Midterm 1 Review

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Contents

1 Linear algebra review **1**
- 1.1 Matrix operations 2
  - 1.1.1 Transpose 2
  - 1.1.2 Vector dot product 3
  - 1.1.3 Matrix multiplication 3
  - 1.1.4 Transpose of matrix multiplication 3

2 Trignometry review **4**

3 Triangle law of vector addition **4**

4 2D Rotation matrix **4**

5 2D Transformation matrix **6**

6 3D Rotation matrix from Euler angles **7**

7 3D Transformation matrix **7**

8 Axis-angle representation **7**

9 Denavit-Hartenberg transformations **7**

10 Camera projection model **7**

11 Linear least squares or Pseudo-inverse **7**

1 Linear algebra review

Definition 1 (Matrix)**.** *A real matrix A with n rows and m columns is defined as a set of real numbers $\{a_{11}, a_{12}, \dots, a_{nm}\}$, arranged in an 2D grid with n rows and m columns :*

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix} \tag{1}$$

The set of all possible real matrices with n rows and m columns is denoted as $\mathbb{R}^{n\times m}$, where $\mathbb{R}$ denotes the set of all real numbers.

Any matrix A with with n rows and m columns is said to lie in the set of $\mathbb{R}^{n\times m}$. $A \in \mathbb{R}^{n\times m}$ is read aloud as "A lies in the set of all n cross m real matrices".

Definition 2 (Vector or Column vector)**.** *A column vector or a vector $\mathbf{x}$ is a matrix with only one column.*

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \tag{2}$$

The set of all possible real vectors with n rows is denoted as $\mathbb{R}^{n\times 1}$ or more simply $\mathbb{R}^n$.

A vector is denoted by bold-font small letter, for example, $\mathbf{x}, \mathbf{y}, \mathbf{z}$. A matrix is denoted by capital letters, A, B, M, P, K.

A matrix $A \in \mathbb{R}^{n\times m}$ is often denoted a set m column vectors of dimension $n \times 1$,

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_m \end{bmatrix},$$
$$\text{where } \mathbf{a}_i = \begin{bmatrix} a_{i1} \\ a_{i2} \\ \vdots \\ a_{in} \end{bmatrix}, \qquad \text{for all } i \in \{1, \dots, m\}. \tag{3}$$

A block matrix is a matrix denoted in terms of other matrices,

$$A = \left[\begin{array}{ccc|ccc} b_{11} & \dots & b_{1q} & c_{11} & \dots & c_{1r} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ b_{p1} & \dots & b_{pq} & c_{1s} & \dots & c_{sr} \\ \hline e_{11} & \dots & e_{1v} & d_{11} & \dots & d_{1x} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ e_{u1} & \dots & e_{uv} & d_{1w} & \dots & d_{wx} \end{array}\right] \tag{4}$$

$$= \begin{bmatrix} B & C \\ E & D \end{bmatrix}, \text{ where } B, C, E, D \text{ are matrices.} \tag{5}$$

Definition 3 (Square matrix)**.** *A matrix is said to be square if its number of columns is same as the number of rows. That is matrix $A \in \mathbb{R}^{n\times m}$ is said to be square matrix if $m = n$.*

Definition 4 (Diagonal of a square matrix)**.** *Let A be a square matrix $A \in \mathbb{R}^{n\times n}$ with entries:*

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \tag{6}$$

The diagonal of a square matrix A is defined to be the vector

$$diag(A) = \begin{bmatrix} a_{11} \\ a_{22} \\ \vdots \\ a_{nn} \end{bmatrix}$$

Definition 5 (Identity matrix)**.** *An identity matrix I of size n is a square matrix with all its diagonal entries as 1 and non-diagonal entries as 0.*

$$I = \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} \tag{7}$$

1.1 Matrix operations

1.1.1 Transpose

Definition 6 (Transpose)**.** *The matrix transpose $A^\top$ of a matrix A is defined as a matrix where rows of matrix A are the columns of $A^\top$ and vice-versa.*

$$A^\top = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{1n} \\ a_{12} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1m} & a_{2m} & \dots & a_{nm} \end{bmatrix} \tag{8}$$

In the matrix as set of m column vectors notation, the transpose is written as m row vectors $\mathbf{a}_i^\top$,

$$A^\top = \begin{bmatrix} \mathbf{a}_1^\top \\ \mathbf{a}_2^\top \\ \vdots \\ \mathbf{a}_m^\top \end{bmatrix}, \quad \mathbf{a}_i^\top = \begin{bmatrix} a_{i1} & a_{i2} & \dots & a_{in} \end{bmatrix}, \quad \text{for all } i \in \{1, \dots, n\}. \tag{9}$$

1. If A has $n$rows and m columns, then $A^\top$ has m rows and n columns. If $A \in \mathbb{R}^{n\times m}$, then $A^\top \in \mathbb{R}^{m\times n}$.

2. The transpose of a transpose is matrix itself. $(A^\top)^\top = A$.

3. The transpose of a block matrix is block-wise transpose of each matrix,

$$\begin{bmatrix} B & C \\ E & D \end{bmatrix}^\top = \begin{bmatrix} B^\top & E^\top \\ C^\top & D^\top \end{bmatrix}$$

Definition 7 (Row vector)**.** *A row vector is Y is matrix with only one row*

$$Y = \begin{bmatrix} y_1 & y_2 & \dots & y_n \end{bmatrix} \tag{10}$$

It is common to denote row vectors as tranpose of a column vector. For example, the matrix Y shown above is typically represented $\mathbf{y}^\top$, where $\mathbf{y}$ is a column vector.

$$Y = \mathbf{y}^\top \qquad \text{where } \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \tag{11}$$

1.1.2 Vector dot product

Before we define general matrix multiplication, it is easier to define matrix multiplication between a row vector and a column vector $\mathbf{x}^\top \in \mathbb{R}^{1\times n}$ and $\mathbf{y} \in \mathbb{R}^{n\times 1}$

$$\mathbf{x}^\top \mathbf{y} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = \sum_{i=1}^{n} x_i y_i \tag{12}$$

where $\mathbf{x}^\top = \begin{bmatrix} x_1 & \dots & x_n \end{bmatrix}$

and $\mathbf{y} = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$.

Note that $\mathbf{x}^\top \mathbf{y}$ is same as the vector dot product or the vector inner-product,

$$\mathbf{x}^\top \mathbf{y} = \mathbf{x} \cdot \mathbf{y} = \|\mathbf{x}\| \|\mathbf{y}\| cos(\theta) = \mathbf{y}^\top \mathbf{x}, \tag{13}$$

where θ is the angle between vectors $\mathbf{x}$ and $\mathbf{y}$ and the vector norm or euclidean norm $\|.\|$ is defined as

$$\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} = \sqrt{\mathbf{x}^\top \mathbf{x}} \tag{14}$$

Definition 8 (Unit vector)**.** *A unit vector, typically denoted with a hat,* $\hat{\mathbf{x}}$ *is a vector with euclidean norm as 1. That is* $\|\hat{\mathbf{x}}\| = 1$ *or equivalently* $\mathbf{x}^\top \mathbf{x} = 1$.

Definition 9 (Orthogonal vectors)**.** *Two vectors,* $\mathbf{x} \in \mathbb{R}^n$ *and* $\mathbf{y} \in \mathbb{R}^n$ *are said to be orthogonal if and only if their dot product is zero* $\mathbf{x}^\top \mathbf{y} = 0$.

Definition 10 (Orthonormal vectors)**.** *A set of vectors,* $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n \in \mathbb{R}^n$ *are said to be orthonormal if and only if they are all unit vectors* $\mathbf{x}_i^\top \mathbf{x}_i = 1$ *and they are pair-wise orthogonal,* $\mathbf{x}_i^\top \mathbf{x}_j = 0$ *for all* $i \neq j$.

1.1.3 Matrix multiplication

The matrix multiplication between matrix $A \in \mathbb{R}^{n\times m}$ and matrix $B \in \mathbb{R}^{m\times p}$ (note that A has m columns while B has m rows; the only case when matrix multiplication is defined) is easier defined if matrix A is written in terms of row vectors while matrix B is written in terms of column vectors. Let the matrix A is written in terms of row vectors $\mathbf{a}_i^\top \in \mathbb{R}^{1\times m}$ and the matrix B is written in terms of column vectors $\mathbf{b}_i \in \mathbb{R}^{m\times 1}$. Then the matrix multiplication $AB \in \mathbb{R}^{n\times p}$ is defined as the matrix,

$$AB = \begin{bmatrix} \mathbf{a}_1^\top \\ \mathbf{a}_2^\top \\ \vdots \\ \mathbf{a}_n^\top \end{bmatrix} \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \dots & \mathbf{b}_p \end{bmatrix} \tag{15}$$

$$= \begin{bmatrix} \mathbf{a}_1^\top \mathbf{b}_1 & \mathbf{a}_1^\top \mathbf{b}_2 & \dots & \mathbf{a}_n^\top \mathbf{b}_p \\ \mathbf{a}_2^\top \mathbf{b}_1 & \mathbf{a}_2^\top \mathbf{b}_2 & \dots & \mathbf{a}_n^\top \mathbf{b}_p \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_n^\top \mathbf{b}_1 & \mathbf{a}_n^\top \mathbf{b}_2 & \dots & \mathbf{a}_n^\top \mathbf{b}_p \end{bmatrix} \tag{16}$$

Block matrix multiplication Block matrix multiplication works in a similar way as scalar multiplication as long as sub-matrix multiplication is properly defined,

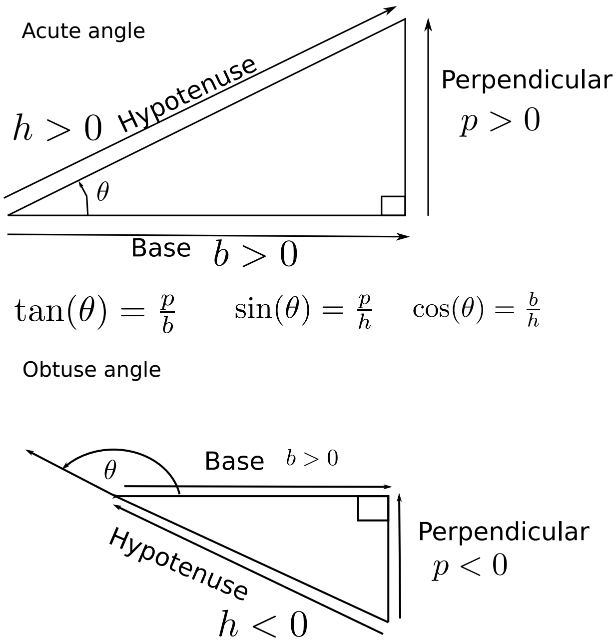
$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} P & Q \\ R & S \end{bmatrix} = \begin{bmatrix} AP + BR & AQ + BS \\ CP + DR & CQ + DS \end{bmatrix} \tag{17}$$

Definition 11 (Orthogonal matrices)**.** *A square matrix* A *is said to be orthogonal if and only if* $A^\top A = I$

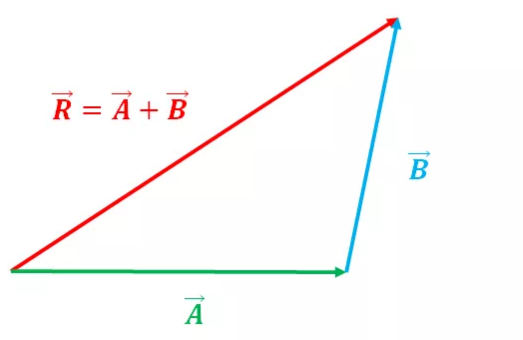
1.1.4 Transpose of matrix multiplication

$$(AB)^\top = B^\top A^\top$$

2 Trignometry review



3 Triangle law of vector addition



4 2D Rotation matrix

Definition 12 (2D Cartesian Coordinate frame)**.** *A 2D cartesian coordinate frame is defined as a set of mutually orthogonal unit vectors $\hat{\mathbf{x}} \in \mathbb{R}^2$ and $\hat{\mathbf{y}} \in \mathbb{R}^2$ called the basis vectors $B = [\hat{\mathbf{x}}, \hat{\mathbf{y}}]$ along with an origin $\mathbf{o} \in \mathbb{R}^2$. Thus the tuple $(B, \mathbf{o})$ form a coordinate frame. A coordinate frame is denoted by curly braces around it, for example, $\{C\}$ or $\{W\}$.*

Example 1 (2D Coordinate frame)**.** *The figure 1 contains two coordinate frames the one shown in red and the one shown in green. Both have the same origin, but different basis vectors. The $\{W\}$ coordinate frame shown in green has basis vectors $B_w = [\hat{\mathbf{x}}_w, \hat{\mathbf{y}}_w]$. The same notation is used for the $\{C\}$ coordinate frame $B_c = [\hat{\mathbf{x}}_c, \hat{\mathbf{y}}_c]$. Note that the basis vectors of $\{C\}$ coordinate frame can be expressed in terms of $\{W\}$ coordinate frame by triangle law of vector addition,*

$$\begin{aligned}\hat{\mathbf{x}}_c &= |\overrightarrow{OA}|\hat{\mathbf{x}}_w + |\overrightarrow{AB}|\hat{\mathbf{y}}_w \\ \hat{\mathbf{y}}_c &= -|\overrightarrow{PQ}|\hat{\mathbf{x}}_w + |\overrightarrow{OP}|\hat{\mathbf{y}}_w\end{aligned} \tag{18}$$

In the triangle ΔOAB (Fig 1),

$$\cos(\theta) = \frac{|\overrightarrow{OA}|}{|\overrightarrow{OB}|} = \frac{|\overrightarrow{OA}|}{\|\hat{\mathbf{x}}_c\|} = |\overrightarrow{OA}| \tag{19}$$

$$\sin(\theta) = \frac{|\overrightarrow{AB}|}{|\overrightarrow{OB}|} = \frac{|\overrightarrow{AB}|}{\|\hat{\mathbf{x}}_c\|} = |\overrightarrow{AB}| \tag{20}$$

Similarly in the right triangle ΔOPQ (Fig 1),

$$\cos(\theta) = \frac{|\overrightarrow{OP}|}{|\overrightarrow{OQ}|} = \frac{|\overrightarrow{OP}|}{\|\hat{\mathbf{y}}_c\|} = |\overrightarrow{OP}| \tag{21}$$

$$\sin(\theta) = \frac{|\overrightarrow{PQ}|}{|\overrightarrow{OQ}|} = \frac{|\overrightarrow{PQ}|}{\|\hat{\mathbf{y}}_c\|} = |\overrightarrow{PQ}| \tag{22}$$

Putting these values back in (18), we get,

$$\begin{aligned}\hat{\mathbf{x}}_c &= cos(\theta)\hat{\mathbf{x}}_w + sin(\theta)\hat{\mathbf{y}}_w \\ \hat{\mathbf{y}}_c &= -sin(\theta)\hat{\mathbf{x}}_w + cos(\theta)\hat{\mathbf{y}}_w\end{aligned} \tag{23}$$

These equations can be written in matrix notation as,

$$\begin{aligned}\hat{\mathbf{x}}_c &= \begin{bmatrix}\hat{\mathbf{x}}_w & \hat{\mathbf{y}}_w\end{bmatrix}\begin{bmatrix}cos(\theta)\\ sin(\theta)\end{bmatrix} = B_w\begin{bmatrix}cos(\theta)\\ sin(\theta)\end{bmatrix} \\ \hat{\mathbf{y}}_c &= \begin{bmatrix}\hat{\mathbf{x}}_w & \hat{\mathbf{y}}_w\end{bmatrix}\begin{bmatrix}-sin(\theta)\\ cos(\theta)\end{bmatrix} = B_w\begin{bmatrix}-sin(\theta)\\ cos(\theta)\end{bmatrix}\end{aligned} \tag{24}$$

The full basis matrix of coordinate frame $\{C\}$ can be written as

$$\begin{aligned} B_c &= \begin{bmatrix}\hat{\mathbf{x}}_c & \hat{\mathbf{y}}_c\end{bmatrix} \\ &= \begin{bmatrix} B_w \begin{bmatrix} cos(\theta) \\ sin(\theta)\end{bmatrix} & B_w \begin{bmatrix} -sin(\theta) \\ cos(\theta)\end{bmatrix}\end{bmatrix} \\ &= B_w \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta)\end{bmatrix} \end{aligned} \tag{25}$$

Definition 13 (2D Coordinates of a point)**.** *The coordinate of a point $\mathbf{p}$ in a given coordinate frame $\{W\}$ with basis vectors $B_w = [\hat{\mathbf{x}}_w, \hat{\mathbf{y}}_w]$ and origin $\mathbf{o}_w = \begin{bmatrix} o_x \\ o_y \end{bmatrix}$ is defined as the vector $\mathbf{p}_w = \begin{bmatrix} p_{wx} \\ p_{wy} \end{bmatrix}$ such that,*

$$\begin{aligned} \mathbf{p} &= (p_{wx} + o_x)\hat{\mathbf{x}}_w + (p_{wy} + o_y)\hat{\mathbf{y}}_w \\ &= \begin{bmatrix}\hat{\mathbf{x}}_w & \hat{\mathbf{y}}_w\end{bmatrix} \left(\begin{bmatrix} p_{wx} \\ p_{wy}\end{bmatrix} + \begin{bmatrix} o_x \\ o_y \end{bmatrix}\right) \\ &= B_w(\mathbf{p}_w + \mathbf{o}_w) \end{aligned} \tag{26}$$

Example 2 (Fig 1)**.** *The point $\mathbf{p}$ can be represented in coordinate frames $\{W\}$ and $\{C\}$. Let the projection on the basis $B_c = [\hat{\mathbf{x}}_c, \hat{\mathbf{y}}_c]$ be $\mathbf{p}_c$, while that on $B_w = [\hat{\mathbf{x}}_w, \hat{\mathbf{y}}_w]$ be $\mathbf{p}_w$. Since both the coordinate frames have same origin, we assume $\mathbf{o}_w = \mathbf{o}_c = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. We have*

$$\mathbf{p} = B_w \mathbf{p}_w = B_c \mathbf{p}_c \tag{27}$$

Theorem 1 (2D Rotation matrix)**.** *In a coordinate transformation as given in Fig 1, the coordinates in frame $\{C\}$, $\mathbf{p}_c$ can be converted into coordinates in frame $\{W\}$, $\mathbf{p}_w$ with the same origin by using a rotation matrix ${}^W R_C(\theta)$,*

$$\begin{aligned} \mathbf{p}_w &= {}^W R_C(\theta)\mathbf{p}_c \\ \text{where } {}^W R_C(\theta) &= \begin{bmatrix} cos(\theta) & -sin(\theta) \\ sin(\theta) & cos(\theta)\end{bmatrix} \end{aligned} \tag{28}$$

Proof. First note that the basis matrix of any coordinate frame $\{W\}$ is orthogonal,

$$\begin{aligned} B_w^\top B_w &= \begin{bmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}}\end{bmatrix}^\top \begin{bmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}}\end{bmatrix} \\ &= \begin{bmatrix}\hat{\mathbf{x}}^\top \\ \hat{\mathbf{y}}^\top\end{bmatrix} \begin{bmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}}\end{bmatrix} \\ &= \begin{bmatrix}\hat{\mathbf{x}}^\top\hat{\mathbf{x}} & \hat{\mathbf{x}}^\top\hat{\mathbf{y}} \\ \hat{\mathbf{y}}^\top\hat{\mathbf{x}} & \hat{\mathbf{y}}^\top\hat{\mathbf{y}}\end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \end{aligned} \tag{29}$$

Left-multiply $B_w^\top$ to both sides of (27)

$$B_w^\top B_w \mathbf{p}_w = B_w^\top B_c \mathbf{p}_c \tag{30}$$

Cancel out $B_w^\top \mathbf{o}$ on both sides. Replace $B_w^\top B_w = I$.

$$I\mathbf{p}_w = B_w^\top B_c \mathbf{p}_c \text{ or } \mathbf{p}_w = B_w^\top B_c \mathbf{p}_c \tag{31}$$

Substitute value of B_c from (25), to get

$$\mathbf{p}_w = B_w^\top \left(B_w \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta)\end{bmatrix}\right) \mathbf{p}_c. \tag{32}$$

Again use $B_w^\top B_w = I$ to get,

$$\mathbf{p}_w = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta)\end{bmatrix} \mathbf{p}_c. \tag{33}$$

Defining ${}^W R_C(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta)\end{bmatrix}$, we get the desired result.

□

Theorem 2 (Orthogonality of 2D Rotation matrices)**.** *All 2D rotation matrices are orthogonal $R^\top R = I$ have determinant as one $\det(R) = 1$. If any square matrix $A \in \mathbb{R}^{2\times 2}$ is orthogonal $A^\top A = I$ and has determinant 1, $\det(A) = 1$, then it is a valid rotation matrix.*

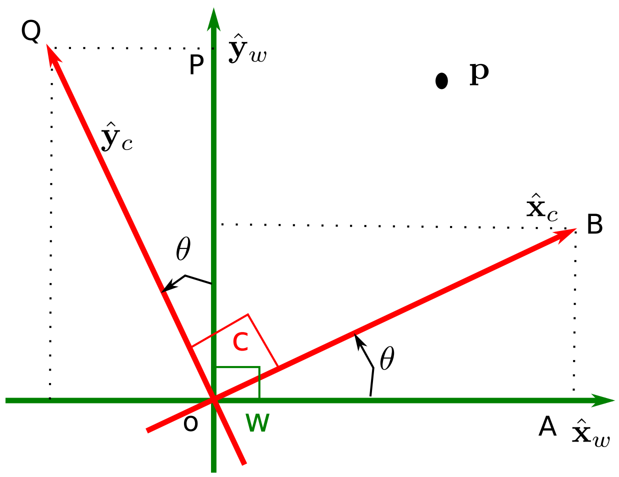


Figure 1: The coordinate frame $\{C\}$ is rotated around origin by an θ from coordinate frame $\{W\}$.

Proof.

$$\begin{aligned} R^\top R &= \begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix}^\top \begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix} \\ &= \begin{bmatrix}\cos(\theta) & \sin(\theta)\\ -\sin(\theta) & \cos(\theta)\end{bmatrix} \begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix} \\ &= \begin{bmatrix}\cos^2(\theta)+\sin^2(\theta) & -\cos(\theta)\sin(\theta)+\sin(\theta)\cos(\theta)\\ -\sin(\theta)\cos(\theta)+\cos(\theta)\sin(\theta) & \sin^2(\theta)+\cos^2(\theta)\end{bmatrix} \\ &= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} \end{aligned} \tag{34}$$

$$\begin{aligned} \det(R) &= \det \begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix} \\ &= \cos^2(\theta)+\sin^2(\theta) = 1 \end{aligned} \tag{35}$$

Denote the columns of square matrix A which is orthogonal with determinant 1 as $A = [\mathbf{a}_1, \mathbf{a}_2]$. Since A is orthogonal, we have

$$\begin{aligned} A^\top A &= \begin{bmatrix}\mathbf{a}_1^\top\\ \mathbf{a}_2^\top\end{bmatrix} \begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2\end{bmatrix} = \begin{bmatrix}\mathbf{a}_1^\top\mathbf{a}_1 & \mathbf{a}_1^\top\mathbf{a}_2\\ \mathbf{a}_2^\top\mathbf{a}_1 & \mathbf{a}_2^\top\mathbf{a}_2\end{bmatrix} \\ &= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}. \end{aligned} \tag{36}$$

This implies that $\mathbf{a}_1$ and $\mathbf{a}_2$ are mutually orthogonal unit vectors. Let $\mathbf{a}_1 = [\cos(\theta), \sin(\theta)]$ because any 2D unit vector can be written in cos,sin form, where $\theta = \text{atan2}(a_{12}, a_{11})$. Next we know that $\mathbf{a}_1^\top\mathbf{a}_2 = 0$ and that $\mathbf{a}_2$ is unit vector. For any unit 2D vector $[u, v]^\top$, there are only two unit vectors perpendicular to it $[-v, u]^\top$ and $[v, -u]^\top$. Then we have only two options for $\mathbf{a}_2$ are either $[-\sin(\theta), \cos(\theta)]$ or $[\sin(\theta), -\cos(\theta)]$. But we also know that the determinant of A is 1. The second option for $\mathbf{a}_2$ leads to determinant of -1.

$$\det\begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2\end{bmatrix} = \det\begin{bmatrix}\cos(\theta) & \sin(\theta)\\ \sin(\theta) & -\cos(\theta)\end{bmatrix} = -1 \tag{37}$$

Hence, we have

$$A = \begin{bmatrix}\mathbf{a}_1 & \mathbf{a}_2\end{bmatrix} = \begin{bmatrix}\cos(\theta) & -\sin(\theta)\\ \sin(\theta) & \cos(\theta)\end{bmatrix} = R(\theta)$$

□

5 2D Transformation matrix

To consider the rotation and translation case, we consider the case shown in Fig 2. We have an intermediate frame $\{I\}$ which has only rotation from $\{C\}$ frame. We assume that basis vectors $\{I\}$ are parallel to $\{W\}$ which make it translation only conversion. We can convert from $\mathbf{p}_c$ to $\mathbf{p}_I$ using the rotation matrix derived in the previous section,

$$\mathbf{p}_I = B_I^{-1} B_c \mathbf{p}_c = R(\theta)\mathbf{p}_c. \tag{38}$$

We can account for the translation of the frame $\mathbf{p}_I$ by noticing that the coordinate frames only differ in origin, such that $B_c\mathbf{o}_c = B_w(\mathbf{o}_w + {}^w\mathbf{t}_c)$, where the translation ${}^w\mathbf{t}_c$ is measured in world coordinate frame.

$$\begin{aligned} &\mathbf{p} = B_c(\mathbf{p}_c + \mathbf{o}_c) = B_w(\mathbf{p}_w + \mathbf{o}_w) \\ \implies &B_c\mathbf{p}_c + B_c\mathbf{o}_c = B_w\mathbf{p}_w + B_w\mathbf{o}_w \\ \implies &B_c\mathbf{p}_c + (B_c\mathbf{o}_c - B_w\mathbf{o}_w) = B_w\mathbf{p}_w \\ \implies &B_c\mathbf{p}_c + B_w{}^w\mathbf{t}_c = B_w\mathbf{p}_w \\ \implies &B_w^\top B_c\mathbf{p}_c + {}^w\mathbf{t}_c = \mathbf{p}_w \\ \implies &\mathbf{p}_w = R(\theta)\mathbf{p}_c + {}^w\mathbf{t}_c \end{aligned} \tag{39}$$

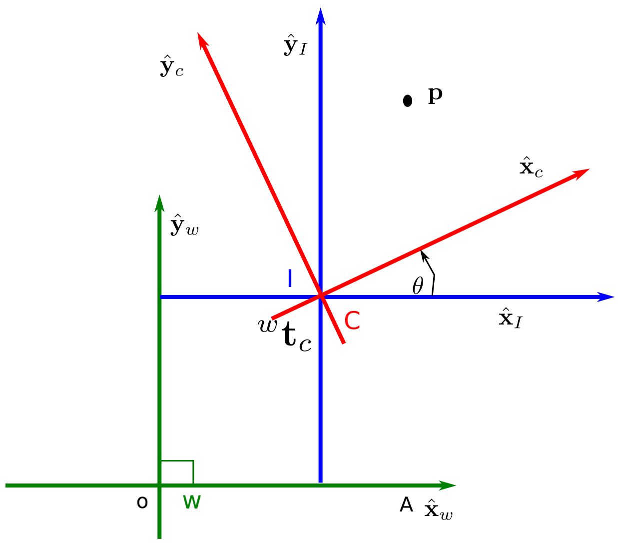


Figure 2: The coordinate frame $\{C\}$ is rotated around origin by an θ from coordinate frame $\{W\}$ and then shifted by translation ${}^{w}\mathbf{t}_c$.

6 3D Rotation matrix from Euler angles

7 3D Transformation matrix

8 Axis-angle representation

9 Denavit-Hartenberg transformations

10 Camera projection model

11 Linear least squares or Pseudo-inverse

List of Theorems

1 Definition (Matrix) 1
2 Definition (Vector or Column vector) . 1
3 Definition (Square matrix) 2
4 Definition (Diagonal of a square matrix) 2
5 Definition (Identity matrix) 2
6 Definition (Transpose) 2
7 Definition (Row vector) 2
8 Definition (Unit vector) 3
9 Definition (Orthogonal vectors) 3
10 Definition (Orthonormal vectors) . . . 3
11 Definition (Orthogonal matrices) . . . 3
12 Definition (2D Cartesian Coordinate frame) 4
13 Definition (2D Coordinates of a point) 5