a Venn diagram.*

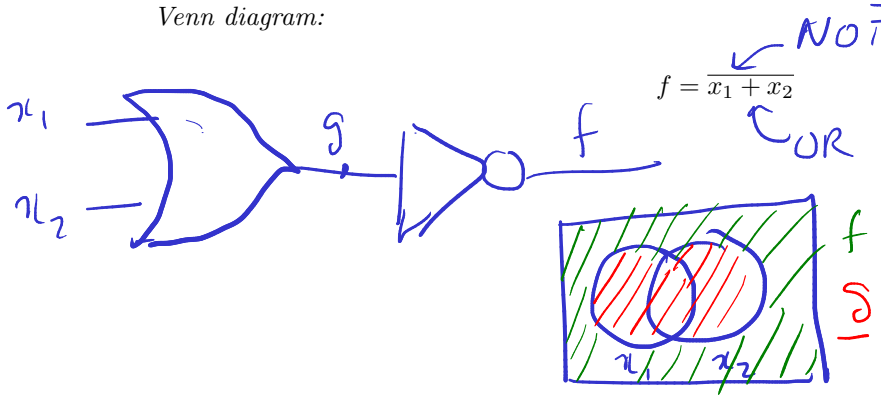


Example 3. *Convert the following Boolean expression into a (ANSI) network, a truth table and a Venn diagram:*

$$f = \overline{x_1 + x_2}$$

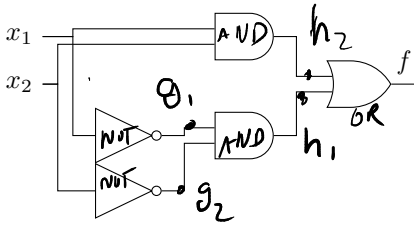
NOT

OR



x_1	x_2	g	$f = \bar{g}$
0	0	0	1
0	1	1	0
1	0	1	0
1	1	1	0

Problem 1 (10 marks)**.** *Convert the following (ANSI) network into a Boolean expression, a truth table and a Venn diagram.*



$g_1 = \bar{x}_1$

$g_2 = \bar{x}_2$

$h_2 = x_1 \cdot x_2$

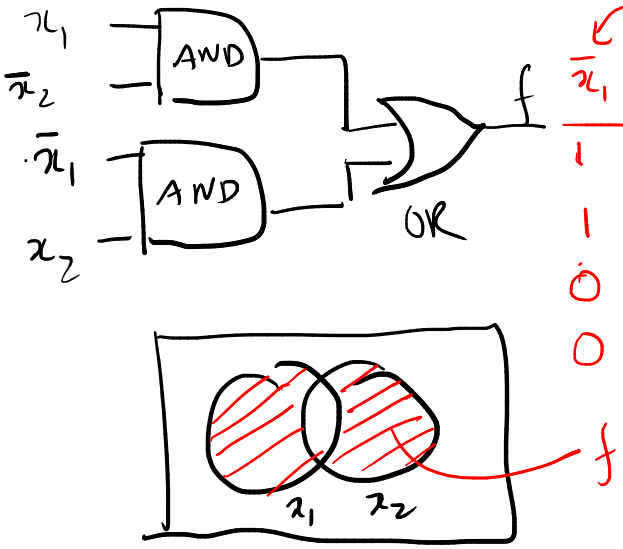
$h_1 = g_1 \cdot g_2$

$f = x_1 x_2 + \bar{x}_1 \cdot \bar{x}_2$

$f = h_1 + h_2 = g_1 \cdot g_2 + x_1 \cdot x_2 = \bar{x}_1 \cdot \bar{x}_2 + x_1 \cdot x_2$

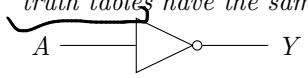
Example 4. *Convert the following Boolean expression into a network, a truth table and a Venn diagram:*

$$f = x_1\bar{x}_2 + \bar{x}_1x_2$$

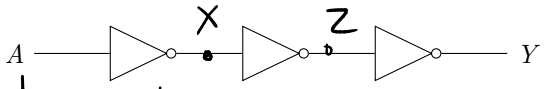


$\bar{x}_1$	$\bar{x}_2$	x_1	x_2	$x_1\bar{x}_2$	$\bar{x}_1x_2$	f
1	1	0	0	0	0	0
1	0	0	1	0	1	1
0	1	1	0	1	0	1
0	0	1	1	0	0	0

Problem 2 (5 marks). *Can two different circuits have the same truth table? Can two different truth tables have the same circuit? Consider the following two circuits for example*



A	Y
0	1
1	0

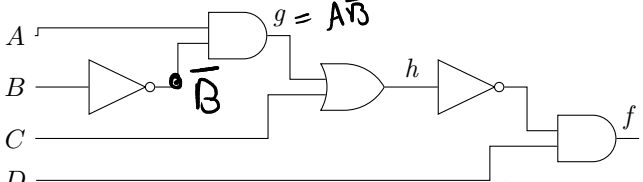


A	X	Z	Y
0	1	0	1
1	0	1	0

Remark 1. *Truth tables and Venn diagrams define* what *the combinational circuit should do. Truth tables define output for every input. Boolean expression and networks define* how *to achieve the desired input output relationship.*

1.5 Multi-input networks

Example 5. *Convert the following (ANSI) network into a Boolean expression and a truth table.*



$h = A\bar{B} + C$

$f = D \cdot (\bar{h})$

$= D \cdot (\overline{A\bar{B} + C})$

A	B	C	D	$\bar{B}$	$A\bar{B}$	$A\bar{B}+C=h$	$\bar{h}$	f
0	0	0	0					
0	0	0	1					
0	0	1	0					
⋮								
1	1	1	1					

A	B	C	D	$\overline{B}$	A $\overline{B}$	A $\overline{B}$ + C = h	$\overline{h}$	f=$\overline{h}$.D
0	0	0	0	1	0	0	1	
0	0	0	1	1	0	0	1	
0	0	1	0	1	0	1	0	
0	0	1	1	1	0	1	0	
0	1	0	0	0	0	0	1	
0	1	0	1	0	0	0	1	
0	1	1	0	0	0	1	0	
0	1	1	1	0	0	1	0	
1	0	0	0	1	1	1	0	
1	0	0	1	1	1	1	0	
1	0	1	0	1	1	1	0	
1	0	1	1	1	1	1	0	
1	1	0	0	0	0	0	1	
1	1	0	1	0	0	0	1	
1	1	1	0	0	0	1	0	
1	1	1	1	0	0	1	0	

Decmal to Binary

Repeated division by 2 and noting all the remainders

$29_{10} = 11101_2$

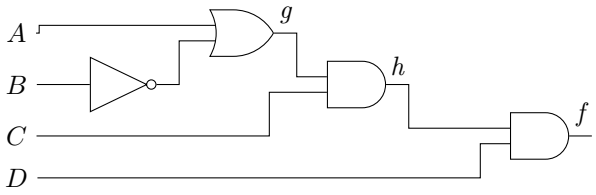
2	29	
2	14	1
2	7	0
2	3	1
2	1	1
	0	1

0	0
1	1
10	2
11	3
100	4
101	5
110	6
111	7
1000	8
1001	9
1010	10
1011	11
1100	12
1101	13
1110	14
1111	15

$+\ \begin{array}{r} 1 \\ 1 \end{array}$

10

Problem 3 (20 marks)**.** *Convert the following (ANSI) network into a Boolean expression and a truth table.*



1.6 Minterms and Maxterms

1.6.1 Minterms

Minterm is a product involving all inputs (or complements) to a function. Every row of a truth table has a corresponding minterm. Minterm is true if and only if the corresponding row in the table is active.

Minterms defined as follows for each row of a two input truth table:

A	B	minterm	minterm name
0	0	$\bar{A}\bar{B}$	m_0
0	1	$\bar{A}B$	m_1
1	0	$A\bar{B}$	m_2
1	1	AB	m_3

Consider a two input circuit whose output Y is given by the truth table:

A	B	Y	minterm	minterm name
0	0	0	$\bar{A}\bar{B}$	m_0
0	1	1	$\bar{A}B$	m_1
1	0	0	$A\bar{B}$	m_2
1	1	1	AB	m_3

then $Y = \bar{A}B + AB = m_1 + m_3 = \sum(1, 3)$.

Minterms and Maxterms

↳ ① Are product terms

② Contain all inputs inverted (NOT gate) or not

③ In the truth table of a minterm one and only one row is 1, all other rows are zero

④ They are denoted by small m with an index of the row number that is 1 in the truth table

Two inputs

	A	B	$m_3 = AB$	$m_0 = \bar{A}\bar{B}$	$m_1 = \bar{A}B$	$m_2 = A\bar{B}$
0	0	0	0	1	0	0
1	0	1	0	0	1	0
2	1	0	0	0	0	1
3	1	1	1	0	0	0

A, B

$m_2 = ?$

Binary → 10

$m_2 = A\bar{B}$

A, B

m_0

Binary → 0 0

$m_0 = \bar{A}\bar{B}$

$m_3 = ?$

Binary → 1 1

$m_3 = AB$

Most significant bit (MSB) x_3 x_2 x_1 x_0 Least significant bit (LSB)

$m_{11} = ?$ for 4 inputs x_3, x_2, x_1, x_0

Binary → 1011_2

$m_{11} = x_3 \bar{x}_2 x_1 x_0$

$m_{13} = ?$ for 4 inputs x_3 x_2 x_1 x_0

A, B, C, D

MSB ↑ LSB

$13_{10} = 1101_2$

$m_{13} = x_3 x_2 \bar{x}_1 x_0$ ✓

$m_{13} = A B \bar{C} D$ ✓

Why minterms?

	A	B	$m_3 = AB$	$m_0 = \bar{A}\bar{B}$	$m_1 = \bar{A}B$	$m_2 = A\bar{B}$	f
0	0	0	0	1	0	0	1
1	0	1	0	0	1	0	0
2	1	0	0	0	0	1	1
3	1	1	1	0	0	0	0

OR

Given

$f = m_0 + m_2$

$= \bar{A}\bar{B} + A\bar{B}$

Minterm notation

	x	y	z	f
0	0	0	0	0
1	0	0	1	1
2	0	1	0	1
3	0	1	1	0
4	1	0	0	1
5	1	0	1	0
6	1	1	0	0
7	1	1	1	1

$f = m_1 + m_2 + m_4 + m_7 = \sum m(1,2,4,7)$

$= \bar{x}\bar{y}z + \bar{x}y\bar{z} + x\bar{y}\bar{z} + xyz$

Sum of products canonical form =/= sum of products form

(Not discussed yet)

Maxterms (Dual of minterm)

	A	B	$M_0 = A+B$	$M_1 = A+\bar{B}$	$M_2 = \bar{A}+B$	$M_3 = \bar{A}+\bar{B}$
0	0	0	0	1	1	1
1	0	1	1	0	1	1
2	1	0	1	1	0	1
3	1	1	1	1	1	0

Maxterm

① Sum of all input inverted or not

② Contains all inputs

③ One and only one row of TT is 0 all others are 1

④ Denote by capital M, subscripted by the row that is 0

Shortcut conversion of Maxterm into boolean expression

$M_2 = ?$

Binary → 10

$\bar{A} + B$

M_{11} for 4 input A, B, C, D

Binary ↓

1011 =

$\bar{A} + B + \bar{C} + \bar{D}$

How to write Boolean expression from TT using Maxterms

	A	B	f
0	0	0	1
1	0	1	0 ✓
2	1	0	1
3	1	1	0 ✓

Maxterm notation

$f = M_1 \cdot M_3 = \prod M(1,3)$

$= (A+\bar{B}) \cdot (\bar{A}+\bar{B})$

Product of sums canonical form

This also gives the *sum of products canonical form*.

Example 6. *What is the minterm m_{13} for a 4-input circuit with inputs x, y, z, w (ordered from MSB to LSB).*

Problem 4 (5 marks)**.** *What is the minterm m_{23} for a 5-input circuit with inputs a, b, c, d, e (ordered from MSB to LSB).*

Example 7. *Convert the following 4-input truth table into sum of minterms and sum of products canonical form.*

minterm name	*A*	*B*	*C*	*D*	*f*
m_0	*0*	*0*	*0*	*0*	*0*
m_1	*0*	*0*	*0*	*1*	*1*
m_2	*0*	*0*	*1*	*0*	*0*
m_3	*0*	*0*	*1*	*1*	*0*
m_4	*0*	*1*	*0*	*0*	*0*
m_5	*0*	*1*	*0*	*1*	*1*
m_6	*0*	*1*	*1*	*0*	*0*
m_7	*0*	*1*	*1*	*1*	*0*
m_8	*1*	*0*	*0*	*0*	*0*
m_9	*1*	*0*	*0*	*1*	*0*
m_{10}	*1*	*0*	*1*	*0*	*0*
m_{11}	*1*	*0*	*1*	*1*	*0*
m_{12}	*1*	*1*	*0*	*0*	*0*
m_{13}	*1*	*1*	*0*	*1*	*1*
m_{14}	*1*	*1*	*1*	*0*	*0*
m_{15}	*1*	*1*	*1*	*1*	*0*

Problem 5 (10 marks)**.** *Convert the following 4-input truth table into sum of minterms and sum of products canonical form.*

minterm name	*A*	*B*	*C*	*D*	*f*
m_0	*0*	*0*	*0*	*0*	*0*
m_1	*0*	*0*	*0*	*1*	*0*
m_2	*0*	*0*	*1*	*0*	*0*
m_3	*0*	*0*	*1*	*1*	*1*
m_4	*0*	*1*	*0*	*0*	*0*
m_5	*0*	*1*	*0*	*1*	*0*
m_6	*0*	*1*	*1*	*0*	*0*
m_7	*0*	*1*	*1*	*1*	*1*
m_8	*1*	*0*	*0*	*0*	*0*
m_9	*1*	*0*	*0*	*1*	*0*
m_{10}	*1*	*0*	*1*	*0*	*0*
m_{11}	*1*	*0*	*1*	*1*	*1*
m_{12}	*1*	*1*	*0*	*0*	*0*
m_{13}	*1*	*1*	*0*	*1*	*1*
m_{14}	*1*	*1*	*1*	*0*	*1*
m_{15}	*1*	*1*	*1*	*1*	*0*

1.6.2 Maxterms

Maxterm is a sum involving all inputs (or complements) to a function. Every row of a truth table has a corresponding maxterm. Minterm is false if and only if the corresponding row in the table is active.

Maxterms are defined as follows for each row of a two input truth table:

A	B	maxterm	maxterm name
0	0	$A + B$	M_0
0	1	$A + \bar{B}$	M_1
1	0	$\bar{A} + B$	M_2
1	1	$\bar{A} + \bar{B}$	M_3

Consider a two input circuit whose output Y is given by the truth table:

A	B	Y	maxterm	maxterm name
0	0	0	$A + B$	M_0
0	1	1	$A + \bar{B}$	M_1
1	0	0	$\bar{A} + B$	M_2
1	1	1	$\bar{A} + \bar{B}$	M_3

then $Y = (A + B)(\bar{A} + B) = M_0 M_2$.

Writing a functional specification in terms of minterms is also called product of sums canonical form.

Example 8. *Write the maxterm M_{11} for 4-input Boolean function with the ordered inputs A, B, C, D.*

Example 9. *Convert the following 4-input truth table into product of maxterms and product of sums canonical form.*

maxterm name	*A*	*B*	*C*	*D*	*f*
M_0	*0*	*0*	*0*	*0*	*0*
M_1	*0*	*0*	*0*	*1*	*0*
M_2	*0*	*0*	*1*	*0*	*0*
M_3	*0*	*0*	*1*	*1*	*1*
M_4	*0*	*1*	*0*	*0*	*0*
M_5	*0*	*1*	*0*	*1*	*0*
M_6	*0*	*1*	*1*	*0*	*0*
M_7	*0*	*1*	*1*	*1*	*1*
M_8	*1*	*0*	*0*	*0*	*0*
M_9	*1*	*0*	*0*	*1*	*0*
M_{10}	*1*	*0*	*1*	*0*	*0*
M_{11}	*1*	*0*	*1*	*1*	*1*
M_{12}	*1*	*1*	*0*	*0*	*0*
M_{13}	*1*	*1*	*0*	*1*	*1*
M_{14}	*1*	*1*	*1*	*0*	*1*
M_{15}	*1*	*1*	*1*	*1*	*0*

Problem 6 (10 marks)**.** *Convert the following 4-input truth table into product of maxterms and products of sums canonical form.*

maxterm name	*A*	*B*	*C*	*D*	*f*
M_0	*0*	*0*	*0*	*0*	*0*
M_1	*0*	*0*	*0*	*1*	*1*
M_2	*0*	*0*	*1*	*0*	*1*
M_3	*0*	*0*	*1*	*1*	*1*
M_4	*0*	*1*	*0*	*0*	*1*
M_5	*0*	*1*	*0*	*1*	*0*
M_6	*0*	*1*	*1*	*0*	*1*
M_7	*0*	*1*	*1*	*1*	*1*
M_8	*1*	*0*	*0*	*0*	*0*
M_9	*1*	*0*	*0*	*1*	*1*
M_{10}	*1*	*0*	*1*	*0*	*1*
M_{11}	*1*	*0*	*1*	*1*	*1*
M_{12}	*1*	*1*	*0*	*0*	*0*
M_{13}	*1*	*1*	*0*	*1*	*1*
M_{14}	*1*	*1*	*1*	*0*	*1*
M_{15}	*1*	*1*	*1*	*1*	*0*

Example 10. *Write the 3-input truth table for the function* $f = m_2 + m_3 + m_7$.

Problem 7 (10 marks)**.** *Write the 3-input truth table for the function* $f = M_4 M_5 M_7$.

Problem 8 (10 marks)**.** *Write the truth table for the function* $f = \bar{A}B\bar{C} + AB\bar{C}$.

1.7 Karnaugh maps

Two input K-maps

Three input K-maps

Four input K-maps

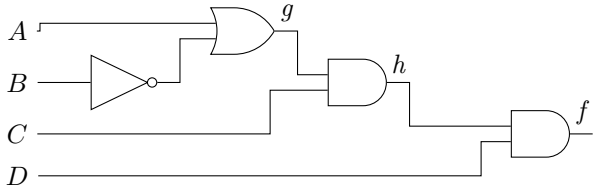
Five input K-maps

1.8 More Gates and notations summary

Name	C/Verilog	Boolean expr.	Truth Table	(ANSI) symbol	K-map
NAND Gate	`Q = ~(x1 & x2)`	$Q = \overline{x_1 \cdot x_2} = \overline{x_1}\,\overline{x_2}$	x_1 x_2 | $\overline{x_1 \cdot x_2}$ 0 0 | 1 0 1 | 1 1 0 | 1 1 1 | 0		
NOR Gate	`Q = ~(x1 | x2)`	$Q = \overline{x_1 + x_2}$	x_1 x_2 | $\overline{x_1 + x_2}$ 0 0 | 1 0 1 | 0 1 0 | 0 1 1 | 0		
XOR Gate	`Q = x1 ^ x2`	$Q = x_1 \oplus x_2$	x_1 x_2 | $x_1 \oplus x_2$ 0 0 | 0 0 1 | 1 1 0 | 1 1 1 | 0		
XNOR Gate	`Q = ~(x1 ^ x2)`	$Q = \overline{x_1 \oplus x_2}$	x_1 x_2 | $\overline{x_1 \oplus x_2}$ 0 0 | 1 0 1 | 0 1 0 | 0 1 1 | 1		

Example 11. *Convert the following Boolean expression into a K-map.* $f = \overline{A\bar{B} + C}D$

Problem 9. *Convert the following logic circuit into a K-map.*



1.9 Boolean Algebra

1.9.1 Axioms of Boolean algebra

1. $0 \cdot 0 = 0$
2. $1 + 1 = 1$

3. $1 \cdot 1 = 1$
4. $0 + 0 = 0$
5. $0 \cdot 1 = 1 \cdot 0 = 0$
6. $\bar{0} = 1$
7. $\bar{1} = 0$
8. $x = 0$ if $x \neq 1$
9. $x = 1$ if $x \neq 0$

1.9.2 Single variable theorems (Prove by drawing K-maps)

1. $x \cdot 0 = 0$

2. $x + 1 = 1$

3. $x \cdot 1 = x$

4. $x + 0 = x$

5. $x \cdot x = x$

6. $x + x = x$

7. $x \cdot \bar{x} = 0$

8. $x + \bar{x} = 1$

9. $\bar{\bar{x}} = x$

Remark 2 (Duality)**.** *Swap* $+$ *with* $\cdot$ *and 0 with 1 to get another theorem*

1.9.3 Two and three variable properties (Prove by K-maps)

1. Commutative: $x \cdot y = y \cdot x$, $x + y = y + x$

2. Associative: $x \cdot (y \cdot z) = (x \cdot y) \cdot z$, $x + (y + z) = (x + y) + z$

3. Distributive: $x \cdot (y + z) = x \cdot y + x \cdot z$, $x + y \cdot z = (x + y) \cdot (y + z)$

4. Absorption: $x + x \cdot y = x$, $x \cdot (x + y) = x$

5. Combining: $x \cdot y + x \cdot \bar{y}$, $(x + y) \cdot (x + \bar{y}) = x$

6. DeMorgan's theorem: $\overline{x \cdot y} = \bar{x} + \bar{y}$, $\overline{x + y} = \bar{x} \cdot \bar{y}$.

7. Concensus:

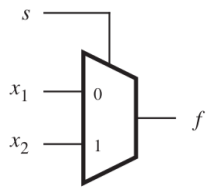
   (a) $x + \bar{x} \cdot y = x + y$

   (b) $x \cdot (\bar{x} + y) = x \cdot y$

   (c) $x \cdot y + y \cdot z + \bar{x} \cdot z = x \cdot y + \bar{x} \cdot z$

   (d) $(x + y) \cdot (y + z) \cdot (\bar{x} + z) = (x + y) \cdot (\bar{x} + z)$

Example 12 (Multiplexer)**.** *Multiplexer is a circuit used to select one of the input lines x_1 and x_2 based only select input s. When $s = 0$, x_1 is selected, x_2 is selected otherwise. Find a boolean expression and a circuit for multiplexer*



s	$f(s, x_1, x_2)$
0	x_1
1	x_2

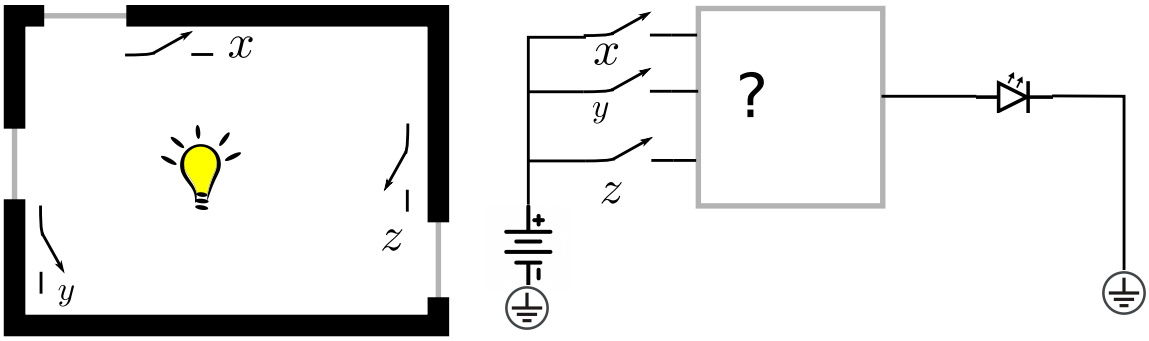
Example 13. *Simplify $f = \bar{A}\bar{B}\bar{C} + A\bar{B}\bar{C} + A\bar{B}\bar{C}$ using boolean algebra.*

Problem 10 (30 marks, Exercise 2.14 [1])**.** *Simplify the following Boolean equations using Boolean theorems.*

$$Y = \bar{A}BC + \bar{A}B\bar{C} \tag{1.1}$$
$$Y = A\bar{B}C + A\bar{B} \tag{1.2}$$
$$Y = ABC\bar{D} + AB\bar{C}D + (A + B \bar{+} C + D) \tag{1.3}$$

Example 14. *Simplify $f = \bar{A}\bar{A}\bar{C} + \bar{A}\bar{B}C$ using K-maps.*

Example 15. *Assume that a large room has three doors and that a switch near each door controls a light in the room. It has to be possible to turn the light on or off by changing the state of any one of the switches.*



Problem 11 (20 marks, Exercise 2.38 [1])**.** *An M-bit thermometer code for the number k consists of k 1's in the least significant bit positions and M – k 0's in all the more significant bit positions. A binary-to-thermometer code converter has N inputs and* 2^{N-1} *outputs. It produces a* 2^N–1 *bit thermometer code for the number specified by the input. For example, if the input is 110, the output should be 0111111. Design a 3:7 binary-to-thermometer code converter. Give a simplified Boolean equation for each output.*

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[1] Sarah L Harris and David Harris. *Digital design and computer architecture*. Morgan Kaufmann, 2022.

[2] Brown Stephen and Vranesic Zvonko. *Fundamentals of digital Logic with Verilog design*. McGraw Hill, 2022.