ECE 417/598: Rotation from P matrix

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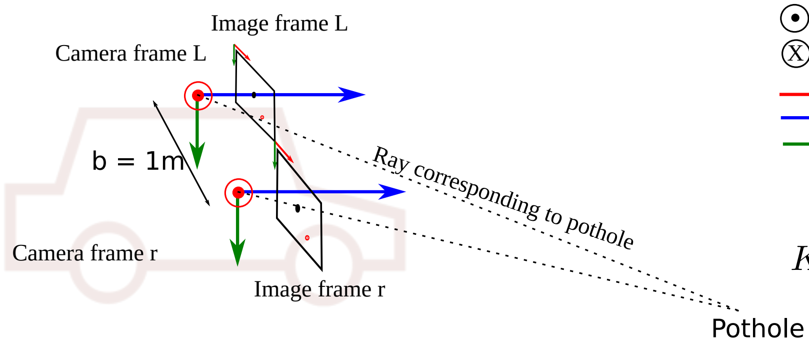
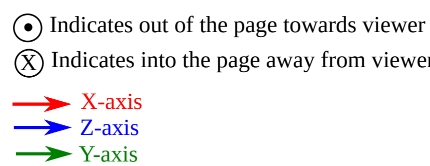
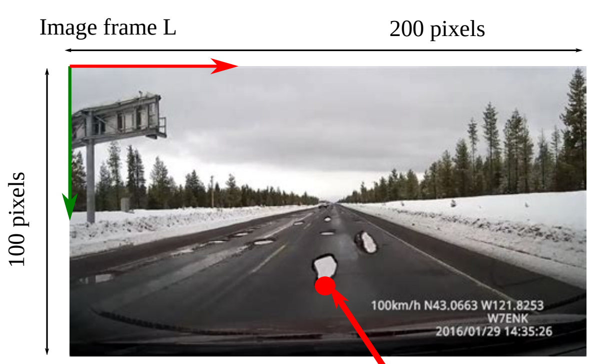
Problem 1 *Find the 3D position of the pothole give that the right camera r is moved along X-axis by $b = 1m$ with respect to the left camera L.*





$$K = \begin{bmatrix} 100 & 0 & 100 \\ 0 & 100 & 50 \\ 0 & 0 & 1 \end{bmatrix}$$



Pothole detected at $\underline{\mathbf{u}}_L = \begin{bmatrix} 105 \\ 70 \\ 1 \end{bmatrix}$ pixels

Pothole detected at $\underline{\mathbf{u}}_r = \begin{bmatrix} 95 \\ 70 \\ 1 \end{bmatrix}$ pixels

1. Find the equation of ray corresponding to the pothole in the left image. In other words, write the equation of the ray corresponding to the point $\underline{\mathbf{u}}_L$, in camera frame L.

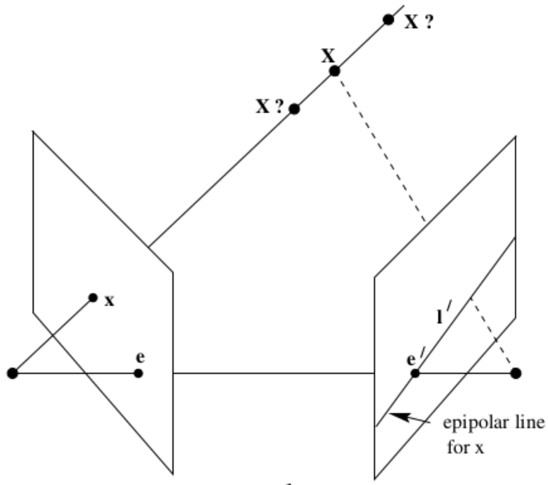
2. Find the rotation matrix ${}^{r}R_L$ and translation vector ${}^{r}\mathbf{t}_L$, so that they transform any point in the coordinate frame L to a point in the coordinate frame r. ($\mathbf{X}_r = {}^{r}R_L \mathbf{X}_L + {}^{r}\mathbf{t}_L$).

3. Transform the equation of ray from coordinate frame L to coordinate frame r.

4. Find the equation of ray corresponding to the pothole in the right image. In other words, write the equation of the ray corresponding to the point $\underline{\mathbf{u}}_r$, in camera frame r.

5. Find the intersection of rays (lines) 4 and 3.

Problem 2 *When the depth corresponding to the point* $\mathbf{x}$ *is unknown, the possible pixels* ($\mathbf{x}'$) *on the right image that can correspond to the point form a line. What is the equation of that line?*



1. Find the equation of ray corresponding to the pothole in the left image. In other words, write the equation of the ray corresponding to the point $\mathbf{x}$, in the first camera frame.

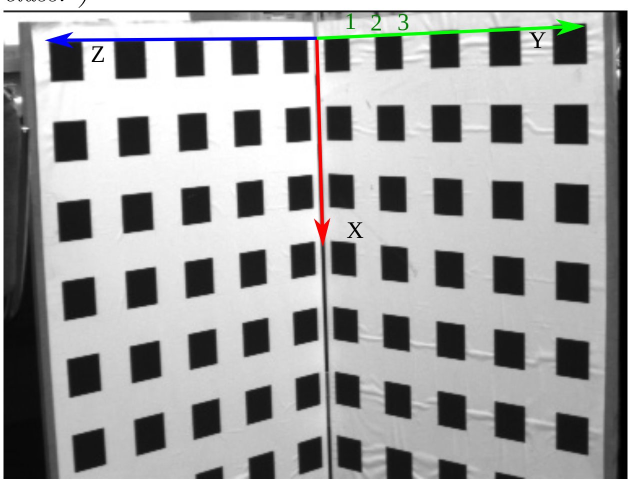
2. Assume the rotation matrix R and translation vector $\mathbf{t}$, so that they transform any point in the left coordinate frame to a point in the right coordinate frame. ($\mathbf{X}' = R\mathbf{X} + \mathbf{t}$). Transform the equation of ray from the left coordinate frame to the right coordinate frame.

3. Project any 3D point on the right camera, call it $\mathbf{x}'$.

4. Find the point $\mathbf{e}'$ on the right camera.

5. Find a line $\mathbf{l}'$ that passes through both $\mathbf{x}'$ and $\mathbf{e}'$.

Problem 3 *Given a set of $n \geq 6$ points $\underline{\mathbf{X}}_i \in \mathbb{P}^3$ for all $i \in \{1, \ldots, n\}$ in 3D projective space, and a set of corresponding points $\underline{\mathbf{u}}_i \in \mathbb{P}^2$ in an image, find the 3D to 2D projective $P \in \mathbb{R}^{3\times4}$ matrix that converts $\mathbf{X}_i$ to $\underline{\mathbf{u}}_i = \lambda_i P \underline{\mathbf{X}}_i$. In other words, convert $\underline{\mathbf{u}}_i \times P\underline{\mathbf{X}}_i = 0$ into a familiar form $A\mathbf{y} = \mathbf{b}$ or $A\mathbf{y} = \mathbf{0}$ so that we can solve for P. For notation purposes, you can denote $\underline{\mathbf{u}}_i = [x_i, y_i, w_i]^\top$ and $P = \begin{bmatrix}\mathbf{p}_1^\top \\ \mathbf{p}_2^\top \\ \mathbf{p}_3^\top\end{bmatrix}$ where $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \in \mathbb{R}^4$ are the rows of P represented as 4-D column vectors. (Practical motivation: We did camera calibration in lab using a single checker board. It is much easier to compute camera calibration using two mutually perpendicular checker boards so that all points do not lie on a single plane (hence linearly independent). One can make a coordinate system attached to the double checker and compute the 3D coordinates of each corner point in that system. Let $\underline{\mathbf{X}}_i \in \mathbb{P}^3$ be such points in 3D on the checker-board. Let $\underline{\mathbf{u}}_i \in \mathbb{P}^2$ be a point detected in the image so that we have one-to-one correspondence between $\underline{\mathbf{X}}_i$ and $\underline{\mathbf{u}}_i$. Finding the projection matrix $P \in \mathbb{R}^{3\times4}$ then reduces to the above problem. We will cover the breakdown of P matrix into $P = K[R, t]$ in class.)*



Solution Watch lecture `https://drive.google.com/file/d/1cY02DTagpckbYl5gS0PYBu569vlZUNN6/view?usp=sharing`

1. Write cross product as a matrix operation

$$[\underline{\mathbf{u}}_i]_\times = \begin{bmatrix} 0 & -w_i & y_i \\ w_i & 0 & -x_i \\ -y_i & x_i & 0 \end{bmatrix}$$

2. Write $P\underline{\mathbf{X}}_i$ in terms of row vectors.

$$P\underline{\mathbf{X}}_i = \begin{bmatrix} \mathbf{p}_1^\top \\ \mathbf{p}_2^\top \\ \mathbf{p}_3^\top \end{bmatrix} \underline{\mathbf{X}}_i = \begin{bmatrix} \mathbf{p}_1^\top \underline{\mathbf{X}}_i \\ \mathbf{p}_2^\top \underline{\mathbf{X}}_i \\ \mathbf{p}_3^\top \underline{\mathbf{X}}_i \end{bmatrix}$$

3. Note that all the three terms like $\mathbf{p}_1^\top \underline{\mathbf{X}}_i$ are scalars hence they are symmetric. Hence $\mathbf{p}_1^\top \underline{\mathbf{X}}_i = \underline{\mathbf{X}}_i^\top \mathbf{p}_1$.

$$P\underline{\mathbf{X}}_i = \begin{bmatrix} \underline{\mathbf{X}}_i^\top \mathbf{p}_1 \\ \underline{\mathbf{X}}_i^\top \mathbf{p}_2 \\ \underline{\mathbf{X}}_i^\top \mathbf{p}_3 \end{bmatrix}$$

4. Substitute these values in the original equation $\underline{\mathbf{u}}_i \times P\underline{\mathbf{X}}_i = \mathbf{0}_{3\times 1}$.

$$\begin{bmatrix} 0 & -w_i & y_i \\ w_i & 0 & -x_i \\ -y_i & x_i & 0 \end{bmatrix} \begin{bmatrix} \underline{\mathbf{X}}_i^\top \mathbf{p}_1 \\ \underline{\mathbf{X}}_i^\top \mathbf{p}_2 \\ \underline{\mathbf{X}}_i^\top \mathbf{p}_3 \end{bmatrix} = \mathbf{0}_{3\times 1}$$

5. Matrix multiply

$$\begin{bmatrix} 0 - w_i \underline{\mathbf{X}}_i^\top \mathbf{p}_2 + y_i \underline{\mathbf{X}}_i^\top \mathbf{p}_3 \\ w_i \underline{\mathbf{X}}_i^\top \mathbf{p}_1 + 0 - x_i \underline{\mathbf{X}}_i^\top \mathbf{p}_3 \\ -y_i \underline{\mathbf{X}}_i^\top \mathbf{p}_1 + x_i \underline{\mathbf{X}}_i^\top \mathbf{p}_2 + 0 \end{bmatrix} = \mathbf{0}_{3\times 1}$$

6. Write the unknowns as a single vector, and the knowns as a matrix multiplication with the unknowns

$$\begin{bmatrix} \mathbf{0}^\top & -w_i \underline{\mathbf{X}}_i^\top & y_i \underline{\mathbf{X}}_i^\top \\ w_i \underline{\mathbf{X}}_i^\top & \mathbf{0}^\top & -x_i \underline{\mathbf{X}}_i^\top \\ -y_i \underline{\mathbf{X}}_i^\top & x_i \underline{\mathbf{X}}_i^\top & \mathbf{0}^\top \end{bmatrix}_{3\times 12} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}_{12\times 1} = \mathbf{0}_{3\times 1}$$

7. Pick only two of the equations as only two are linearly independent.

$$\begin{bmatrix} \mathbf{0}^\top & -w_i \underline{\mathbf{X}}_i^\top & y_i \underline{\mathbf{X}}_i^\top \\ w_i \underline{\mathbf{X}}_i^\top & \mathbf{0}^\top & -x_i \underline{\mathbf{X}}_i^\top \end{bmatrix}_{2\times 12} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}_{12\times 1} = \mathbf{0}_{2\times 1}$$

8. Collect all the equations from n pairs of corresponding points $\underline{\mathbf{u}}_1, \dots, \underline{\mathbf{u}}_n$ and $\underline{\mathbf{X}}_1, \dots, \underline{\mathbf{X}}_n$.

$$\begin{bmatrix} \mathbf{0}^\top & -w_1 \underline{\mathbf{X}}_1^\top & y_1 \underline{\mathbf{X}}_1^\top \\ w_1 \underline{\mathbf{X}}_1^\top & \mathbf{0}^\top & -x_1 \underline{\mathbf{X}}_1^\top \\ \vdots & \vdots & \vdots \\ \mathbf{0}^\top & -w_n \underline{\mathbf{X}}_n^\top & y_n \underline{\mathbf{X}}_n^\top \\ w_n \underline{\mathbf{X}}_n^\top & \mathbf{0}^\top & -x_n \underline{\mathbf{X}}_n^\top \end{bmatrix}_{2n\times 12} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}_{12\times 1} = \mathbf{0}_{2n\times 1}$$

9. P matrix has rank rank$(P) = 11$ because it has 12 elements and equivalence upto a scale factor. So the solution of the above equation can be computed from SVD by choosing the right singular vector corresponding to the smallest singular value.

$$A = \begin{bmatrix} \mathbf{0}^\top & -w_1\underline{\mathbf{X}}_1^\top & y_1\underline{\mathbf{X}}_1^\top \\ w_1\underline{\mathbf{X}}_1^\top & \mathbf{0}^\top & -x_1\underline{\mathbf{X}}_1^\top \\ \vdots & \vdots & \vdots \\ \mathbf{0}^\top & -w_n\underline{\mathbf{X}}_n^\top & y_n\underline{\mathbf{X}}_n^\top \\ w_n\underline{\mathbf{X}}_n^\top & \mathbf{0}^\top & -x_n\underline{\mathbf{X}}_n^\top \end{bmatrix} = U\Sigma V^T$$

Let $V = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$, then

$$\begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix} = \mathbf{v}_n$$

Now we can write the P matrix as

$$P = \begin{bmatrix} \mathbf{p}_1^\top \\ \mathbf{p}_2^\top \\ \mathbf{p}_3^\top \end{bmatrix}$$

Problem 4 *Given to vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *find a set of unit vectors that are mutually orthonormal and write* $\mathbf{a}$, $\mathbf{b}$. *in terms of those mutually orthonormal vectors.*

Problem 5 *Repeat the process for 3 vectors,* $\mathbf{a}$*,* $\mathbf{b}$ *and* $\mathbf{c}$*.*

Problem 6 *Repeat the process for* n *vectors,* $\mathbf{a}_1$*,* $\mathbf{a}_2$ *and* $\mathbf{a}_3$*.*