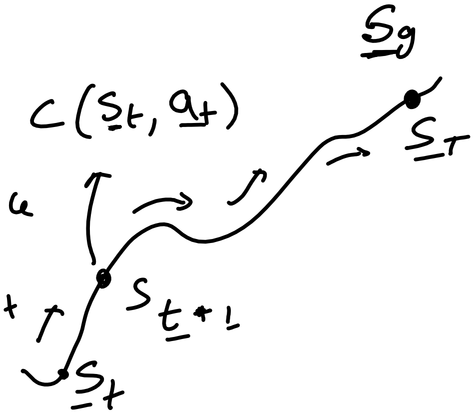
Trajectory in Terms of state space

Control : Proportional Integral Derivative (PID)

2. LQR: Linear Quadratic Regulator

↑

Planning / Control → Trajectory in terms of action space



$$\min_{\{a_t\}_{t=1}^{T}} \sum_{t=1}^{T} c(\underline{s}_t, \underline{a}_t)$$

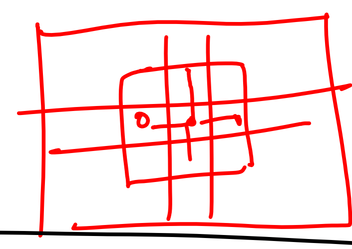
subject to $\underline{s}_{t+1} = f(\underline{s}_t, a_t)$

Examples

$$\underbrace{\begin{bmatrix} x_{t+1} \\ y_{t+1} \\ \theta_{t+1} \end{bmatrix}}_{s_{t+1}} = \underbrace{\begin{bmatrix} v_t \cos\theta_t \Delta t + x_t \\ v_t \sin\theta_t \Delta t + y_t \\ \omega_t \Delta t + \theta_t \end{bmatrix}}_{f(\underline{s}_t, \underline{a}_t)}$$

$$\underline{c(\underline{s}_t, \underline{a}_t)} = \|\underline{s}_g - \underline{s}_t\|^2 + \frac{1}{2} m v^2 + \frac{1}{2} I_m \omega^2$$

$$\underline{s}_t = \begin{bmatrix} x_t \\ y_t \\ \theta_t \end{bmatrix} \quad \underline{a}_t = \begin{bmatrix} v_t \\ \omega_t \end{bmatrix}$$



System dynamics to be Linear

Cost function to be Quadratic

$$\underline{s}_{t+1} = f(\underline{s}_t, \underline{a}_t) = \underset{n\times n}{A_t} \underset{n\times 1}{\underline{s}_t} + \underset{n\times m}{B_t} \underset{m\times 1}{\underline{a}_t}$$

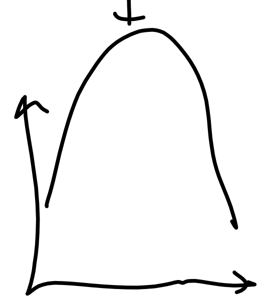
optional

$$c(s_t, a_t) = \underline{s}_t^T \underset{n\times n}{Q} \underline{s}_t + \underline{a}_t^T \underset{m\times m}{R} \underline{a}_t + \underline{b}^T \underline{s}_t + \underline{c}^T \underline{a}_t + \underline{s}_t^T N \underline{a}_t$$

$$\underline{x}^T A \underline{x} = x^2 a_{11} + y^2 a_{22} + a_{12} xy + \ldots$$

$+2x^2 - 5x + 6$

$-2x^2 + 5x + 6$



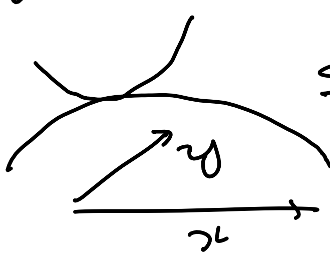
$Q \succ 0$

Positive definite

$Q \prec 0$

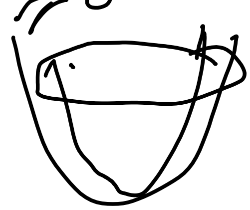
negative definite

Q can be indefinite

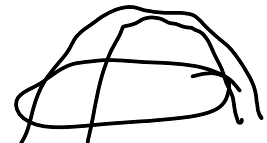


saddle point

$Q \succeq 0$



Positive semi definite



$Q \preceq 0$

Negative semi definite

Defn : $Q \succ 0$

for any $\underline{x}$

$\underline{x}^T Q \underline{x} > 0$

Test : all eigenvalues > 0

$\underline{x}^T E \Lambda E^T \underline{x} > 0$

matrix of eigen vector

Diagonal matrix of eigen values

$\lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2 = \underline{y}^T \Lambda \underline{y}$

$\underline{y} = E \underline{x}$

LQR

$$\min_{\{a_t\}_{t=1}^{T}} \Big(\sum_{t=1}^{T} \underbrace{\underset{\text{how far are you from goal}}{s_t^T Q s_t} + \underset{\text{energy}}{a_t^T R a_t}}_{\text{cost function}} \Big) + s_{T+1}^T Q s_{T+1}$$

$$\text{s.t.} \quad \underline{s_{t+1} = A_t s_t + B_t a_t}$$

Dynamic programming (mathematical Induction)

$$\min_{\{a_t\}_{t=1}^{T-1}} \Big(\sum_{t=1}^{T-1} s_t^T Q s_t + a_t^T R a_t \Big) + \Big(\min_{a_T} s_T^T Q s_T + a_T^T R a_T \Big)$$

Cost to go

$$V(s_1) = \min_{\{a_t\}_{t=1}^{T}} \Big(\sum_{t=1}^{T} \underbrace{\underset{\text{how far are you from goal}}{s_t^T Q s_t} + \underset{\text{energy}}{a_t^T R a_t}}_{\text{cost function}} \Big) + s_{T+1}^T Q s_{T+1}$$

cost to go

$$\text{s.t.} \quad \underline{s_{t+1} = A_t s_t + B_t a_t}$$

$t = T$

$$V(s_k) = \min_{\{a_t\}_{t=k}^{T}} \sum_{t=k}^{T} \left(s_t^T Q s_t + a_t^T R a_t \right) + s_{T+1}^T Q s_{T+1}$$

$$\text{s.t.} \quad s_{t+1} = A_t s_t + B_T a_t \quad \text{for all } t \in \{k \ldots T\}$$

A correction to the cost function

$$\underline{\qquad\qquad} + s_T^T Q s_T + a_T^T R a_T$$

$$s_{T+1} = A_T s_T + B_T a_T$$

$$V(s_T) = \min_{\{a_t\}_{t=T}^{T}} \sum_{t=T}^{T} \underline{s}_T^T Q \underline{s}_T + \underline{a}_T^T R a_T$$
$$+ s_{T+1}^T Q s_{T+1}$$

$$\text{s.t.} \quad \underline{s}_{T+1} = A_T \underline{s}_T + B_T \underline{a}_T$$

$$= \min_{a_T} \underline{s}_{T+1}^T Q s_{T+1} + a_T^T R a_T + \underline{s}_T^T Q \underline{s}_T$$

$$\underline{s}_{T+1} = A_T \underline{s}_T + B_T \underline{a}_T$$

$$V(s_T) = \min_{\underline{a}_T} (A_T s_T + B_T \underline{a}_T)^T Q (A_T \underline{s}_T + B_T a_T)$$
$$+ \underline{a}_T^T R \underline{a}_T$$

$$M\underline{a} = \begin{bmatrix} -\underline{m}_1^T \underline{a}- \\ -\underline{m}_2^T \underline{a}- \\ \vdots \\ -\underline{m}_n^T \underline{a}- \end{bmatrix}$$

$$J_{a_T}[\underline{a}_T^T M \underline{a}_T] = 2\underline{a}_T^T M$$

$$J_{\underline{a}_T}[\underline{m}^T \underline{a}_T] = \underline{m}^T \quad / \quad J_{a_T}[M\underline{a}_T] = M \qquad \begin{cases} Q = Q^T \\ R = R^T \end{cases}$$

$$J_{\underline{a}_T}\left\{(A\underline{s}_T + B\, a_T)^T Q (A\underline{s}_T + B\underline{a}_T) + \underline{a}_T^T R a_T\right\} = \underline{0}^T$$

$$2(A\underline{s}_T + B\underline{a}_T)^T Q J_{\underline{a}_T}\{(A\underline{s}_T + B\underline{a}_T)\} + 2a_T^T R = \underline{0}^T$$

$$2(A\underline{s}_T + B\underline{a}_T)^T Q B + 2a_T^T R = \underline{0}^T$$

$$\underline{a}_T^T = \quad -\underline{s}_T^T A^T Q B (R + B^T Q B)^{-1}$$

$$\underline{a}_T = -(R + B^T Q B)^{-T} B^T Q A \underline{s}_T$$

$$\underline{a}_T = -\underbrace{(R + B_T^T Q B_T)^{-1} B_T^T Q A_T}_{K_T} \underline{s}_T$$

$$\min_{\{a_{T-1}, a_T\}} \sum_{t=T-1}^{T} \left(\underline{s}_t^T Q \underline{s}_t + a_t^T R \underline{a}_t\right) + s_{T+1}^T Q s_{T+1}$$

$$\text{s.t.} \quad \underline{s}_{t+1} = A \underline{s}_t + B \underline{a}_t$$

$$\min_{a_{T-1}} s_{T-1}^T Q s_{T-1} + a_{T-1}^T R s_{T-1} + \underbrace{V(s_T)}_{\min_{a_T}}$$

$$\text{s.t} \quad \underline{s}_T = A \underline{s}_{T-1} + B \underline{a}_{T-1}$$

Is $V(s_T)$ quadratic in s_T? $\quad \underline{P_{T+1} = Q}$

$$V(s_{T+1}) = \underline{s}_{T+1}^T Q \underline{s}_{T+1} = \underline{s}_{T+1}^T P_{T+1} \underline{s}_{T+1}$$

$$V(s_T) = \min_{a_T} \underline{s}_T^T Q \underline{s}_T + \underline{a}_T^T R \underline{a}_T + \underbrace{s_{T+1}^T Q s_{T+1}}$$

$$= \min_{a_T} \quad '' \quad + \quad '' \quad + s_{T+1}^T P_{T+1} s_{T+1}$$

$$\underline{a}_T = -K_T \underline{s}_T$$

$$V(\underline{s}_T) = s_T^T Q s_T + \underline{s}_T^T K_T^T R K_T \underline{s}_T + s_{T+1}^T P_{T+1} s_{T+1}$$

$$s_T^T P_T s_T = \underbrace{s_T^T (Q + K_T^T R K_T) s_T}_{} + (A s_T + B a_T)^T P_{T+1} \underbrace{(A s_T + B a_T)}_{}$$

$$= \underline{\qquad\qquad} + (A s_T - B K_T s_T)^T P_{T+1} (A s_T - B K_T s_T)$$

$$V(s_T) = s_T^T (Q + K_T^T R K_T) s_T + s_T^T \left\{ (A - B K_T)^T P_{T+1} (A - B K_T) \right\} s_T$$

This is Quadratic in s_T

$$V(s_T) = s_T^T P_T s_T =$$

$$\underline{P_T = Q + K_T^T R K_T + (A - B K_T)^T P_{T+1} (A - B K_T)}$$

$K_{T-1} = ?$

$P_{T-1} = ?$

$\vdots$

$$\min_{a_T} (A_T s_T + B_T a_T)^T Q (A_T s_T + B_T a_T) + a_T^T R a_T$$

$$\min_{a_{T-1}} \underbrace{s_{T-1}^T Q s_{T-1}}_{\times} + a_{T-1}^T R a_{T-1} + \underbrace{V(s_T)}_{s_T^T P_T s_T}$$

$$\min_{a_{T-1}} a_{T-1}^T R a_{T-1} + (A_{T-1} s_{T-1} + B_{T-1} a_{T-1})^T P_T (A_{T-1} s_{T-1} + B_{T-1} a_{T-1})$$

$$\underline{a}_{T-1} = -K_{T-1}\underline{s}_{T-1} =$$

$$K_{T-1} = \left(R + B_{T-1}^{T} P_T B_{T-1}\right)^{-1} B_{T-1} P_T A_{T-1}$$

Discrete Algebraic Riccatti Equations for LQR

$$\begin{cases} P_T = \underbrace{Q + K_T^T R K_T + (A - BK_T)^T P_{T+1} (A - BK_T)} \\ K_T = \left(R + B_T^T P_{T+1} B_T\right)^{-1} B_T P_{T+1} A_T \end{cases}$$

① Dynamic programming

② Quadratic optimization can be solved by taking derivatives and equating them to zero

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Is the unicyle model Linear?

$$\underline{s}_{t+1} = \underbrace{\begin{bmatrix} x_{t+1} \\ y_{t+1} \\ \theta_{t+1} \end{bmatrix}}_{\underline{s}_{t+1}} = \underbrace{\begin{bmatrix} x_t + v_t \cos(\theta_t)\Delta t \\ y_t + v_t \sin(\theta_t)\Delta t \\ \theta_t + \omega_t \Delta t \end{bmatrix}}_{\underline{s}_t} \quad \ldots \quad \underline{a}_t = \begin{bmatrix} v_t \\ \omega_t \end{bmatrix}$$

$$\underline{s}_{t+1} = A\underline{s}_t + B\underline{a}_t$$

Non-linear in $\underline{\theta}_t, \underline{v}_t$

Closest Linear model

single integrator)

$$\underline{a}_t = \begin{bmatrix} v_{x_t} \\ v_{y_t} \end{bmatrix}$$

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \\ \theta_{t+1} \end{bmatrix} = \begin{bmatrix} x_t + v_{x_t}\Delta t \\ y_t + v_{y_t}\Delta t \\ \theta_t + \omega_t \Delta t \end{bmatrix} \qquad \underline{a}_t = \begin{bmatrix} v_{x_t} \\ v_{y_t} \\ \omega_t \end{bmatrix}$$

$$\|\underline{s}_t - \underline{s}_g\|^2 = \underbrace{(\underline{s}_{T+1} - \overset{\text{constant}}{\underset{\text{zero}}{\underline{s}_g}})^T Q_{T+1} (\underline{s}_{T+1} - \underline{s}_g)}$$

$$\tilde{\underline{s}}_t = \begin{bmatrix} \underline{s}_t \\ 1 \end{bmatrix}$$

$$\tilde{\underline{s}}_t^T \tilde{Q}_t \tilde{\underline{s}} = [\underline{s}_t^T \;\; 1] \overbrace{\begin{bmatrix} Q_t & \underline{a} \\ \underline{a}^T & b \end{bmatrix}}^{\tilde{Q}_t} \begin{bmatrix} s_t \\ 1 \end{bmatrix}$$

$$= \underline{s}_t^T Q_t \underline{s}_t + \underset{\sim}{2\underline{a}^T} Q_t \underline{s}_t + \underset{\sim}{b}$$

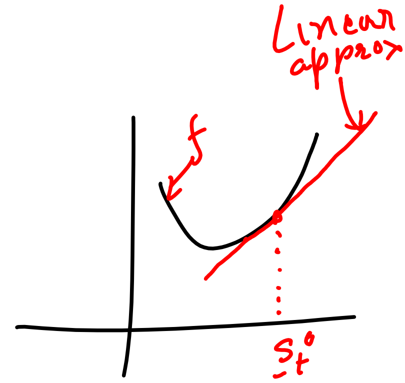
$$(\underline{s}_t - s_g)^T Q (\underline{s}_t - \underline{s}_g)$$

$$= \underline{s}_t^T Q \underline{s}_T - \underset{\sim}{2\underline{s}_g^T} Q \underline{s}_t + \underset{\sim}{\underline{s}_g^T Q \underline{s}_g}$$

$$\tilde{Q}_t = \begin{bmatrix} Q_t & -s_g \\ \underline{s}_g^T & \underline{s}_g^T Q_t \underline{s}_g \end{bmatrix} \qquad \tilde{\underline{s}}_t = \begin{bmatrix} s_t \\ 1 \end{bmatrix}$$

iLQR : Incremental LQR

non-Linear System Dynamics



$$\underline{s}_{t+1} = f(\underline{s}_t, \underline{a}_t)$$

$\downarrow$ Taylor series approximation to convert to Linear System

$$\underline{s}_{t+1} = f(\underline{s}_t^0, \underline{a}_t^0) + \underbrace{J_{\underline{s}_t}\left[f(\underline{s}_t^0, \underline{a}_t^0)\right]}_{A_t}(\underline{s}_t - \underline{s}_t^0)$$

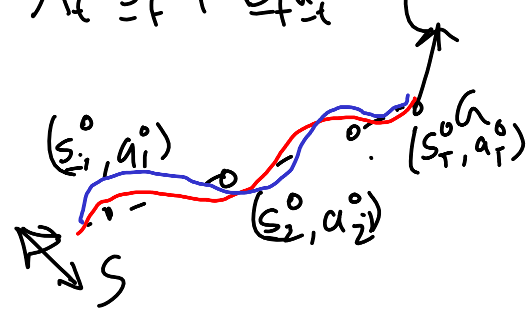
$$+ \underbrace{J_{\underline{a}_t}\left[f(\underline{s}_t^0, \underline{a}_t^0)\right]}_{B_t}(\underline{a}_t - \underline{a}_t^0)$$

$$+ \ldots\ldots$$

$$\underline{s}_{t+1} = A_t \underline{s}_t + B_t \underline{a}_t + \underbrace{\left(f(\underline{s}_t^0, \underline{a}_t^0) - A_t \underline{s}_t^0 - B_t \underline{a}_t^0\right)}_{\text{constant term}}$$

$$\{\underline{a}_t^{1}\}_{t=1}^{T} = \min_{\{a_t\}_{t=1}^{T}} \sim\sim\sim$$

$$\text{s.t} \quad \underline{s}_{t+1} = A_t \underline{s}_t + B_t \underline{a}_t + (\qquad \underline{s}_t^0, \underline{a}_t^0)$$



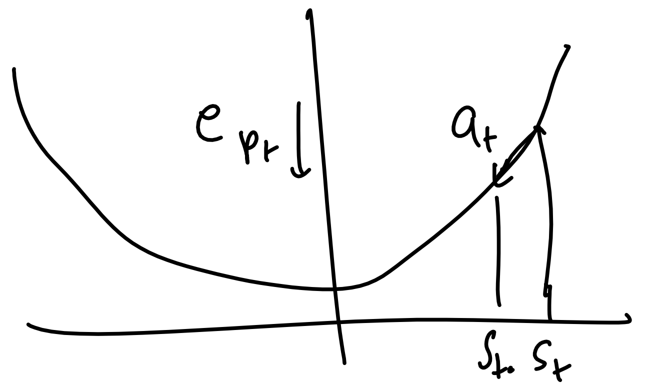
MPC : Model Predictive control

cost to go

$$\left\{ \min_{\{a_t\}_{t=1}^{5}} \underbrace{\sum_{t=1}^{5} c(\underline{s}_t, a_t)} + \underbrace{V(\underline{s}_{t+6})}_{\underline{s}_{t+6}^{T} P_{t+6}\, s_t} \right.$$

$$\underline{s}_{t+1} = f(\underline{s}_t, a_t)$$

$$\underline{a}_t = -K_p \, e_{pt}$$



PID vs LQR vs ILQR

Ethics in Robotics

Tesla
Postives:
1. Hopefully less accidents in the future?

Negatives:
1. Who is responsible, accountable for accidents?
2. Privacy issues?
3. Development and testing issues?
4. Unemployment issues?