ECE 417/598 Midterm 2 2024

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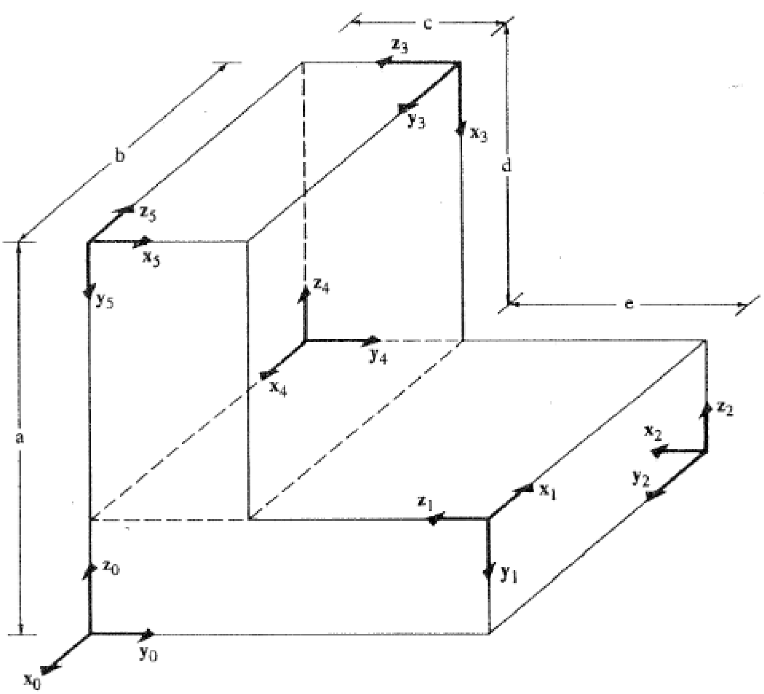
About the exam

1. There are total 5 problems. You must attempt all 5.
2. Maximum marks: 50.
3. Maximum time allotted: 50 min
4. Calculators are allowed.
5. One 8x11 in cheat sheet (both-sides) is allowed.

Problem 1 *Find the 4x4 transformation matrix $^0T_2$ that transforms coordinates from coordinate frame 2 to coordinate frame 0 (5 marks).*



Solution: I am going to write transform from frame 1 to frame 0. $^0T_2$

$$^0T_2 = \begin{bmatrix} 0 & 1 & 0 & -b \\ -1 & 0 & 0 & c+e \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}$$

The rotation matrix is obtained by writing the basis vectors of the destination coordinate system in the source coordinate system.

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Problem 2 *In your own words, prove using trignometry that the 2D rotation matrix is given by (10 marks)*

$$R = \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}. \tag{2}$$

Solution:

We can express original coordinates (x, y) in polar coordinates,

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} r\cos(\alpha) \\ r\sin(\alpha) \end{bmatrix}, \tag{3}$$

where $r = \sqrt{x^2 + y^2}$ is the length of the vector and $\alpha = \arctan2(y, x)$ is the angle between the vector and X-axis.

The rotated points (x', y') have the same length r but a different angle

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} r\cos(\alpha + \alpha) \\ r\sin(\alpha + \alpha) \end{bmatrix}. \tag{4}$$

Using the trignometric identities we can write,

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} r\cos(\alpha)\cos(\alpha) - r\sin(\alpha)\sin(\alpha) \\ r\sin(\alpha)\cos(\alpha) + r\cos(\alpha)\sin(\alpha) \end{bmatrix}. \tag{5}$$

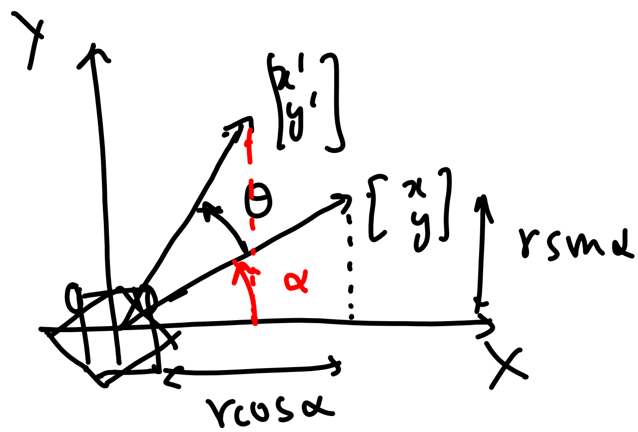


Figure 1: Rotation of points $[x, y]$ to $[x', y']$ by an angle α

Substituting $r\cos(\alpha) = x$ and $r\sin(\alpha) = y$, we get

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x\cos(\alpha) - y\sin(\alpha) \\ x\sin(\alpha) + y\cos(\alpha) \end{bmatrix}. \tag{6}$$

Writing the right hand side (RHS) as a matrix vector product,

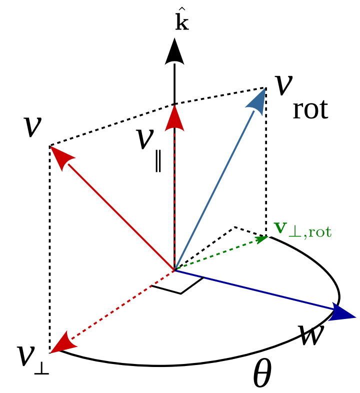
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & +\cos(\alpha) \end{bmatrix}}_{R} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{7}$$

The matrix being multiplied here is the 2D rotation matrix.

Problem 3 *(Rodrigues formula) In the figure below, we rotate a point* $\mathbf{v}$ *around axis unit-vector* $\hat{\mathbf{k}}$ *by an angle* θ*. A unit vector* $\hat{\mathbf{w}}$ *is perpendicular to the both* $\mathbf{v}$ *and* $\hat{\mathbf{k}}$*. Another vector* $\mathbf{v}_\perp$ *is the projection of* $\mathbf{v}$ *onto a plane that is perpendicular to* $\hat{\mathbf{k}}$*. Note that* $\mathbf{v}_\perp$ *is perpendicular to both* $\hat{\mathbf{w}}$ *and* $\hat{\mathbf{k}}$ *(10 marks).*

a. *write the unit-vector* $\hat{\mathbf{w}}$ *in terms of* $\mathbf{v}$ *and* $\hat{\mathbf{k}}$ *.*

b. *Then write the vector (including the correct magnitude)* $\mathbf{v}_\perp$ *in terms of* $\mathbf{v}$ *and* $\hat{\mathbf{k}}$ *(5 marks).*



Solution:

$$\hat{\mathbf{w}} = \frac{\hat{\mathbf{k}} \times \mathbf{v}}{\|\hat{\mathbf{k}} \times \mathbf{v}\|} \tag{8}$$

Note that $\hat{\mathbf{k}} \times \mathbf{v}$ has the magnitude $\|\hat{\mathbf{k}} \times \mathbf{v}\| = |\hat{\mathbf{k}}||\mathbf{v}|\sin(\alpha) = |\mathbf{v}|\sin(\alpha)$ where α is the angle between $\mathbf{v}$ and $\hat{\mathbf{k}}$.

$$\mathbf{v}_\perp = -\hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times \mathbf{v}) \tag{9}$$

The magnitude of $\mathbf{v}_\perp$ is same as RHS because both $|\mathbf{v}_\perp| = |\mathbf{v}|\sin(\alpha)$ and $|-\hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\mathbf{v})| = |\mathbf{v}||\hat{\mathbf{k}}|^2 \sin(\alpha)\sin(90°) = |\mathbf{v}|\sin(\alpha)$ where α is the angle between $\mathbf{v}$ and $\hat{\mathbf{k}}$.

Problem 4 *Consider a unit-vector $\hat{\mathbf{k}} = (k_x, k_y, k_z)^\top$ and the corresponding cross-product matrix,*

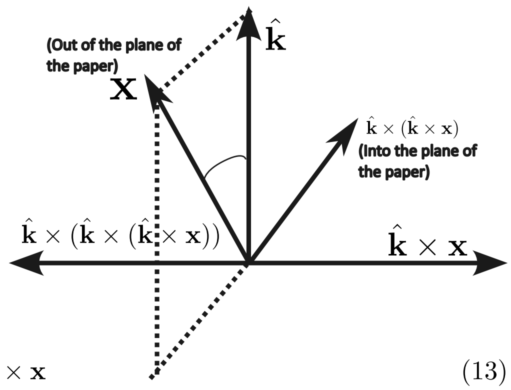
$$K = [\hat{\mathbf{k}}]_\times = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix}. \tag{10}$$

Using the geometry of the cross-product to prove that $K^3 = -K$. Hint on next page. (5 marks)

Solution: Geometric method:

Choose any vector $\mathbf{x} \neq \gamma\hat{\mathbf{k}}$. Draw the following three vectors: (1) $\mathbf{a} = \hat{\mathbf{k}}\times\mathbf{x}$, (2) $\mathbf{b} = \hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\mathbf{x})$, (3) $\mathbf{c} = \hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\mathbf{x}))$. Compare vectors $\mathbf{c}$ and $\mathbf{a}$. Note that the direction of $\mathbf{c}$ is opposite to $\mathbf{a}$. In other words, $\mathbf{c} = -|\beta|\mathbf{a}$ where $|\beta|$ is some scalar.

What is magnitude of $\mathbf{c}$ and $\mathbf{a}$?



$$|\mathbf{c}| = |\hat{\mathbf{k}}|^3|\mathbf{x}|\sin(\alpha)\sin(90)\sin(90) = |\mathbf{x}|\sin(\alpha) \tag{11}$$

$$|\mathbf{a}| = |\hat{\mathbf{k}}||\mathbf{x}|\sin(\alpha) = |\mathbf{x}|\sin(\alpha), \tag{12}$$

where α is the angle between $\hat{\mathbf{k}}$ and $\mathbf{x}$. Thus $|\mathbf{c}| = |\mathbf{a}|$.

Because $|\mathbf{c}| = |\mathbf{a}|$ and $\mathbf{c} = -|\beta|\mathbf{a}$, therefore $\mathbf{c} = -\mathbf{a}$, or

$$\hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times \mathbf{x})) = -\hat{\mathbf{k}} \times \mathbf{x} \tag{13}$$

$$\implies K^3\mathbf{x} = -K\mathbf{x}. \tag{14}$$

Since $K^3\mathbf{x} = -K\mathbf{x}$ for any $\mathbf{x} \neq \gamma\hat{\mathbf{k}}$, hence $K^3 = K$. (When $\mathbf{x} = \gamma\hat{\mathbf{k}}$, then $K\mathbf{x} = \mathbf{0} = K^3\mathbf{x}$).

Proof by algebra

This was not allowed because I asked to use geometry of the cross-product. But it is possible to prove this using algebra as well.

$$K^2 = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix} \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix} = \begin{bmatrix} -(k_z^2 + k_y^2) & k_y k_x & k_z k_x \\ k_x k_y & -(k_x^2 + k_z)^2 & k_z k_y \\ k_x k_z & k_y k_z & -(k_y^2 + k_x^2) \end{bmatrix} \tag{15}$$

$$\implies K^3 = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix} \begin{bmatrix} -(k_z^2 + k_y^2) & k_y k_x & k_z k_x \\ k_x k_y & -(k_x^2 + k_z)^2 & k_z k_y \\ k_x k_z & k_y k_z & -(k_y^2 + k_x^2) \end{bmatrix} \tag{16}$$

$$= \begin{bmatrix} -k_z k_y k_x + k_y k_x k_z, & k_z(k_x^2 + k_z^2) + k_y^2 k_z, & -k_z^2 k_y - k_y(k_y^2 + k_x^2) \\ -k_z(k_z^2 + k_y^2) - k_x^2 k_z, & -k_z k_y k_x + k_y k_x k_z, & k_z^2 k_x + k_x(k_y^2 + k_x^2) \\ k_y(k_z^2 + k_y^2) + k_x^2 k_y & -k_y^2 k_x - k_x(k_x^2 + k_z^2) & -k_z k_y k_x + k_y k_x k_z \end{bmatrix} \tag{17}$$

$$= \begin{bmatrix} 0, & k_z(k_x^2 + k_z^2 + k_y^2), & -k_y(k_z^2 + k_y^2 + k_x^2) \\ -k_z(k_z^2 + k_y^2 + k_x^2), & 0, & k_x(k_z^2 + k_y^2 + k_x^2) \\ k_y(k_z^2 + k_y^2 + k_x^2), & -k_x(k_y^2 + k_x^2 + k_z^2), & 0 \end{bmatrix}. \tag{18}$$

Use the fact, $k_x^2 + k_y^2 + k_z^2 = \|\hat{\mathbf{k}}\|^2 = 1$.

$$K^3 = \begin{bmatrix} 0 & k_z & -k_y \\ -k_z & 0 & k_x \\ k_y & -k_x & 0 \end{bmatrix} = -\begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix} = -K \tag{19}$$

(Hint for Problem 4: Choose any vector $\mathbf{x} \neq \gamma\hat{\mathbf{k}}$. Draw the following three vectors: (1) $\mathbf{a} = \hat{\mathbf{k}} \times \mathbf{x}$, (2) $\mathbf{b} = \hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times \mathbf{x})$, (3) $\mathbf{c} = \hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times \mathbf{x}))$. Compare vectors $\mathbf{c}$ and $\mathbf{a}$. What do you observe?)

Problem 5 *Derive the rotation matrix corresponding to the Euler angles (α, β, γ) representation $R = R_x(\alpha)R_y(\beta)R_z(\gamma)$. Also derive an expression to convert the rotation matrix back to Euler angles. (20 marks).*

Solution:

$$R = R_x(\alpha)R_y(\beta)R_z(\gamma) \tag{20}$$

$$\implies \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha \\ 0 & s_\alpha & c_\alpha \end{bmatrix} \begin{bmatrix} c_\beta & 0 & s_\beta \\ 0 & 1 & 0 \\ -s_\beta & 0 & c_\beta \end{bmatrix} \begin{bmatrix} c_\gamma & -s_\gamma & 0 \\ s_\gamma & c_\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{21}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\alpha & -s_\alpha \\ 0 & s_\alpha & c_\alpha \end{bmatrix} \begin{bmatrix} c_\beta c_\gamma & -c_\beta s_\gamma & s_\beta \\ s_\gamma & c_\gamma & 0 \\ -s_\beta c_\gamma & s_\beta s_\gamma & c_\beta \end{bmatrix} \tag{22}$$

$$\implies \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c_\beta c_\gamma & -c_\beta s_\gamma & s_\beta \\ c_\alpha s_\gamma + s_\alpha s_\beta c_\gamma & c_\alpha c_\gamma - s_\alpha s_\beta s_\gamma & -s_\alpha c_\beta \\ s_\alpha s_\gamma - c_\alpha s_\beta c_\gamma & s_\alpha c_\gamma + c_\alpha s_\beta s_\gamma & c_\alpha c_\beta \end{bmatrix} \tag{23}$$

$$\beta = \sin^{-1}(-r_{13}) \in [-\pi/2, \pi/2] \tag{24}$$

$$\gamma = \text{arctan2}(-r_{12}, r_{11}) \tag{25}$$

$$\alpha = \text{arctan2}(-r_{23}, r_{33}) \tag{26}$$