Denavit Hartenberg parameters (convention) Standardize the choice and description of robotic arms

Kinematic chains

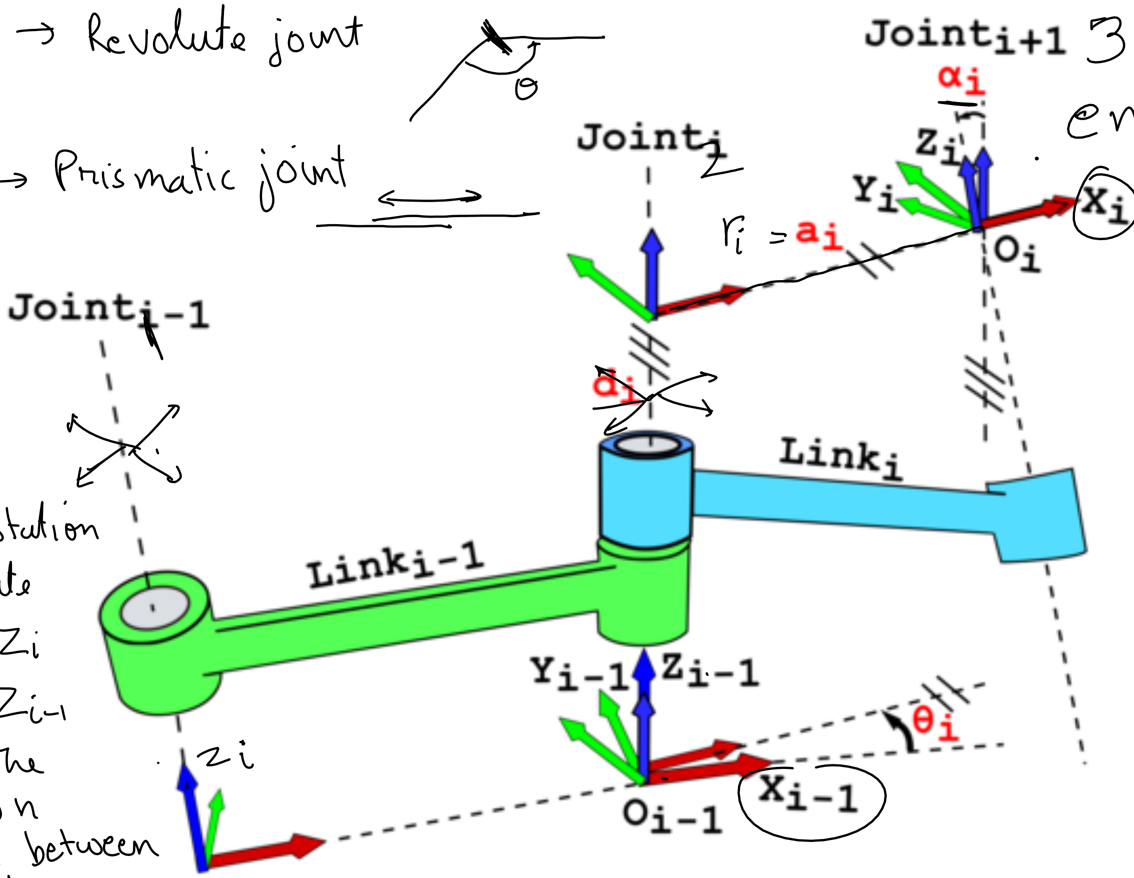
https://www.youtube.com/watch?v=rA9tm0gTln8

Robotic arm joints are mostly of two types

→ Revolute joint θ

→ Prismatic joint

Ball and socket joint



end effector

(1) Axis of rotation of revolute joints $= Z_i$
$= Z_{i-1}$

(2) Find the common normal between Z_i and Z_{i-1}
$= x_i$

(3) x_{i-1} depends upon the previous link or is arbitrary

$$\underline{y}_{i-1} = \underline{Z}_{i-1} \times \underline{x}_{i-1} \quad , \quad \underline{y}_i = Z_i - \underline{x}_i$$

(5) Rotation (1) and translation (2) along the x-axis $= \alpha_i, r_i$ (x_i-axis)

" (3) " " (4) " " Z-axis $= \theta_i, d_i$ (z_{i-1} axis)

$${}^{i-1}_{i}R_{x_i}(\alpha_i) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha_i) & -\sin(\alpha_i) \\ 0 & \sin(\alpha_i) & \cos(\alpha_i) \end{bmatrix}_{3\times 3} \rightarrow {}^{i-1}_{i}T^{R}_{x_i}(\alpha_i) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\alpha_i & -s\alpha_i & 0 \\ 0 & s\alpha_i & c\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= {}^{i-1}_{i}Rot_{x_i}(\alpha_i)$$

$$ {}^{i-1}_{\;\;i}T^{t}_{x_i}(r_i) = \begin{bmatrix} 1 & 0 & 0 & r_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^{i-1}_{\;\;i}Trans_{x_i}(r_i) $$

$$ {}^{i-1}_{\;\;i}T_x(\alpha_i, r_i) = \underbrace{{}^{i-1}_{\;\;i}Trans_{x_i}(r_i)} \; \underbrace{{}^{i-1}_{\;\;i}Rot_{x_i}(\alpha_i)} $$

$$ = \begin{bmatrix} 1 & 0 & 0 & r_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\alpha_i & -s\alpha_i & 0 \\ 0 & s\alpha_i & c\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ = \begin{bmatrix} 1 & 0 & 0 & r_i \\ 0 & c\alpha_i & -s\alpha_i & 0 \\ 0 & s\alpha_i & c\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{i-1}_{\;\;i}T_z(\theta_i; d_i) = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & 0 \\ s\theta_i & c\theta_i & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{i-1}_{\;\;i}T(\theta_i, d_i, \alpha_i, r_i) = {}^{i-1}_{\;\;i}T_z(\theta_i, d_i) \; {}^{i-1}_{\;\;x}T(\alpha_i, r_i) $$

Classical $\underbrace{\text{DH}}_{\text{Denavit Hartenberg}}$ parameters

For a robotic arm with
n-links, a D-H table is typically provided

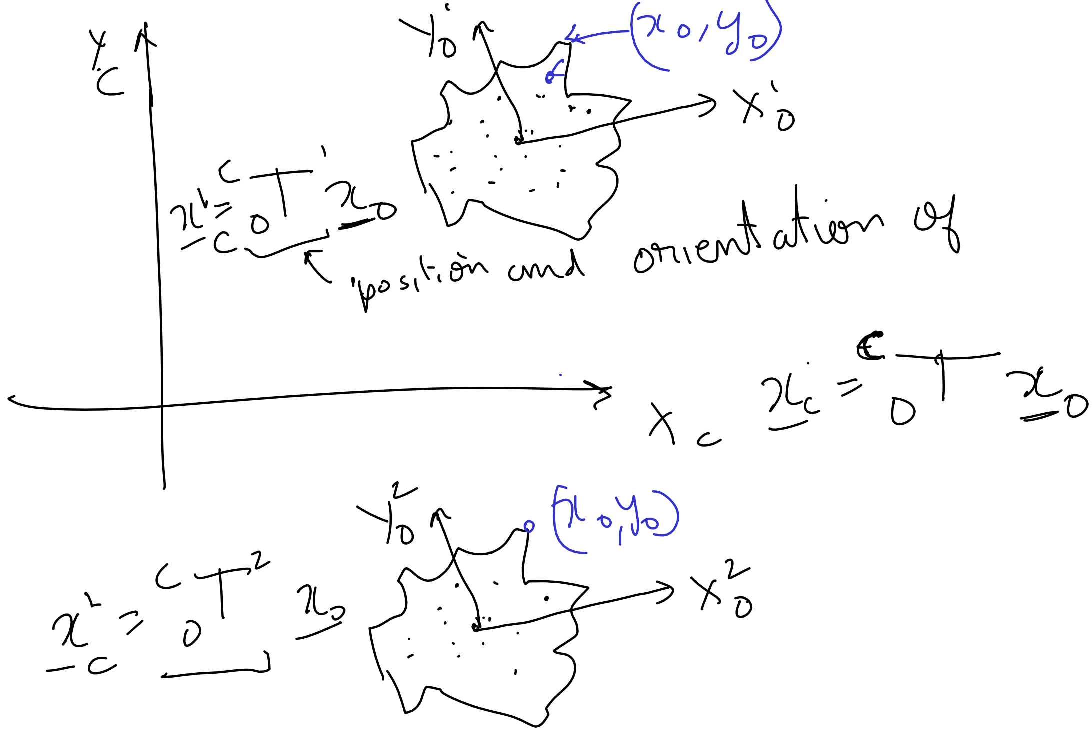
	θ_i	d_i	α_i	r_i	
1			var		revolute joints
2	var				
3					
.		var			← prismatic joints
.					
n-1					

n-1 row

$$ {}^{0}_{n}T = {}^{0}_{1}T(\theta_1)\;{}^{1}_{2}T(\theta_2)\;\dots\;{}^{n-1}_{n}T(\theta_n) $$

Forward Kinematics

Why Transformation matrices also describe position + orientation



321 Kinematic Structure

rotation orientation

$$\boxed{{}^0_3T = {}^0_1T(\theta_1)\,{}^1_2T(\theta_2)\,{}^2_3T(\theta_3)}$$

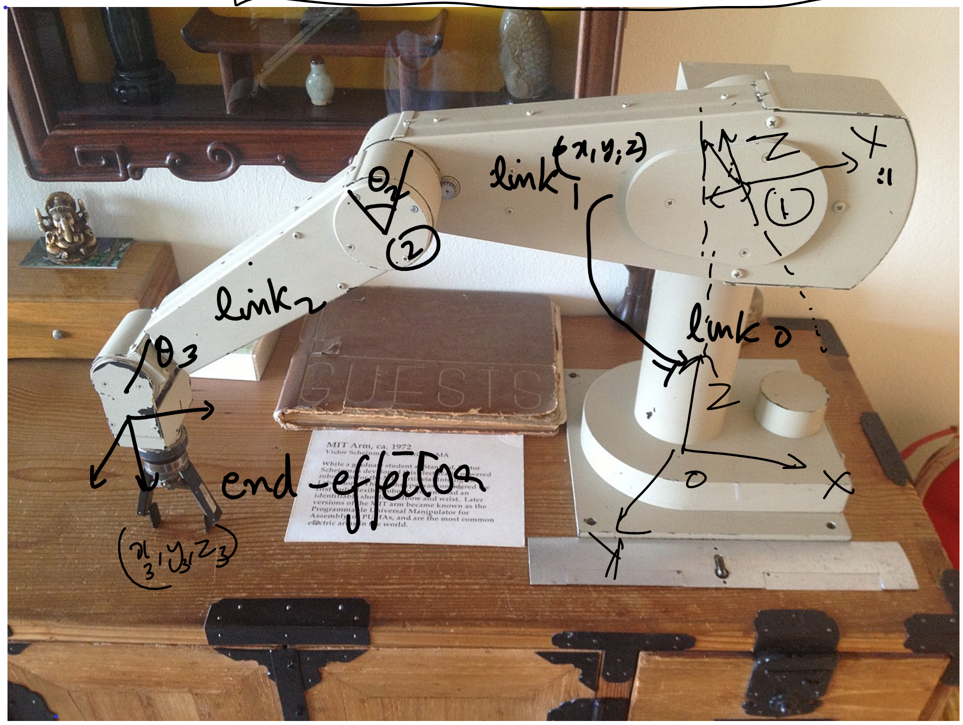
$$\boxed{x_0 = {}^0_3T\, x_3}$$

src

9Rot 3trans

12

4x4



description of robotic arm

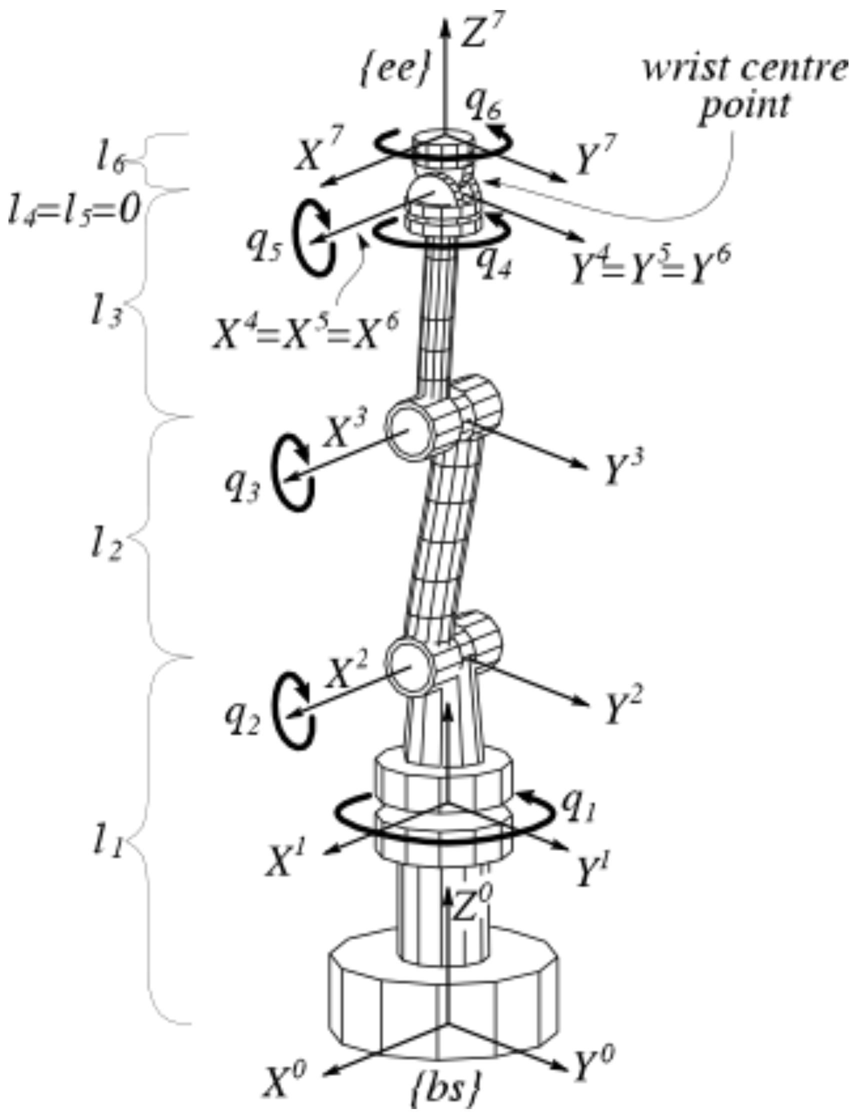
dst

$$\boxed{x_0 = \boxed{{}^0_1T(\theta_1)\,{}^1_2T(\theta_2)\,{}^2_3T(\theta_3)}\,x_3}$$

src

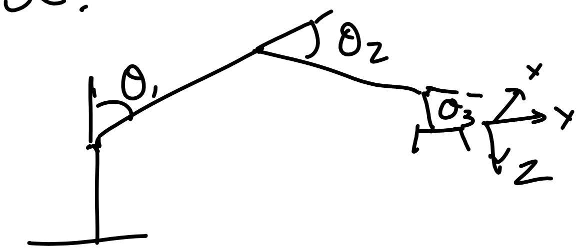
Forward Kinematics

is the problem of finding the end-effector pose (pos + orientat) in the base coordinate system when the joint angles (joint states) are given.



Numerical solutions to IK problems: Jacobian inverse technique

Inverse Kinematics: The problem of finding motor angle (joint states) so that the end effector achieves a given pose.



Skip ① Closed form solutions for simple arms (2-DOF, 6-DOF)

Do FK analytically → Solve systems of eqns

② Numerical or iterative solutions

→ Small changes to motor angles (joint states) that move the end-effector towards desired pose

Given: Position of end effector $\underline{p} \in \mathbb{R}^{3\times 1}$

Find: motor angle/joint states $\underline{\theta} \in \mathbb{R}^{n\times 1}$

$$\underline{p} = {}^{0}_{n}T(\underline{\theta})\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \underline{p}(\underline{\theta}) \leftarrow$$

position of end-effector as a function of $\underline{\theta}$

origin of end-effector

Taylor series approximation

Scalar valued functions $f(x)$

$$f(x+\Delta x) = f(x) + \frac{\Delta x}{1!} f'(x) + \frac{1}{2!}\Delta x^2 f''(x) + \dots$$

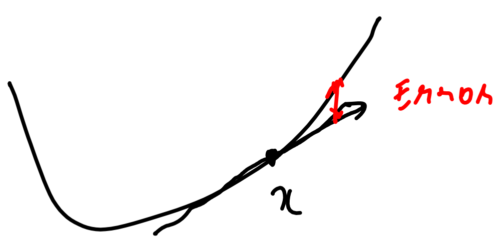
Vector-valued vector functions

$\underline{f}(\underline{x}) \in \mathbb{R}^{m\times 1}$ $\qquad$ $\underline{x} \in \mathbb{R}^{n\times 1}$

$$\underline{f}(\underline{x}+\Delta\underline{x}) = \underline{f}(\underline{x}) + \frac{J_x \underline{f}(\underline{x})\Delta \underline{x}}{1!} + \dots O(\Delta x^2)$$

$$\underline{f}(\underline{x}+\Delta\underline{x}) \approx \underline{f}(\underline{x}) + J_x f(x)\Delta \underline{x}$$

for $\|\Delta \underline{x}\| \ll 1$



Inverse Kinematics

$\underline{p}(\underline{\theta}) \in \mathbb{R}^{3\times 1}$

$\underline{\theta} \in \mathbb{R}^{n\times 1}$

$J_\theta p(\underline{\theta}) \in \mathbb{R}^{?}$

$$\underline{p}(\underline{\theta}+\Delta\underline{\theta}) \approx \underbrace{\underline{p}(\underline{\theta})}_{3\times 1} + \underbrace{\underbrace{\boxed{J_\theta p(\underline{\theta})}}_{3\times n}\underbrace{\Delta\underline{\theta}}_{n\times 1}}_{3\times 1}$$

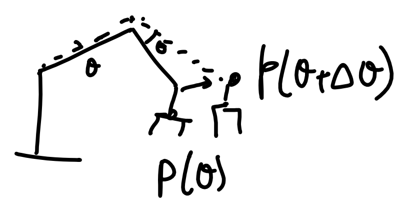
$$J_\theta \underline{p}(\underline{\theta}) = {\scriptstyle \text{rows}\downarrow} \overset{\text{cols}\ \longrightarrow}{\begin{bmatrix} \frac{\partial p_1(\underline{\theta})}{\partial\theta_1} & \frac{\partial p_1(\underline{\theta})}{\partial\theta_2} & \cdots & \frac{\partial p_1(\underline{\theta})}{\partial\theta_n} \\ \frac{\partial p_2(\underline{\theta})}{\partial\theta_1} & & & \\ \vdots & & & \\ \frac{\partial p_3(\underline{\theta})}{\partial\theta_1} & & & \frac{\partial p_3(\underline{\theta})}{\partial\theta_n} \end{bmatrix}} \in \mathbb{R}^{3\times n}$$

$$\underline{p}(\underline{\theta}) = \begin{bmatrix} p_1(\underline{\theta}) \\ p_2(\underline{\theta}) \\ p_3(\underline{\theta}) \end{bmatrix} \qquad \underline{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{bmatrix}$$

$$\underline{p}(\underline{\theta}+\underline{\Delta\theta}) \approx \underline{p}(\underline{\theta}) + J_\theta \underline{p}(\underline{\theta})\underline{\Delta\theta}.$$

$$J_\theta \underline{p}(\underline{\theta})\,\underline{\Delta\theta} = \underline{p}(\underline{\theta}+\underline{\Delta\theta}) - \underline{p}(\underline{\theta})$$

$$\boxed{\underline{\Delta\theta} = \left[J_\theta \underline{p}(\underline{\theta})\right]^{\dagger}\left(\underline{p}(\underline{\theta}+\underline{\Delta\theta}) - \underline{p}(\underline{\theta})\right)}$$



① How to find $J_\theta \underline{p}(\theta)$

② What is $[\ ]^{\dagger}$

dagger symbol $\dagger$

Pseudo inverse

Inverse of a matrix is only defined for square matrices

Pseudo inverse of a matrix $A_{m\times n}$ is $A^{\dagger}_{n\times m}$ iff

$$\underset{n\times m}{A^{+}} \underbrace{\underset{m\times n}{A}\ \underset{n\times m}{A^{+}}}_{I} = \underset{n\times m}{A^{+}}$$

$$\underset{m\times n}{A} \underbrace{\underset{n\times m}{A^{+}}\ \underset{m\times n}{A}}_{I} = \underset{m\times n}{A}$$

Capital letter = matrices
small letters = vector or scalars

$$A\underline{x} = \underline{b}$$
$$\Rightarrow \quad \underline{x} = \underbrace{A^{+}}\underline{b}$$

Solution to a system of Linear equations

$$m\left\{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ a_{m1} & \cdots & & a_{mn} \end{bmatrix}\right. \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{n\times 1} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$m\times n$ $\qquad$ $m\times 1$

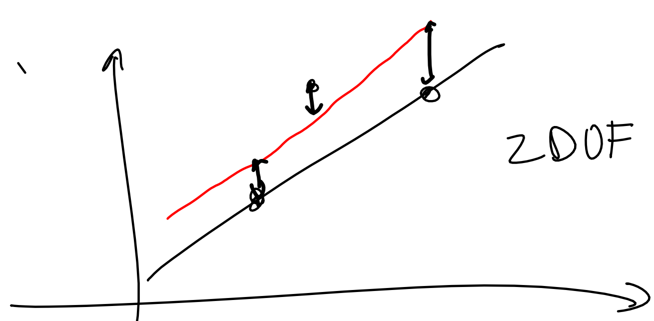
If the number of equations $\underline{m} > n$

the number of unknowns

how many $\underline{\text{exact}}$ solution = 0
×

if want approximate solution

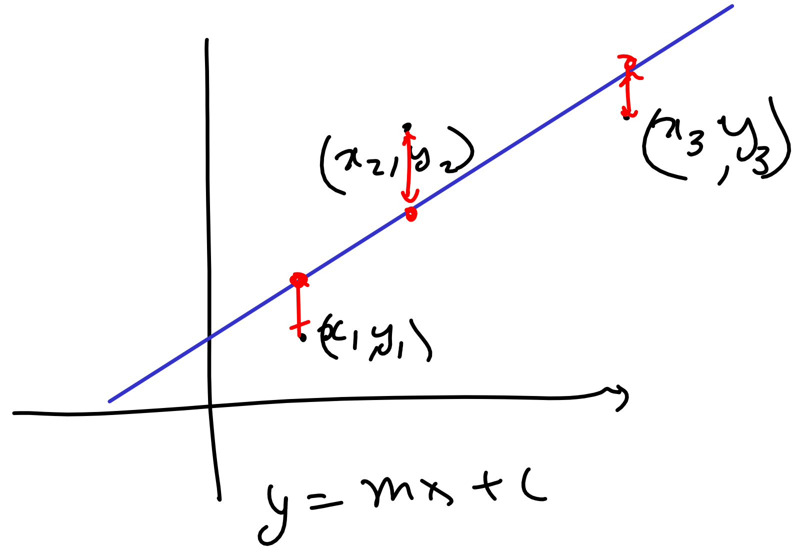
then you can minimize an error



$$A\underline{x} = \underline{b}$$

$$\min_{\underline{x}} \|A\underline{x} - \underline{b}\|^2$$

$$y_1 = m x_1 + c$$
$$y_2 = m x_2 + c$$
$$y_3 = m x_3 + c$$



$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{\underline{b}} = \underbrace{\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ x_3 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} m \\ c \end{bmatrix}}_{\underline{x}}$$

$$\|x\|_2^2 = \underline{x}^T \underline{x}$$

$$\|x\|_2^2 = x_1^2 + x_2^2 + \cdots x_n^2$$

$$\underline{x}^T \underline{x} = [x_1, x_2 \ldots x_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x_1^2 + x_2^2 + \ldots x_n$$

$$\|A\underline{x} - \underline{b}\|^2 = (A\underline{x} - b)^T (A\underline{x} - b)$$

$$= (\underline{x}^T A^T - \underline{b}^T)(A\underline{x} - \underline{b})$$

$$= \underbrace{\underline{x}^T A^T A \underline{x}}_{\text{quadratic form in } \underline{x}} \underbrace{- \underline{b}^T A \underline{x}^{①} - \underline{x}^T A^T \underline{b}^{②}}_{\text{linear term in } x} + \underline{b}^T \underline{b}$$

completing the squares

$$\underbrace{(x - \underline{y})^T M (x - \underline{y})}_{\to \text{ when } x \doteq y} + \underbrace{\quad}_{\text{not containing } x}$$

$$\underline{b}^T A \underline{x}^{①} = \underline{x}^T A^T \underline{b}^{②}$$

$$\underbrace{\underset{1\times m}{\underline{b}^T} \underset{m\times n}{A} \underset{n\times 1}{\underline{x}}}_{1\times 1} \quad \underbrace{\underset{1\times n}{\underline{x}^T} \underset{n\times m}{A^T} \underset{m\times 1}{\underline{b}}}_{1\times 1}$$

why is this true

$$A\underline{x} = \underline{b}$$
$$A \in \mathbb{R}^{m\times n}$$
$$\underline{x} \in \mathbb{R}^{n\times 1}$$
$$\underline{b} \in \mathbb{R}^{m\times 1}$$

$$(\underline{b}^T A \underline{x})^T = (\underline{b}^T A \underline{x})$$

$\hookrightarrow \underline{x}^T A^T \underline{b} = \underline{b}^T A \underline{x}$

$(PQR)^T = R^T Q^T P^T$

$$\|A\underline{x}-\underline{b}\|^2 = \underline{x}^T A^T A \underline{x} - 2\underline{b}^T A \underline{x} + \underline{b}^T \underline{b}$$

$$= x^T \underbrace{(A^TA)(A^TA)^{-1}(A^TA)}_{I} x$$

$$-2\underline{b}^T A \underbrace{(A^TA)^{-1}(A^TA)}_{I} \underline{x}$$

$$+ \underline{b}^T A (A^TA)^{-1} A^T \underline{b}$$

$$- \underline{b}^T A (A^TA)^{-1} A^T \underline{b}$$

$$+ b b$$

$$\left(x^2 - \frac{2b}{a}x + \frac{c}{a}\right)$$

$$= \left(x^2 - \frac{2b}{a}x + \boxed{\frac{b^2}{a^2}}\right) - \boxed{\frac{b^2}{a^2}} + \frac{c}{a}$$

$$= \left(x - \frac{b}{a}\right)\left(x - \frac{b}{a}\right) - \frac{b^2}{a^2} + \frac{c}{a}$$

inverse

$(PQR)^{-1} = R^{-1}Q^{-1}P^{-1}$

$$\cancel{\underline{x}^T \underbrace{(A^TA)(A^TA)^{-1}}(A^TA)\, x} = \left(x^T (A^TA) \underbrace{A^{-1}}\right)\left(A^{-T}(A^TA)\underline{x}\right)$$

$\times$ not defined

$$\cancel{-2\underline{b}^T A (A^TA)^{-1}(A^TA)\underline{x}} = -2\underline{b}^T \underbrace{A A^{-1}} \left(A^{-T}(A^TA)\underline{x}\right)$$

cannot do this because A^{-1} is not defined

$$(A^TA\underline{x} - \underline{y})^T (A^TA)^{-1} (A^TA\underline{x} - \underline{y})$$

$$= (\underline{x}^T A^T A - \underline{y}^T)(A^TA)^{-1}(A^TA\underline{x} - \underline{y})$$

$$= x(A^TA)(A^TA)^{-1}(A^TA)x - 2y^T(A^TA)^{-1}(A^TA)x + y^T(A^TA)^{-1}y$$

$$= x^T(A^TA)\underbrace{(A^TA)^{-1}(A^TA)}_{I}x$$

$$y = A^Tb$$

will complete the square

$$-2b^TA\underbrace{(A^TA)^{-1}(A^TA)}_{I}x$$

$$+ b^TA(A^TA)^{-1}A^Tb$$

$$\|Ax - b\|^2 = \overbrace{(A^TAx - A^Tb)^T(A^TA)^{-1}(A^TAx - A^Tb)}^{\text{depends upon } x \geqslant 0} - b^TA(A^TA)^{-1}A^Tb + b^Tb$$

If we make first term 0, then the whole thing is minimised

$$A^TAx - A^Tb = 0$$

$$\Rightarrow \quad x = \underbrace{\boxed{(A^TA)^{-1}A^T}}_{A^+ \text{ for } m > n} b$$

$$A^T_{n\times m}A_{m\times n}$$

$$(A^TA)_{n\times n}$$

for $m < n$

$$\underline{x} = \underbrace{A^T (AA^T)^{-1}}_{A^+ \text{ for } m<n} \underline{b}$$

Easier way of completing the square

$$\|A\underline{x}-\underline{b}\|^2 = \underline{x}^T A^T A \underline{x} - 2\underline{b}^T A \underline{x} + \underline{b}^T \underline{b}$$

$$= \underline{x}^T A^T A \underline{x} - 2\underline{b}^T A \underbrace{(A^TA)^{-1}(A^TA)}_{I} \underline{x}$$

$$+ b^T A^T (A^TA)^{-1} A \underline{b}$$

$$- b^T A^T (A^TA)^{-1} A \underline{b}$$

$$+ \underline{b}^T \underline{b}$$

$$= \boxed{(\underline{x} - (A^TA)^{-1}A^T\underline{b})^T (A^TA) (\underline{x} - (A^TA)^{-1}A^T\underline{b})}$$

Quadratic form

$$- \underline{b}^T A^T (A^TA)^{-1} A \underline{b} + \underline{b}^T \underline{b}$$

$$ax^2 - 2bx + c$$
$$= a\left(x^2 - \frac{2b}{a}x + \frac{c}{a}\right)$$
$$= a\left(x^2 - \frac{2b}{a}x + \frac{b^2}{a}\right)$$
$$+ a\left(-\frac{b^2}{a^2} + \frac{c}{a}\right)$$
$$= a\left(x - \frac{b}{a}\right)^2$$

$$\underline{b}^T A \overbrace{(A^TA)^{-1}(A^TA)}(A^TA)^{-1}A^T\underline{b}$$
$$= b^T A (A^TA)^{-1} A^T \underline{b}$$

$$\underline{x} = \underbrace{(A^TA)^{-1}A^T}_{A^+}\underline{b}$$

$$A\underline{x} = b$$

Quadratic form := $\text{vector}^T (\text{Matrix})\, \text{vector} > 0$ for all vector

Matrix has to be Positive definite is called

A matrix Q is Positive definite if and only if for vectors $\underline{x}$

$$\underline{x}^T Q \underline{x} > 0$$

$Q \in \mathbb{R}^{n \times n}$

$\underline{x} \in \mathbb{R}^{n \times 1}$

Matrix of the form $Q = P^T P$ are always positive semi definite. Why?

$P \in \mathbb{R}^{m \times n}$

$\Rightarrow Q \in \mathbb{R}^{n \times n}$

$\underline{x} \in \mathbb{R}^{n \times 1}$

$\underline{y} = P\underline{x} \in \mathbb{R}^{m \times 1}$

$$\underline{x}^T Q \underline{x} = \underline{x}^T P^T P \underline{x}$$

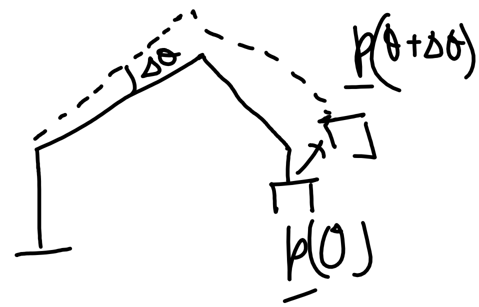
$$= (P\underline{x})^T (P\underline{x})$$

$$= \underline{y}^T \underline{y}$$

$$= \|\underline{y}\|^2 \geq 0$$

for all $\underline{x}$ or $Q = P^T P$ is always Positive semi definite

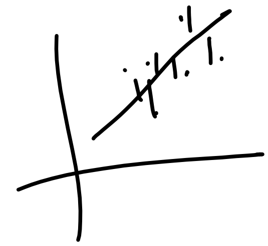
$$\underline{\Delta\theta} = \underbrace{\left[J_\theta\, p(\theta)\right]}_{}^{\dagger}\left[\underline{p}(\theta+\Delta\theta) - \underline{p}(\theta)\right]$$



$A^{\dagger} = (A^TA)^{-1}A^T$ if $A \in \mathbb{R}^{m\times n}$

$\underbrace{m > n}_{}$

tall matrix



$= A^T(AA^T)^{-1}$ if $m < n$

less equations

more unknowns

how many exact solutions? ∞ solution

Solve $A\underline{x} = \underline{b}$, $\Rightarrow \infty$ solutions

pick solution with

smallest $\|\underline{x}\|^2$

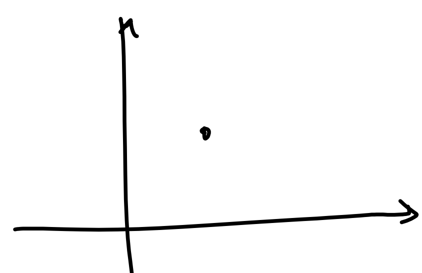
$3x + 4y + 6z = 10$

$7x + 5y + 7z = 5$

$$\min_{\underline{x}} \|x\|^2$$

such that $A\underline{x} = b$

This is a constrained optimization problem, that can be solved via Lagrange multipliers

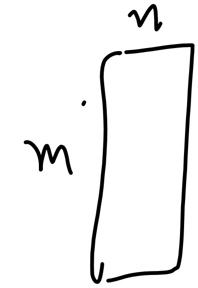


$y = mx + c$

$y = \underbrace{(m \;\; c)}_{}\begin{bmatrix} x \\ 1 \end{bmatrix}$

pick the solution with smallest parameters - smallest $m^2 + c^2$

$$A^{+} = \begin{cases} (A^TA)^{-1}A^T & \text{if } A \in \mathbb{R}^{m \times n} \\ & \underbrace{m > n} \\ & \text{tall matrix} \\ A^T(AA^T)^{-1} & \text{if } \underbrace{m < n} \text{ fat matrix} \\ & \text{less equations} \\ & \text{more unknowns} \end{cases}$$



A^{+} via SVD = Singular value Decomposition

numpy.linalg.svd

$$A = U \Sigma V^T$$

Any matrix A can be decompose via SVD into U, Σ, V^T $\quad A = U\Sigma V^T$

where U is orthonormal $\quad U^TU = I$

V is orthonormal $\quad V^TV = I$

Σ is diagonal

Singular values

$$\Sigma = \begin{bmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ 0 & & \ddots & \lambda_n \\ \cdots & & & \\ & 0 & & \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & & 0 \\ & \lambda_2 & & \\ 0 & & \ddots & \\ & & & \lambda_n \end{bmatrix}$$

Typically $\lambda_1 \geq \lambda_2 \geq \lambda_3 \ldots \geq \lambda_n$

$$A^{+} = V \cdot \Sigma^{+} U^{T}$$

from SVD of A

$$\Sigma^{+} = \begin{bmatrix} 1/\lambda_1 & & & 0 \\ & 1/\lambda_2 & & \\ & & \ddots & \\ 0 & & & 1/\lambda_n \end{bmatrix}$$

Take elementwise inverse of singular values

A^{+} = numpy.linalg.pinv

$$\underline{\Delta\theta} = \underbrace{\left[J_\theta\, p(\theta)\right]}^{+}\left[\underline{p}(\theta+\Delta\theta) - \underline{p}(\theta)\right]$$

$$\underline{\Delta\theta} = \text{numpy.linalg.lstsq}\left(\underline{J_\theta p(\theta)},\ p(\underline{\theta}+\underline{\Delta\theta}) - p(\underline{\theta})\right)$$

Least square solutions

To find A^{+}

$$\min_{\underline{x}} \|A\underline{x} - \underline{b}\|_2^2 \quad \text{for tall matrix}$$

$$\min_{\underline{x}} \|x\|_2^2 \quad \text{s.t } A\underline{x} = b \quad \text{for fat matrix}$$

$$\|\underline{x}\|_2^2 = x_1^2 + x_2^2 + x_3^2 + \dots + x_n^2$$

sum of squares

$$\|x\|_1 = |x_1| + |x_2| + \dots + |x_n|$$

Least absolute