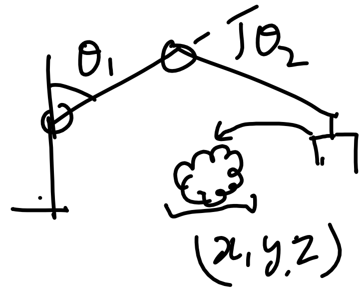
Forward and $\boxed{\text{inverse kinematics}}$

What should the joint ~~angles~~ state/conf. of the robot be so that the end-effector reaches a desired pose?



How to move the end-effector to a desired pose [position + orientation]

← end-effector

gripper or suction cup

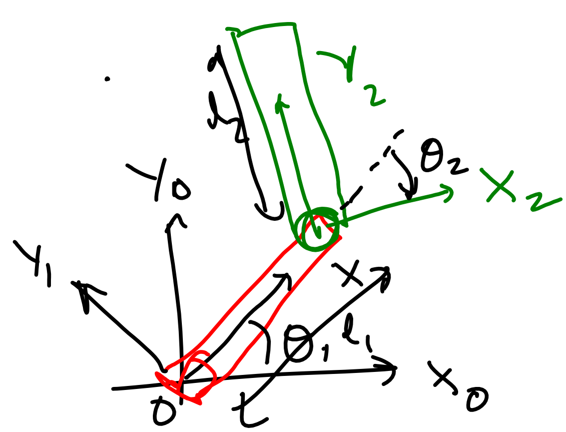
Forward kine.

If my joint ~~angles~~ state/conf. are given what would the pose of end-effector be?

Forward kinematics

$$^0T_2 = {}^0T_1(\theta_1, l_1)\,{}^1T_2(\theta_2, l_2)$$

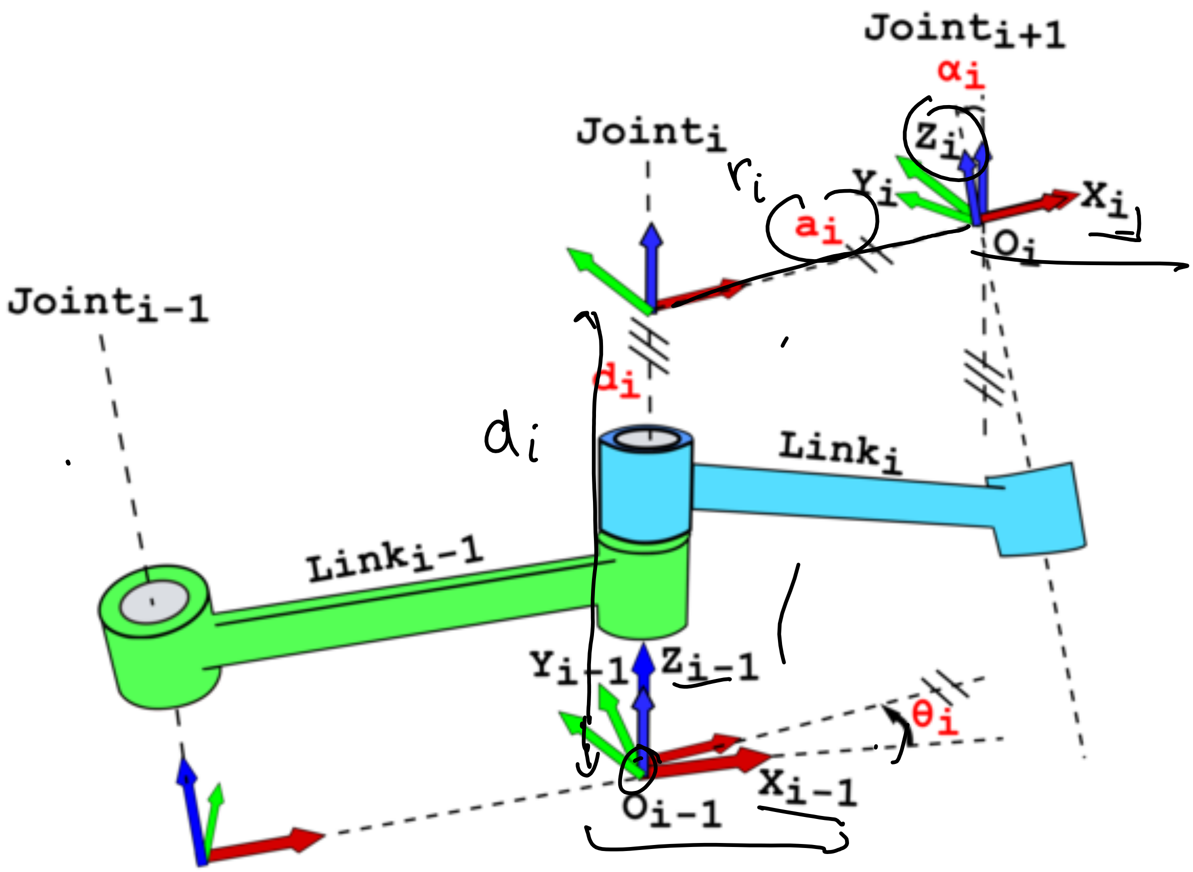
in terms of $\underbrace{\theta_1 \text{ and } \theta_2}_{\text{Given}}$



(Denavit Hartenberg Parameters/Convention)

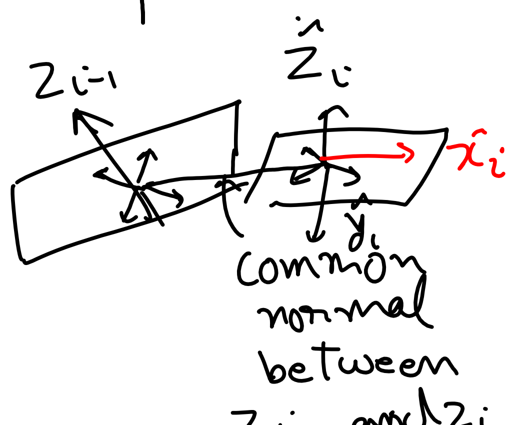
Denavit Hartenberg parameters

https://www.youtube.com/watch?v=rA9tm0gTln8



① $\hat{z}_{i-1}, \hat{z}_i$ aligned along the axis of rotation

② Choose $\hat{x}_i$ along the common normal between $\hat{z}_{i-1}, \hat{z}_i$



③ $\hat{y}_i = \hat{z}_i \times \hat{x}_i$

a) θ_i = Rotation along z_{i-1} (to align x_{i-1} with x_i)

b) d_i = translation along z_{i-1} (to align the origins)

c) α_i = Rotation along x_i (to align z_{i-1} with z_i)

d) r_i/a_i = translation along x_i (to align the origin)

(a) and (b) can be swapped
(c) and (d)

But Tranformation along z goes first
followed by " " x

$$\underset{\text{target}}{\overset{i-1}{\uparrow}} T_{\underset{\text{source}}{\underset{\uparrow}{i}}} = {}^{i-1}T_{zi} \; {}^{i-1}T_{xi}$$

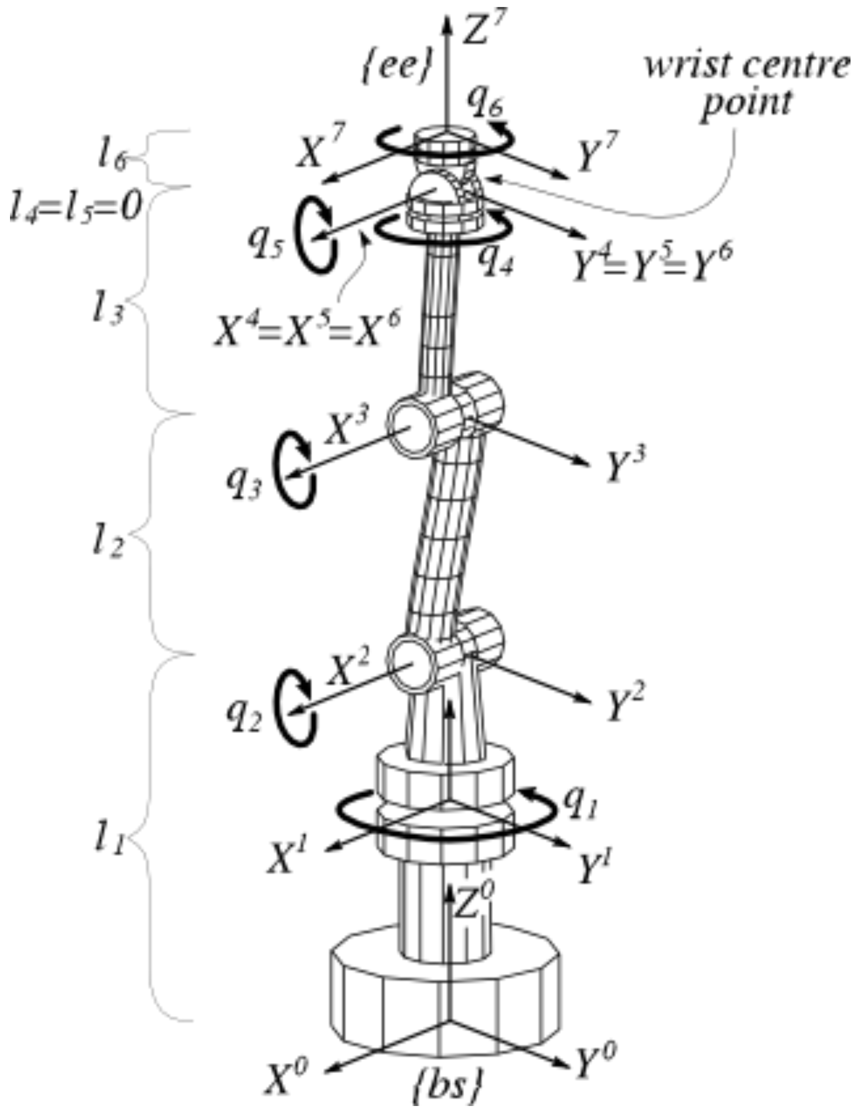
$\xleftarrow{\hspace{3em}}$ Tranformations are applied right to left

$${}^{i-1}T_{xi} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & r_i \\ 0 & \cos\alpha_i & -\sin\alpha_i & 0 \\ 0 & \sin\alpha_i & \cos\alpha_i & 0 \\ \hline 0 & 0 & 0 & 1 \end{array}\right]_{4\times 1}$$

3x3   3x1

$${}^{i-1}T_{zi} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 & 0 \\ \sin\theta_i & \cos\theta_i & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times 1}$$

θ_i, d_i



Z^7
$\{ee\}$
wrist centre point
q_6
l_6
X^7
Y^7
$l_4=l_5=0$
q_5
q_4
$Y^4=Y^5=Y^6$
l_3
$X^4=X^5=X^6$
X^3
q_3
Y^3
l_2
X^2
q_2
Y^2
q_1
l_1
X^1
Y^1
Z^0
X^0
$\{bs\}$
Y^0

Numerical solutions to IK problems: Jacobian inverse technique

Forward and inverse kinematics

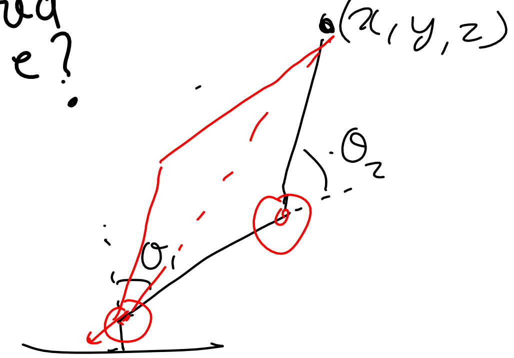
What should the joint ~~angles~~ state/conf. of the robot be so that the end-effector reaches a desired pose?

Inverse kinematics

↳ closed form solution

↳ Numerical/Iterative solutions

only polynomials of degree ≤ 5 have closed form solutions



~~$\theta_1 = x_1^2 + y_2 + \cos^{-1}\left(\frac{z}{y}\right)$~~

$\cos(\theta_1)$ $\quad$ $\sin(\theta_1)$

$\sqrt{1 - x_1^2}$ $\quad$ x_1

Newton-Raphson method (Gradient descent) (optimization solution)

Suppose a function $y = f(x)$ to find

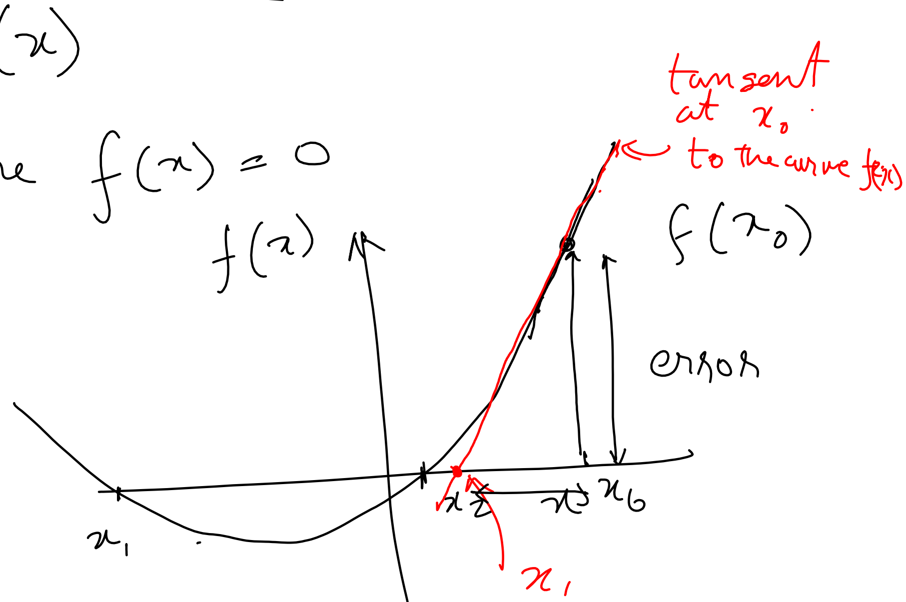
we want ~~all~~ any x where $f(x) = 0$

① Initial guess

x_0

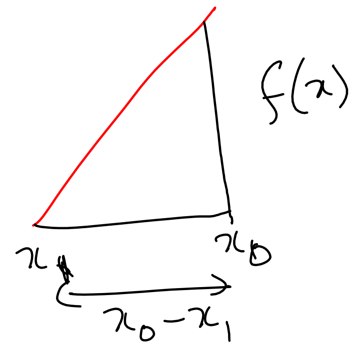
↑

iteration



② Improve the initial guess

$$f'(x)\Big|_{x_0} = \frac{f(x_0)}{x_0 - x_1}$$



$$\Rightarrow \quad x_0 - x_1 = \left[f'(x_0)\right]^{-1} f(x_0)$$

$$x_1 = x_0 - \left[f'(x_0)\right]^{-1} f(x_0)$$

② Repeat

$$x_2 = x_1 - \left[f'(x_1)\right]^{-1} f(x_1)$$

$$x_n = x_{n-1} - \left[f'(x_{n-1})\right]^{-1} f(x_{n-1})$$

---

find the square root of 2

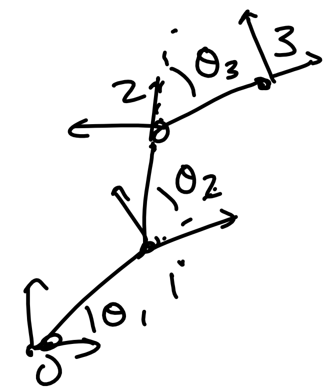
$$f(z) = z^2 - 2 = 0$$

$$z_n = z_{n-1} - \frac{(z_{n-1}^2 - 2)}{2 z_{n-1}}$$

Forward Kinematics

$$\underbrace{{}^0T_3}_{\text{Given}} = {}^0T_1(\theta_1)\ {}^1T_2(\theta_2)\ {}^2T_3(\theta_3)$$

Find



$$\underbrace{({}^0T_3)^{-1}\ {}^0T_1(\theta_1)\ {}^1T_2(\theta_2)\ {}^2T_3(\theta_3) - I_{4\times 4}} = O_{4\times 4}$$

$$\underbrace{F_{4\times 4}(\theta_1, \theta_2, \theta_3)} = O_{4\times 4}$$

$$\underline{f}_{16\times 1}\left(\underbrace{\begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{bmatrix}_{3\times 1}}_{\text{vector}}\right) = \begin{bmatrix}0\\ \vdots \\ 0\end{bmatrix}_{16\times 1}$$

$f(\underline{\theta})$ is a vector function

$$\underline{f}_{16\times 1}(\underline{\theta}) = \begin{bmatrix} f_1(\underline{\theta}) \\ f_2(\underline{\theta}) \\ \vdots \\ f_{16}(\underline{\theta}) \end{bmatrix}$$

vector-valued vector function

What's a Jacobian matrix

is ~~a~~ derivative of vector-valued vector function
many

$$J\left[\underline{f}_{16\times1}(\underline{\theta}_{3\times1})\right] = \begin{array}{cc} & \text{inputs} \\ \text{outputs} & \begin{bmatrix} \frac{\partial f_1}{\partial\theta_1} & \frac{\partial f_1}{\partial\theta_2} & \cdots & \frac{\partial f_1}{\partial\theta_3} \\ \frac{\partial f_2}{\partial\theta_1} & \ddots & & \\ \vdots & & \ddots & \\ \frac{\partial f_{16}}{\partial\theta_1} & & & \ddots \end{bmatrix} \end{array}$$

Hand-waviness

scalar Newton Raphson

generalizes to (vector Newton - Raphson) ~~x~~

Gauss Newton method

(1) Initial guess $\underline{\theta}_0$

Psuedo-inverse dagger

(2) $$\underline{\theta}_1 = \underline{\theta}_0 - J\left[\underline{f}_{16\times1}(\underline{\theta}_{3\times1})\right]^{\dagger} \underline{f}_{16\times1}(\underline{\theta}_{3\times1})$$

$$x_1 = x_0 - \left[f'(x_0)\right]^{-1} f(x_0)$$

$$③ \quad \underline{\theta}_n = \underline{\theta}_{n-1} - J\left[\underline{f}(\underline{\theta}_{n-1})\right]^{+} \underline{f}(\underline{\theta}_{n-1})$$

① What is a Psuedo inverse?

→② How can we compute the Jacobian?

③ What is the relationship (similarities and differences) b/w Newton-Raphson, Gauss Newton, Gradient descents

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Problem 4 of Midterm helps in computing derivative of rotation matrices

$$K^3 = -K$$

$$R(\theta, \hat{k}) = I_{3\times 3} + \sin\theta\, K + (1-\cos\theta) K^2$$

$$\frac{\partial}{\partial\theta} R(\theta, \hat{k}) = 0 + \cos\theta\, \underset{\sim}{K} + (0 + \sin\theta) K^2$$

$$= -\cos\theta\, K^3 + \sin\theta\, K^2$$

$$= K(-\cos\theta\, K^2 + \sin\theta\, K)$$

$$= K(I - I - \cos\theta\, K^2 + \sin\theta\, K)$$

$$= K\left(I - \cos\theta K^2 + \sin\theta K\right) - K$$

$$= K\left(I + K^2 - K^2 - \cos\theta K^2 + \sin\theta K\right) - K$$

$$= K\left(I + (1-\cos\theta)K^2 + \sin\theta K\right) - K - K^3$$

$$= K R(\theta, \hat{k}) - \cancel{K} + \cancel{K}$$

$$\boxed{\frac{\partial}{\partial\theta} R(\theta, \hat{k}) = K R(\theta, \hat{k})} \qquad \boxed{\frac{d}{dx} f(x) = a f(x)}$$

scalar matrix

$$R(\theta, \hat{k}) = \exp(\theta K) = \frac{I}{1} + \frac{\theta K^1}{1!} + \frac{\theta^2 K^2}{2!} + \frac{\theta^3 K^3}{3!} + \dots \infty$$

Matrix exponention

$$\frac{\partial}{\partial\theta} R(\theta, \hat{k}) = 0 + \frac{K^1}{1!} + \frac{2\theta K^2}{2!} + \frac{3\theta^2 K^3}{3!} + \dots$$

$$= K\left(\frac{I}{1!} + \frac{\theta K}{1!} + \frac{\theta^2 K^2}{2!} + \dots\right)$$