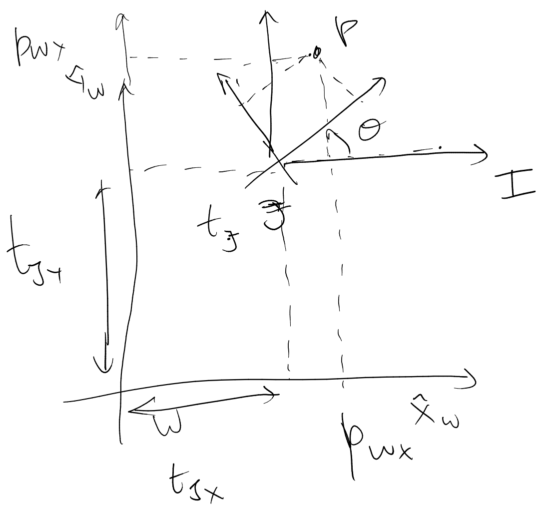
Review of topics covered so far

1. Rotation in 2D
2. Translation in 2D
3. Transformation (rotation+translation) in 2D
4. Rotation in 3D
5. Euler angles: Euler angle to rotation matrix
6. Axis angle representation: Rodrigues formula

$$\underline{p}_J = \begin{bmatrix} p_{Jx} \\ p_{Jy} \end{bmatrix}$$

$$\underline{p}_I = {}^{I}R_J(\theta)\,\underline{p}_J$$



$$\underline{p}_W = \underline{t}_J + \underline{p}_I$$

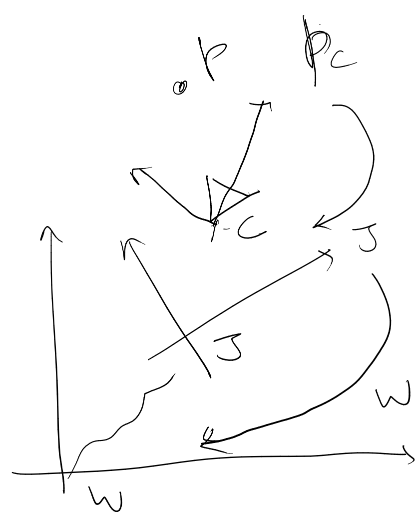
$$J \xrightarrow{\text{Rot only}} I \xrightarrow{\text{trans only}} W$$

$$\boxed{\underset{2\times 1}{\underline{p}_W} = {}^{W}R_J(\theta)\,\underline{p}_J + \underline{t}_J}$$

Homogeneous

$$\begin{bmatrix} p_W \\ 1 \end{bmatrix}_{3\times 1} = \underbrace{\begin{bmatrix} {}^{W}R_J(\theta) & t_J \\ \underline{0}^T & 1 \end{bmatrix}}_{{}^{W}T_J} \begin{bmatrix} \underline{p}_J \\ 1 \end{bmatrix}$$

$$\boxed{\vec{\underline{p}}_W = {}^{W}T_J\,\vec{\underline{p}}_J}$$



$$\vec{p}_J = {}^{J}T_C\,\vec{\underline{p}}_C$$

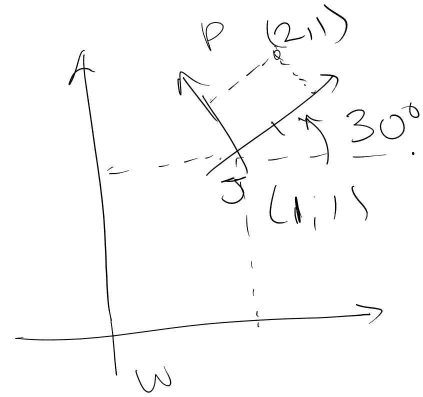
$$\vec{\underline{p}}_W = {}^{W}T_J\,\vec{\underline{p}}_J$$

$$\vec{p}_w = \underbrace{{}^{w}T_{J}\;{}^{J}T_{c}}\;\underline{\vec{p}_c}$$

$$\vec{p}_w = \boxed{{}^{w}T_{c}}\;\underline{\vec{p}_c} \qquad 3\times 3 \text{ in 2D}$$

$$\text{transformation} = \begin{cases} \text{position} \\ + \\ \text{orientation} \end{cases}$$

$$\vec{p}_J = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$



$$\theta = 30^\circ$$

$$t_J = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\vec{p}_w = ?$$

$${}^{w}_{J}R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} \cos(30) & -\sin(30) \\ \sin(30) & \cos(30) \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}$$

$$\begin{bmatrix} \vec{p}_w \\ \hline 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 & 1 \\ 1/2 & \sqrt{3}/2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \sqrt{3} - 1/2 + 1 \\ 1 + \sqrt{3}/2 + 1 \\ 1 \end{bmatrix}$$

0.866

$$= \begin{bmatrix} 2.232 \\ 2.866 \\ 1 \end{bmatrix} \qquad \underline{\vec{p}_w} = \begin{bmatrix} 2.232 \\ 2.866 \end{bmatrix}$$

3D → 3DOF Rotation + 3DOF translat

↓

Euler angles

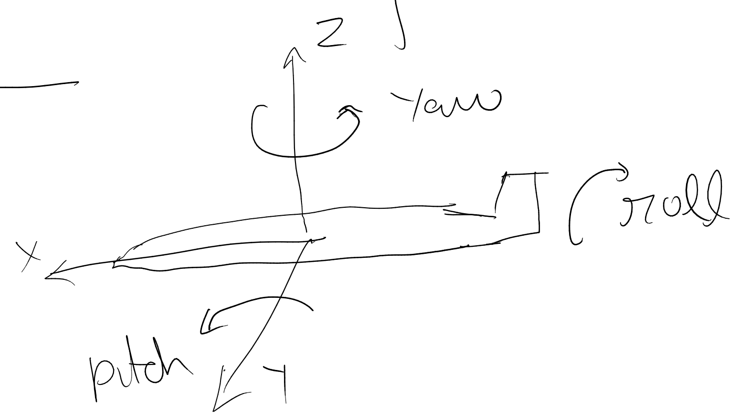
XYZ XZY XYX

YZX XZX

ZXY YXY

$\underbrace{}_{6}$ $\underbrace{}_{6}$

12 variation



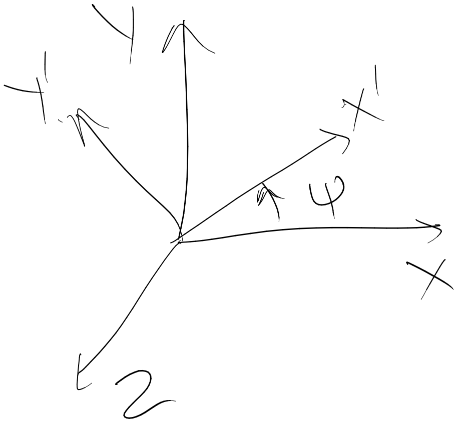
XYX $\underline{XYZ}$

roll pitch yaw

θ ϕ ψ

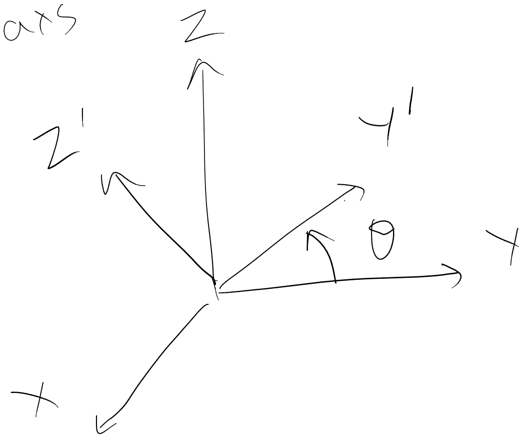
Rot in 2D → $\underline{\text{Rot in 3D along}}$ axis

Principal Rotation matrice

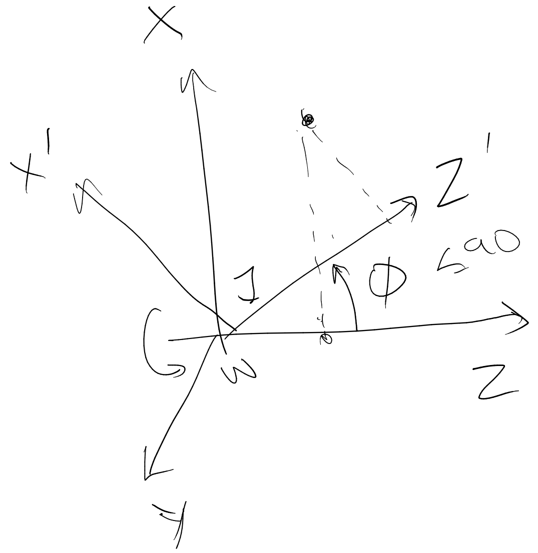


$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{R_z(\psi)} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix}$$

x-axis



$$\begin{bmatrix} X' \\ Y' \\ Z' \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix}}_{R_x(\theta)} \begin{bmatrix} X \\ y \\ Z \end{bmatrix}$$



$$\begin{bmatrix} x' \\ y' \\ z \end{bmatrix} = \begin{bmatrix} \cos(\phi) & 0 & \sin(\phi) \\ 0 & 1 & 0 \\ -\sin(\phi) & 0 & \cos(\phi) \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$R_Y(\phi)$

Roll - pitch - yaw

$$\boxed{R(\theta, \phi, \psi) = R_Z(\psi)\, R_Y(\phi)\, R_X(\theta)}$$

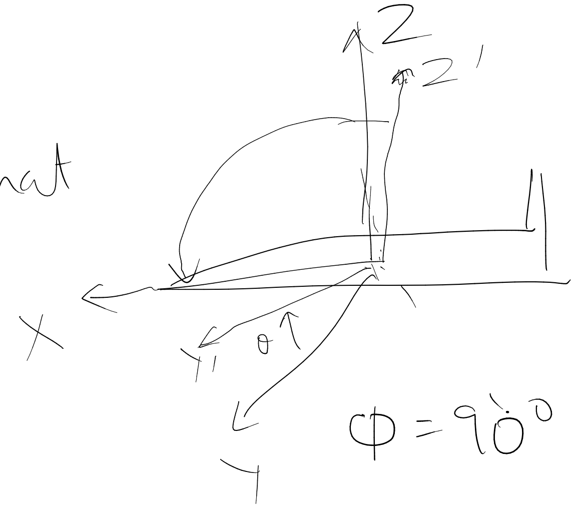
Euler angle XYZ (roll pitch yaw) $\rightarrow$ Rotation matrix

Limitation? : Gimbal lock

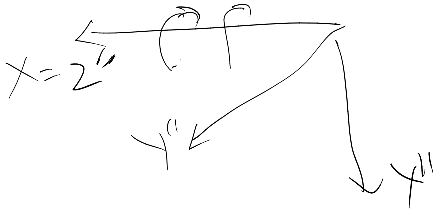
X Y Z $\quad\quad$ X Y Z
$\uparrow$ $\quad\quad$ $\uparrow$
$\pi/2$ $\quad\quad$ $\pi/2$

Euler angle $\rightarrow$ Rot mat

$\leftarrow$



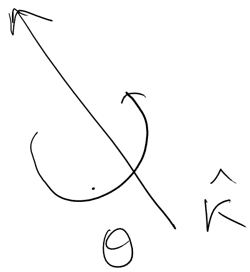
$\phi = 90°$



✓ Axis-angle

$$\theta, \quad \hat{k} \qquad = \theta \hat{k}$$

$$= \underline{k}$$



$$\|\underline{k}\| = \theta$$

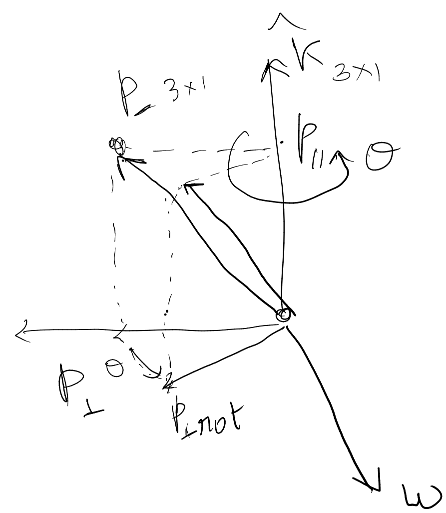
$$\hat{k} = \frac{\underline{k}}{\|k\|}$$

✓ Quaternion

$$\left[\underbrace{\cos\tfrac{\theta}{2}}_{1},\ \underbrace{\sin\tfrac{\theta}{2}\,\hat{k}}_{3}\right]^T_{4\times 1}$$

Axis angle $\longrightarrow$ Rot matrix

Rodrigue formula



Cross product

$\underline{c} = \underline{a} \times \underline{b}$ $\qquad \underline{c} \perp \underline{a}$

$\qquad \underline{b} \perp \underline{c}$

$$|\underline{c}| = |a||b| \sin\theta$$

$$\boxed{\underline{c} = \lfloor a \rfloor_{\times} b}$$

$$\underline{p} = p_{\parallel} + p_{\perp}$$