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# Chapter 1

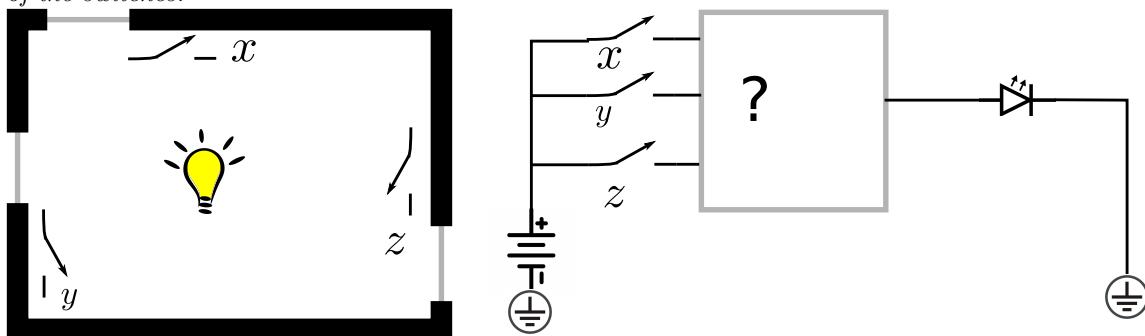
## Boolean Algebra

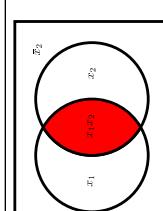
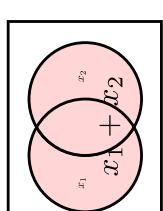
### 1.1 Learning objectives

1. Representing digital circuits
2. Converting between different notations: Boolean expression, logic networks and switching circuits
3. Converting between different logic network specifications: truth table, minterm, maxterms, product of sums canonical form and sum of product canonical form.
4. Introduce truth tables as Behavioral Verilog
5. This handout has 11 homework problems totaling to 140 marks

### 1.2 Motivating Problem

**Example 1.1.** Assume that a large room has three doors and that a switch near each door controls a light in the room. It has to be possible to turn the light on or off by changing the state of any one of the switches.



| Name     | C/Verilog   | Boolean expr.                | Truth Table                                                                                                                                                                                                                                                                                                                       | Switching circuit | (ANSI) symbol | Venn diagram    |   |   |   |                                                                                     |          |                                                                                     |   |   |   |   |   |   |                                                                                   |               |                                                                                   |
|----------|-------------|------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|-------------------|---------------|-----------------|---|---|---|-------------------------------------------------------------------------------------|----------|-------------------------------------------------------------------------------------|---|---|---|---|---|---|-----------------------------------------------------------------------------------|---------------|-----------------------------------------------------------------------------------|
| AND Gate | L = x1 & x2 | $L = x_1 \cdot x_2 = x_1x_2$ | <table border="1"> <thead> <tr> <th><math>x_1</math></th><th><math>x_2</math></th><th><math>x_1 \cdot x_2</math></th></tr> </thead> <tbody> <tr> <td>0</td><td>0</td><td>0</td></tr> <tr> <td>0</td><td>1</td><td>0</td></tr> <tr> <td>1</td><td>0</td><td>0</td></tr> <tr> <td>1</td><td>1</td><td>1</td></tr> </tbody> </table> | $x_1$             | $x_2$         | $x_1 \cdot x_2$ | 0 | 0 | 0 | 0                                                                                   | 1        | 0                                                                                   | 1 | 0 | 0 | 1 | 1 | 1 |  | $L(x_1, x_2)$ |  |
| $x_1$    | $x_2$       | $x_1 \cdot x_2$              |                                                                                                                                                                                                                                                                                                                                   |                   |               |                 |   |   |   |                                                                                     |          |                                                                                     |   |   |   |   |   |   |                                                                                   |               |                                                                                   |
| 0        | 0           | 0                            |                                                                                                                                                                                                                                                                                                                                   |                   |               |                 |   |   |   |                                                                                     |          |                                                                                     |   |   |   |   |   |   |                                                                                   |               |                                                                                   |
| 0        | 1           | 0                            |                                                                                                                                                                                                                                                                                                                                   |                   |               |                 |   |   |   |                                                                                     |          |                                                                                     |   |   |   |   |   |   |                                                                                   |               |                                                                                   |
| 1        | 0           | 0                            |                                                                                                                                                                                                                                                                                                                                   |                   |               |                 |   |   |   |                                                                                     |          |                                                                                     |   |   |   |   |   |   |                                                                                   |               |                                                                                   |
| 1        | 1           | 1                            |                                                                                                                                                                                                                                                                                                                                   |                   |               |                 |   |   |   |                                                                                     |          |                                                                                     |   |   |   |   |   |   |                                                                                   |               |                                                                                   |
| OR Gate  | L = x1   x2 | $L = x_1 + x_2$              | <table border="1"> <thead> <tr> <th><math>x_1</math></th><th><math>x_2</math></th><th><math>x_1 + x_2</math></th></tr> </thead> <tbody> <tr> <td>0</td><td>0</td><td>0</td></tr> <tr> <td>0</td><td>1</td><td>1</td></tr> <tr> <td>1</td><td>0</td><td>1</td></tr> <tr> <td>1</td><td>1</td><td>1</td></tr> </tbody> </table>     | $x_1$             | $x_2$         | $x_1 + x_2$     | 0 | 0 | 0 | 0                                                                                   | 1        | 1                                                                                   | 1 | 0 | 1 | 1 | 1 | 1 |  | $L(x_1, x_2)$ |  |
| $x_1$    | $x_2$       | $x_1 + x_2$                  |                                                                                                                                                                                                                                                                                                                                   |                   |               |                 |   |   |   |                                                                                     |          |                                                                                     |   |   |   |   |   |   |                                                                                   |               |                                                                                   |
| 0        | 0           | 0                            |                                                                                                                                                                                                                                                                                                                                   |                   |               |                 |   |   |   |                                                                                     |          |                                                                                     |   |   |   |   |   |   |                                                                                   |               |                                                                                   |
| 0        | 1           | 1                            |                                                                                                                                                                                                                                                                                                                                   |                   |               |                 |   |   |   |                                                                                     |          |                                                                                     |   |   |   |   |   |   |                                                                                   |               |                                                                                   |
| 1        | 0           | 1                            |                                                                                                                                                                                                                                                                                                                                   |                   |               |                 |   |   |   |                                                                                     |          |                                                                                     |   |   |   |   |   |   |                                                                                   |               |                                                                                   |
| 1        | 1           | 1                            |                                                                                                                                                                                                                                                                                                                                   |                   |               |                 |   |   |   |                                                                                     |          |                                                                                     |   |   |   |   |   |   |                                                                                   |               |                                                                                   |
| NOT Gate | L = ~ x1    | $L = \bar{x}_1 = x'_1$       | <table border="1"> <thead> <tr> <th><math>x_1</math></th><th><math>\bar{x}_1</math></th></tr> </thead> <tbody> <tr> <td>0</td><td>1</td></tr> <tr> <td>1</td><td>0</td></tr> </tbody> </table>                                                                                                                                    | $x_1$             | $\bar{x}_1$   | 0               | 1 | 1 | 0 |  | $L(x_1)$ |  |   |   |   |   |   |   |                                                                                   |               |                                                                                   |
| $x_1$    | $\bar{x}_1$ |                              |                                                                                                                                                                                                                                                                                                                                   |                   |               |                 |   |   |   |                                                                                     |          |                                                                                     |   |   |   |   |   |   |                                                                                   |               |                                                                                   |
| 0        | 1           |                              |                                                                                                                                                                                                                                                                                                                                   |                   |               |                 |   |   |   |                                                                                     |          |                                                                                     |   |   |   |   |   |   |                                                                                   |               |                                                                                   |
| 1        | 0           |                              |                                                                                                                                                                                                                                                                                                                                   |                   |               |                 |   |   |   |                                                                                     |          |                                                                                     |   |   |   |   |   |   |                                                                                   |               |                                                                                   |

## Boolean algebra and gates

3 basic gates

AND gate

Boolean Algebra ( $x \in \{0,1\}$ )  
 $x \cdot y$

OR gate

$x + y$

NOT gate

$\bar{x}$

$0 \equiv \text{FALSE}$

$1 \equiv \text{TRUE}_x$

(UMAINE IS A TOP SCHOOL)

(UMAINE SPORTS IS THE BEST)

Aside :  $\{0,1\}$   
 set

$\in \leftarrow$  in / is an element of

$x \in \{0,1\}$

The variable  $x$  is an element of the set  $\{0,1\}$

$$x \text{ AND } y = x \cdot y .$$

| Truth table |     |             |
|-------------|-----|-------------|
| $x$         | $y$ | $x \cdot y$ |
| 0           | 0   | 0           |
| 0           | 1   | 0           |
| 1           | 0   | 0           |
| 1           | 1   | 1           |

Truth Table for OR gate

| $x$ | $y$ | $x + y$ |
|-----|-----|---------|
| 0   | 0   | 0       |
| 0   | 1   | 1       |
| 1   | 0   | 1       |
| 1   | 1   | 1       |

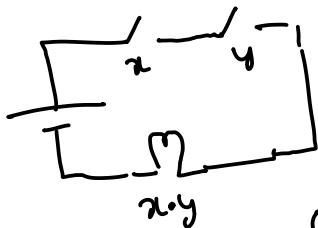
Truth table for NOT gate

| $x$ | $\bar{x}$ |
|-----|-----------|
| 0   | 1         |
| 1   | 0         |

NOT (UMAINE IS A TOP SCHOOL)

# Digital circuits

## AND / OR / NOT

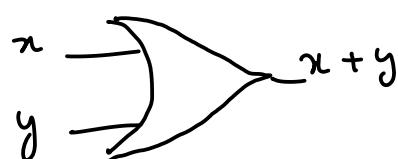
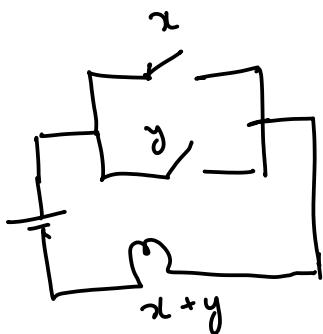


The bulb should turn on only if both switches are on (AND gate). Should we connect the switches in series or parallel?

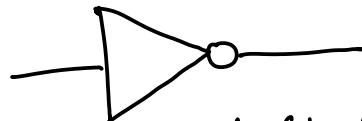
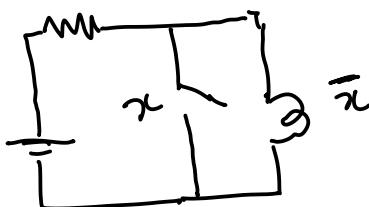
One possible implementation of AND gate



ANSI symbol for AND gate



ANSI symbol for OR gate

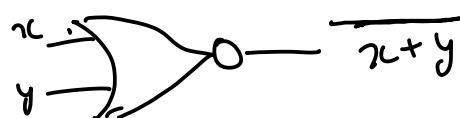


ANSI symbol for NOT gate

OR



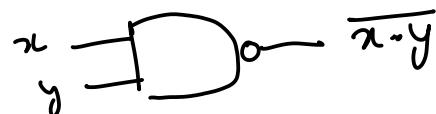
NOR



AND



NAND



| $x$ | $y$ | $x+y$ | $\overline{x+y}$ |
|-----|-----|-------|------------------|
| 0   | 0   | 0     | 1                |
| 0   | 1   | 1     | 0                |
| 1   | 0   | 1     | 0                |
| 1   | 1   | 1     | 0                |

| $x$ | $y$ | $x \cdot y$ | $\overline{x \cdot y}$ |
|-----|-----|-------------|------------------------|
| 0   | 0   | 0           | 1                      |
| 0   | 1   | 0           | 1                      |
| 1   | 0   | 0           | 1                      |
| 1   | 1   | 1           | 0                      |

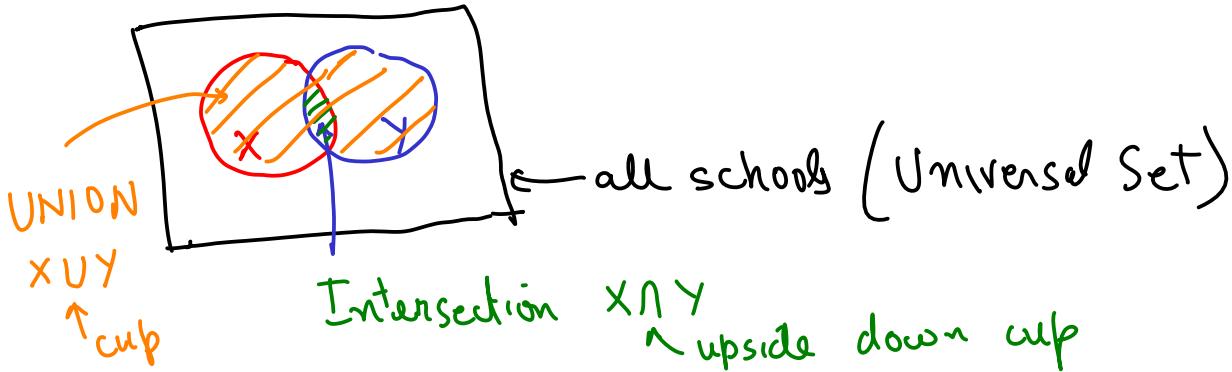
|                      |                   |                                 |
|----------------------|-------------------|---------------------------------|
| $C / System Verilog$ | $x \& y$          | bitwise<br>logical<br>$x \&& y$ |
| AND                  | $x   y$           | $x    y$                        |
| NOT                  | $\sim x$<br>tilde | $\overline{x}$                  |

Venn Diagram  $\rightarrow$  Karnaugh Maps = (Truth table + Venn Diagram)

$x$  is a top school

$x \in X$  is a set of all top schools  $\rightarrow x$  is a top school

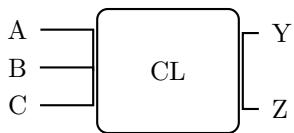
$y \in Y$  is a set of all top sports teams  $\rightarrow y$  has a top sports team



Intersections correspond to AND operations

Unions correspond to OR operations

### 1.3 Digital circuits or networks

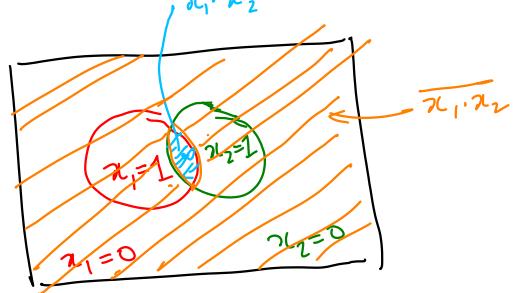
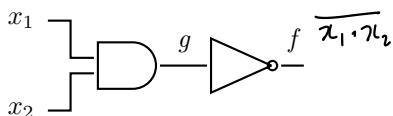


$$Y = F(A, B, C) \quad Z = G(A, B, C)$$

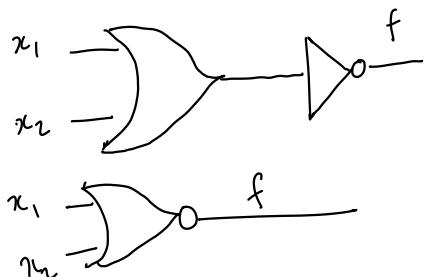


### 1.4 Two input networks

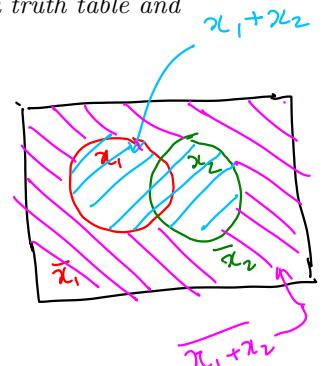
**Example 1.2.** Convert the following (ANSI) network into a Boolean expression, a truth table and a Venn diagram.



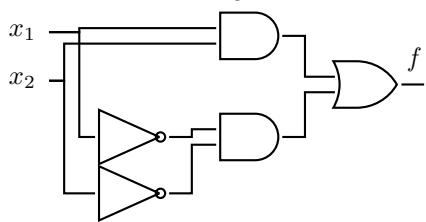
**Example 1.3.** Convert the following Boolean expression into a (ANSI) network, a truth table and a Venn diagram:



| $x_1$ | $x_2$ | $x_1 + x_2$ | $\overline{x_1 + x_2}$ |
|-------|-------|-------------|------------------------|
| 0     | 0     | 0           | 1                      |
| 0     | 1     | 1           | 0                      |
| 1     | 0     | 1           | 0                      |
| 1     | 1     | 1           | 0                      |



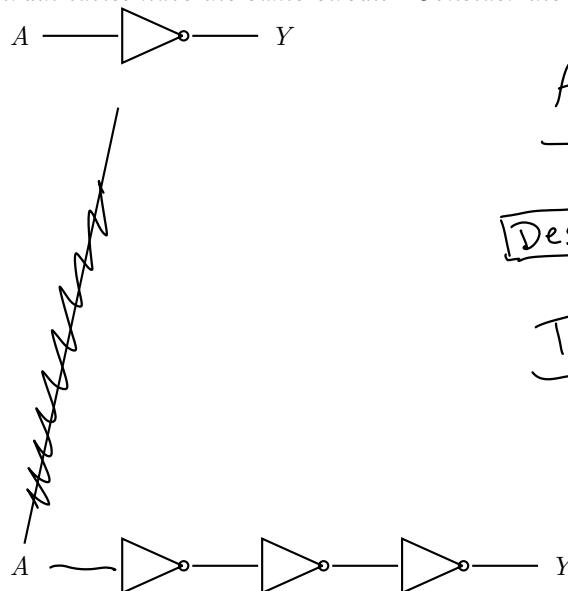
**Problem 1.1** (10 marks). Convert the following (ANSI) network into a Boolean expression, a truth table and a Venn diagram.



**Example 1.4.** Convert the following Boolean expression into a network, a truth table and a Venn diagram:

$$f = x_1 \bar{x}_2 + \bar{x}_1 x_2$$

**Problem 1.2** (5 marks). Can two different circuits have the same truth table? Can two different truth tables have the same circuit? Consider the following two circuits for example

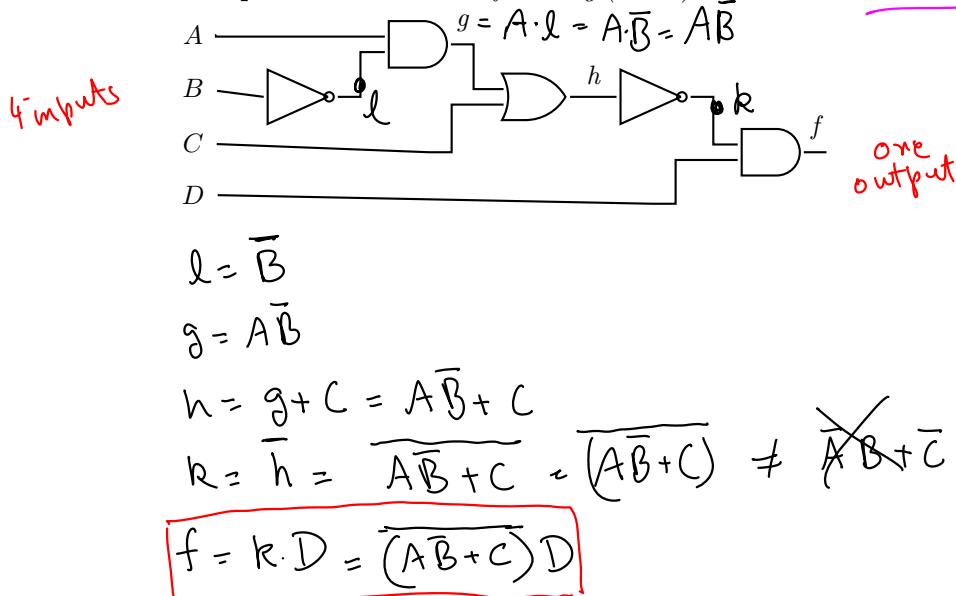


ANSI networks / Boolean expression  
 Design      How      Analysis  
 Truth table / Venn Diagram  
 what  
 Output / Input relationship

**Remark 1.1.** Truth tables and Venn diagrams define what the combinational circuit should do. Truth tables define output for every input. Boolean expression and networks define how to achieve the desired input output relationship.

## 1.5 Multi-input networks

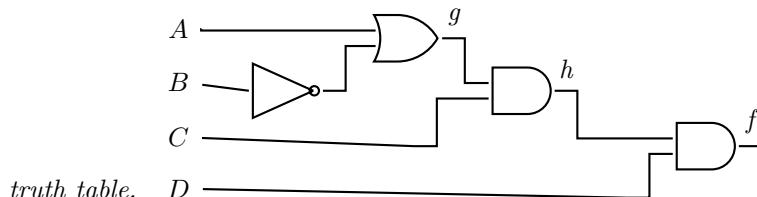
**Example 1.5.** Convert the following (ANSI) network into a Boolean expression and a truth table.



Truth table

| #  | A | B | C | D | $\bar{B}$ | $A \bar{B}$ | $A \bar{B} + C$ | $A \bar{B} + C \bar{D}$ |
|----|---|---|---|---|-----------|-------------|-----------------|-------------------------|
| 0  | 0 | 0 | 0 | 0 | 1         | 0           | 0               | 0                       |
| 1  | 0 | 0 | 0 | 1 | 1         | 0           | 0               | 0                       |
| 2  | 0 | 0 | 1 | 0 | 1         | 0           | 1               | 1                       |
| 3  | 0 | 0 | 1 | 1 | 1         | 0           | 1               | 1                       |
| 4  | 0 | 1 | 0 | 0 | 0         | 0           | 0               | 0                       |
| 5  | 0 | 1 | 0 | 1 | 0         | 0           | 0               | 0                       |
| 6  | 0 | 1 | 1 | 0 | 0         | 0           | 0               | 0                       |
| 7  | 0 | 1 | 1 | 1 | 0         | 0           | 0               | 0                       |
| 8  | 1 | 0 | 0 | 0 | 1         | 1           | 1               | 1                       |
| 9  | 1 | 0 | 0 | 1 | 1         | 1           | 1               | 1                       |
| 10 | 1 | 0 | 1 | 0 | 1         | 0           | 1               | 1                       |
| 11 | 1 | 0 | 1 | 1 | 1         | 0           | 1               | 1                       |
| 12 | 1 | 1 | 0 | 0 | 0         | 0           | 0               | 0                       |
| 13 | 1 | 1 | 0 | 1 | 0         | 0           | 0               | 0                       |
| 14 | 1 | 1 | 1 | 0 | 0         | 0           | 0               | 0                       |
| 15 | 1 | 1 | 1 | 1 | 0         | 0           | 0               | 0                       |

**Problem 1.3** (20 marks). Convert the following (ANSI) network into a Boolean expression and a



Truth table

| #  | A | B | C | D | $\bar{B}$ | $A\bar{B}$ | $A\bar{B}+C$ | $A\bar{B}C$ | $(A\bar{B}+C)\cdot D$ |
|----|---|---|---|---|-----------|------------|--------------|-------------|-----------------------|
| 0  | 0 | 0 | 0 | 0 | 1         | 0          | 0            | 0           | 0                     |
| 1  | 0 | 0 | 0 | 1 | 1         | 0          | 1            | 0           | 0                     |
| 2  | 0 | 0 | 1 | 0 | 1         | 0          | 1            | 0           | 0                     |
| 3  | 0 | 0 | 1 | 1 | 1         | 0          | 1            | 0           | 0                     |
| 4  | 0 | 1 | 0 | 0 | 0         | 0          | 0            | 0           | 0                     |
| 5  | 0 | 1 | 0 | 1 | 0         | 0          | 0            | 0           | 0                     |
| 6  | 0 | 1 | 1 | 0 | 0         | 0          | 1            | 0           | 0                     |
| 7  | 0 | 1 | 1 | 1 | 0         | 0          | 0            | 0           | 0                     |
| 8  | 1 | 0 | 0 | 0 | 1         | 1          | 1            | 0           | 0                     |
| 9  | 1 | 0 | 0 | 1 | 1         | 1          | 1            | 0           | 0                     |
| 10 | 1 | 0 | 1 | 0 | 1         | 0          | 0            | 0           | 0                     |
| 11 | 1 | 0 | 1 | 1 | 0         | 0          | 0            | 0           | 0                     |
| 12 | 1 | 1 | 0 | 0 | 0         | 0          | 0            | 0           | 0                     |
| 13 | 1 | 1 | 0 | 1 | 0         | 0          | 1            | 0           | 0                     |
| 14 | 1 | 1 | 1 | 0 | 0         | 0          | 1            | 0           | 0                     |
| 15 | 1 | 1 | 1 | 1 | 0         | 0          | 1            | 0           | 0                     |

## 1.6 Minterms and Maxterms

### 1.6.1 Minterms

Minterm is a product involving all inputs (or complements) to a function. Every row of a truth table has a corresponding minterm. Minterm is true if and only if the corresponding row in the table is active.

Minterms defined as follows for each row of a two input truth table:

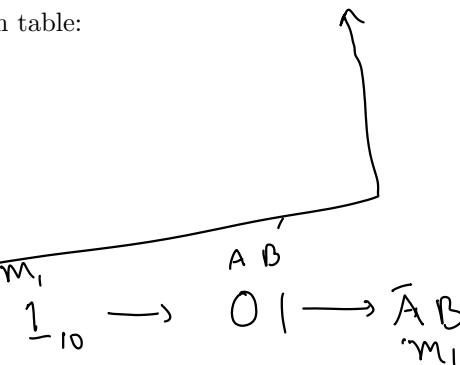
|   | A | B | minterm          | minterm name |
|---|---|---|------------------|--------------|
| 0 | 0 | 0 | $\bar{A}\bar{B}$ | $m_0$        |
| 1 | 0 | 1 | $\bar{A}B$       | $m_1$        |
| 2 | 1 | 0 | $A\bar{B}$       | $m_2$        |
| 3 | 1 | 1 | $AB$             | $m_3$        |

| A | B | $\bar{A}\bar{B} = m_1$ | $m_3$ | $m_1 + m_3$ |
|---|---|------------------------|-------|-------------|
| 0 | 0 | 0                      | 0     | 0           |
| 0 | 1 | 1                      | 0     | 1           |
| 1 | 0 | 0                      | 0     | 0           |
| 1 | 1 | 0                      | 1     | 1           |

Consider a two input circuit whose output  $Y$  is given by the truth table:

| A | B | Y | minterm          | minterm name |
|---|---|---|------------------|--------------|
| 0 | 0 | 0 | $\bar{A}\bar{B}$ | $m_0$        |
| 0 | 1 | 1 | $\bar{A}B$       | $m_1$        |
| 1 | 0 | 0 | $A\bar{B}$       | $m_2$        |
| 1 | 1 | 1 | $AB$             | $m_3$        |

then  $Y = \bar{A}B + AB = m_1 + m_3 = \sum(1, 3)$   
This also gives the sum of products canonical form.



**Example 1.6.** What is the minterm  $m_{13}$  for a 4-input circuit with inputs  $x, y, z, w$  (ordered from MSB to LSB)?

$$(13)_{10} \rightarrow (1101)_2 \leftarrow LSB$$

MSB

$$m_{13} = xy\bar{z}w$$

MSB  
Most Significant Bit

LSB

Least Significant Bit

**Problem 1.4** (5 marks). What is the minterm  $m_{23}$  for a 5-input circuit with inputs  $a, b, c, d, e$  (ordered from MSB to LSB).

$$m_{23} = \bar{a}\bar{b}cde$$

**Example 1.7.** Convert the following 4-input truth table into sum of minterms and sum of products canonical form.

$$\boxed{\bar{A}\bar{B}\bar{C} \neq \bar{A}\bar{B}C}$$

Sum of minterms

$$f = m_1 + m_5 + m_{13} = \sum (1, 5, 13)$$

$$= \sum m (1, 5, 13)$$

Sum of products canonical form

$$f = \overline{A} \cdot \overline{B} \cdot \overline{C} \cdot D + \overline{A} \cdot B \cdot \overline{C} \cdot D + A \cdot B \cdot \overline{C} \cdot D$$

$$5_{10} \rightarrow \begin{matrix} 0 & 1 & 0 & 1 \\ AB\bar{C}D \end{matrix}$$

| minterm name | A | B | C | D | f |
|--------------|---|---|---|---|---|
| $m_0$        | 0 | 0 | 0 | 0 | 0 |
| $m_1$        | 0 | 0 | 0 | 1 | 1 |
| $m_2$        | 0 | 0 | 1 | 0 | 0 |
| $m_3$        | 0 | 0 | 1 | 1 | 0 |
| $m_4$        | 0 | 1 | 0 | 0 | 0 |
| $m_5$        | 0 | 1 | 0 | 1 | 1 |
| $m_6$        | 0 | 1 | 1 | 0 | 0 |
| $m_7$        | 0 | 1 | 1 | 1 | 0 |
| $m_8$        | 1 | 0 | 0 | 0 | 0 |
| $m_9$        | 1 | 0 | 0 | 1 | 0 |
| $m_{10}$     | 1 | 0 | 1 | 0 | 0 |
| $m_{11}$     | 1 | 0 | 1 | 1 | 0 |
| $m_{12}$     | 1 | 1 | 0 | 0 | 0 |
| $m_{13}$     | 1 | 1 | 0 | 1 | 1 |
| $m_{14}$     | 1 | 1 | 1 | 0 | 0 |
| $m_{15}$     | 1 | 1 | 1 | 1 | 0 |

**Problem 1.5** (10 marks). Convert the following 4-input truth table into sum of minterms and sum of products canonical form.

| minterm name | A | B | C | D | f |
|--------------|---|---|---|---|---|
| $m_0$        | 0 | 0 | 0 | 0 | 0 |
| $m_1$        | 0 | 0 | 0 | 1 | 0 |
| $m_2$        | 0 | 0 | 1 | 0 | 0 |
| $m_3$        | 0 | 0 | 1 | 1 | 1 |
| $m_4$        | 0 | 1 | 0 | 0 | 0 |
| $m_5$        | 0 | 1 | 0 | 1 | 0 |
| $m_6$        | 0 | 1 | 1 | 0 | 0 |
| $m_7$        | 0 | 1 | 1 | 1 | 1 |
| $m_8$        | 1 | 0 | 0 | 0 | 0 |
| $m_9$        | 1 | 0 | 0 | 1 | 0 |
| $m_{10}$     | 1 | 0 | 1 | 0 | 0 |
| $m_{11}$     | 1 | 0 | 1 | 1 | 1 |
| $m_{12}$     | 1 | 1 | 0 | 0 | 0 |
| $m_{13}$     | 1 | 1 | 0 | 1 | 1 |
| $m_{14}$     | 1 | 1 | 1 | 0 | 1 |
| $m_{15}$     | 1 | 1 | 1 | 1 | 0 |

### 1.6.2 Maxterms (Dual of minterms)

Maxterm is a sum involving all inputs (or complements) to a function. Every row of a truth table has a corresponding maxterm. Minterm is false if and only if the corresponding row in the table is active.

Maxterm

Maxterms are defined as follows for each row of a two input truth table:

| A | B | maxterm             | maxterm name |
|---|---|---------------------|--------------|
| 0 | 0 | $A + B$             | $M_0$        |
| 0 | 1 | $A + \bar{B}$       | $M_1$        |
| 1 | 0 | $\bar{A} + B$       | $M_2$        |
| 1 | 1 | $\bar{A} + \bar{B}$ | $M_3$        |

| A | B | $M_0$   | $M_2$         |
|---|---|---------|---------------|
|   |   | $A + B$ | $\bar{A} + B$ |
| 0 | 0 | 0       | 1             |
| 0 | 1 | 1       | 1             |
| 1 | 0 | 1       | 0             |
| 1 | 1 | 1       | 1             |

Consider a two input circuit whose output  $Y$  is given by the truth table:

| A | B | Y | maxterm             | maxterm name |
|---|---|---|---------------------|--------------|
| 0 | 0 | 0 | $A + B$             | $M_0$        |
| 0 | 1 | 1 | $A + \bar{B}$       | $M_1$        |
| 1 | 0 | 0 | $\bar{A} + B$       | $M_2$        |
| 1 | 1 | 1 | $\bar{A} + \bar{B}$ | $M_3$        |

$$\begin{aligned} M_2 &= \overline{A} + B \\ (2)_{10} &\rightarrow (10)_2 \end{aligned}$$

$$\text{then } Y = (A + B)(\bar{A} + B) = M_0M_2.$$

Writing a functional specification in terms of minterms is also called product of sums canonical form.

**MSB** → **Example 1.8.** Write the maxterm  $M_{11}$  for 4-input Boolean function with the ordered inputs  $A, B, C, D$ .  $\left(\underline{11}\right)_{10} = (1011)_2$   $M_{11} = \overline{A} + B + \overline{C} + \overline{D}$

**Example 1.9.** Convert the following 4-input truth table into product of maxterms and product of sums canonical form.

| maxterm name | A | B | C | D | f |
|--------------|---|---|---|---|---|
| $M_0$        | 0 | 0 | 0 | 0 | 0 |
| $M_1$        | 0 | 0 | 0 | 1 | 0 |
| $M_2$        | 0 | 0 | 1 | 0 | 0 |
| $M_3$        | 0 | 0 | 1 | 1 | 1 |
| $M_4$        | 0 | 1 | 0 | 0 | 0 |
| $M_5$        | 0 | 1 | 0 | 1 | 0 |
| $M_6$        | 0 | 1 | 1 | 0 | 0 |
| $M_7$        | 0 | 1 | 1 | 1 | 1 |
| $M_8$        | 1 | 0 | 0 | 0 | 0 |
| $M_9$        | 1 | 0 | 0 | 1 | 0 |
| $M_{10}$     | 1 | 0 | 1 | 0 | 0 |
| $M_{11}$     | 1 | 0 | 1 | 1 | 1 |
| $M_{12}$     | 1 | 1 | 0 | 0 | 0 |
| $M_{13}$     | 1 | 1 | 0 | 1 | 1 |
| $M_{14}$     | 1 | 1 | 1 | 0 | 1 |
| $M_{15}$     | 1 | 1 | 1 | 1 | 0 |

$$\begin{aligned} f &= M_0 M_1 M_2 M_4 M_5 M_6 M_7 M_9 M_{10} M_{12} M_{15} \\ &= \prod M(0, 1, 2, 4, 5, 6, 8, 9, 10, 12, 15) \end{aligned}$$

$$\begin{aligned} f &= (A + B + C + D)(A + B + C + \bar{D})(A + B + \bar{C} + D)(A + \bar{B} + C + D) \\ &\quad (A + \bar{B} + C + \bar{D})(A + \bar{B} + \bar{C} + D)(\bar{A} + B + C + D)(\bar{A} + B + C + \bar{D}) \\ &\quad (\bar{A} + B + \bar{C} + D)(\bar{A} + \bar{B} + C + D)(\bar{A} + \bar{B} + \bar{C} + D) \end{aligned}$$

Product of sums canonical form

**Problem 1.6** (10 marks). Convert the following 4-input truth table into product of maxterms and products of sums canonical form.

| maxterm<br>name | A | B | C | D | f |
|-----------------|---|---|---|---|---|
| $M_0$           | 0 | 0 | 0 | 0 | 0 |
| $M_1$           | 0 | 0 | 0 | 1 | 1 |
| $M_2$           | 0 | 0 | 1 | 0 | 1 |
| $M_3$           | 0 | 0 | 1 | 1 | 1 |
| $M_4$           | 0 | 1 | 0 | 0 | 1 |
| $M_5$           | 0 | 1 | 0 | 1 | 0 |
| $M_6$           | 0 | 1 | 1 | 0 | 1 |
| $M_7$           | 0 | 1 | 1 | 1 | 1 |
| $M_8$           | 1 | 0 | 0 | 0 | 0 |
| $M_9$           | 1 | 0 | 0 | 1 | 1 |
| $M_{10}$        | 1 | 0 | 1 | 0 | 1 |
| $M_{11}$        | 1 | 0 | 1 | 1 | 1 |
| $M_{12}$        | 1 | 1 | 0 | 0 | 0 |
| $M_{13}$        | 1 | 1 | 0 | 1 | 1 |
| $M_{14}$        | 1 | 1 | 1 | 0 | 1 |
| $M_{15}$        | 1 | 1 | 1 | 1 | 0 |

**Example 1.10.** Write the 3-input truth table for the function  $f = m_2 + m_3 + m_7$ .

| # | x | y | z | f |
|---|---|---|---|---|
| 0 | 0 | 0 | 0 | 0 |
| 1 | 0 | 0 | 1 | 0 |
| 2 | 0 | 1 | 0 | 1 |
| 3 | 0 | 1 | 1 | 1 |
| 4 | 1 | 0 | 0 | 0 |
| 5 | 1 | 0 | 1 | 0 |
| 6 | 1 | 1 | 0 | 0 |
| 7 | 1 | 1 | 1 | 1 |

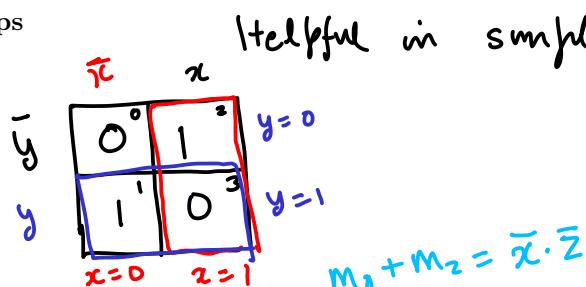
**Problem 1.7** (10 marks). Write the 3-input truth table for the function  $f = M_4M_5M_7$ .

**Problem 1.8** (10 marks). Write the truth table for the function  $f = \bar{A}B\bar{C} + AB\bar{C}$ .

## 1.7 Karnaugh maps (Truth table + Venn Diagram)

Two input K-maps

| $x, y$    | $f$ |
|-----------|-----|
| $m_0$ 0 0 | 0   |
| $m_1$ 0 1 | 1   |
| $m_2$ 1 0 | 1   |
| $m_3$ 1 1 | 0   |

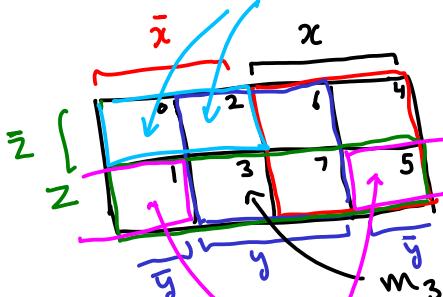


It helps in simplifying Boolean expressions  
Digital circuits

MSB

Three input K-maps

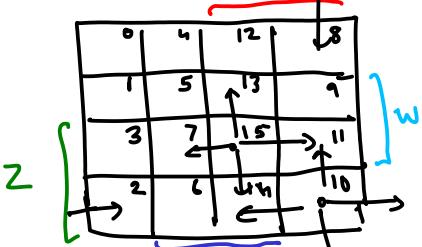
| $x \ y \ z$ | $f$ |
|-------------|-----|
| 0 0 0       | 0   |
| 0 0 1       | 0   |
| 0 1 0       | 1   |
| 0 1 1       | 0   |
| 1 0 0       | 0   |



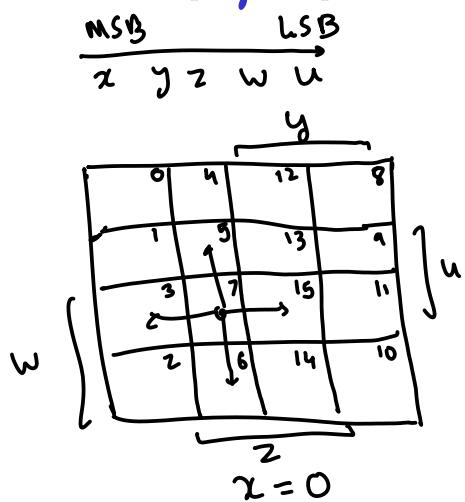
Each rectangular grouping of  $2, 4, 8, 16, \dots$  corresponds to a product term

MSB → LSB

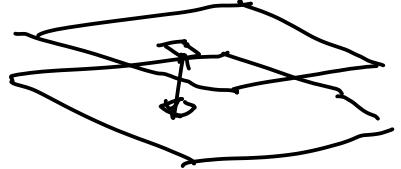
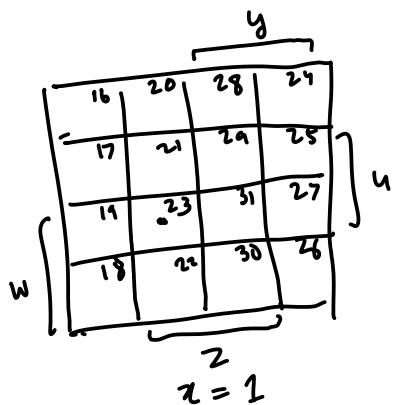
Four input K-maps



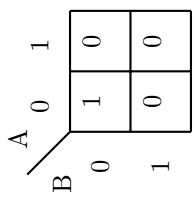
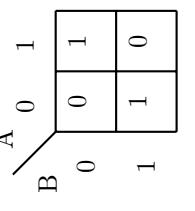
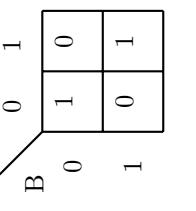
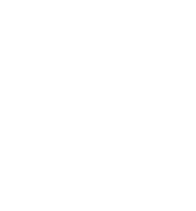
Five input K-maps



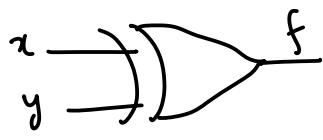
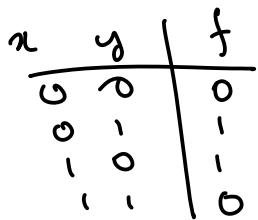
In input K-map must have 4-neighbors for each cell  
K-map wraps around



## 1.8 More Gates and notations summary

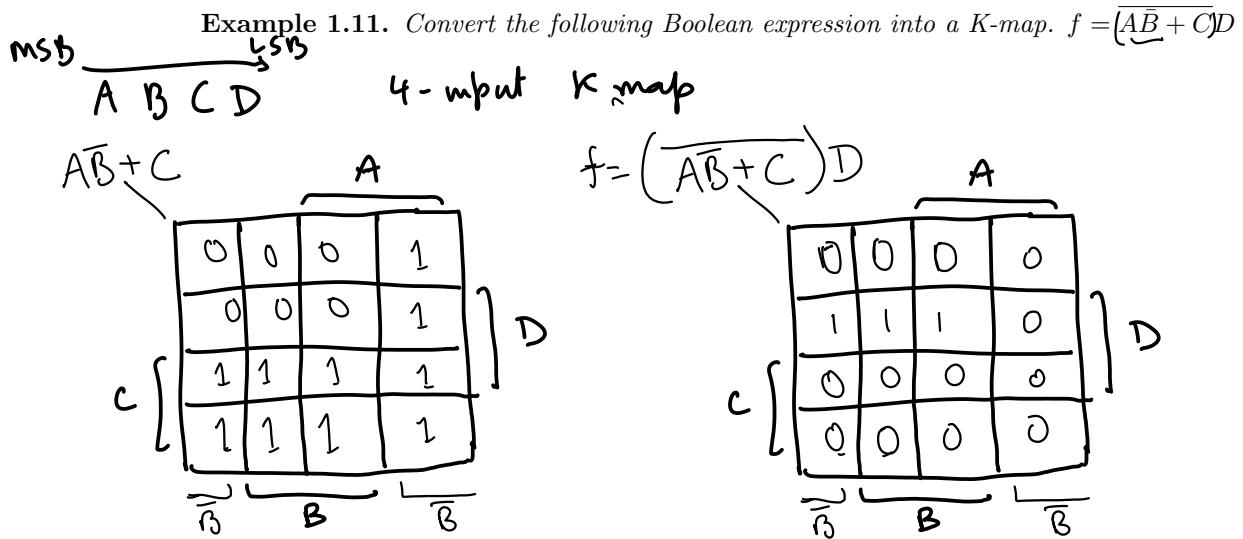
| Name      | C/Verilog                | Boolean expr.                                                  | Truth Table                                                                                                                                                                                                                                                                                                                                   | (ANSI) symbol | K-map |                             |   |   |   |   |   |   |   |   |   |   |   |   |                                                                                     |                                                                                                                                                       |   |   |   |   |   |   |  |   |   |
|-----------|--------------------------|----------------------------------------------------------------|-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|---------------|-------|-----------------------------|---|---|---|---|---|---|---|---|---|---|---|---|-------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------------------------------------------------------------|---|---|---|---|---|---|--|---|---|
|           |                          |                                                                |                                                                                                                                                                                                                                                                                                                                               |               |       |                             |   |   |   |   |   |   |   |   |   |   |   |   |                                                                                     |                                                                                                                                                       |   |   |   |   |   |   |  |   |   |
| NAND Gate | $Q = \sim(x_1 \& x_2)$   | $Q = \overline{x_1 \cdot x_2} = \overline{x_1} \overline{x_2}$ | <table border="1"> <thead> <tr> <th><math>x_1</math></th><th><math>x_2</math></th><th><math>\overline{x_1 \cdot x_2}</math></th></tr> </thead> <tbody> <tr> <td>0</td><td>0</td><td>1</td></tr> <tr> <td>0</td><td>1</td><td>1</td></tr> <tr> <td>1</td><td>0</td><td>1</td></tr> <tr> <td>1</td><td>1</td><td>0</td></tr> </tbody> </table>  | $x_1$         | $x_2$ | $\overline{x_1 \cdot x_2}$  | 0 | 0 | 1 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 |    | <table border="1"> <tr> <td>A</td><td>0</td><td>1</td></tr> <tr> <td>B</td><td>0</td><td>1</td></tr> <tr> <td></td><td>1</td><td>0</td></tr> </table> | A | 0 | 1 | B | 0 | 1 |  | 1 | 0 |
| $x_1$     | $x_2$                    | $\overline{x_1 \cdot x_2}$                                     |                                                                                                                                                                                                                                                                                                                                               |               |       |                             |   |   |   |   |   |   |   |   |   |   |   |   |                                                                                     |                                                                                                                                                       |   |   |   |   |   |   |  |   |   |
| 0         | 0                        | 1                                                              |                                                                                                                                                                                                                                                                                                                                               |               |       |                             |   |   |   |   |   |   |   |   |   |   |   |   |                                                                                     |                                                                                                                                                       |   |   |   |   |   |   |  |   |   |
| 0         | 1                        | 1                                                              |                                                                                                                                                                                                                                                                                                                                               |               |       |                             |   |   |   |   |   |   |   |   |   |   |   |   |                                                                                     |                                                                                                                                                       |   |   |   |   |   |   |  |   |   |
| 1         | 0                        | 1                                                              |                                                                                                                                                                                                                                                                                                                                               |               |       |                             |   |   |   |   |   |   |   |   |   |   |   |   |                                                                                     |                                                                                                                                                       |   |   |   |   |   |   |  |   |   |
| 1         | 1                        | 0                                                              |                                                                                                                                                                                                                                                                                                                                               |               |       |                             |   |   |   |   |   |   |   |   |   |   |   |   |                                                                                     |                                                                                                                                                       |   |   |   |   |   |   |  |   |   |
| A         | 0                        | 1                                                              |                                                                                                                                                                                                                                                                                                                                               |               |       |                             |   |   |   |   |   |   |   |   |   |   |   |   |                                                                                     |                                                                                                                                                       |   |   |   |   |   |   |  |   |   |
| B         | 0                        | 1                                                              |                                                                                                                                                                                                                                                                                                                                               |               |       |                             |   |   |   |   |   |   |   |   |   |   |   |   |                                                                                     |                                                                                                                                                       |   |   |   |   |   |   |  |   |   |
|           | 1                        | 0                                                              |                                                                                                                                                                                                                                                                                                                                               |               |       |                             |   |   |   |   |   |   |   |   |   |   |   |   |                                                                                     |                                                                                                                                                       |   |   |   |   |   |   |  |   |   |
| NOR Gate  | $Q = \sim(x_1 \mid x_2)$ | $Q = \overline{x_1 + x_2}$                                     | <table border="1"> <thead> <tr> <th><math>x_1</math></th><th><math>x_2</math></th><th><math>\overline{x_1 + x_2}</math></th></tr> </thead> <tbody> <tr> <td>0</td><td>0</td><td>1</td></tr> <tr> <td>0</td><td>1</td><td>0</td></tr> <tr> <td>1</td><td>0</td><td>0</td></tr> <tr> <td>1</td><td>1</td><td>0</td></tr> </tbody> </table>      | $x_1$         | $x_2$ | $\overline{x_1 + x_2}$      | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 1 | 0 |   | <table border="1"> <tr> <td>A</td><td>0</td><td>1</td></tr> <tr> <td>B</td><td>0</td><td>1</td></tr> <tr> <td></td><td>1</td><td>0</td></tr> </table> | A | 0 | 1 | B | 0 | 1 |  | 1 | 0 |
| $x_1$     | $x_2$                    | $\overline{x_1 + x_2}$                                         |                                                                                                                                                                                                                                                                                                                                               |               |       |                             |   |   |   |   |   |   |   |   |   |   |   |   |                                                                                     |                                                                                                                                                       |   |   |   |   |   |   |  |   |   |
| 0         | 0                        | 1                                                              |                                                                                                                                                                                                                                                                                                                                               |               |       |                             |   |   |   |   |   |   |   |   |   |   |   |   |                                                                                     |                                                                                                                                                       |   |   |   |   |   |   |  |   |   |
| 0         | 1                        | 0                                                              |                                                                                                                                                                                                                                                                                                                                               |               |       |                             |   |   |   |   |   |   |   |   |   |   |   |   |                                                                                     |                                                                                                                                                       |   |   |   |   |   |   |  |   |   |
| 1         | 0                        | 0                                                              |                                                                                                                                                                                                                                                                                                                                               |               |       |                             |   |   |   |   |   |   |   |   |   |   |   |   |                                                                                     |                                                                                                                                                       |   |   |   |   |   |   |  |   |   |
| 1         | 1                        | 0                                                              |                                                                                                                                                                                                                                                                                                                                               |               |       |                             |   |   |   |   |   |   |   |   |   |   |   |   |                                                                                     |                                                                                                                                                       |   |   |   |   |   |   |  |   |   |
| A         | 0                        | 1                                                              |                                                                                                                                                                                                                                                                                                                                               |               |       |                             |   |   |   |   |   |   |   |   |   |   |   |   |                                                                                     |                                                                                                                                                       |   |   |   |   |   |   |  |   |   |
| B         | 0                        | 1                                                              |                                                                                                                                                                                                                                                                                                                                               |               |       |                             |   |   |   |   |   |   |   |   |   |   |   |   |                                                                                     |                                                                                                                                                       |   |   |   |   |   |   |  |   |   |
|           | 1                        | 0                                                              |                                                                                                                                                                                                                                                                                                                                               |               |       |                             |   |   |   |   |   |   |   |   |   |   |   |   |                                                                                     |                                                                                                                                                       |   |   |   |   |   |   |  |   |   |
| XOR Gate  | $Q = x_1 \sim x_2$       | $Q = x_1 \oplus x_2$                                           | <table border="1"> <thead> <tr> <th><math>x_1</math></th><th><math>x_2</math></th><th><math>x_1 \oplus x_2</math></th></tr> </thead> <tbody> <tr> <td>0</td><td>0</td><td>0</td></tr> <tr> <td>0</td><td>1</td><td>1</td></tr> <tr> <td>1</td><td>0</td><td>1</td></tr> <tr> <td>1</td><td>1</td><td>0</td></tr> </tbody> </table>            | $x_1$         | $x_2$ | $x_1 \oplus x_2$            | 0 | 0 | 0 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 |  | <table border="1"> <tr> <td>A</td><td>0</td><td>1</td></tr> <tr> <td>B</td><td>0</td><td>1</td></tr> <tr> <td></td><td>1</td><td>0</td></tr> </table> | A | 0 | 1 | B | 0 | 1 |  | 1 | 0 |
| $x_1$     | $x_2$                    | $x_1 \oplus x_2$                                               |                                                                                                                                                                                                                                                                                                                                               |               |       |                             |   |   |   |   |   |   |   |   |   |   |   |   |                                                                                     |                                                                                                                                                       |   |   |   |   |   |   |  |   |   |
| 0         | 0                        | 0                                                              |                                                                                                                                                                                                                                                                                                                                               |               |       |                             |   |   |   |   |   |   |   |   |   |   |   |   |                                                                                     |                                                                                                                                                       |   |   |   |   |   |   |  |   |   |
| 0         | 1                        | 1                                                              |                                                                                                                                                                                                                                                                                                                                               |               |       |                             |   |   |   |   |   |   |   |   |   |   |   |   |                                                                                     |                                                                                                                                                       |   |   |   |   |   |   |  |   |   |
| 1         | 0                        | 1                                                              |                                                                                                                                                                                                                                                                                                                                               |               |       |                             |   |   |   |   |   |   |   |   |   |   |   |   |                                                                                     |                                                                                                                                                       |   |   |   |   |   |   |  |   |   |
| 1         | 1                        | 0                                                              |                                                                                                                                                                                                                                                                                                                                               |               |       |                             |   |   |   |   |   |   |   |   |   |   |   |   |                                                                                     |                                                                                                                                                       |   |   |   |   |   |   |  |   |   |
| A         | 0                        | 1                                                              |                                                                                                                                                                                                                                                                                                                                               |               |       |                             |   |   |   |   |   |   |   |   |   |   |   |   |                                                                                     |                                                                                                                                                       |   |   |   |   |   |   |  |   |   |
| B         | 0                        | 1                                                              |                                                                                                                                                                                                                                                                                                                                               |               |       |                             |   |   |   |   |   |   |   |   |   |   |   |   |                                                                                     |                                                                                                                                                       |   |   |   |   |   |   |  |   |   |
|           | 1                        | 0                                                              |                                                                                                                                                                                                                                                                                                                                               |               |       |                             |   |   |   |   |   |   |   |   |   |   |   |   |                                                                                     |                                                                                                                                                       |   |   |   |   |   |   |  |   |   |
| XNOR Gate | $Q = \sim(x_1 \sim x_2)$ | $Q = \overline{x_1 \oplus x_2}$                                | <table border="1"> <thead> <tr> <th><math>x_1</math></th><th><math>x_2</math></th><th><math>\overline{x_1 \oplus x_2}</math></th></tr> </thead> <tbody> <tr> <td>0</td><td>0</td><td>1</td></tr> <tr> <td>0</td><td>1</td><td>0</td></tr> <tr> <td>1</td><td>0</td><td>0</td></tr> <tr> <td>1</td><td>1</td><td>1</td></tr> </tbody> </table> | $x_1$         | $x_2$ | $\overline{x_1 \oplus x_2}$ | 0 | 0 | 1 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 1 | 1 |  | <table border="1"> <tr> <td>A</td><td>0</td><td>1</td></tr> <tr> <td>B</td><td>0</td><td>1</td></tr> <tr> <td></td><td>1</td><td>0</td></tr> </table> | A | 0 | 1 | B | 0 | 1 |  | 1 | 0 |
| $x_1$     | $x_2$                    | $\overline{x_1 \oplus x_2}$                                    |                                                                                                                                                                                                                                                                                                                                               |               |       |                             |   |   |   |   |   |   |   |   |   |   |   |   |                                                                                     |                                                                                                                                                       |   |   |   |   |   |   |  |   |   |
| 0         | 0                        | 1                                                              |                                                                                                                                                                                                                                                                                                                                               |               |       |                             |   |   |   |   |   |   |   |   |   |   |   |   |                                                                                     |                                                                                                                                                       |   |   |   |   |   |   |  |   |   |
| 0         | 1                        | 0                                                              |                                                                                                                                                                                                                                                                                                                                               |               |       |                             |   |   |   |   |   |   |   |   |   |   |   |   |                                                                                     |                                                                                                                                                       |   |   |   |   |   |   |  |   |   |
| 1         | 0                        | 0                                                              |                                                                                                                                                                                                                                                                                                                                               |               |       |                             |   |   |   |   |   |   |   |   |   |   |   |   |                                                                                     |                                                                                                                                                       |   |   |   |   |   |   |  |   |   |
| 1         | 1                        | 1                                                              |                                                                                                                                                                                                                                                                                                                                               |               |       |                             |   |   |   |   |   |   |   |   |   |   |   |   |                                                                                     |                                                                                                                                                       |   |   |   |   |   |   |  |   |   |
| A         | 0                        | 1                                                              |                                                                                                                                                                                                                                                                                                                                               |               |       |                             |   |   |   |   |   |   |   |   |   |   |   |   |                                                                                     |                                                                                                                                                       |   |   |   |   |   |   |  |   |   |
| B         | 0                        | 1                                                              |                                                                                                                                                                                                                                                                                                                                               |               |       |                             |   |   |   |   |   |   |   |   |   |   |   |   |                                                                                     |                                                                                                                                                       |   |   |   |   |   |   |  |   |   |
|           | 1                        | 0                                                              |                                                                                                                                                                                                                                                                                                                                               |               |       |                             |   |   |   |   |   |   |   |   |   |   |   |   |                                                                                     |                                                                                                                                                       |   |   |   |   |   |   |  |   |   |

$x_{\text{C}\text{O}\text{R}} = \text{Exclusive OR gate}$

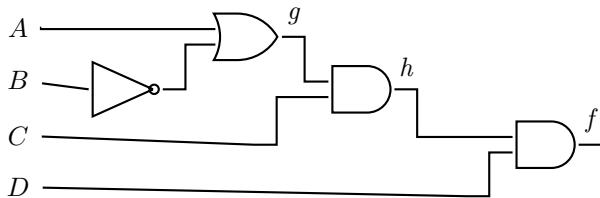


$$f = x \oplus y$$

$f = x \sim y$  in C/verilog



**Problem 1.9** (10 marks). Convert the following logic circuit into a K-map.



## 1.9 Boolean Algebra

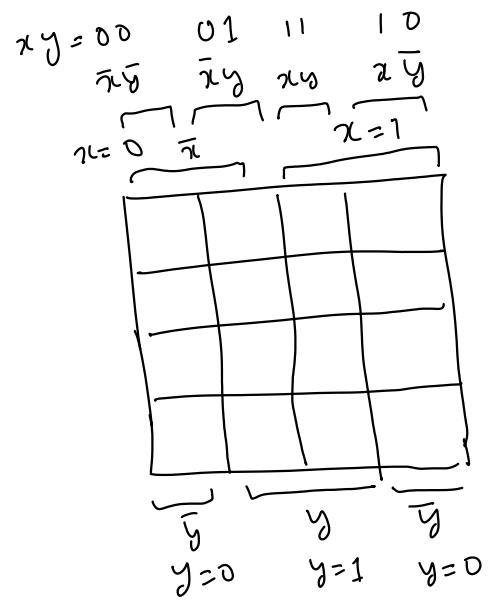
### 1.9.1 Axioms of Boolean algebra

$$1. 0 \cdot 0 = 0$$

$$2. 1 + 1 = 1$$

Definitions

≠ Axioms are fundamental truths  
that you take to be true  
and do not ask proofs for.



3.  $1 \cdot 1 = 1$
4.  $0 + 0 = 0$
5.  $0 \cdot 1 = 1 \cdot 0 = 0$
6.  $\bar{0} = 1$
7.  $\bar{1} = 0$
8.  $x = 0$  if  $x \neq 1$
9.  $x = 1$  if  $x \neq 0$

### 1.9.2 Single variable theorems (Prove by drawing K-maps)

$\cancel{x}$

|              |                 |              |
|--------------|-----------------|--------------|
| $\cancel{1}$ | $1 \cdot 0 = 0$ | by axiom (5) |
| $\cancel{0}$ | $0 \cdot 0 = 0$ | by axiom (1) |

2.  $x + 1 = 1$

3.  $x \cdot 1 = x$

4.  $x + 0 = x$

5.  $x \cdot x = x$

6.  $x + x = x$

7.  $x \cdot \bar{x} = 0$

$$8. x + \bar{x} = 1$$

$$9. \bar{\bar{x}} = x$$

**Remark 1.2** (Duality). Swap  $+$  with  $\cdot$  and  $0$  with  $1$  to get another theorem

### 1.9.3 Two and three variable properties (Prove by K-maps)

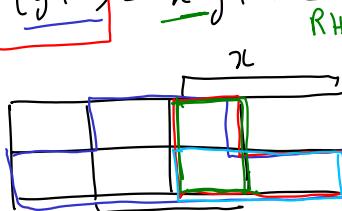
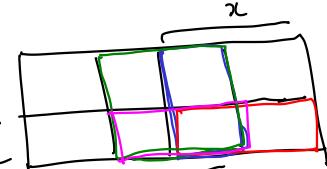
$$1. \text{ Commutative: } x \cdot y = y \cdot x, x + y = y + x$$

$$2. \text{ Associative: } x \cdot (y \cdot z) = (x \cdot y) \cdot z, x + (y + z) = (x + y) + z$$

3. Distributive:  $x \cdot (y + z) = x \cdot y + x \cdot z$

*Dual*

$LHS = x \cdot (y + z)$        $RHS = (x + y) \cdot (x + z)$

$LHS = x \cdot (y + z) = x \cdot y + x \cdot z$

$RHS = (x + y) \cdot (x + z) = x \cdot (y + z) + y \cdot (y + z) = x \cdot y + x \cdot z + y \cdot y + y \cdot z = x \cdot y + x \cdot z + y + y \cdot z = y(x + 1 + z) + x \cdot z = y \cdot 1 + x \cdot z = y + x \cdot z = RHS$

4. Absorption:  $x + x \cdot y = x, x \cdot (x + y) = x$

$LHS = x + x \cdot y = x$

$LHS = x \cdot (x + y) = x \cdot x + x \cdot y = x + x \cdot y = x$

5. Combining:  $x \cdot y + x \cdot \bar{y}$ ,  $(x + y) \cdot (x + \bar{y}) = x$

$$\cancel{x} + \cancel{x} = 1$$

$$LHS = x \cdot y + x \cdot \bar{y} = x \cdot (y + \bar{y}) = x \cdot 1 = x$$

6. DeMorgan's theorem:  $\overline{x \cdot y} = \bar{x} + \bar{y}$ ,  $\overline{x + y} = \bar{x} \cdot \bar{y}$ .

7. Concensus:

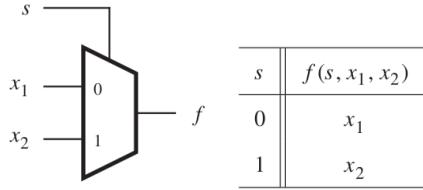
(a)  $x + \bar{x} \cdot y = x + y$

(b)  $x \cdot (\bar{x} + y) = x \cdot y$

(c)  $x \cdot y + y \cdot z + \bar{x} \cdot z = x \cdot y + \bar{x} \cdot z$

(d)  $(x + y) \cdot (y + z) \cdot (\bar{x} + z) = (x + y) \cdot (\bar{x} + z)$

**Example 1.12** (Multiplexer). *Multiplexer is a circuit used to select one of the input lines  $x_1$  and  $x_2$  based only select input  $s$ . When  $s = 0$ ,  $x_1$  is selected,  $x_2$  is selected otherwise. Find a boolean expression and a circuit for multiplexer*



**Example 1.13.** Simplify  $f = \bar{A}\bar{B}\bar{C} + A\bar{B}\bar{C} + A\bar{B}\bar{C}$  using boolean algebra.

**Problem 1.10** (30 marks, Exercise 2.14 [1]). Simplify the following Boolean equations using Boolean theorems.

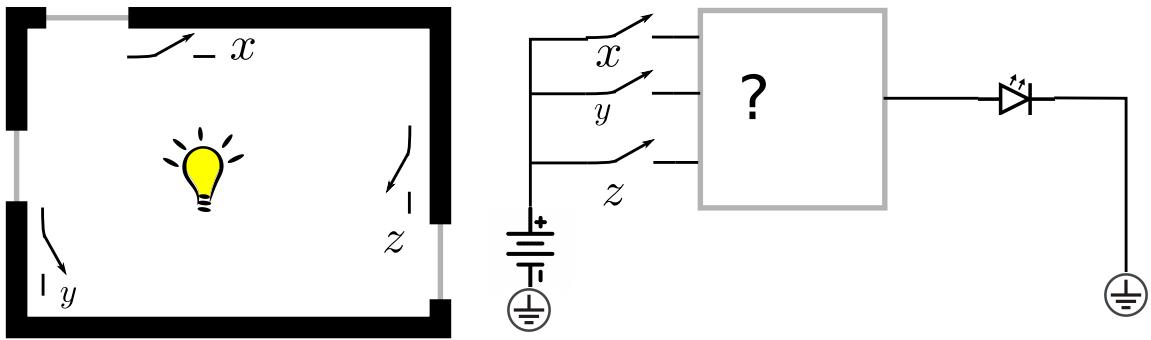
$$Y = \bar{A}BC + \bar{A}B\bar{C} \quad (1.1)$$

$$Y = \overline{ABC} + A\bar{B} \quad (1.2)$$

$$Y = ABC\bar{D} + A\overline{BCD} + (\overline{A + B + C + D}) \quad (1.3)$$

**Example 1.14.** Simplify  $f = \bar{A}\bar{A}\bar{C} + \bar{A}\bar{B}C$  using K-maps.

**Example 1.15.** Assume that a large room has three doors and that a switch near each door controls a light in the room. It has to be possible to turn the light on or off by changing the state of any one of the switches.



**Problem 1.11** (20 marks, Exercise 2.38 [1]). An  $M$ -bit thermometer code for the number  $k$  consists of  $k$  1's in the least significant bit positions and  $M - k$  0's in all the more significant bit positions. A binary-to-thermometer code converter has  $N$  inputs and  $2^{N-1}$  outputs. It produces a  $2^{N-1}$  bit thermometer code for the number specified by the input. For example, if the input is 110, the output should be 0111111. Design a 3:7 binary-to-thermometer code converter. Give a simplified Boolean equation for each output.