



# Chapter 1

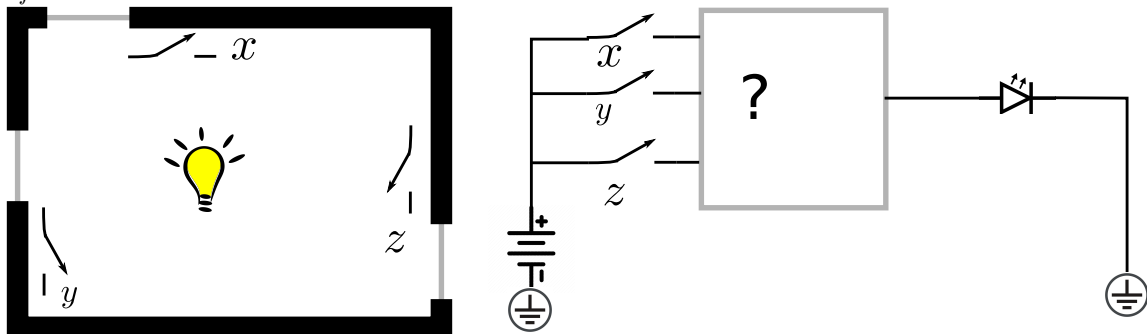
## Boolean Algebra

### 1.1 Learning objectives

1. Representing digital circuits
2. Converting between different notations: Boolean expression, logic networks and switching circuits
3. Converting between different logic network specifications: truth table, minterm, maxterms, product of sums canonical form and sum of product canonical form.
4. Introduce truth tables as Behavioral Verilog
5. This handout has 11 homework problems totaling to 140 marks

### 1.2 Motivating Problem

**Example 1.1.** Assume that a large room has three doors and that a switch near each door controls a light in the room. It has to be possible to turn the light on or off by changing the state of any one of the switches.



Name	C/Verilog	Boolean expr.	Truth Table	Switching circuit	(ANSI) symbol	Venn diagram															
AND Gate	L = x1 & x2	$L = x_1 \cdot x_2 = x_1x_2$	<table><tr><th><math>x_1</math></th><th><math>x_2</math></th><th><math>x_1 \cdot x_2</math></th></tr><tr><td>0</td><td>0</td><td>0</td></tr><tr><td>0</td><td>1</td><td>0</td></tr><tr><td>1</td><td>0</td><td>0</td></tr><tr><td>1</td><td>1</td><td>1</td></tr></table>	$x_1$	$x_2$	$x_1 \cdot x_2$	0	0	0	0	1	0	1	0	0	1	1	1			
$x_1$	$x_2$	$x_1 \cdot x_2$																			
0	0	0																			
0	1	0																			
1	0	0																			
1	1	1																			
OR Gate	L = x1   x2	$L = x_1 + x_2$	<table><tr><th><math>x_1</math></th><th><math>x_2</math></th><th><math>x_1 + x_2</math></th></tr><tr><td>0</td><td>0</td><td>0</td></tr><tr><td>0</td><td>1</td><td>1</td></tr><tr><td>1</td><td>0</td><td>1</td></tr><tr><td>1</td><td>1</td><td>1</td></tr></table>	$x_1$	$x_2$	$x_1 + x_2$	0	0	0	0	1	1	1	0	1	1	1	1			
$x_1$	$x_2$	$x_1 + x_2$																			
0	0	0																			
0	1	1																			
1	0	1																			
1	1	1																			
NOT Gate	L = ~ x1	$L = \bar{x}_1 = x'_1$	<table><tr><th><math>x_1</math></th><th><math>\bar{x}_1</math></th></tr><tr><td>0</td><td>1</td></tr><tr><td>1</td><td>0</td></tr></table>	$x_1$	$\bar{x}_1$	0	1	1	0												
$x_1$	$\bar{x}_1$																				
0	1																				
1	0																				

# Boolean algebra and gates

3 basic gates

AND gate

OR gate

NOT gate

Boolean Algebra  $\left( \begin{matrix} x \in \{0,1\} \\ y \in \{0,1\} \end{matrix} \right)$

$$x \cdot y$$

$$x + y$$

$$\overline{x}$$

$$0 \equiv \text{FALSE}$$

$$1 \equiv \text{TRUE } x$$

$(\text{UMAINE IS A TOP SCHOOL})$   
 $(\text{UMAINE SPORTS IS THE BEST})$   
 $y$

$$x \text{ AND } y = x \cdot y$$

Aside :  $\underbrace{\{0,1\}}_{\text{set}}$   
 $\in \leftarrow$  in / is an element of  
 $x \in \{0,1\}$   
 The variable  $x$  is an element the set  $\{0,1\}$

Truth table

$x$	$y$	$x \cdot y$
0	0	0
0	1	0
1	0	0
1	1	1

Truth Table for OR gate

$x$	$y$	$x + y$
0	0	0
0	1	1
1	0	1
1	1	1

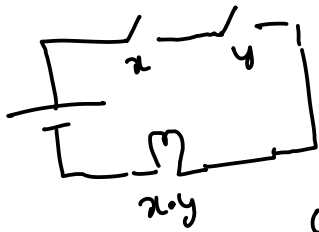
Truth table for NOT gate

$x$	$\overline{x}$
0	1
1	0

NOT (UMAINE IS A TOP SCHOOL)

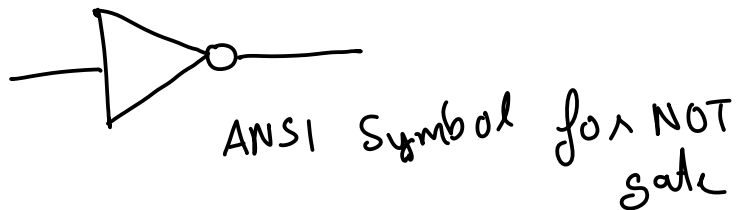
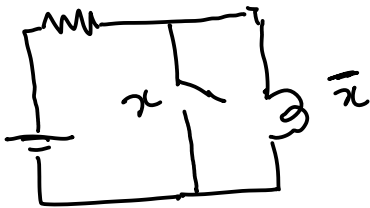
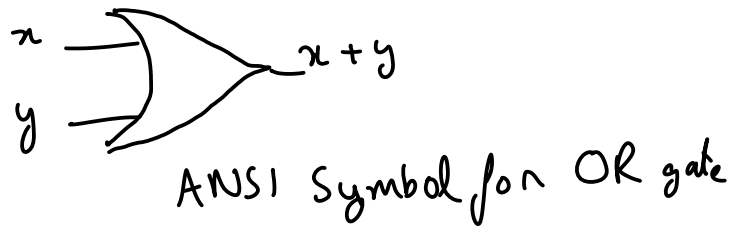
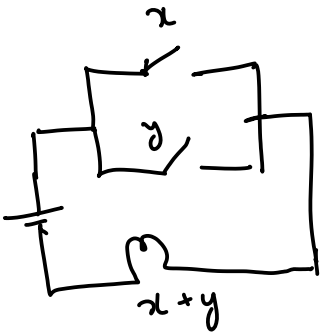
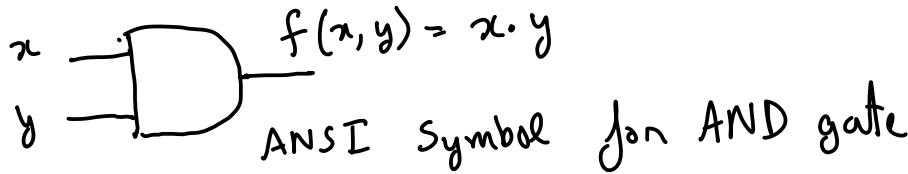
# Digital circuits

## AND/OR/NOT

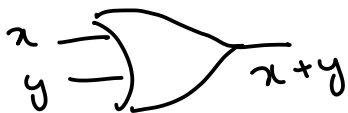


The bulb should turn on only if both switches are on (AND gate). Should we connect the switches in series or parallel?

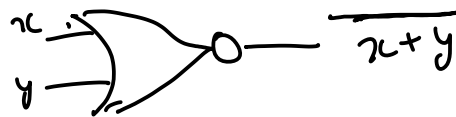
One possible implementation of AND gate



OR



NOR

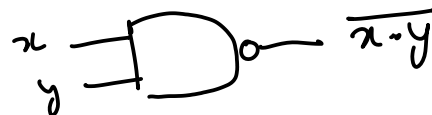


x	y	x+y	x+y-bar
0	0	0	1
0	1	1	0
1	0	1	0
1	1	1	0

AND



NAND



x	y	x.y	x.y-bar
0	0	0	1
0	1	0	1
1	0	0	1
1	1	1	0

C / System Verilog

AND

$x \& y$  bitwise

$x \&\& y$  logical

OR

$x | y$

$x || y$

NOT

$\sim x$   
↑ tilde

$! x$   
↓

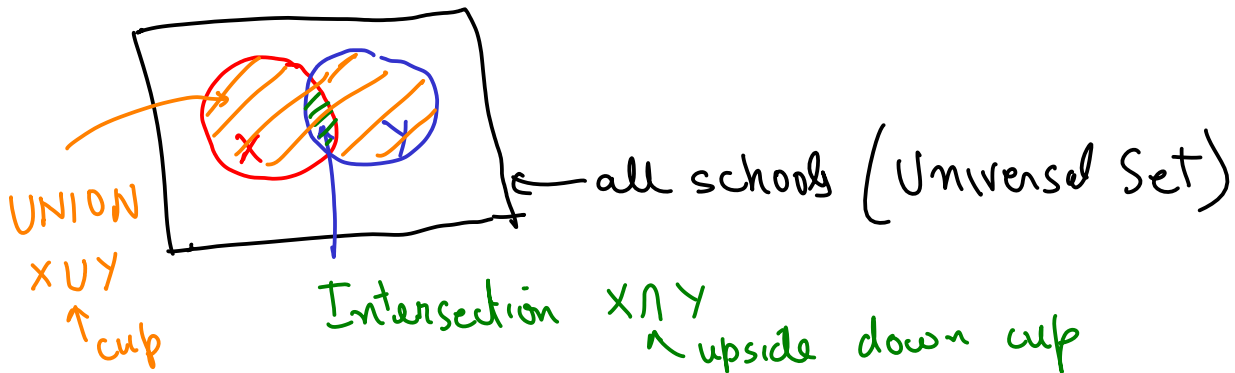
---

Venn Diagram  $\rightarrow$  Karnaugh Maps = (Truth table + Venn Diagram)

$x$  is a top school

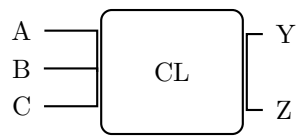
$x \in X$  is a set of all top schools  $\rightarrow$   $x$  is a top school

$y \in Y$  is a set of all top sports teams  $\rightarrow$   $y$  has a top sports team

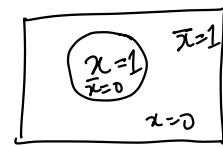


Intersections correspond to AND operations  
Unions correspond to OR operations

### 1.3 Digital circuits or networks

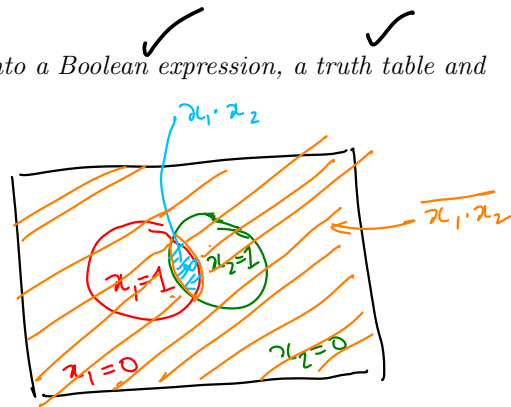
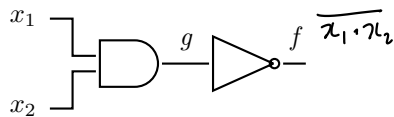


$$Y = F(A, B, C) \quad Z = G(A, B, C)$$



### 1.4 Two input networks

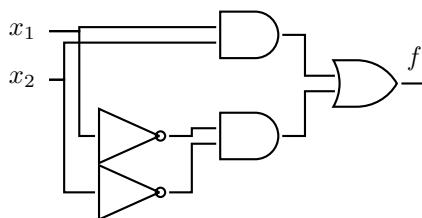
**Example 1.2.** Convert the following (ANSI) network into a Boolean expression, a truth table and a Venn diagram.



**Example 1.3.** Convert the following Boolean expression into a (ANSI) network, a truth table and a Venn diagram:

$$f = \overline{x_1 + x_2}$$

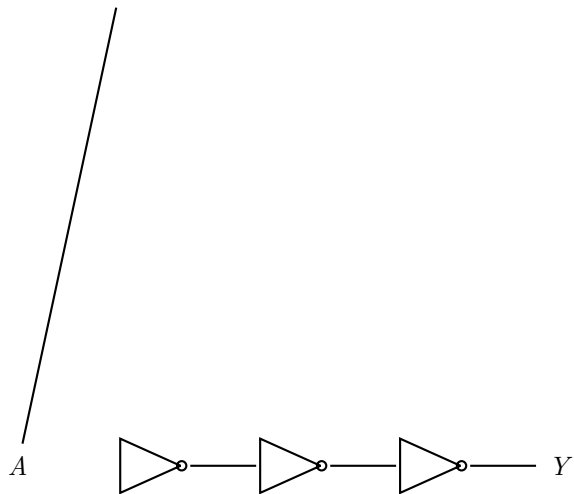
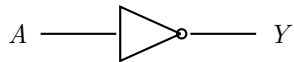
**Problem 1.1** (10 marks). Convert the following (ANSI) network into a Boolean expression, a truth table and a Venn diagram.



**Example 1.4.** Convert the following Boolean expression into a network, a truth table and a Venn diagram:

$$f = x_1\bar{x}_2 + \bar{x}_1x_2$$

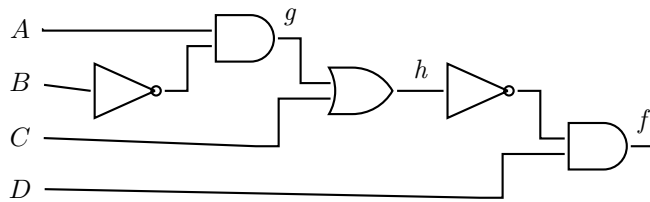
**Problem 1.2** (5 marks). Can two different circuits have the same truth table? Can two different truth tables have the same circuit? Consider the following two circuits for example



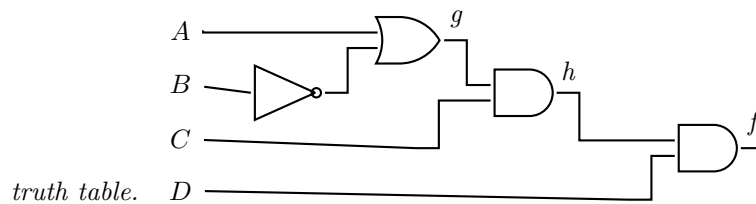
**Remark 1.1.** Truth tables and Venn diagrams define what the combinational circuit should do. Truth tables define output for every input. Boolean expression and networks define how to achieve the desired input output relationship.

## 1.5 Multi-input networks

**Example 1.5.** Convert the following (ANSI) network into a Boolean expression and a truth table.



**Problem 1.3** (20 marks). Convert the following (ANSI) network into a Boolean expression and a





## 1.6 Minterms and Maxterms

### 1.6.1 Minterms

Minterm is a product involving all inputs (or complements) to a function. Every row of a truth table has a corresponding minterm. Minterm is true if and only if the corresponding row in the table is active.

Minterms defined as follows for each row of a two input truth table:

A	B	minterm	minterm name
0	0	$\bar{A}\bar{B}$	$m_0$
0	1	$\bar{A}B$	$m_1$
1	0	$A\bar{B}$	$m_2$
1	1	$AB$	$m_3$

Consider a two input circuit whose output  $Y$  is given by the truth table:

A	B	Y	minterm	minterm name
0	0	0	$\bar{A}\bar{B}$	$m_0$
0	1	1	$\bar{A}B$	$m_1$
1	0	0	$A\bar{B}$	$m_2$
1	1	1	$AB$	$m_3$

then  $Y = \bar{A}B + AB = m_1 + m_3 = \sum(1, 3)$ .

This also gives the *sum of products canonical form*.

**Example 1.6.** What is the minterm  $m_{13}$  for a 4-input circuit with inputs  $x, y, z, w$  (ordered from MSB to LSB).

**Problem 1.4** (5 marks). What is the minterm  $m_{23}$  for a 5-input circuit with inputs  $a, b, c, d, e$  (ordered from MSB to LSB).

**Example 1.7.** Convert the following 4-input truth table into sum of minterms and sum of products canonical form.

<i>minterm name</i>	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>f</i>
$m_0$	0	0	0	0	0
$m_1$	0	0	0	1	1
$m_2$	0	0	1	0	0
$m_3$	0	0	1	1	0
$m_4$	0	1	0	0	0
$m_5$	0	1	0	1	1
$m_6$	0	1	1	0	0
$m_7$	0	1	1	1	0
$m_8$	1	0	0	0	0
$m_9$	1	0	0	1	0
$m_{10}$	1	0	1	0	0
$m_{11}$	1	0	1	1	0
$m_{12}$	1	1	0	0	0
$m_{13}$	1	1	0	1	1
$m_{14}$	1	1	1	0	0
$m_{15}$	1	1	1	1	0

**Problem 1.5** (10 marks). *Convert the following 4-input truth table into sum of minterms and sum of products canonical form.*

<i>minterm name</i>	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>f</i>
$m_0$	0	0	0	0	0
$m_1$	0	0	0	1	0
$m_2$	0	0	1	0	0
$m_3$	0	0	1	1	1
$m_4$	0	1	0	0	0
$m_5$	0	1	0	1	0
$m_6$	0	1	1	0	0
$m_7$	0	1	1	1	1
$m_8$	1	0	0	0	0
$m_9$	1	0	0	1	0
$m_{10}$	1	0	1	0	0
$m_{11}$	1	0	1	1	1
$m_{12}$	1	1	0	0	0
$m_{13}$	1	1	0	1	1
$m_{14}$	1	1	1	0	1
$m_{15}$	1	1	1	1	0

### 1.6.2 Maxterms

Maxterm is a sum involving all inputs (or complements) to a function. Every row of a truth table has a corresponding maxterm. Minterm is false if and only if the corresponding row in the table is active.

Maxterms are defined as follows for each row of a two input truth table:

A	B	maxterm	maxterm name
0	0	$A + B$	$M_0$
0	1	$A + \bar{B}$	$M_1$
1	0	$\bar{A} + B$	$M_2$
1	1	$\bar{A} + \bar{B}$	$M_3$

Consider a two input circuit whose output  $Y$  is given by the truth table:

A	B	Y	maxterm	maxterm name
0	0	0	$A + B$	$M_0$
0	1	1	$A + \bar{B}$	$M_1$
1	0	0	$\bar{A} + B$	$M_2$
1	1	1	$\bar{A} + \bar{B}$	$M_3$

then  $Y = (A + B)(\bar{A} + B) = M_0M_2$ .

Writing a functional specification in terms of minterms is also called product of sums canonical form.

**Example 1.8.** Write the maxterm  $M_{11}$  for 4-input Boolean function with the ordered inputs  $A, B, C, D$ .

**Example 1.9.** Convert the following 4-input truth table into product of maxterms and product of sums canonical form.

maxterm name	A	B	C	D	f
$M_0$	0	0	0	0	0
$M_1$	0	0	0	1	0
$M_2$	0	0	1	0	0
$M_3$	0	0	1	1	1
$M_4$	0	1	0	0	0
$M_5$	0	1	0	1	0
$M_6$	0	1	1	0	0
$M_7$	0	1	1	1	1
$M_8$	1	0	0	0	0
$M_9$	1	0	0	1	0
$M_{10}$	1	0	1	0	0
$M_{11}$	1	0	1	1	1
$M_{12}$	1	1	0	0	0
$M_{13}$	1	1	0	1	1
$M_{14}$	1	1	1	0	1
$M_{15}$	1	1	1	1	0

**Problem 1.6** (10 marks). Convert the following 4-input truth table into product of maxterms and products of sums canonical form.

<i>masterm name</i>	<i>A</i>	<i>B</i>	<i>C</i>	<i>D</i>	<i>f</i>
$M_0$	0	0	0	0	0
$M_1$	0	0	0	1	1
$M_2$	0	0	1	0	1
$M_3$	0	0	1	1	1
$M_4$	0	1	0	0	1
$M_5$	0	1	0	1	0
$M_6$	0	1	1	0	1
$M_7$	0	1	1	1	1
$M_8$	1	0	0	0	0
$M_9$	1	0	0	1	1
$M_{10}$	1	0	1	0	1
$M_{11}$	1	0	1	1	1
$M_{12}$	1	1	0	0	0
$M_{13}$	1	1	0	1	1
$M_{14}$	1	1	1	0	1
$M_{15}$	1	1	1	1	0

**Example 1.10.** Write the 3-input truth table for the function  $f = m_2 + m_3 + m_7$ .

**Problem 1.7** (10 marks). Write the 3-input truth table for the function  $f = M_4M_5M_7$ .

**Problem 1.8** (10 marks). Write the truth table for the function  $f = \bar{A}B\bar{C} + AB\bar{C}$ .

## 1.7 Karnaugh maps


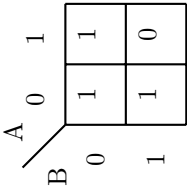

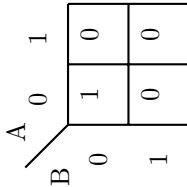

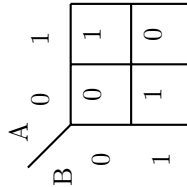
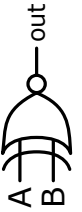
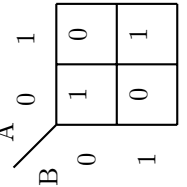
Two input K-maps

Three input K-maps

Four input K-maps

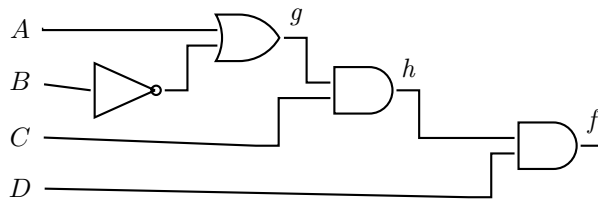
Five input K-maps

## 1.8 More Gates and notations summary

Name	C/Verilog	Boolean expr.	Truth Table	(ANSI) symbol	K-map															
NAND Gate	Q = ~(x1 & x2)	$Q = \overline{x_1 \cdot x_2} = \overline{x_1}x_2 + x_1\overline{x_2}$	<table><tr><th>x<sub>1</sub></th><th>x<sub>2</sub></th><th><math>\overline{x_1 \cdot x_2}</math></th></tr><tr><td>0</td><td>0</td><td>1</td></tr><tr><td>0</td><td>1</td><td>1</td></tr><tr><td>1</td><td>0</td><td>1</td></tr><tr><td>1</td><td>1</td><td>0</td></tr></table>	x <sub>1</sub>	x <sub>2</sub>	$\overline{x_1 \cdot x_2}$	0	0	1	0	1	1	1	0	1	1	1	0		
x <sub>1</sub>	x <sub>2</sub>	$\overline{x_1 \cdot x_2}$																		
0	0	1																		
0	1	1																		
1	0	1																		
1	1	0																		
NOR Gate	Q = ~(x1   x2)	$Q = \overline{x_1 + x_2}$	<table><tr><th>x<sub>1</sub></th><th>x<sub>2</sub></th><th><math>\overline{x_1 + x_2}</math></th></tr><tr><td>0</td><td>0</td><td>1</td></tr><tr><td>0</td><td>1</td><td>0</td></tr><tr><td>1</td><td>0</td><td>0</td></tr><tr><td>1</td><td>1</td><td>0</td></tr></table>	x <sub>1</sub>	x <sub>2</sub>	$\overline{x_1 + x_2}$	0	0	1	0	1	0	1	0	0	1	1	0		
x <sub>1</sub>	x <sub>2</sub>	$\overline{x_1 + x_2}$																		
0	0	1																		
0	1	0																		
1	0	0																		
1	1	0																		
XOR Gate	Q = x1 ^ x2	$Q = x_1 \oplus x_2$	<table><tr><th>x<sub>1</sub></th><th>x<sub>2</sub></th><th><math>x_1 \oplus x_2</math></th></tr><tr><td>0</td><td>0</td><td>0</td></tr><tr><td>0</td><td>1</td><td>1</td></tr><tr><td>1</td><td>0</td><td>1</td></tr><tr><td>1</td><td>1</td><td>0</td></tr></table>	x <sub>1</sub>	x <sub>2</sub>	$x_1 \oplus x_2$	0	0	0	0	1	1	1	0	1	1	1	0		
x <sub>1</sub>	x <sub>2</sub>	$x_1 \oplus x_2$																		
0	0	0																		
0	1	1																		
1	0	1																		
1	1	0																		
XNOR Gate	Q = ~(x1 ^ x2)	$Q = \overline{x_1 \oplus x_2}$	<table><tr><th>x<sub>1</sub></th><th>x<sub>2</sub></th><th><math>\overline{x_1 \oplus x_2}</math></th></tr><tr><td>0</td><td>0</td><td>1</td></tr><tr><td>0</td><td>1</td><td>0</td></tr><tr><td>1</td><td>0</td><td>0</td></tr><tr><td>1</td><td>1</td><td>1</td></tr></table>	x <sub>1</sub>	x <sub>2</sub>	$\overline{x_1 \oplus x_2}$	0	0	1	0	1	0	1	0	0	1	1	1		
x <sub>1</sub>	x <sub>2</sub>	$\overline{x_1 \oplus x_2}$																		
0	0	1																		
0	1	0																		
1	0	0																		
1	1	1																		

**Example 1.11.** Convert the following Boolean expression into a K-map.  $f = \overline{AB} + CD$

**Problem 1.9** (10 marks). Convert the following logic circuit into a K-map.



## 1.9 Boolean Algebra

### 1.9.1 Axioms of Boolean algebra

1.  $0 \cdot 0 = 0$
2.  $1 + 1 = 1$

3.  $1 \cdot 1 = 1$

4.  $0 + 0 = 0$

5.  $0 \cdot 1 = 1 \cdot 0 = 0$

6.  $\bar{0} = 1$

7.  $\bar{1} = 0$

8.  $x = 0$  if  $x \neq 1$

9.  $x = 1$  if  $x \neq 0$

### 1.9.2 Single variable theorems (Prove by drawing K-maps)

1.  $x \cdot 0 = 0$

2.  $x + 1 = 1$

3.  $x \cdot 1 = x$

4.  $x + 0 = x$

5.  $x \cdot x = x$

6.  $x + x = x$

7.  $x \cdot \bar{x} = 0$



8.  $x + \bar{x} = 1$

9.  $\bar{\bar{x}} = x$

**Remark 1.2** (Duality). *Swap  $+$  with  $\cdot$  and  $0$  with  $1$  to get another theorem*

### 1.9.3 Two and three variable properties (Prove by K-maps)

1. Commutative:  $x \cdot y = y \cdot x$ ,  $x + y = y + x$

2. Associative:  $x \cdot (y \cdot z) = (x \cdot y) \cdot z$ ,  $x + (y + z) = (x + y) + z$

3. Distributive:  $x \cdot (y + z) = x \cdot y + x \cdot z$ ,  $x + y \cdot z = (x + y) \cdot (y + z)$

4. Absorption:  $x + x \cdot y = x$ ,  $x \cdot (x + y) = x$

5. Combining:  $x \cdot y + x \cdot \bar{y}, (x + y) \cdot (x + \bar{y}) = x$

6. DeMorgan's theorem:  $\overline{x \cdot y} = \bar{x} + \bar{y}, \overline{x + y} = \bar{x} \cdot \bar{y}.$

7. Concensus:

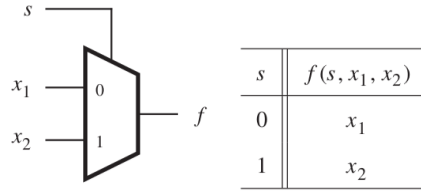
(a)  $x + \bar{x} \cdot y = x + y$

(b)  $x \cdot (\bar{x} + y) = x \cdot y$

(c)  $x \cdot y + y \cdot z + \bar{x} \cdot z = x \cdot y + \bar{x} \cdot z$

(d)  $(x + y) \cdot (y + z) \cdot (\bar{x} + z) = (x + y) \cdot (\bar{x} + z)$

**Example 1.12** (Multiplexer). *Multiplexer is a circuit used to select one of the input lines  $x_1$  and  $x_2$  based only select input  $s$ . When  $s = 0$ ,  $x_1$  is selected,  $x_2$  is selected otherwise. Find a boolean expression and a circuit for multiplexer*



**Example 1.13.** *Simplify  $f = \bar{A}\bar{B}\bar{C} + A\bar{B}\bar{C} + A\bar{B}C$  using boolean algebra.*

**Problem 1.10** (30 marks, Exercise 2.14 [1]). *Simplify the following Boolean equations using Boolean theorems.*

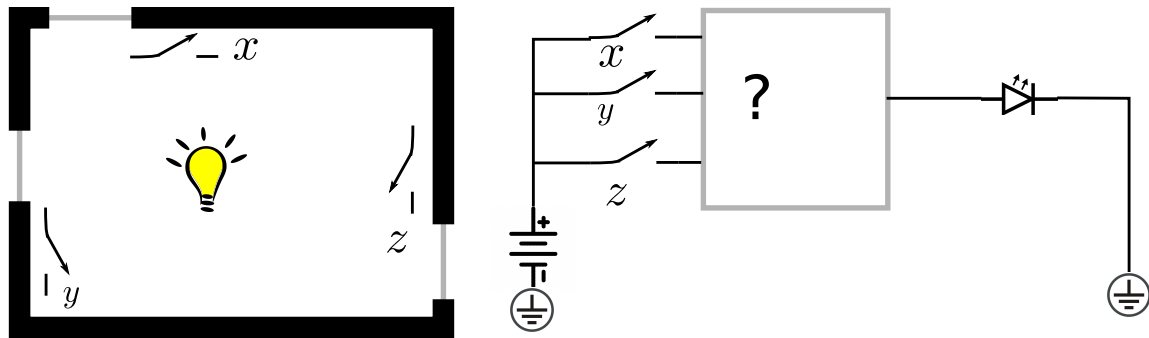
$$Y = \bar{A}BC + \bar{A}B\bar{C} \quad (1.1)$$

$$Y = \overline{ABC} + A\bar{B} \quad (1.2)$$

$$Y = ABC\bar{D} + \overline{ABCD} + \overline{(A + B + C + D)} \quad (1.3)$$

**Example 1.14.** *Simplify  $f = \bar{A}\bar{A}\bar{C} + \bar{A}\bar{B}C$  using K-maps.*

**Example 1.15.** *Assume that a large room has three doors and that a switch near each door controls a light in the room. It has to be possible to turn the light on or off by changing the state of any one of the switches.*



**Problem 1.11** (20 marks, Exercise 2.38 [1]). An  $M$ -bit thermometer code for the number  $k$  consists of  $k$  1's in the least significant bit positions and  $M - k$  0's in all the more significant bit positions. A binary-to-thermometer code converter has  $N$  inputs and  $2^N - 1$  outputs. It produces a  $2^N - 1$  bit thermometer code for the number specified by the input. For example, if the input is 110, the output should be 0111111. Design a 3:7 binary-to-thermometer code converter. Give a simplified Boolean equation for each output.