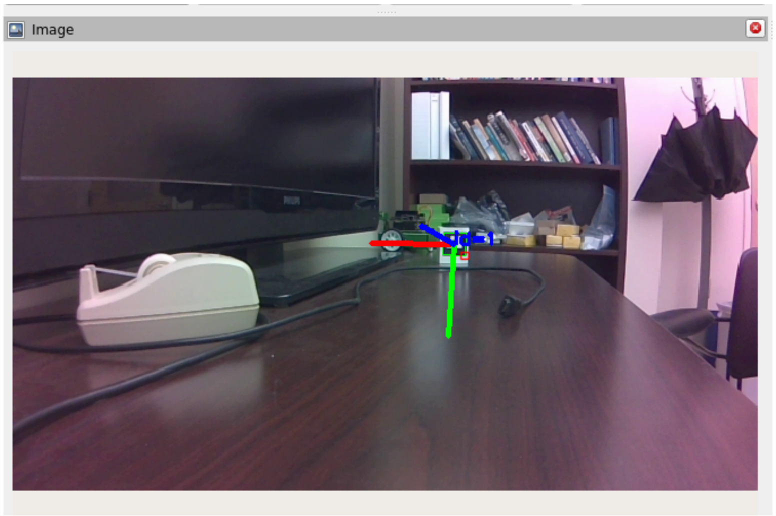
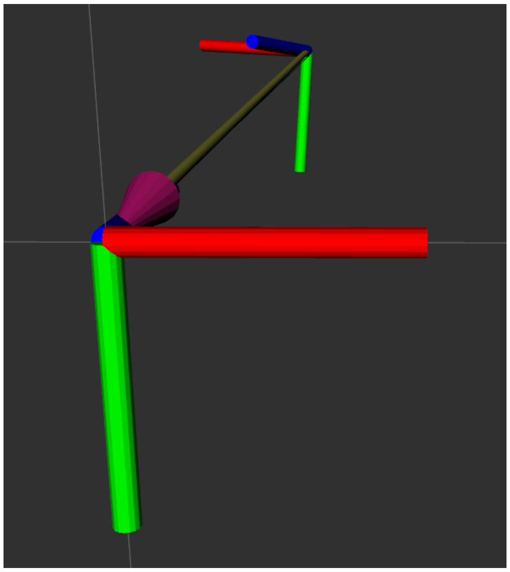
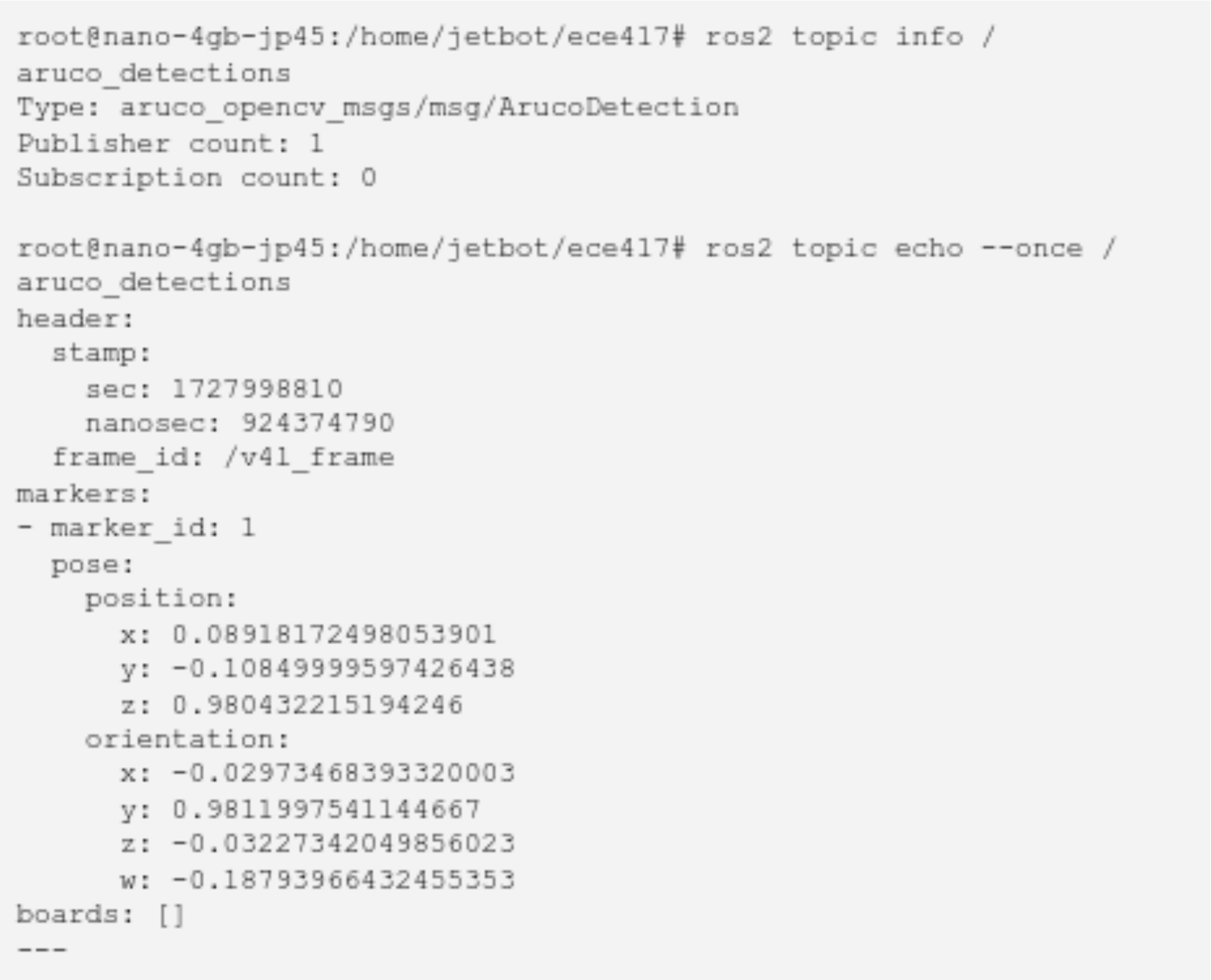
Rotations and translations/Coordinate transformations







```
root@nano-4gb-jp45:/home/jetbot/ece417# ros2 topic info /
aruco_detections
Type: aruco_opencv_msgs/msg/ArucoDetection
Publisher count: 1
Subscription count: 0

root@nano-4gb-jp45:/home/jetbot/ece417# ros2 topic echo --once /
aruco_detections
header:
  stamp:
    sec: 1727998810
    nanosec: 924374790
  frame_id: /v4l_frame
markers:
- marker_id: 1
  pose:
    position:
      x: 0.08918172498053901
      y: -0.10849999597426438
      z: 0.980432215194246
    orientation:
      x: -0.02973468393320003
      y: 0.9811997541144667
      z: -0.03227342049856023
      w: -0.18793966432455353
boards: []
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```
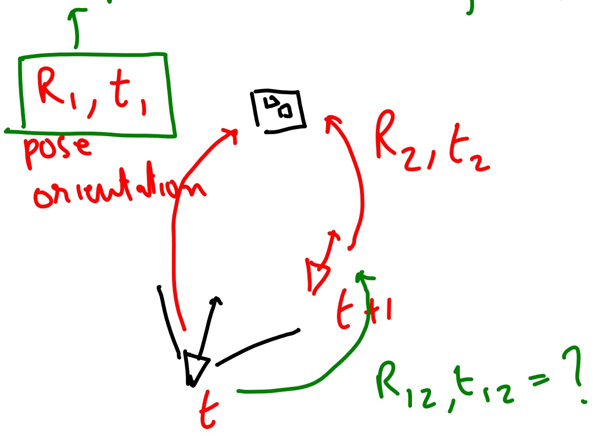

joint representation = Transformation matrix

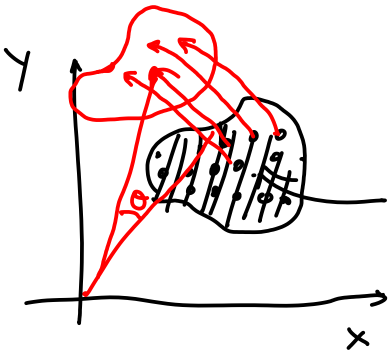


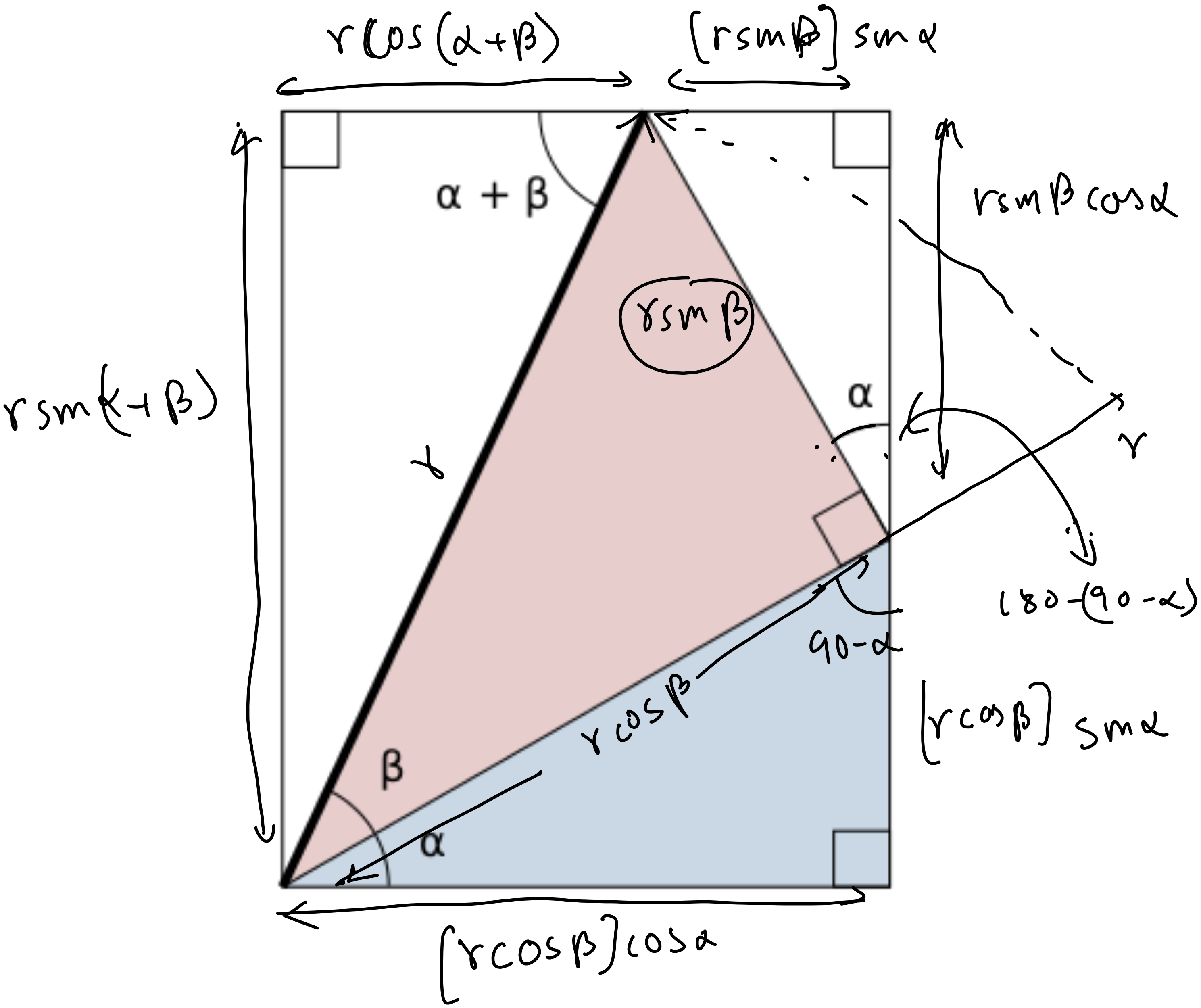
① Rigid bodies are sets of points

↳ Rotate

↳ Translate

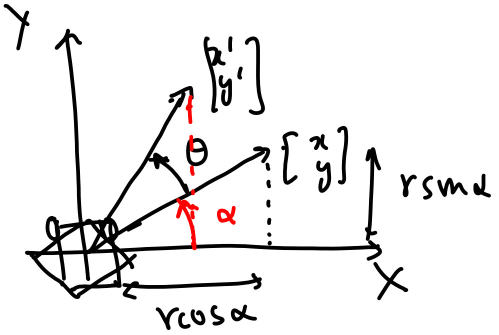
All points in the rigid body Rotate and translate by the same amount





2D Rotation

$\begin{bmatrix}x'\\y'\end{bmatrix} = ?$ in terms of θ and $\begin{bmatrix}x\\y\end{bmatrix}$



$x = r\cos\alpha$

$y = r\sin\alpha$ $\qquad r = \sqrt{x^2+y^2}$

$\underline{x'} = r\cos(\theta+\alpha)$

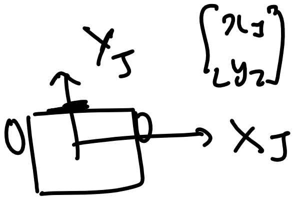
$= \underline{r}\cos\theta\,\underline{\cos\alpha} - \underline{r}\sin\theta\,\underline{\sin\alpha}$

$= x\cos\theta - y\sin\theta$

$\underline{y'} = r\sin(\theta+\alpha) = r\sin\theta\cos\alpha + r\cos\theta\sin\alpha = x\sin\theta + y\cos\theta$

$$\underbrace{\begin{bmatrix}x'\\y'\end{bmatrix}_{2\times1}}_{\underline{p}'} = \begin{bmatrix}x\cos\theta - y\sin\theta\\ x\sin\theta + y\cos\theta\end{bmatrix} = \underbrace{\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}_{2\times2}}_{R(\theta)}\underbrace{\begin{bmatrix}x\\y\end{bmatrix}_{2\times1}}_{\underline{p}}$$

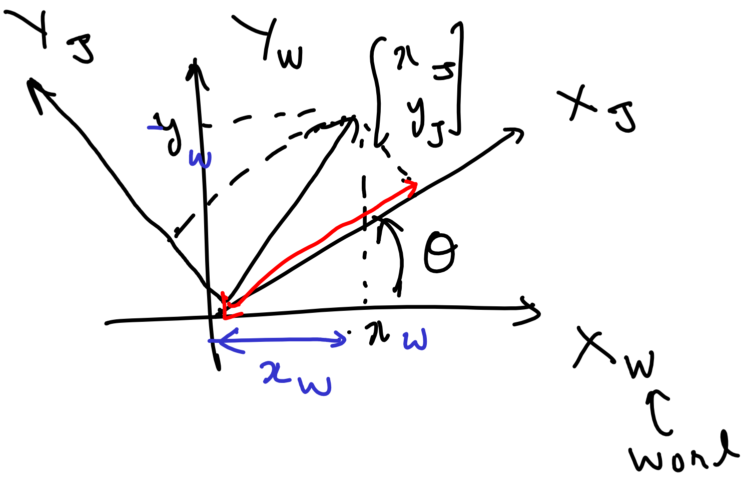
$$\underline{p}' = R(\theta)\,\underline{p}$$



$$\underbrace{\begin{bmatrix}x_w\\y_w\end{bmatrix}}_{\underline{p}_w} = \underbrace{\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}}_{{}^{w}R_{J}(\theta)}\underbrace{\begin{bmatrix}x_J\\y_J\end{bmatrix}}_{\underline{p}_J}$$

SOURCE

DESTINATION



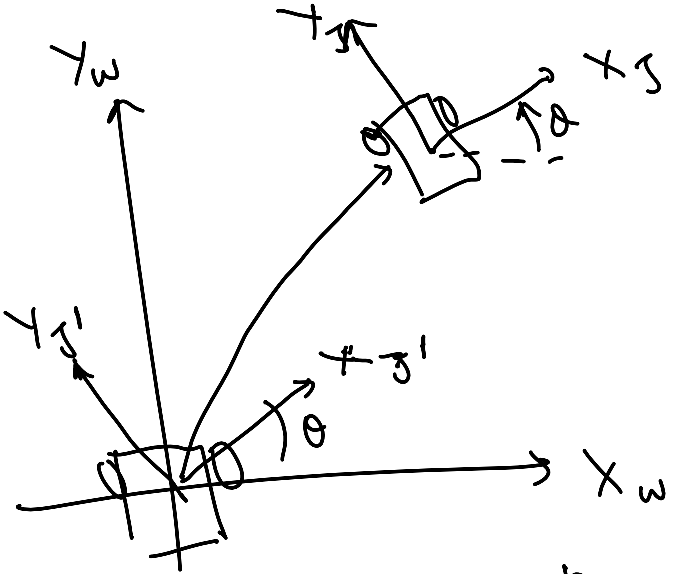
$$\boxed{\underline{p}_w = {}^{w}R_{J}(\theta)\,\underline{p}_J}$$

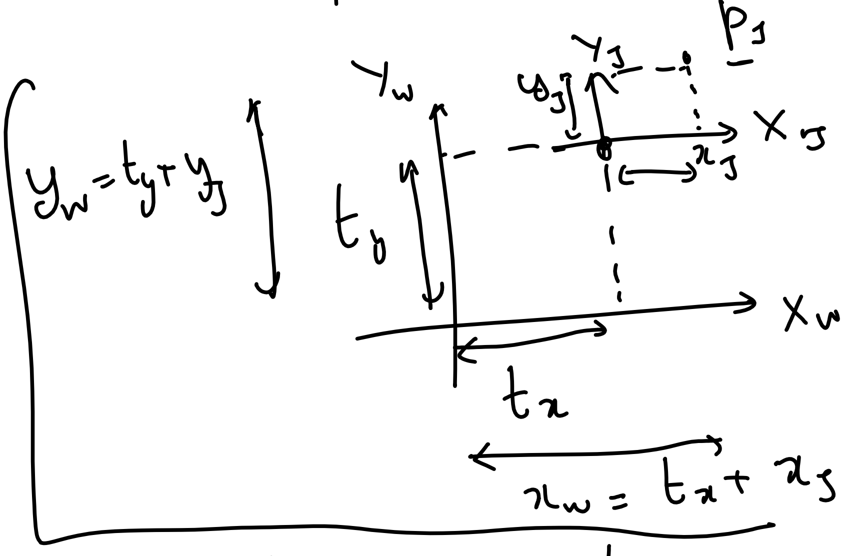
$$\underline{p}_{J'} = {}^{J'}R_J \, \underline{p}_J$$

$$\underline{p}_W = \underline{p}_{J'} + {}^{W}\underline{t}_{J'}$$

$$\underline{p}_W = {}^{J'}R_J \, \underline{p}_J + {}^{W}\underline{t}_{J'}$$

$$\boxed{\underline{p}_W = {}^{W}R_J \, \underline{p}_J + {}^{W}\underline{t}_J}$$





$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \begin{bmatrix} x_J + t_x \\ y_J + t_y \end{bmatrix}$$

$$= \begin{bmatrix} x_J \\ y_J \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

$$\underline{p}_w = \underline{p}_J + {}^{w}\underline{t}_J$$

Matrices = capital letters

Vectors = small letters with underscore

Scalars = small letter without underscor

Transformation matrix

$$\underline{p}_W = \underset{\substack{\uparrow \\ \text{orientation}}}{{}^W R_J}\, \underline{p}_J + \underset{\substack{\nwarrow \\ \text{position}}}{{}^W \underline{t}_J} \qquad \underbrace{{}^W R_J ,\ {}^W \underline{t}_J}$$

$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \begin{bmatrix} \underline{r_{11}} & \underline{r_{12}} \\ r_{21} & r_{22} \end{bmatrix}_{2\times 1} \begin{bmatrix} x_J \\ y_J \end{bmatrix} + \begin{bmatrix} \underline{t_1} \\ t_2 \end{bmatrix}_{2\times 1}$$

$$\begin{bmatrix} x_w \\ y_w \\ 1 \end{bmatrix}_{3\times 1} = \underbrace{\left[\begin{array}{cc|c} r_{11} & r_{12} & t_1 \\ r_{21} & r_{22} & t_2 \\ \hline 0 & 0 & 1 \end{array}\right]_{3\times 3}}_{{}^W T_J} \underset{\substack{\Leftarrow \text{homo} \\ \text{geneous} \\ \text{coordinate}}}{\begin{bmatrix} x_J \\ y_J \\ 1 \end{bmatrix}_{3\times 1}}$$

$$\begin{bmatrix} \underline{p}_w \\ 1 \end{bmatrix} = \left[\begin{array}{c|c} {}^W R_J & {}^W \underline{t}_J \\ \hline \underline{0}^T_{2\times 1} & 1 \end{array}\right] \begin{bmatrix} \underline{p}_J \\ 1 \end{bmatrix}$$

Properties of a Rotation matrix

① What is the <u>inverse</u> of a Rotation matrix

$$AA^{-1} = A^{-1}A = I$$

Prereq

$$\underbrace{U = \begin{bmatrix} | & & | \\ \underline{u}_1 & \cdots & \underline{u}_n \\ | & & | \end{bmatrix}}_{\text{orthonormal matrix}}$$

when $\underline{u}_i^T \underline{u}_i = 1$

$\underline{u}_i^T \underline{u}_j = 0 \quad i \neq j$

$$R = \begin{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} & \begin{matrix} -\sin\theta \\ \cos\theta \end{matrix} \end{bmatrix}$$

is an orthonormal matrix

$$\begin{bmatrix} \cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \cos^2\theta + \sin^2\theta$$

$$= 1$$

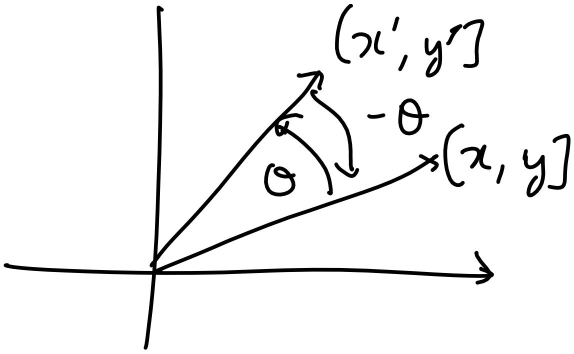
$$\begin{bmatrix}\cos\theta & \sin\theta\end{bmatrix}\begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix} = -\cos\theta\sin\theta + \sin\theta\cos\theta$$

$$= 0$$

For all orthonormal matrices, their inverse is their transpose $U^{-1}=U^{T}$

$$\underbrace{R^{-1}=R^{T}}$$

$$\begin{bmatrix}x'\\ y'\end{bmatrix} = \begin{bmatrix} & R & \end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$$



$$\begin{bmatrix}x\\ y\end{bmatrix} = R^{-1}\begin{bmatrix}x'\\ y'\end{bmatrix}$$

$$R^{-1} = \begin{bmatrix}\cos(-\theta) & -\sin(-\theta)\\ \sin(-\theta) & \cos(-\theta)\end{bmatrix} = \underbrace{\begin{bmatrix}\cos\theta & \sin(\theta)\\ -\sin\theta & \cos\theta\end{bmatrix}}_{R^{T}}$$

① Rotation matrices are orthonormal / Rotation matrix inverse is its transpose

$$RR^{T} = R^{T}R = I$$

Are all orthonormal matrices rotation matrices?

Answer :: No

② $\det(R) = 1$

In general an orthonormal matrix, U

$$\det(U) \in \{-1, +1\}$$

Reflection matrix $\nearrow -1$ $+1 \nwarrow$ Rotation matrix

$$\det\left(\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\right) = \cos^2\theta - (-\sin^2\theta)$$
$$= 1$$

2D transformation matrix

$$\begin{bmatrix} \underline{p}_w \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} {}^wR_J & {}^w\underline{t}_J \\ \underline{0}^T & 1 \end{bmatrix}}_{{}^wT_J} \begin{bmatrix} \underline{p}_J \\ 1 \end{bmatrix}$$

$${}^JT_w = ? = {}^wT_J^{-1}$$

$$\underline{p}_w = {}^wR_J\,\underline{p}_J + {}^w\underline{t}_J$$

$$\underline{p}_w - {}^w\underline{t}_J = {}^wR_J\,\underline{p}_J$$

$$\boxed{\underline{p}_J = \frac{\underline{p}_w - {}^wt_J}{{}^wR_J}}$$

BLASPHEMY ✗

$\boxed{\text{Left}}$ multiply by ${}^{w}R_J^T$

$${}^{w}R_J^T\left(\underline{p}_w - {}^{w}t_J\right) = \underbrace{\left({}^{w}R_J^T\right){}^{w}R_J}_{I}\,\underline{p}_J$$

$${}^{w}R_J^T\,\underline{p}_w - {}^{w}R_J^T\,{}^{w}\underline{t}_J = \underline{p}_J$$

$$\underbrace{\begin{bmatrix} \underline{{}^{w}R_J^T} & \underline{\left(-{}^{w}R_J^T\,{}^{w}\underline{t}_J\right)} \\ \underline{0}_{2\times1}^T & 1 \end{bmatrix}}_{{}^{w}T_J^{-1}} \begin{pmatrix} \underline{p}_w \\ 1 \end{pmatrix} = \begin{pmatrix} \underline{p}_J \\ 1 \end{pmatrix}$$

$${}^{J}T_w = \begin{bmatrix} {}^{w}R_J^T & -{}^{w}R_J^T\,{}^{w}\underline{t}_J \\ 0_{2\times1}^T & 1 \end{bmatrix}$$

Intuitive interpretation

① Rotation ⟶ ② translation

$\longleftarrow \qquad\qquad \longleftarrow$

$$\underbrace{\underline{p}' = R\,\underline{p} + \underline{t}}_{\text{Rotation first}} \neq \underbrace{\underline{p}' = R\left(\underline{p} + \underline{t}\right)}_{\text{translation first}}$$

$\underline{SO(2)}$

Special Orthogonal group

$$\{R_{2\times2} : R^T R = \underline{I}, \det(R)=1\}$$

$\underbrace{O(2)}$

Orthogonal $\underbrace{\text{group}}$

$$\{U_{2\times2} : U^T_{2\times2} U_{2\times2} = \underline{I}_{2\times2} \}$$

$SE(2)$

Special Euclidean group

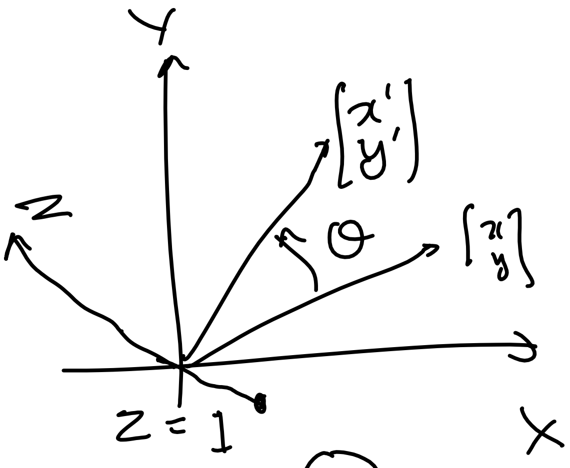
$$\{R_{2\times2}, \underline{t}_{2\times1} : R_{2\times2} \in \underline{SO(2)}, \underline{t} \in \boxed{\mathbb{R}^2}\}$$

2D vectors
in real space

3D Rotations

2D Rotation around Z-axis

$$\begin{bmatrix}x'\\y'\\z'\end{bmatrix} = \underbrace{\begin{bmatrix}\boxed{R(\theta_z)} & \begin{matrix}0\\0\end{matrix}\\ 0 \quad 0 & 1\end{bmatrix}}_{R_z(\theta_z)}\begin{bmatrix}x\\y\\z\end{bmatrix}$$



$\underset{-s\theta}{X} \to Y \to$ Ⓩ

Right hand coordinate frame

X → Y	:	Z
Y → Z	:	X
Z → X	:	Y
curl finger		thumb

around X-axis

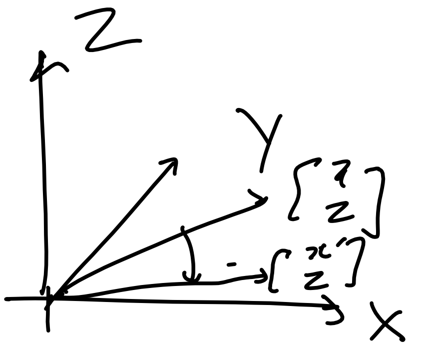
$$\begin{bmatrix}x'\\y'\\z'\end{bmatrix} = \underbrace{\begin{bmatrix}1 & 0 & 0\\ 0 & c\theta_x & -s\theta_x\\ 0 & s\theta_x & c\theta_x\end{bmatrix}}_{R_x(\theta_x)}\begin{bmatrix}x\\y\\z\end{bmatrix}$$

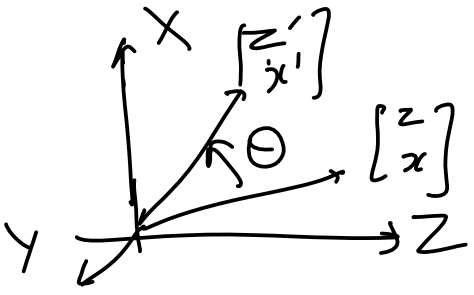
Ⓧ → $\boxed{\underset{-s\theta}{Y} \to Z}$

around Y-axis

$$\begin{bmatrix}x'\\y'\\z'\end{bmatrix} = \underbrace{\begin{bmatrix}c\theta & 0 & s\theta\\ 0 & 1 & 0\\ -s\theta & 0 & c\theta\end{bmatrix}}_{R_Y(\theta_Y)}\begin{bmatrix}x\\y\\z\end{bmatrix}$$

$$\begin{tikzcd} X \arrow[r] & \boxed{Y} \arrow[r] & Z^{-s\theta} \arrow[ll, bend left] \end{tikzcd}$$





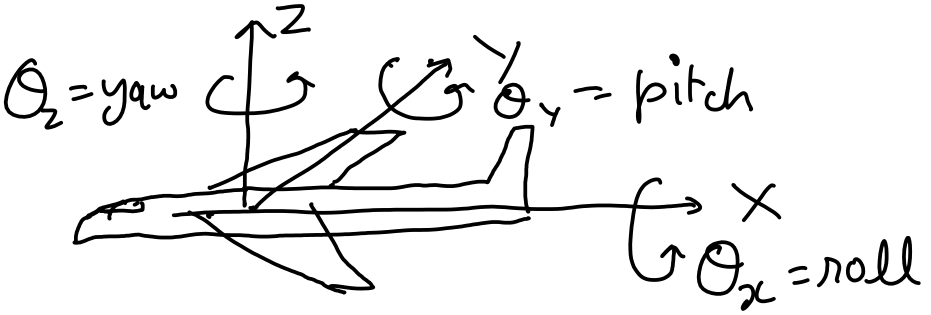
$$\underset{3\times3}{R}\,\underline{p} = \left[\underset{z}{R}(\theta_z)\left[\underset{y}{R}(\theta_y)\left[\underset{x}{R}(\theta_x)_{3\times3}\,\underline{p}\right]\right]\right]$$

All 3D rotations can be decomposed into $\underbrace{\text{3 axis aligned rotions}}$

three angles of rotions are called Euler angles

12 possible combination

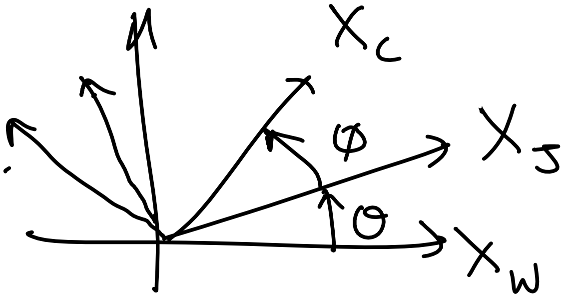
Roll - Pitch - Yaw
① ② ③



$$R = \underleftarrow{R_z(\theta_z)\, R_y(\theta_y)\; R_x(\theta_x)}$$

Application of Rotation matrices is from right to left

Composition of Rotations / Transformations



$$\underline{p}_J = {}^J R_c \, \underline{p}_c$$

$$\underline{p}_w = {}^w R_J \, \underline{p}_J$$

$$\boxed{{}^w R_c = ?}$$

$$\underline{p}_w = {}^w R_J \Big[\underbrace{{}^J R_c \, \underline{p}_c}_{\underline{p}_J} \Big]$$

$$\underline{p}_w = \underbrace{{}^w R_J \, {}^J R_c}_{{}^w R_c} \, \underline{p}_c$$

$$\boxed{{}^w R_c = {}^w R_J \, {}^J R_c}$$

Is

$(R_1 R_2)$

orthonormal ?

$$\det(R_1 R_2) \overset{?}{=} 1$$

$$\underbrace{(R_1 R_2)^{-1}} = R_2^{-1} R_1^{-1} = R_2^T R_1^T = \underbrace{(R_1 R_2)^T}$$

$$\det(R_1 R_2) = \det(R_1)\det(R_2) = 1$$

$${}^{W}T_{C} = {}^{W}T_{J}\;{}^{J}T_{C}$$

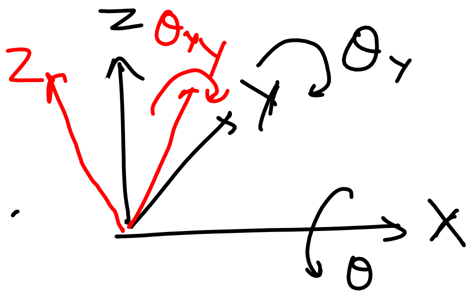
why are there 12 possible Euler angles?

$$R = \left.\begin{matrix} R_Z\, R_Y\, R_X \\ R_Y\, R_Z\, R_X \\ \\ \\ \end{matrix}\right\} \; 6 = 3! \text{ possibilites permutations}$$

when chaining rotations there are two possibilities

① either rotate along the new axis after first rotation.

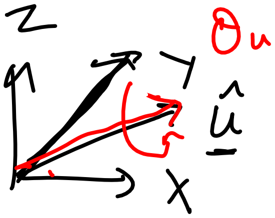
② or rotate along the original axis



3D Rotation representation

- $\underbrace{3\times3,\ \det(R)=1,\ R^TR=I}$ ≡ $\underbrace{\text{Rotation matrix}}$
- 3 Euler angles $(\theta_x, \theta_y, \theta_z)$
- Quaternions $(\underbrace{x, y, z, w})$
- Axis-angle representation $(\hat{\underline{u}}, \theta_u)$

  (Rodrigues formula)



$$A = \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}_{3\times3} = 9\text{ DOF}$$

$A^T = A$ if A is symmetric $= 6$ DOF

$U_{3\times3}$ is orthonormal $\rightarrow$ 3 DOF

$U = [\underline{u}_1\ \ \underline{u}_2\ \ \underline{u}_3]$

$\underline{u}_1^T\underline{u}_2 = 0$ $\quad$ $\underline{u}_1^T\underline{u}_1 = 1$

$\underline{u}_2^T\underline{u}_3 = 0$ $\quad$ $\underline{u}_2^T\underline{u}_2 = 1$

$\underline{u}_3^T\underline{u}_1 = 0$ $\quad$ $\underline{u}_3^T\underline{u}_3 = 1$

Conversions

Roll-pitch-Yaw ⟶ Rotation matrix

$$R = R_z(\theta_z)\, R_y(\theta_y)\, R_x(\theta_x)$$

$$= \underbrace{\begin{bmatrix} c\theta_z & -s\theta_z & 0 \\ s\theta_z & c\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta_y & 0 & s\theta_y \\ 0 & 1 & 0 \\ -s\theta_y & 0 & c\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta_x & -s\theta_x \\ 0 & s\theta_x & c\theta_x \end{bmatrix}}_{}$$

Rotation matrix ⟶ Roll pitch Yaw angles

$$R = \left[\begin{array}{c|c|c} c(\theta_z)\,^{\circ}c(\theta_y) & \cdots & \cdots \\ \hline s(\theta_z)\,^{\circ}c(\theta_y) & \cdots & \cdots \\ \hline \underbrace{-s(\theta_y)}_{} & \underbrace{\overset{\circ}{c}(\theta_y)}_{}\, s(\theta_x) & \underbrace{c(\theta_y)}_{}\,^{\circ}c(\theta_x) \end{array}\right]$$

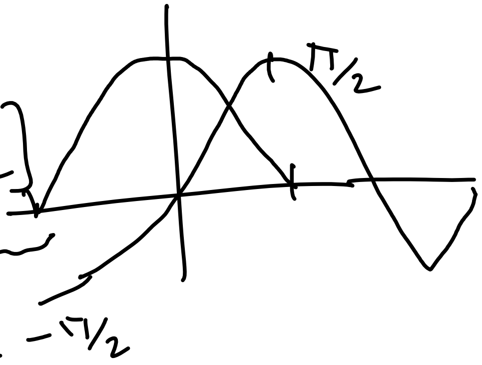
$$= \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\times \left[\tan(\theta_x) = \frac{r_{32}}{r_{33}}\right] \Rightarrow \theta_x = \underbrace{\tan^{-1}\left(\frac{r_{32}}{r_{33}}\right)}_{\in(-\pi/2,\ \pi/2)}$$

$$\rightarrow \theta_x = \arctan2(r_{33}, r_{32}) \in (-\pi, \pi]$$

$$-\sin(\theta_y) = r_{31}$$

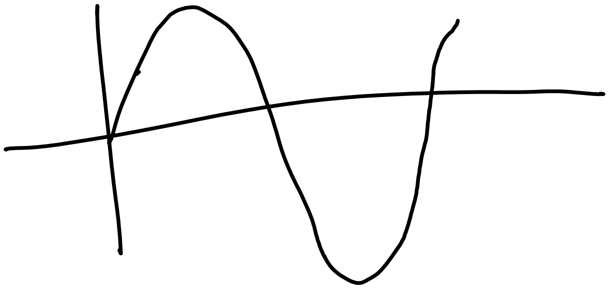
$$\underbrace{\theta_y = \sin^{-1}(-r_{31}) \in [-\pi/2, \pi/2]}$$



$$\theta_x = \arctan2\Big(\overbrace{\mathrm{sign}(\cos(\theta_y))}^{+1} r_{33},$$
$$\underbrace{\mathrm{sign}(\cos(\theta_y))}_{+1} r_{32}\Big)$$

$$\theta_z = \arctan2\Big(\overbrace{\mathrm{sign}(\cos(\theta_y))} r_{21},$$
$$\overbrace{\mathrm{sign}(\cos(\theta_y))} r_{11}\Big)$$

$$\begin{tikzcd} & \text{Euler angles} \\ R \arrow[ur] \arrow[dr] & \\ & \text{Euler angles} \end{tikzcd}$$



$$\sin(\pi-\theta) = \sin(\theta)$$

✓ $\boxed{\theta_y = \sin^{-1}(-r_{31}) \in [-\pi/2, \pi/2]}$ ✓

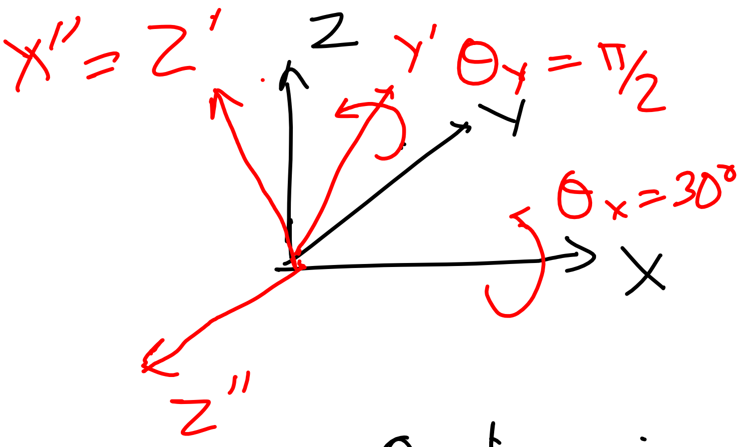
$\theta_y' =$ ✓ $\pi - \theta$

Gimbal Lock (Problem with Euler angles)

$\theta_Y = 90^\circ \approx \pi/2$

$\cos(\theta_Y) = 0$

Then $\underline{\theta_X}$ and $\underline{\theta_Z}$ can be swapped

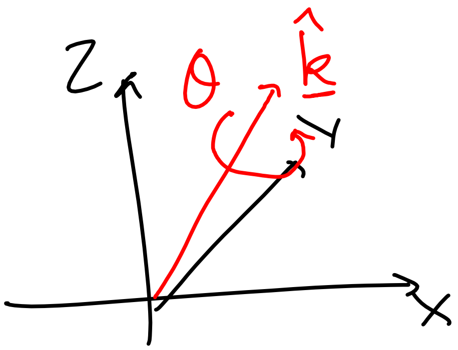


Other representations
- → Quaternions
- → Axis-angle representation

Axis-angle represention

$\underline{\hat{k}} = [k_x, k_y, k_z]$

$||\underline{\hat{k}}|| = \sqrt{k_x^2 + k_y^2 + k_z^2} = 1$



θ 3DOF, 4 numbers, 1 constraint

Axis angle $\xrightarrow[\text{formula}]{\text{Rodrigues}}$ Rot Matrix

$$R = \underline{I} + \sin\theta \underbrace{[\hat{\underline{k}}\times]} + (1-\cos\theta)[\hat{\underline{k}}\times]^2$$

$$[\hat{k}\times] = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix} : \text{Cross product matrix}$$

<u>Cross product matrix</u>

$$\underline{a} \times \underline{b} = \underbrace{[\underline{a}\times]}_{\substack{\text{Matrix}\\ \text{Cross product}}} \underline{b}$$

$$\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix} = \hat{i}(a_y b_z - b_y a_z) + \hat{j}(a_z b_x - a_z b_z)$$

$$+ \hat{k}(a_x b_y - b_x a_y)$$

$$= \begin{bmatrix} a_y b_z - b_y a_z \\ a_z b_x - a_x b_z \\ a_x b_y - b_x a_y \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}_{3\times3}} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

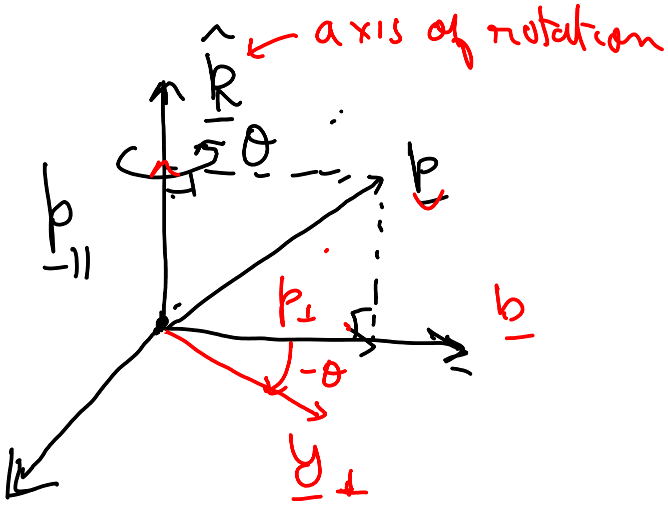
$A = \underbrace{\left[\underline{a}\times\right]}_{}$ cross-product matrix of $\underline{a}$

$\underbrace{A^T = -A}_{}$ Skew-symmetric matrices

$$R\,\underline{p} = \left(\underline{I} + \sin\theta \underbrace{\left[\hat{\underline{k}}\times\right]}_{} + (1-\cos\theta)\left[\hat{\underline{k}}\times\right]^2\right)\underline{p}$$

$$= \underline{p} + \sin\theta\left(\hat{\underline{k}}\times\underline{p}\right) + (1-\cos\theta)\,\hat{\underline{k}}\times\left(\hat{\underline{k}}\times\underline{p}\right)$$

$\underline{y} = ?$ after rotating point $\underline{p}$ around $\hat{\underline{k}}$ by angle θ



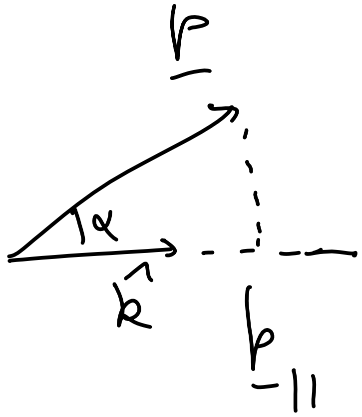
① Split $\underline{p}$ into $\underline{p}_\parallel + \boxed{\underline{p}_\perp}$

$\underline{y} = \underline{p}_\parallel + \boxed{\underline{y}_\perp}$

② Rotate $\underline{p}_\perp$ by angle θ to get $\underline{y}_\perp$

③ Combine $\underline{p}_\parallel$ and $\underline{y}_\perp$ to get $\underline{y}$

$$\underline{p}_\parallel = \underbrace{(\hat{\underline{k}}^T \underline{p})}_{\|\underline{p}|\cos(\alpha)} \hat{\underline{k}}$$



Find $\underline{a}$ that is $\underline{a} \perp \hat{\underline{k}}$ and $\underline{a} \perp \underline{p}$ } 3D

$\underline{a} = \underline{p} \times \hat{\underline{k}} = -(\hat{\underline{k}} \times \underline{p})$, $\underline{b} = ?$

Find $\underline{b}$ that is $\underline{b} \perp \hat{\underline{k}}$ and $\underline{b} \perp \underline{a}$

$\underline{b} = \hat{\underline{k}} \times \underline{a} = \hat{\underline{k}} \times (\underline{p} \times \hat{\underline{k}}) = -\hat{\underline{k}} \times (\hat{\underline{k}} \times \underline{p})$

$$\underline{a} \times (\underline{b} \times \underline{c}) \neq (\underline{a} \times \underline{b}) \times \underline{c}$$

Cross product is not associative

$$\hat{\underline{k}} \times \hat{\underline{k}} = ?$$

0

$$\underline{b} = -\hat{\underline{k}} \times \underbrace{(\hat{\underline{k}} \times \underline{p})}$$

$$\underline{b} = -K(K\underline{p}) = -K^2\underline{p}$$

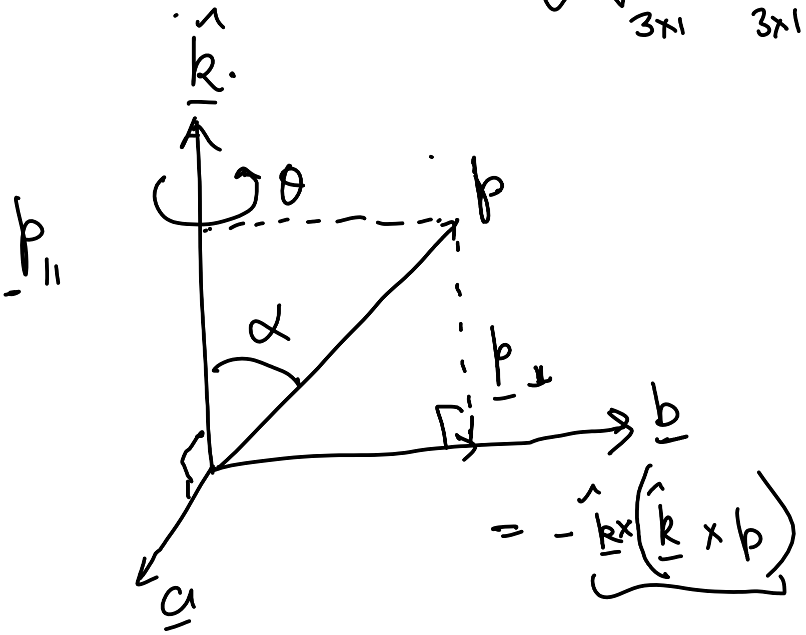
$$[\hat{\underline{k}}_\times] = K$$

$$\cancel{\underline{b} = -(\underbrace{\hat{\underline{k}} \times \hat{\underline{k}}}) \times \underline{p}}$$

$$\cancel{= -\underbrace{(\underset{3\times3}{K}\ \underset{3\times1}{\hat{\underline{k}}})}_{3\times1}\ \underbrace{\underline{p}}_{3\times1}}$$

Magnitude of $\underline{p}_\perp = ?$

in terms of α



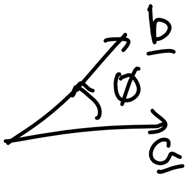
$$|\underline{p}_\perp| = \underbrace{|\underline{p}|\sin\alpha}$$

$$|\underline{b}| = ?$$

$$|\underline{a}| = |(\hat{\underline{k}} \times \underline{p})| = |\hat{\underline{k}}||\underline{p}|\sin\alpha$$

$$= \underbrace{|\underline{p}|\sin\alpha}$$

$$|\underline{a} \times \underline{b}| = |\underline{a}||\underline{b}|\sin\phi$$



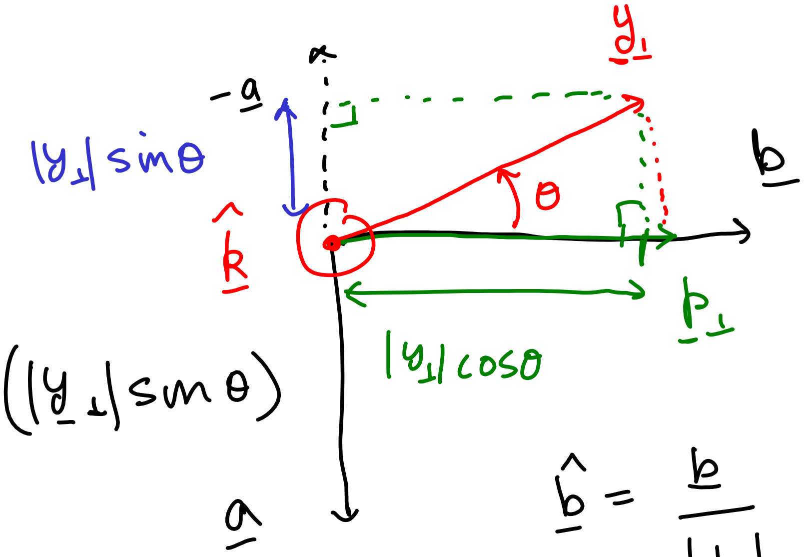
$$\underbrace{|\underline{b}|} = |\hat{\underline{k}} \times \underline{a}| = |\hat{\underline{k}}||\underline{a}|\sin(90^\circ) = \underbrace{|\underline{p}|\sin\alpha} = |\underline{p}_\perp|$$

$$\underline{p}_\perp = \underline{b} = -\hat{\underline{k}} \times (\hat{\underline{k}} \times \underline{p})$$

① $\underline{p} = \underline{p}_{\parallel} + \underline{p}_\perp$

$$= (\hat{\underline{k}}^T \underline{p})\hat{\underline{k}} - \hat{\underline{k}} \times (\hat{\underline{k}} \times \underline{p})$$

② Rotate $\underline{p}_\perp$ by θ in the plane made by



$$|\underline{y}_\perp| = |\underline{p}_\perp|$$

$$\underline{y}_\perp = (|\underline{y}_\perp| \cos\theta)\, \hat{\underline{b}} + (-\hat{\underline{a}})\,(|\underline{y}_\perp| \sin\theta)$$

$$\hat{\underline{b}} = \frac{\underline{b}}{|\underline{b}|}$$

$$= (|\underline{p}_\perp| \cos\theta)\, \hat{\underline{b}} + (-\hat{\underline{a}})\,(|\underline{p}_\perp| \sin\theta)$$

$$\hat{\underline{a}} = \frac{\underline{a}}{|\underline{a}|}$$

$$= \underbrace{(|\underline{p}_\perp|\, \hat{\underline{b}})} \cos\theta + (-\hat{\underline{a}})(|\underline{p}_\perp| \sin\theta)$$

$$\boxed{\begin{aligned} (\hat{\underline{a}})|\underline{a}| &= \underline{a} \\ |\underline{p}_\perp| &= |\underline{p}| \sin\alpha \\ |\underline{a}| &= |\underline{p}| \sin\alpha \end{aligned}}$$

$$= \underline{b} \cos\theta - \underline{a} \sin\theta$$

$$= -\hat{\underline{k}} \times (\hat{\underline{k}} \times \underline{p}) \cos\theta + (\hat{\underline{k}} \times \underline{p}) \sin\theta$$

$$\underline{y} = \underline{p}_{\parallel} + \underline{y}_{\perp}$$

$$= \underbrace{(\hat{\underline{k}}^T \underline{p})\hat{k}} + (\hat{\underline{k}} \times \underline{p}) \sin\theta - \hat{\underline{k}} \times (\hat{\underline{k}} \times \underline{p}) \cos\theta$$

$$\underline{p} = \underline{p}_{\parallel} + \underline{p}_{\perp} \Rightarrow \underline{p}_{\parallel} = \underline{p} - \underline{p}_{\perp}$$

$$= \underline{p} + \hat{k} \times (\hat{k} \times \underline{p})$$

$$\underline{y} = \underline{p} + \underbrace{\hat{\underline{k}} \times (\underline{k} \times \underline{p})} + (\hat{k} \times p) \sin\theta - \underbrace{\hat{\underline{k}} \times (\hat{\underline{k}} \times \hat{\underline{p}})} \cos\theta$$

$$= \underline{p} + (\hat{k} \times \underline{p}) \sin\theta + (1 - \cos\theta)\ \hat{\underline{k}} \times (\underline{k} \times \hat{p})$$

$$[\hat{\underline{k}}_{\times}] = K$$ cross-product matrix

$$= \underline{p} + K \underline{p} \sin\theta + (1 - \cos\theta) K^2 \underline{p}$$

$$= \underbrace{\left[\underline{I}_{3\times 3} + K \sin\theta + (1 - \cos\theta) K^2 \right]}_{R(\underset{\text{axis}}{\hat{\underline{k}}},\ \overset{\text{angle}}{\theta})} \underline{p}$$

$$\begin{tikzcd} \underbrace{\text{Axis angle representation}}_{(\hat{k},\,\theta)} \arrow[r] & R(\hat{k},\theta) \arrow[l, bend left, "?"] \end{tikzcd}$$

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{32} \end{bmatrix} = \left[I_{3\times3} + K\sin\theta + (1-\cos\theta)K^2 \right]$$

$$\hat{k} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} \Rightarrow K = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix}$$

$$K^2 = \begin{bmatrix} -(k_z^2+k_y^2) & k_y k_x & k_z k_x \\ k_y k_x & -(k_x^2+k_z^2) & k_z k_y \\ k_z k_x & k_z k_y & -(k_x^2+k_y^2) \end{bmatrix} \qquad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\boxed{I + K\sin\theta + (1-\cos\theta)K^2}$$

$$\underbrace{\begin{bmatrix} r_{11} & & \\ & r_{22} & \\ & & r_{23} \end{bmatrix}} = \begin{bmatrix} 1+0-(1-\cos\theta)(k_z^2+k_y^2) & - & - \\ & 1-(1-\cos\theta)(k_x^2+k_z^2) & - \\ - & - & 1-(1-\cos\theta)(k_x^2+k_y^2) \end{bmatrix}$$

$$\text{trace}(R) = r_{11} + r_{22} + r_{33} = 2\cos\theta\,(k_x^2 + k_y^2 + k_z^2)$$

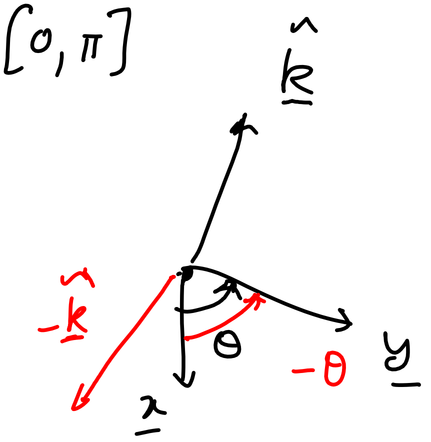
$$\cancel{= 2\cos\theta}$$

$$trace(R) = r_{11}+r_{22}+r_{33} = 3 - (1-\cos\theta)\underbrace{\left(k_x^2+k_y^2+k_z^2\right)}_{|\hat{\underline{k}}|^2=1} \times 2$$

$$= 3 - 2(1-\cos\theta)$$

$$= 1 + 2\cos\theta$$

$$\Rightarrow \quad \theta = \cos^{-1}\left(\frac{trace(R) - 1}{2}\right) \in [0,\pi]$$



$$R(\hat{\underline{k}}, \underline{\theta}) = R(-\hat{\underline{k}}, -\theta)$$

$$\begin{tikzcd} R \arrow[r] \arrow[dr] & \theta \Rightarrow \cos^{-1}\left(\frac{trR - 1}{2}\right) \\ & \hat{\underline{k}} = ? \end{tikzcd}$$

Eigen values and eigen vectors for a matrix A

Definition?

$$\boxed{A\underline{v} = \lambda\underline{v}}$$

square

for a given sq matrix A, the solutions to the above equation will have

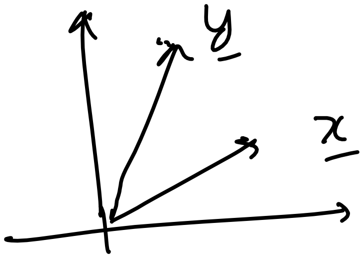
$\underline{v}$ as the eigen vector

λ " " " value

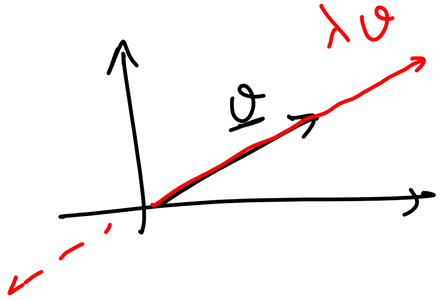
$$\underbrace{A_{n\times n}\,x_{n\times 1}} = \underline{y}_{n\times 1}$$

linear

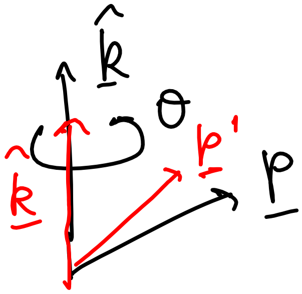
The matrix A transforms a vector to another vector



$A\underline{v} = \lambda\underline{v}$ for eigen vectors



$\det(A) = \lambda_1 \lambda_2 \ldots \lambda_n$



$R\hat{\underline{k}} = \hat{\underline{k}}$

[$\hat{\underline{k}}$ is an eigen vector of Rotation matrix with an real eigen value of $\underline{1}$]

$$\times \begin{cases} A\underline{v} = \lambda\underline{v} \Rightarrow (A-\lambda I)\underline{v} = 0 \\ \Rightarrow \det(A-\lambda I) = 0 \end{cases}$$

eigen value

$\underline{V}, \underline{\lambda}$ = numpy.linalg.eig(A)

Eigen vectors $= [\circled{\underline{v}}_1, \underline{v}_2, \ldots, \underline{v}_n]$, $\underline{\lambda} = [\boxed{\lambda_1}, \ldots, \lambda_n]$ $\overset{}{=1}$

$$\begin{cases} \underline{\hat{k}} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} = \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} \dfrac{1}{2 \sin\theta} & \text{when } \theta \neq 0, 180^\circ \\ \text{if } \theta = 180^\circ \\ \underline{\hat{k}} = \pm \begin{bmatrix} \sqrt{(r_{11}+1)}/2 \\ \sqrt{(r_{22}+1)}/2 \\ \sqrt{(r_{33}+1)}/2 \end{bmatrix} \end{cases}$$

Quaternions

$$\underline{q} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

$$\underline{q} = \overset{\text{scalar part}}{\underline{w}} + \overbrace{ix + jy + kz}^{\text{vector}}$$

$$i^2 = -1, \; j^2 = -1, \; k^2 = -1$$

$$ijk = -1$$

Quaternions ⟶ Axis-angle representation

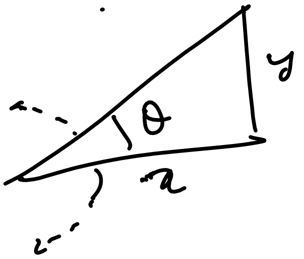
↓

Rotation matrices

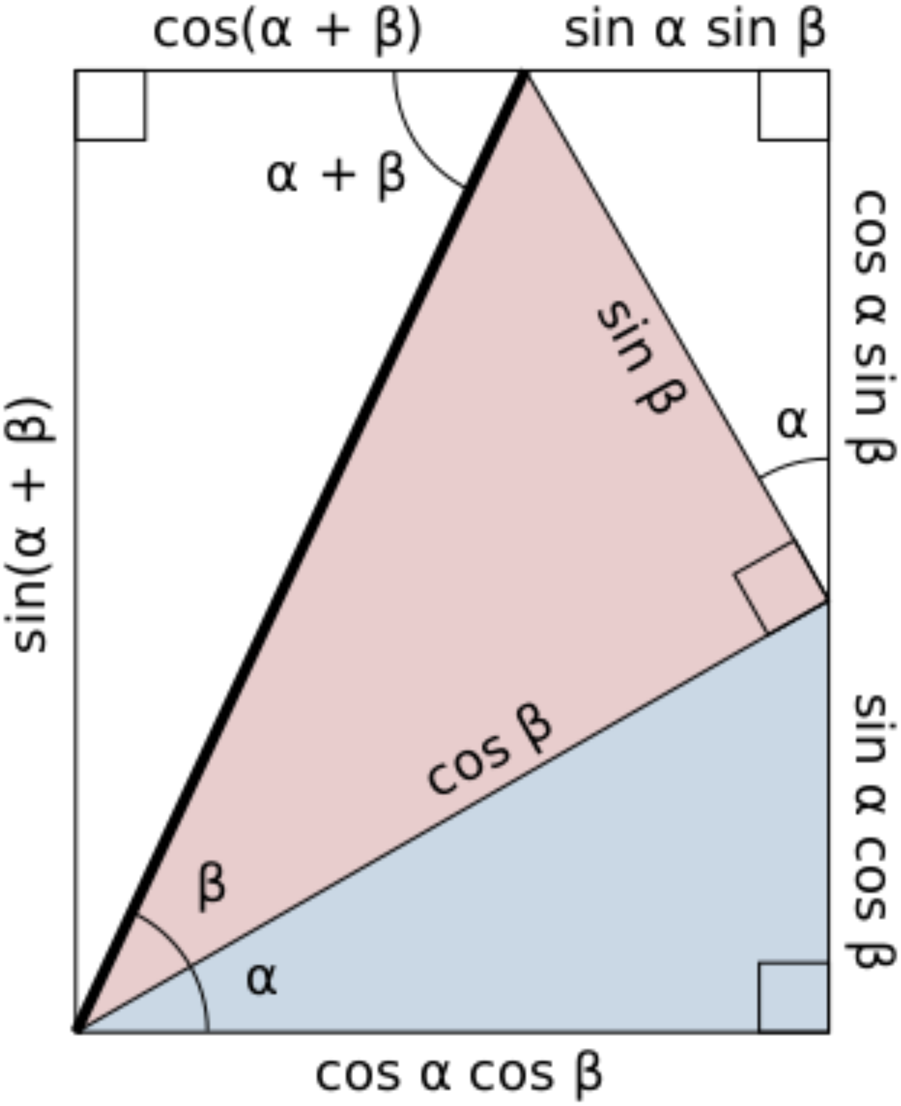
$$\underline{q} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos(\theta/2) \\ k_x \sin(\theta/2) \\ k_y \sin(\theta/2) \\ k_z \sin(\theta/2) \end{bmatrix}$$

$$\theta = 2 * \arctan2\left(\sqrt{x^2 + y^2 + z^2}, w\right) \qquad \underline{\hat{k}} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \frac{1}{\sin(\theta/2)}$$

$$\begin{cases} \tan^{-1}\left(\frac{y}{x}\right) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \\ \text{if } x < 0 \\ \qquad \tan^{-1}\left(\frac{y}{x}\right) \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right) \end{cases}$$



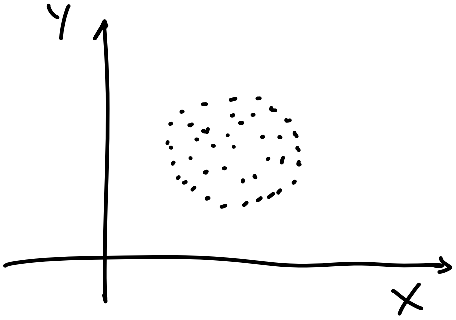
$$\text{arctan2}(y, x) \in [-\pi, \pi)$$

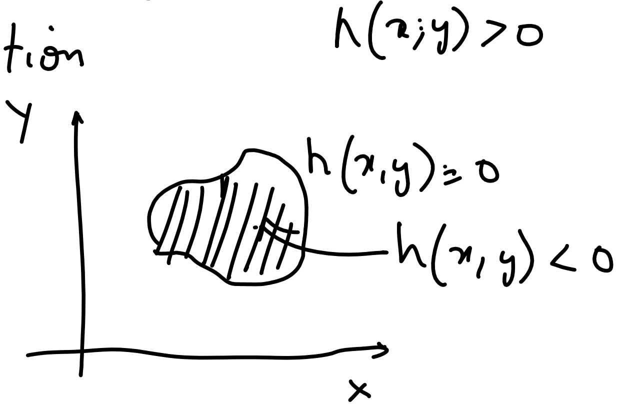

cos(α + β)
sin α sin β
α + β
sin β
α
cos α sin β
sin(α + β)
cos β
β
α
sin α cos β
cos α cos β

2D coordinate transforms

Rotations + translations

Orientation + position



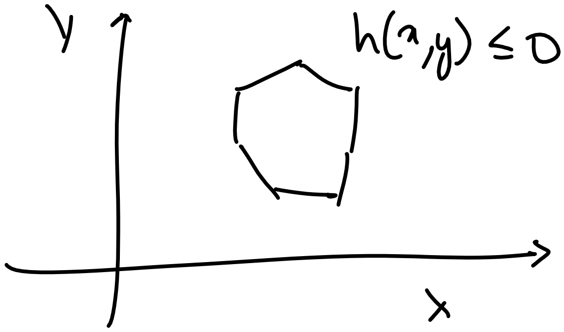


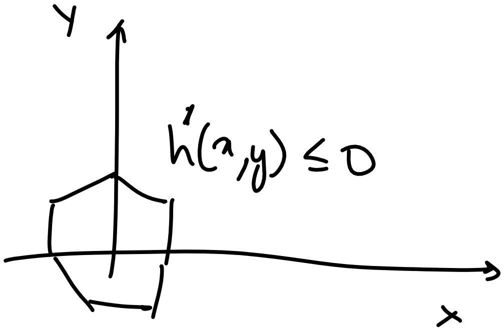
$$\text{Object} = \{(x,y)\in\mathbb{R}^2 : h(x,y) \le 0\}$$

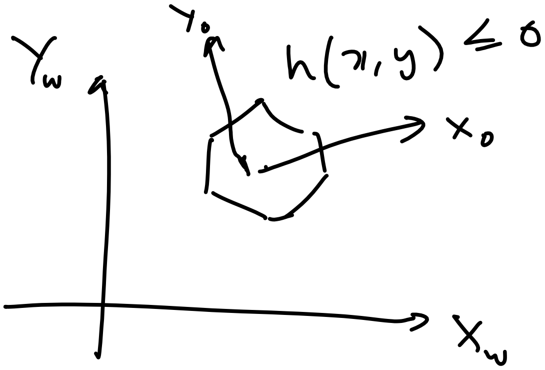
Set of all points $(x,y)\in\mathbb{R}^2$ ← set of all real numbers

such that

$h(x,y)\le 0$



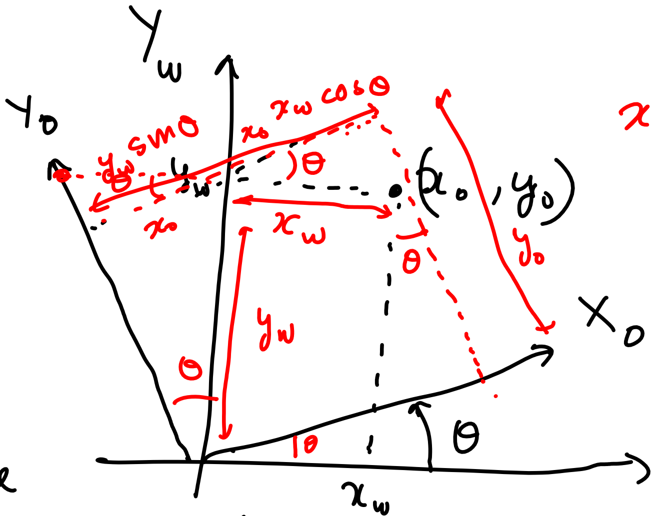




2D Rotations

① Rotation

② Translation



$$x_0 = x_w\cos\theta + y_w\sin\theta$$

Problem (x_0, y_0) is given in (X_0, Y_0) coordinate frame. $(X_0 Y_0)$ has been rotated by angle θ w.r.t. (X_w, Y_w)

Find (x_w, y_w) in world coordinate frame

Proof using Basis vectors

In Linear algebra, Basis vectors are set of orthonormal unit vectors that span the entire space

Span is the set of all vectors that can be obtained by linear combinations of a given set of vectors

$$\text{Span}\{\underbrace{\underline{a}, \underline{b}}_{\in \mathbb{R}^n}\} = \{\underbrace{\alpha \underline{a} + \beta \underline{b}}_{\in \mathbb{R}^n}, \begin{matrix} \alpha \in \mathbb{R} \\ \beta \in \mathbb{R} \end{matrix}\}$$

Standard Basis vector.

For example, in $\mathbb{R}^2$ $\quad \hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \hat{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

in $\mathbb{R}^3$ $\quad \hat{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \hat{j} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

in $\mathbb{R}^n$ $\quad \hat{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n \times 1} \quad \ldots \quad \hat{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}_{n \times 1}$

Basis vectors for $\mathbb{R}^n$

↳① All vectors must be perpendicular/orthogonal to each other

↳② They must be unit vectors

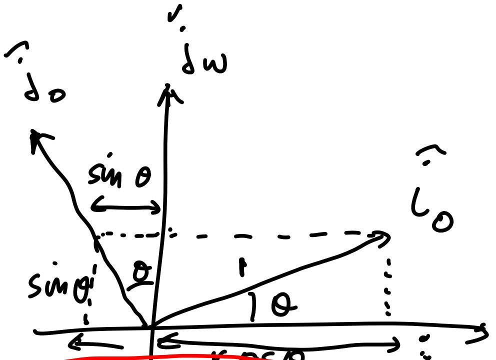
↳③ They must span the entire space $\mathbb{R}^n$

Let Basis vector for (X_w, Y_w) be standard basis vector

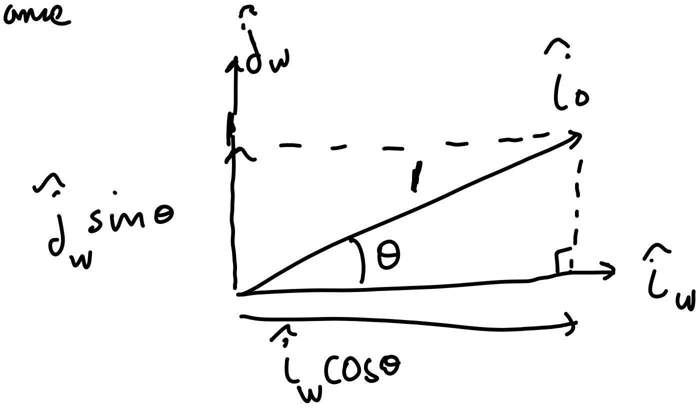
$\hat{i}_w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \hat{j}_w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

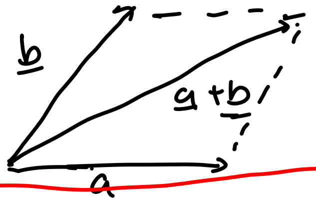
Any point in world coordinate frame $$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = x_w \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\hat{i}_w} + y_w \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\hat{j}_w}$$

Any point in the object coordinate frame $$\begin{bmatrix} x_o \\ y_o \end{bmatrix} = x_o \hat{i}_o + y_o \hat{j}_o$$



$$\boxed{\hat{i}_o = \cos\theta\, \hat{i}_w + \sin\theta\, \hat{j}_w}$$





$$\boxed{\hat{j}_o = -\hat{i}_w \sin\theta + \hat{j}_w \cos\theta}$$

$$\overbrace{x_w \hat{i}_w + y_w \hat{j}_w}^{\text{world}} = \overbrace{x_o \hat{i}_o + y_o \hat{j}_o}^{\text{object}}$$

$$\underbrace{\begin{bmatrix} x_w \\ y_w \end{bmatrix}}_{\text{Because we are using standard basis for world coordinate frame}} = x_o \left[\cos\theta\, \hat{i}_w + \sin\theta\, \hat{j}_w\right] + y_o \left[-\hat{i}_w \sin\theta + \hat{j}_w \cos\theta\right]$$

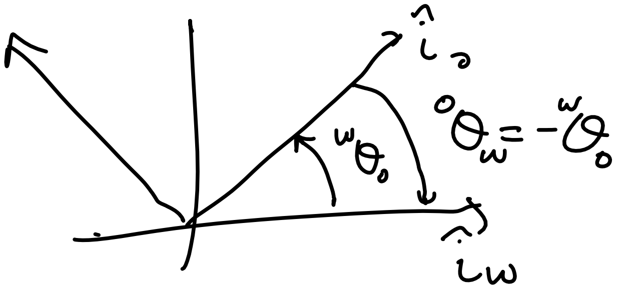
$$= \left[x_o \cos\theta - y_o \sin\theta\right] \underset{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}{\underset{\downarrow}{\hat{i}_w}} + \left[x_o \sin\theta + y_o \cos\theta\right] \underset{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}{\underset{\downarrow}{\hat{j}_w}}$$

$$= \begin{bmatrix} x_o \cos\theta - y_o \sin\theta \\ x_o \sin\theta + y_o \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_o \\ y_o \end{bmatrix}$$

$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \begin{bmatrix} {}^{w}R_{o}(\theta) \end{bmatrix} \begin{bmatrix} x_o \\ y_o \end{bmatrix}$$

$${}^{w}R_{o}(\theta) := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



$$\begin{bmatrix} x_o \\ y_o \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}}_{{}^{o}R_{w}(\theta) = {}^{w}R_{o}(-\theta) = {}^{w}R_{o}^{T}(\theta)} \begin{bmatrix} x_w \\ y_w \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & +\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_w \\ y_w \end{bmatrix}$$

$$\begin{bmatrix} {}^{w}R_{o}^{T} \end{bmatrix} {}^{w}R_{o} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

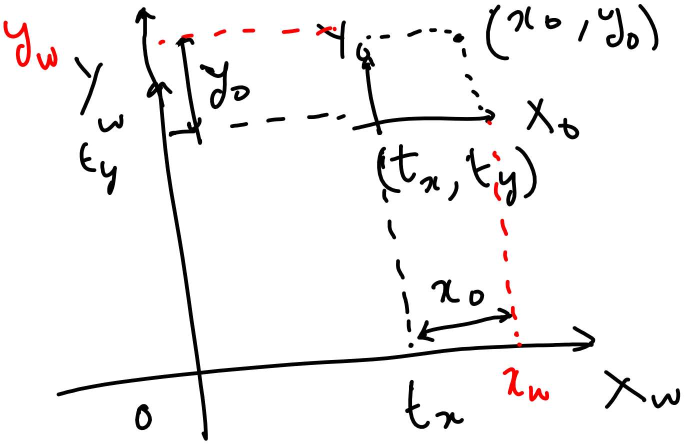
$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -c(\theta)s(\theta) + s(\theta)c(\theta) \\ -s(\theta)c(\theta) + c(\theta)s(\theta) & s^2(\theta) + c^2(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_{2\times 2}$$

$$\underbrace{R^T R = I}_{(A^{-1})A = I} \longrightarrow R^{-1} = R^T$$

2D Translation

$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \underbrace{\begin{bmatrix} t_x \\ t_y \end{bmatrix}}_{\text{translation vector}} + \begin{bmatrix} x_o \\ y_o \end{bmatrix}$$

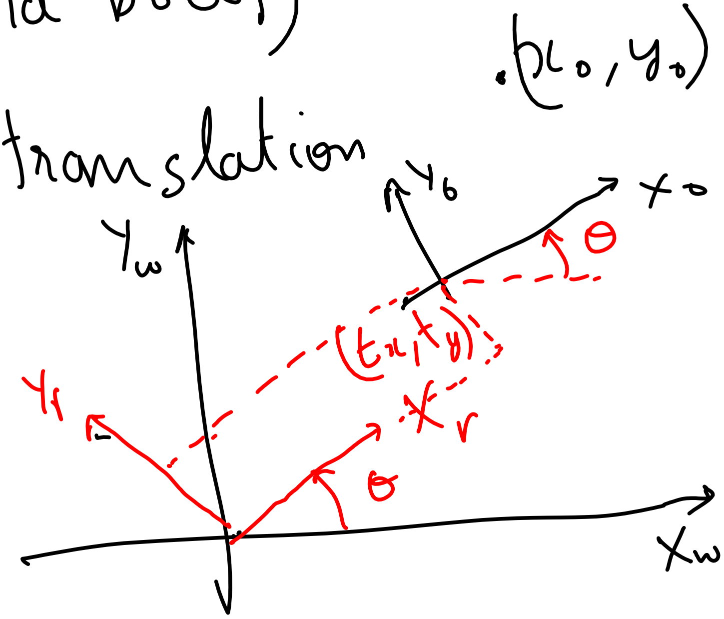


2D transformation (Rigid body)

Rotation followed by translation

$$\begin{bmatrix} x_r \\ y_r \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_o \\ y_o \end{bmatrix}$$

$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \begin{bmatrix} t_x \\ t_y \end{bmatrix} + \begin{bmatrix} x_r \\ y_r \end{bmatrix}$$



$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}} \begin{bmatrix} x_o \\ y_o \end{bmatrix} + \underbrace{\begin{bmatrix} t_x \\ t_y \end{bmatrix}}_{\underline{t}}$$

$$\boxed{\underline{x}_w = {}^{w}_{o}R(\theta)\, \underline{x}_o + {}^{w}_{o}\underline{t}}$$

$$\underline{x}_o = ?(\underline{x}_w)$$

$$\hookrightarrow \underline{x}_w - {}^{w}_{o}\underline{t} = {}^{w}_{o}R\,\underline{x}_o$$

$${}^{w}_{o}R = {}^{w}_{o}R(\theta)$$

Multiply on the <u>left</u> of both sides by ${}^{w}_{o}R^T$

$$\hookrightarrow {}^{w}_{o}R^T\left(\underline{x}_w - {}^{w}_{o}\underline{t}\right) = \left({}^{w}_{o}R^T\,{}^{w}_{o}R\right)\underline{x}_o$$

$$\frac{\underline{x}_w - {}^{w}_{o}\underline{t}}{{}^{w}_{o}R} = \underline{x}_o$$

NEVER EVER DO THIS

$$\underline{x}_o = {}^{w}_{o}R^T\underline{x}_w - {}^{w}_{o}R^T\,{}^{w}_{o}\underline{t} \quad \text{—} ①$$

$$\underline{x}_o = {}^{o}_{w}R\cdot\underline{x}_w + {}^{o}_{w}\underline{t} \quad \text{—} ②$$

Compare ① and ②

$$\boxed{{}^{o}_{w}R = {}^{w}_{o}R^T} \quad \text{and} \quad \boxed{{}^{o}_{w}\underline{t} = -{}^{w}_{o}R^T\,{}^{w}_{o}\underline{t}}$$

$$\underline{x}_w = {}^{w}_{o}R\,\underline{x}_o + {}^{w}_{o}\underline{t}$$

$$\begin{bmatrix} \underline{x}_w \\ 1 \end{bmatrix} = \begin{bmatrix} x_w \\ y_w \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} {}^w_o R_{2\times 2} & {}^w_o \underline{t}_{2\times 1} \\ 0^T_{1\times 2} & 1 \end{bmatrix}}_{{}^w_o T_{3\times 3}} \begin{bmatrix} \underline{x}_{o\,2\times 1} \\ 1 \end{bmatrix}$$

Block matrix

Block Matrix multiplication

✓ $$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{bmatrix}$$

$$\begin{bmatrix} {}^w_o R_{2\times 2} & {}^w_o \underline{t}_{2\times 1} \\ 0^T_{1\times 2} & 1 \end{bmatrix} \begin{bmatrix} \underline{x}_{o\,2\times 1} \\ 1 \end{bmatrix} = \begin{bmatrix} {}^w_o R \, \underline{x}_o + {}^w \underline{t}_o \\ \underline{0}^T \underline{x}_o + 1 \end{bmatrix}$$

$$= \begin{bmatrix} {}^w_o R \, \underline{x}_o + {}^w \underline{t}_o \\ 1 \end{bmatrix} = \begin{bmatrix} \underline{x}_w \\ 1 \end{bmatrix}$$

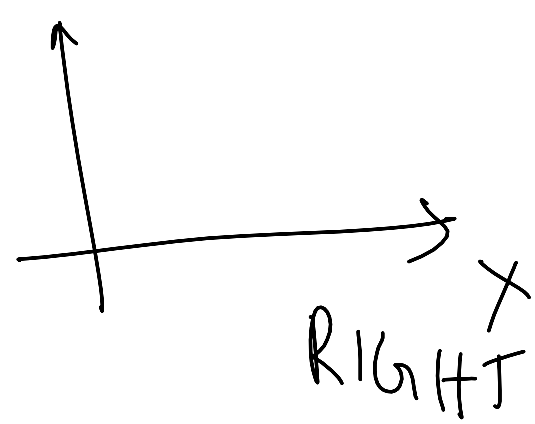
$$\underline{x}_w = {}^w_o T \, \underline{x}_o$$

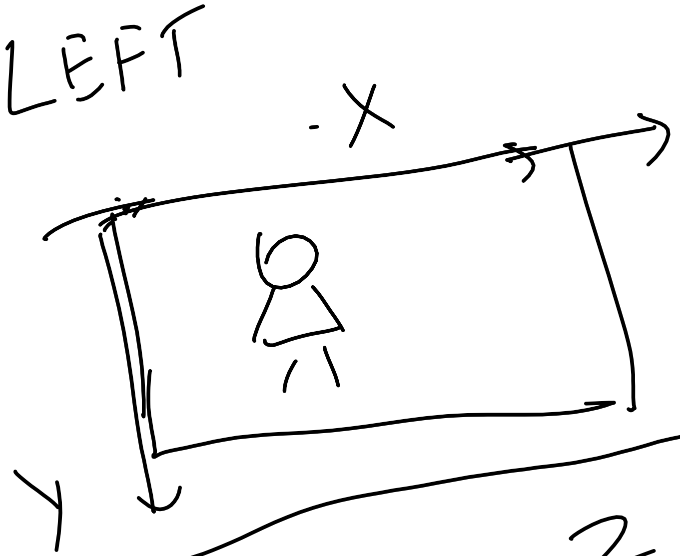
$${}^w T_o = \begin{bmatrix} {}^w_o R_{2\times 2} & {}^w_o \underline{t}_{2\times 1} \\ 0^T_{1\times 2} & 1 \end{bmatrix}$$

Transformation matrix

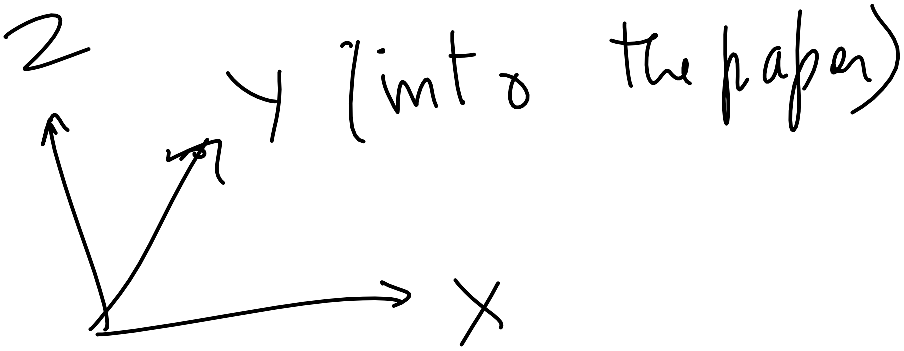
In 2D and 3D

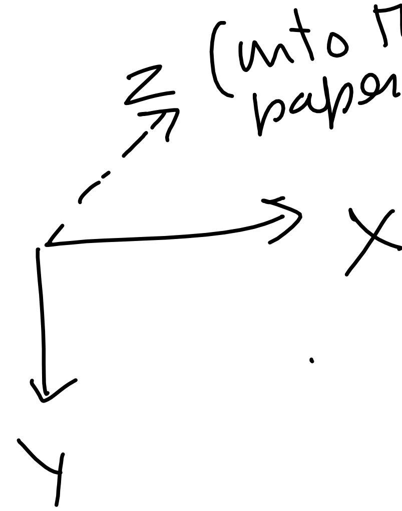
- Right hand ✓
- Left hand ✗

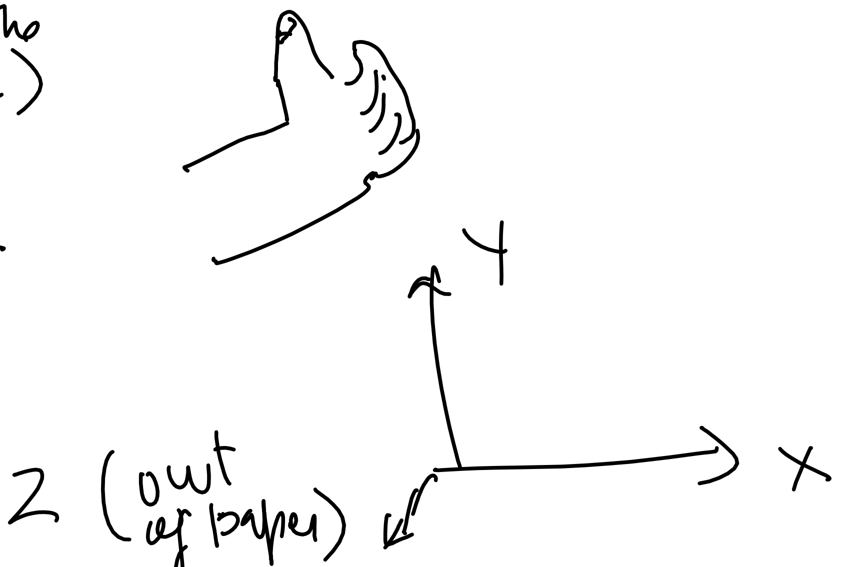




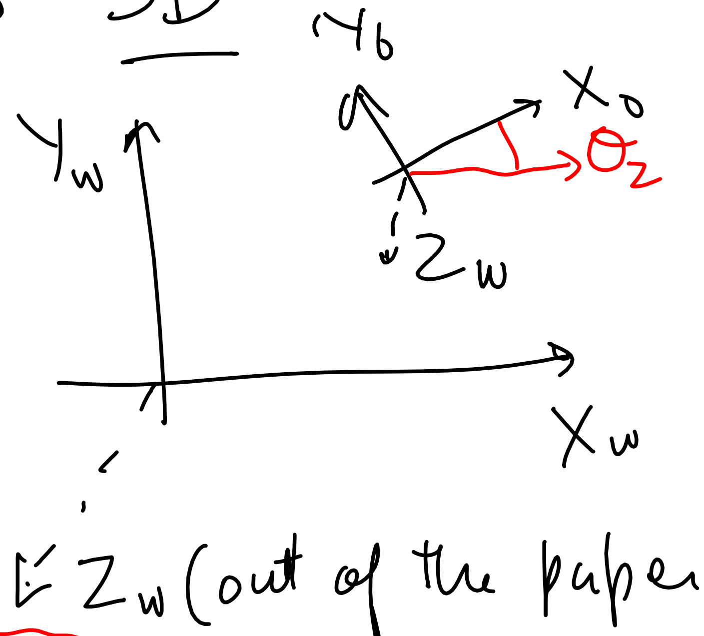
In 3D







Extending 2D to 3D

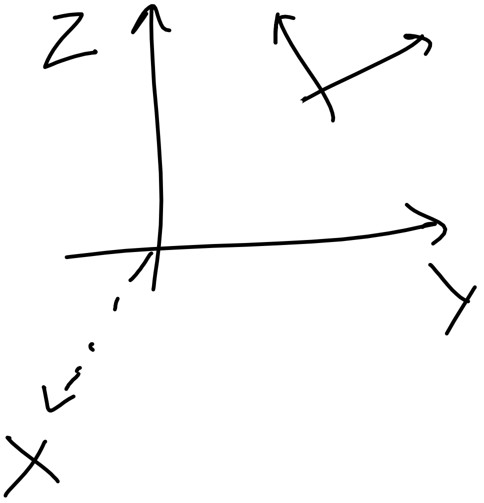


$$\begin{bmatrix} x_w \\ y_w \\ Z_w = 0 \end{bmatrix} = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_o \\ y_o \\ Z_o \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ 0 \end{bmatrix}$$

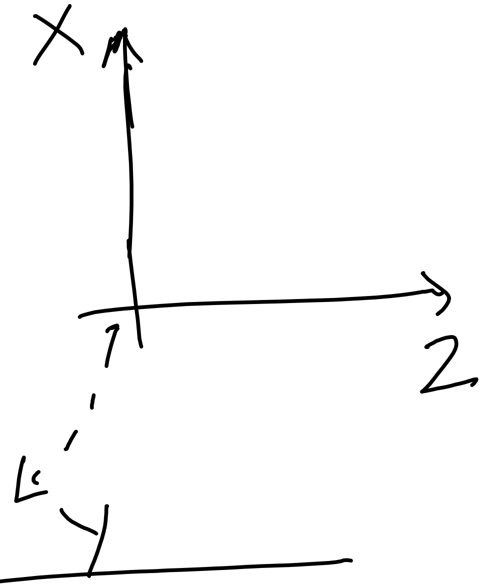
Rotation along Z-axis changes only X-Y coordinates

$$R(\theta_z) = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R(\theta_x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \underline{-\sin\theta} \\ 0 & \underline{\sin\theta} & \cos\theta \end{bmatrix}$$

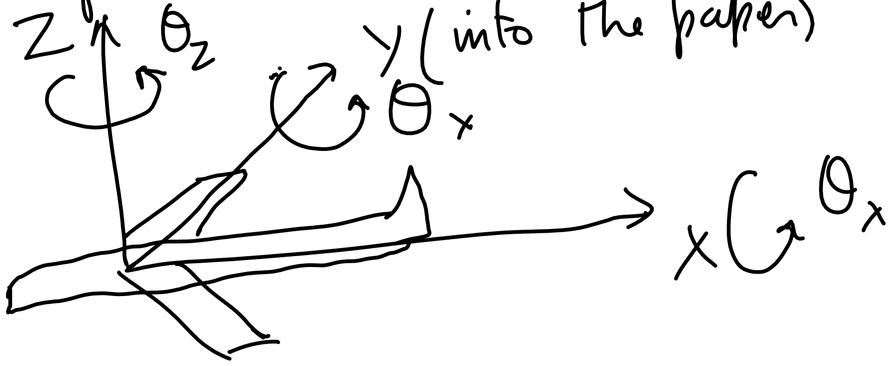


$$R(\theta_y) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$



3D Rotation representations

Aeronautics



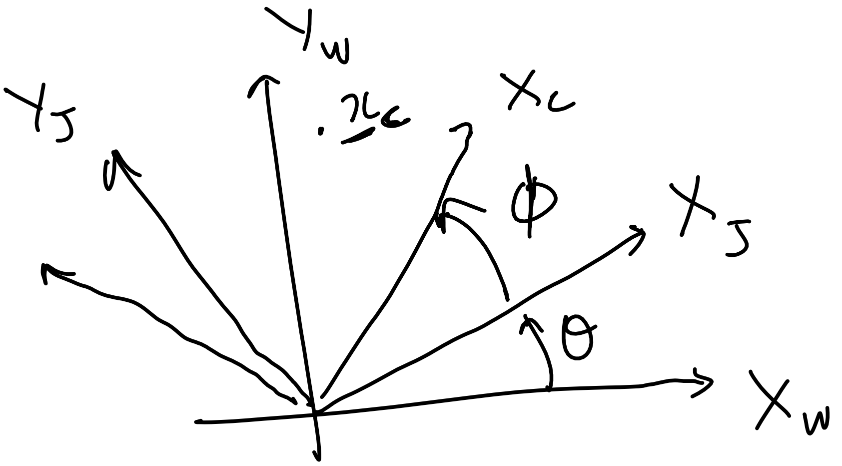
θ_x = roll

θ_y = pitch

θ_z = yaw

$$\boxed{R = \underset{\text{Yaw}}{R(\theta_z)}\,\underset{\text{Pitch}}{R(\theta_y)}\,\underset{\text{Roll}}{R(\theta_x)}}$$

Chain rotation, translation, transformations



$$\underline{x}_J = {}^J_C R(\phi)\, \underline{x}_c$$

$$\underline{x}_w = {}^w_J R(\theta)\, \underline{x}_J$$

$$= R(\theta)\left[R(\phi)\underline{x}_c\right]$$

$$= \underbrace{\left(R(\theta)\ R(\phi)\right)}_{{}^w_c R} \underline{x}_c$$

$$\boxed{{}^w_c R = {}^w_{\cancel{J}} R(\theta)\ {}^{\cancel{J}}_c R(\phi)}$$

$$\boxed{{}^w_c T = {}^w_J T\ {}^J_c T}$$

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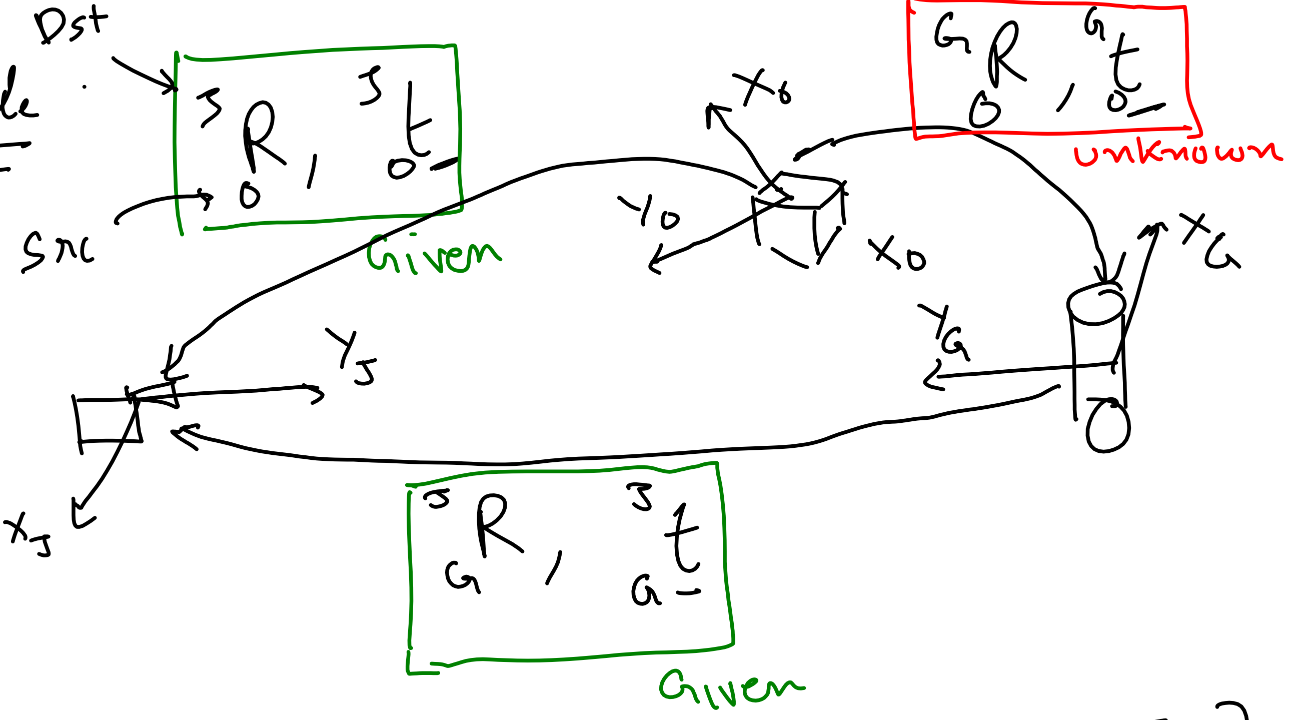
Example :



$${}^J_O T = \begin{bmatrix} {}^J_O R & {}^J_O \underline{t} \\ 0^T & 1 \end{bmatrix}, \quad {}^J_G T = \begin{bmatrix} {}^J_G R & {}^J \underline{t} \\ 0^T & 1 \end{bmatrix}$$

$${}^G_O T = \begin{bmatrix} {}^G_O R & {}^G_O \underline{t} \\ 0^T & 1 \end{bmatrix} \qquad {}^G_O T = \left({}^J_G T\right)^{-1} {}^J_O T$$

$$= {}^G_J T \, {}^J_O T \, ..$$

$$\underbrace{{}^G_O T \, \underline{x}_G}_{\underline{x}_G} = \underbrace{\left({}^G_J T \underbrace{\left({}^J_O \underline{T} \quad \underline{x}_O\right)}_{x_J}\right))}_{\underline{x}_G}$$

①

$$R_{3D} = \underset{\underset{\text{yaw}}{\uparrow}}{R(\theta_z)}\,\underset{\underset{\text{pitch}}{\uparrow}}{R(\theta_y)}\,\underset{\underset{\text{roll}}{\uparrow}}{R(\theta_x)}$$

$$\xleftarrow{\hspace{6em}}$$

$$\dot{\theta}_x \xrightarrow{\text{then}} \theta_y \xrightarrow{\text{then}} \theta_z$$

This sequences

$$\overrightarrow{XYZ}$$

$$ZYX$$

$$\vdots$$

6 possibl = 3!

Euler angle representation of 3D rotation is a sequence of rotation around standard axis

Euler representation with $\overrightarrow{XYZ}$ then

$$R_{3D} = \underleftarrow{R(\theta_z)R(\theta_y)R(\theta_x)}$$

Conversion from Euler angles to Rotation matrix

How to do the opposite?

convert from Rotation matrix to Euler angles?

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}$$

$$\theta = ? = \tan^{-1}\left(\frac{r_{21}}{r_{11}}\right) \in \left[\frac{-\pi}{2}, \pi/2\right]$$

$$\underbrace{\theta = \operatorname{arctan2}(r_{21}, r_{11})} \in [-\pi, \pi)$$

3D

$$R = \overset{\overset{z}{\downarrow}}{R(\psi)}\,\overset{\overset{y}{\downarrow}}{R(\phi)}\,\overset{\overset{x}{\downarrow}}{R(\theta)}$$

$$= \begin{bmatrix} c(\psi) & -s(\psi) & 0 \\ s(\psi) & c(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c(\phi) & 0 & s(\phi) \\ 0 & 1 & 0 \\ -s(\phi) & 0 & c(\phi) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c(\theta) & -s(\theta) \\ 0 & s(\theta) & c(\theta) \end{bmatrix}$$

$$= \left[\begin{array}{c|c|c} \boxed{c(\psi)c(\phi)} & \begin{array}{c} c(\psi)c(\phi)s(\theta) \\ -s(\psi)c(\theta) \end{array} & \begin{array}{c} c(\psi)s(\phi)c(\theta) \\ +s(\psi)s(\theta) \end{array} \\ \hline \boxed{s(\psi)c(\phi)} & \begin{array}{c} s(\psi)s(\phi)s(\theta) \\ +c(\psi)c(\theta) \end{array} & \begin{array}{c} s(\psi)s(\phi)c(\theta) \\ -c(\psi)s(\theta) \end{array} \\ \hline \boxed{-s(\phi)} & \boxed{c(\phi)s(\theta)} & \boxed{c(\phi)c(\theta)} \end{array}\right]$$

$$= \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & V_{32} & r_{33} \end{bmatrix}$$

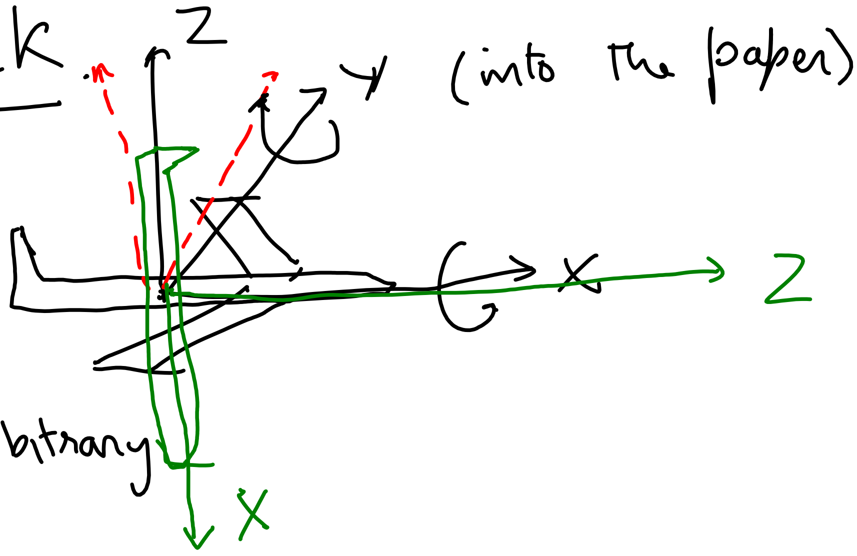
$$\phi = -\sin^{-1}(r_{31}) \in [0, \pi]$$

$$\frac{r_{21}}{r_{11}} = \frac{\sin(\psi)\,\cancel{c(\phi)}}{\cos(\psi)\,\cancel{c(\phi)}} \Rightarrow \psi = \operatorname{arctan2}(r_{21}, r_{11})$$

$$\theta = \operatorname{arctan2}(r_{32}, r_{33})$$

Conversion from Rotation matrix to Euler angles

Gimbal lock



$\theta_x = 30°$ ← arbitrary

$[\theta_y = 90°]$

$\theta_z = 45°$ ← arbitrary

$$\begin{tikzcd}\text{Euler angles} \arrow[r, shift left, "\text{deterministically}"] & \text{Rot mat} \arrow[l, shift left, "\text{multiple solutions}"]\end{tikzcd}$$

Other representations. It is impossible to unambiguously represent 3D rotation with only 3 number

Degree of freedom

3D rot = 3 DOF

but needs 4 numbers + 1 constraint to represent it

---

② Axis-angle representation ←

↓

Rot mats

↓

Algebra

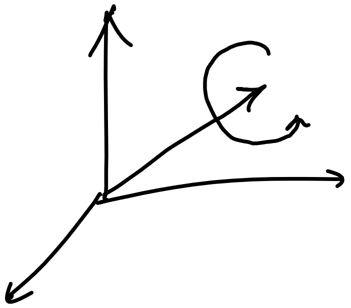
③ Quaternions

Quaternion algebra

≡ complex numbers

---

② Axis-angle representation



Any 3D rotation can be represented as a unit vector (axis) and rotation angle around it.

axis = $\underline{a} = [a_x, a_y, a_z]$

angle = θ.

Constraint : $\|a\|_2 = \sqrt{a_x^2 + a_y^2 + a_z^2} = 1$

Free scalars = 4
constraint = 1
} Degree of freedom = 3

DOF of 2D Rot matrix = ?

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

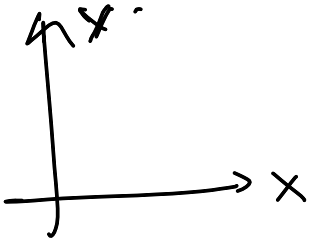
Free scalars = 4

True for N-D rotations

Two vector constraints : $\left\{ \boxed{\begin{aligned} R^T R = I = R R^T \\ \det(R) = +1 \end{aligned}} \right.$ $\begin{aligned} &\text{2 scalar constraints} \\ &\text{1 scalar constraint} \end{aligned}$

Check for valid rotation

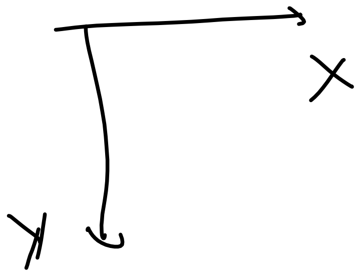
Reflection also matrices satisfy



not Rotation

reflection

det(Reflection) = −1



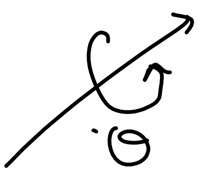
Rodrigues rotation formula

Axis angle $(\theta, \hat{k})$ → Rot matrix

$$\underline{R = I + \sin\theta [k_x] + (1-\cos\theta)[k_x]^2}$$

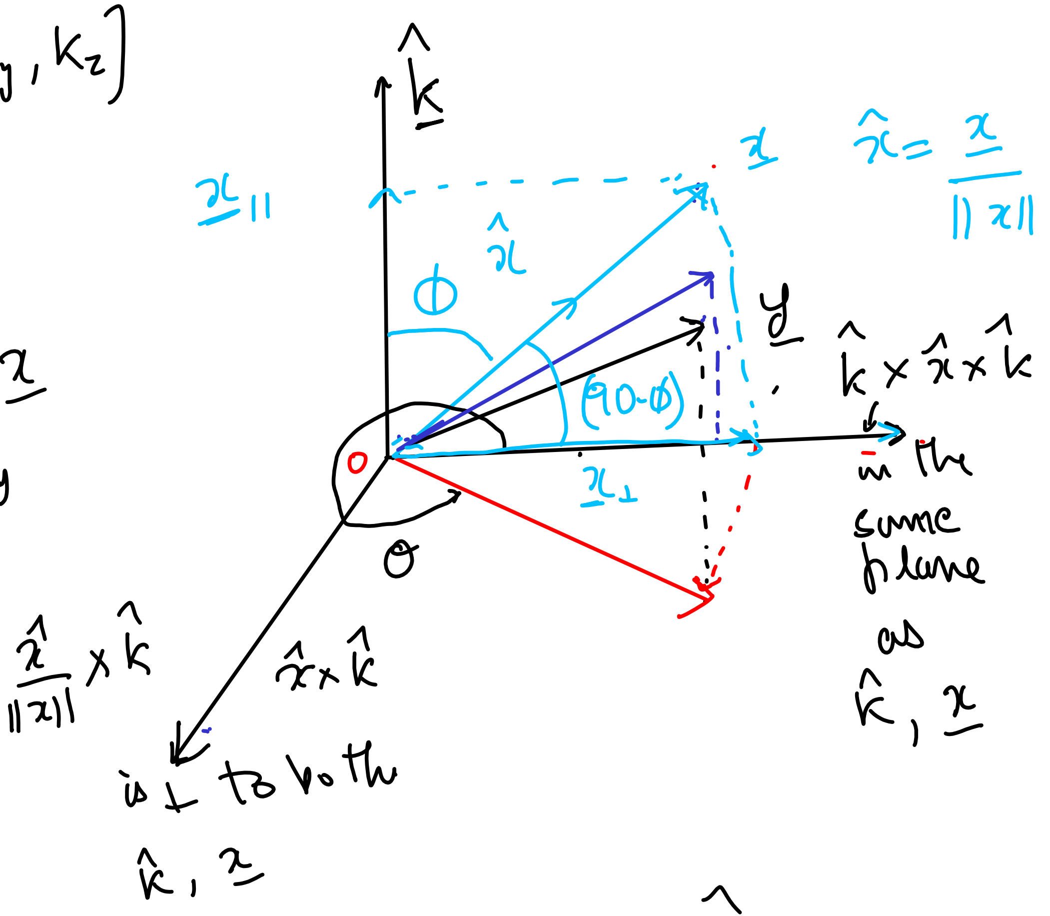
axis angle representation → Rotation matrix (3D)

Rodrigues rotation formula



$\underline{\hat{k}} = [k_x, k_y, k_z]$

$\underline{y}$ = by rotating $\underline{x}$ around $\underline{\hat{k}}$ by angle θ



In the plane of $\underline{\hat{k}}$ and $\underline{\hat{k} \times \hat{x} \times \hat{k}}$,

$\underline{x}$ can be projected into two component

$$\underline{x} = \underline{x}_{\|} + \underline{x}_{\perp}$$

$$\underline{x}_{\|} = (\hat{k} \cdot \underline{x})\, \hat{k}$$

$$\underline{x}_{\perp} = \left[\left(\hat{k} \times (\hat{\underline{x}} \times \hat{k})\right) \cdot \underline{x}\right] (\hat{k} \times \hat{x} \times \hat{k})$$

$$\underline{x}_{\perp} = \hat{k} \times (\underline{x} \times \hat{k}) = \hat{k} \times (-\hat{k} \times \underline{x}) = -\hat{k} \times (\hat{k} \times \underline{x}$$

$$\hat{k} \cdot \underline{x} = |\hat{k}|\,|x| \cos\phi$$

$$= |x| \cos\phi$$

→ $|x \times \hat{k}| = |x||k| \sin\phi$

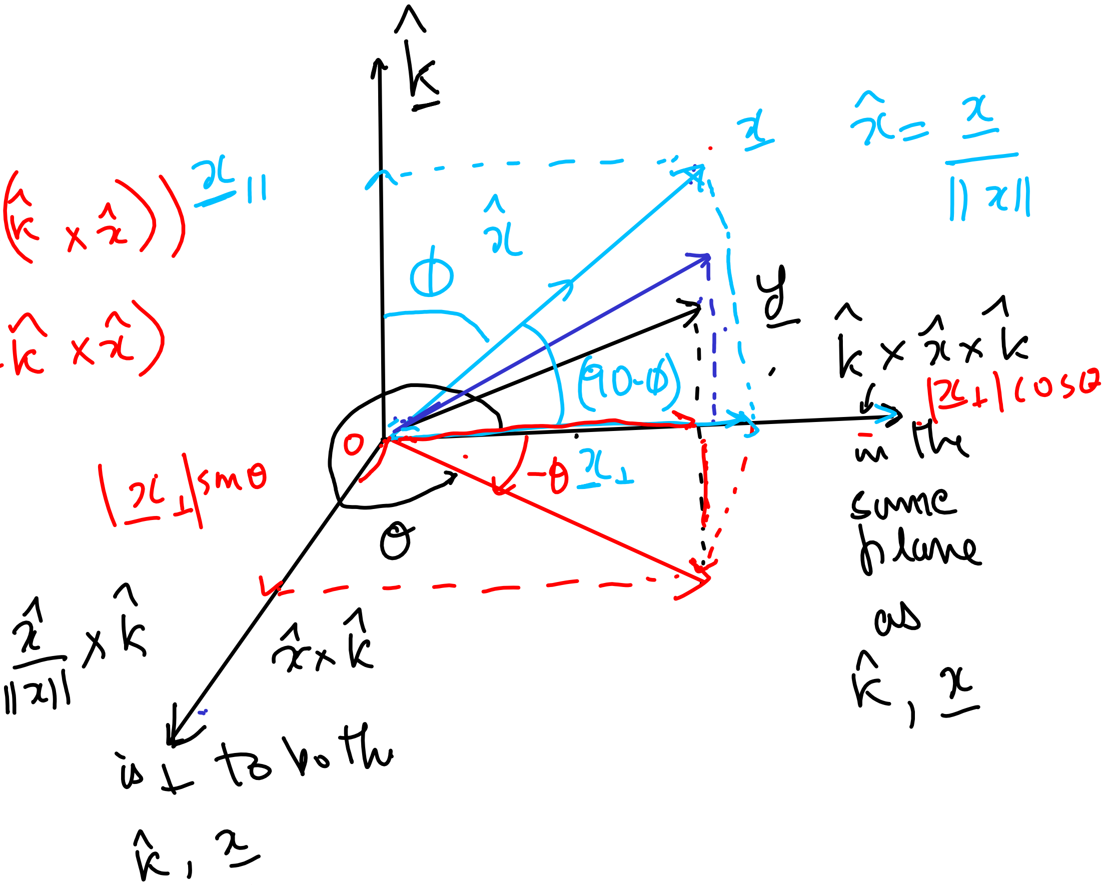
→ $x \times \hat{k} = (\hat{\underline{x}} \times \hat{k})\, (|x| \sin\phi)$

$$\boxed{|x| \sin\phi} = |x| \sqrt{1 - \cos^2\phi}$$

$$\underline{y} = \underline{x_{\parallel}} + \underline{x_{\perp rot}}$$

$$\underline{x_{\perp rot}} = |\underline{x_\perp}|\cos(\theta)\left(-\hat{k} \times \left(\hat{k} \times \hat{x}\right)\right)$$

$$+ |\underline{x_\perp}| \sin(-\theta)\left(-\hat{k} \times \hat{x}\right)$$



$$\underline{x_{\perp rot}} = \underbrace{|\underline{x_\perp}|}\cos\theta\left(-\hat{k} \times \underbrace{\left(\hat{k} \times \hat{x}\right)}\right)$$

$$- \underbrace{|\underline{x_\perp}|}\sin\theta\left(\underbrace{-\hat{k} \times \hat{x}}\right)$$

$$= \cos\theta\left(-\hat{k} \times \left(\hat{k} \times \underline{x}\right)\right)$$

$$- \sin\theta\left(-\hat{k} \times \underline{x}\right)$$

$$= \sin\theta\left(\hat{k} \times \underline{x}\right)$$

$$- \cos\theta\left(\hat{k} \times \left(\hat{k} \times \underline{x}\right)\right)$$

$$\left(\hat{k} \times \hat{x}\right)|\underline{x_\perp}| =$$

$$= \underline{\left(\hat{k} \times \hat{x}\right)|x| \sin\phi} = \underline{\hat{k} \times \underline{x}}$$

$$|\underline{a} \times \underline{b}| = |a||b| \sin\phi$$

$$\underline{y} = x_{\parallel} + \underline{x_\perp} \, rot$$

Rodrigues formula

$$\boxed{\underline{y} = \left(\hat{k} \cdot \underline{x}\right)\hat{k} + \sin\theta\left(\hat{k} \times \underline{x}\right) - \cos\theta\left(\hat{k} \times \hat{k} \times \underline{x}\right)}$$

$$x_{\parallel} = (\hat{k}\cdot\underline{x})\,\hat{k} \qquad\qquad \underline{x} = \underline{x}_{\parallel} + \underline{x}_{\perp}$$

$$= \underline{x} - x_{\perp}$$

$$= \underline{x} - \left(-\hat{k}\times(\hat{k}\times\underline{x})\right)$$

$$= x + \hat{k}\times(\hat{k}\times\underline{x})$$

$$\underline{y} = \underline{x} + \underbrace{\hat{k}\times(\hat{k}\times\underline{x})} + \sin\theta\,(\hat{k}\times\underline{x}) - \cos\theta\,(\hat{k}\times(\hat{k}\times\underline{x}))$$

$$\boxed{\underline{y} = \underline{x} + \sin\theta\,(\underline{\hat{k}}\times\underline{x}) + (1-\cos(\theta))\,(\hat{k}\times(\hat{k}\times\underline{x}))}$$

$$\underline{y} = \underbrace{R(\theta,\hat{k})}_{\text{matrix}}\,\underline{x}$$

Writing cross product using matrix notation

$$\underline{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

$$\underline{a}\times\underline{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \qquad \underline{b} = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

$$= \hat{i}(a_y b_z - b_y a_z) \qquad = \begin{bmatrix} a_y b_z - b_y a_z \\ -a_x b_z + b_x a_z \\ a_x b_y + b_x a_y \end{bmatrix}$$

$$+ j($$

$$\underbrace{\begin{bmatrix} a_y b_z - b_y a_z \\ -a_x b_z + b_x a_z \\ a_x b_y + b_x a_y \end{bmatrix}}_{\underline{a} \times \underline{b}} = \boxed{\underbrace{\begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ a_y & a_x & 0 \end{bmatrix}}_{\lfloor a_\times \rfloor}} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

Cross product matrix of vector

$$\hat{k} \times \underline{x} = \underbrace{\lfloor k_\times \rfloor}_{K} \underline{x} = K \underline{x}$$

$$\hat{k} \times (\hat{k} \times \underline{x}) = \lfloor K_\times \rfloor (\lfloor K_\times \rfloor \underline{x}) = K(K \underline{x})$$

$$= K^2 \underline{x}$$

$$\underline{y} = \underline{x} + \sin\theta \, (K \underline{x}) + (1 - \cos\theta) K^2 \underline{x}$$

$$= \underbrace{\left[I_{3 \times 3} + \sin\theta \, K + (1 - \cos\theta) K^2 \right]}_{R(\theta, \hat{k})} \underline{x}$$

Oct 25

Convert from Rot matrix to axis angle representatio

$$\boxed{\alpha \underline{y} = R \underline{y}}$$

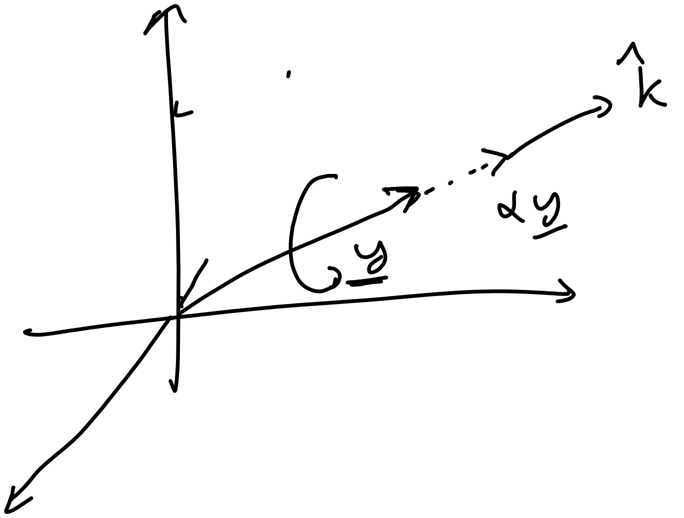
R is 3 × 3 matrix

$\underline{y}$ is 3×1 vector

α is scalar

$$A\underline{v} = \lambda \underline{v}$$

Eigen vectors of a matrix A are all the solution for $\underline{v}$ from the above equation



Corresponding solutions of λ are called eigen values

$$A\underline{v} - \lambda \underline{v} = 0$$

$$\Rightarrow \underbrace{(A - \lambda I)}_{\text{matrix}} \underbrace{\underline{v}}_{\text{vector}} = 0$$

$\left.\det(A - \lambda I) = 0 \right\} \rightarrow$ solve for eigen value

---

The axis of rotation is an eigen vector of the rotation matrix.

$$\boxed{\underline{y} = R\underline{x} \Rightarrow \|y\| = \|x\|}$$

$$\|\underline{y}\| = \sqrt{y_1^2 + y_2^2 + \dots + y_n^2}$$

$$= \sqrt{\begin{bmatrix} y_1 & y_2 & & y_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}$$

$$= \sqrt{\underline{y}^T \underline{y}}$$

$$\underline{y}^T \underline{y} = (R\underline{x})^T (R\underline{x})$$

$$= (\underline{x}^T R^T)(R\underline{x})$$

$$= \underline{x}^T (R^T R)\underline{x}$$

$$= x^T x$$

$$\underset{\text{eigen value}}{1}\,\underset{\text{eigen vector}}{\underline{k}} = R\,\underset{\text{eigen vector}}{\underline{k}}$$

if $\underline{k}$ is along the axis of rotation for R

The axis of rotation is the eigen vector of the rotation matrix corresponding to eigen value 1.

$$\boxed{\det(R - \lambda I) = 0}$$

Use numpy.linalg.eig( ) to find eigen value and eigen vector

For 3 x 3 Rotation matrix

if $\underline{\theta} = 0^\circ$ or 180°

special cases

else ($\underline{\theta} \neq 0, \theta \neq 180^\circ$)

$$R = \begin{bmatrix} r_{11} & r_{12} & \\ r_{21} & & \\ & & r_{33} \end{bmatrix}$$

$$\underline{\hat{k}} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} = \begin{pmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{pmatrix} \frac{1}{(2\sin\theta)}$$

How to compute angle θ in axis-angle

$$R = I + K\sin\theta + (1-\cos\theta)K^2$$

$$K^2 = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix} \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix}$$

A is a symmetric matrix if $A^T = A$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & - & - \\ a_{13} & - & - \end{bmatrix}$$

A is a skew-symmetric matrix if $A^T = -A$

$$\begin{bmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & \\ -a_{13} & 0 & 0 \end{bmatrix}$$

$$K^2 = \begin{bmatrix} -(k_z^2 + k_y^2) & k_y k_x & k_z k_x \\ k_y k_x & -(k_x^2 + k_z^2) & k_z k_y \\ k_z k_x & k_z k_y & -(k_x^2 + k_y^2) \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & & \\ & 0 & \\ & & 0 \end{bmatrix} \sin\theta + $$

$$\begin{bmatrix} -(k_z^2 + k_y^2) & k_y k_x & k_z k_x \\ k_y k_x & -(k_x^2 + k_z^2) & k_z k_y \\ k_z k_x & k_z k_y & -(k_x^2 + k_y^2) \end{bmatrix} (1-\cos\theta)$$

$\hat{k} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix}$ is a unit vector

$$k_x^2 + k_y^2 + k_z^2 = 1$$

$$R = \begin{bmatrix} r_{11} & & \\ & r_{22} & \\ & & r_{33} \end{bmatrix}$$

$$\text{trace}(R) = r_{11} + r_{22} + r_{33} = 1+1+1 - 2(\underbrace{k_x^2 + k_y^2 + k_z^2}_{1})(1-\cos\theta)$$

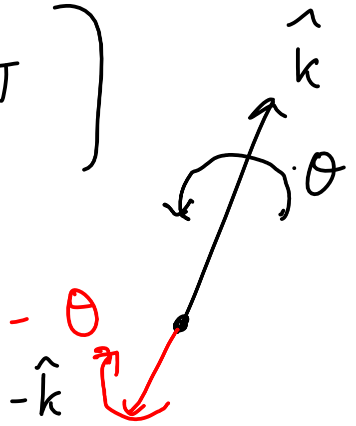
$$\Rightarrow tr(R) = r_{11} + r_{22} + r_{33} = 3 - 2 + 2\cos\theta$$

$$\Rightarrow tr(R) = 1 + 2\cos\theta$$

$$\Rightarrow \boxed{\underline{\theta} = \cos^{-1}\left(\frac{tr(R) - 1}{2}\right)} \in [0, \pi]$$

If $\theta = \frac{3\pi}{4} = 270°$ rotated around $\hat{k}$

this is same as $\theta = \frac{\pi}{4} = 90°$ " " $-\hat{k}$



if $\underline{\theta} = 180°$

$$k_x = \pm\sqrt{(r_{11}+1)/2}$$
$$k_y = \pm\sqrt{(r_{22}+1)/2}$$
$$k_z = \pm\sqrt{(r_{33}+1)/2}$$

$$r_{11} = 1 + 0 + \underset{\underset{\cos(-\pi)}{\uparrow}}{(1-(-1))}\left(-k_x^2+k_y^2\right)$$
$$r_{12} = 1 + 0 +$$

$$R = \begin{bmatrix} r_{11} & r_{12} & \\ r_{21} & & \\ & & r_{33} \end{bmatrix}$$

else $(\underline{\theta} \neq 0, \theta \neq 180°)$

$$\underline{\hat{k}} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} = \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} \frac{1}{(2\sin\theta)}$$

Quaternions

$$\underline{q} = w + ix + jy + kz$$

$$\boxed{\begin{array}{l} i^2 = -1,\ j^2 = -1,\ k^2 = -1, \\ ijk = -1 \end{array}}$$

complex numbers

$$\hat{k} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} \quad \theta$$

Axis angle to quat

$$\underline{q} = \left[\underset{\cos}{\cancel{\sin}}\left(\frac{\theta}{2}\right),\ k_x \underset{\sin}{\cancel{\cos}}\left(\frac{\theta}{2}\right),\ k_y \underset{\sin}{\cancel{\cos}}\left(\frac{\theta}{2}\right),\ k_z \underset{\sin}{\cancel{\cos}}\left(\frac{\theta}{2}\right)\right]$$

$w =$ $\qquad x \qquad y \qquad z$

$$\underline{q} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \quad \text{OR} \quad q = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

$$q = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

$$\boxed{\cancel{\theta = \sin^{-1}(2w)}}$$

$$\boxed{\hat{k} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Big/ \underset{\sin}{\cancel{\cos}}(\theta/2)}$$

$$\sqrt{x^2+y^2+z^2} = \underset{\sin}{\cancel{\cos}}\left(\frac{\theta}{2}\right)$$

$$w = \underset{\cos}{\cancel{\sin}}\,\frac{\theta}{2}$$

$$\boxed{\theta = \cancel{2}\,\mathrm{arctan}2\left(w, \sqrt{x^2+y^2+z^2}\right)}$$

$$\theta = 2\,\mathrm{arctan}2\left(\sqrt{x^2+y^2+z^2}, w\right)$$