# ECE 417/598: Homework 2

Max marks: 90 marks. ETA: 90 min

Due on Feb 7th, 2021, midnight, 11:59 PM.

You can also use the following template to fill in your answers: hw2.cpp

## 1 Jan 26 Lecture

**Problem 1** *In class we proved the Rodrigues formula that converts from axis-angle representation $(\theta, \hat{\mathbf{k}})$, where $\theta$ is the angle of rotation and $\hat{\mathbf{k}}$ is the axis of rotation ($\|\hat{\mathbf{k}}\| = 1$). Let $\mathbf{K} = [\hat{\mathbf{k}}]_\times$ be the cross product matrix of $\hat{\mathbf{k}}$. The cross product matrix of $\hat{\mathbf{k}} = [k_x, k_y, k_z]^\top$ (such that $k_x^2 + k_y^2 + k_z^2 = 1$) is defined as,*

$$\mathbf{K} = [\hat{\mathbf{k}}]_\times = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix} \tag{1}$$

*The corresponding rotation matrix is given by,*

$$R(\theta, \hat{\mathbf{k}}) = \mathbf{I} + \sin\theta\mathbf{K} + (1 - cos\theta)\mathbf{K}^2. \tag{2}$$

*An exponential of a square matrix $\mathbf{M}$ is defined as*

$$\exp(\mathbf{M}) = \sum_{n=0}^{\infty} \frac{1}{n!}\mathbf{M}^k = \mathbf{I} + \frac{1}{1!}\mathbf{M} + \frac{1}{2!}\mathbf{M}^2 + \dots \tag{3}$$

*Recall the series expansion of $\sin\theta$, and $\cos\theta$,*

$$\sin\theta = \theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots \tag{4}$$

$$\cos\theta = 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \dots \tag{5}$$

1. *First prove that $\mathbf{K}^3 = -\mathbf{K}$. (15 marks, 15 minutes)*
2. *As a result note that $\mathbf{K}^4 = -\mathbf{K}^2$, $\mathbf{K}^5 = \mathbf{K}$, and so on. In general, $\mathbf{K}^{2n+1} = (-1)^n\mathbf{K}$ and $\mathbf{K}^{2n+2} = (-1)^n\mathbf{K}^2$. Using the expansion of $\sin\theta$ and $\cos\theta$, prove that $R(\theta, \hat{\mathbf{k}}) = \exp(\theta\mathbf{K})$. (30 marks, 30 minutes)*

**Solution** Expanding $\mathbf{K}^2$

$$\mathbf{K}^2 = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix} \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix} \tag{6}$$

$$= \begin{bmatrix} -k_z^2 - k_y^2 & k_y k_x & k_z k_x \\ k_x k_y & -k_z^2 - k_x^2 & k_z k_y \\ k_x k_z & k_y k_z & -k_y^2 - k_x^2 \end{bmatrix}. \tag{7}$$

Expanding $\mathbf{K}^3$,

$$\mathbf{K}^3 = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix} \begin{bmatrix} -k_z^2 - k_y^2 & k_y k_x & k_z k_x \\ k_x k_y & -k_z^2 - k_x^2 & k_z k_y \\ k_x k_z & k_y k_z & -k_y^2 - k_x^2 \end{bmatrix} \tag{8}$$

$$= \begin{bmatrix} -k_x k_y k_z + k_x k_y k_z & k_z(k_z^2 + k_x^2 + k_y^2) & -k_y(k_z^2 + k_x^2 + k_y^2) \\ -k_z(k_z^2 + k_x^2 + k_y^2) & -k_x k_y k_z + k_x k_y k_z & k_x(k_z^2 + k_x^2 + k_y^2) \\ k_y(k_z^2 + k_x^2 + k_y^2) & -k_x(k_z^2 + k_x^2 + k_y^2) & -k_x k_y k_z + k_x k_y k_z \end{bmatrix} \tag{9}$$

We know that $k_x^2 + k_y^2 + k_z^2 = 1$. Applying that we get the desired result, $\mathbf{K}^3 = -\mathbf{K}$.

Expanding Rodrigues formula using expansion of $\sin\theta$ and $\cos\theta$, we get

$$R(\theta, \hat{\mathbf{k}}) = \mathbf{I} + \mathbf{K}(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \dots) + \mathbf{K}^2(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \dots) \tag{10}$$

Now, note that $\mathbf{K}^3 = -\mathbf{K}$, $\mathbf{K}^4 = -\mathbf{K}^2$, $\mathbf{K}^5 = \mathbf{K}$. Moving $\mathbf{K}$ inside the series with $\theta$, we get

$$R(\theta, \hat{\mathbf{k}}) = \mathbf{I} + (\theta\mathbf{K} + \frac{(\theta\mathbf{K})^3}{3!} + \frac{(\theta\mathbf{K})^5}{5!} + \dots) + (\frac{(\theta\mathbf{K})^2}{2!} + \frac{(\theta\mathbf{K})^4}{4!} + \dots) \tag{11}$$

This is exactly the series expansion of $\exp(\theta\mathbf{K})$ by definition.

**Problem 2** *Write a pair of functions in C++ that converts rotation matrix from axis-angle representation and vice versa. Recall that*

$$R(\theta, \hat{\mathbf{k}}) = \mathbf{I} + \sin\theta \mathbf{K} + (1 - cos\theta)\mathbf{K}^2. \qquad (12)$$

*and to get axis-angle back from a given rotation matrix*

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}, \qquad (13)$$

*we have*

$$\theta = \cos^{-1}\left(\frac{tr(R) - 1}{2}\right) \qquad (14)$$

$$\hat{\mathbf{k}} = \frac{1}{2\sin\theta}\begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} \quad \textit{if } \theta \neq 0 \textit{ or } \pi. \qquad (15)$$

*If $\theta = 0$ or $\pi$, then*

$$\hat{\mathbf{k}} = \pm\begin{bmatrix} \sqrt{(r_{11}+1)/2} \\ \sqrt{(r_{22}+1)/2} \\ \sqrt{(r_{33}+1)/2} \end{bmatrix} \qquad (16)$$

*(30 marks. Estimated time: 30 min)*

## 2 Jan 31 Lecture

**Problem 3** *Recall the definition of Denavit-Hartenberg parameters from the video. Recall that transformation between two joints for the defined parameters $d, \theta, r, \alpha$ is given by,*

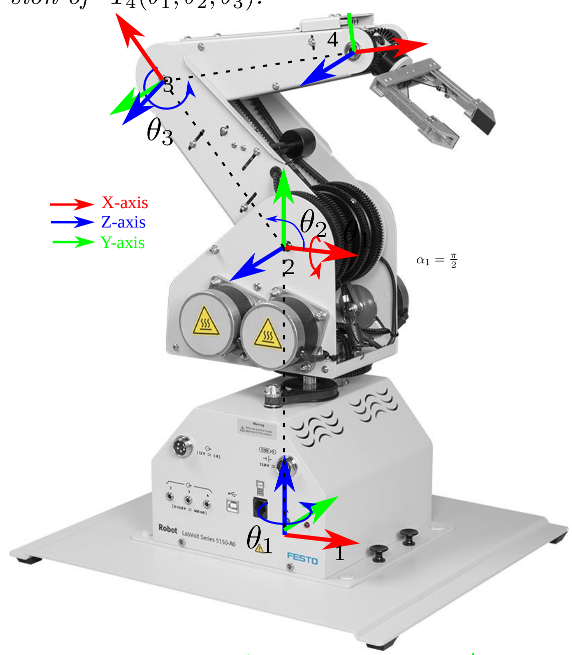
$$T = T_z(\theta, d)T_x(\alpha, r), \qquad (17)$$

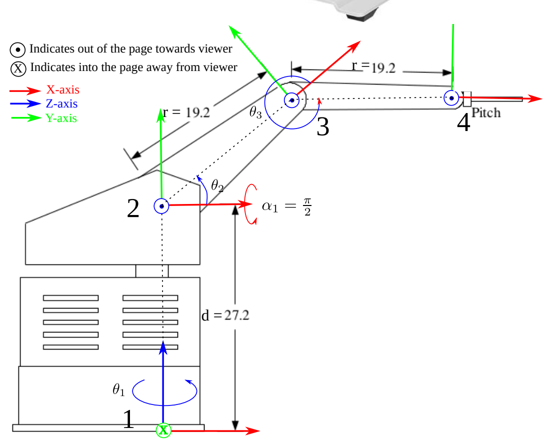
*where*

$$T_x(\alpha, r) = \begin{bmatrix} 1 & 0 & 0 & r \\ 0 & \cos(\alpha) & -\sin(\alpha) & 0 \\ 0 & \sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (18)$$

$$T_z(\theta, d) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 & 0 \\ \sin(\theta) & \cos(\theta) & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (19)$$

*For the robot given below find transformation matrix from joint 4 to joint 1 assuming the joint angles to be $\theta_1$, $\theta_2$, $\theta_3$ respectively. Write the expression for ${}^3T_4(\theta_3)$, ${}^2T_3(\theta_2)$, ${}^1T_2(\theta_1)$ and then ${}^1T_4(\theta_1, \theta_2, \theta_3)$ in terms of the first three transformations. You do not need to expand the expression of ${}^1T_4(\theta_1, \theta_2, \theta_3)$.*




*(15 marks. 15 min)*

**Solution**

$$ {}^3T_4(\theta_3) = \begin{bmatrix} cos(\theta_3) & -\sin(\theta_3) & 0 & 0 \\ sin(\theta_3) & cos(\theta_3) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 & 19.2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
$$= \begin{bmatrix} cos(\theta_3) & -\sin(\theta_3) & 0 & 19.2\cos(\theta_3) \\ sin(\theta_3) & cos(\theta_3) & 0 & 19.2\sin(\theta_3) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (20)$$

$$
{}^2T_3(\theta_2) = \begin{bmatrix} cos(\theta_2) & -\sin(\theta_2) & 0 & 0 \\ sin(\theta_2) & cos(\theta_2) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 19.2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} cos(\theta_2) & -\sin(\theta_2) & 0 & 19.2\cos(\theta_2) \\ sin(\theta_2) & cos(\theta_2) & 0 & 19.2\sin(\theta_2) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{21}
$$

$$
{}^1T_2(\theta_1) = \begin{bmatrix} cos(\theta_1) & -\sin(\theta_1) & 0 & 0 \\ sin(\theta_1) & cos(\theta_1) & 0 & 0 \\ 0 & 0 & 1 & 27.2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} cos(\theta_1) & 0 & sin(\theta_1) & 0 \\ sin(\theta_1) & 0 & -cos(\theta_1) & 0 \\ 0 & 1 & 0 & 27.2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{22}
$$

$$
{}^1T_4(\theta_1, \theta_2, \theta_3) = {}^1T_2(\theta_1){}^2T_3(\theta_2){}^3T_4(\theta_3) \tag{23}
$$

# 3 ECE 598 only

Write a short review of the following paper On continuity of rotation representations in Neural networks. We have not covered all the concepts covered in this paper; you can skip the parts that you do not understand. In the review answer the following questions evaluating the paper,

1. Problem: What problem is the paper trying to solve?
2. Approach: What is the proposed approach to solve the problem?
3. Contribution: What is the paper's novel contribution?
4. Evidence: Do they any experiments or proof that their approach/contributions work?
5. Results: Are the results of the paper justified by evidence and a direct result of the contibutions?

(Ungraded. 3-5 hrs)