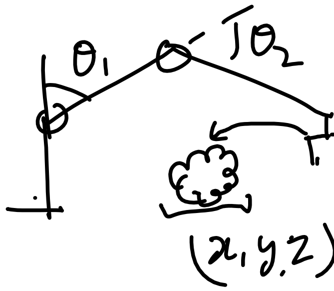
Forward and $\boxed{\text{inverse kinematics}}$

What should the joint ~~angles~~ state/conf. of the robot be so that the end-effector reaches a desired pose?



[How to move the end-effector to a desired pose [position + orientation]]

← end-effector

gripper or suction cup

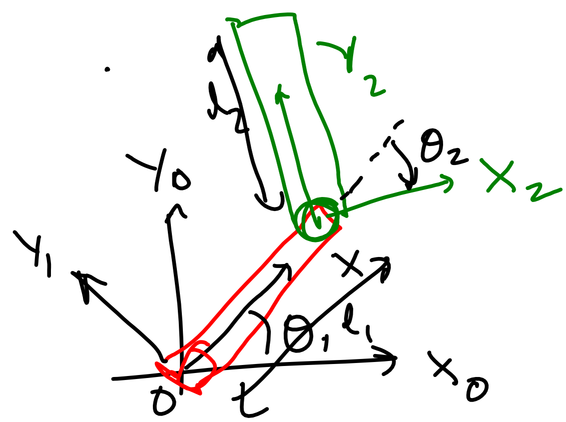
Forward kine.

If my joint ~~angles~~ state/conf. are given what would the pose of end-effector be?

Forward kinematics

$$ {}^0T_2 = {}^0T_1(\theta_1, l_1)\,{}^1T_2(\theta_2, l_2) $$

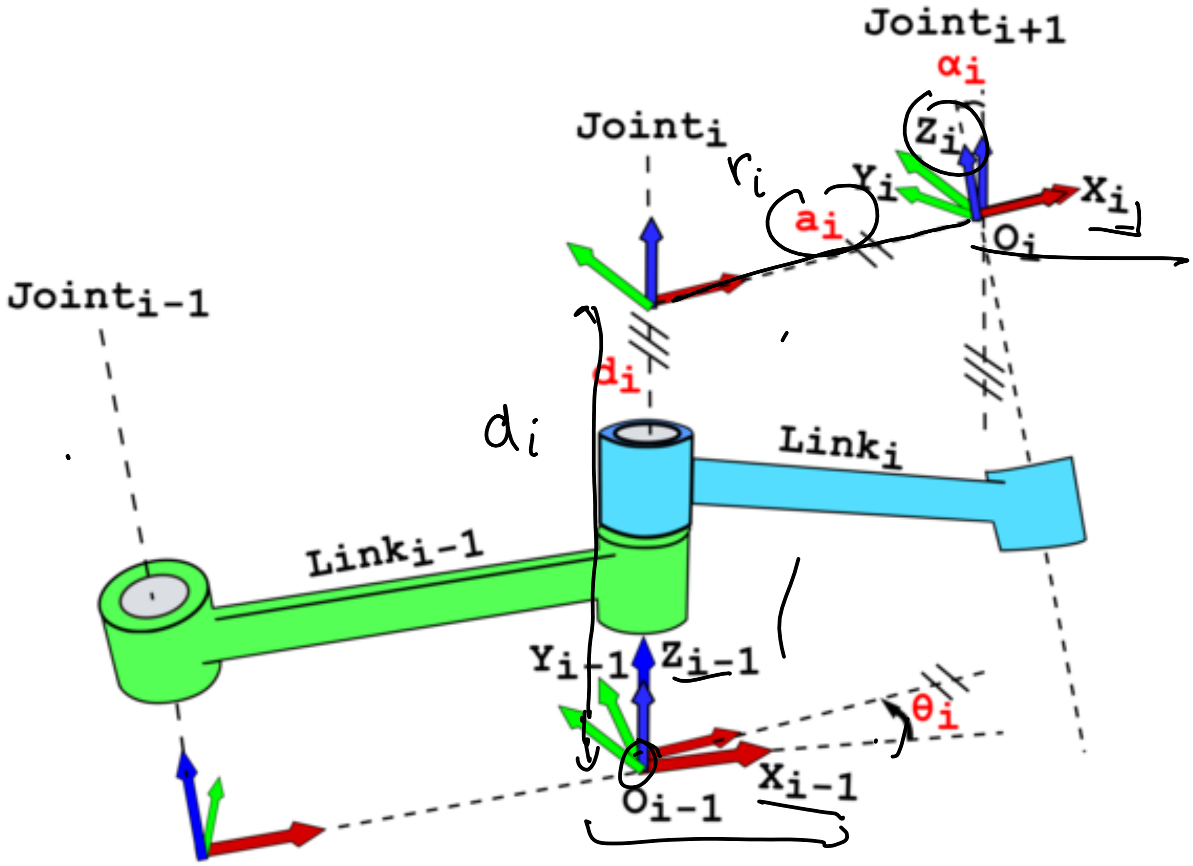
in terms of $\underbrace{\theta_1 \text{ and } \theta_2}_{\text{Given}}$



[Denavit Hartenberg Parameters/Convention]

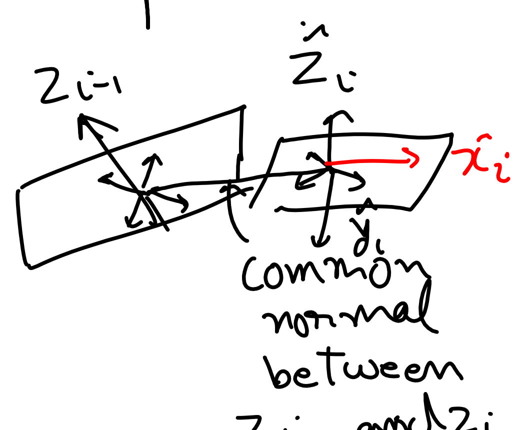
Denavit Hartenberg parameters

https://www.youtube.com/watch?v=rA9tm0gTln8



① $\hat{z}_{i-1}, \hat{z}_i$ aligned along the axis of rotation

② Choose $\hat{x}_i$ along the common normal between $\hat{z}_{i-1}, \hat{z}_i$



③ $\hat{y}_i = \hat{z}_i \times \hat{x}_i$

a) θ_i = Rotation along z_{i-1} (to align x_{i-1} with x_i)

b) d_i = translation along z_{i-1} (to align the origins)

c) α_i = Rotation along x_i (to align z_{i-1} with z_i)

d) r_i/a_i = translation along x_i (to align the origin)

(a) and (b) can be swapped
(c) and (d)

But Tranformation along z goes first
followed by " " x

$$\underbrace{{}^{i-1}}_{\text{target}} T_{\underbrace{i}_{\text{source}}} = {}^{i-1}T_{zi} \; {}^{i-1}T_{xi}$$

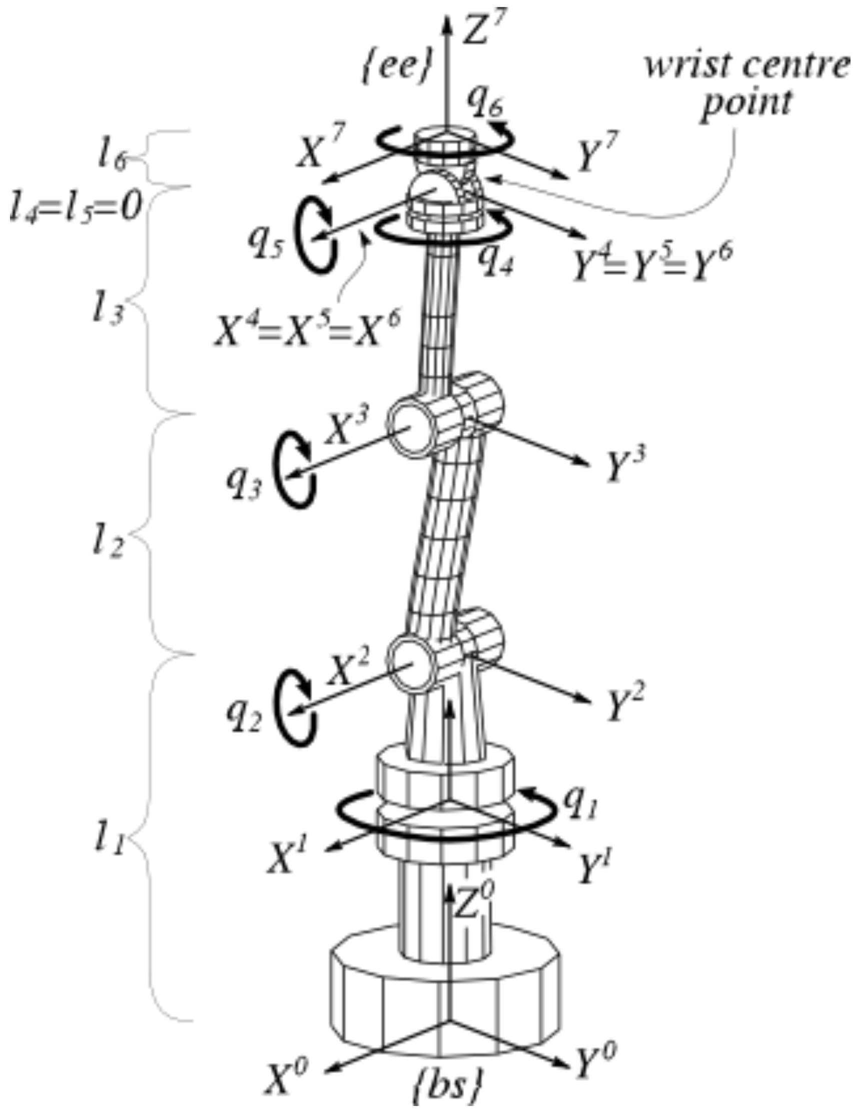
Tranformations are applied right to left

$${}^{i-1}T_{xi} = \begin{bmatrix} \begin{array}{ccc|c} 1 & 0 & 0 & r_i \\ 0 & \cos\alpha_i & -\sin\alpha_i & 0 \\ 0 & \sin\alpha_i & \cos\alpha_i & 0 \\ \hline 0 & 0 & 0 & 1 \end{array} \end{bmatrix}_{4\times 1}$$

3×3 $\quad$ 3×1

$${}^{i-1}T_{zi} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 & 0 \\ \sin\theta_i & \cos\theta_i & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times 1}$$

θ_i, d_i



Z^7
{ee}
wrist centre point
q_6
l_6
X^7
Y^7
$l_4=l_5=0$
q_5
q_4
$Y^4=Y^5=Y^6$
l_3
$X^4=X^5=X^6$
X^3
q_3
Y^3
l_2
X^2
q_2
Y^2
q_1
l_1
X^1
Y^1
Z^0
X^0
Y^0
{bs}

Numerical solutions to IK problems: Jacobian inverse technique

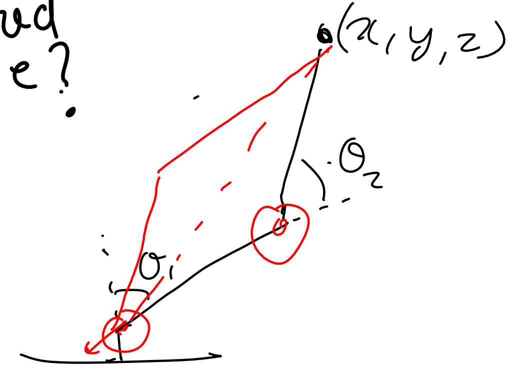
Forward and inverse kinematics

Inverse kinematics

↳ closed form solution

↳ Numerical/Iterative solutions

What should the joint ~~angles~~ state/conf. of the robot be so that the end-effector reaches a desired pose?



only polynomials of degree ≤ 5 have closed form solutions

$\cancel{\theta_1 = x_1^2 + y_2 + \cos^{-1}\left(\frac{z}{y}\right)}$

$\cos(\theta_1)$ $\quad$ $\underbrace{\sin(\theta_1)}_{x_1}$

$\sqrt{1 - x_1^2}$

Newton-Raphson method (Gradient descent) (optimization solution)

Suppose a function to find $y = f(x)$

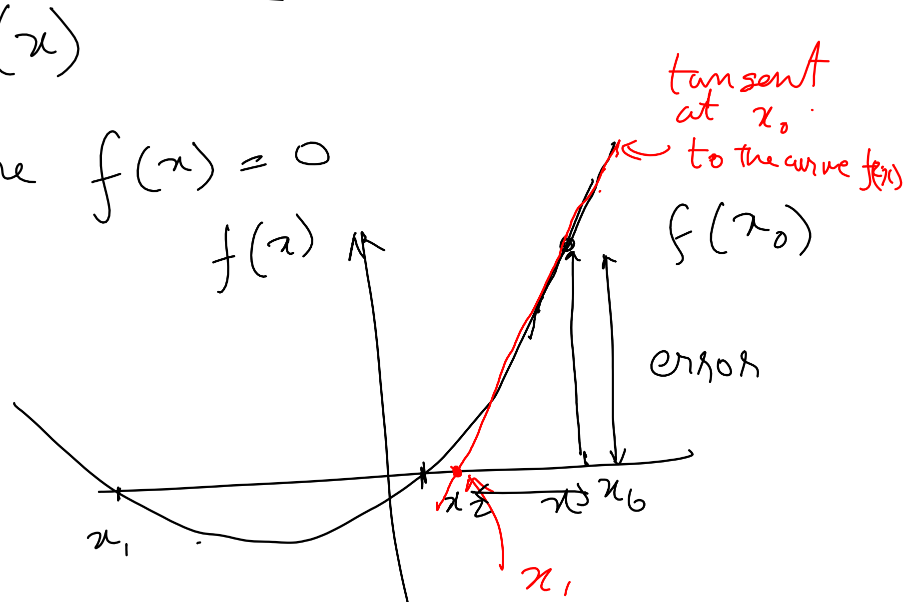
we want ~~all~~ any x where $f(x) = 0$

① Initial guess

x_0

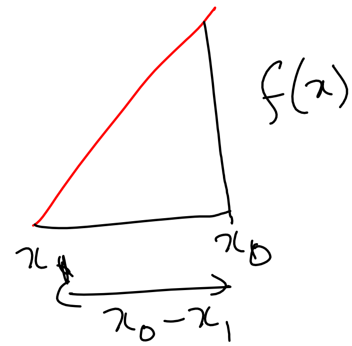
↑

iteration



② Improve the initial guess

$$f'(x)\Big|_{x_0} = \frac{f(x_0)}{x_0 - x_1}$$



$$\Rightarrow \quad x_0 - x_1 = \left[f'(x_0)\right]^{-1} f(x_0)$$

$$x_1 = x_0 - \left[f'(x_0)\right]^{-1} f(x_0)$$

② Repeat

$$x_2 = x_1 - \left[f'(x_1)\right]^{-1} f(x_1)$$

$$x_n = x_{n-1} - \left[f'(x_{n-1})\right]^{-1} f(x_{n-1})$$

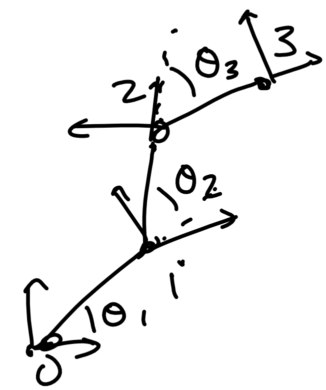
---

find the square root of 2

$$f(z) = z^2 - 2 = 0$$

$$z_n = z_{n-1} - \frac{\left(z_{n-1}^2 - 2\right)}{2 z_{n-1}}$$

Forward Kinematics



$$\underbrace{{}^0T_3}_{\text{Given}} = {}^0T_1(\theta_1)\;{}^1T_2(\theta_2)\;{}^2T_3(\theta_3)$$

Find

$$\underbrace{({}^0T_3)^{-1}\,{}^0T_1(\theta_1)\;{}^1T_2(\theta_2)\;{}^2T_3(\theta_3) - I_{4\times4}} = 0_{4\times4}$$

$$\underbrace{F_{4\times4}(\theta_1, \theta_2, \theta_3)} = 0_{4\times4}$$

$$\underline{f}_{16\times1}\left(\underbrace{\begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{bmatrix}_{3\times1}}_{\text{vector}}\right) = \begin{bmatrix}0\\ \\ 0\end{bmatrix}_{16\times1}$$

$f(\underline{\theta})$ is a vector function

$$\underline{f}_{16\times1}(\underline{\theta}) = \begin{bmatrix} f_1(\underline{\theta})\\ f_2(\underline{\theta})\\ \vdots\\ f_{16}(\underline{\theta})\end{bmatrix}$$

vector-valued vecto function

What's a Jacobian matrix

is ~~a~~ derivative of vector-valued vector function

many

$$
J\left[\underline{f}_{16\times 1}(\underline{\theta}_{3\times 1})\right] =
\begin{array}{cc}
 & \text{inputs} \\
\text{outputs} &
\begin{bmatrix}
\frac{\partial f_1}{\partial \theta_1} & \frac{\partial f_1}{\partial \theta_2} & \cdots & \frac{\partial f_1}{\partial \theta_3} \\
\frac{\partial f_2}{\partial \theta_1} & \ddots & & \\
\vdots & & \ddots & \\
\frac{\partial f_{16}}{\partial \theta_1} & & & \ddots
\end{bmatrix}
\end{array}
$$

Hand-waviness

scalar Newton Raphson

generalizes to (vector Newton – Raphson) ✗

Gauss Newton method

(1) Initial guess $\underline{\theta}_0$

Psuedo-inverse dagger

(2) $\underline{\theta}_1 = \underline{\theta}_0 - J\left[\underline{f}_{16\times 1}(\underline{\theta}_{3\times 1})\right]^{\dagger} \underline{f}_{16\times 1}(\underline{\theta}_{3\times 1})$

$$x_1 = x_0 - \left[f'(x_0)\right]^{-1} f(x_0)$$

(3) $$\underline{\theta}_n = \underline{\theta}_{n-1} - J\left[\underline{f}(\underline{\theta}_{n-1})\right]^{+} \underline{f}(\underline{\theta}_{n-1})$$

(1) What is a Psudo inverse?

→ (2) How can we compute the Jacobian?

(3) What is the relationship (similarities and differences) b/w Newton-Raphson, Gauss Newton, Gradient descents

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Problem 4 of Midterm helps in computing derivative of rotation matrices

$$K^3 = -K$$

$$R(\theta, \hat{k}) = I_{3\times3} + \sin\theta\, K + (1-\cos\theta)K^2$$

$$\frac{\partial}{\partial\theta} R(\theta, \hat{k}) = 0 + \cos\theta\, \underset{\sim}{K} + (0 + \sin\theta)K^2$$

$$= -\cos\theta\, K^3 + \sin\theta\, K^2$$

$$= K(-\cos\theta\, K^2 + \sin\theta\, K)$$

$$= K(I - I - \cos\theta\, K^2 + \sin\theta\, K)$$

$$= K\left(I - \cos\theta K^2 + \sin\theta K\right) - K$$

$$= K\left(I + K^2 - K^2 - \cos\theta K^2 + \sin\theta K\right) - K$$

$$= K\left(I + (1-\cos\theta)K^2 + \sin\theta K\right) - K - K^3$$

$$= K R(\theta, \hat{k}) - \cancel{K} + \cancel{K}$$

$$\boxed{\frac{\partial}{\partial\theta} R(\theta, \hat{k}) = K R(\theta, \hat{k})}$$

$$\boxed{\frac{d}{dx} f(x) = a f(x)}$$

$$\downarrow \uparrow$$

scalar matrix

$$R(\theta, \hat{k}) = \exp(\theta K) = \frac{I}{1} + \frac{\theta K^1}{1!} + \frac{\theta^2 K^2}{2!} + \frac{\theta^3 K^3}{3!} + \dots \infty$$

matrix exponential

$$\frac{\partial}{\partial\theta} R(\theta, \hat{k}) = 0 + \frac{K^1}{1!} + \frac{2\theta K^2}{2!} + \frac{3\theta^2 K^3}{3!} + \dots$$

$$= K\left(\frac{I}{1!} + \frac{\theta K}{1!} + \frac{\theta^2 K^2}{2!} + \dots\right)$$

$$T = \begin{bmatrix} R(\theta, \hat{k})_{3\times3} & \underline{t}_{3\times1} \\ \underline{0}^T & 1 \end{bmatrix}$$

$$\frac{\partial T}{\partial\theta} = \begin{bmatrix} K R(\theta, \hat{k}) & 0 \\ 0 & 0 \end{bmatrix}$$

$$K = [\hat{\underline{k}}]_\times$$

Jacobian of forward kinematics

$$ {}^0T_3 = \underbrace{{}^0T_1(\theta_1)\ {}^1T_2(\theta_2)\ {}^2T_3(\theta_3)} $$

$$ \boxed{\frac{\partial\, {}^0T_3}{\partial\theta_1} = \left[\frac{\partial\, {}^0T_1(\theta_1)}{\partial\theta_1}\right] {}^1T_2(\theta_2)\ {}^2T_3(\theta_3)} $$

$$ \frac{\partial}{\partial\theta} A(\theta)\, B(\theta) = \left[\frac{\partial A(\theta)}{\partial\theta}\right] B(\theta) + A(\theta)\left[\frac{\partial B(\theta)}{\partial\theta}\right] $$

scalar

$$ C(\theta) = A(\theta)\, B(\theta) $$

$$ \begin{bmatrix} c_{11} & \cdots \\ \vdots & \end{bmatrix} = \begin{bmatrix} a_{11}(\theta) & \cdots & a_{1n}(\theta) \\ & & \end{bmatrix} \begin{bmatrix} b_{11}(\theta) & \\ \vdots & \\ b_{n1}(\theta) & \end{bmatrix} $$

$$ c_{11} = a_{11}(\theta)\, b_{11}(\theta) + a_{12}(\theta)\, b_{21}(\theta) + \dots + a_{1n}(\theta)\, b_{n1}(\theta) $$

$$ \frac{\partial c_{11}}{\partial\theta} = \left\{ \left[\frac{\partial a_{11}(\theta)}{\partial\theta}\right] b_{11}(\theta) + a_{11}(\theta)\left[\frac{\partial}{\partial\theta} b_{11}(\theta)\right] \right\} + \dots $$

$$\left\{\frac{\partial ({}^{0}T_3)}{\partial \theta_2}\right\}_{4\times 4} = {}^{0}T_1(\theta_1)\left[\frac{\partial}{\partial \theta_2}\, {}^{1}T_2(\theta_2)\right] {}^{2}T_3(\theta_3)$$

$$\mathrm{vec}({}^{0}T_3)_{4\times 4} = [\ ]_{16\times 1} = \text{row-wise vectorization operator}$$

$$\underset{\underline{\theta}}{J}\left[\mathrm{vec}({}^{0}T_3)\right] = \begin{bmatrix} | & | & | \\ \frac{\partial\, \mathrm{vec}({}^{0}T_3)}{\partial \theta_1} & \frac{\partial\, \mathrm{vec}({}^{0}T_3)}{\partial \theta_2} & \frac{\partial\, \mathrm{vec}({}^{0}T_3)}{\partial \theta_3} \\ | & | & | \end{bmatrix}_{16\times 3}$$

12×3

$$\frac{\partial}{\partial \theta_1}\, {}^{0}T_1(\theta_1) = \left[\frac{\partial T_z(\theta, d)}{\partial \theta_1}\right]\left[T_x(r,\alpha)\right]$$

$$= \begin{bmatrix} K_z R_z(\theta) & \underline{0}'_{3\times} \\ \boxed{\underline{0}^T_{1\times 3} \quad 0'} \end{bmatrix} T_x(r,\alpha)$$

$$ {}^{0}T_3 = \begin{bmatrix} & \\ 0^T_{1\times 3} & 1 \end{bmatrix}$$

we know how to compute Jacobian ✓

$$\underline{\theta}_n = \underline{\theta}_{n-1} - J\left[\underline{f}(\underline{\theta}_{n-1})\right]^{\dagger} \underbrace{\underline{f}(\underline{\theta}_{n-1})}_{\text{error}}$$

$$\underline{f}(\underline{\theta}_{n-1}) = \mathrm{vec}\left({}^{0}\underline{T}_3(\underline{\theta}_{n-1}) - \boxed{{}^{0}T_3^{*}}\right)$$

Desired end-effector pose

equivalent in terms of roots

$$\underline{f}(\underline{\theta}_{n-1}) = \mathrm{vec}\left(\left[{}^{0}T_3^{*}\right]^{-1}\, {}^{0}T_3(\underline{\theta}_{n-1}) - \underline{I}_{4\times4}\right)$$

$$J\left[\underline{f}(\underline{\theta}_{n-1})\right] = J\left[\mathrm{vec}\left[{}^{0}T_3(\underline{\theta}_{n-1})\right]\right]$$

$$\underline{\theta}_n = \underline{\theta}_{n-1} - J\left[\mathrm{vec}\left({}^{0}T_3(\theta_{n-1})\right)\right]^{\dagger} \mathrm{vec}\left({}^{0}\underline{T}_3(\underline{\theta}_{n-1}) - \boxed{{}^{0}T_3^{*}}\right)$$

Pseudo Inverse

systems of eqns in multiple variable

(1) $\boxed{\begin{aligned} 2x + 3y &= 4 \\ 5x + 7y &= 9 \end{aligned}}$ $\begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \end{bmatrix}$ 1 solution

$A\underline{x} = \underline{b}$ $\qquad \underline{x} = A^{-1}\underline{b}$

No. of eqns = No of unknowns

(2) No of eqns > No of unknowns] no solution

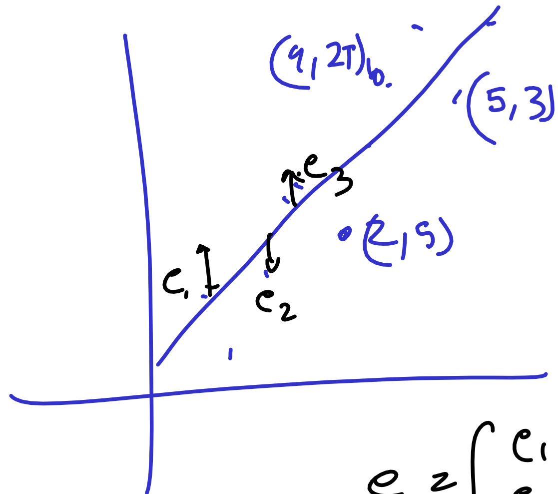
(3) " < "] multiple solution

(2) No of eqns > No of unknows

$$\begin{bmatrix} 2x + 3y \\ 5x + 7y \\ 6x + 13y \end{bmatrix} \overset{?}{=} \begin{bmatrix} 4 \\ 9 \\ 13 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 2 & 3 \\ 5 & 7 \\ 6 & 13 \end{bmatrix}}_{A_{3\times 2}} \begin{bmatrix} x \\ y \end{bmatrix} \overset{?}{=} \begin{bmatrix} 4 \\ 9 \\ 13 \end{bmatrix}$$

$$A_{3\times 2}\,\underline{x}_{2\times 1} = \underline{b}_{3\times 1}$$



$$\underline{e} = \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ \vdots \end{bmatrix}$$

$$\left.\begin{aligned} 5 &= 2m + c \\ 27 &= 9m + c \\ 3 &= 5m + c \end{aligned}\right] \longrightarrow$$ Least square linear regression

---

# of eqns, # of unknowns

$$A_{m\times n}\,\underline{x}_{n\times 1} \approx \underline{b}_{m\times 1} \qquad m > n$$

Linear Least square

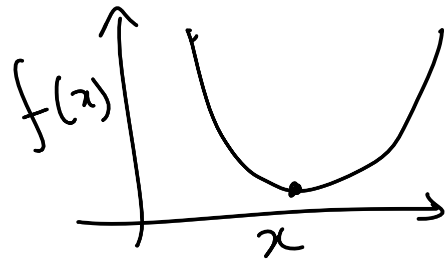
$$\begin{bmatrix} & \\ & \\ & \end{bmatrix}_{m\times n}$$

tall matrix

$$\boxed{\min_{\underline{x}} \|\underbrace{A\underline{x} - \underline{b}}_{\underline{e}}\|^2}$$

$$|\underline{e}|^2 = e_1^2 + e_2^2 + e_3^2 + \ldots$$

Background



① $\frac{d}{dx} f(x) = 0$

$\downarrow$ vector

Scalar valued vector function $f(\underline{x})$

$$J_{\underline{x}}\left[f(\underline{x})\right] = \underbrace{\left[\frac{\partial}{\partial x_1} f(\underline{x}), \; \cdots\cdots \; \frac{\partial}{\partial x_n} f(\underline{x})\right]}_{\text{Gradient}} = \underline{0}^T$$

② $\|\underline{c}\| = \sqrt{\underline{c}^T \underline{c}}$

③ Chain rule with Jacobians

$$\frac{d}{dx} f(g(x)) = \frac{d}{dg} f(g) \, \frac{d}{dx} g(x)$$

$$f(\underline{y}) : \mathbb{R}^{n} \longrightarrow \mathbb{R}$$

$$\underline{g}(\underline{x}) : \mathbb{R}^{m} \longrightarrow \mathbb{R}^{n}$$

$$J_{\underline{x}}\left[f(\underline{g}(\underline{x}))\right] = \underbrace{J_{\underline{g}}\left[f(\underline{g})\right]}_{1 \times n} \; \underbrace{J_{\underline{x}}\left[\underline{g}(\underline{x})\right]}_{n \times m}$$

Example $n = 2$, $m = 1$

$$J_x\left[f(\underline{g}(x))\right] = \left[\frac{\partial}{\partial x} f(\underline{g}(x))\right]_{1\times 1}$$

$$\to \left[\frac{\partial}{\partial x} f\left(\begin{bmatrix} g_1(x) \\ g_2(x) \end{bmatrix}\right)\right] = \frac{\partial}{\partial x} f(g_1(x), g_2(x))$$

$$= \frac{\partial}{\partial g_1} f(\underline{g}(x)) \frac{\partial g_1(x)}{\partial x} + \frac{\partial}{\partial g_2} f(\underline{g}(x)) \frac{\partial g_2(x)}{\partial x}$$

$$= \underbrace{\begin{bmatrix} \frac{\partial}{\partial g_1} f(\underline{g}(x)) & \frac{\partial}{\partial g_2} f(\underline{g}(x)) \end{bmatrix}}_{J_{\underline{g}}[f(\underline{g})]} \underbrace{\begin{bmatrix} \frac{\partial g_1(x)}{\partial x} \\ \frac{\partial g_2(x)}{\partial x} \end{bmatrix}}_{J_x[\underline{g}(x)]}$$

(4) $$J_{\underline{x}}\left[\underset{\uparrow\, f(\underline{x})}{\underline{a}^T \underline{x}}\right] = \left[\frac{\partial(\underline{a}^T \underline{x})}{\partial x_1} \cdots\cdots \frac{\partial \underline{a}^T \underline{x}}{\partial x_n}\right]_{1\times n} \qquad \underline{x} \in \mathbb{R}^{n\times 1}, \quad \underline{a} \in \mathbb{R}^{n\times 1}$$

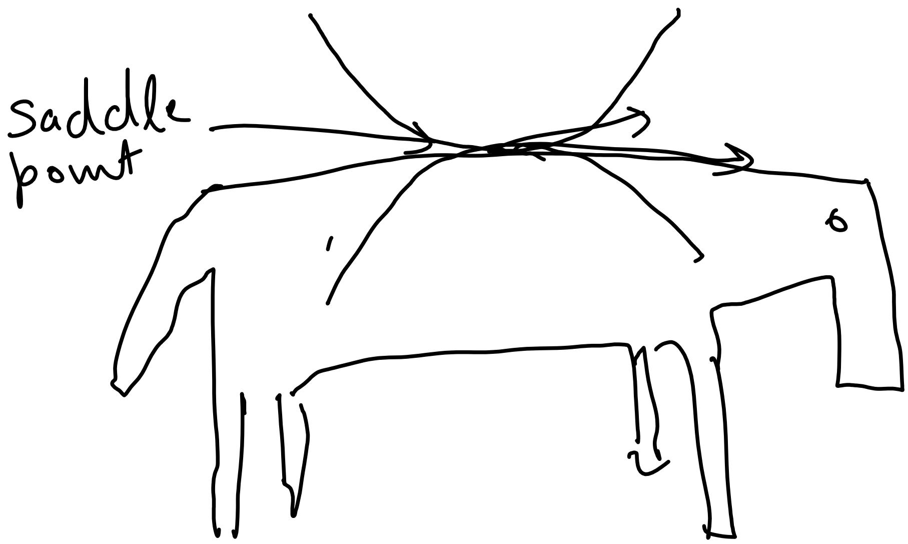
$$\underline{a}^T \underline{x} = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$$

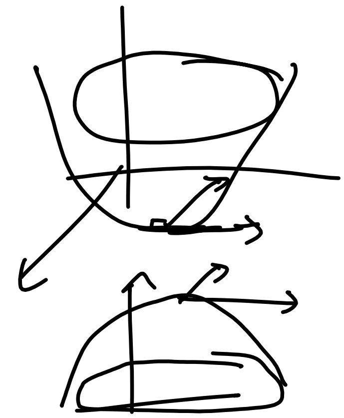
$$J_{\underline{x}}[\underline{a}^T \underline{x}] = [a_1, a_2 \; \cdots \; a_n] = \underline{a}^T$$

$$J_{\underline{x}}[A\underline{x}] = J_{\underline{x}}\begin{bmatrix} \underline{a}_1^T \underline{x} \\ \vdots \\ \underline{a}_m^T \cdot \underline{x} \end{bmatrix} = \begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix} = A \qquad A = \begin{bmatrix} - \underline{a}_1^T - \\ - a_2^T - \\ \vdots \\ - a_m^T - \end{bmatrix}$$

(1) The minimum/maximum/saddle points of a function $f(\underline{x})$ occur when

$$J_{\underline{x}}[f(\underline{x})] = \underline{0}^T$$





(2) $\|\underline{e}\| = \sqrt{\underline{e}^T \underline{e}}$

(3) $J_{\underline{x}} f(\underline{g}(\underline{x})) = J_{\underline{g}}[f(\underline{g})] \, J_{\underline{x}}[\underline{g}(\underline{x})]$

(4) $J_{\underline{x}}[A\underline{x}] = A$

$$\min_{\underline{x}} \|A\underline{x} - b\|^2$$

$$\min_{\underline{x}} (A\underline{x} - b)^T (A\underline{x} - b)$$

Using (2)

(5) $J_{\underline{x}}[\underline{x}^T A \underline{x}] = \underline{x}^T (A + A^T)$

Quadratic form

Proof left as an exercise

$$\underline{x}^T A \underline{x} = \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= a_{11} x^2 + a_{12} xy + a_{21} xy + a_{22} y^2$$

if

A is symmetric matrix, $A^T = A$

$$J_x [\underline{x}^T A \underline{x}] = x^T (A^T + A) = 2\underline{x}^T A$$

. Special case

$$J_{\underline{x}} [\underline{x}^T \underline{x}] = J_{\underline{x}} [\underline{x}^T \underline{I} \underline{x}] = 2\underline{x}^T$$

—

$$\min_{\underline{x}} \underbrace{(A\underline{x} - b)}_{\underline{e}(\underline{x})}^T (A\underline{x} - b)$$

$$\min_{\underline{x}} [\underline{e}(\underline{x})^T \underline{e}(\underline{x})]$$ where $e(\underline{x}) = A\underline{x} - \underline{b}$

$$J_{\underline{x}} [\underline{e}(\underline{x})^T \underline{e}(\underline{x})] = \underline{0}^T$$

$$J_{\underline{e}} [\underline{e}(\underline{x})^T \underline{e}(\underline{x})] \, J_{\underline{x}} [\underline{e}(\underline{x})] = \underline{0}^T$$

$$2\underline{e}(\underline{x})^T J_{\underline{x}} [A\underline{x} - \underline{b}] = \underline{0}^T$$

$$2\underline{e}(\underline{x})^T (J_{\underline{x}} [A\underline{x}] - J_{\underline{x}} [\underline{b}]) = \underline{0}^T$$

$$2\,\underline{e}(\underline{x})^T\left[A - 0_{m\times n}\right] = \underline{0}^T$$

$$2(A\underline{x} - \underline{b})^T A = \underline{0}^T$$

$$\Rightarrow 2\left((A\underline{x})^T - \underline{b}^T\right) A = \underline{0}^T$$

$$\Rightarrow 2\left(\underline{x}^T A^T - \underline{b}^T\right) A = \underline{0}^T$$

$$\Rightarrow 2\left(\underline{x}^T A^T A - b^T A\right) = \underline{0}^T$$

$$\Rightarrow x^T \underbrace{A^T A} = b^T A$$

$$\Rightarrow \underline{x}^T = b^T A (A^T A)^{-1}$$

$$\Rightarrow \boxed{\underline{x} = \underbrace{(A^T A)^{-1} A^T}_{A^{+}} \underline{b}}$$

for $m > n$

$A \in \mathbb{R}^{m\times n}$

$b \in \mathbb{R}^{m\times 1}$

$x \in \mathbb{R}^{n\times 1}$ $\quad m > n$

$5 = 2m + c$

$10 = 4m + 2c$