ECE 417/598: Line fitting using null space

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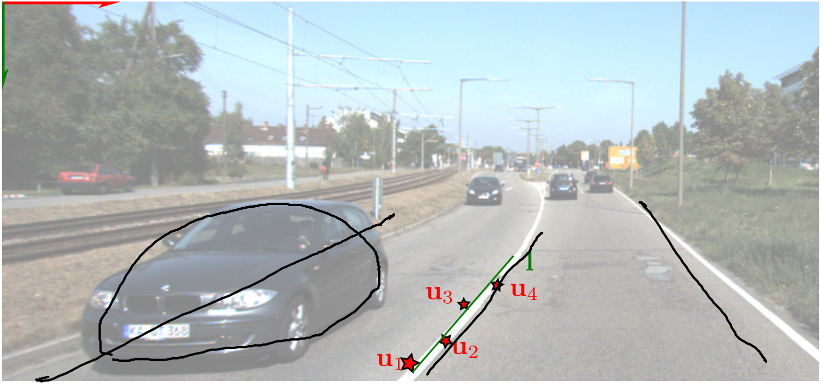


$$\underline{\mathbf{u}}_1 = [100, 98, 1]^\top$$
$$\underline{\mathbf{u}}_2 = [105, 95, 1]^\top$$
$$\underline{\mathbf{u}}_3 = [107, 90, 1]^\top$$
$$\underline{\mathbf{u}}_4 = [110, 85, 1]^\top$$

Find the line $\mathbf{l}$ such that it is the "closest line" passing through $\mathbf{u}_1, \ldots, \mathbf{u}_4$.

$$U = \begin{bmatrix} \mathbf{u}_1^\top \\ \mathbf{u}_2^\top \\ \mathbf{u}_3^\top \\ \mathbf{u}_4^\top \end{bmatrix}$$

We want to solve for $\mathbf{l}$ such that

$$U\mathbf{l} = 0$$

Singular Value Decomposition (SVD)

$V^T V = V V^T = I$

$m \times n$ $\quad$ $m \times m$ $\quad$ $n \times n$

$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^\top$$

orthonormal

orthonormal

$U U^T = U^T U = I$

$$A^\top A = V \Sigma^2 V^{-1}$$

$$A^\top A \mathbf{v}_i = \lambda_i \mathbf{v}_i \qquad \lambda_i = \sigma_i^2$$

$$\boxed{\mathbf{u}_i = \frac{A\mathbf{v}_i}{\sigma_i}}$$

$$U = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_m \end{bmatrix}$$

$A = U \Sigma V^T$

$p^T (x_w - x_{w0}) = 0$

$$A = U \begin{bmatrix} \Sigma_r & 0 \\ 0 & 0 \end{bmatrix} V^\top$$

$q^T (x_w - x_{w,0}) = 0$

$\begin{bmatrix} p^T \\ q^T \end{bmatrix} \underline{(x_n - x_{w,0})} = 0$

If $A \in \mathbb{R}^{m \times n}$

$p \times q$

$$\underline{A = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_m \end{bmatrix}} \begin{array}{c|c} & n \\ m & \begin{bmatrix} \sigma_1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & \sigma_2 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & \ddots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sigma_r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \begin{bmatrix} \mathbf{v}_1^\top \\ \mathbf{v}_2^\top \\ \vdots \\ \mathbf{v}_n^\top \end{bmatrix} \quad (2)$$

v_r

$v_{r+1} \dots v_n$

$$V = \begin{bmatrix} \underline{v_1} \dots v_r, \underbrace{v_{\underline{r+1}} \dots v_{\underline{n}}} \end{bmatrix}$$

$$A = \begin{bmatrix} \underline{u}_1 & \cdots & \underline{u}_r, & \underline{u}_{r+1} & \cdots & \underline{u}_m \end{bmatrix} \begin{bmatrix} \sigma_1 & & & 0 & 0 \\ 0 & \ddots & \sigma_r & & \\ & & & 0 & \\ & 0 & & & 0 \end{bmatrix} \begin{bmatrix} \begin{matrix} \underline{v}_1^T \\ \vdots \\ \underline{v}_r \end{matrix} \\ \begin{bmatrix} \underline{v}_{r+1} \\ \vdots \\ \underline{v}_n \end{bmatrix} \end{bmatrix}$$

$$A = \begin{array}{cc} & r \\ m & \begin{bmatrix} \underline{u}_1 & \cdots & \underline{u}_r \end{bmatrix} \end{array} \begin{array}{cc} & r \\ r & \begin{bmatrix} \sigma_1 & & 0 \\ & \ddots & \\ 0 & & \sigma_r \end{bmatrix} \end{array} \begin{array}{cc} & n \\ r & \begin{bmatrix} \underline{v}_1^T \\ \vdots \\ \underline{v}_r^T \end{bmatrix} \end{array}$$

Coding example

$V = 3 \begin{bmatrix} \overset{3}{\overline{2}} & 1 \end{bmatrix}$ ← Nullsp

$A\underline{l} = 0$

$A =$ $4 \left\{ \underbrace{\begin{matrix} \underline{\mathbf{u}}_1 = [100, 98, 1]^\top \\ \underline{\mathbf{u}}_2 = [105, 95, 1]^\top \\ \underline{\mathbf{u}}_3 = [107, 90, 1]^\top \\ \underline{\mathbf{u}}_4 = [110, 85, 1]^\top \end{matrix}}_{} \right.$ 3

Find the line $\mathbf{l}$ such that it is the "closest line" passing through $\mathbf{u}_1, \ldots, \mathbf{u}_4$.

https://github.com/wecacuee/ECE417-Mobile-Robots/tree/master/docs/slides/03-09-svd-null-space

$$A = \begin{bmatrix} \mathbf{u}_1^\top \\ \mathbf{u}_2^\top \\ \mathbf{u}_3^\top \\ \mathbf{u}_4^\top \end{bmatrix}$$

We want to solve for $\mathbf{l}$ such that

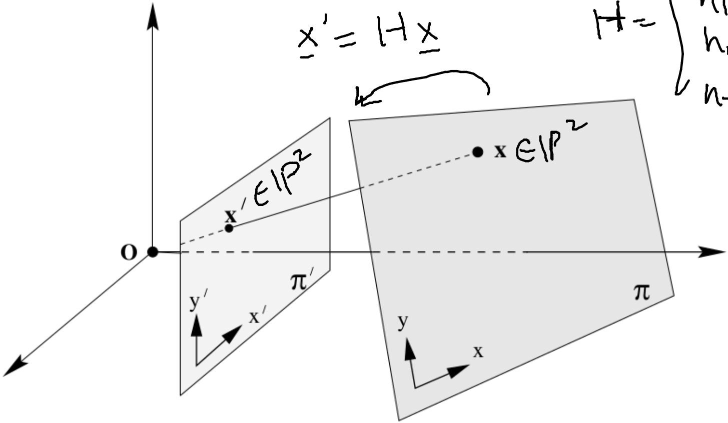
$$U\mathbf{l} = 0$$

Homography $\in \mathbb{P}^2$

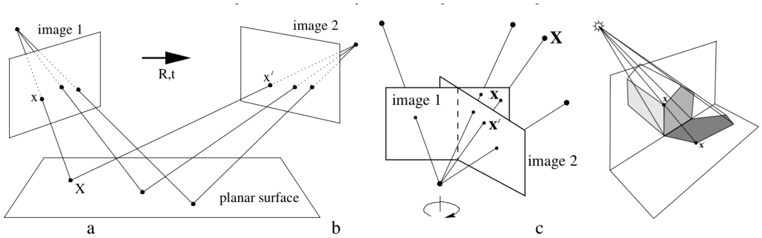
linea transformation

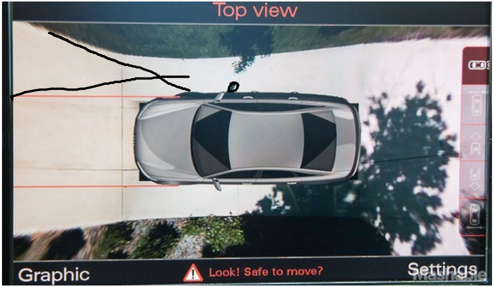
$$\underline{x}' = H\underline{x}$$

$$H = \begin{pmatrix} h_1 & h_2 & h_3 \\ h_4 & h_5 & h_6 \\ h_7 & h_8 & h_9 \end{pmatrix}$$

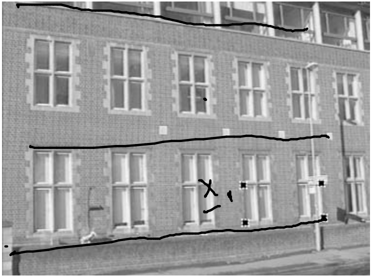


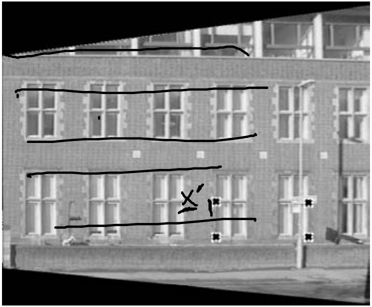
Examples of Homography




Top view
Graphic
Look! Safe to move?
Settings

Computing Homography





$$\begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \\ 14 & 16 & 18 \end{bmatrix}$$

DOF of H ?

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}}$$

$$H \overset{3}{=} \begin{bmatrix} h_1^3, & h_2 h_3 \end{bmatrix}$$

9 DOF

-1 DOF

$\boxed{8 \text{ DOF}}$

$\underline{x}' \in \mathbb{P}^2$

$\underline{x} \in \mathbb{P}^2$

$$\underline{x}' = H \underline{x}$$

$$\underline{x}' = \lambda H \underline{x}$$

$$x' = \lambda x'$$

To solve for H
we need 8 equations

$x_1' = H x_1$ $\begin{bmatrix} x' \\ y' \\ 1 \end{bmatrix} = \begin{bmatrix} & H & \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$

each point correspondanc $\longrightarrow$ 2 equatun

Computing Homography

Solving for Homography derivation

Direct Linear Transformation (DLT) algorithm

Objective

Given $n \geq 4$ 2D to 2D point correspondences $\{\mathbf{x}_i \leftrightarrow \mathbf{x}'_i\}$, determine the 2D homography matrix $\mathtt{H}$ such that $\mathbf{x}'_i = \mathtt{H}\mathbf{x}_i$.

Algorithm

(i) For each correspondence $\mathbf{x}_i \leftrightarrow \mathbf{x}'_i$ compute the matrix $\mathtt{A}_i$ from (4.1). Only the first two rows need be used in general.

(ii) Assemble the n 2×9 matrices $\mathtt{A}_i$ into a single $2n \times 9$ matrix $\mathtt{A}$.

(iii) Obtain the SVD of $\mathtt{A}$ (section A4.4(p585)). The unit singular vector corresponding to the smallest singular value is the solution $\mathbf{h}$. Specifically, if $\mathtt{A} = \mathtt{UDV}^\mathsf{T}$ with $\mathtt{D}$ diagonal with positive diagonal entries, arranged in descending order down the diagonal, then $\mathbf{h}$ is the last column of $\mathtt{V}$.

(iv) The matrix $\mathtt{H}$ is determined from $\mathbf{h}$ as in (4.2).

2D homography

Given a set of points $\mathbf{x}_i \in \mathbb{P}^2$ and a corresponding set of points $\mathbf{x}'_i \in \mathbb{P}^2$, compute the projective transformation that takes each $\mathbf{x}_i$ to $\mathbf{x}'_i$. In a practical situation, the points $\mathbf{x}_i$ and $\mathbf{x}'_i$ are points in two images (or the same image), each image being considered as a projective plane $\mathbb{P}^2$.

3D to 2D camera projection matrix estimation

Given a set of points $\mathbf{X}_i$ in 3D space, and a set of corresponding points $\mathbf{x}_i$ in an image, find the 3D to 2D projective $\mathbf{P}$ mapping that maps $\mathbf{X}_i$ to $\mathbf{x}_i = \mathbf{P}\mathbf{X}_i$.