Content from Drew Bagnell, Pieter Abeel

Iterative LQR & Model Predictive Control

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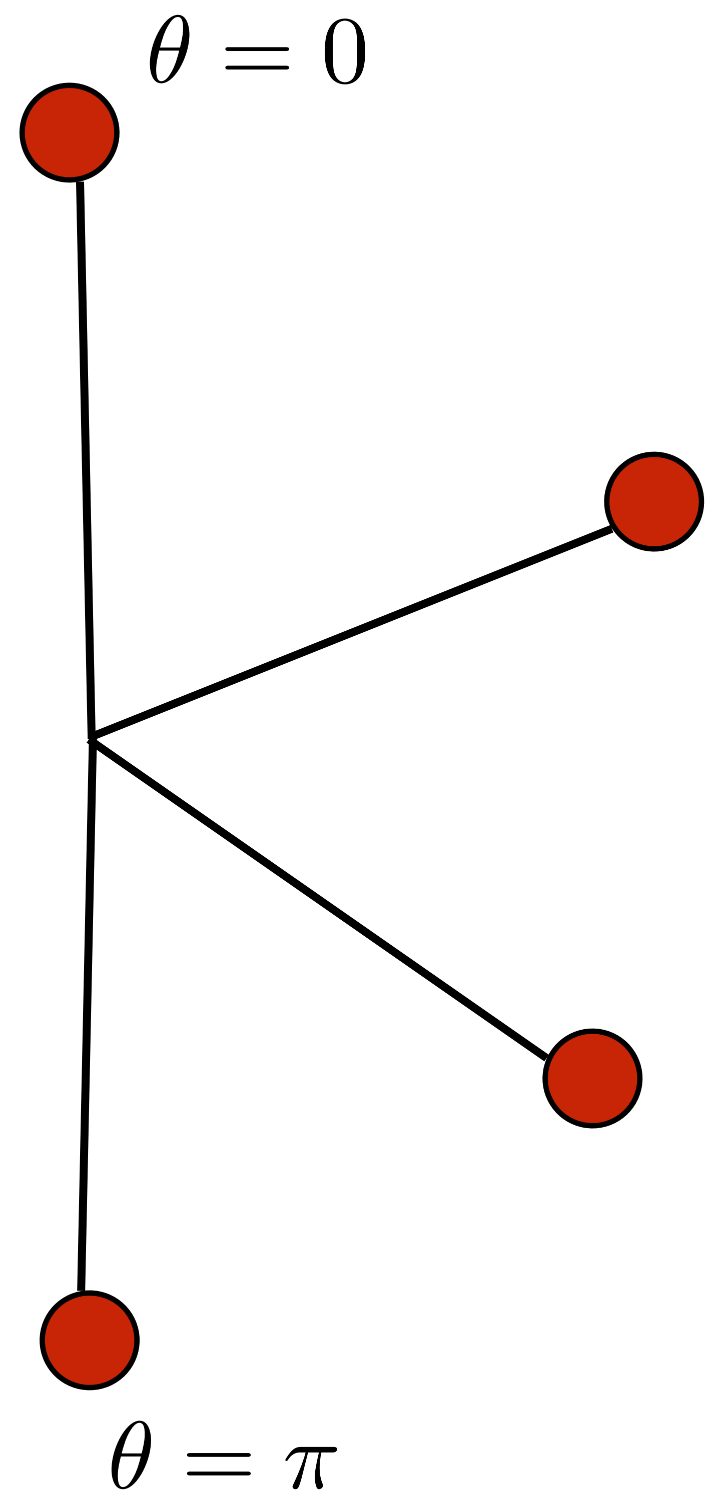
Table of Controllers

	Control Law	Uses model	Stability Guarantee	Minimize Cost
PID		No	No	No
Pure Pursuit		Circular arcs	Yes - with assumptions	No
Lyapunov		Non-linear	Yes	No
LQR		Linear	Yes	Quadratic

Can we use LQR to swing up a pendulum?



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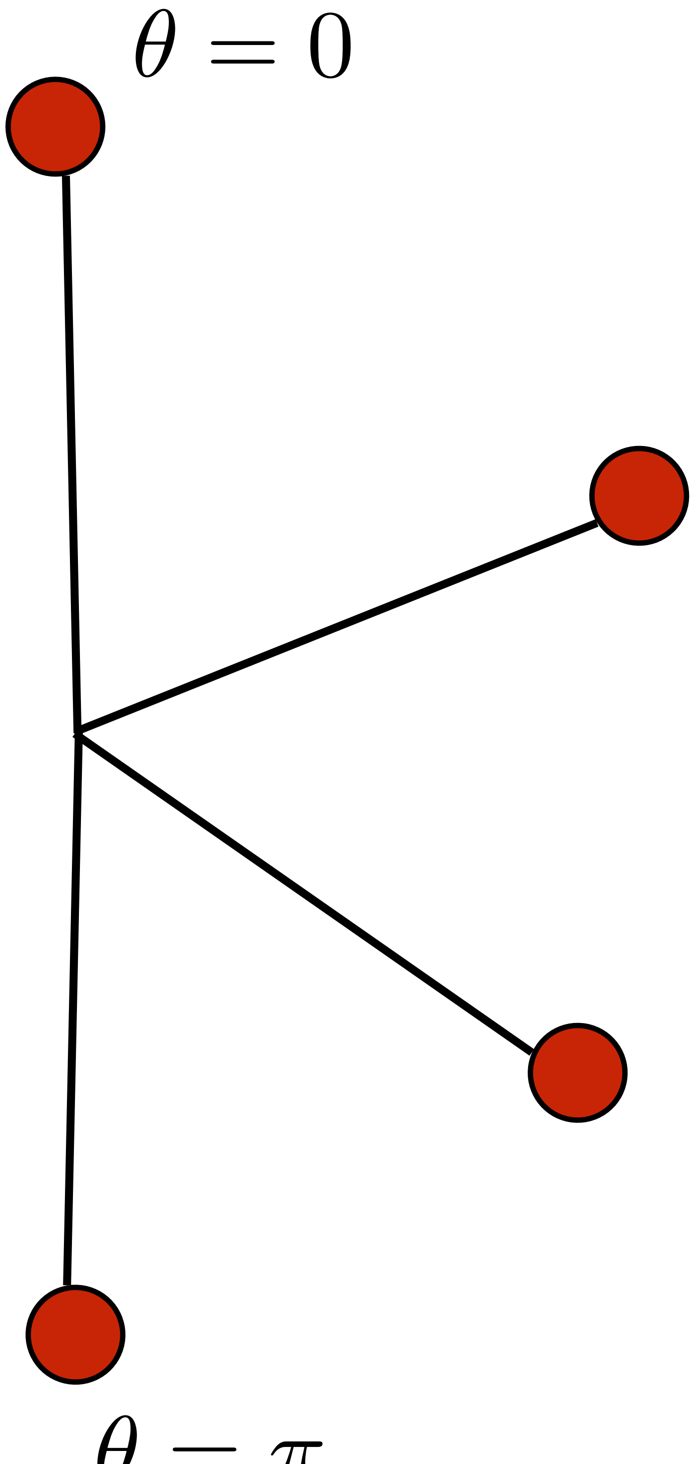
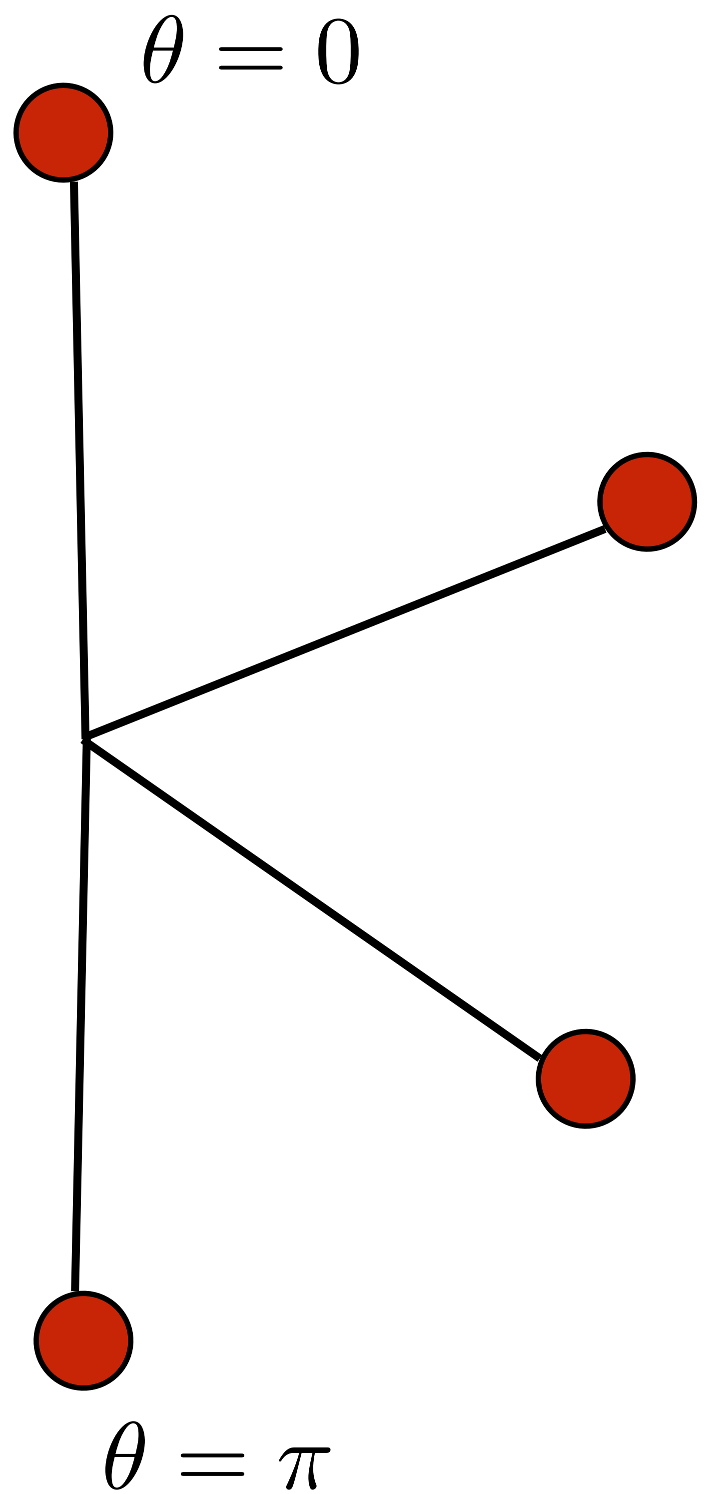


Can we use LQR to swing up a pendulum?

No!
(Large angles imply
large linearization error)

But we can track
a reference swing up trajectory
(small linearization error)



But, first we need to talk about time-varying systems

Today's objectives

1. LQR for time-varying systems
2. Trajectory following with iLQR
3. General nonlinear trajectory optimization with iLQR
4. Model predictive control (MPC)

LQR for Time-Varying Dynamical Systems

$$x_{t+1} = A_t x_t + B_t u_t$$

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Straight forward to get LQR equations

$$K_t = -(R_t + B_t^T V_{t+1} B_t)^{-1} B_t^T V_{t+1} A_t$$

$$V_t = Q_t + K_t^T R_t K_t + (A_t + B_t K_t)^T V_{t+1} (A_t + B_t K_t)$$

Why do we care about time-varying?

Ans: Linearization about a trajectory

Trajectory tracking for stationary rolls?

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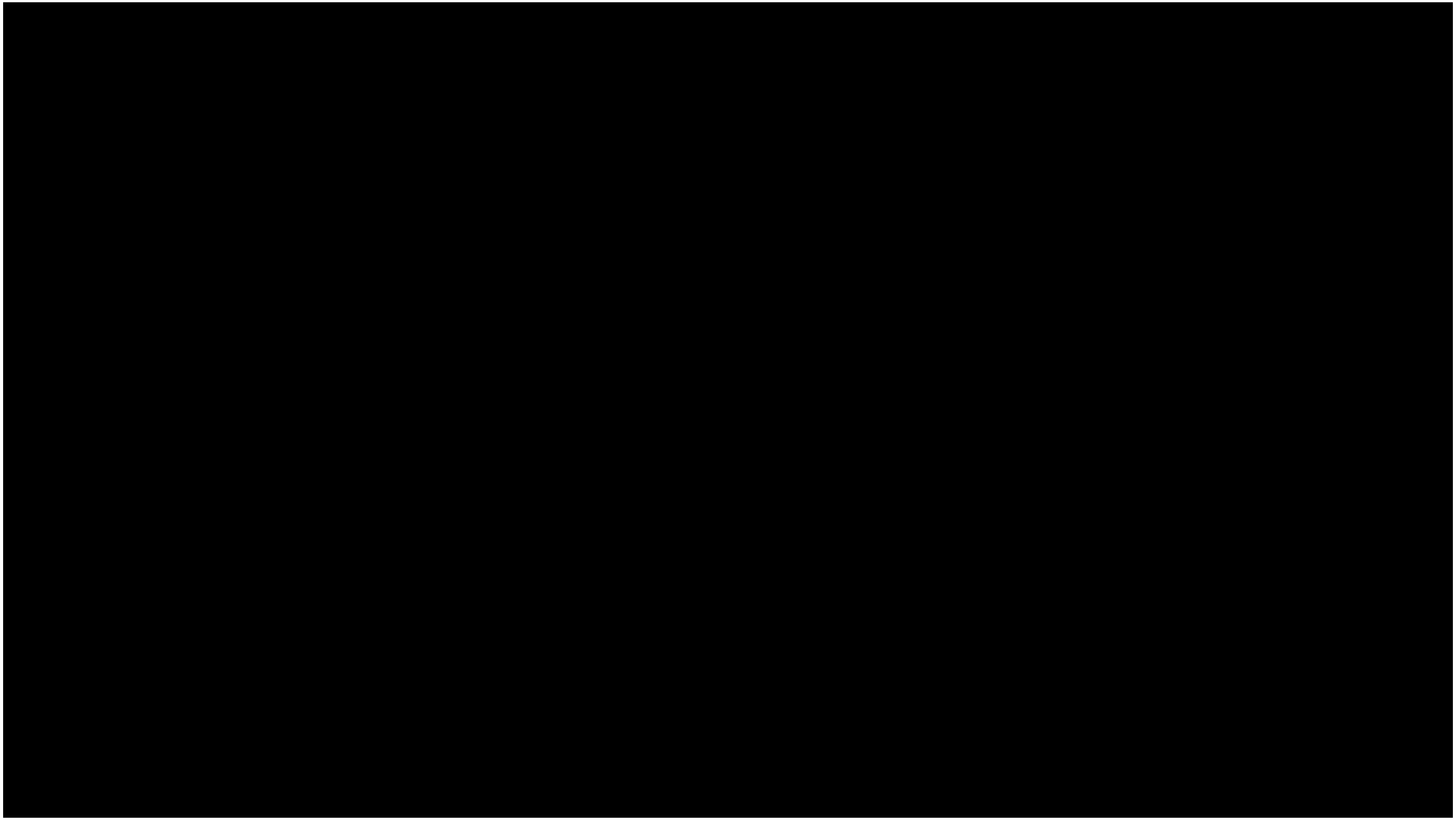
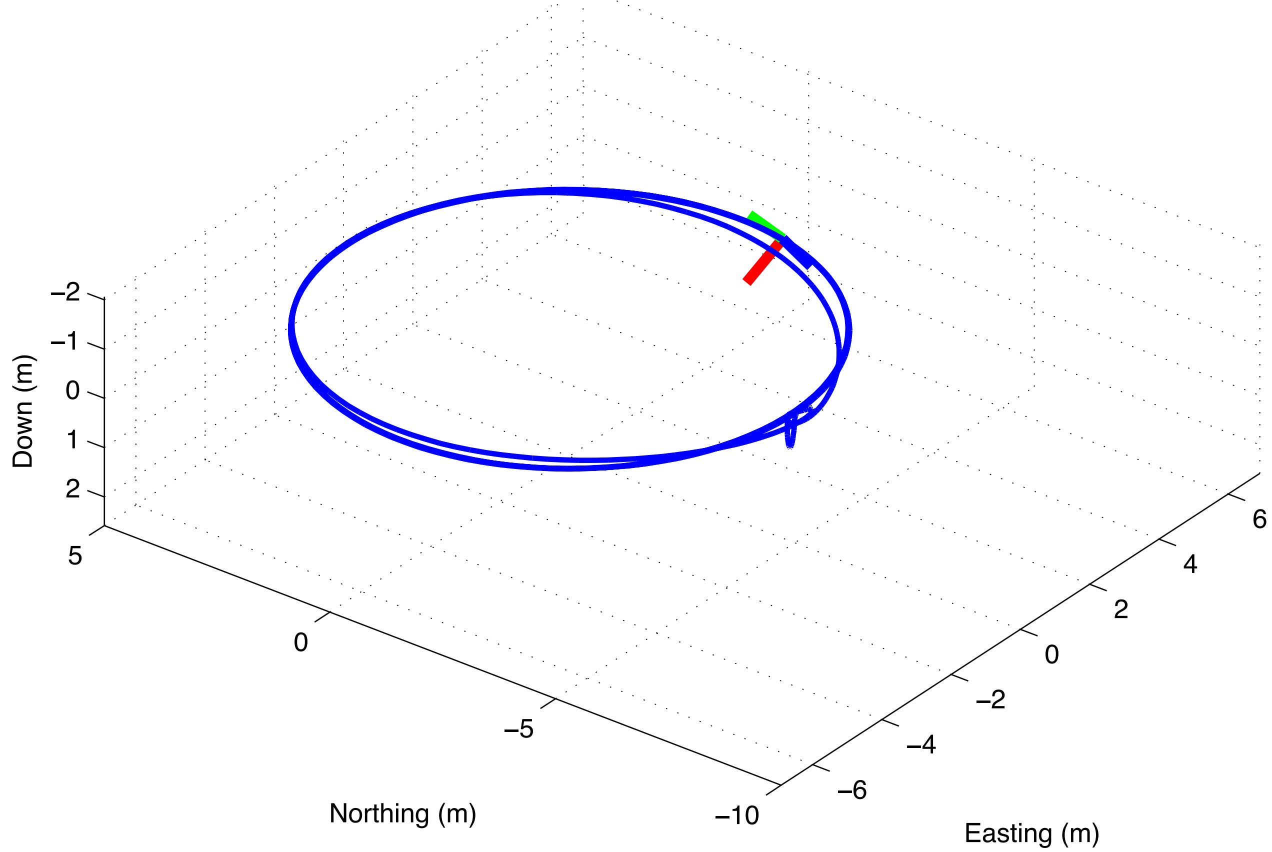
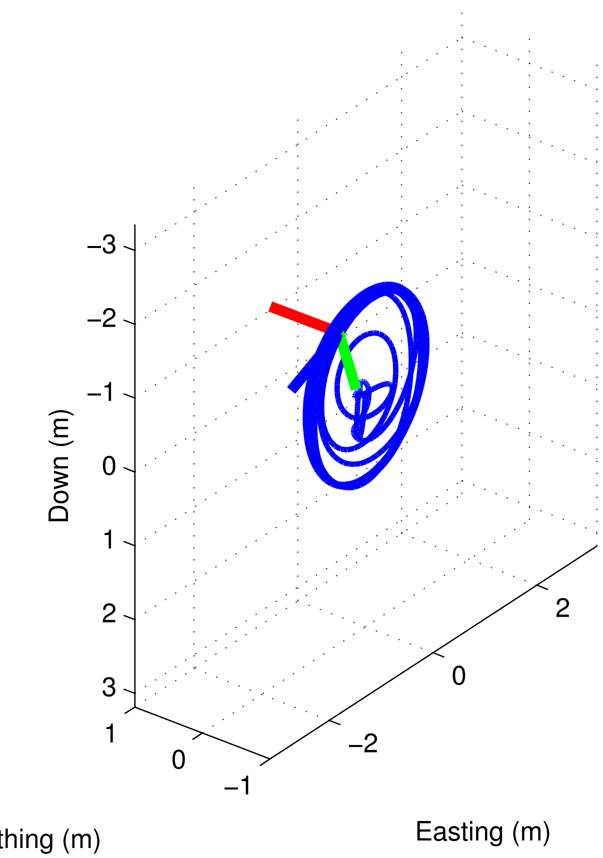


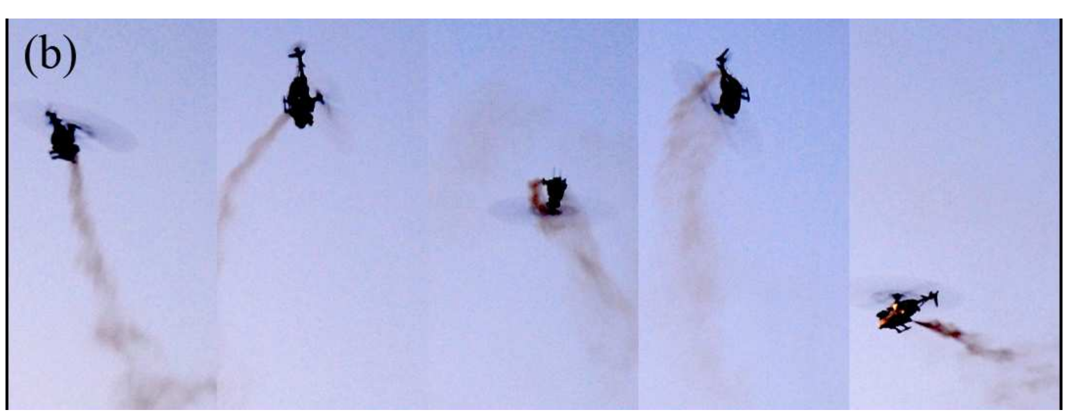
How do we get such behaviors?

Nose-in funnel



Stationary rolls





Task: Minimize tracking error

$$\min_{u_0, u_1, \ldots, u_{T-1}} \sum_{t=0}^{T-1} c(x_t, u_t)$$

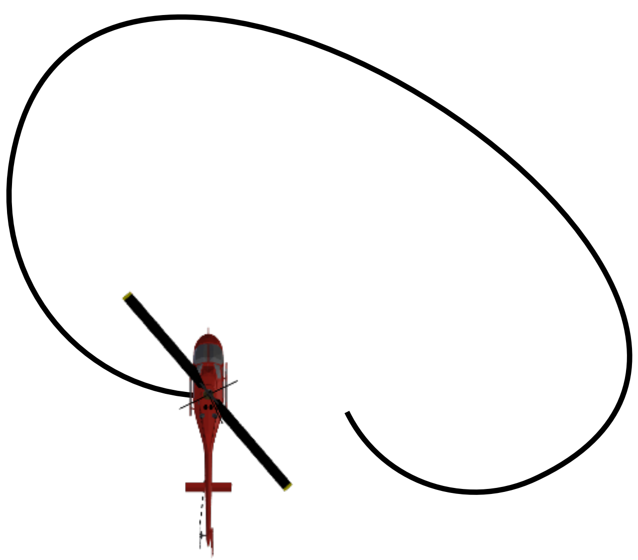
$$\text{subject to } \; x_{t+1} = f(x_t, u_t) \; \forall t$$

In this scenario, cost is simply a quadratic tracking cost

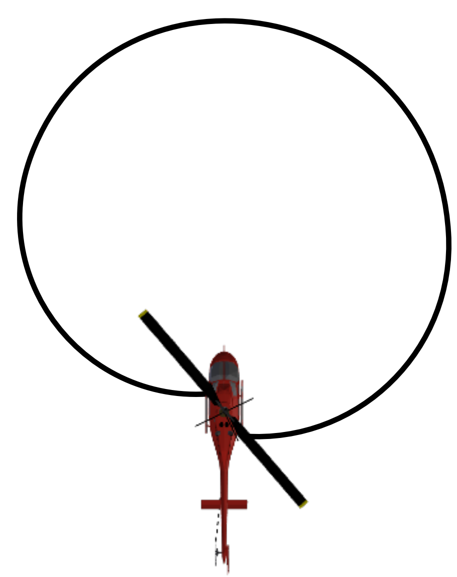
Why is this a hard optimization problem?

Iterative LQR (iLQR)

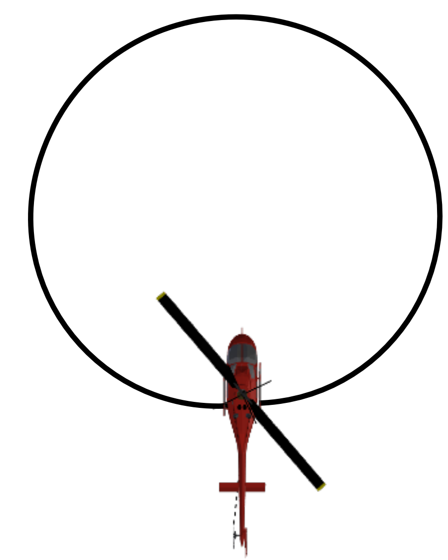
Start by guessing a control sequence, Forward simulate dynamics, Linearize about trajectory, Solve for new control sequence and repeat!



i=0



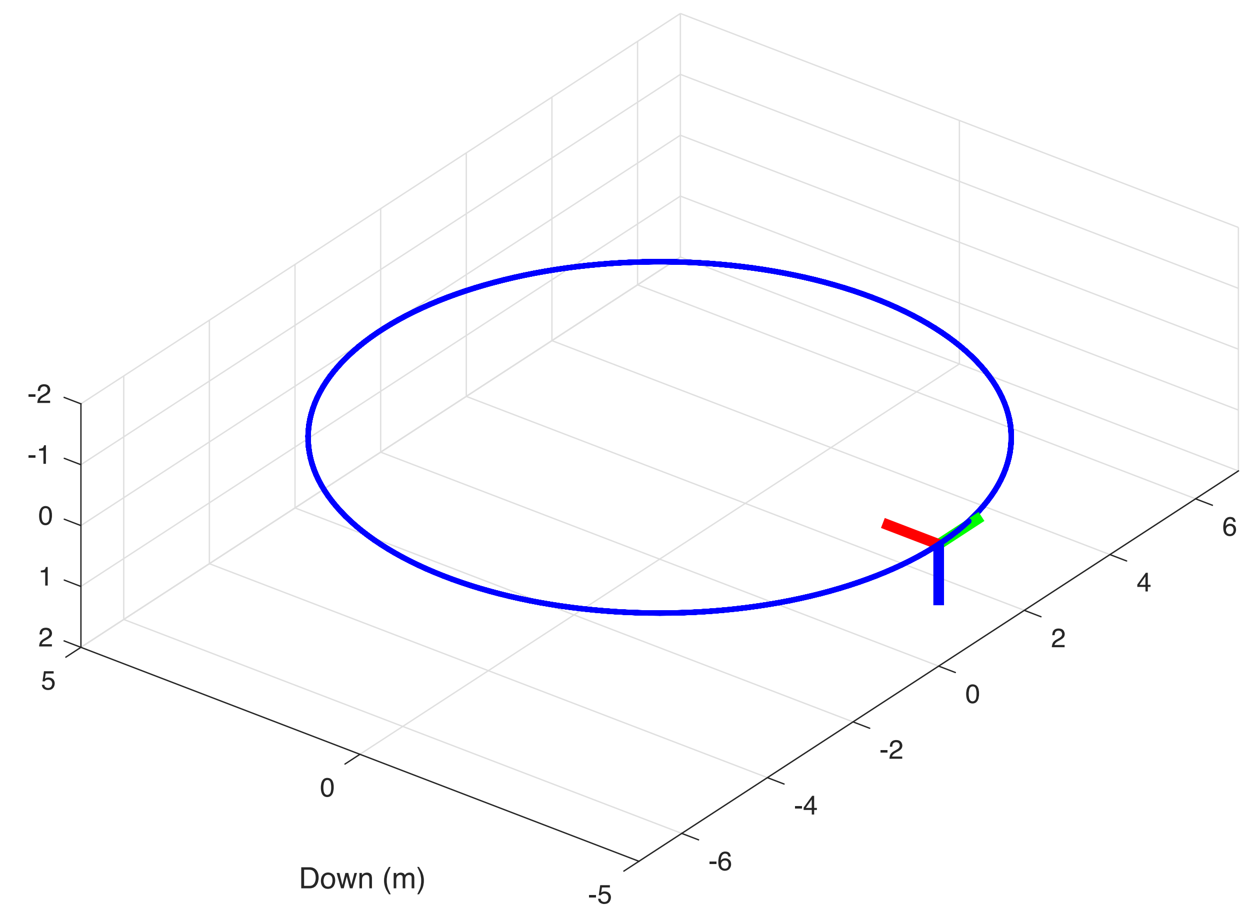
i=10



i=100

Step 1: Get a reference trajectory

$$x_0^{ref}, u_0^{ref}, x_1^{ref}, u_1^{ref}, \ldots, x_{T-1}^{ref}, u_{T-1}^{ref}$$



Note: Simply executing open loop trajectory wont work!

Step 2: Initialize your algorithm

Choose initial trajectory at iteration 0 to linearize about

$$x^0(t), u^0(t) = \{x_0^0, u_0^0, x_1^0, u_1^0, \ldots, x_{T-1}^0, u_{T-1}^0\}$$

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It's a good idea to choose the reference trajectory

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It's a good idea to choose the reference trajectory

Initialization is very important!

We will be perturbing this initial trajectory

Step 3: Linearize your dynamics!

At a given iteration i, we are going to linearize about

$$x_0^i, u_0^i, x_1^i, \dots$$

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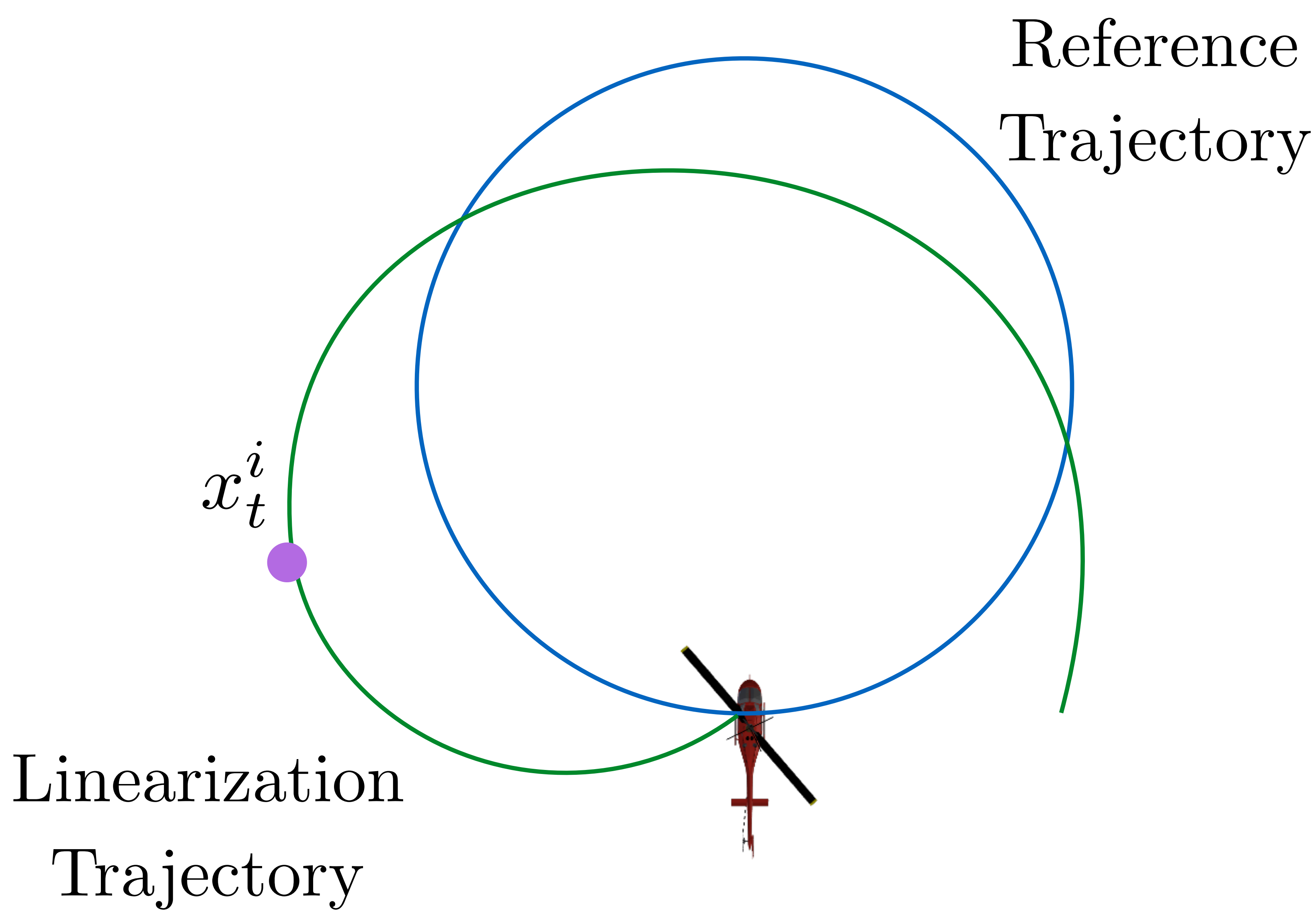
$$x_0^i, u_0^i, x_1^i, \ldots$$

Change of variable - we will track the delta perturbations

$$\delta x_t = x_t - x_t^i$$

$$\delta u_t = u_t - u_t^i$$

Step 3: Linearize your dynamics!



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$$\delta x_{t+1} = A_t \delta x_t + B_t \delta u_t + \left(f(x_t^i, u_t^i) - x_{t+1}^i\right)$$

$$A_t = \left.\frac{\partial f}{\partial x}\right|_{x_t^i} \qquad B_t = \left.\frac{\partial f}{\partial u}\right|_{u_t^i}$$

Step 3: Linearize your dynamics!

$$\delta x_t = x_t - x_t^i \qquad \delta u_t = u_t - u_t^i$$

This is an affine system, not linear

$$\delta x_{t+1} = A_t \delta x_t + B_t \delta u_t + \left(f(x_t^i, u_t^i) - x_{t+1}^i\right)$$

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Step 3: Linearize your dynamics!

Homogenous coordinate system $\begin{bmatrix} \delta x_t \\ 1 \end{bmatrix}$

Affine dynamics is now linear!

$$\begin{bmatrix} \delta x_{t+1} \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} A_{t+1} & f(x_t^i, u_t^i) - x_{t+1}^i \\ 0 & 1 \end{bmatrix}}_{\tilde{A}_t} \begin{bmatrix} \delta x_t \\ 1 \end{bmatrix} + \underbrace{\begin{bmatrix} B_{t+1} \\ 0 \end{bmatrix}}_{\tilde{B}_t} \delta u_t$$

Step 4: Quadricize cost about trajectory

Our cost function is already quadratic, otherwise we would apply Taylor expansion

$$c(x_t, u_t) = (x_t - x_t^{ref})^T Q (x_t - x_t^{ref}) + (u_t - u_t^{ref})^T R (u_t - u_t^{ref})$$

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$$= \begin{bmatrix} \delta x_t \\ 1 \end{bmatrix}^T \underbrace{\begin{bmatrix} Q & Q(x_t^i - x_t^{ref}) \\ (x_t^i - x_t^{ref})^T Q & (x_t^i - x_t^{ref})^T (x_t^i - x_t^{ref}) \end{bmatrix}}_{\tilde{Q}_t} \begin{bmatrix} \delta x_t \\ 1 \end{bmatrix}$$

$$+$$

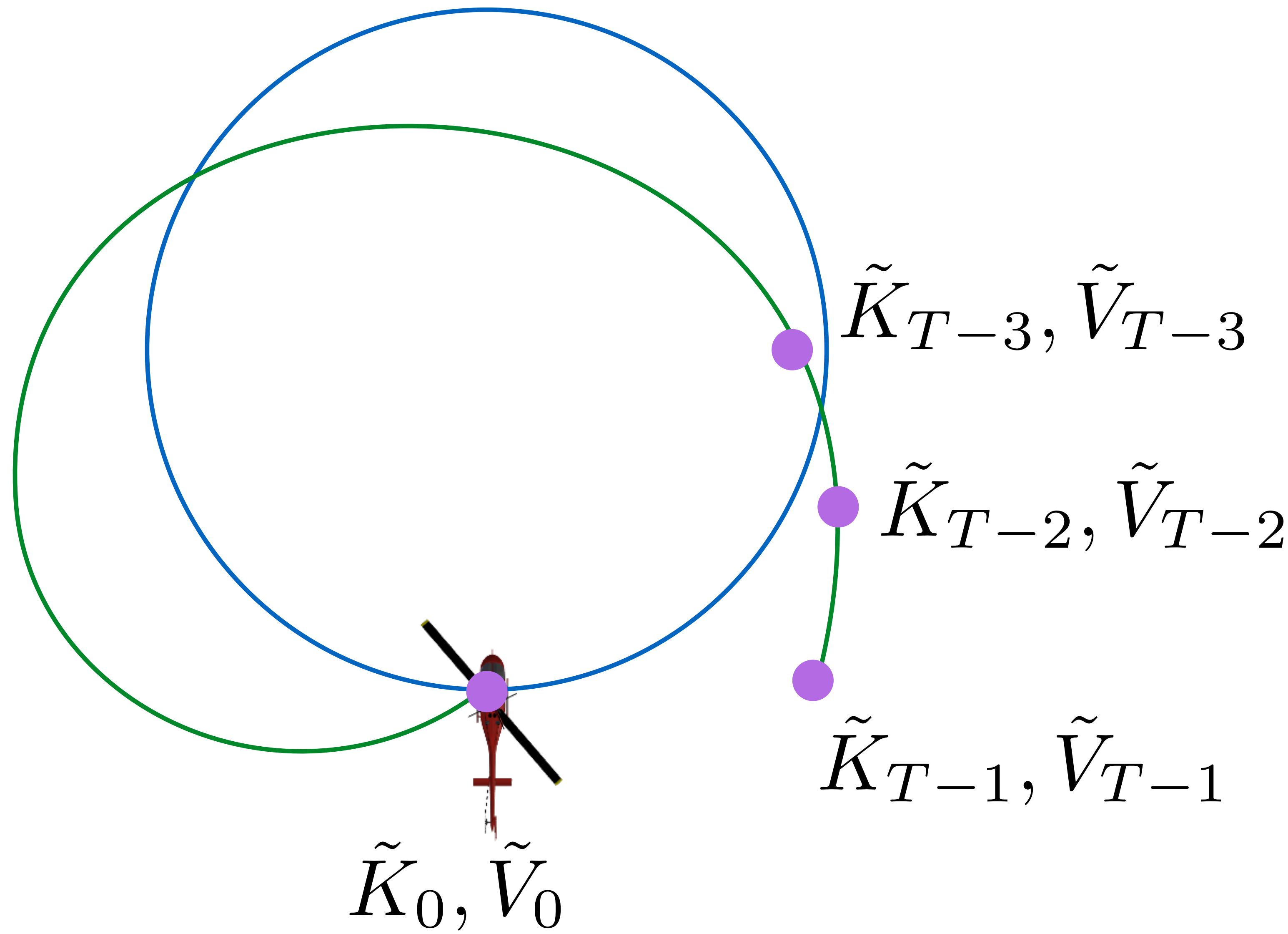
$$\begin{bmatrix} \delta u_t \\ 1 \end{bmatrix}^T \underbrace{\begin{bmatrix} R & R(u_t^i - u_t^{ref}) \\ (u_t^i - u_t^{ref})^T R & (u_t^i - u_t^{ref})^T (u_t^i - u_t^{ref}) \end{bmatrix}}_{\tilde{R}_t} \begin{bmatrix} \delta u_t \\ 1 \end{bmatrix}$$

We have all the ingredients to call LQR!

$$\tilde{K}_t = -(\tilde{R}_t + \tilde{B}_t^T \tilde{V}_{t+1} \tilde{B}_t)^{-1} \tilde{B}_t^T \tilde{V}_{t+1} \tilde{A}_t$$

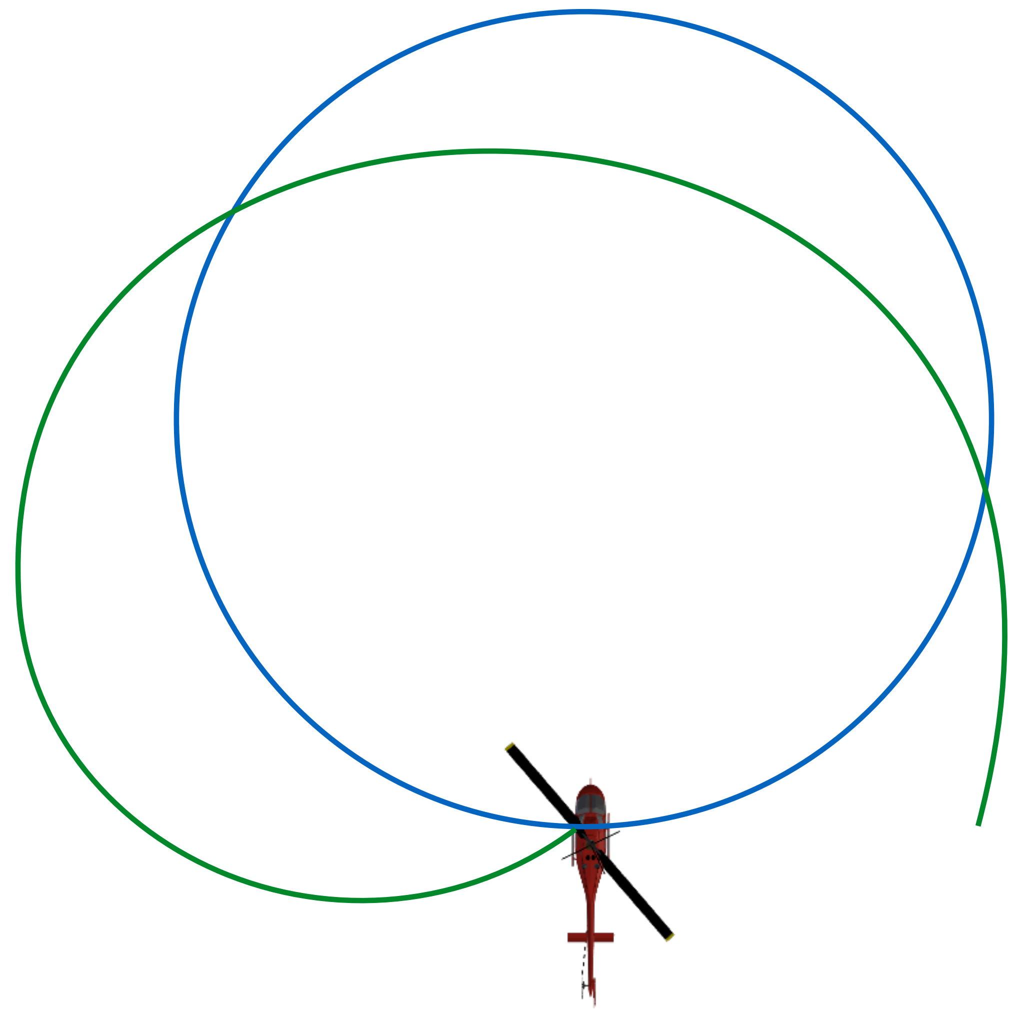
similarly calculate the value function ...

Step 5: Do a backward pass



Calculate controller gains for all time steps

Step 6: Do a forward pass to get new trajectory



Compute
control action

$$u_t^{i+1} = u_t^i + \tilde{K}_t \begin{bmatrix} x_t^{i+1} - x_t^i \\ 1 \end{bmatrix}$$

Apply dynamics

$$x_t^{i+1} = f(x_t^{i+1}, u_t^{i+1})$$

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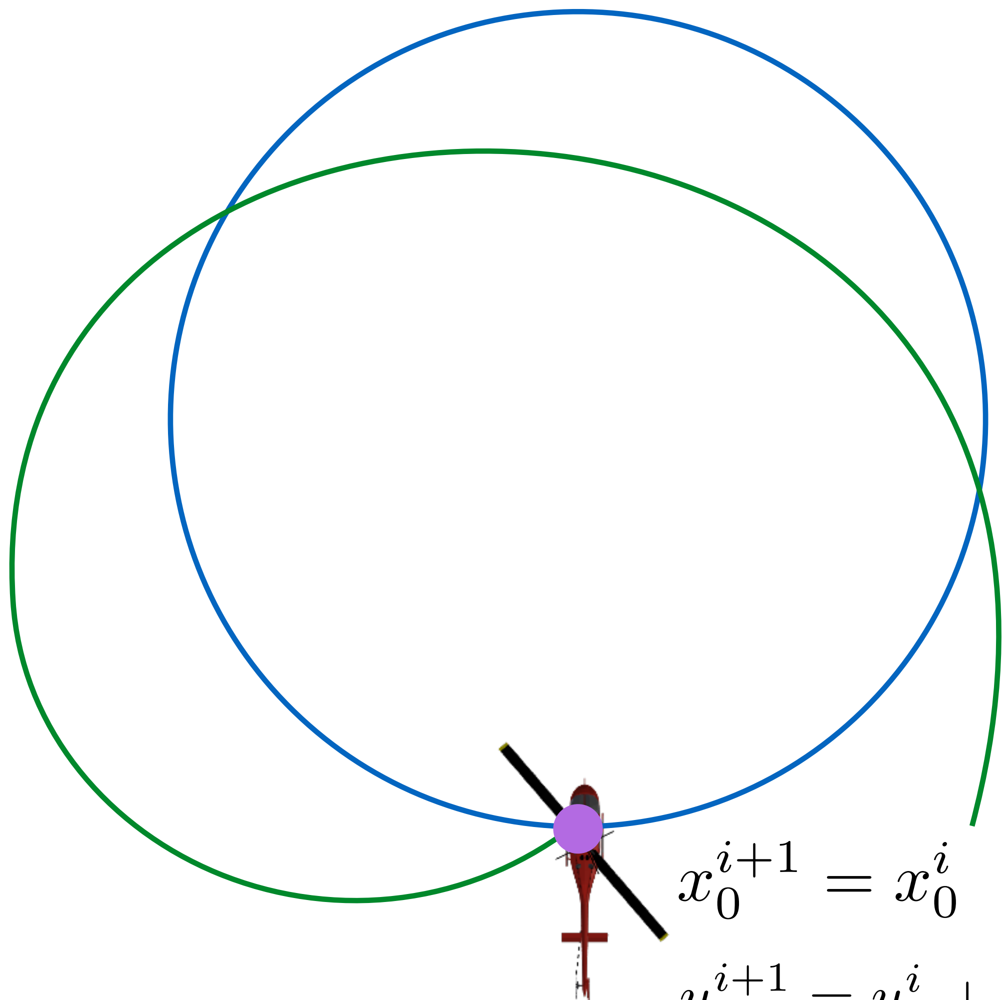

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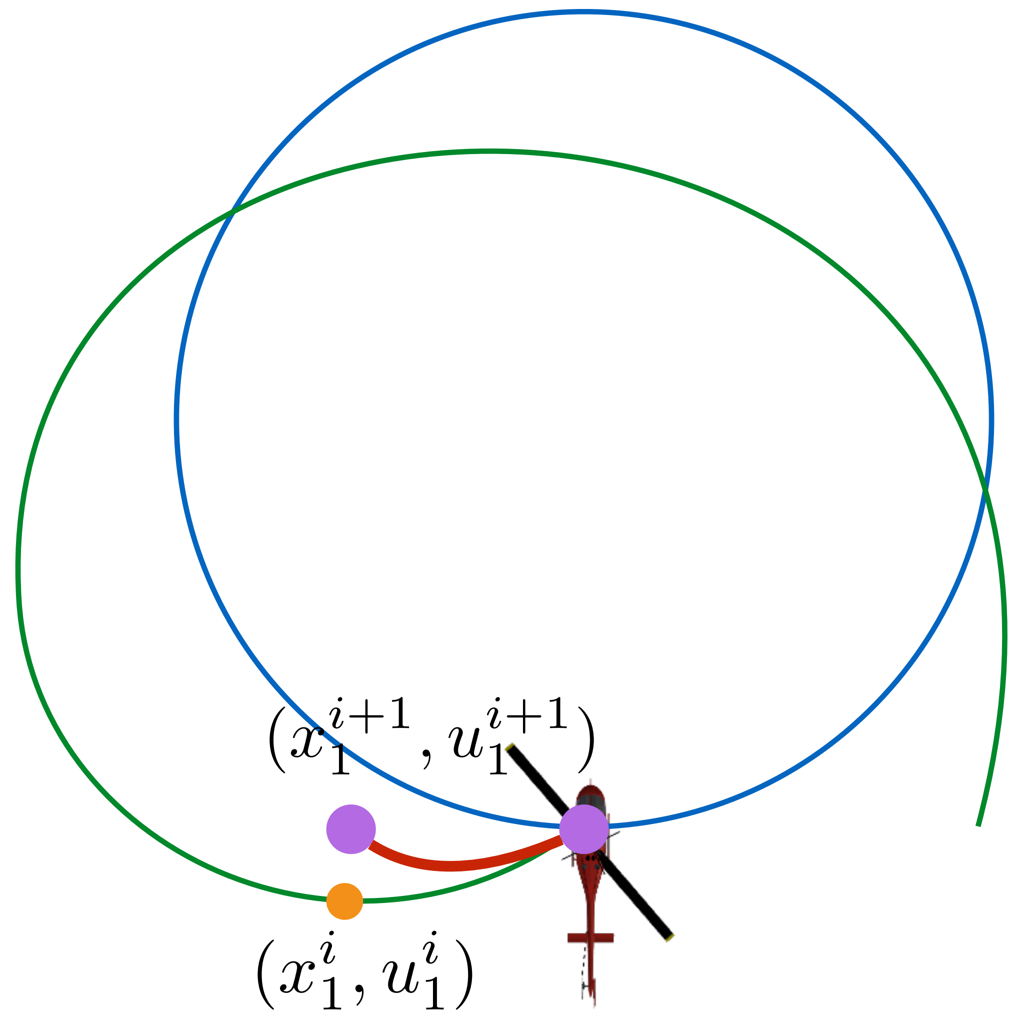

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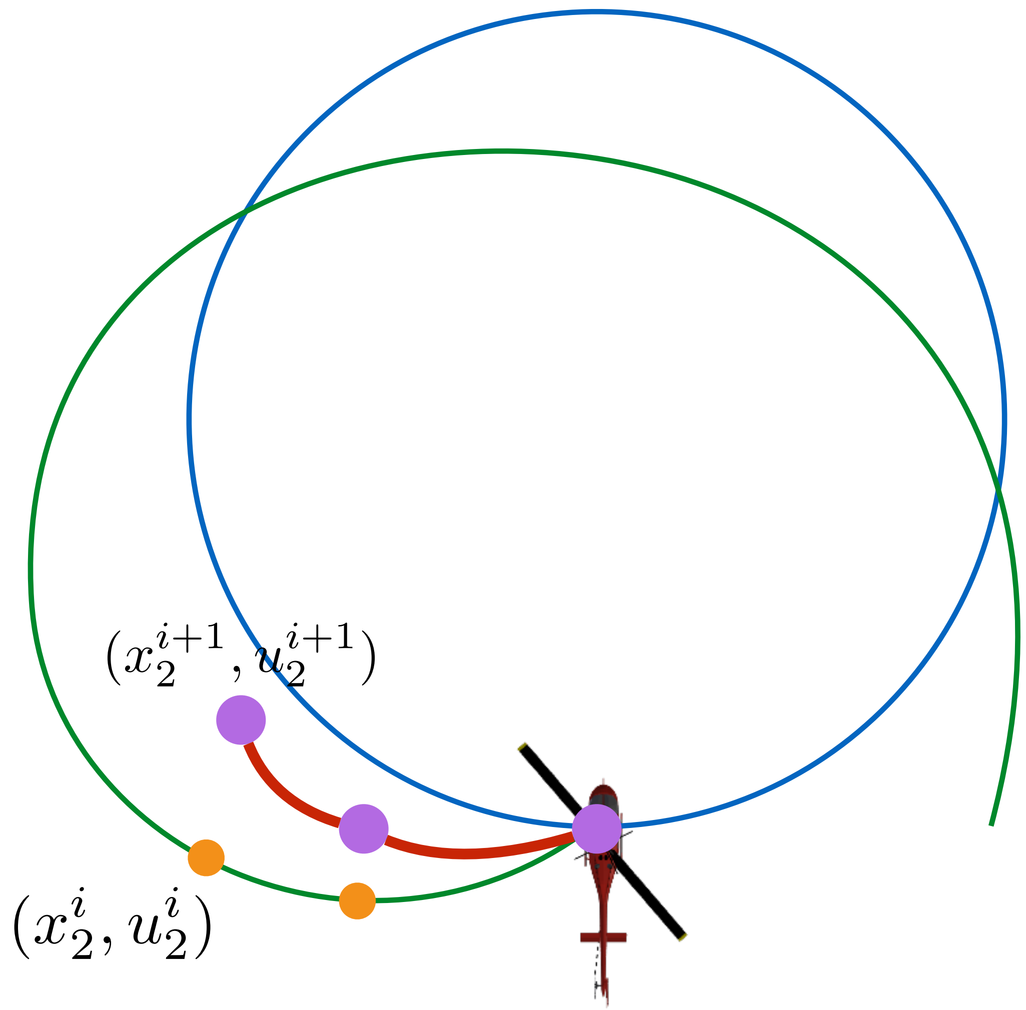

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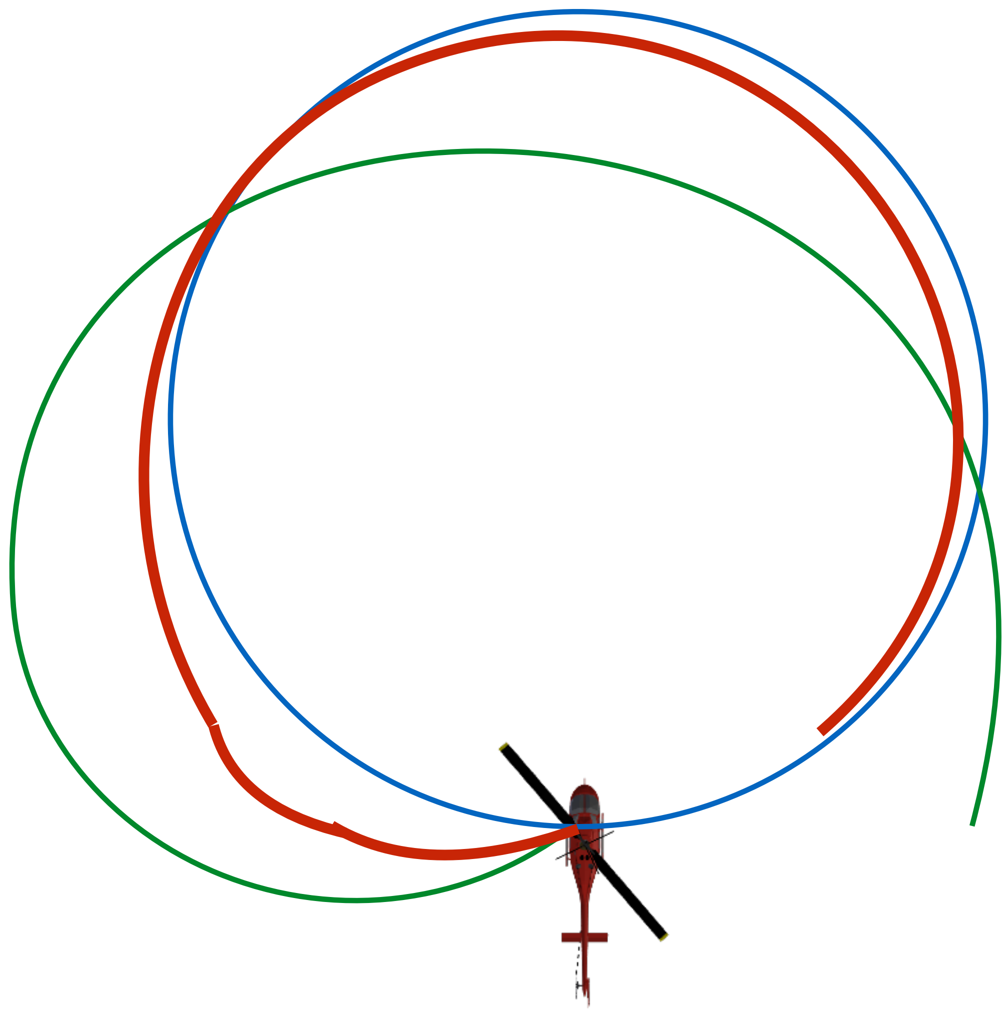

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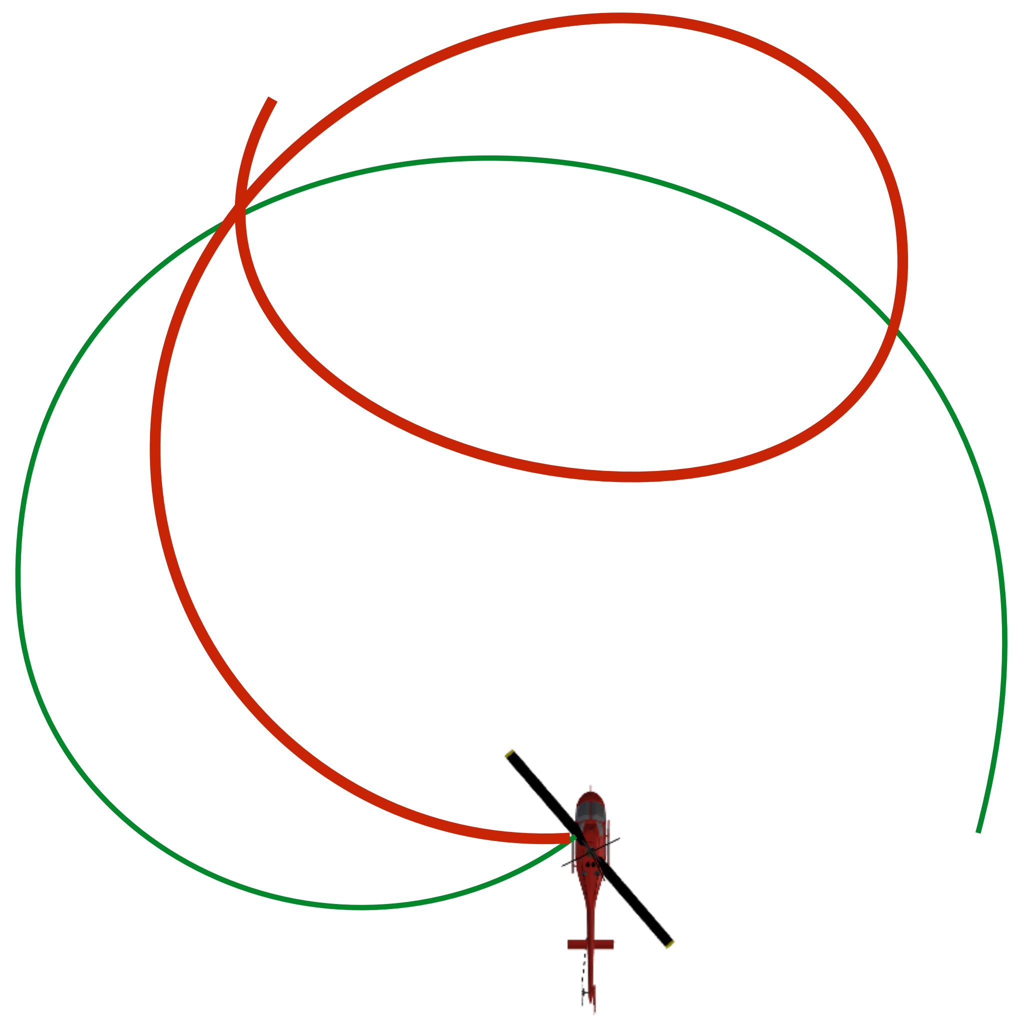

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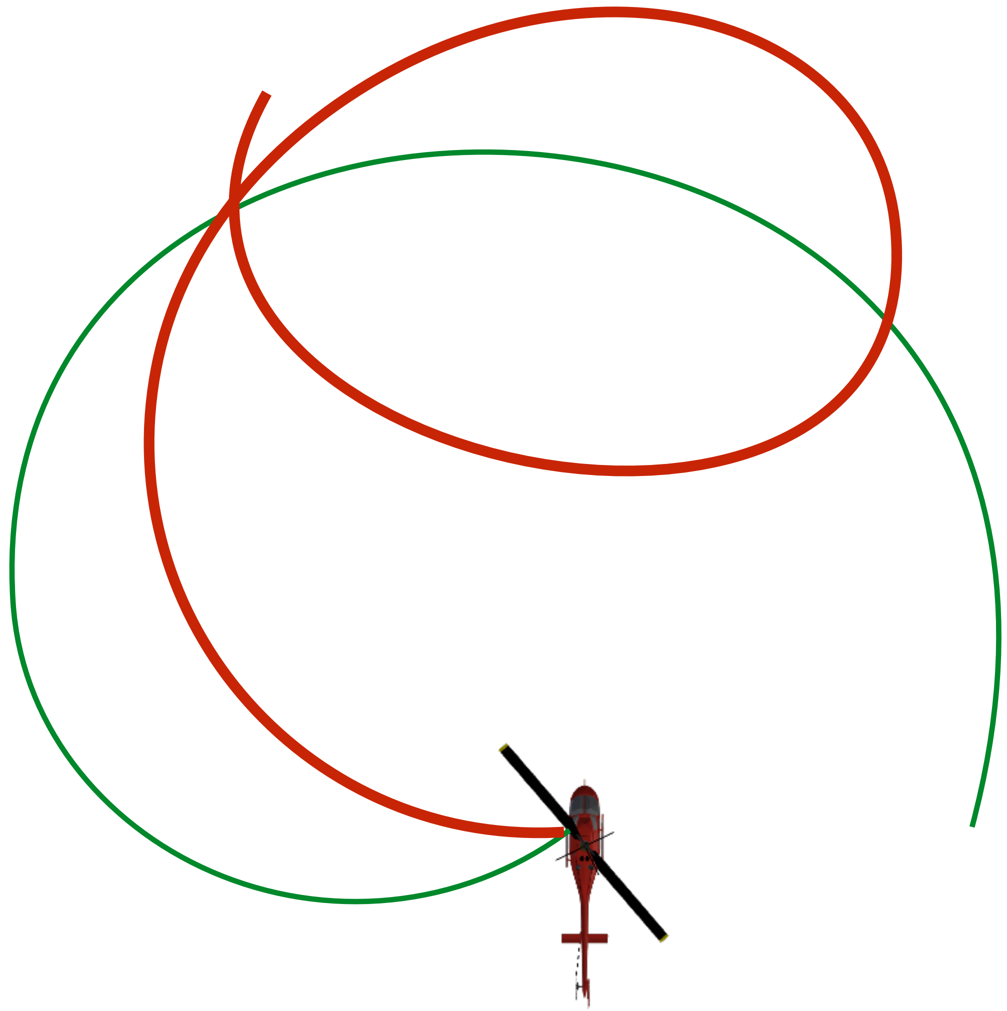

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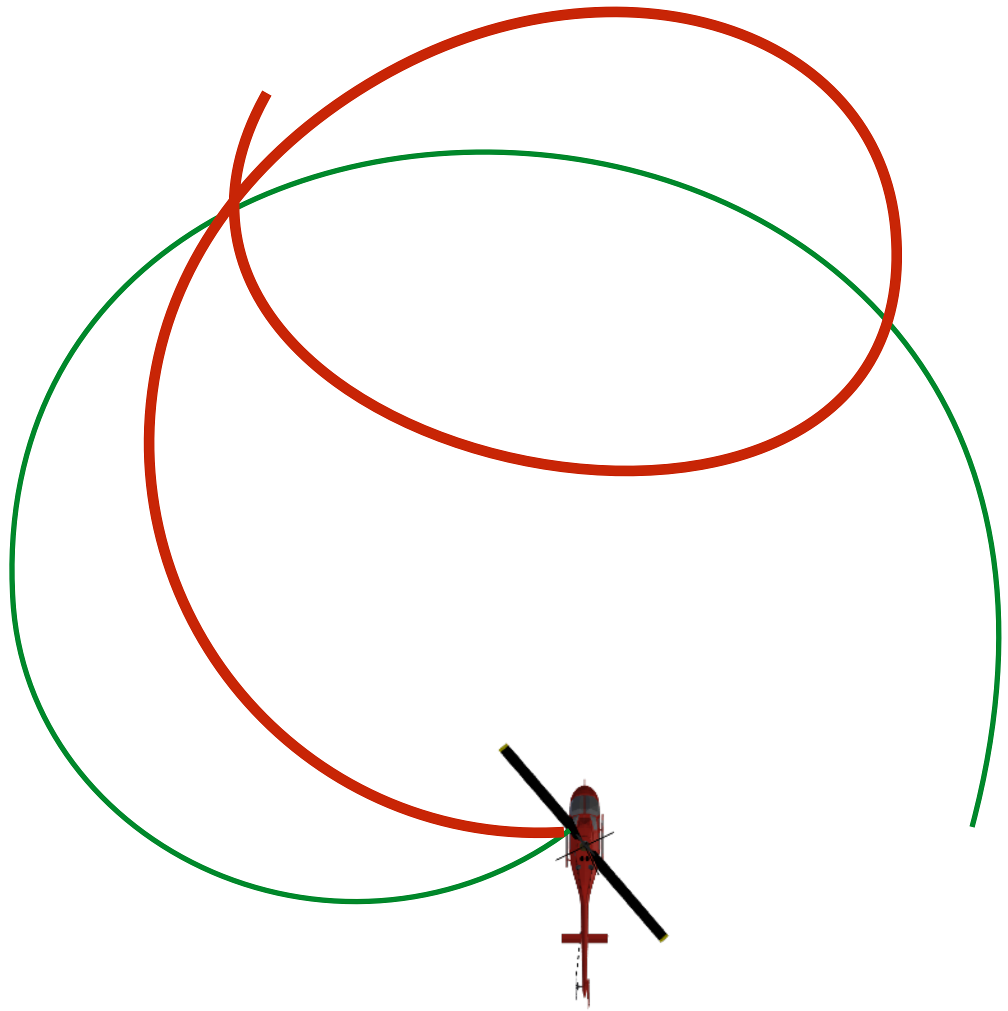

Linearization error gets bigger and bigger and bigger

Remedies: Change cost function to penalize deviation from linearization

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4. Can we handle noisy dynamics?

    Yes! Gaussian noise does not change the answer

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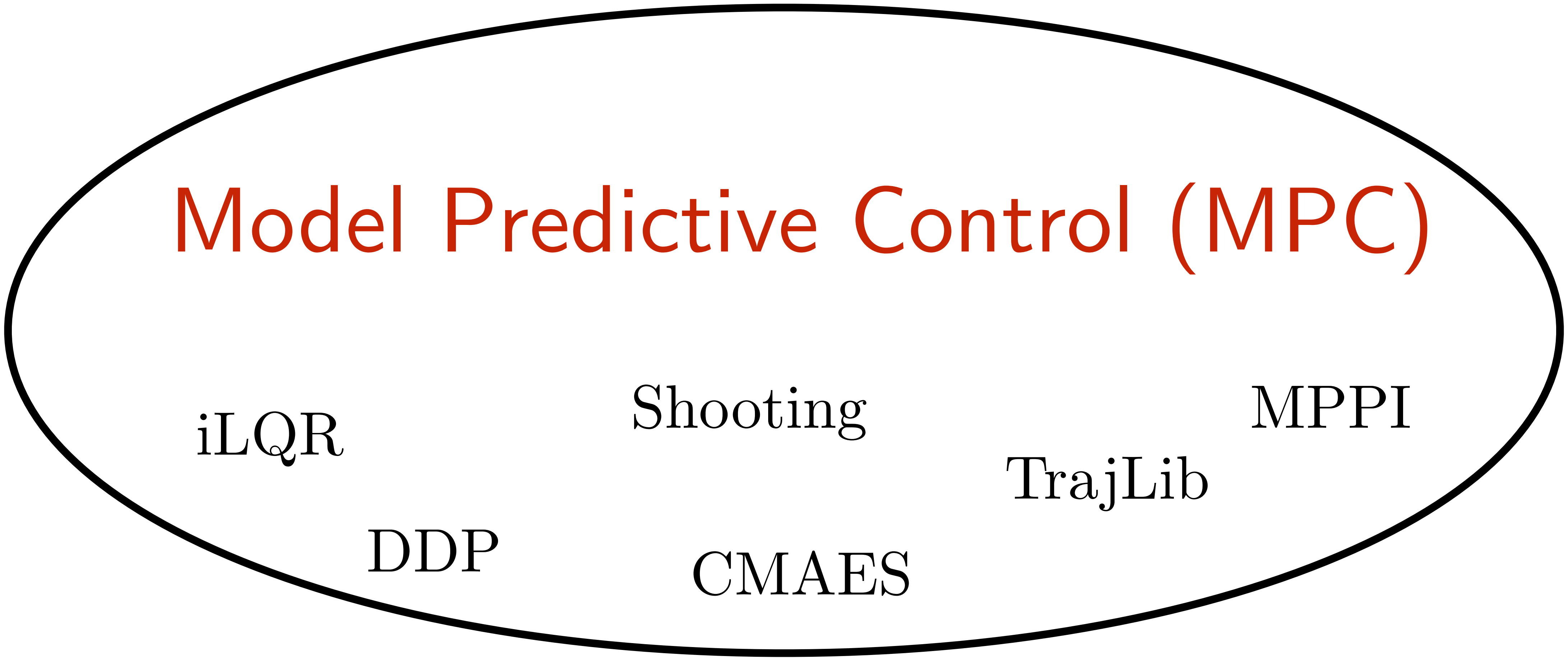
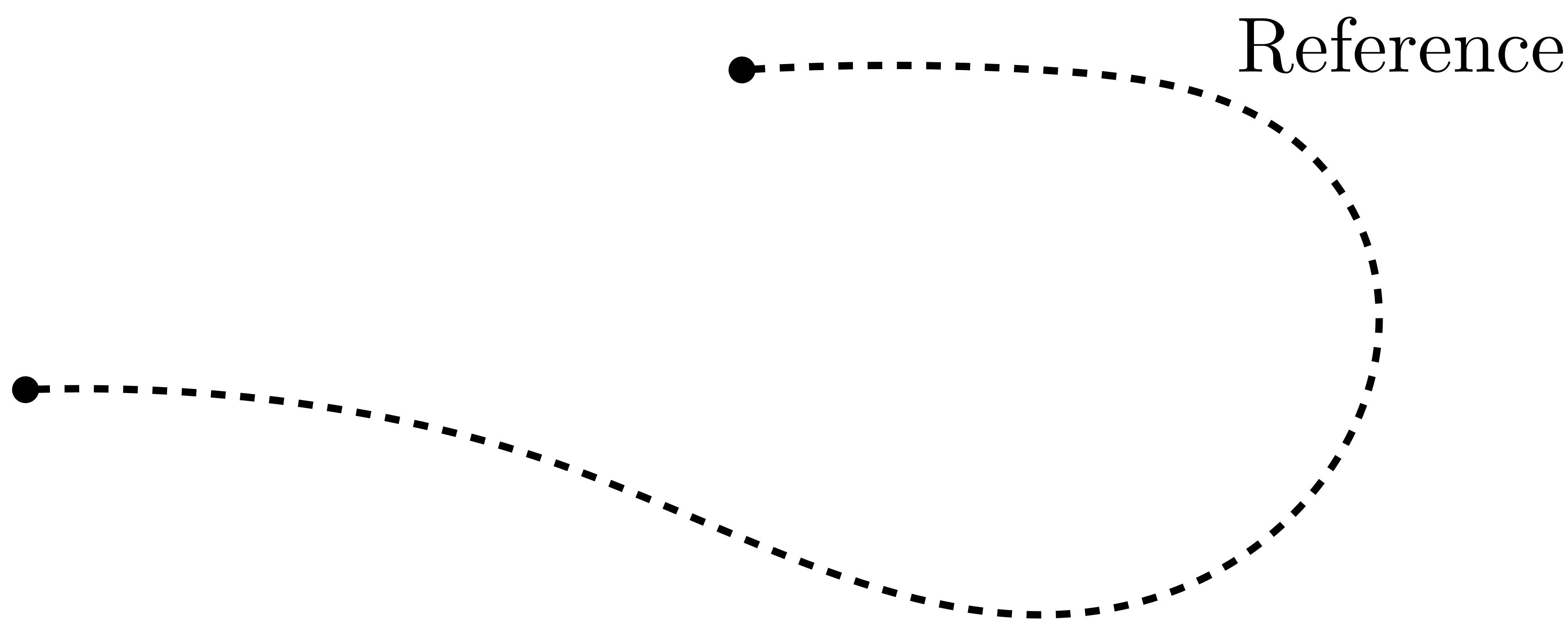
Lyapunov		Non-linear	Yes	No
LQR		Linear	Yes	Quadratic
iLQR		Non-linear	Yes	Yes

iLQR is just one technique

It's far from perfect - can't deal with model errors / constraints ...

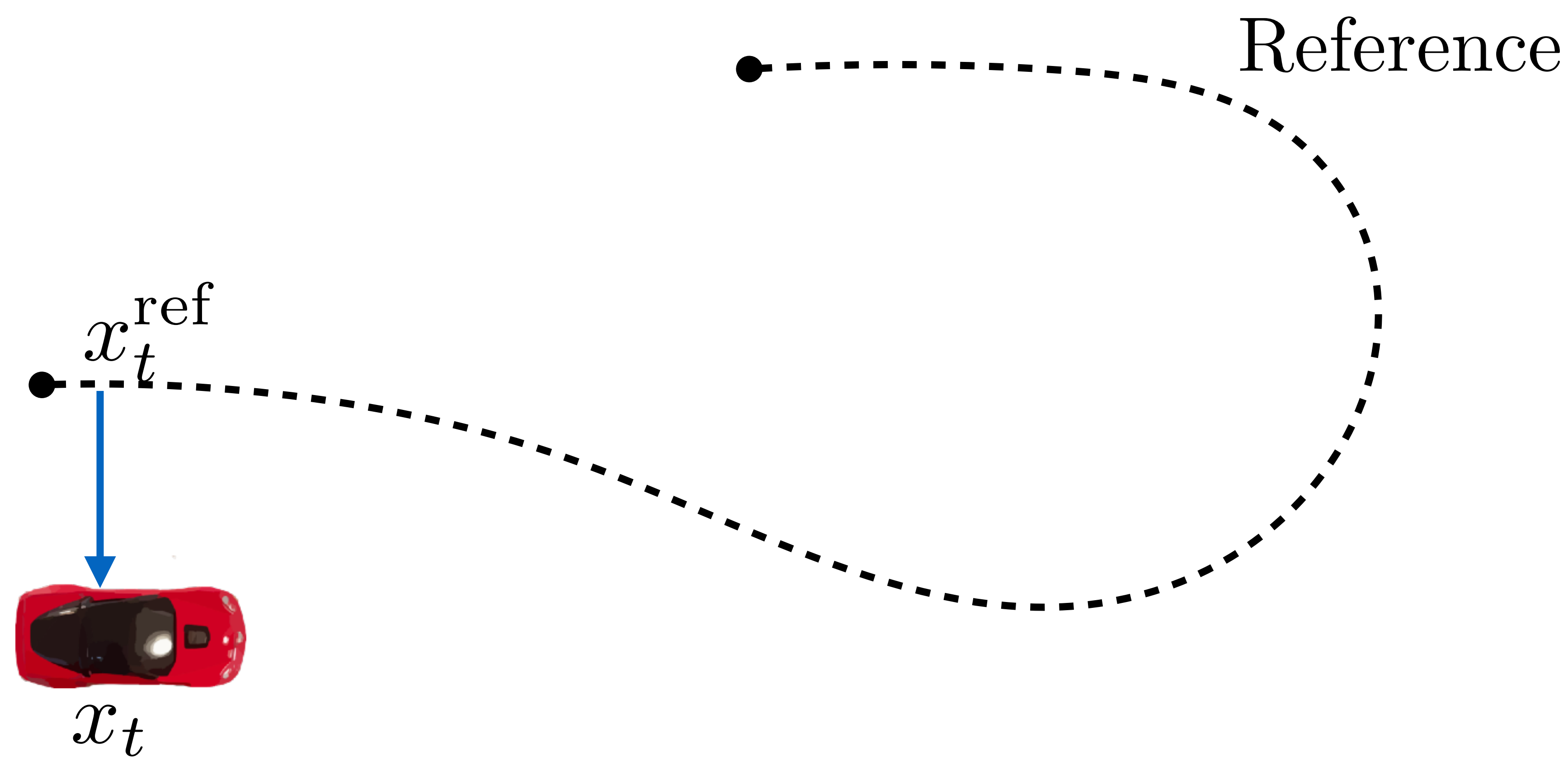


Recap: Feedback control framework



Look at current state error and compute control actions

Recap: Feedback control framework



$$u_t = \pi(x_t, x_t^{\text{ref}})$$

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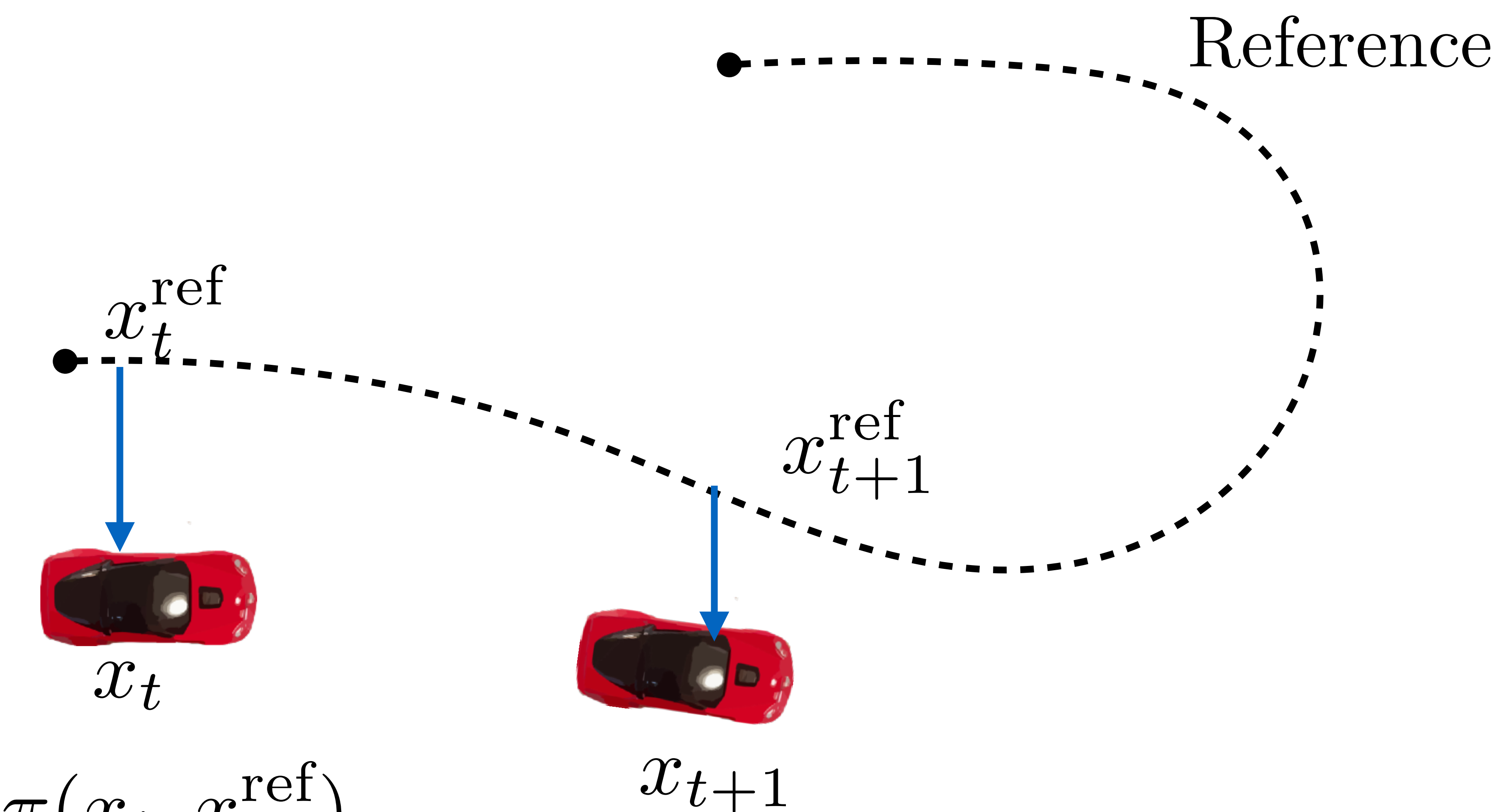

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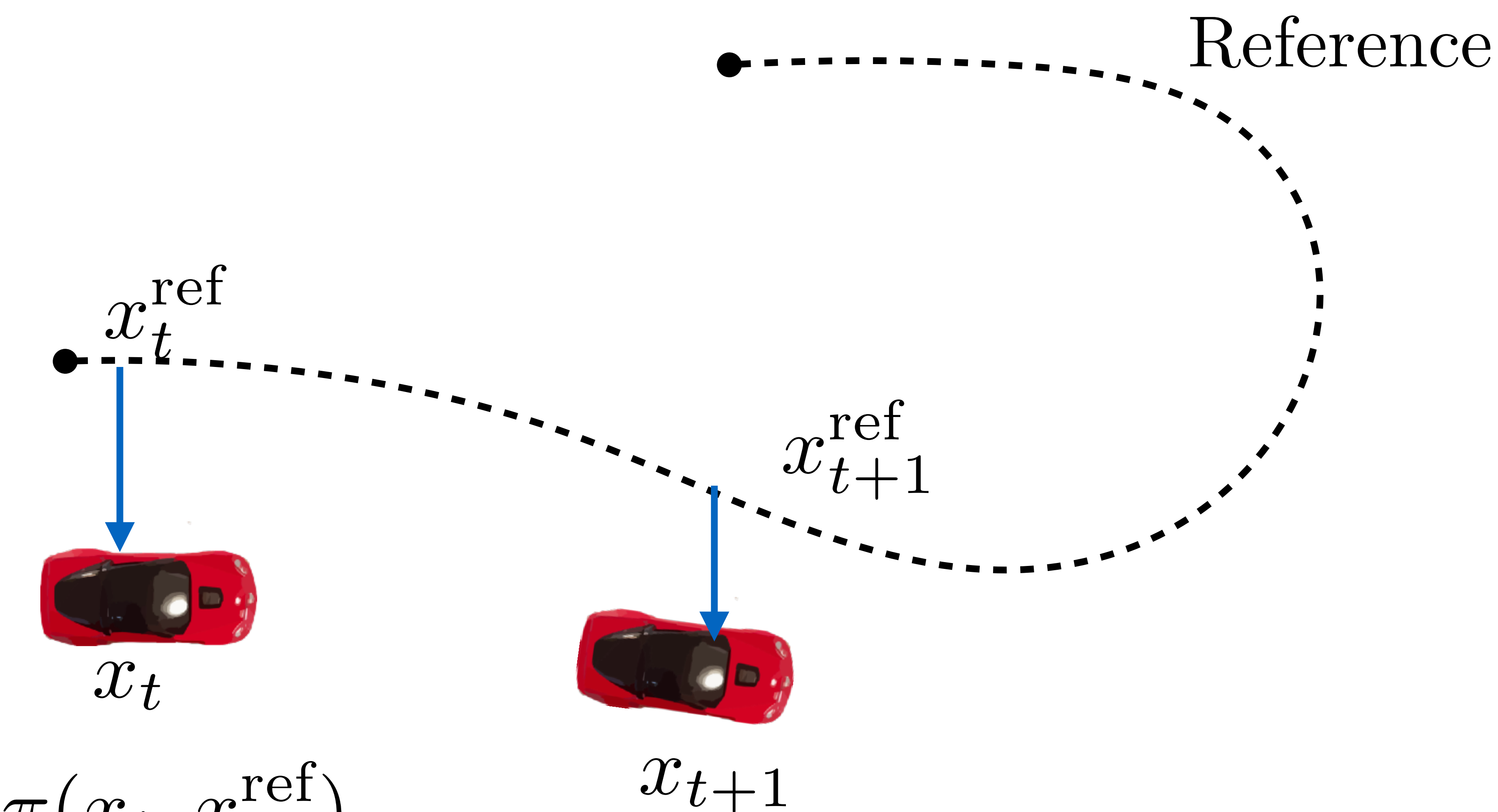
Recap: Feedback control framework



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Look at current state error and compute control actions

Goal: To drive error to 0 … to optimally drive it to 0

Limitations of this framework

A fixed control law that looks at instantaneous feedback

$$u_t = \underbrace{\pi}_{\text{Fixed}}(x_t, \underbrace{x_t^{\text{ref}}}_{\text{Reference}})$$

Why is it so difficult to create a magic control law?

Problem 1: What if we have constraints?

Simple scenario: Car tracking a straight line

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Small error,
control within
steering constraints

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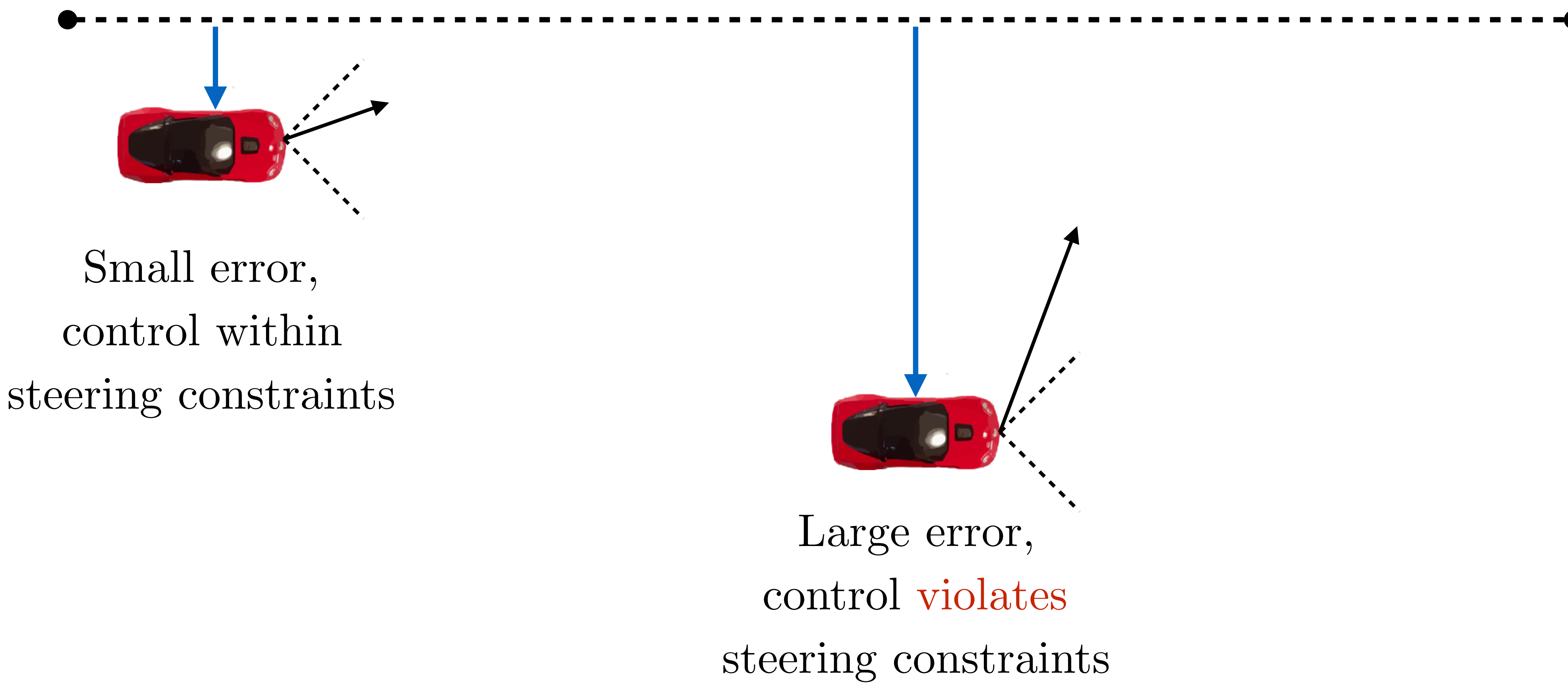

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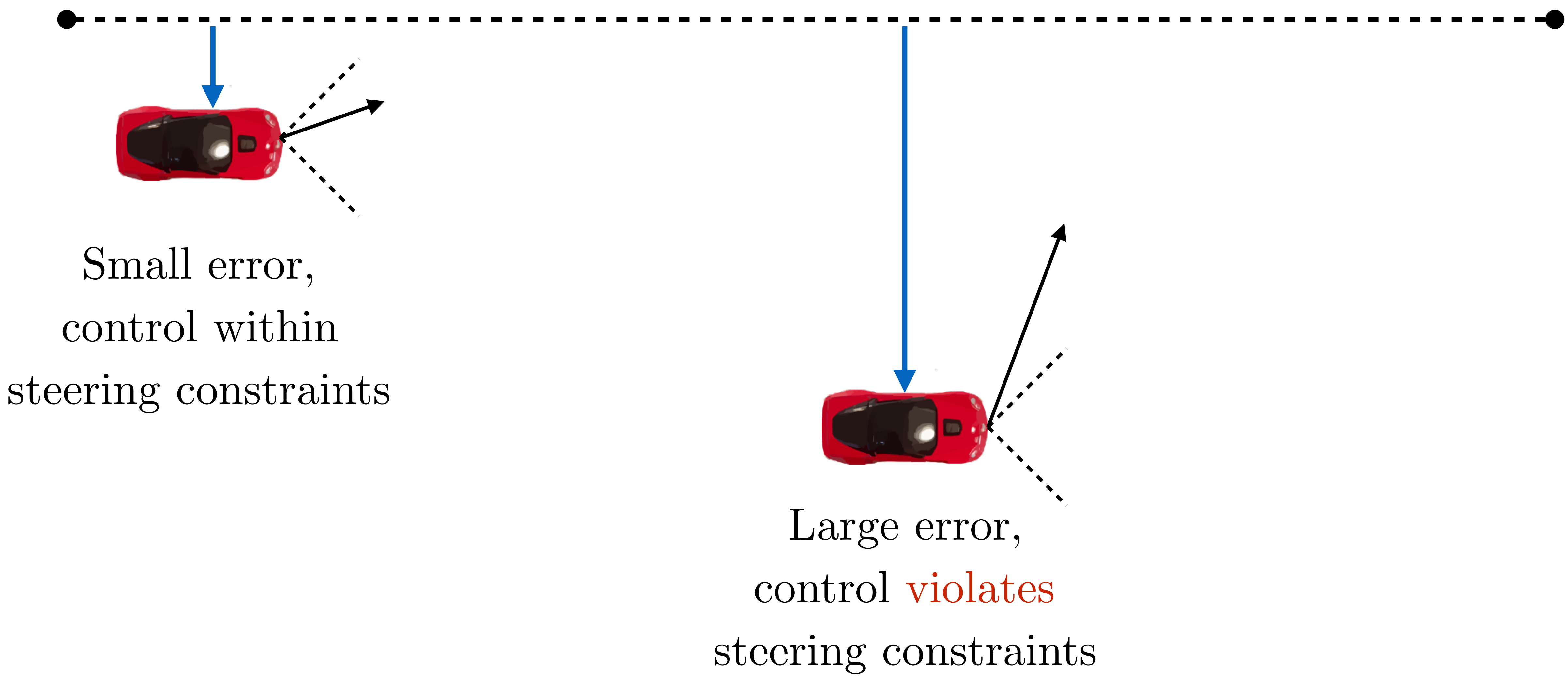


We could "clamp control command" ...
but what are the implications?

General problem: Complex models

Dynamics $$x_{t+1} = f(x_t, u_t)$$

Constraints $$g(x_t, u_t) \leq 0$$

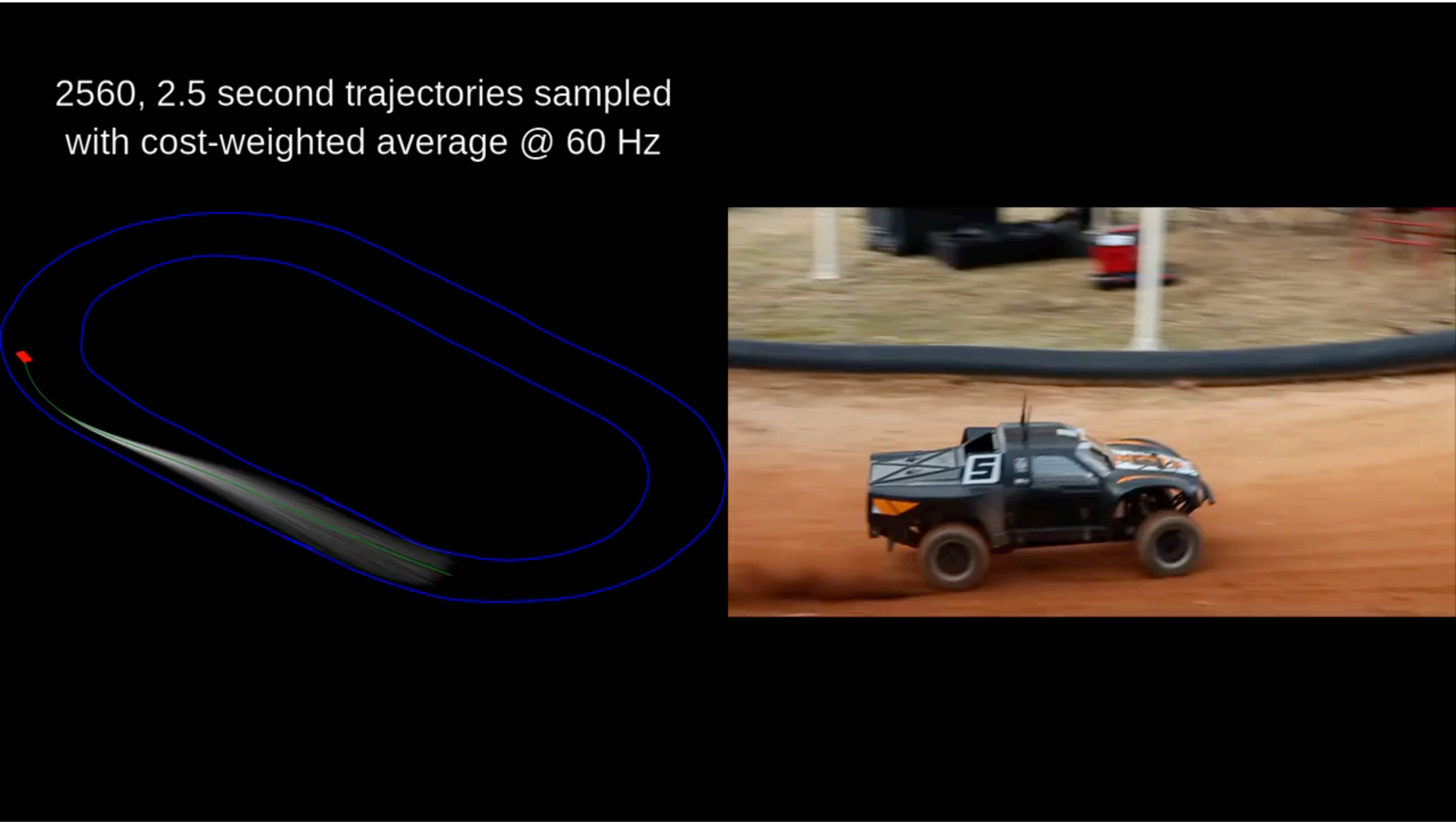
Such complex models imply we need to:

1. Predict the implications of control actions

2. Do corrections NOW that would affect the future

3. It may not be possible to find one law - might need to predict

Example: Rough terrain mobility



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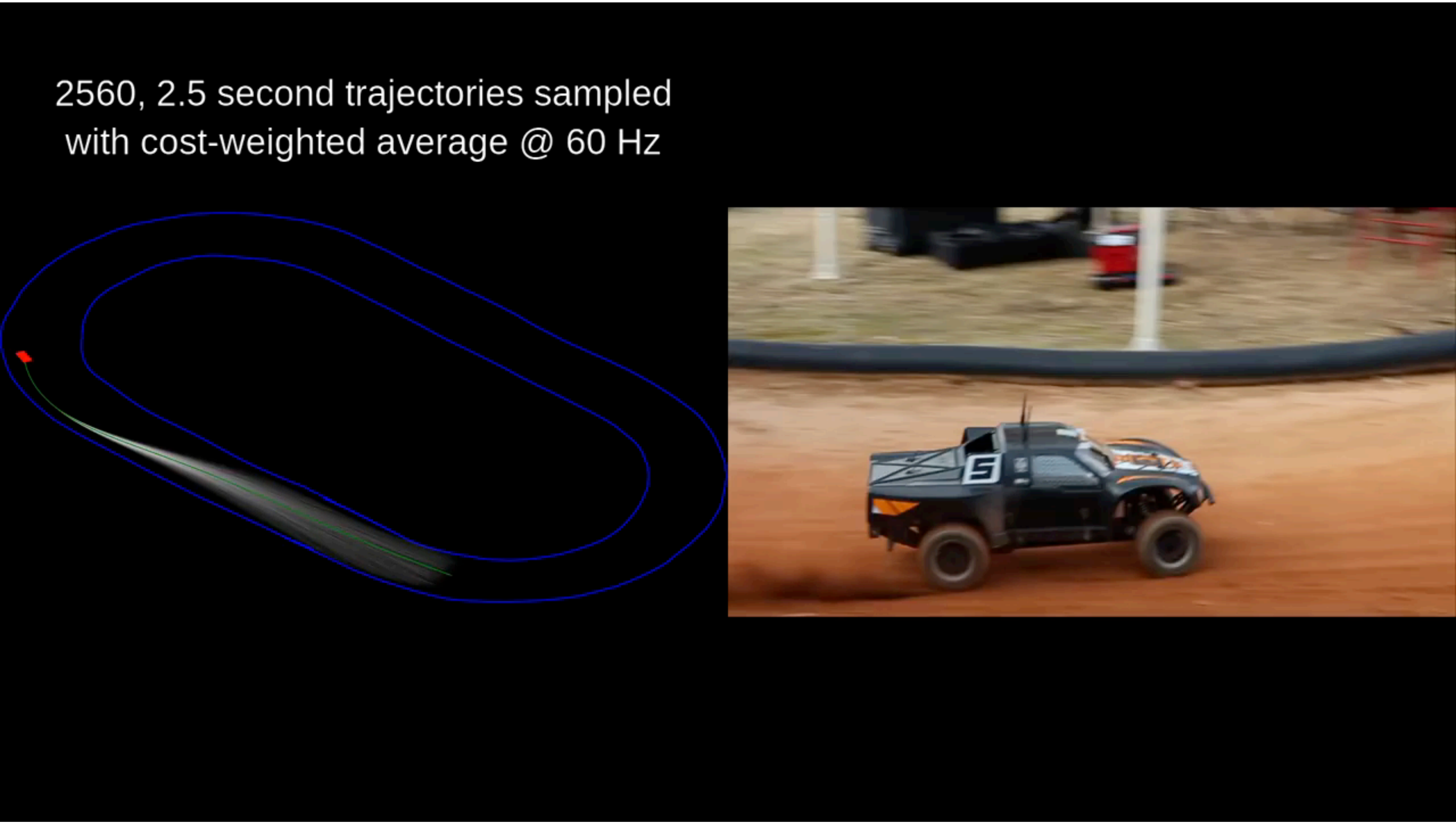
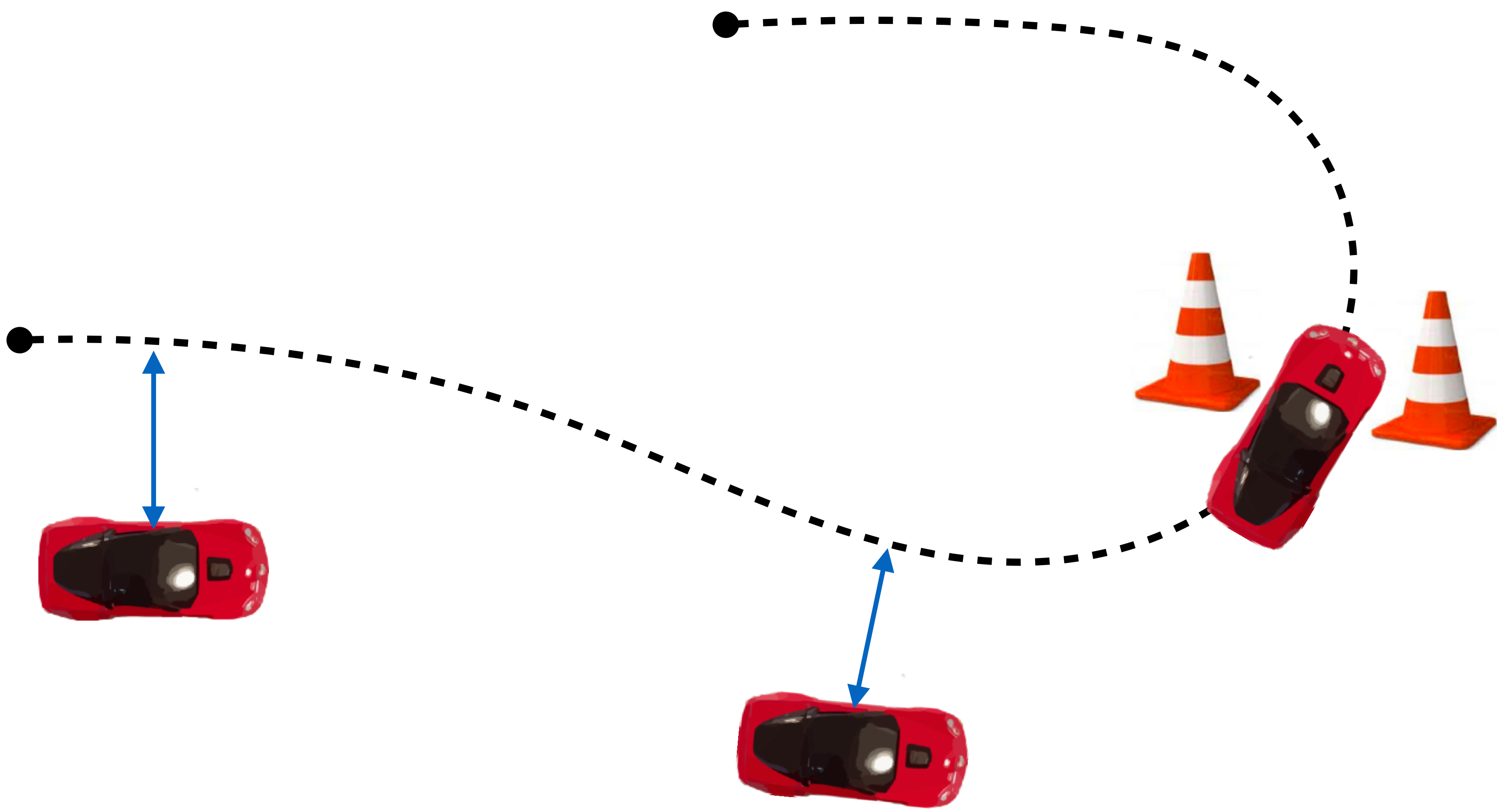


Problem 2: What if some errors are worse than others?



We need a cost function that penalizes states non-uniformly

Key Idea:

Frame control as an optimization problem

Model predictive control (MPC)

1. Plan a sequence of control actions

2. Predict the set of next states unto a horizon H

3. Evaluate the cost / constraint of the states and controls

4. Optimize the cost

Model predictive control (MPC)

$$\min_{u_{t+1},\ldots u_{t+H}} \sum_{k=t}^{t+H-1} J(x_k, u_{k+1})$$

(plan till horizon H) (Cost)

$$x_{k+1} = f(x_k, u_{k+1})$$

(Predict next state with dynamics)

$$g(x_k, u_{k+1}) \leq 0$$

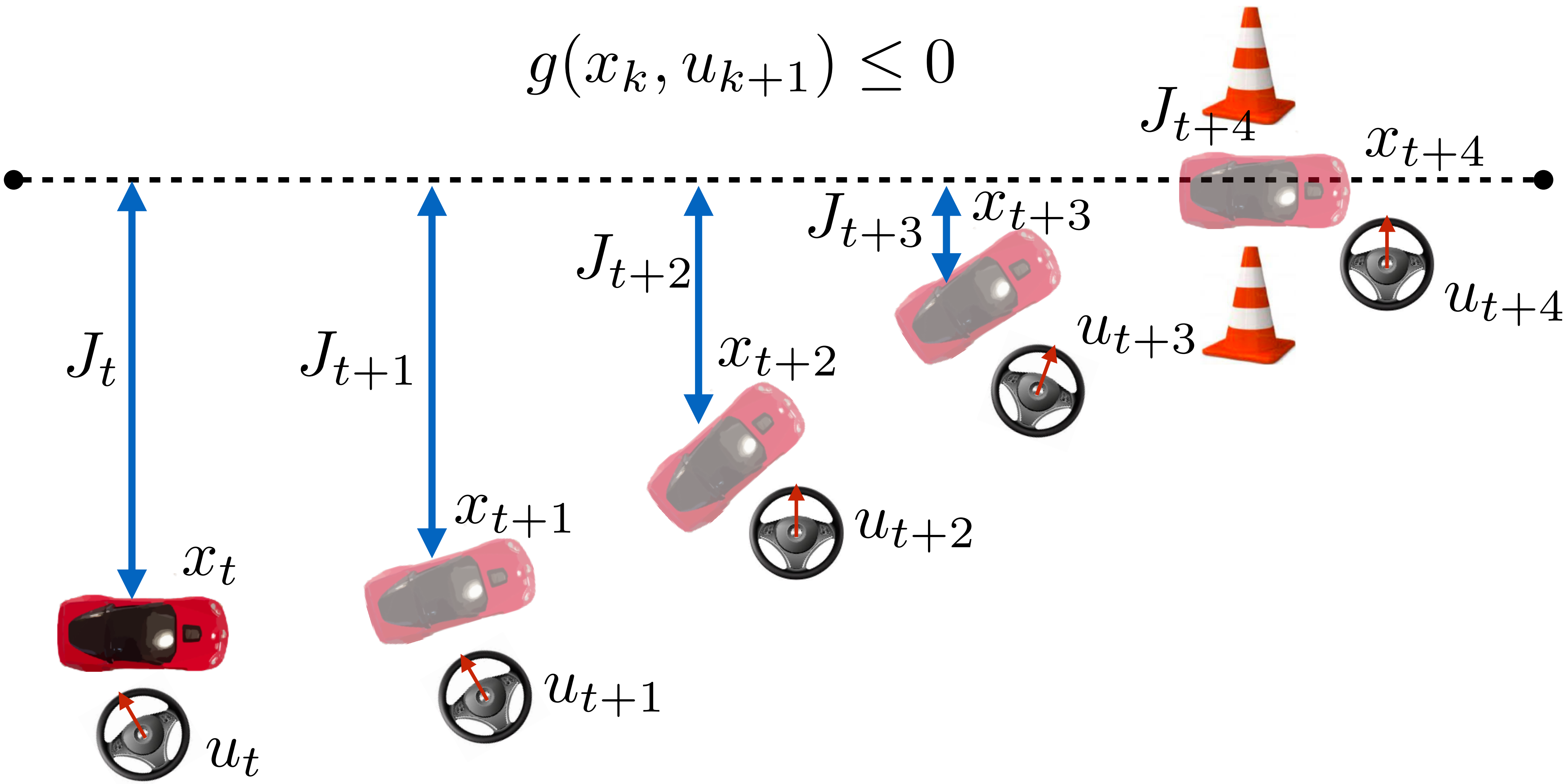
(Constraints)

Model predictive control (MPC)

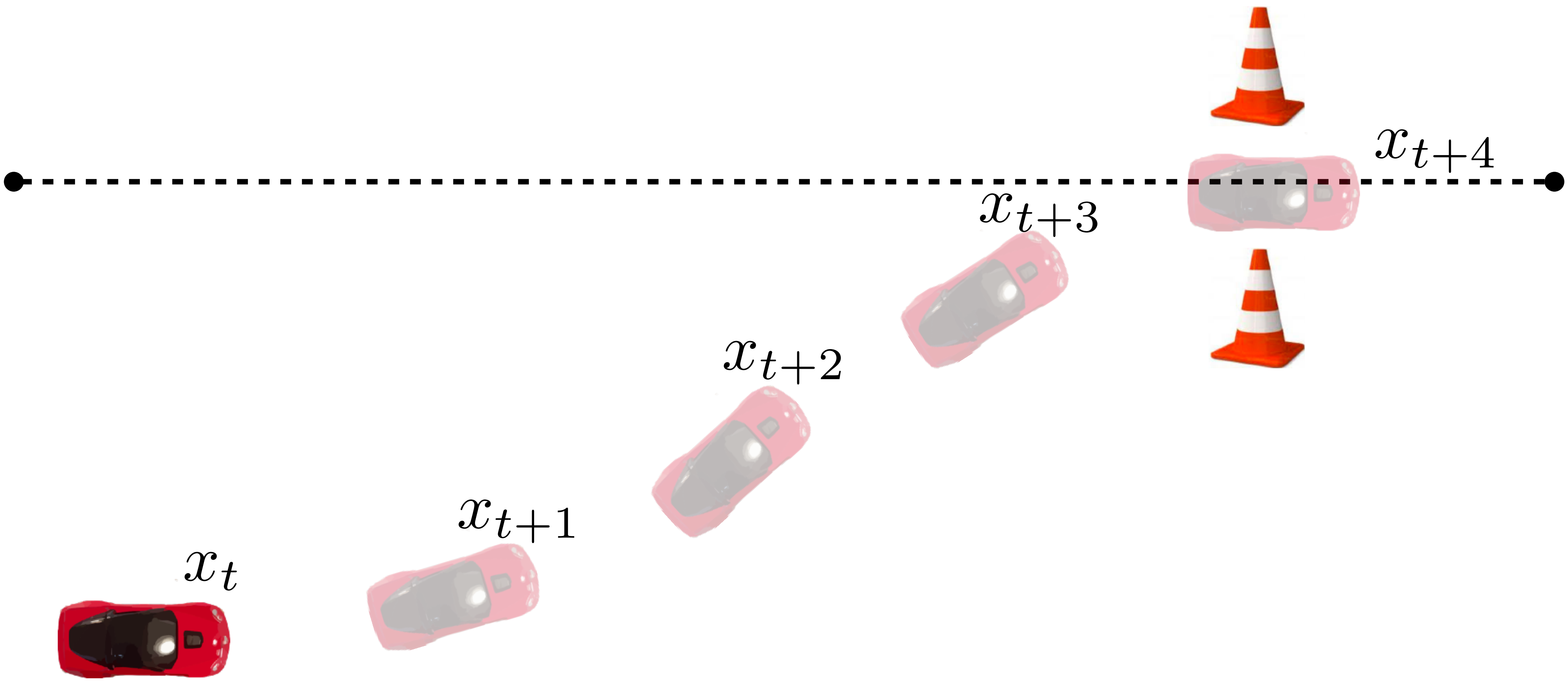
$$\min_{u_{t+1},\ldots u_{t+H}} \sum_{k=t}^{t+H-1} J(x_k, u_{k+1})$$

$$x_{k+1} = f(x_k, u_{k+1})$$

$$g(x_k, u_{k+1}) \leq 0$$



How are the controls executed?



Step 1: Solve optimization problem to a horizon

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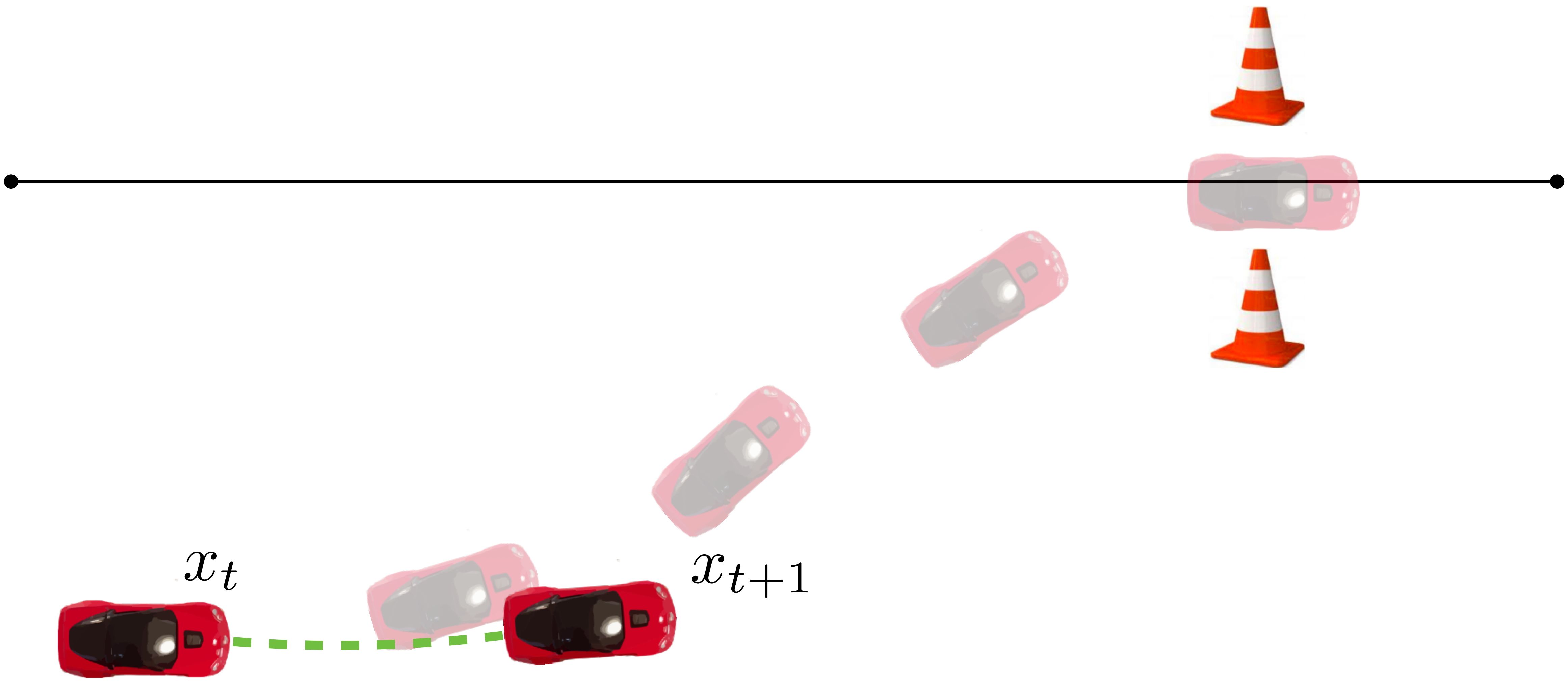

Step 1: Solve optimization problem to a horizon

Step 2: Execute the first control

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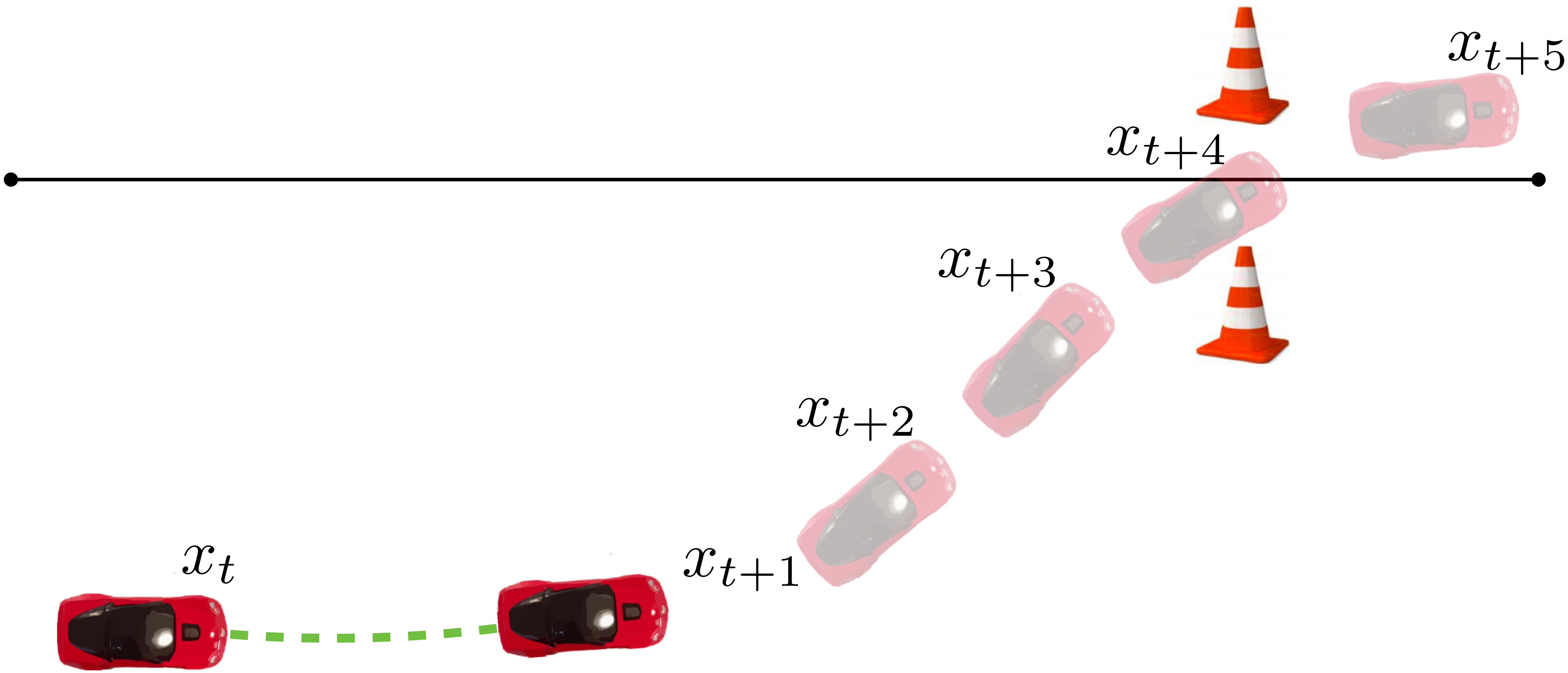
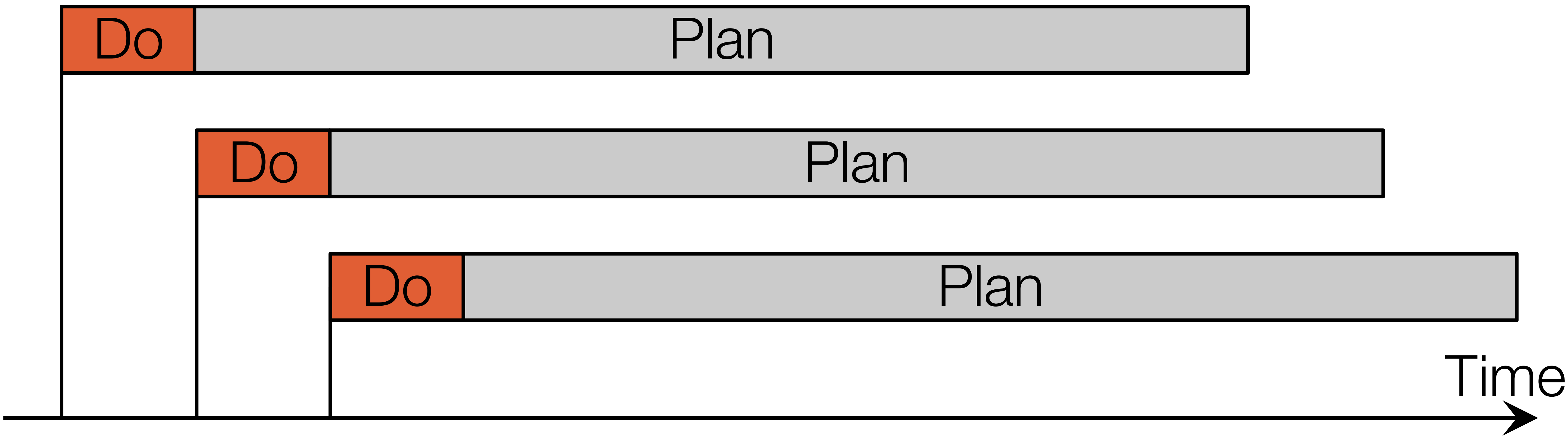


Step 1: Solve optimization problem to a horizon

Step 2: Execute the first control

Step 3: Repeat!

MPC is a framework

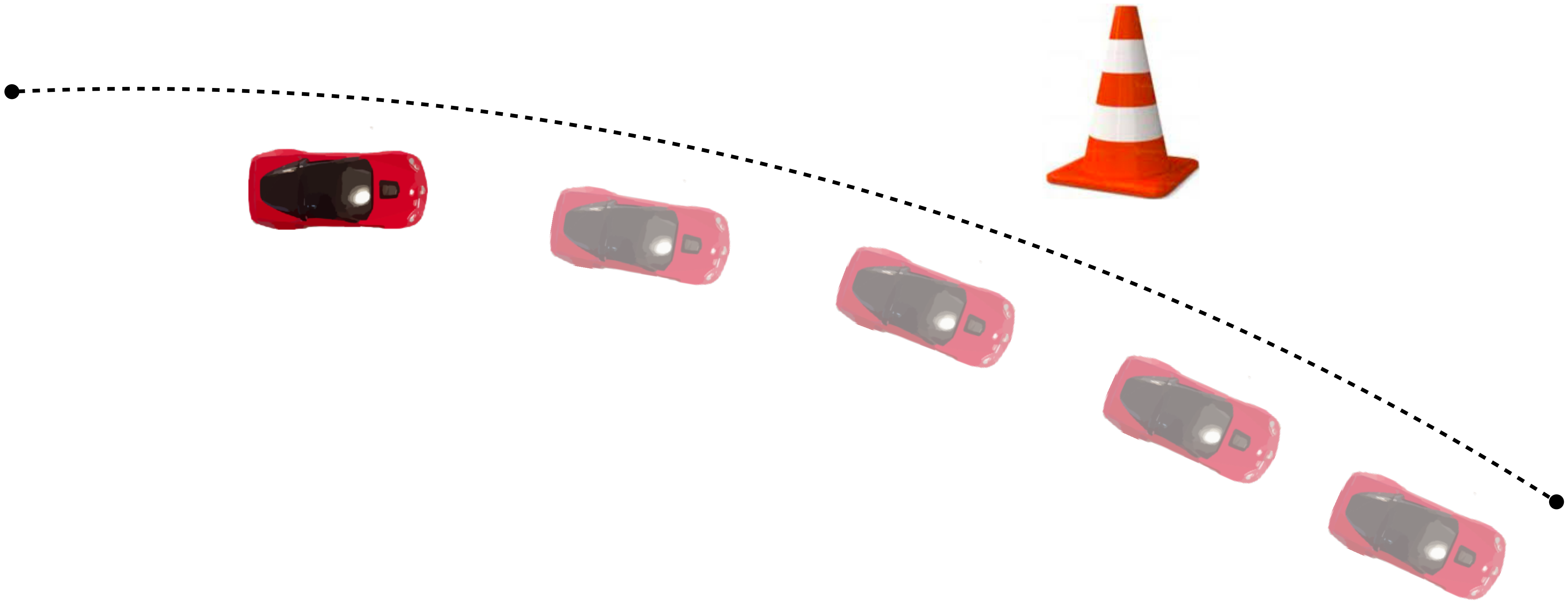


Step 1: Solve optimization problem to a horizon

Step 2: Execute the first control

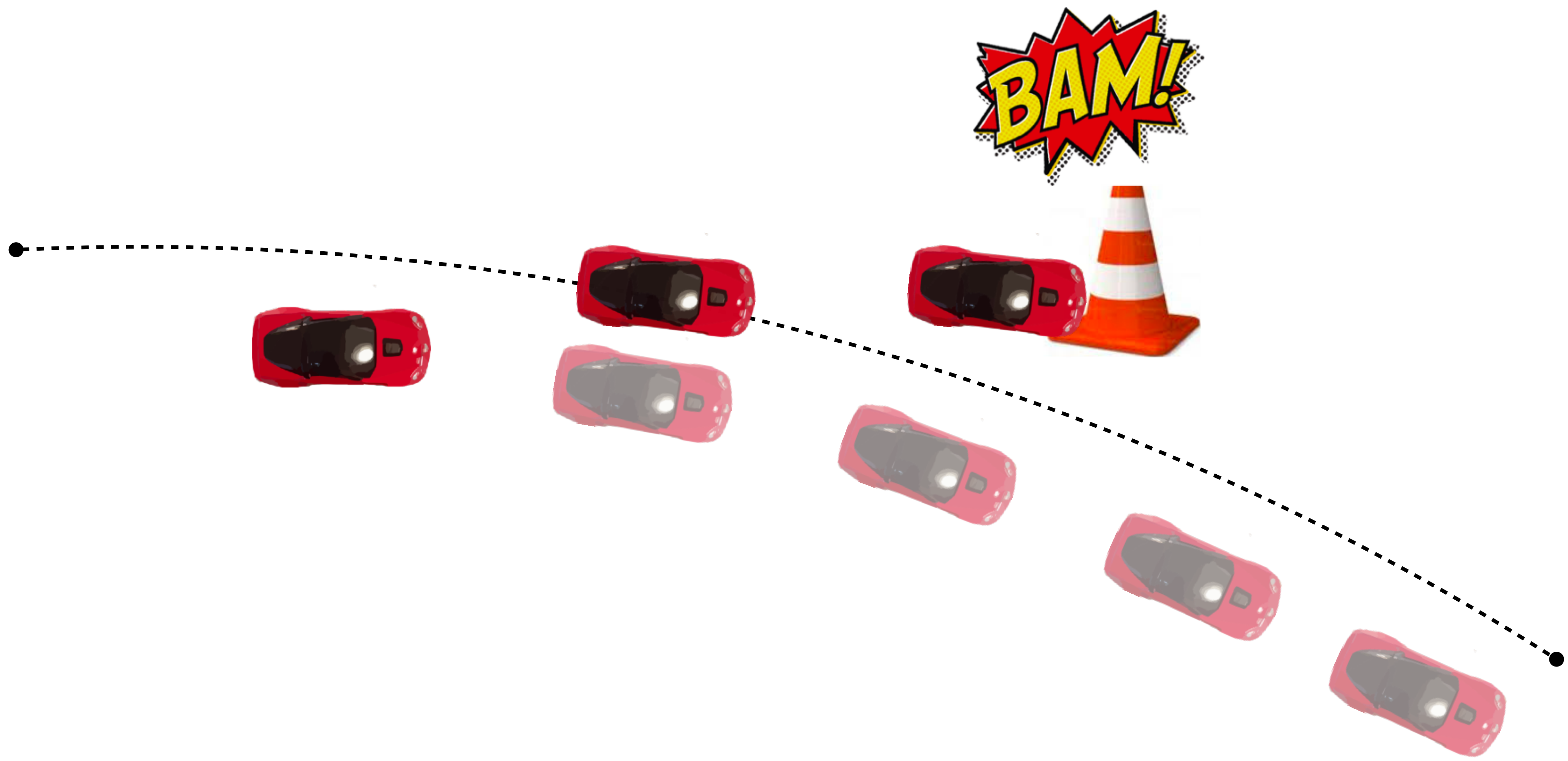
Step 3: Repeat!

Why do we need to replan?



What happens if the controls are planned once and executed?

Why do we need to replan?



What happens if the controls are planned once and executed?

Model predictive control (MPC)

$$\min_{u_{t+1},\ldots u_{t+H}} \sum_{k=t}^{t+H-1} J(x_k, u_{k+1})$$

(plan till horizon H) (Cost)

$$x_{k+1} = f(x_k, u_{k+1})$$

(Predict next state with dynamics)

$$g(x_k, u_{k+1}) \leq 0$$

(Constraints)