ECE 417/598: Direct Linear Transform

Vikas Dhiman.

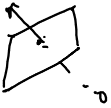
March 23, 2022

Implicit equation and parameteric representation of 3D plane

Implicit equation of 3D plane

$$\underline{\underline{\mathbf{p}}}^\top \underline{\mathbf{x}} = 0 \qquad \mathbf{p} \in \mathbb{P}^4, \underline{\mathbf{x}} \in \mathbb{P}^4$$

$$\mathbf{p} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \end{bmatrix} \qquad \underline{x} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} \Rightarrow$$



$$\underbrace{\begin{bmatrix} x_1^T \\ x_2^T \\ x_3^T \\ \vdots \\ x_n^T \end{bmatrix}}_{A} \underline{p} = 0 \rightarrow$$

Parameteric representation of 3D plane

$$\lambda_2 \underline{v}_2 + \lambda_3 \underline{v}_3 + \lambda_4 \underline{v}_4$$

$$\underline{\mathbf{x}} = \underline{\underline{\mathbf{v}_2}} + t_1 \underline{\underline{\mathbf{v}_3}} + t_2 \underline{\underline{\mathbf{v}_4}}$$

where $t_1, t_2 \in \mathbb{R}$ are the free parameters.

$$\underbrace{p^T}_{A_{1\times4}} = U \Sigma V^T$$

$$V_{4\times4} = [\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4]$$

$$(v_1, \ldots, v_4)$$

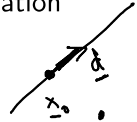
$$N(A) \Rightarrow R(A^T)$$

Implicit equation and parameteric representation of a 3D line

Parameter representation of a 3D line

$\underline{x} = \lambda$

$$\underline{\mathbf{x}} = \lambda \underline{\mathbf{d}} + \underline{\mathbf{x}}_0,$$

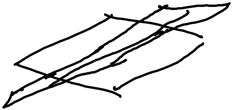


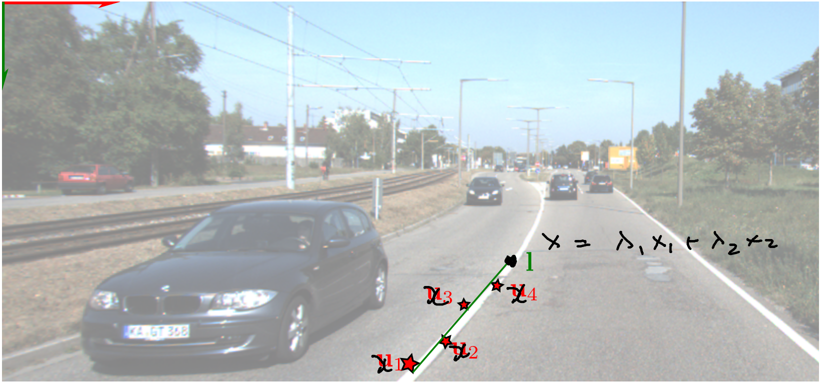
where $\lambda \in \mathbb{R}$ is the free parameter, $\underline{\mathbf{x}}_0 \in \mathbb{P}^3$ is a point on the line and $\underline{\mathbf{d}} \in \mathbb{P}^3$ is the direction of the line.

Implicit equations of a 3D line

$$\mathbf{p}_1^\top \underline{\mathbf{x}} = 0, \quad \mathbf{p}_2^\top \underline{\mathbf{x}} = 0, \quad (1)$$

where $\mathbf{p}_1, \mathbf{p}_2, \underline{\mathbf{x}} \in \mathbb{P}^3$.





$$\underline{\mathbf{x}}_1 = [100, 98, 45, 1]^\top$$
$$\underline{\mathbf{x}}_2 = [105, 95, 46, 1]^\top$$
$$\underline{\mathbf{x}}_3 = [107, 90, 47, 1]^\top$$
$$\underline{\mathbf{x}}_4 = [110, 85, 43, 1]^\top$$

$A = [\underline{x}_1, \underline{x}_2, x_3, x_4]$

column space

Find the 3D line such that it is the "closest line" passing through $\mathbf{x}_1, \ldots, \mathbf{x}_4 \in \mathbf{P}^3$.

Parameteric representation through Range space

$$\underline{l} = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \\ l_4 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} \underline{x}_1^T \\ \underline{x}_2^T \\ \underline{x}_3^T \\ \vdots \\ \underline{x}_4^T \end{bmatrix}}_{A} \underline{l} = 0$$

$$A \underline{l} = 0$$

$$A = U \Sigma V^T$$

$$V = [\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4]$$

$$\underline{l} = \lambda_1 \underline{v}_3 + \lambda \underline{v}_4$$

$$\underline{p}_1 = \underline{v}_3 \quad , \quad \underline{p}_2 = \underline{v}_4$$

$$\begin{bmatrix} \underline{v}_3^T \underline{x} = 0 \\ \underline{v}_4^T \underline{x} = 0 \end{bmatrix}$$ implicit eqn of line in 3D

$r = \text{rank}(A)$

$R(A^T) = \text{Range of } A^T$

$= \text{row space of } A^T$ $\quad A \in \mathbb{R}^{m \times n}$

$$\begin{bmatrix} \underset{R(A)}{U_{1..r}} & \overset{N(A^T)}{U_{m-r..m}} \end{bmatrix} \Sigma \begin{bmatrix} \overset{R(A^T)}{V_{1..r}^T} \\ \vdots \\ \underset{N(A)}{V_{n-r..n}^T} \end{bmatrix}$$

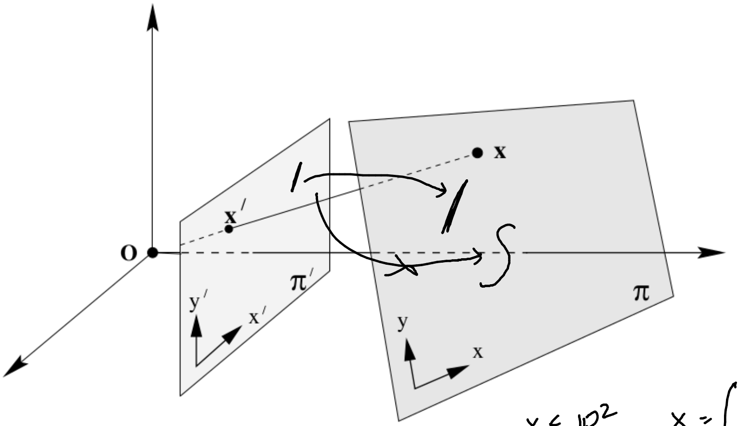
$V_{1..r} = R(A^T)$

$$\underline{x} = \lambda_1 \underline{v}_1 + \lambda_2 \underline{v}_2$$

$$\underline{x} = \underline{v}_1 + \frac{\lambda_2}{\lambda_1} \underline{v}_2$$

$$\underline{x} = \underline{v}_1 + t \underline{v}_2 \qquad t \in \mathbb{R}$$

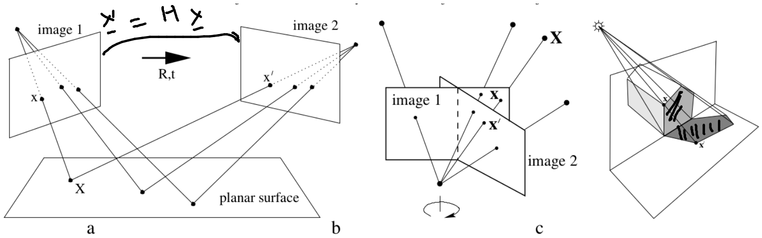
Homography

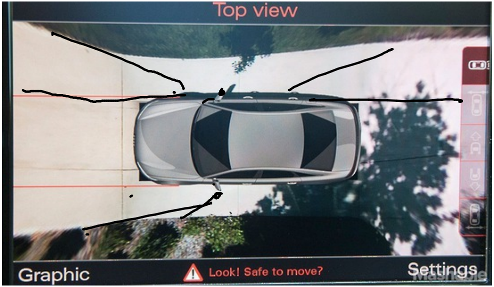


$$\underline{x}' = H\,\underline{x}$$

$\underline{x} \in \mathbb{P}^2$ $\quad \underline{x} = \begin{bmatrix} x \\ y \\ w \end{bmatrix}$

Examples of Homography

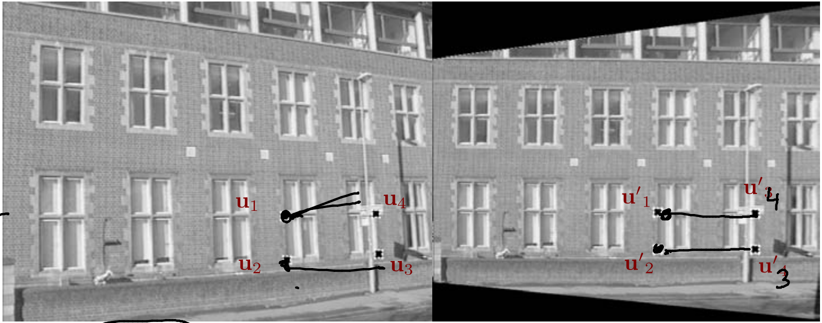



Top view
Graphic
Look! Safe to move?
Settings

Computing Homography

Given → Desired

98



x-axis

y-axis

Lab $\boxed{K, R, t}$

H?

$$\underline{\mathbf{u}}_1 = [100, 98, 1]^\top \qquad \underline{\mathbf{u}}_2 = [102, 95, 1]^\top$$

$$\underline{\mathbf{u}}_3 = [107, 90, 1]^\top \qquad \underline{\mathbf{u}}_4 = [110, 85, 1]^\top$$

$$\underline{\mathbf{u}}'_1 = [100, 98, 1]^\top \qquad \underline{\mathbf{u}}'_2 = [102, 95, 1]^\top$$

$$\underline{\mathbf{u}}'_3 = [107, 98, 1]^\top \qquad \underline{\mathbf{u}}'_4 = [110, 85, 1]^\top$$

$\underline{U}' = H \underline{u}$

Find H such that $\underline{\mathbf{u}}' = H\underline{\mathbf{u}}$ for any point on one image to another image.

2D homography

Given a set of points $\underline{\mathbf{u}}_i \in \mathbb{P}^2$ and a corresponding set of points $\underline{\mathbf{u}}'_i \in \mathbb{P}^2$, compute the projective transformation that takes each $\underline{\mathbf{u}}_i$ to $\underline{\mathbf{u}}'_i$. In a practical situation, the points $\underline{\mathbf{u}}_i$ and $\underline{\mathbf{u}}'_i$ are points in two images (or the same image), each image being considered as a projective plane $\mathbb{P}^2$.

find

$$A\underline{x} = \underline{b}$$

given

given

linear system of equations

$$\underline{u}_1' = H\underline{u}$$

perspective transform

find

$$\underline{v}_1' = H\underline{u}$$

given

given

$$\underline{u}_1 = \begin{bmatrix} x \\ y \\ w \end{bmatrix} \qquad \underline{u}_1' = \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} h_1 & h_2 & h_3 \\ \vdots & & \\ \vdots & & \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

Solving for Homography

$h_i = ?$

Linear

$$\begin{cases} x' = h_1 x + h_2 y + h_3 w \\ y' = h_4 x + h_5 y + h_6 w \\ w' = h_7 x + h_8 y + h_9 w \end{cases}$$

$\left.\begin{matrix} x, x' \\ y, y' \\ w, w' \end{matrix}\right\}$ given

polynomial $\boxed{x^2, y^2, xy}$, xyz ; $x^3, y^3 z$

Linear h_1, h_2, h_3 x xy, yz

Quadratic

unknown

$A\underline{x} = \underline{b}$

$$x' = h_1 x + h_2 y + h_3 w$$

$$y' = h_4 x + h_5 y + h_6 w$$

$$w' = h_7 x + h_8 y + h_9 w$$

$$\overset{9}{\longleftrightarrow}$$

$$\underbrace{\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix}}_{b} = \begin{bmatrix} \underbrace{x \quad y \quad w}_{u_1^T} & 0 & 0 & 0 \\ 0 \quad 0 \quad 0 & \underbrace{x \quad y \quad w}_{u_2^T} & & \\ & & \ddots & \end{bmatrix}_{x\,y\,w} \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ \vdots \\ h_9 \end{bmatrix}$$

because $\underline{u} \in \mathbb{P}^2$

3 eqn 9 unknowns

$\downarrow$

2 eqn 9 unknowns

$\downarrow$ because $H \in \mathbb{P}^2 \times \mathbb{P}^2$

2 eqn 8 unknowns

per point $\underline{u}_1$

$$\left.\begin{aligned} \underline{u}_1' &= H\,\underline{u}_1 \\ \underline{u}_2' &= H\,\underline{u}_2 \\ \underline{u}_3' &= H\,u_3 \\ \underline{u}_4 &= H\,u_4 \end{aligned}\right\} \Rightarrow \begin{bmatrix} \underline{u}_1' = \begin{matrix}1\\2\\3\end{matrix} \\ \\ \underline{u}_2' = \\ \\ \\ \end{bmatrix} \begin{bmatrix} u_1^T & & \\ & u_1^T & \\ & & u_1^T \\ u_2^T & & \\ & u_2^T & \\ & & u_2^T \\ & & \end{bmatrix} \begin{bmatrix} h_1 \\ \vdots \\ h_9 \end{bmatrix}$$

$$
\begin{array}{r}
\underline{u}_1' \left\{ \vphantom{\begin{matrix} x \\ y \end{matrix}} \right. \\
\underline{u}_2' \left\{ \vphantom{\begin{matrix} x \\ y \end{matrix}} \right. \\
\underline{u}_3' \left\{ \vphantom{\begin{matrix} x \\ y \end{matrix}} \right. \\
\vphantom{\begin{matrix} x \\ y \end{matrix}}
\end{array}
\underbrace{\begin{bmatrix} x_1' \\ y_1' \\ x_2' \\ y_2' \\ x_3' \\ y_3' \\ x_4' \\ y_4' \end{bmatrix}}_{\underline{b}}
=
\underbrace{\begin{bmatrix} \underline{u}_1^T & 0 \\ 0 & u_1^T \\ \underline{u}_2^T & 0 \\ 0 & \underline{u}_2^T \\ u_3^T & 0 \\ 0 & u_3^T \\ u_4^T & 0 \\ 0 & u_4^T \end{bmatrix}}_{A}
\underbrace{\begin{bmatrix} h_1 \\ \vdots \\ h_9 \end{bmatrix}}_{\underline{x}}
$$

3D to 2D camera projection matrix estimation

Given a set of points $\mathbf{X}_i$ in 3D space, and a set of corresponding points $\mathbf{x}_i$ in an image, find the 3D to 2D projective $\mathbf{P}$ mapping that maps $\mathbf{X}_i$ to $\mathbf{x}_i = \mathbf{P}\mathbf{X}_i$.