Digital circuit design notes

Vikas Dhiman for ECE275[1]

February 5, 2025

[1]The notes are from the following books [1, 2]. Not intended for distribution outside the class.

Contents

Chapter 1

Boolean Algebra

1.1 Learning objectives

1. Representing digital circuits

2. Converting between different notations: Boolean expression, logic networks and switching circuits

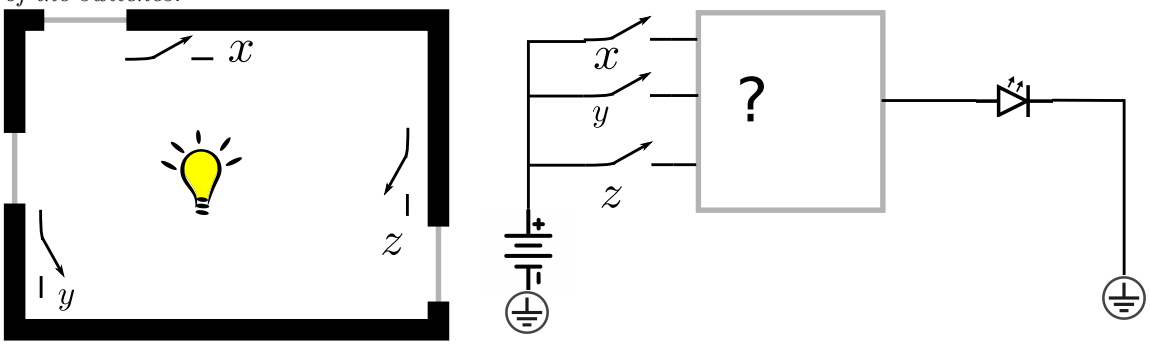
3. Converting between different logic network specifications: truth table, minterm, maxterms, product of sums canonical form and sum of product canonical form.

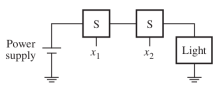
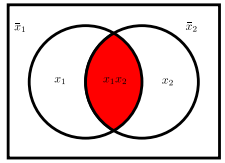
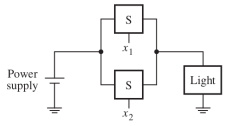
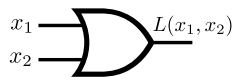
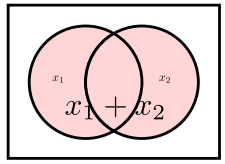
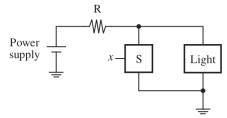
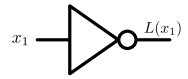
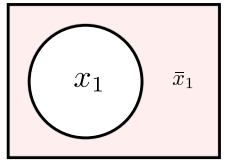
4. Introduce truth tables as Behavioral Verilog

5. This handout has 11 homework problems totaling to 140 marks

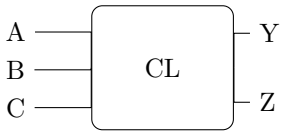
1.2 Motivating Problem

Example 1.1. *Assume that a large room has three doors and that a switch near each door controls a light in the room. It has to be possible to turn the light on or off by changing the state of any one of the switches.*



Name	C/Verilog	Boolean expr.	Truth Table	Switching circuit	(ANSI) symbol	Venn diagram
AND Gate	`L = x1 & x2`	$L = x_1 \cdot x_2 = x_1 x_2$	x_1 x_2 | $x_1 \cdot x_2$ 0 0 | 0 0 1 | 0 1 0 | 0 1 1 | 1			
OR Gate	`L = x1 | x2`	$L = x_1 + x_2$	x_1 x_2 | $x_1 + x_2$ 0 0 | 0 0 1 | 1 1 0 | 1 1 1 | 1			
NOT Gate	`L = ~ x1`	$L = \bar{x}_1 = x_1'$	x_1 | $\bar{x}_1$ 0 | 1 1 | 0			

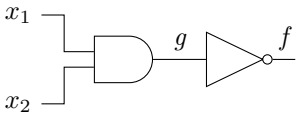
1.3 Digital circuits or networks



$$Y = F(A, B, C) \qquad Z = G(A, B, C)$$

1.4 Two input networks

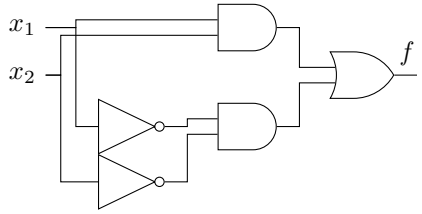
Example 1.2. *Convert the following (ANSI) network into a Boolean expression, a truth table and a Venn diagram.*



Example 1.3. *Convert the following Boolean expression into a (ANSI) network, a truth table and a Venn diagram:*

$$f = \overline{x_1 + x_2}$$

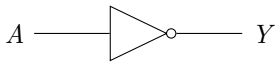
Problem 1.1 (10 marks)**.** *Convert the following (ANSI) network into a Boolean expression, a truth table and a Venn diagram.*

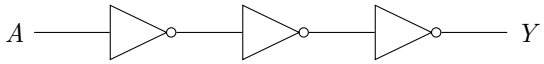


Example 1.4. *Convert the following Boolean expression into a network, a truth table and a Venn diagram:*

$$f = x_1\bar{x}_2 + \bar{x}_1x_2$$

Problem 1.2 (5 marks)**.** *Can two different circuits have the same truth table? Can two different truth tables have the same circuit? Consider the following two circuits for example*

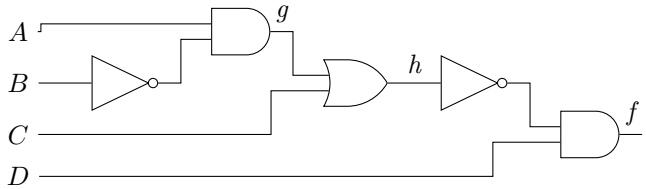




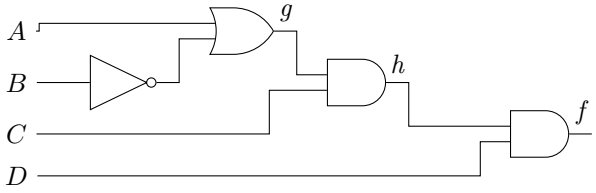
Remark 1.1. *Truth tables and Venn diagrams define* what *the combinational circuit should do. Truth tables define output for every input. Boolean expression and networks define* how *to achieve the desired input output relationship.*

1.5 Multi-input networks

Example 1.5. *Convert the following (ANSI) network into a Boolean expression and a truth table.*



Problem 1.3 (20 marks)**.** *Convert the following (ANSI) network into a Boolean expression and a truth table.*



1.6 Minterms and Maxterms

1.6.1 Minterms

Minterm is a product involving all inputs (or complements) to a function. Every row of a truth table has a corresponding minterm. Minterm is true if and only if the corresponding row in the table is active.

Minterms defined as follows for each row of a two input truth table:

A	B	minterm	minterm name
0	0	$\bar{A}\bar{B}$	m_0
0	1	$\bar{A}B$	m_1
1	0	$A\bar{B}$	m_2
1	1	AB	m_3

Consider a two input circuit whose output Y is given by the truth table:

A	B	Y	minterm	minterm name
0	0	0	$\bar{A}\bar{B}$	m_0
0	1	1	$\bar{A}B$	m_1
1	0	0	$A\bar{B}$	m_2
1	1	1	AB	m_3

then $Y = \bar{A}B + AB = m_1 + m_3 = \sum(1, 3)$.

This also gives the *sum of products canonical form.*

Example 1.6. *What is the minterm m_{13} for a 4-input circuit with inputs x, y, z, w (ordered from MSB to LSB).*

Problem 1.4 (5 marks)**.** *What is the minterm m_{23} for a 5-input circuit with inputs a, b, c, d, e (ordered from MSB to LSB).*

Example 1.7. *Convert the following 4-input truth table into sum of minterms and sum of products canonical form.*

minterm name	*A*	*B*	*C*	*D*	*f*
m_0	*0*	*0*	*0*	*0*	*0*
m_1	*0*	*0*	*0*	*1*	*1*
m_2	*0*	*0*	*1*	*0*	*0*
m_3	*0*	*0*	*1*	*1*	*0*
m_4	*0*	*1*	*0*	*0*	*0*
m_5	*0*	*1*	*0*	*1*	*1*
m_6	*0*	*1*	*1*	*0*	*0*
m_7	*0*	*1*	*1*	*1*	*0*
m_8	*1*	*0*	*0*	*0*	*0*
m_9	*1*	*0*	*0*	*1*	*0*
m_{10}	*1*	*0*	*1*	*0*	*0*
m_{11}	*1*	*0*	*1*	*1*	*0*
m_{12}	*1*	*1*	*0*	*0*	*0*
m_{13}	*1*	*1*	*0*	*1*	*1*
m_{14}	*1*	*1*	*1*	*0*	*0*
m_{15}	*1*	*1*	*1*	*1*	*0*

Problem 1.5 (10 marks)**.** *Convert the following 4-input truth table into sum of minterms and sum of products canonical form.*

minterm name	*A*	*B*	*C*	*D*	*f*
m_0	*0*	*0*	*0*	*0*	*0*
m_1	*0*	*0*	*0*	*1*	*0*
m_2	*0*	*0*	*1*	*0*	*0*
m_3	*0*	*0*	*1*	*1*	*1*
m_4	*0*	*1*	*0*	*0*	*0*
m_5	*0*	*1*	*0*	*1*	*0*
m_6	*0*	*1*	*1*	*0*	*0*
m_7	*0*	*1*	*1*	*1*	*1*
m_8	*1*	*0*	*0*	*0*	*0*
m_9	*1*	*0*	*0*	*1*	*0*
m_{10}	*1*	*0*	*1*	*0*	*0*
m_{11}	*1*	*0*	*1*	*1*	*1*
m_{12}	*1*	*1*	*0*	*0*	*0*
m_{13}	*1*	*1*	*0*	*1*	*1*
m_{14}	*1*	*1*	*1*	*0*	*1*
m_{15}	*1*	*1*	*1*	*1*	*0*

1.6.2 Maxterms

Maxterm is a sum involving all inputs (or complements) to a function. Every row of a truth table has a corresponding maxterm. Minterm is false if and only if the corresponding row in the table is active.

Maxterms are defined as follows for each row of a two input truth table:

A	B	maxterm	maxterm name
0	0	$A + B$	M_0
0	1	$A + \bar{B}$	M_1
1	0	$\bar{A} + B$	M_2
1	1	$\bar{A} + \bar{B}$	M_3

Consider a two input circuit whose output Y is given by the truth table:

A	B	Y	maxterm	maxterm name
0	0	0	$A + B$	M_0
0	1	1	$A + \bar{B}$	M_1
1	0	0	$\bar{A} + B$	M_2
1	1	1	$\bar{A} + \bar{B}$	M_3

then $Y = (A + B)(\bar{A} + B) = M_0 M_2$.
Writing a functional specification in terms of minterms is also called product of sums canonical form.

Example 1.8. *Write the maxterm M_{11} for 4-input Boolean function with the ordered inputs A, B, C, D.*

Example 1.9. *Convert the following 4-input truth table into product of maxterms and product of sums canonical form.*

maxterm name	*A*	*B*	*C*	*D*	*f*
M_0	*0*	*0*	*0*	*0*	*0*
M_1	*0*	*0*	*0*	*1*	*0*
M_2	*0*	*0*	*1*	*0*	*0*
M_3	*0*	*0*	*1*	*1*	*1*
M_4	*0*	*1*	*0*	*0*	*0*
M_5	*0*	*1*	*0*	*1*	*0*
M_6	*0*	*1*	*1*	*0*	*0*
M_7	*0*	*1*	*1*	*1*	*1*
M_8	*1*	*0*	*0*	*0*	*0*
M_9	*1*	*0*	*0*	*1*	*0*
M_{10}	*1*	*0*	*1*	*0*	*0*
M_{11}	*1*	*0*	*1*	*1*	*1*
M_{12}	*1*	*1*	*0*	*0*	*0*
M_{13}	*1*	*1*	*0*	*1*	*1*
M_{14}	*1*	*1*	*1*	*0*	*1*
M_{15}	*1*	*1*	*1*	*1*	*0*

Problem 1.6 (10 marks)**.** *Convert the following 4-input truth table into product of maxterms and products of sums canonical form.*

maxterm name	*A*	*B*	*C*	*D*	*f*
M_0	*0*	*0*	*0*	*0*	*0*
M_1	*0*	*0*	*0*	*1*	*1*
M_2	*0*	*0*	*1*	*0*	*1*
M_3	*0*	*0*	*1*	*1*	*1*
M_4	*0*	*1*	*0*	*0*	*1*
M_5	*0*	*1*	*0*	*1*	*0*
M_6	*0*	*1*	*1*	*0*	*1*
M_7	*0*	*1*	*1*	*1*	*1*
M_8	*1*	*0*	*0*	*0*	*0*
M_9	*1*	*0*	*0*	*1*	*1*
M_{10}	*1*	*0*	*1*	*0*	*1*
M_{11}	*1*	*0*	*1*	*1*	*1*
M_{12}	*1*	*1*	*0*	*0*	*0*
M_{13}	*1*	*1*	*0*	*1*	*1*
M_{14}	*1*	*1*	*1*	*0*	*1*
M_{15}	*1*	*1*	*1*	*1*	*0*

Example 1.10. *Write the 3-input truth table for the function* $f = m_2 + m_3 + m_7$.

Problem 1.7 (10 marks)**.** *Write the 3-input truth table for the function* $f = M_4 M_5 M_7$.

Problem 1.8 (10 marks)**.** *Write the truth table for the function* $f = \bar{A}B\bar{C} + AB\bar{C}$.

1.7 Karnaugh maps

Two input K-maps

Three input K-maps

Four input K-maps

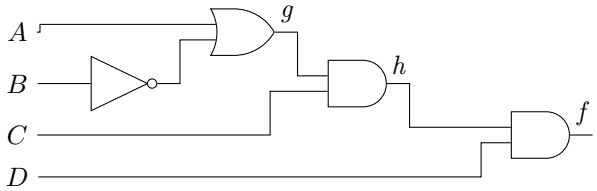
Five input K-maps

1.8 More Gates and notations summary

Name	C/Verilog	Boolean expr.	Truth Table	(ANSI) symbol	K-map
NAND Gate	`Q = ~(x1 & x2)`	$Q = \overline{x_1 \cdot x_2} = \overline{x_1} x_2$	x_1 x_2 | $\overline{x_1 \cdot x_2}$ 0 0 | 1 0 1 | 1 1 0 | 1 1 1 | 0		A \ B: 0 1 0: 1 1 1: 1 0
NOR Gate	`Q = ~(x1 | x2)`	$Q = \overline{x_1 + x_2}$	x_1 x_2 | $\overline{x_1 + x_2}$ 0 0 | 1 0 1 | 0 1 0 | 0 1 1 | 0		A \ B: 0 1 0: 1 0 1: 0 0
XOR Gate	`Q = x1 ^ x2`	$Q = x_1 \oplus x_2$	x_1 x_2 | $x_1 \oplus x_2$ 0 0 | 0 0 1 | 1 1 0 | 1 1 1 | 0		A \ B: 0 1 0: 0 1 1: 1 0
XNOR Gate	`Q = ~(x1 ^ x2)`	$Q = \overline{x_1 \oplus x_2}$	x_1 x_2 | $\overline{x_1 \oplus x_2}$ 0 0 | 1 0 1 | 0 1 0 | 0 1 1 | 1		A \ B: 0 1 0: 1 0 1: 0 1

Example 1.11. *Convert the following Boolean expression into a K-map.* $f = \overline{A\bar{B} + CD}$

Problem 1.9 (10 marks)**.** *Convert the following logic circuit into a K-map.*



1.9 Boolean Algebra

1.9.1 Axioms of Boolean algebra

1. $0 \cdot 0 = 0$
2. $1 + 1 = 1$

3. $1 \cdot 1 = 1$
4. $0 + 0 = 0$
5. $0 \cdot 1 = 1 \cdot 0 = 0$
6. $\bar{0} = 1$
7. $\bar{1} = 0$
8. $x = 0$ if $x \neq 1$
9. $x = 1$ if $x \neq 0$

1.9.2 Single variable theorems (Prove by drawing K-maps)

1. $x \cdot 0 = 0$

2. $x + 1 = 1$

3. $x \cdot 1 = x$

4. $x + 0 = x$

5. $x \cdot x = x$

6. $x + x = x$

7. $x \cdot \bar{x} = 0$

8. $x + \bar{x} = 1$

9. $\bar{\bar{x}} = x$

Remark 1.2 (Duality)**.** *Swap + with · and 0 with 1 to get another theorem*

1.9.3 Two and three variable properties (Prove by K-maps)

1. Commutative: $x \cdot y = y \cdot x$, $x + y = y + x$

2. Associative: $x \cdot (y \cdot z) = (x \cdot y) \cdot z$, $x + (y + z) = (x + y) + z$

3. Distributive: $x \cdot (y + z) = x \cdot y + x \cdot z$, $x + y \cdot z = (x + y) \cdot (y + z)$

4. Absorption: $x + x \cdot y = x$, $x \cdot (x + y) = x$

5. Combining: $x \cdot y + x \cdot \bar{y}$, $(x + y) \cdot (x + \bar{y}) = x$

6. DeMorgan's theorem: $\overline{x \cdot y} = \bar{x} + \bar{y}$, $\overline{x + y} = \bar{x} \cdot \bar{y}$.

7. Concensus:

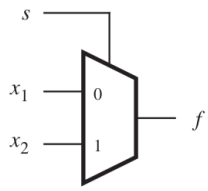
   (a) $x + \bar{x} \cdot y = x + y$

   (b) $x \cdot (\bar{x} + y) = x \cdot y$

   (c) $x \cdot y + y \cdot z + \bar{x} \cdot z = x \cdot y + \bar{x} \cdot z$

   (d) $(x + y) \cdot (y + z) \cdot (\bar{x} + z) = (x + y) \cdot (\bar{x} + z)$

Example 1.12 (Multiplexer)**.** *Multiplexer is a circuit used to select one of the input lines x_1 and x_2 based only select input s. When $s = 0$, x_1 is selected, x_2 is selected otherwise. Find a boolean expression and a circuit for multiplexer*



s	$f(s, x_1, x_2)$
0	x_1
1	x_2

Example 1.13. *Simplify $f = \bar{A}\bar{B}\bar{C} + A\bar{B}\bar{C} + A\bar{B}\bar{C}$ using boolean algebra.*

Problem 1.10 (30 marks, Exercise 2.14 [1])**.** *Simplify the following Boolean equations using Boolean theorems.*

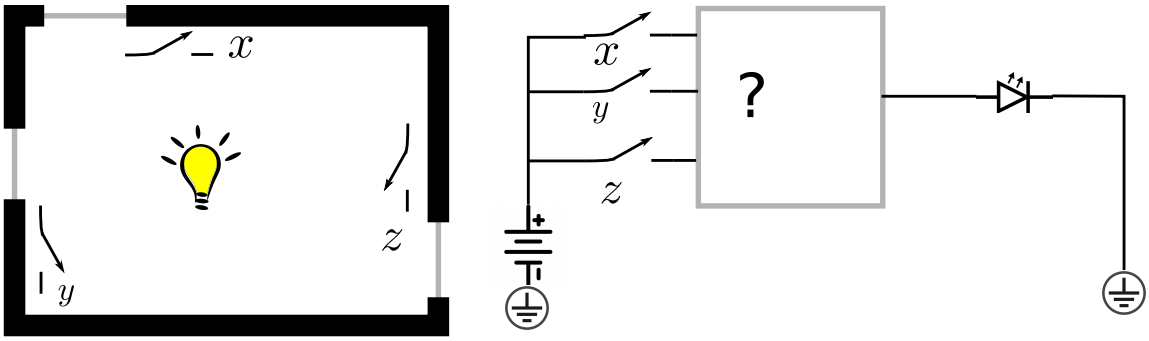
$$Y = \bar{A}BC + \bar{A}B\bar{C} \tag{1.1}$$

$$Y = \overline{ABC} + A\bar{B} \tag{1.2}$$

$$Y = ABC\bar{D} + A\overline{BCD} + (\overline{A + B + C + D}) \tag{1.3}$$

Example 1.14. *Simplify $f = \bar{A}\bar{A}\bar{C} + \bar{A}\bar{B}C$ using K-maps.*

Example 1.15. *Assume that a large room has three doors and that a switch near each door controls a light in the room. It has to be possible to turn the light on or off by changing the state of any one of the switches.*



Problem 1.11 (20 marks, Exercise 2.38 [1]). *An M-bit thermometer code for the number k consists of k 1's in the least significant bit positions and M – k 0's in all the more significant bit positions. A binary-to-thermometer code converter has N inputs and 2^{N-1} outputs. It produces a 2^N–1 bit thermometer code for the number specified by the input. For example, if the input is 110, the output should be 0111111. Design a 3:7 binary-to-thermometer code converter. Give a simplified Boolean equation for each output.*

Chapter 2

Logic minimization

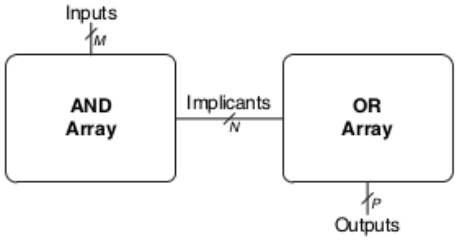
2.1 Logic minimization

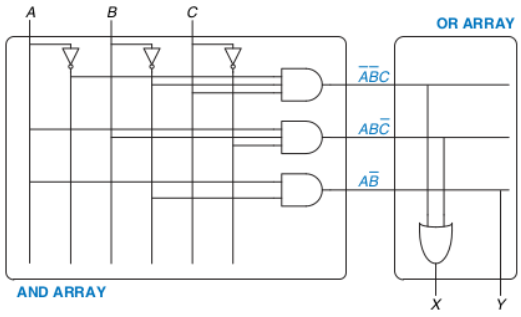
A general optimization criteria for multi-level logic are to Minimize some combination of:

1. Area occupied by the logic gates and interconnect;
2. the Critical Path Delay of the longest path through the logic;
3. the Degree of Testability of the circuit, measured in terms of the percentage of faults covered by a specified set of test vectors, for an appropriate fault model (Eg., single stuck faults, multiple stuck faults, etc.);
4. Power consumed by the logic gates.

In this course, we will start with two-level multi-input circuits and a criteria based on the number of gates/transistors/diodes.

2.2 Programmable Logic Arrays





2.3 Two-level circuits

The cost that we are going to consider in this class depend upon:

1. Number of gates.
2. Number of input to the gates.

More gates need more transistors, more area on the chip. More-inputs the gate need more transistors within each gate. Number of gate inputs can be considered secondary criterion to the number of gates.

Example 2.1. *Find the cost of the following Boolean expression* $X = \bar{A}\bar{B}C + AB\bar{C} + A\bar{B}$.

Problem 2.1 (5 marks)**.** *Find the cost of the following Boolean expression* $X = A\bar{B}C + \bar{A}B\bar{C} + \bar{B}C$.

2.4 Terminology for K-maps

Running Example: $f = \sum m(0, 1, 2, 3, 7) = \bar{x}_1 + x_1x_2x_3$.

Literal A single variable or its complement. Example: $\bar{x}, x_1, x_2, x_3$

Implicant A product term which is true for a function. All minterms are implicants. Example: $x_1x_2x_3$, $\bar{x}_1$, $m_0 = \bar{x}_1\bar{x}_2\bar{x}_3$, $\bar{x}_1x_3$, $\bar{x}_1\bar{x}_3$.

Prime Implicant An implicant that cannot be combined into fewer literals. Example: $\bar{x}_1, x_2x_3$.

Essential Prime Implicant An implicant that cannot be combined into fewer literals. Example: x_2x_3.

Cover : List of Prime Implicants that account for all $f = 1$.

Cost : Number of gates (excluding not gate on literals) and number of inputs to each gate.

Example 2.2. *Find minimum cost expression for the function* $f(x_1, x_2, x_3) = \prod M(4, 5, 6)$

Problem 2.2 (10 marks)**.** *Find minimum cost expression for the function* $f(x_1, x_2, x_3) = \prod M(2, 5, 6)$

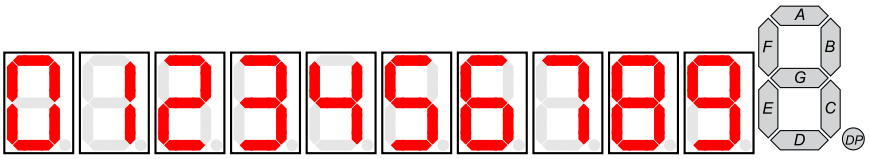


Figure 2.1: 7 Segment Representations of Each Integer

2.4.1 Incompletely specified functions or Don't cares

BCD Value				LED Segment
D_3	D_2	D_1	D_0	E
0	0	0	0	0
0	0	0	1	1
0	0	1	0	0
0	0	1	1	1
0	1	0	0	1
0	1	0	1	1
0	1	1	0	0
0	1	1	1	1
1	0	0	0	0
1	0	0	1	1
1	0	1	0	d
1	0	1	1	d
1	1	0	0	d
1	1	0	1	d
1	1	1	0	d
1	1	1	1	d

Example 2.3. *Find minimum cost expression for the function*

$$f(x_1, \ldots, x_4) = \sum m(2, 4, 5, 6, 10) + D(12, 13, 14, 15)$$

Problem 2.3 (10 marks)**.** *Find minimum cost expression for the function*

$$f(x_1, \ldots, x_4) = \sum m(0, 2, 4, 6, 7, 8, 9, 13) + D(1, 12, 15)$$

2.5 A few more Boolean problems

Example 2.4. *Simplify the following Boolean expression:*

$$f = x_1\bar{x}_3\bar{x}_4 + x_2\bar{x}_3\bar{x}_4 + x_1\bar{x}_2\bar{x}_3$$

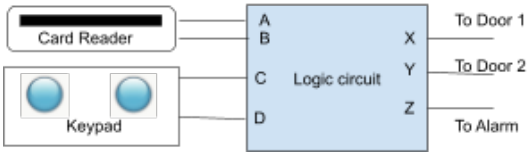
Problem 2.4 (20 marks)**.** *A simple security system for two doors consists of a card reader and a keypad.*

A person may open a particular door if he or she has a card containing the corresponding code and enters an authorized keypad code for that card. Note that card-code and keypad-code are different. The outputs from the card reader are given in the table below.

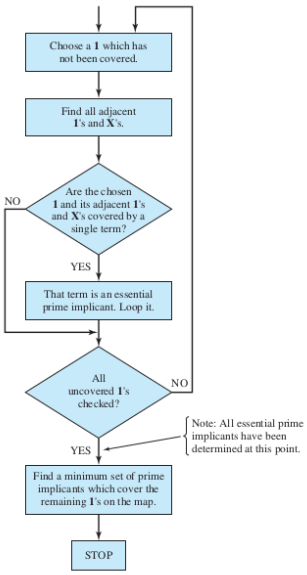
To unlock a door, a person must hold down the proper keys on the keypad and, then, insert the card in the reader. The authorized keypad code for door 1 is 10, and the authorized keypad code for door 2 is 11. If the card has an invalid code or if the wrong keypad code is entered, the alarm will ring when the card is inserted. If the correct keypad code is entered, the corresponding door will be unlocked when the card is inserted.

Design the logic circuit for this simple security system. Your circuit's inputs will consist of a card code AB, and a keypad code CD. The circuit will have three outputs XYZ (if X is 1, door 1 will be opened; if Y is 1, door 2 will be opened; if Z 1, the alarm will sound).

Find the minimal cost two-level circuit using K-maps for X, Y, Z. Provide the minimal cost. (It can be either of SOP/POS forms)



	A	B
No card inserted	0	0
Valid card-code for door 1	0	1
Valid card-code for door 2	1	1
Invalid card code	1	0



Row	x_1	x_2	x_3	f
0	0	0	0	0
1	0	0	1	1
2	0	1	0	1
3	0	1	1	0
4	1	0	0	1
5	1	0	1	0
6	1	1	0	0
7	1	1	1	1

Table 2.1: Truth table for a 3-way light switch

Example 2.5.

CD \ *AB*	*00*	*01*	*11*	*10*
00	*d*	*1*		*1*
01		*1*	*1*	*1*
11		*d*	*d*	
10		*1*		*1*

Problem 2.5 (10 marks)**.** *Find the minimum SOP (sum of products) and POS (product of sum) expression for the function* $f(a,b,c,d) = \prod M(5,7,13,14,15) \cdot \prod D(1,2,3,9)$

Problem 2.6 (10 marks)**.** *If the Sum of Products (SOP) form for* $\bar{f} = AB\bar{C} + \bar{A}\bar{B}$*, then give the Product of Sums (POS) form for* f*.*

Problem 2.7 (10 marks)**.** *Use DeMorgan's Theorem to find* f *if* $\bar{f} = (A + \bar{B}C)D + EF$*.*

Example 2.6. *Implement the function in Table 2.1 using only NAND gates.*

Problem 2.8 (10 marks)**.** *Implement the function in Table 2.1 using only NOR gates.*

Problem 2.9 (10 marks)**.** *Find the minimum-cost Sum of Products (SOP) and Product of Sums (POS) forms for the function* $f(x_1,x_2,x_3) = m(1,3,4,5)$*. Chose the minimum-cost expression by comparing Product of Sums (POS) and Sum of Products (SOP) forms.*

Problem 2.10 (10 marks)**.** *Find the minimum-cost Sum of Products (SOP) and Product of Sums (POS) forms for the function* $f(x_1,x_2,x_3) = \sum m(1,5,7) + D(2,4)$*.*

Problem 2.11 (10 marks)**.** *Find the minimum-cost Sum of Products (SOP) and Product of Sums (POS) forms for the function* $f(x_1,x_2,x_3,x_4) = \prod M(1,2,4,5,7,8,9,10,12,14,15)$*. Chose the minimum-cost expression by comparing Product of Sums (POS) and Sum of Products (SOP) forms.*

Problem 2.12 (10 marks)**.** *Derive a minimum-cost realization of the four-variable function that is equal to 1 if exactly two or exactly three of its variables are equal to 1; otherwise it is equal to 0.*

Bibliography

[1] Sarah L Harris and David Harris. *Digital design and computer architecture*. Morgan Kaufmann, 2022.

[2] Brown Stephen and Vranesic Zvonko. *Fundamentals of digital Logic with Verilog design*. McGraw Hill, 2022.