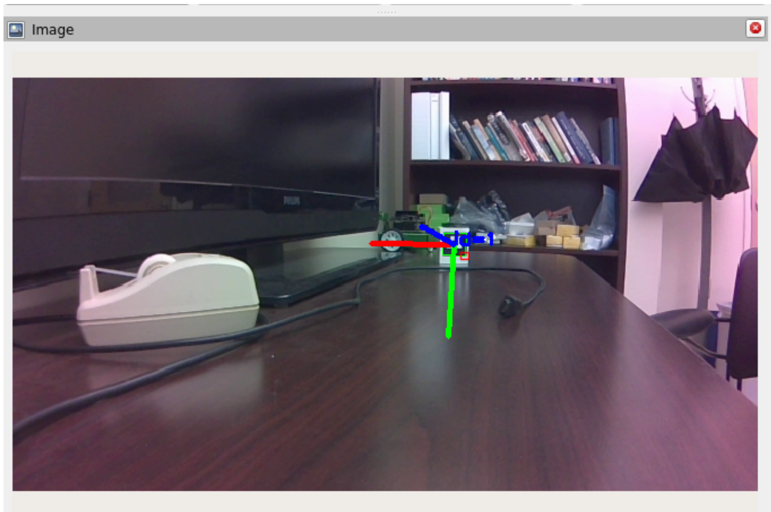
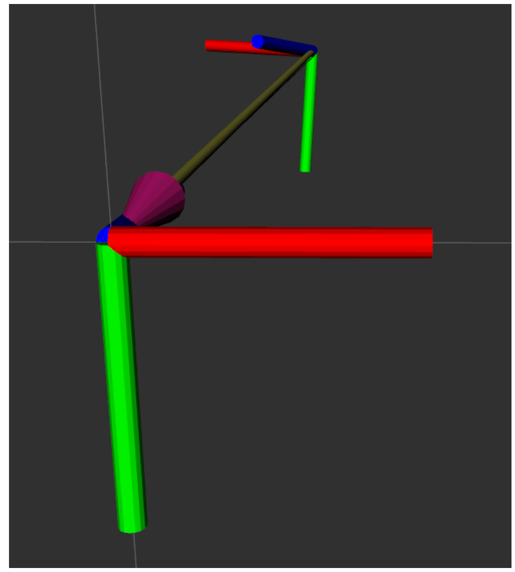
Rotations and translations/Coordinate transformations

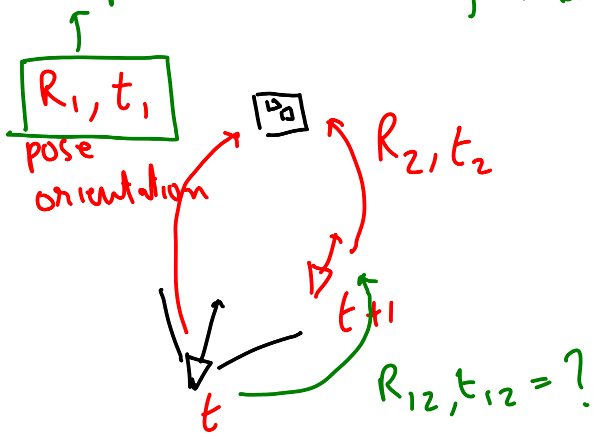




```
root@nano-4gb-jp45:/home/jetbot/ece417# ros2 topic info /
aruco_detections
Type: aruco_opencv_msgs/msg/ArucoDetection
Publisher count: 1
Subscription count: 0

root@nano-4gb-jp45:/home/jetbot/ece417# ros2 topic echo --once /
aruco_detections
header:
  stamp:
    sec: 1727998810
    nanosec: 924374790
  frame_id: /v4l_frame
markers:
- marker_id: 1
  pose:
    position:
      x: 0.08918172498053901
      y: -0.10849999597426438
      z: 0.980432215194246
    orientation:
      x: -0.02973468393320003
      y: 0.9811997541144667
      z: -0.03227342049856023
      w: -0.18793966432455353
boards: []
---
```

joint representation = Transformation matrix

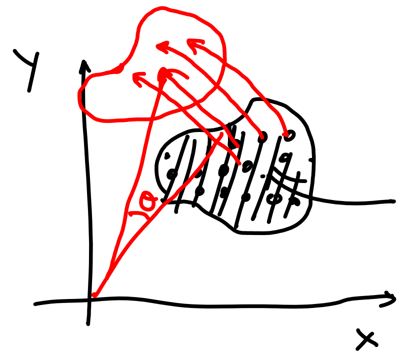


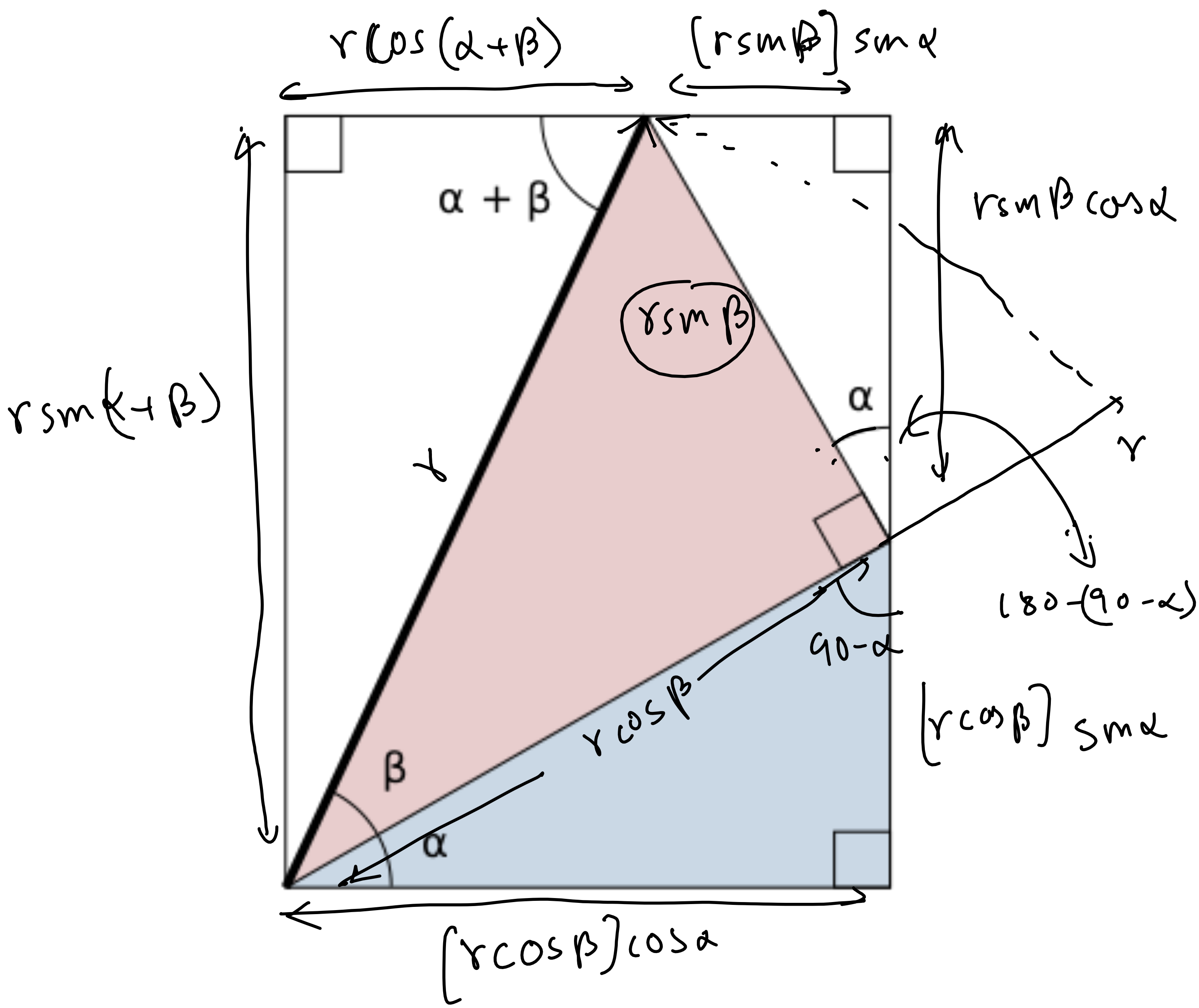
① Rigid bodies are sets of points

↳ Rotate

↳ Translate

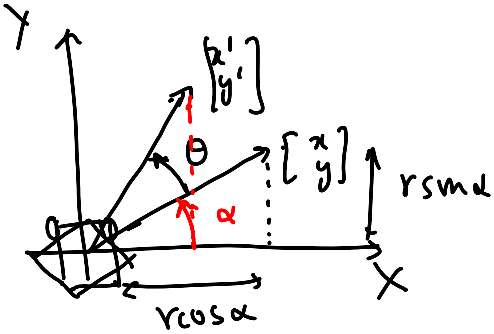
All points in the rigid body Rotate and translate by the same amount





2D Rotation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = ? \quad \text{in terms of} \quad \theta \text{ and } \begin{bmatrix} x \\ y \end{bmatrix}$$



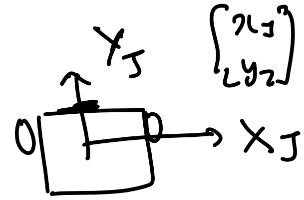
$$x = r\cos\alpha$$
$$y = r\sin\alpha \qquad r = \sqrt{x^2+y^2}$$

$$\begin{aligned} x' &= r\cos(\theta+\alpha) \\ &= r\cos\theta\cos\alpha - r\sin\theta\sin\alpha \\ &= x\cos\theta - y\sin\theta \end{aligned}$$

$$y' = r\sin(\theta+\alpha) = r\sin\theta\cos\alpha + r\cos\theta\sin\alpha = x\sin\theta + y\cos\theta$$

$$\underbrace{\begin{bmatrix} x' \\ y' \end{bmatrix}_{2\times1}}_{p'} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}_{2\times2}}_{R(\theta)} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}_{2\times1}}_{p}$$

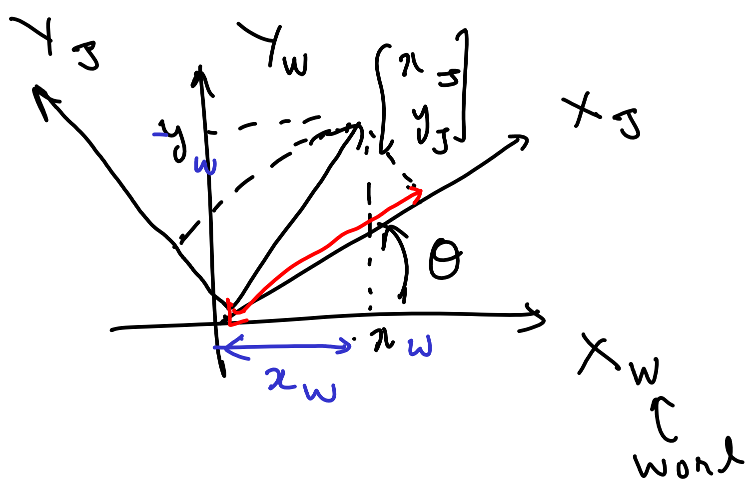
$$p' = R(\theta)\, p$$



$$\underbrace{\begin{bmatrix} x_w \\ y_w \end{bmatrix}}_{p_w} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{{}^W R_J(\theta)} \underbrace{\begin{bmatrix} x_J \\ y_J \end{bmatrix}}_{p_J}$$

SOURCE

DESTINATION



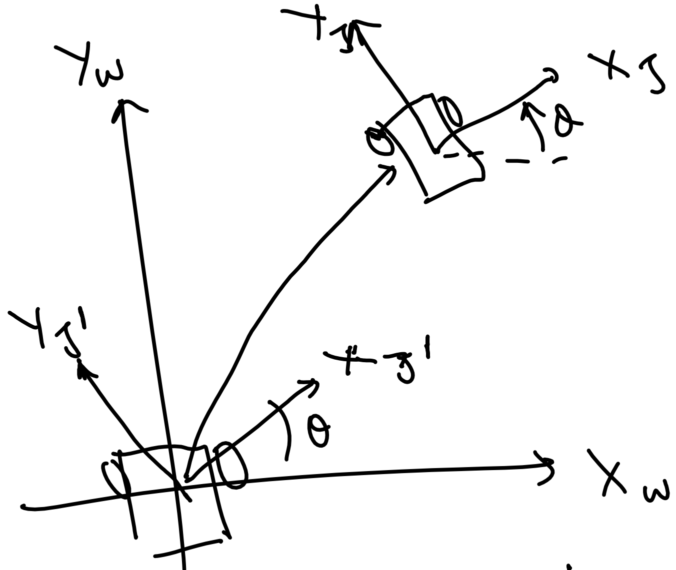
$$\boxed{p_w = {}^W R_J(\theta)\, p_J}$$

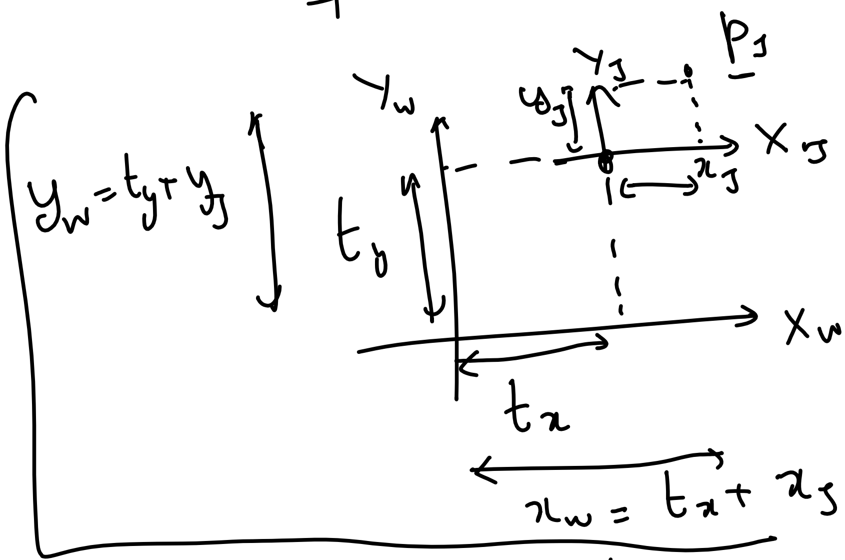
$$\underline{p}_{J'} = {}^{J'}R_J \, \underline{p}_J$$

$$\underline{p}_W = \underline{p}_{J'} + {}^{W}\underline{t}_{J'}$$

$$\underline{p}_W = {}^{J'}R_J \, \underline{p}_J + {}^{W}\underline{t}_{J'}$$

$$\boxed{\underline{p}_W = {}^{W}R_J \, \underline{p}_J + {}^{W}\underline{t}_J}$$





$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \begin{bmatrix} x_J + t_x \\ y_J + t_y \end{bmatrix}$$

$$= \begin{bmatrix} x_J \\ y_J \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix}$$

$$\underline{p}_w = \underline{p}_J + {}^{W}\underline{t}_J$$

Matrices = capital letters

Vectors = small letters with underscore

Scalars = small letter without underscore

Transformation matrix

$$\underline{p}_w = {}^{w}R_J\, \underline{p}_J + {}^{w}\underline{t}_J$$

orientation $\quad$ position

$$\underbrace{{}^{w}R_J,\ {}^{w}\underline{t}_J}$$

$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \begin{bmatrix} \underline{r_{11}} & \underline{r_{12}} \\ r_{21} & r_{22} \end{bmatrix}_{2\times 1} \begin{bmatrix} x_J \\ y_J \end{bmatrix} + \begin{bmatrix} \underline{t_1} \\ t_2 \end{bmatrix}_{2\times 1}$$

$$\begin{bmatrix} x_w \\ y_w \\ 1 \end{bmatrix}_{3\times 1} = \underbrace{\left[\begin{array}{cc|c} r_{11} & r_{12} & t_1 \\ r_{21} & r_{22} & t_2 \\ \hline 0 & 0 & 1 \end{array}\right]_{3\times 3}}_{{}^{w}T_J} \begin{bmatrix} x_J \\ y_J \\ 1 \end{bmatrix}_{3\times 1}$$

homogeneous coordinate

$$\begin{bmatrix} \underline{p}_w \\ 1 \end{bmatrix} = \left[\begin{array}{c|c} {}^{w}R_J & {}^{w}\underline{t}_J \\ \hline \underline{0}^T_{2\times 1} & 1 \end{array}\right] \begin{bmatrix} \underline{p}_J \\ 1 \end{bmatrix}$$

Properties of a Rotation matrix

(1) What is the $\underline{\text{inverse}}$ of a Rotation matrix

$$AA^{-1} = A^{-1}A = I$$

Prereq

$$\underbrace{U = \begin{bmatrix} | & & | \\ \underline{u}_1 & \cdots & \underline{u}_n \\ | & & | \end{bmatrix}}$$

orthonormal matrix

when $\underline{u}_i^T \underline{u}_i = 1$

$\underline{u}_i^T \underline{u}_j = 0 \quad i \neq j$

$$R = \begin{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} & \begin{matrix} -\sin\theta \\ \cos\theta \end{matrix} \end{bmatrix}$$

is an orthonormal matrix

$$\begin{bmatrix} \cos\theta & \sin\theta \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = \cos^2\theta + \sin^2\theta$$
$$= 1$$

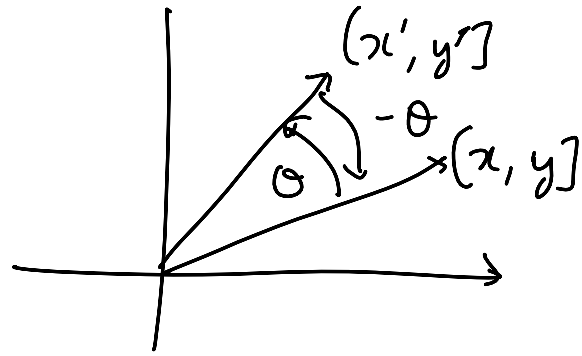
$$\begin{bmatrix}\cos\theta & \sin\theta\end{bmatrix}\begin{bmatrix}-\sin\theta\\ \cos\theta\end{bmatrix} = -\cos\theta\sin\theta + \sin\theta\cos\theta$$

$$= 0$$

For all orthonormal matrices, their inverse is their transpose $U^{-1} = U^T$

$$\underbrace{R^{-1} = R^T}$$

$$\begin{bmatrix}x'\\ y'\end{bmatrix} = \begin{bmatrix} R \end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$$



$$\begin{bmatrix}x\\ y\end{bmatrix} = R^{-1}\begin{bmatrix}x'\\ y'\end{bmatrix}$$

$$R^{-1} = \begin{bmatrix}\cos(-\theta) & -\sin(-\theta)\\ \sin(-\theta) & \cos(-\theta)\end{bmatrix} = \underbrace{\begin{bmatrix}\cos\theta & \sin(\theta)\\ -\sin\theta & \cos\theta\end{bmatrix}}_{R^T}$$

① Rotation matrices are orthonormal / Rotation matrix inverse is its transpose

$$RR^T = R^T R = \underline{I}$$

Are all orthonormal matrices rotation matrices?

Answer : No

② $\det(R) = 1$

In general an orthonormal matrix, U

$$\det(U) \in \{-1, +1\}$$

Reflection matrix (-1)   Rotation matrix ($+1$)

$$\det\left(\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\right) = \cos^2\theta - (-\sin^2\theta)$$
$$= 1$$

2D transformation matrix

$$\begin{bmatrix} \underline{p}_W \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} {}^W R_J & {}^W \underline{t}_J \\ \underline{0}^T & 1 \end{bmatrix}}_{{}^W T_J} \begin{bmatrix} \underline{p}_J \\ 1 \end{bmatrix}$$

$${}^J T_W = ? = {}^W T_J^{\,-1}$$

$$\underline{p}_W = {}^W R_J \, \underline{p}_J + {}^W \underline{t}_J$$

$$\underline{p}_W - {}^W \underline{t}_J = {}^W R_J \, \underline{p}_J$$

$$\boxed{\underline{p}_J = \frac{\underline{p}_W - {}^W t_J}{{}^W R_J}}$$

BLASPHEMY ✗

$\boxed{\text{Left}}$ multiply by ${}^{w}R_J^T$

$${}^{w}R_J^T\left(\underline{p}_w - {}^{w}t_J\right) = \underbrace{\left({}^{w}R_J\right)^T {}^{w}R_J}_{I}\ \underline{p}_J$$

$${}^{w}R_J^T\,\underline{p}_w - {}^{w}R_J^T\,{}^{w}\underline{t}_J = \underline{p}_J$$

$$\underbrace{\begin{bmatrix} {}^{w}\underline{R}_J^T & \left(-{}^{w}R_J^T\,{}^{w}\underline{t}_J\right) \\ \underline{0}_{2\times 1}^T & 1 \end{bmatrix}}_{{}^{w}T_J^{-1}} \begin{bmatrix} \underline{p}_w \\ 1 \end{bmatrix} = \begin{bmatrix} \underline{p}_J \\ 1 \end{bmatrix}$$

$${}^{J}T_w = \begin{bmatrix} {}^{w}R_J^T & -{}^{w}R_J^T\,{}^{w}\underline{t}_J \\ 0_{2\times 1}^T & 1 \end{bmatrix}$$

Intuitive interpretation

(1) Rotation $\longrightarrow$ (2) translation

$$\underbrace{\underline{p}' = R\,\underline{p} + \underline{t}}_{\text{Rotation first}} \neq \underline{p}' = R\underbrace{(\underline{p} + \underline{t})}_{\text{translation first}}$$

$\underline{SO(2)}$

Special Orthogonal group

$\{R_{2\times 2} : R^T R = I, \det(R) = 1\}$

$\underbrace{O(2)}$

Orthogonal $\underbrace{\text{group}}$

$\{U_{2\times 2} : U_{2\times 2}^T U_{2\times 2} = I_{2\times 2} \}$

$SE(2)$

Special Euclidean group

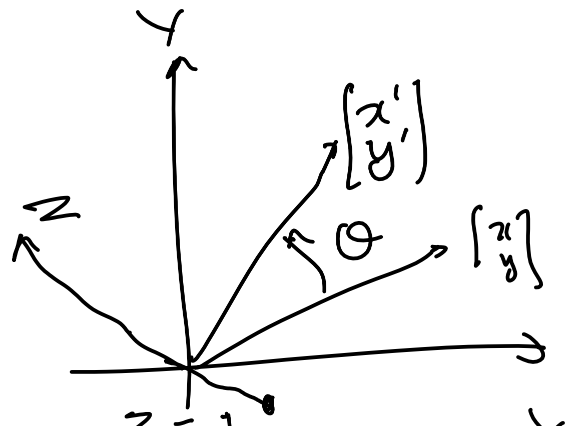
$\{R_{2\times 2}, \underline{t}_{2\times 1} : R_{2\times 2} \in \underline{SO(2)}, \underline{t} \in \boxed{\mathbb{R}^2}\}$

2D vectors in real space

3D Rotations

2D Rotation around Z-axis

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \underbrace{\begin{bmatrix} \boxed{R(\theta_z)} & \begin{matrix} 0 \\ 0 \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{bmatrix}}_{R_z(\theta_z)} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$



$\underset{-s\theta}{X} \to Y \to Z$

Right hand coordinate frame

$X \to Y : Z$

$Y \to Z : X$

$Z \to X : Y$

$\underbrace{}_{\text{curl finger}}$ $\underbrace{}_{\text{thumb}}$

around X-axis

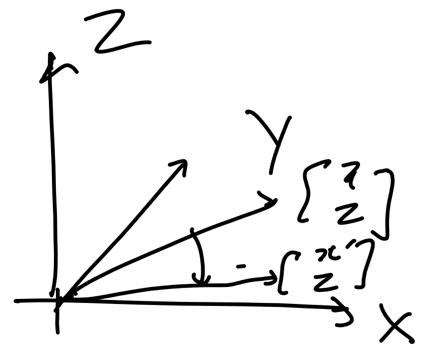
$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta_x & -s\theta_x \\ 0 & s\theta_x & c\theta_x \end{bmatrix}}_{R_x(\theta_x)} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

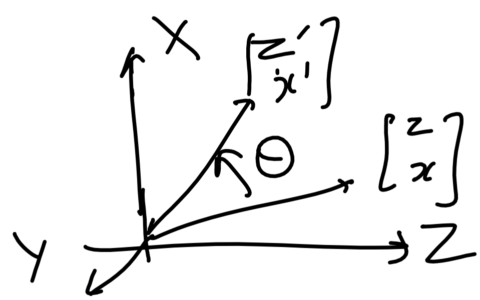
$R_x(\theta_x)$ $X \to \boxed{\underset{-s\theta}{Y} \to Z}$

around Y-axis

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \underbrace{\begin{bmatrix} c\theta & 0 & s\theta \\ 0 & 1 & 0 \\ -s\theta & 0 & c\theta \end{bmatrix}}_{R_y(\theta_y)} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$R_y(\theta_y)$ $X \to \boxed{Y} \to \overset{-s\theta}{Z}$





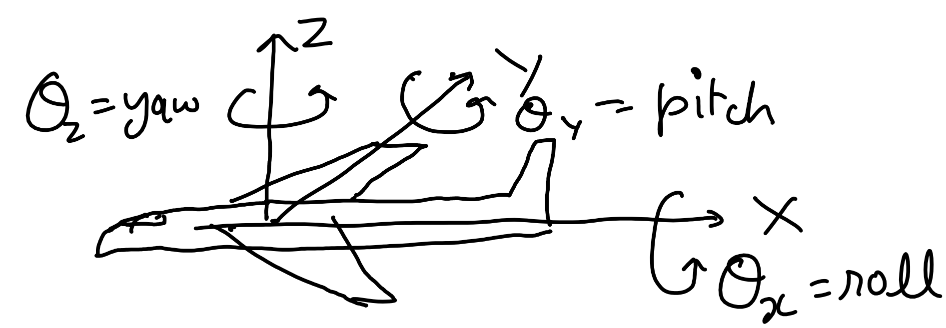
$$\underset{3\times3}{R}\,p = \left(\underset{z}{R}(\theta_z)\left[\underset{y}{R}(\theta_y)\left[\underset{x}{R}(\theta_x)_{3\times3}\,\underline{p}\right]\right]\right)$$

All 3D rotations can be decomposed into $\underbrace{\text{3 axis aligned rotions}}$

three angles of rotions are called $\underline{\text{Euler angles}}$

12 possible combination

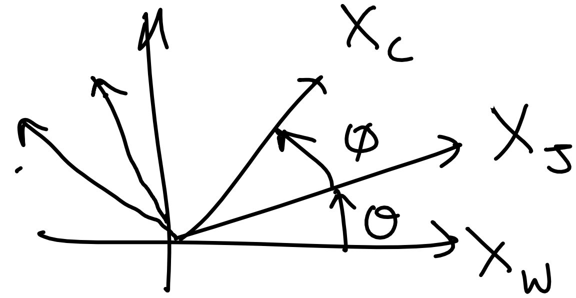
Roll - Pitch - Yaw
(1) (2) (3)



$$R = \underleftarrow{R_z(\theta_z)\,R_y(\theta_y)\;R_x(\theta_x)}$$

Application of Rotation matrices is from right to left

Composition of Rotations / Transformations



$\underline{p}_J = {}^J R_C \, \underline{p}_C$

$\underline{p}_W = {}^W R_J \, \underline{p}_J$

$$\boxed{{}^W R_C = \,?}$$

$$\underline{p}_W = {}^W R_J \left[\underbrace{{}^J R_C \, \underline{p}_C}_{\underline{p}_J} \right]$$

I_3

$(R_1 R_2)$

orthonormal ?

$\det(R_1 R_2) \stackrel{?}{=} 1$

$$\underline{p}_W = \underbrace{{}^W R_J \, {}^J R_C}_{{}^W R_C} \, \underline{p}_C$$

$$\boxed{{}^W R_C = {}^W R_J \, {}^J R_C}$$

$$\underbrace{(R_1 R_2)}^{-1} = R_2^{-1} R_1^{-1}$$

$$\det(R_1 R_2) = \det(R_1)\det(R_2) = 1$$

$$= R_2^T R_1^T = \underbrace{(R_1 R_2)}^{T}$$

$${}^{W}T_{C} = {}^{W}T_{J}\,{}^{J}T_{C}$$

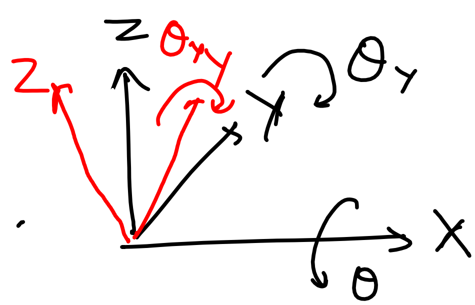
why are there 12 possible Euler angles?

$$R = \begin{array}{l} R_Z \; R_Y \; R_X \\ R_Y \; R_Z \; R_X \end{array} \Bigg\} \quad 6 = 3! \text{ possibilities permutations}$$

when chaining rotations there are two possibilities

✓

① either rotate along the new axis after first rotation.

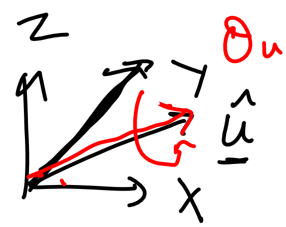


② or rotate along the original axis

3D Rotation representation

- → 3×3, $\underbrace{\det(R) = 1, \quad R^T R = I}$ $\equiv$ Rotation matrix
- → 3 Euler angles $(\theta_x, \theta_y, \theta_z)$
- → Quaternions $(\underbrace{x, y, z, w})$
- → Axis-angle representation $(\hat{\underline{u}}, \theta_u)$

(Rodrigues formula)



$$A = \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}_{3\times 3} = 9 \text{ DOF}$$

$A^T = A$ if A is symmetric $=$ 6 DOF

$U_{3\times 3}$ is orthonormal → 3 DOF

$$U = \begin{bmatrix} \underline{u}_1 & \underline{u}_2 & \underline{u}_3 \end{bmatrix}$$

$\underline{u}_1^T \underline{u}_2 = 0 \qquad \underline{u}_1^T \underline{u}_1 = 1$

$\underline{u}_2^T \underline{u}_3 = 0 \qquad \underline{u}_2^T \underline{u}_2 = 1$

$\underline{u}_3^T \underline{u}_1 = 0 \qquad \underline{u}_3^T \underline{u}_3 = 1$

Conversions

Roll-pitch-Yaw $\longrightarrow$ Rotation matrix

$$R = R_z(\theta_z)\, R_y(\theta_y)\, R_x(\theta_x)$$

$$= \underbrace{\begin{bmatrix} c\theta_z & -s\theta_z & 0 \\ s\theta_z & c\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta_y & 0 & s\theta_y \\ 0 & 1 & 0 \\ -s\theta_y & 0 & c\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta_x & -s\theta_x \\ 0 & s\theta_x & c\theta_x \end{bmatrix}}$$

Rotation matrix $\longrightarrow$ Roll pitch Yaw angles

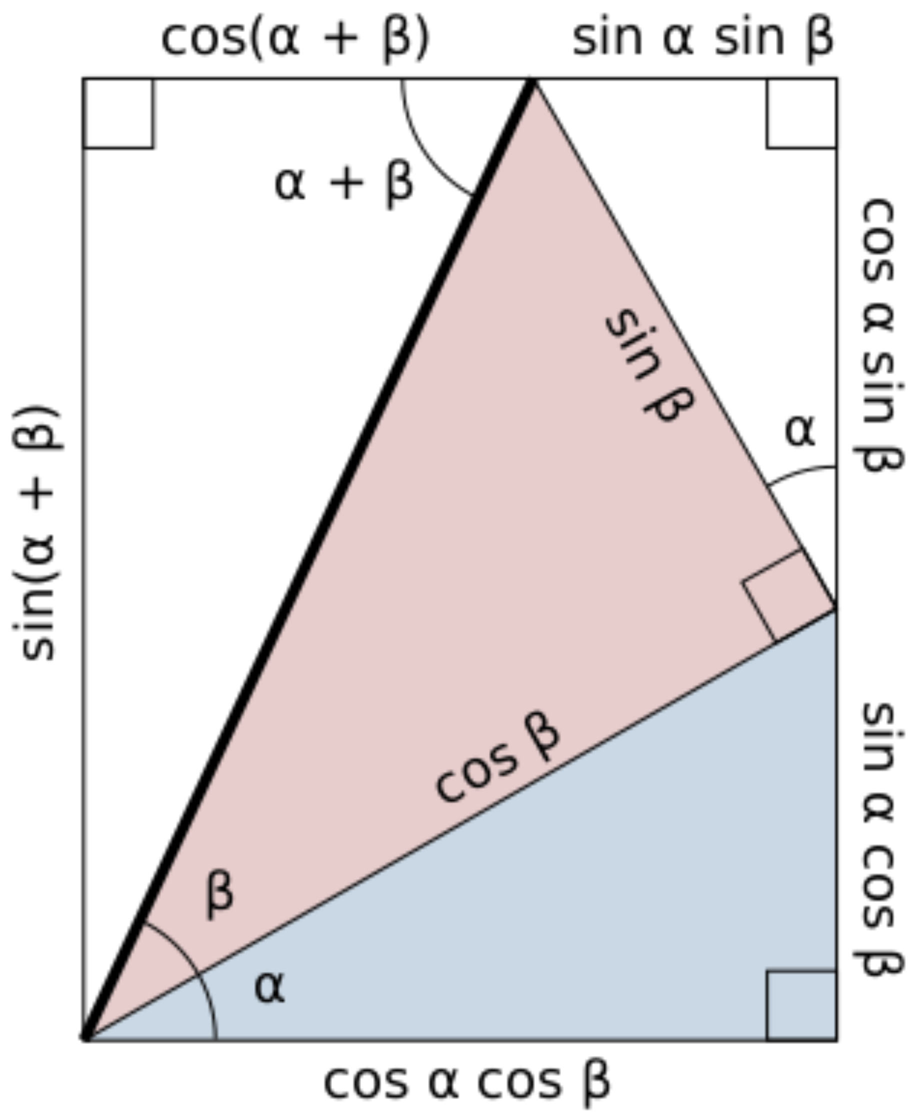
$$R = \left[\begin{array}{c|c|c} c(\theta_z)\,c(\theta_y) & \cdots & \cdots \\ \hline s(\theta_z)\,c(\theta_y) & \cdots & \cdots \\ \hline -s(\theta_y) & c(\theta_y)\,s(\theta_x) & c(\theta_y)\,c(\theta_x) \end{array}\right]$$

$$= \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\times \left[\tan(\theta_x) = \frac{r_{32}}{r_{33}} \right] \Rightarrow \theta_x = \tan^{-1}\left(\frac{r_{32}}{r_{33}}\right)$$

$\in (-\pi/2, \pi/2)$

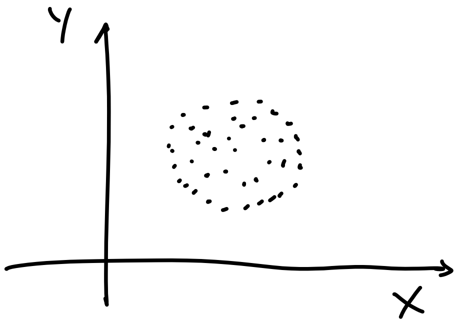
$$\theta_x = \arctan 2\left(r_{33}, r_{32} \right) \in (-\pi, \pi]$$

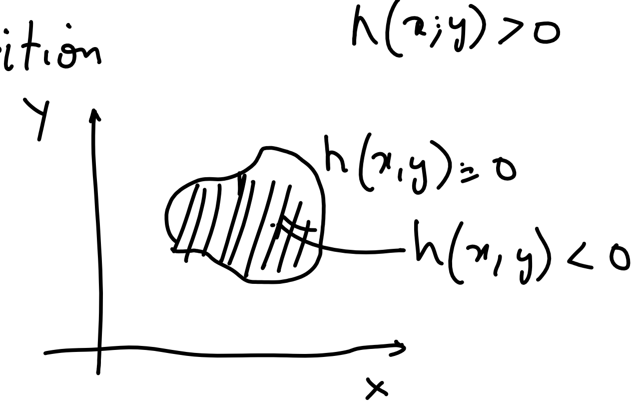

cos(α + β)
sin α sin β
α + β
sin β
α
cos α sin β
sin(α + β)
cos β
β
α
sin α cos β
cos α cos β

2D coordinate transforms

Rotations + translations

Orientation + position



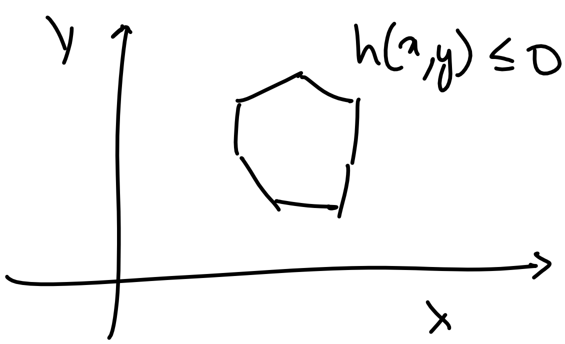


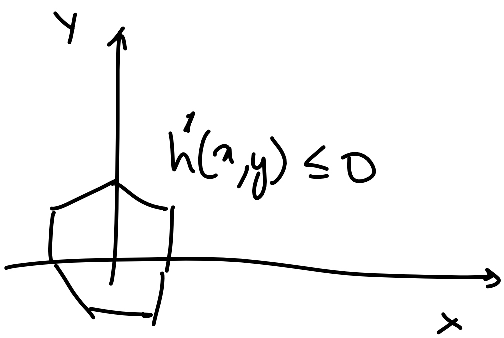
$$\text{Object} = \{(x,y) \in \mathbb{R}^2 : h(x,y) \le 0\}$$

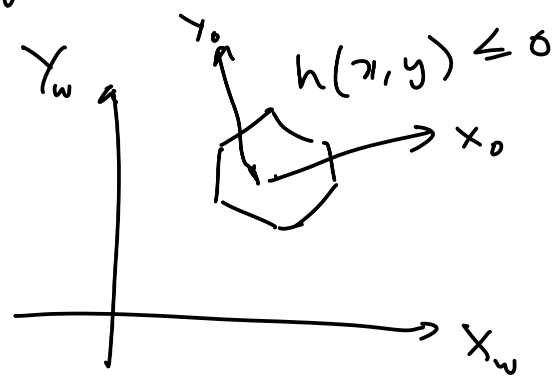
Set of all points $(x,y) \in \mathbb{R}^2$ ← set of all real numbers

such that

$h(x,y) \le 0$



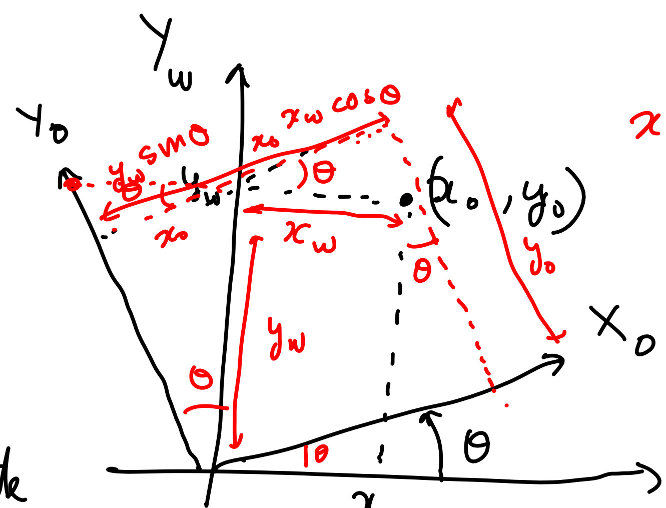




2D Rotations

① Rotation

② Translation



$$x_o = x_w \cos\theta + y_w \sin\theta$$

Problem: (x_o, y_o) is given in (X_o, Y_o) coordinate frame. $(X_o Y_o)$ has been rotated by angle θ w.r.t. (X_w, Y_w)

Find (x_w, y_w) in world coordinate frame

Proof using <u>Basis vectors</u>

In Linear algebra, <u>Basis vectors</u> are set of orthonormal unit vectors that <u>span</u> the entire <u>space</u>

<u>Span</u> is the set of all vectors that can be obtained by linear combinations of a given set of vectors

$$\text{Span}\{\underline{a}, \underline{b}\} = \{\underbrace{\alpha \underline{a} + \beta \underline{b}}_{\in \mathbb{R}^n}, \ \alpha \in \mathbb{R}, \beta \in \mathbb{R}\}$$

$\underline{a}, \underline{b} \in \mathbb{R}^n$

<u>Standard Basis vector</u>.

For example, in $\mathbb{R}^2$ $\quad \hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \hat{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

in $\mathbb{R}^3$ $\quad \hat{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

in $\mathbb{R}^n$ $\quad \hat{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n \times 1} \quad \dots \quad \hat{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}_{n \times 1}$

<u>Basis vectors</u> for $\mathbb{R}^n$

↳ ① All vectors must be perpendicular/orthogonal to each other

↳ ② They must be unit vectors

↳ ③ They must span the entire space $\mathbb{R}^n$

Let Basis vector for (X_w, Y_w) be standard basis vector

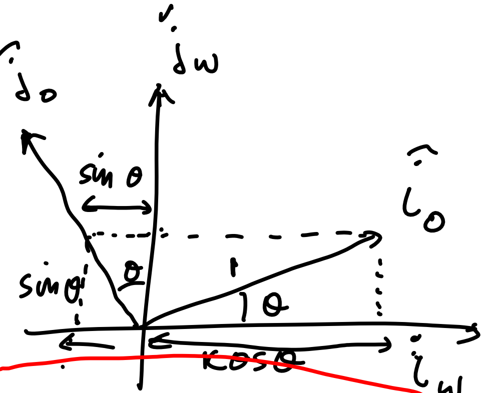
$\hat{i}_w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \hat{j}_w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Any point in world coordinate frame

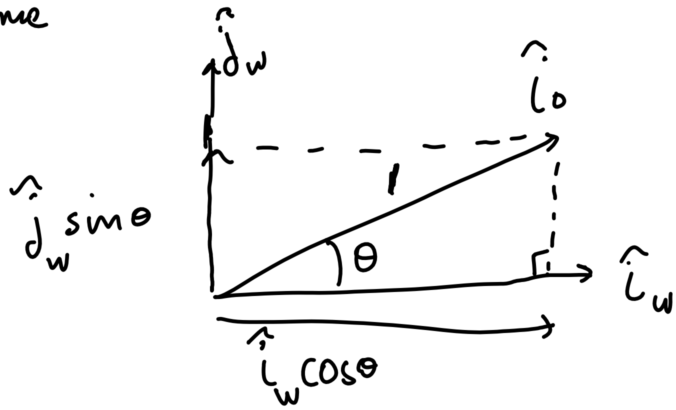
$$\begin{bmatrix}x_w\\ y_w\end{bmatrix} = x_w \underbrace{\begin{bmatrix}1\\ 0\end{bmatrix}}_{\hat{i}_w} + y_w \underbrace{\begin{bmatrix}0\\ 1\end{bmatrix}}_{\hat{j}_w}$$

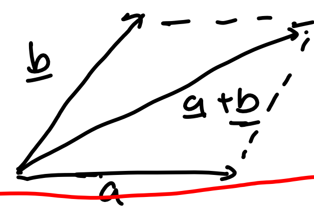
Any point in the object coordinate frame

$$\begin{bmatrix}x_o\\ y_o\end{bmatrix} = x_o \hat{i}_o + y_o \hat{j}_o$$



$$\boxed{\hat{i}_o = \cos\theta\, \hat{i}_w + \sin\theta\, \hat{j}_w}$$





$$\boxed{\hat{j}_o = -\hat{i}_w \sin\theta + \hat{j}_w \cos\theta}$$

$$\overbrace{x_w \hat{i}_w + y_w \hat{j}_w}^{\text{world}} = \overbrace{x_o \hat{i}_o + y_o \hat{j}_o}^{\text{object}}$$

$$\underbrace{\begin{bmatrix}x_w\\ y_w\end{bmatrix}}_{\substack{\text{Because we are using}\\ \text{standard basis for world}\\ \text{coordinate frame}}} = x_o\left[\cos\theta\, \hat{i}_w + \sin\theta\, \hat{j}_w\right] + y_o\left[-\hat{i}_w \sin\theta + \hat{j}_w \cos\theta\right]$$

$$= \left[x_o\cos\theta - y_o\sin\theta\right]\hat{i}_w + \left[x_o\sin\theta + y_o\cos\theta\right]\hat{j}_w$$

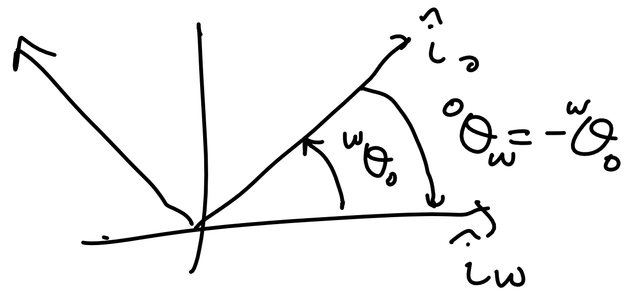
$$\hat{i}_w \to \begin{bmatrix}1\\ 0\end{bmatrix} \qquad \hat{j}_w \to \begin{bmatrix}0\\ 1\end{bmatrix}$$

$$= \begin{bmatrix} x_o\cos\theta - y_o\sin\theta \\ x_o\sin\theta + y_o\cos\theta \end{bmatrix}$$

$$\begin{bmatrix}x_w\\ y_w\end{bmatrix} = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x_o\\ y_o\end{bmatrix}$$

$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \begin{bmatrix} {}^{w}R_{o}(\theta) \end{bmatrix} \begin{bmatrix} x_o \\ y_o \end{bmatrix}$$

$${}^{w}R_{o}(\theta) := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



$$\begin{bmatrix} x_o \\ y_o \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}} \begin{bmatrix} x_w \\ y_w \end{bmatrix}$$

$${}^{o}R_{w}(\theta) = {}^{w}R_{o}(-\theta) = {}^{w}R_{o}^{T}(\theta)$$

$$= \begin{bmatrix} \cos\theta & +\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_w \\ y_w \end{bmatrix}$$

$$\begin{bmatrix} {}^{w}R_{o}^{T} \end{bmatrix} {}^{w}R_{o} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -c(\theta)s(\theta) + s(\theta)c(\theta) \\ -s(\theta)c(\theta) + c(\theta)s(\theta) & s^2(\theta) + c^2(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_{2\times 2}$$

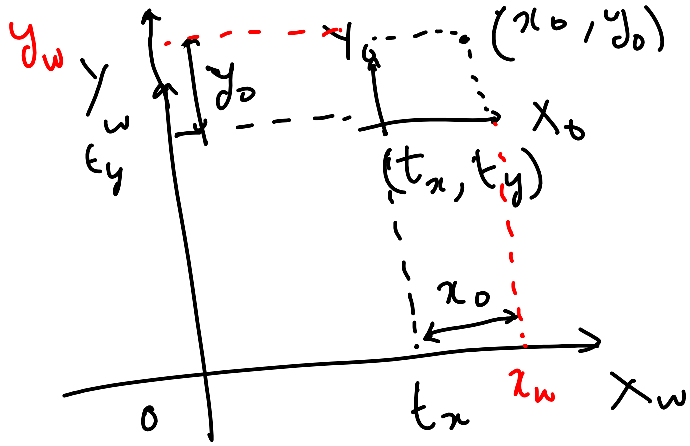
$$\underbrace{R^T R = I}$$

$$(A^{-1})A = I$$

$$R^{-1} = R^T$$

2D Translation

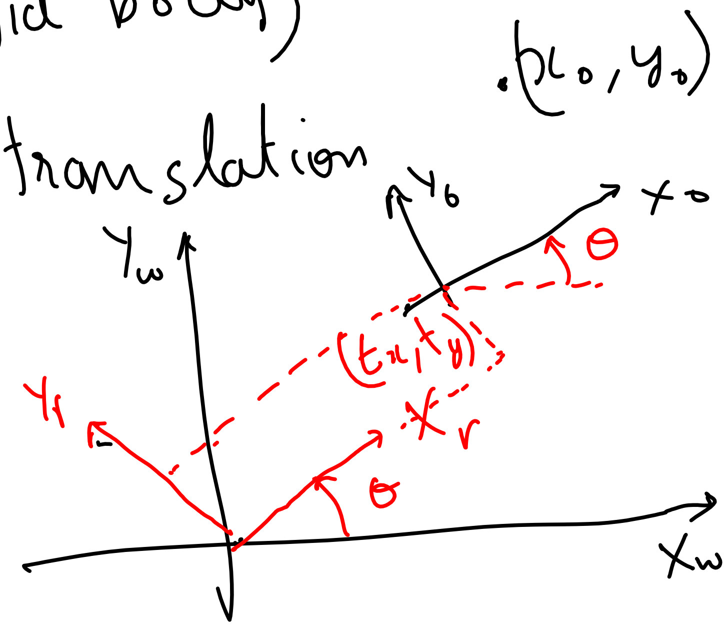
$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \underbrace{\begin{bmatrix} t_x \\ t_y \end{bmatrix}}_{\text{translation vector}} + \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$



2D transformation (Rigid body)

Rotation followed by translation

$$\begin{bmatrix} x_r \\ y_r \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$



$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \begin{bmatrix} t_x \\ t_y \end{bmatrix} + \begin{bmatrix} x_r \\ y_r \end{bmatrix}$$

$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \underbrace{\begin{bmatrix} t_x \\ t_y \end{bmatrix}}_{\underline{t}}$$

$$\boxed{\underline{x}_w = {}^{w}_{0}R(\theta)\,\underline{x}_0 + {}^{w}_{0}\underline{t}}$$

$$\underline{x}_o = ?(\underline{x}_w)$$

$$\underline{x}_w - {}^{w}_{o}\underline{t} = {}^{w}_{o}R\,\underline{x}_o \qquad {}^{w}_{o}R = {}^{w}_{o}R(\theta)$$

Multiply on the left of both sides by ${}^{w}_{o}R^T$

$${}^{w}_{o}R^T\left(\underline{x}_w - {}^{w}_{o}\underline{t}\right) = \left({}^{w}_{o}R^T\,{}^{w}_{o}R\right)\underline{x}_o$$

$$\frac{\underline{x}_w - {}^{w}_{o}\underline{t}}{{}^{w}_{o}R} = \underline{x}_o$$

NEVER EVER DO THIS

$$\underline{x}_o = {}^{w}_{o}R^T\,\underline{x}_w - {}^{w}_{o}R^T\,{}^{w}_{o}\underline{t} \quad \text{— (1)}$$

$$\underline{x}_o = {}^{o}_{w}R\cdot\underline{x}_w + {}^{o}_{w}\underline{t} \quad \text{— (2)}$$

Compare (1) and (2)

$$\boxed{{}^{o}_{w}R = {}^{w}_{o}R^T} \quad \text{and} \quad \boxed{{}^{o}_{w}\underline{t} = -{}^{w}_{o}R^T\,{}^{w}_{o}\underline{t}}$$

$$\underline{x}_w = {}^{w}_{o}R\,\underline{x}_o + {}^{w}_{o}\underline{t}$$

$$\begin{bmatrix} \underline{x}_w \\ 1 \end{bmatrix} = \begin{bmatrix} x_w \\ y_w \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} {}^w_o R_{2\times 2} & {}^w_o \underline{t}_{2\times 1} \\ 0^T_{1\times 2} & 1 \end{bmatrix}}_{{}^w_o T_{3\times 3}} \begin{bmatrix} \underline{x}_{o\,2\times 1} \\ 1 \end{bmatrix}$$

Block matrix

Block Matrix multiplication

$$\checkmark \quad \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} AE+BG & AF+BH \\ CE+DG & CF+DH \end{bmatrix}$$

$$\begin{bmatrix} {}^w_o R_{2\times 2} & {}^w_o \underline{t}_{2\times 1} \\ 0^T_{1\times 2} & 1 \end{bmatrix} \begin{bmatrix} \underline{x}_{o\,2\times 1} \\ 1 \end{bmatrix} = \begin{bmatrix} {}^w_o R\, \underline{x}_o + {}^w\underline{t}_o \\ \underline{0}^T \underline{x}_o + 1 \end{bmatrix}$$

$$= \begin{bmatrix} {}^w_o R\, \underline{x}_o + {}^w\underline{t}_o \\ 1 \end{bmatrix} = \begin{bmatrix} \underline{x}_w \\ 1 \end{bmatrix}$$

$$\underline{x}_w = {}^w_o T\, \underline{x}_o$$

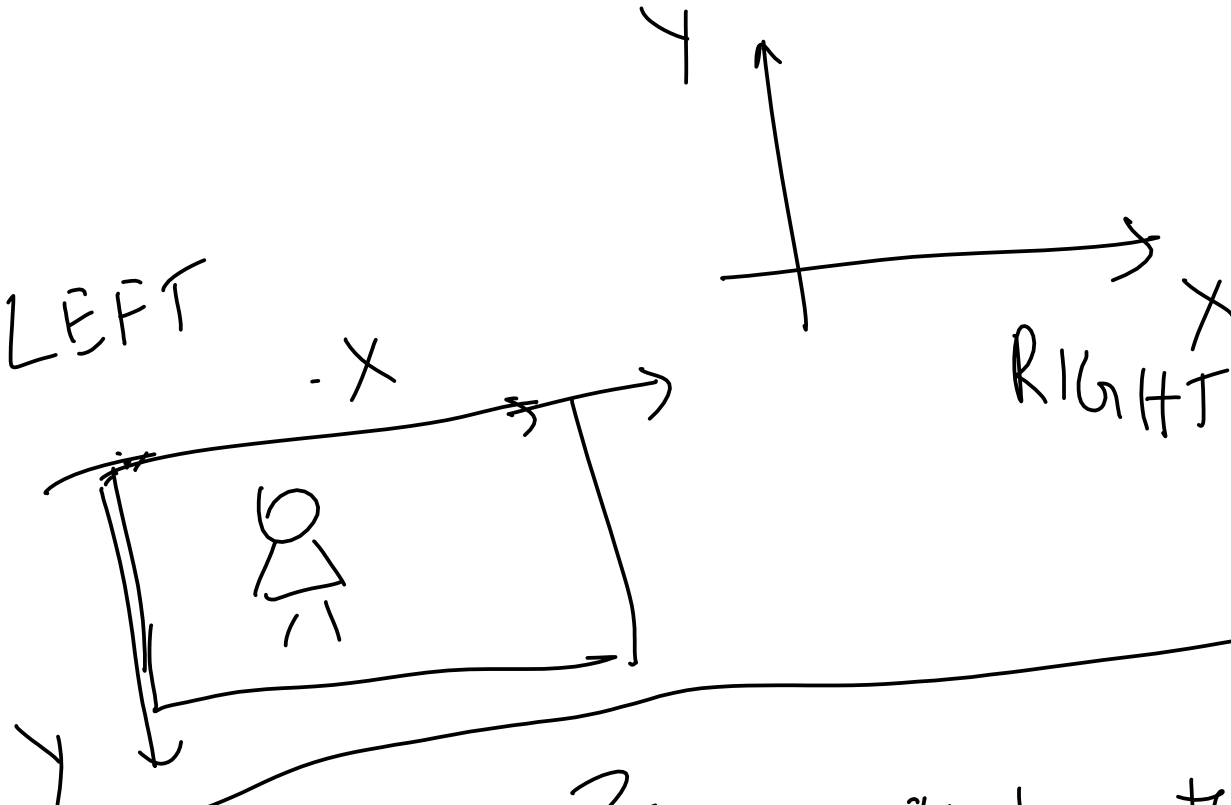
Transformation matrix

$${}^w T_o = \begin{bmatrix} {}^w_o R_{2\times 2} & {}^w_o \underline{t}_{2\times 1} \\ 0^T_{1\times 2} & 1 \end{bmatrix}$$

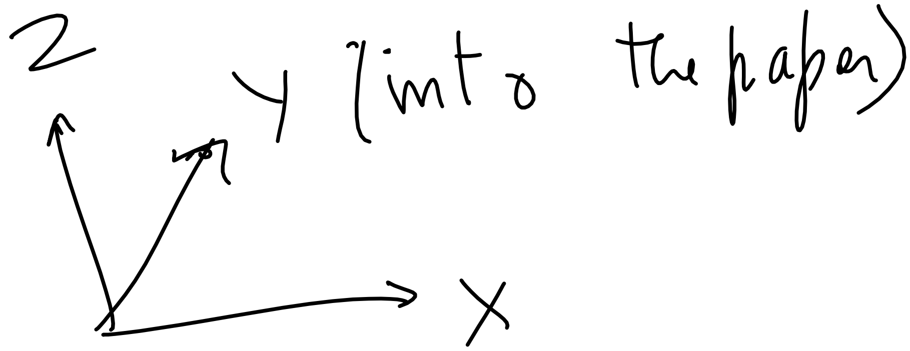
In 2D and 3D

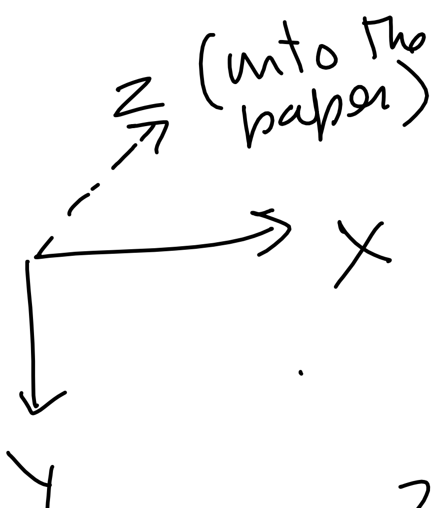
$\boxed{\text{Right hand}}$ ✓

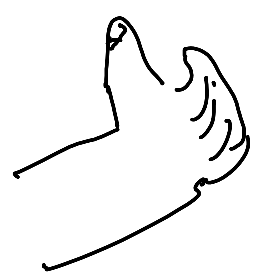
Left hand ✗



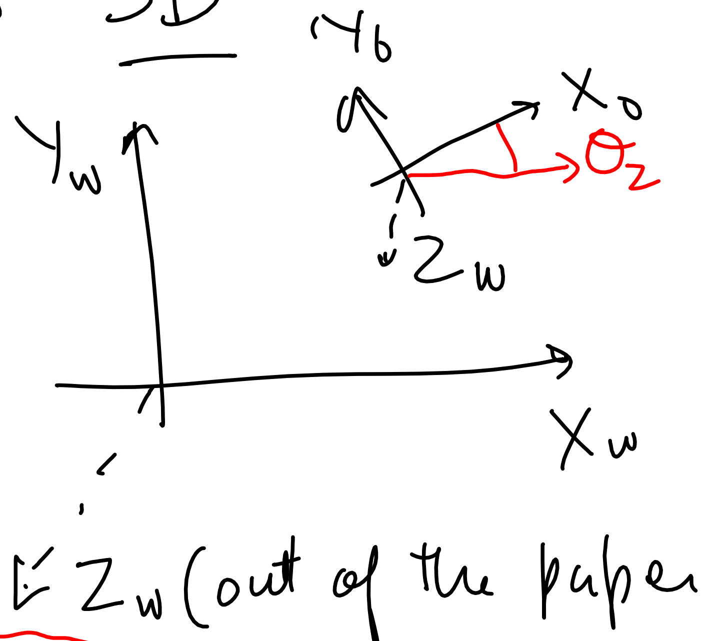
In 3D







Extending 2D to 3D

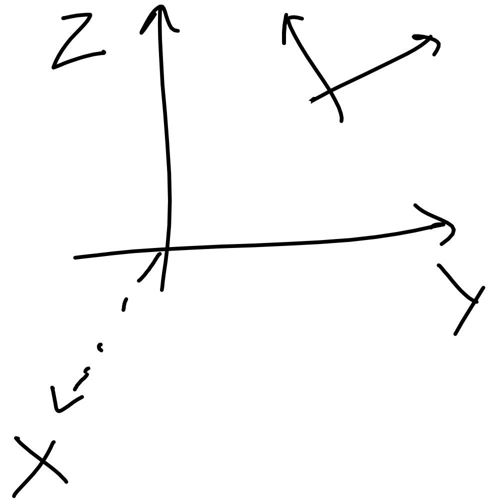


$$\begin{bmatrix} x_w \\ y_w \\ z_w = 0 \end{bmatrix} = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ 0 \end{bmatrix}$$

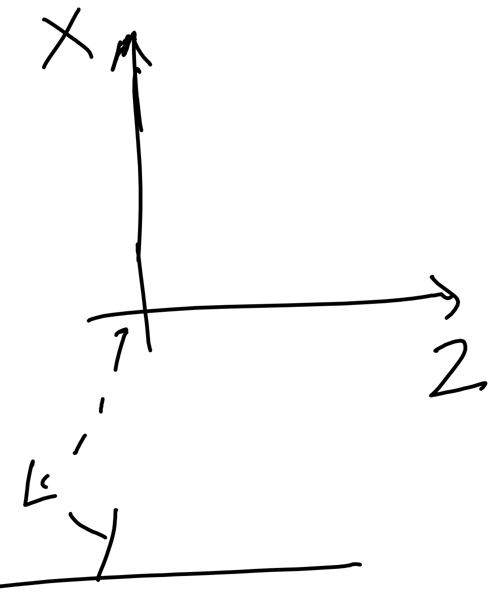
Rotation along Z-axis changes only X-Y coordinates

$$R(\theta_2) = \begin{bmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R(\theta_x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \underline{-\sin\theta} \\ 0 & \underline{\sin\theta} & \cos\theta \end{bmatrix}$$



$$R(\theta_y) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$



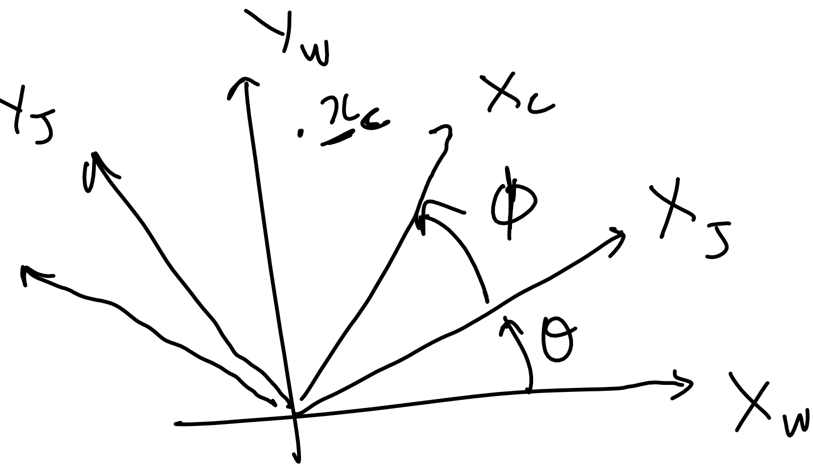
3D Rotation representations

Aeronautics

θ_x = roll
θ_y = pitch
θ_z = yaw

$$\boxed{R = \underbrace{R(\theta_z)}_{\text{Yaw}}\underbrace{R(\theta_y)}_{\text{Pitch}}\underbrace{R(\theta_x)}_{\text{Roll}}}$$

Chain rotation, translation, transformations



$$\underline{x}_J = {}^{J}_{C}R(\phi)\,\underline{x}_c$$

$$\underline{x}_w = {}^{w}_{J}R(\theta)\,\underline{x}_J$$

$$= R(\theta)\left[R(\phi)\underline{x}_c\right]$$

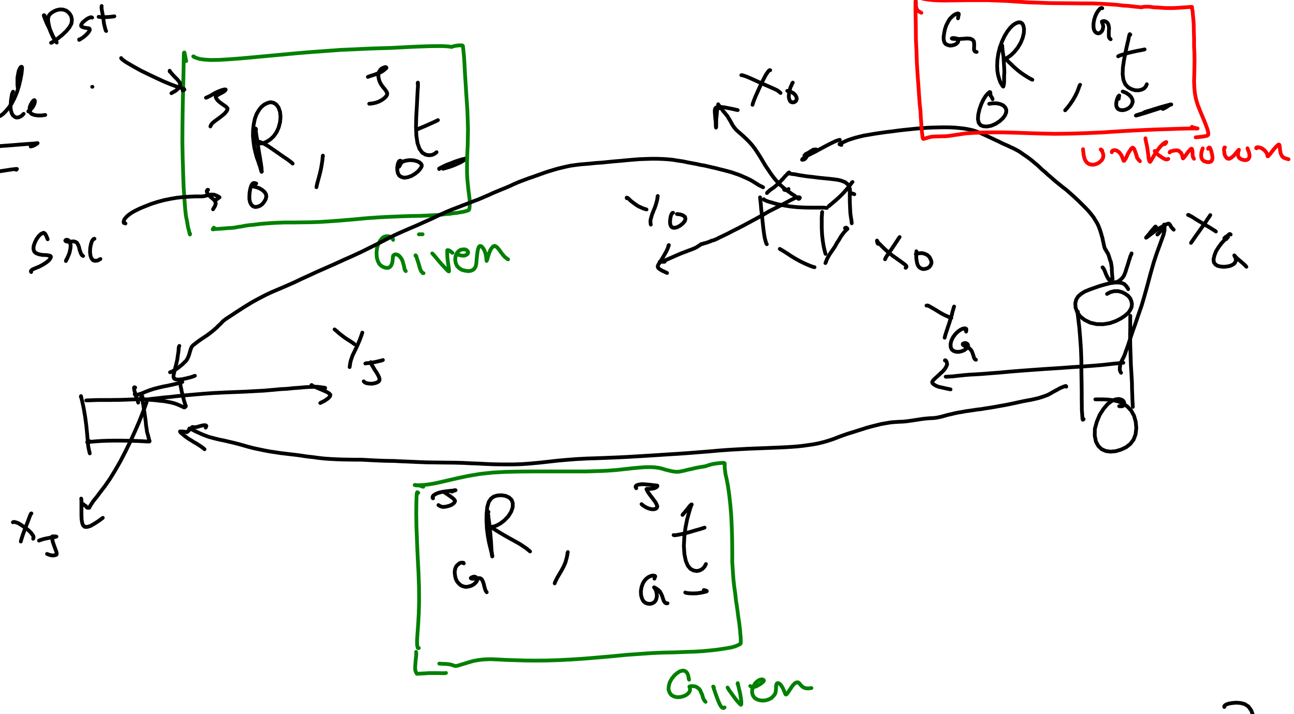
$$= \underbrace{\left(R(\theta)\,R(\phi)\right)}_{{}^{w}_{c}R}\underline{x}_c$$

$$\boxed{{}^{w}_{c}R = {}^{w}_{\cancel{J}}R(\theta)\;{}^{\cancel{J}}_{c}R(\phi)}$$

$$\boxed{{}^{w}_{c}\vec{T} = {}^{w}_{J}T\;{}^{J}_{c}T}$$

10/20

<u>Example</u>



$$^J_O T = \begin{bmatrix} ^J_O R & ^J_O \underline{t} \\ 0^T & 1 \end{bmatrix}, \quad ^J_G T = \begin{bmatrix} ^J_G R & ^J_G \underline{t} \\ 0^T & 1 \end{bmatrix}$$

$$^G_O T = \begin{bmatrix} ^G_O R & ^G_O \underline{t} \\ 0^T & 1 \end{bmatrix} \qquad ^G_O T = \left(^J_G T\right)^{-1} \, ^J_O T$$

$$= \, ^G_J T \, ^J_O T \ldots$$

$$\underbrace{^G_O T \, \underline{x}_o}_{\underline{x}_G} = \underbrace{\left(^G_J T \underbrace{\left(^J_O T \, \underline{x}_o \right)}_{x_J} \right)}_{\underline{x}_G}$$

① $R_{3D} = R(\theta_z) R(\theta_y) R(\theta_x)$

$\uparrow$ yaw $\quad \uparrow$ pitch $\quad \uparrow$ roll

$\dot{\theta}_x \xrightarrow{\text{then}} \theta_y \xrightarrow{\text{then}} \theta_z$

This sequences

$\overrightarrow{XYZ}$

ZYX

$\vdots$

6 possibl $= 3!$

Euler angle representation of 3D rotation is a sequence of rotation around standard axis

Euler representation with $\overrightarrow{XYZ}$ then

$$R_{3D} = \overleftarrow{R(\theta_z) R(\theta_y) R(\theta_x)}$$

Conversion from Euler angles to Rotation matrix

How to do the opposite?

convert from Rotation matrix to Euler angles?

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}$$

$$\theta = ? = \tan^{-1}\left(\frac{r_{21}}{r_{11}}\right) \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\underline{\theta = \operatorname{arctan2}(r_{21}, r_{11})} \in [-\pi, \pi)$$

3D

$$R = \underset{z}{R(\psi)}\,\underset{y}{R(\phi)}\,\underset{x}{R(\theta)}$$

$$= \begin{bmatrix} c(\psi) & -s(\psi) & 0 \\ s(\psi) & c(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c(\phi) & 0 & s(\phi) \\ 0 & 1 & 0 \\ -s(\phi) & 0 & c(\phi) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c(\theta) & -s(\theta) \\ 0 & s(\theta) & c(\theta) \end{bmatrix}$$

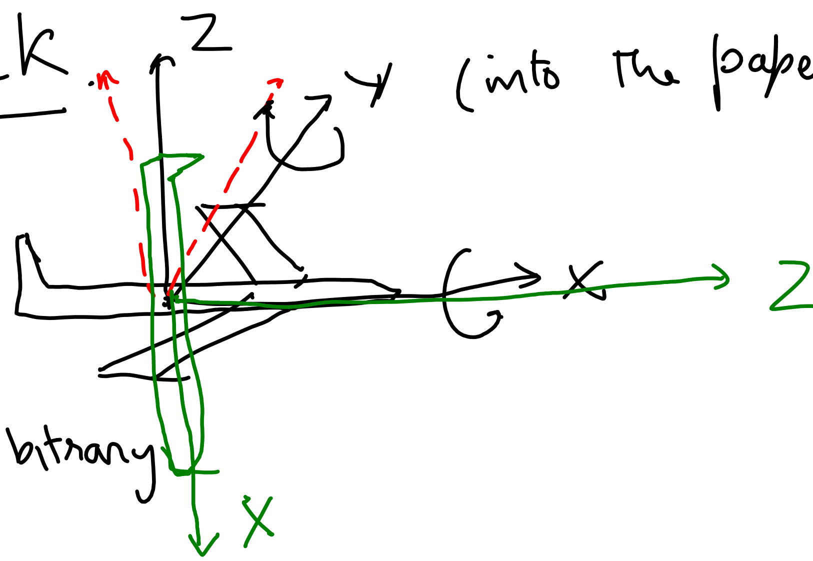
$$= \begin{bmatrix} \boxed{c(\psi)c(\phi)} & c(\psi)c(\phi)s(\theta) - s(\psi)c(\theta) & c(\psi)s(\phi)c(\theta) + s(\psi)s(\theta) \\ \boxed{s(\psi)c(\phi)} & s(\psi)s(\phi)s(\theta) + c(\psi)c(\theta) & s(\psi)s(\phi)c(\theta) - c(\psi)s(\theta) \\ \boxed{-s(\phi)} & \boxed{c(\phi)s(\theta)} & \boxed{c(\phi)c(\theta)} \end{bmatrix}$$

$$= \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\begin{cases} \phi = -\sin^{-1}(r_{31}) \in [0, \pi] \\ \dfrac{r_{21}}{r_{11}} = \dfrac{\sin(\psi)}{\cos(\psi)} \dfrac{\cancel{c(\phi)}}{\cancel{c(\phi)}} \Rightarrow \psi = \arctan 2(r_{21}, r_{11}) \\ \theta = \arctan 2(r_{32}, r_{33}) \end{cases}$$

Conversion from Rotation matrix to Euler angles

Gimbal lock



$\theta_x = 30°$ ← arbitrary

$[\theta_y = 90°]$

$\theta_z = 45°$ ← arbitrary

Euler angles $\xrightarrow{\text{deterministically}}$ Rot mat

Euler angles $\xleftarrow[\text{multiple solutions}]{}$ Rot mat

Other representations. It is impossible to unambiguously represent 3D rotation with only 3 number

Degree of freedom
3D rot = 3 DOF

but needs 4 numbers
+ 1 constraint
to represent it

② Axis-angle representation ← ③ Quaternions

↓

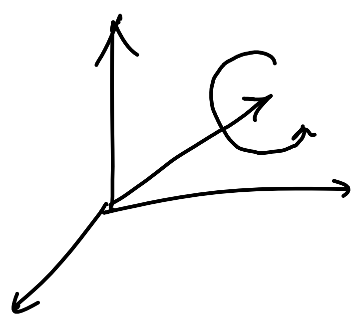
Rot mats

↓

Algebra

Quaternion algebra ≡ complex numbers

② Axis-angle representation



Any 3D rotation can be represented as a unit vector (axis) and rotation angle around it.

axis = $\underline{a} = [a_x, a_y, a_z]$

angle = θ.

constraint : $\|a\|_2 = \sqrt{a_x^2 + a_y^2 + a_z^2} = 1$

$$\left.\begin{array}{ll}\text{Free scalars} & = 4 \\ \text{constraint} & = 1\end{array}\right\}\ \text{Degree of freedom} = 3$$

DOF of 2D Rot matrix = ?

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Free scalars = 4

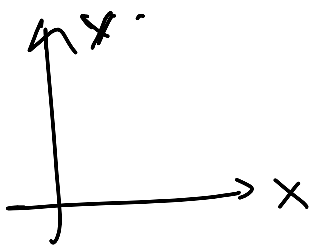
True for N-D rotations

Two vector constraints:

$$\left\{ \boxed{\begin{array}{l} R^T R = \underline{I} = R R^T \\ \det(R) = +1 \end{array}} \right. \quad \begin{array}{l} \text{2 scalar constraints} \\ \text{1 scalar constraint} \end{array}$$

Check for valid rotation

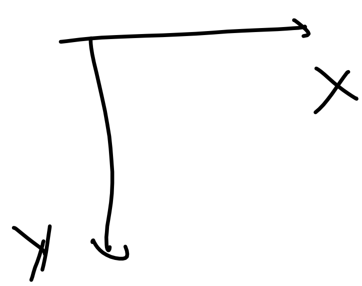
Reflection matrices also satisfy



not Rotation

reflection

det(Reflection) = −1



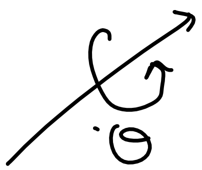
Rodrigues rotation formula

Axis angle $(\theta, \hat{k})$ → Rot matrix

$$R = I + \sin\theta [k_\times] + (1-\cos\theta)[k_\times]^2$$

axis angle representation $\longrightarrow$ Rotation matrix (3D)

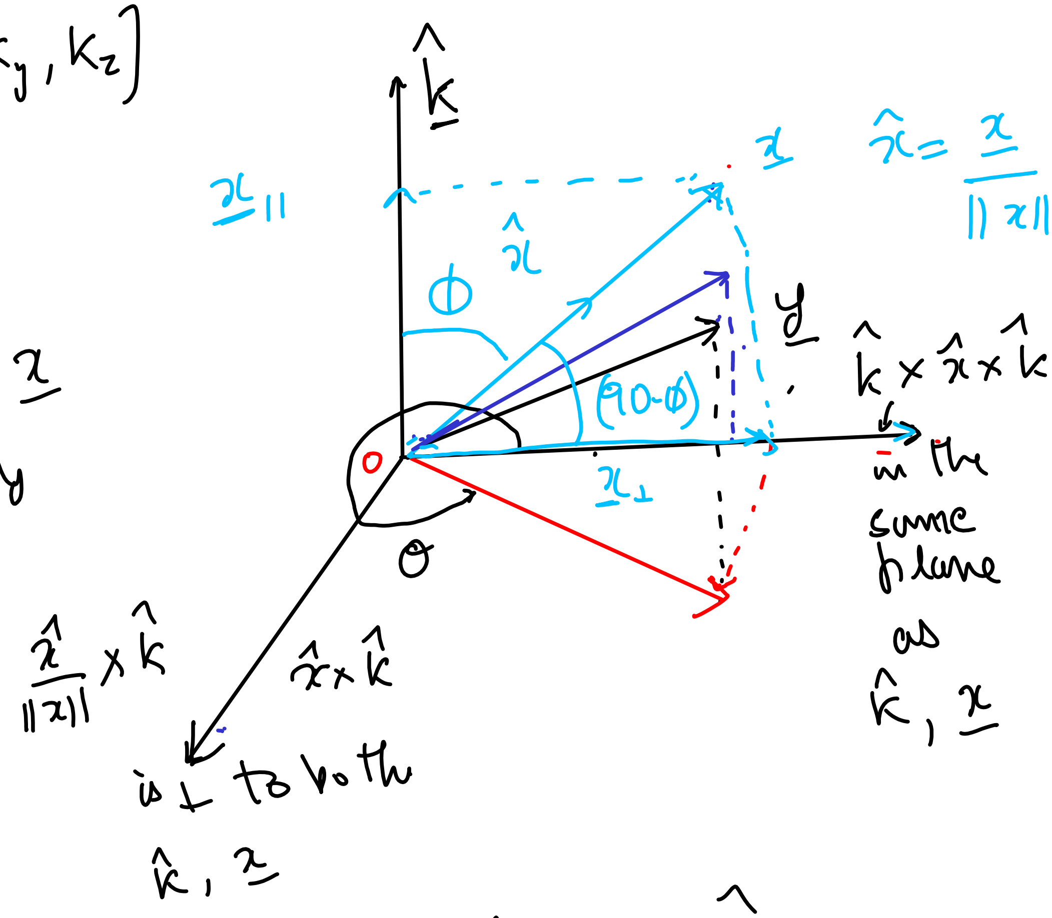
Rodrigues rotation formula



$\hat{k} = [k_x, k_y, k_z]$

θ

$\underline{y}$ = by rotating $\underline{x}$ around $\hat{\underline{k}}$ by angle θ



In the plane of $\hat{\underline{k}}$ and $\underline{\hat{k} \times \hat{x} \times \hat{k}}$, $\underline{x}$ can be projected into two component

$$\underline{x} = \underline{x}_{\|} + \underline{x}_{\perp}$$

$$\underline{x}_{\|} = (\hat{k} \cdot \underline{x})\,\hat{k}$$

$$\underline{x}_{\perp} = \left[\left(\hat{k} \times (\hat{\underline{x}} \times \hat{k})\right) \cdot \underline{x}\right] (\hat{k} \times \hat{x} \times \hat{k})$$

$$\underline{x}_{\perp} = \hat{k} \times (\underline{x} \times \hat{k}) = \hat{k} \times (-\hat{k} \times \underline{x}) = -\hat{k} \times (\hat{k} \times \underline{x})$$

$$\hat{k} \cdot \underline{x} = |\hat{k}|\,|x| \cos\phi = |x| \cos\phi$$

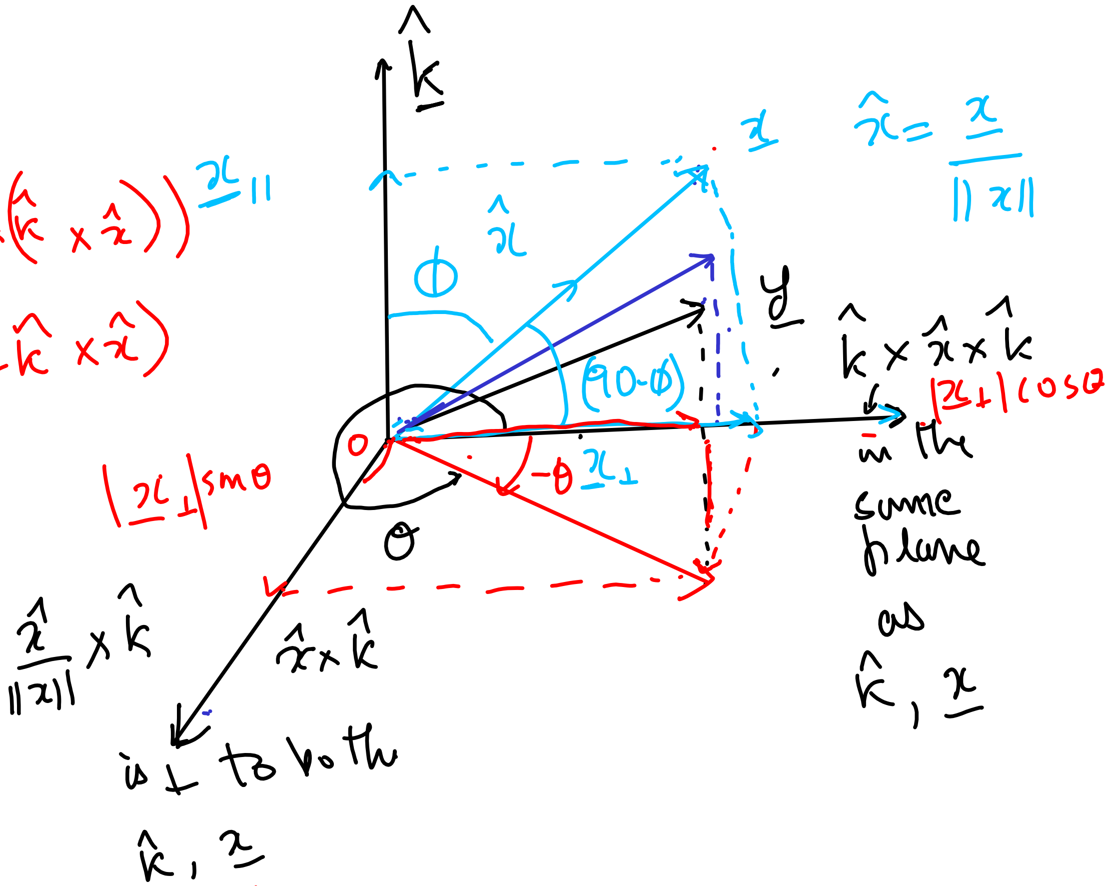
$$\rightarrow |x \times \hat{k}| = |x|\,|k| \sin\phi$$

$$\rightarrow x \times \hat{k} = (\hat{x} \times \hat{k})\,(|x| \sin\phi)$$

$$\boxed{|x| \sin\phi} = \boxed{|x|}\sqrt{1 - \cos^2\phi}$$

$$\underline{y} = \underline{x}_{\parallel} + \underline{x}_{\perp rot}$$

$$\underline{x}_{\perp rot} = |\underline{x}_{\perp}|\cos(-\theta)\left(-\hat{k}\times(\hat{k}\times\hat{x})\right) + |\underline{x}_{\perp}|\sin(-\theta)\left(-\hat{k}\times\hat{x}\right)$$



$$\underline{x}_{\perp rot} = |\underline{x}_{\perp}|\cos\theta\left(-\hat{k}\times(\hat{k}\times\hat{x})\right) - |\underline{x}_{\perp}|\sin\theta\left(-\hat{k}\times\hat{x}\right)$$

$$= \cos\theta\left(-\hat{k}\times(\hat{k}\times\underline{x})\right) - \sin\theta\left(-\hat{k}\times\underline{x}\right)$$

$$= \sin\theta(\hat{k}\times\underline{x}) - \cos\theta\left(\hat{k}\times(\hat{k}\times\underline{x})\right)$$

$$(\hat{k}\times\hat{x})|\underline{x}_\perp| = (\hat{k}\times\hat{x})|x|\sin\phi = \hat{k}\times\underline{x}$$

$$|\underline{a}\times\underline{b}| = |a||b|\sin\phi$$

$$\underline{y} = \underline{x}_{\parallel} + \underline{x}_{\perp rot}$$

Rodrigues formula

$$\boxed{\underline{y} = (\hat{k}\cdot\underline{x})\hat{k} + \sin\theta(\hat{k}\times\underline{x}) - \cos\theta(\hat{k}\times\hat{k}\times\underline{x})}$$

$$x_{\parallel} = (\hat{k}\cdot\underline{x})\,\hat{k} \qquad \underline{x} = \underline{x}_{\parallel} + \underline{x}_{\perp}$$

$$= \underline{x} - x_{\perp}$$

$$= \underline{x} - \left(-\hat{k}\times(\hat{k}\times\underline{x})\right)$$

$$= x + \hat{k}\times(\hat{k}\times\underline{x})$$

$$\underline{y} = \underline{x} + \underbrace{\hat{k}\times(\hat{k}\times\underline{x})} + \sin\theta\,(\hat{k}\times\underline{x}) - \cos\theta\,(\hat{k}\times(\hat{k}\times\underline{x}))$$

$$\boxed{\underline{y} = \underline{x} + \sin\theta\,(\underline{\hat{k}}\times\underline{x}) + (1-\cos(\theta))(\hat{k}\times(\hat{k}\times\underline{x}))}$$

$$\underline{y} = \underbrace{R(\theta,\hat{k})}_{\text{matrix}}\,\underline{x}$$

Writing cross product using matrix notation

$$\underline{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

$$\underline{a}\times\underline{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \qquad \underline{b} = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

$$= \hat{i}(a_y b_z - b_y a_z) + j(\qquad = \begin{bmatrix} a_y b_z - b_y a_z \\ -a_x b_z + b_x a_z \\ a_x b_y + b_x a_y \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} a_y b_z - b_y a_z \\ -a_x b_z + b_x a_z \\ a_x b_y + b_x a_y \end{bmatrix}}_{\underline{a} \times \underline{b}} = \boxed{\underbrace{\begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ a_y & a_x & 0 \end{bmatrix}}_{\lfloor a_\times \rfloor}} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

Cross product matrix of vector

$$\hat{k} \times \underline{x} = \underbrace{\lfloor k_\times \rfloor}_{K} \underline{x} = K \underline{x}$$

$$\hat{k} \times (\hat{k} \times \underline{x}) = \lfloor k_\times \rfloor (\lfloor k_\times \rfloor \underline{x}) = K(K \underline{x})$$
$$= K^2 \underline{x}$$

$$\underline{y} = \underline{x} + \sin\theta (K \underline{x}) + (1 - \cos\theta) K^2 \underline{x}$$
$$= \underbrace{\left[I_{3\times3} + \sin\theta \, K + (1 - \cos\theta) K^2 \right]}_{R(\theta, \hat{k})} \underline{x}$$

Oct 25

Convert from Rot matrix to axis angle representation

$$\boxed{\alpha \underline{y} = R \underline{y}}$$

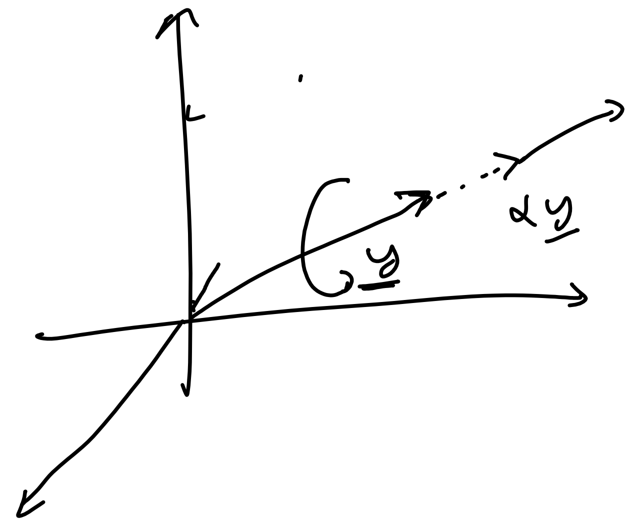
R is 3×3 matrix

$\underline{y}$ is 3×1 vector

α is scalar

$$A\underline{v} = \lambda \underline{v}$$

Eigen vectors of a matrix A are all the solution for $\underline{v}$ from the above equation



Corresponding solutions of λ are called eigen values

$$A\underline{v} - \lambda \underline{v} = 0$$

$$\Rightarrow \underbrace{(A - \lambda \underline{I})}_{\text{matrix}} \underbrace{\underline{v}}_{\text{vector}} = 0$$

$$\det(A - \lambda \underline{I}) = 0 \rightarrow \text{solve for eigen value}$$

The axis of rotation is $\underbrace{\text{an eigen vector}}$ of the rotation matrix.

$$\boxed{\underline{y} = R\underline{x} \Rightarrow \|y\| = \|x\|}$$

$$\|\underline{y}\| = \sqrt{y_1^2 + y_2^2 + \dots + y_n^2}$$

$$= \sqrt{\begin{bmatrix} y_1 & y_2 & \cdots & y_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}$$

$$= \sqrt{\underline{y}^T \underline{y}}$$

$$\underline{y}^T \underline{y} = (R\underline{x})^T (R\underline{x})$$

$$= (\underline{x}^T R^T)(R\underline{x})$$

$$= \underline{x}^T (R^T R)\, \underline{x}$$

$$= \underline{x}^T \underline{x}$$

eigen value ↘ ↙ eigen vector ↓

$$1\,\underline{k} = R\,\underline{k}$$

y $\underline{k}$ is along the axis of rotation for R

The axis of rotation is the eigen vector of the rotation matrix corresponding to eigen value 1.

$$\boxed{\det(R - \lambda I) = 0}$$

Use numpy.linalg.eig( ) to find eigen value and eigen vector

For 3 x 3 Rotation matrix

if $\underline{\theta} = 0°$ or $180°$

special cases

else $(\underline{\theta} \neq 0, \theta \neq 180°)$

$$\hat{\underline{k}} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} = \begin{pmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{pmatrix} \Big/ (2\sin\theta)$$

$$R = \begin{bmatrix} r_{11} & r_{12} & \\ r_{21} & & \\ & & r_{33} \end{bmatrix}$$

How to compute angle θ in axis-angle

$$R = I + K\sin\theta + (1-\cos\theta) K^2$$

$$K^2 = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix} \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix}$$

A is a symmetric matrix if $A^T = A$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & & \\ a_{13} & & \end{bmatrix}$$

A is a skew-symmetric matrix if $A^T = -A$

$$\begin{bmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & \\ -a_{13} & & 0 \end{bmatrix}$$

$$K^2 = \begin{bmatrix} -(k_z^2 + k_y^2) & k_y k_x & k_z k_x \\ k_y k_x & -(k_x^2 + k_z^2) & k_z k_y \\ k_z k_x & k_z k_y & -(k_x^2 + k_y^2) \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & & \\ & 0 & \\ & & 0 \end{bmatrix} \sin\theta +$$

$$\begin{bmatrix} -(k_z^2 + k_y^2) & k_y k_x & k_z k_x \\ k_y k_x & -(k_x^2 + k_z^2) & k_z k_y \\ k_z k_x & k_z k_y & -(k_x^2 + k_y^2) \end{bmatrix} (1-\cos\theta)$$

$\hat{k} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix}$ is a unit vector

$$k_x^2 + k_y^2 + k_z^2 = 1$$

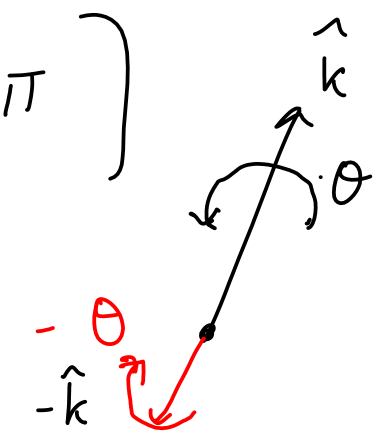
$$R = \begin{bmatrix} r_{11} & & \\ & r_{22} & \\ & & r_{33} \end{bmatrix}$$

$$\text{trace}(R) = r_{11} + r_{22} + r_{33} = 1+1+1 - 2\underbrace{(k_x^2 + k_y^2 + k_z^2)}_{1}(1-\cos\theta)$$

$$\Rightarrow tr(R) = r_{11} + r_{22} + r_{33} = 3 - 2 + 2\cos\theta$$

$$\Rightarrow tr(R) = 1 + 2\cos\theta$$

$$\Rightarrow \boxed{\theta = \cos^{-1}\left(\frac{tr(R) - 1}{2}\right)} \in [0, \pi]$$



If $\theta = \frac{3\pi}{4} = 270°$ rotated around $\hat{k}$

this is same as $\theta = \frac{\pi}{4} = 90°$ " " $-\hat{k}$

if $\underline{\theta} = 180°$

$r_{11} = 1 + 0 + \underbrace{(1-(-1))}_{\cos(-\pi)} (-k_x^2 + k_y^2)$

$r_{12} = 1 + 0 +$

$k_x = \pm\sqrt{(r_{11}+1)/2}$

$k_y = \pm\sqrt{(r_{22}+1)/2}$

$k_z = \pm\sqrt{(r_{33}+1)/2}$

$$R = \begin{bmatrix} r_{11} & r_{12} & \\ r_{21} & & \\ & & r_{33} \end{bmatrix}$$

else $(\underline{\theta} \neq 0, \theta \neq 180°)$

$$\hat{\underline{k}} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} = \begin{pmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{pmatrix} \frac{1}{(2\sin\theta)}$$

Quaternions

$\underline{q} = w + ix + jy + kz$

$\boxed{i^2 = -1, j^2 = -1, k^2 = -1, ijk = -1}$

complex numbers

$\hat{k} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} \quad \theta$

Axis angle to quat

$$\underline{q} = \left[\cancel{\sin}\ \cos\left(\frac{\theta}{2}\right), \; k_x \cancel{\cos}\ \sin\left(\frac{\theta}{2}\right), \; k_y \cancel{\cos}\ \sin\left(\frac{\theta}{2}\right), k_z \cancel{\cos}\ \sin\left(\frac{\theta}{2}\right) \right]$$

$w = \quad x \quad y \quad z$

$$\underline{q} = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix} \quad \text{OR} \quad q = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

$$q = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

$\boxed{\cancel{\theta = \sin^{-1}(2w)}}$

$\boxed{\hat{k} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} / \cancel{\cos}\ \sin(\theta/2)}$

$$\sqrt{x^2+y^2+z^2} = \underset{\sin}{\cancel{\cos}}\left(\frac{\theta}{2}\right)$$

$$w = \underset{\cos}{\cancel{\sin}}\,\frac{\theta}{2}$$

$$\boxed{\theta = \cancel{2}\,\arctan 2\left(w, \sqrt{x^2+y^2+z^2}\right)}$$

$$\theta = 2\arctan 2\left(\sqrt{x^2+y^2+z^2}, w\right)$$