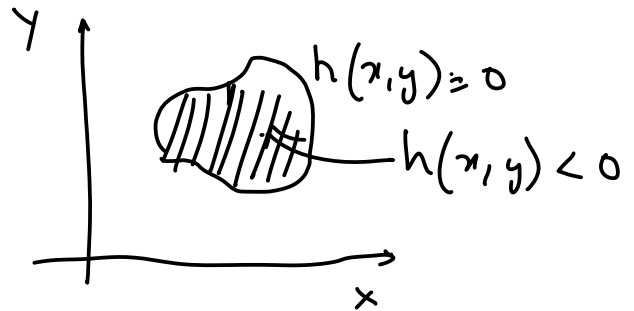
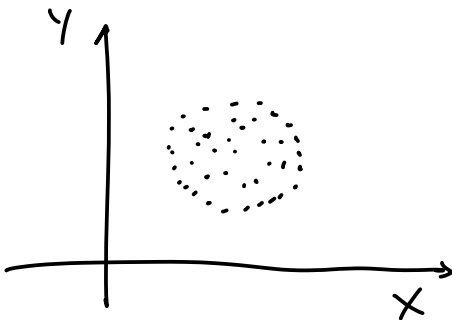


2D coordinate transforms

Rotations + translations
Orientation + position

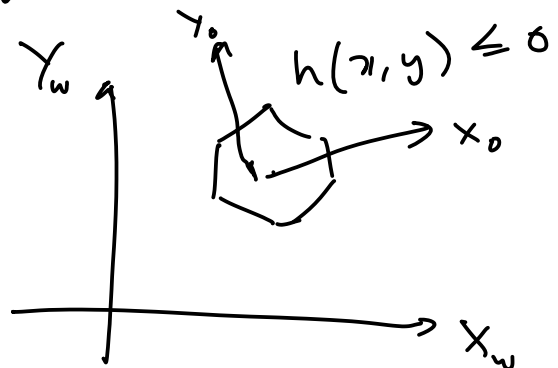
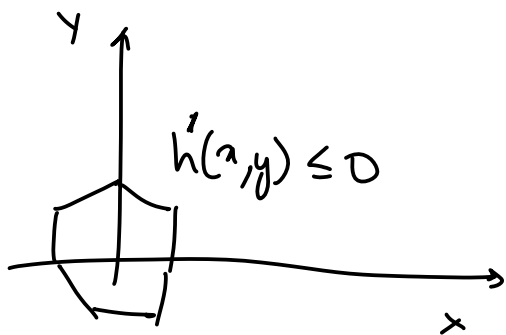
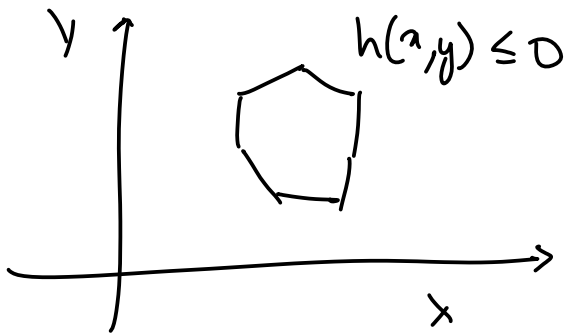
$$h(x,y) > 0$$



$$\text{Object} = \{ (x,y) \in \mathbb{R}^2 : h(x,y) \leq 0 \}$$

Set of all points $(x,y) \in \mathbb{R}^2$
such that $h(x,y) \leq 0$

set of all real numbers

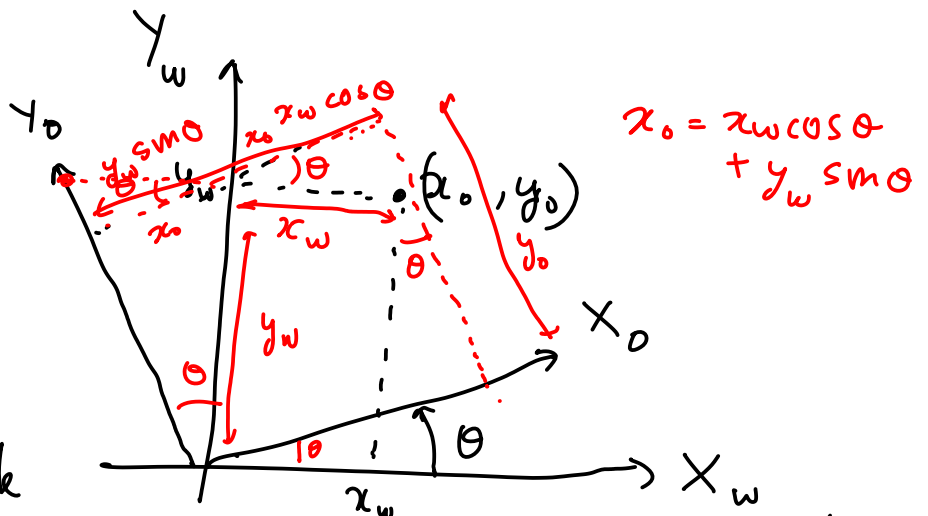


2D Rotations

- ① Rotation
- ② Translation

Problem

(x_0, y_0) is given in (x_0, y_0) coordinate frame. (x_0, y_0) has been rotated by angle θ w.r.t. (x_w, y_w)



Find (x_w, y_w) in world coordinate frame

Proof using Basis vectors

In Linear algebra, Basis vectors are set of orthonormal unit vectors that span the entire space

Span is the set of all vectors that can be obtained by linear combinations of a given set of vectors

$$\text{Span} \{ \underset{\substack{\uparrow \\ \in \mathbb{R}^n}}{\underline{a}}, \underset{\substack{\uparrow \\ \in \mathbb{R}^n}}{\underline{b}} \} = \{ \underset{\substack{\uparrow \\ \in \mathbb{R}^n}}{\underline{\alpha a + \beta b}}, \underset{\substack{\uparrow \\ \in \mathbb{R}^n}}{\beta \in \mathbb{R}} \}$$

Standard Basis vector.

For example, in \mathbb{R}^2
in \mathbb{R}^3

$$\hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \hat{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$\hat{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \hat{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{in } \mathbb{R}^n \quad \hat{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n \times 1} \dots \hat{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}_{n \times 1}$$

Basis vectors for \mathbb{R}^n

↳ ① All vectors must be perpendicular/orthogonal to each other

↳ ② They must be unit vectors

↳ ③ They must span the entire space \mathbb{R}^n

Let Basis vector for (x_w, y_w) be standard basis vector

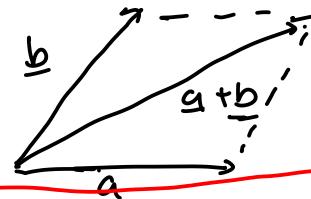
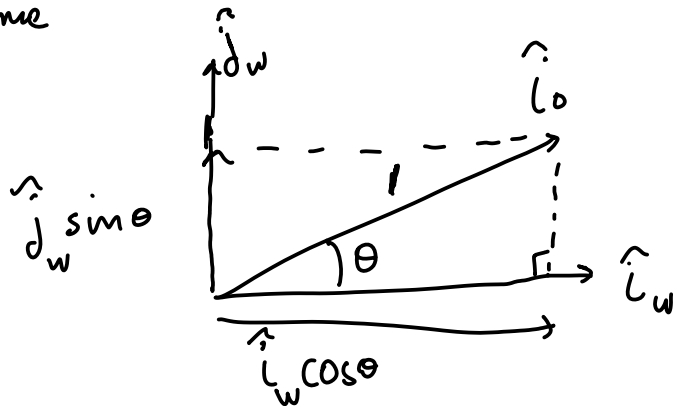
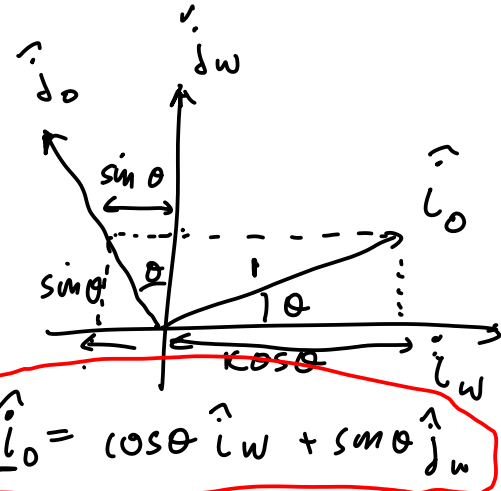
$$\hat{i}_w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \hat{j}_w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Any point
in world
coordinate
frame

$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = x_w \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\hat{i}_w} + y_w \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\hat{j}_w}$$

Any point
in the object
coordinate
frame

$$\begin{bmatrix} x_o \\ y_o \end{bmatrix} = x_o \hat{i}_o + y_o \hat{j}_o$$



$$\hat{j}_o = -\hat{i}_w \sin \theta + \hat{j}_w \cos \theta$$

world

object

$$x_w \hat{i}_w + y_w \hat{j}_w = x_o \hat{i}_o + y_o \hat{j}_o$$

$$= x_o [\cos \theta \hat{i}_w + \sin \theta \hat{j}_w] + y_o [-\hat{i}_w \sin \theta + \hat{j}_w \cos \theta]$$

$$= [x_o \cos \theta - y_o \sin \theta] \hat{i}_w + [x_o \sin \theta + y_o \cos \theta] \hat{j}_w$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

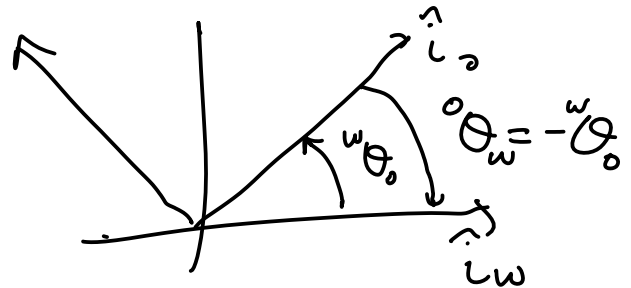
$$= \begin{bmatrix} x_o \cos \theta - y_o \sin \theta \\ x_o \sin \theta + y_o \cos \theta \end{bmatrix}$$

$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x_o \\ y_o \end{bmatrix}$$

Because
we are
using
standard
basis
for world
coordinate
frame

$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \begin{bmatrix} {}^w R_0(\theta) \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$${}^w R_0(\theta) := \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$



$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_w \\ y_w \end{bmatrix}$$

$${}^0 R_w(\theta) = {}^w R_0(-\theta) = {}^w R_0^T(\theta)$$

$$= \begin{bmatrix} \cos \theta & +\sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x_w \\ y_w \end{bmatrix}$$

$$\begin{bmatrix} {}^w R_0^T \end{bmatrix} {}^w R_0 = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 \theta + \sin^2 \theta & -c(\theta)s(\theta) + s(\theta)c(\theta) \\ -s(\theta)c(\theta) + c(\theta)s(\theta) & s^2(\theta) + c^2(\theta) \end{bmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \underline{I}_{2 \times 2}$$

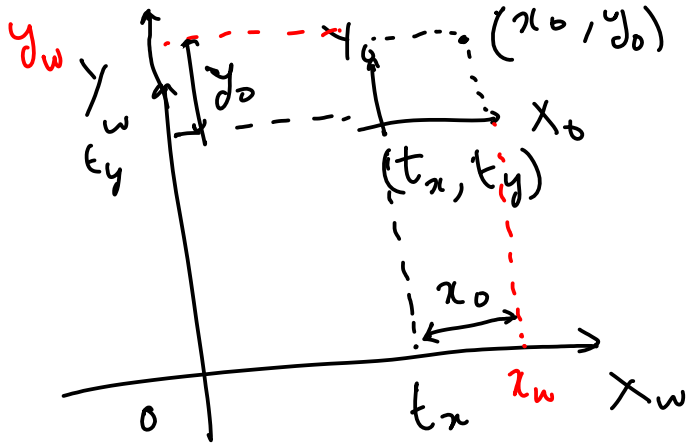
$$\underline{R^T R = I}$$

$$(A^{-1})A = I$$

$$R^{-1} = R^T$$

2D Translation

$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \underbrace{\begin{bmatrix} t_x \\ t_y \end{bmatrix}}_{\text{translation vector}} + \begin{bmatrix} x_o \\ y_o \end{bmatrix}$$

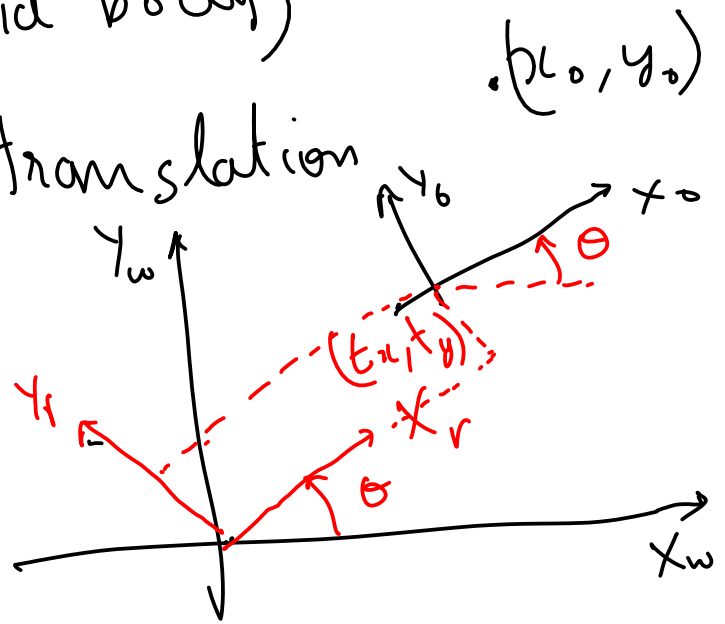


2D transformation (Rigid body)

Rotation followed by translation

$$\begin{bmatrix} x_r \\ y_r \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x_o \\ y_o \end{bmatrix}$$

$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \begin{bmatrix} t_x \\ t_y \end{bmatrix} + \begin{bmatrix} x_r \\ y_r \end{bmatrix}$$



$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x_o \\ y_o \end{bmatrix} + \underbrace{\begin{bmatrix} t_x \\ t_y \end{bmatrix}}_{\underline{t}}$$

$$\underline{x}_w = {}^w_0 R(\theta) \underline{x}_o + {}^w_0 \underline{t}$$

$$\underline{x}_0 = ? (\underline{x}_w)$$

$$\rightarrow \underline{x}_w - {}^w_0 \underline{t} = {}^w_0 R \underline{x}_0$$

$${}^w_0 R = {}^w_0 R(0)$$

Multiply on the left of both sides by ${}^w_0 R^T$

$$\rightarrow {}^w_0 R^T (\underline{x}_w - {}^w_0 \underline{t}) = ({}^w_0 R^T {}^w_0 R) \underline{x}_0$$

$$\frac{\underline{x}_w - {}^w_0 \underline{t}}{{}^w_0 R} = \underline{x}_0$$

NEVER EVER
DO THIS

$$\underline{x}_0 = {}^w_0 R^T \underline{x}_w - {}^w_0 R^T {}^w_0 \underline{t} \quad \text{--- (1)}$$

$$\underline{x}_0 = {}^0_w R \cdot \underline{x}_w + {}^0_w \underline{t} \quad \text{--- (2)}$$

Compare (1) and (2)

$${}^0_w R = {}^w_0 R^T$$

and

$${}^0_w \underline{t} = - {}^w_0 R^T {}^w_0 \underline{t}$$

$$\underline{x}_w = {}^w_0 R \underline{x}_0 + {}^w_0 \underline{t}$$

$$\begin{bmatrix} \underline{x}_w \\ 1 \end{bmatrix} = \begin{bmatrix} x_w \\ y_w \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} {}^wR_{2 \times 2} & {}^w\vec{t}_{2 \times 1} \\ O_{1 \times 2}^T & 1 \end{bmatrix}}_{\substack{{}^wT_o \\ 3 \times 3}} \begin{bmatrix} \underline{x}_o_{2 \times 1} \\ 1 \end{bmatrix}$$

Block matrix

Block Matrix multiplication

$$\checkmark \quad \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{bmatrix}$$

$$\begin{bmatrix} {}^wR_{2 \times 2} & {}^w\vec{t}_{2 \times 1} \\ O_{1 \times 2}^T & 1 \end{bmatrix} \begin{bmatrix} \underline{x}_o_{2 \times 1} \\ 1 \end{bmatrix} = \begin{bmatrix} {}^wR \underline{x}_o + {}^w\vec{t}_o \\ O^T \underline{x}_o + 1 \end{bmatrix} = \begin{bmatrix} {}^wR \underline{x}_o + {}^w\vec{t}_o \\ 1 \end{bmatrix} = \begin{bmatrix} \underline{x}_w \\ 1 \end{bmatrix}$$

$$\underline{x}_w = {}^wT_o \underline{x}_o$$

$${}^wT_o = \begin{bmatrix} {}^wR_{2 \times 2} & {}^w\vec{t}_{2 \times 1} \\ O_{1 \times 2}^T & 1 \end{bmatrix}$$

Transformation matrix

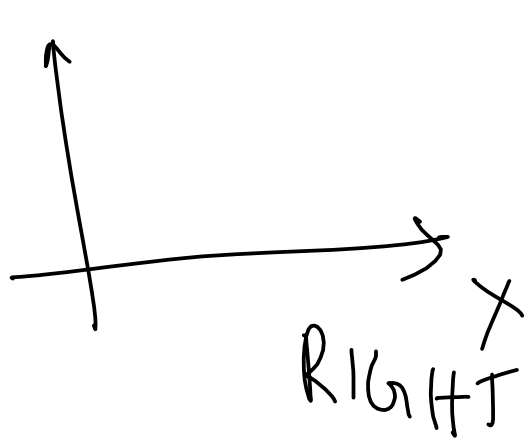
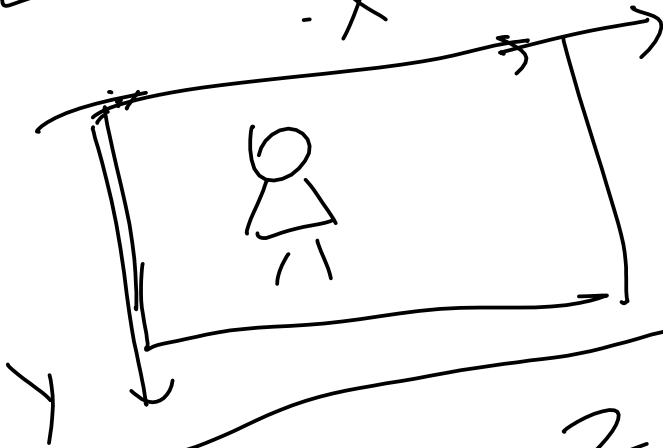
In 2D
and
3D

Right hand ✓

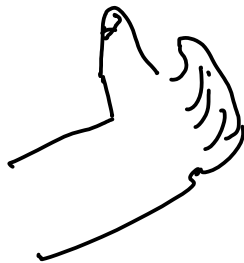
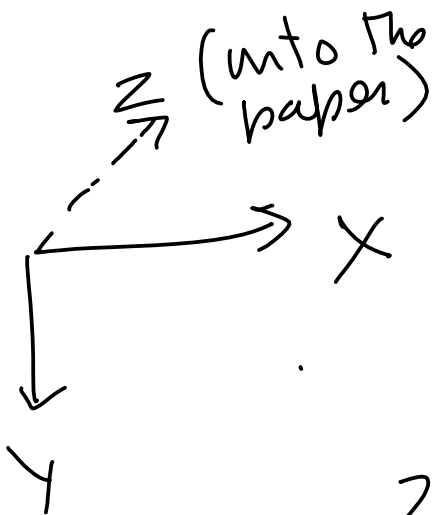
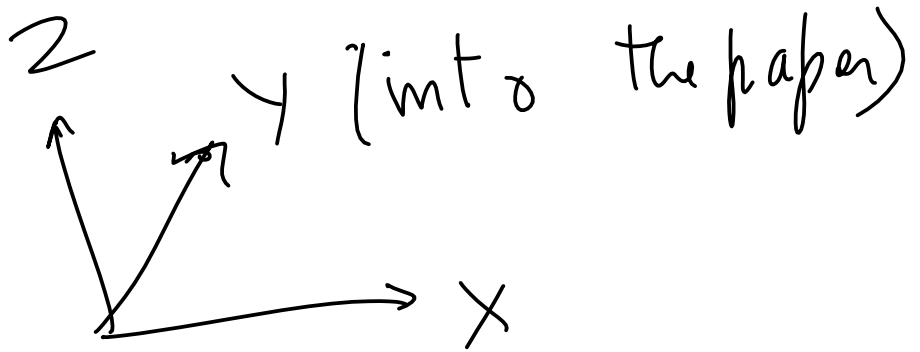
Left hand ✗

LEFT

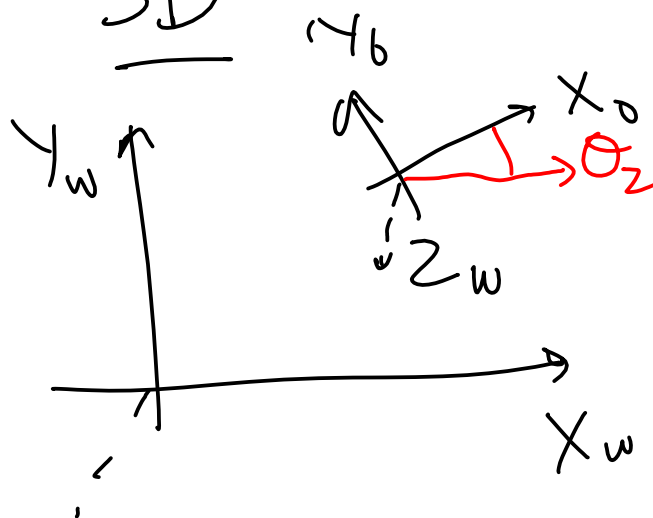
-X



In 3D



Extending 2D to 3D



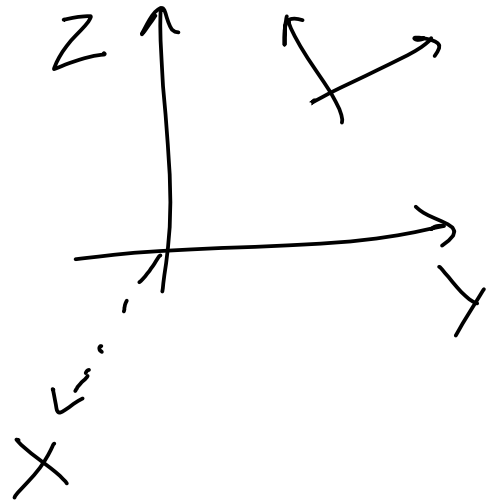
$\nwarrow z_w$ (out of the paper)

$$\begin{bmatrix} x_w \\ y_w \\ z_w = 0 \end{bmatrix} = \begin{bmatrix} \cos \theta_2 & -\sin \theta_2 & 0 \\ \sin \theta_2 & \cos \theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ 0 \end{bmatrix}$$

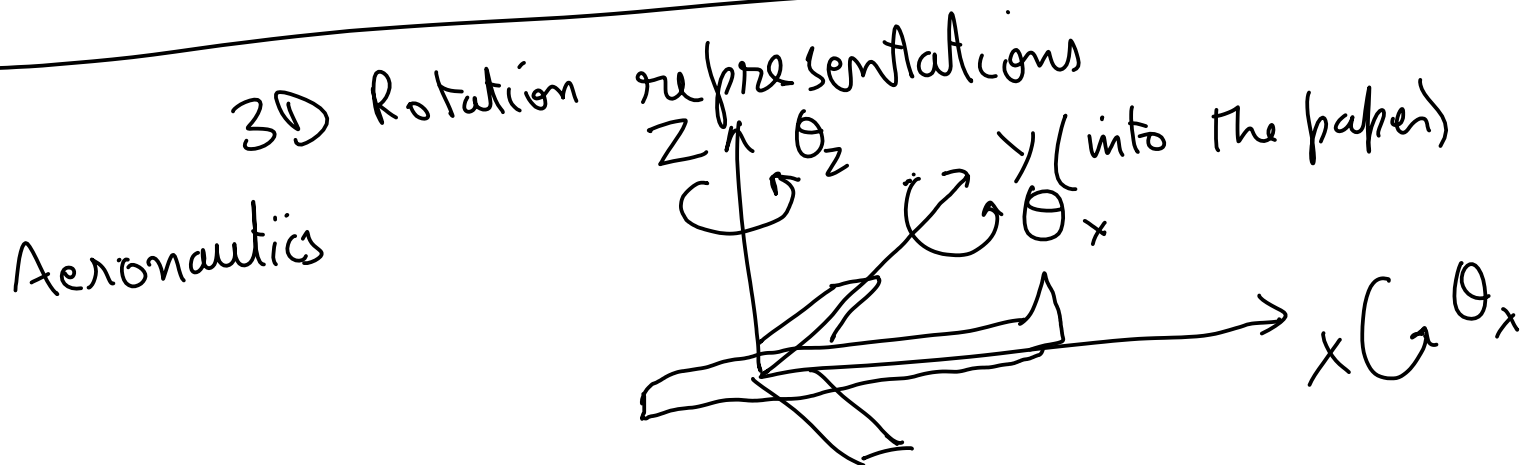
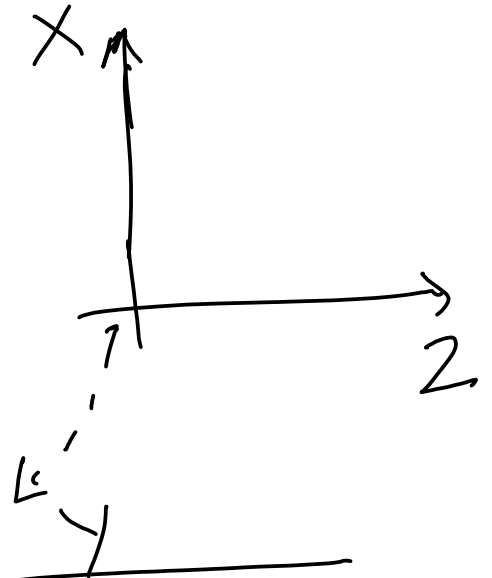
Rotation along Z-axis changes
only X-Y coordinates

$$R(\theta_2) = \begin{bmatrix} \cos \theta_2 & -\sin \theta_2 & 0 \\ \sin \theta_2 & \cos \theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R(\theta_x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}$$



$$R(\theta_y) = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$$

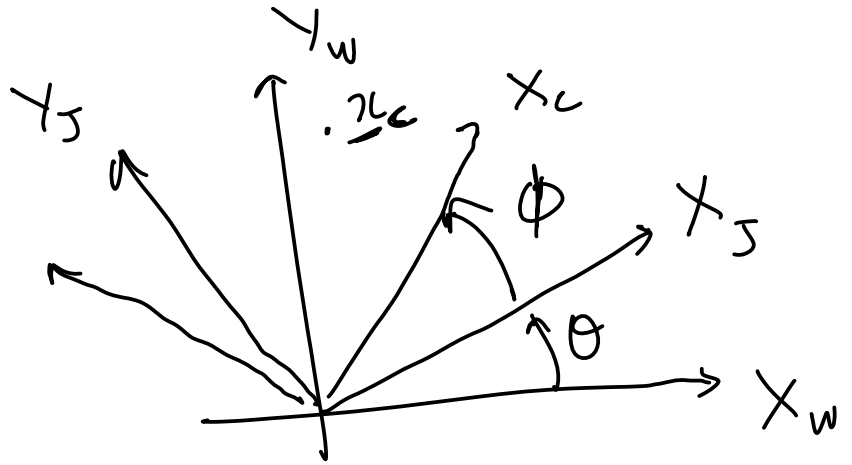


$\theta_x = \text{roll}$
 $\theta_y = \text{pitch}$
 $\theta_z = \text{yaw}$

$$R = R(\theta_z) R(\theta_y) R(\theta_x)$$

\uparrow \uparrow \uparrow
 Yaw Pitch Roll

Chain rotation, translation, transformations



$$\underline{x}_j = {}^j_c R(\phi) \underline{x}_c$$

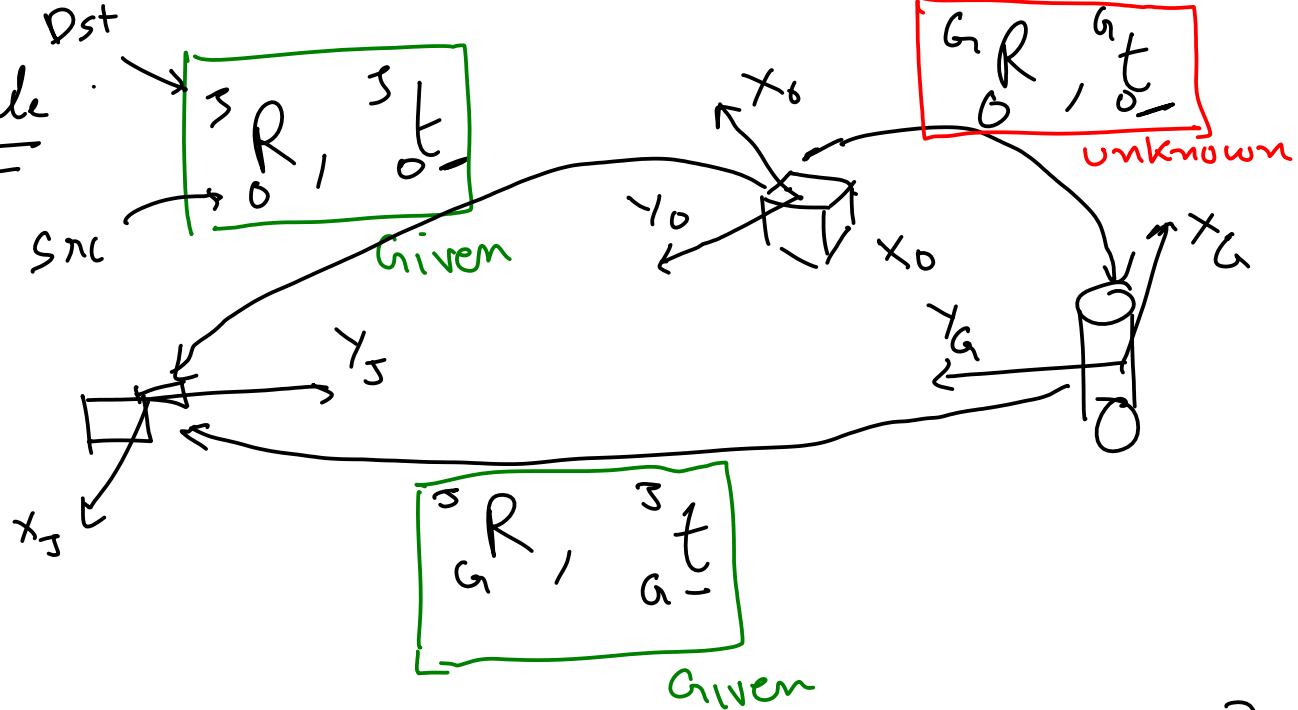
$$\underline{x}_w = {}^w_j R(\theta) \underline{x}_j$$

$$= R(\theta) \left[R(\phi) \underline{x}_c \right]$$

$$= \underbrace{\left(R(\theta) R(\phi) \right)}_{{}^w_c R} \underline{x}_c$$

$${}^w_c R = {}^w_j R(\theta) {}^j_c R(\phi)$$

$${}^w_c T = {}^w_j T {}^j_c T$$

Example

$${}^0T_1 = \begin{bmatrix} {}^0R_1 & {}^0t_1 \\ 0^T & 1 \end{bmatrix}, \quad {}^1T_2 = \begin{bmatrix} {}^1R_2 & {}^1t_2 \\ 0^T & 1 \end{bmatrix}$$

$${}^0T_2 = \begin{bmatrix} {}^0R_2 & {}^0t_2 \\ 0^T & 1 \end{bmatrix}, \quad {}^0T_2 = ({}^0T_1^{-1}) {}^1T_2$$

$$= {}^0T_1 {}^1T_2 \dots$$

$${}^0T_2 \underline{x}_2 = \left({}^0T_1 \left({}^1T_2 \underline{x}_2 \right) \right)$$

\underline{x}_1

\underline{x}_2

\underline{x}_3

① $R_{3D} = R(\theta_z) R(\theta_y) R(\theta_x)$

$\begin{matrix} & \uparrow & \uparrow & \uparrow \\ & \text{yaw} & \text{pitch} & \text{roll} \end{matrix}$

This sequence

$\begin{matrix} XYZ \\ \hline ZYX \\ \vdots \\ \vdots \\ \vdots \end{matrix}$

6 possible
= 3!

$\begin{matrix} \dot{\theta}_x & \xrightarrow{\text{then}} & \theta_y \\ & & \downarrow \text{then} \\ & & \theta_z \end{matrix}$

Euler angle representation of 3D rotation is a sequence of rotation around standard axis

Euler representation with XYZ then

$R_{3D} = R(\theta_z) R(\theta_y) R(\theta_x)$

Conversion from Euler angles to Rotation matrix

How to do The opposite?

convert from Rotation matrix to Euler angles?

$$R(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}$$

$$\theta = ? = \tan^{-1} \left(\frac{r_{21}}{r_{11}} \right) \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$$

$$\theta = \arctan 2(r_{21}, r_{11}) \in (-\pi, \pi)$$

$$\begin{array}{c}
 \text{3D} \quad \downarrow^z \quad \downarrow^y \quad \downarrow^x \\
 R = R(\psi) R(\phi) R(\theta) \\
 = \begin{bmatrix} c(\psi) & -s(\psi) & 0 \\ s(\psi) & c(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c(\phi) & 0 & s(\phi) \\ 0 & 1 & 0 \\ -s(\phi) & 0 & c(\phi) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c(\theta) & -s(\theta) \\ 0 & s(\theta) & c(\theta) \end{bmatrix}
 \end{array}$$

$$= \begin{bmatrix} \boxed{c(\psi)c(\phi)} & c(\psi)c(\phi)s(\theta) & c(\psi)s(\phi)c(\theta) + s(\psi)s(\theta) \\ \boxed{s(\psi)c(\phi)} & s(\psi)s(\phi)s(\theta) + c(\psi)c(\theta) & s(\psi)s(\phi)c(\theta) - c(\psi)s(\theta) \\ \boxed{-s(\phi)} & \boxed{c(\phi)s(\theta)} & \boxed{c(\phi)c(\theta)} \end{bmatrix}$$

$$= \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

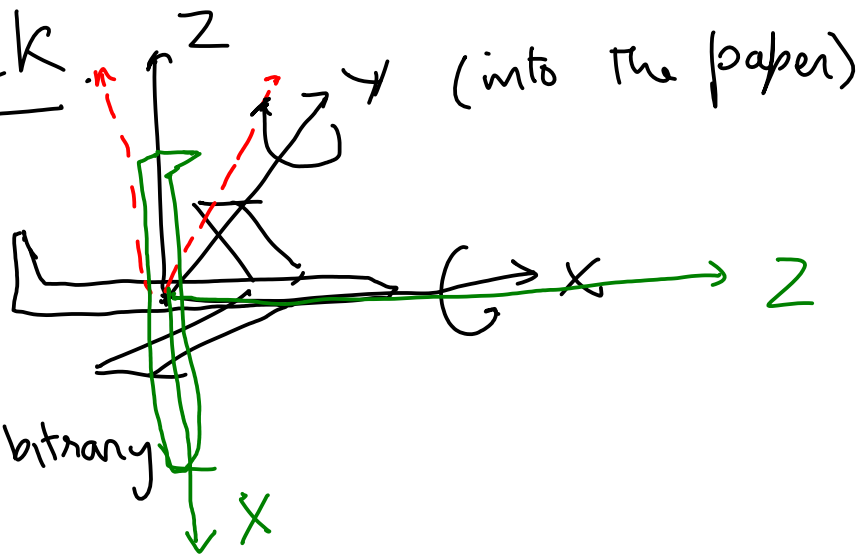
$$\phi = -\sin^{-1}(r_{31}) \in [0, \pi]$$

$$\frac{r_{21}}{r_{11}} = \frac{\sin(\psi) \cancel{c(\phi)}}{\cos(\psi) \cancel{c(\phi)}} \Rightarrow \psi = \arctan 2(r_{21}, r_{11})$$

$$\theta = \arctan 2(r_{32}, r_{33})$$

conversion from Rotation matrix to Euler angles

Gimbal lock



$$\theta_x = 30^\circ \leftarrow \text{arbitrary}$$

$$[\theta_y = \underline{90^\circ}]$$

$$\theta_z = 45^\circ \leftarrow \text{arbitrary}$$

Euler angles $\xrightarrow{\text{deterministically}}$ Rot mat
 $\xleftarrow{\text{multiple solutions}}$

Other representations. It is impossible to unambiguously represent 3D rotation with only 3 numbers

Degree of freedom

but needs 4 numbers
+ 1 constraint
to represent it

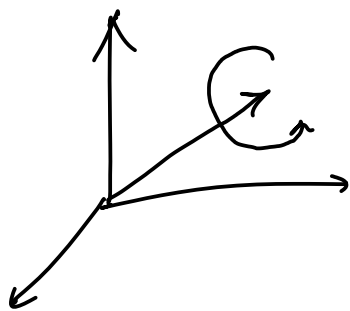
3D rot = 3 DOF

② Axis-angle representation ← (3) Quaternions

↓
Rot mats
↓
Algebra

Quaternions
[Quaternion algebra]
≡ [complex numbers]

② Axis-angle representation



Any 3D rotation can be represented
as a unit vector (axis) and rotation angle
around it.

$$\text{Axis} = \underline{a} = [a_x, a_y, a_z]$$

$$\text{angle} = \theta$$

$$\text{constraint: } \|a\|_2 = \sqrt{a_x^2 + a_y^2 + a_z^2} = 1$$

$$\left. \begin{array}{l} \text{Free scalars} = 4 \\ \text{Constraint} = 1 \end{array} \right\} \text{Degree of freedom} = 3$$

DOF of 2D Rot matrix = ?

$$R = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

Free scalars = 4

Two vector constraints :

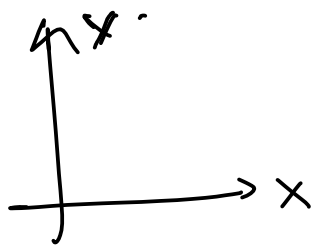
$$\left[\begin{array}{l} R^T R = I = R R^T \\ \det(R) = +1 \end{array} \right]$$

2 scalar constraints

1 scalar constraint

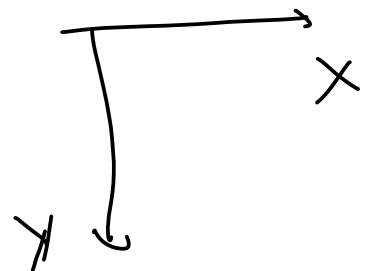
Reflection matrices also satisfy

Check for valid rotation



not Rotation

$$\begin{array}{c} \xrightarrow{\text{reflection}} \\ \det(\text{Reflection}) \\ = -1 \end{array}$$



Rodrigues rotation formula

Axis angle \rightarrow Rot matrix
 (θ, \hat{k})

$$R = I + \sin \theta [k_x] + (1 - \cos \theta) [k_x]^2$$