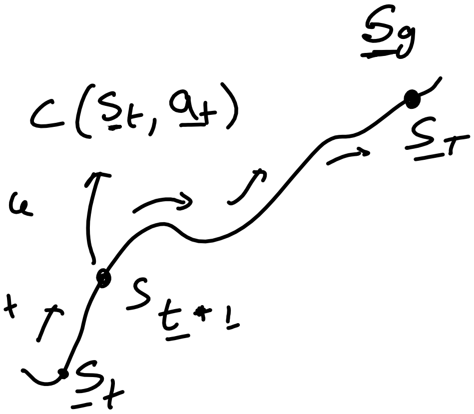
Trajectory in Terms of state space

Control : Proportional Integral Derivative (PID)

2. LQR: Linear Quadratic Regulator

$\uparrow$

Planning/Control $\longrightarrow$ Trajectory in terms of action space



$$\min_{\{a_t\}_{t=1}^{T}} \sum_{t=1}^{T} c(\underline{s}_t, \underline{a}_t)$$

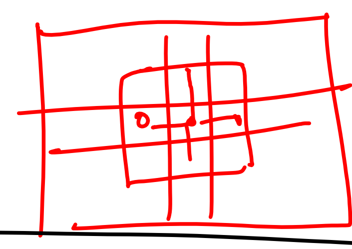
subject to $\underline{s}_{t+1} = f(\underline{s}_t, a_t)$

Examples

$$\overbrace{\begin{bmatrix} x_{t+1} \\ y_{t+1} \\ \theta_{t+1} \end{bmatrix}}^{s_{t+1}} = \overbrace{\begin{bmatrix} v_t \cos\theta_t \Delta t + x_t \\ v_t \sin\theta_t \Delta t + y_t \\ \omega_t \Delta t + \theta_t \end{bmatrix}}^{f(\underline{s}_t, \underline{a}_t)}$$

$$\underline{c(\underline{s}_t, \underline{a}_t)} = \|\underline{s}_g - \underline{s}_t\|^2 + \frac{1}{2} m v^2 + \frac{1}{2} I_m \omega^2$$

$$\underline{s}_t = \begin{bmatrix} x_t \\ y_t \\ \theta_t \end{bmatrix} \quad \underline{a}_t = \begin{bmatrix} v_t \\ \omega_t \end{bmatrix}$$



System dynamics to be Linear

Cost function to be Quadratic

$$\underline{s}_{t+1} = f(\underline{s}_t, \underline{a}_t) = \underset{n\times n}{A_t} \underset{n\times 1}{\underline{s}_t} + \underset{n\times m}{B_t} \underset{m\times 1}{\underline{a}_t}$$

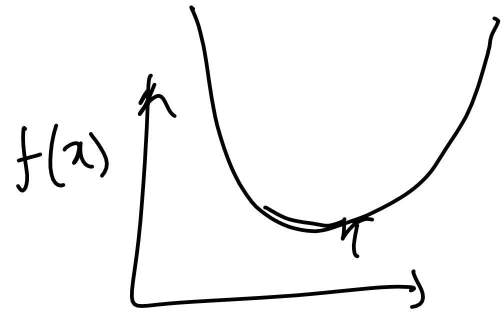
$$c(s_t, a_t) = \underline{s}_t^T \underset{n\times n}{Q} \underline{s}_t + \underline{a}_t^T \underset{m\times m}{R} \underline{a}_t + \underline{b}^T \underline{s}_t + \underline{c}^T \underline{a}_t \cdot \underline{s}_t^T N \underline{a}_t$$

optional

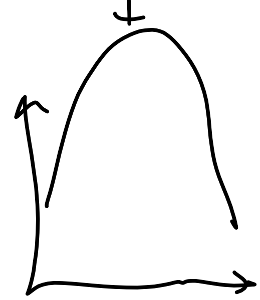
$\underline{x}^T A \underline{x}$

$= x^2 a_{11} + y^2 a_{22} + a_{12} xy +$

$+2x^2 - 5x + 6$



$-2x^2 + 5x + 6$



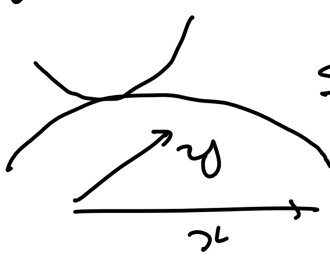
$Q \succ 0$

Positive definite

$Q \prec 0$

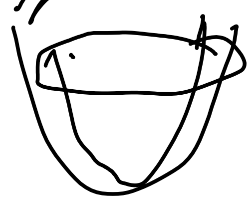
negative definite

Q can be indefinite

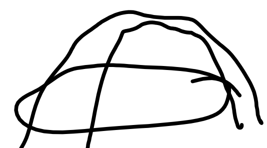


saddle point

$Q \succeq 0$



Positive semi definite



$Q \preceq 0$

Negative semi definite

Defn :

$Q \succ 0$

for any $\underline{x}$

$\underline{x}^T Q \underline{x} > 0$

Test : all eigenvalues > 0

$\underline{x}^T E \Lambda E^T \underline{x} > 0$

matrix of eigen vector

Diagonal matrix of eigen values

$\lambda_1 y_1^2 + \lambda_2 y_2^2 + \dots + \lambda_n y_n^2 = \underline{y}^T \Lambda \underline{y}$

$\underline{y} = E \underline{x}$

LQR

$$\min_{\{a_t\}_{t=1}^{T}} \overbrace{\left(\sum_{t=1}^{T} s_t^T Q s_t + a_t^T R a_t\right)}^{\text{cost functio}} + s_{T+1}^T Q s_{T+1}$$

energy

how far are you from goal

$$\text{s.t.} \quad \underline{s_{t+1} = A_t s_t + B_t a_t}$$

Dynamic programming (mathematical Induction)

$$\min_{\{a_t\}_{t=1}^{T-1}} \left(\sum_{t=1}^{T-1} s_t^T Q s_t + a_t^T R a_t\right) + \left[\min_{a_T} s_T^T Q s_T + a_T^T R a_t\right]$$

Cost to go

$$V(s_1) = \min_{\{a_t\}_{t=1}^{T}} \overbrace{\left(\sum_{t=1}^{T} s_t^T Q s_t + a_t^T R a_t\right)}^{\text{cost functio}} + s_{T+1}^T Q s_{T+1}$$

energy

how far are you from goal

cost to go

$$\text{s.t.} \quad \underline{s_{t+1} = A_t s_t + B_t a_t}$$

$$t = T$$

$$V(s_k) = \min_{\{a_t\}_{t=k}^{T}} \sum_{t=k}^{T} \left(s_t^T Q s_t + a_t^T R a_t\right) + s_{T+1}^T Q s_{T+1}$$

$$\left.\text{s.t.} \quad s_{t+1} = A_t s_t + B_T a_t \right] \text{ for all } t \in \{k \ldots T\}$$

A correction to the cost function

$$\underline{\qquad\qquad} + s_T^T Q s_T + a_T^T R a_T$$

$$s_{T+1} = A_T s_T + B_T a_T$$

$$V(s_T) = \min_{\{a_t\}_{t=T}^{T}} \sum_{t=T}^{T} \underline{s}_T^T Q \underline{s}_T + \underline{a}_T^T R a_T + s_{T+1}^T Q s_{T+1}$$

$$\text{s.t.} \quad \underline{s}_{T+1} = A_T \underline{s}_T + B_T \underline{a}_T$$

$$= \min_{a_T} \underline{s}_{T+1}^T Q s_{T+1} + a_T^T R a_T + \underline{s}_T^T Q \underline{s}_T$$

$$\underline{s}_{T+1} = A_T \underline{s}_T + B_T \underline{a}_T$$

$$V(s_T) = \min_{\underline{a}_T} (A_T s_T + B_T \underline{a}_T)^T Q (A_T \underline{s}_T + B_T a_T) + \underline{a}_T^T R \underline{a}_T$$

$$M\underline{a} = \begin{bmatrix} - \underline{m}_1^T \underline{a} - \\ - \underline{m}_2^T \underline{a} - \\ \vdots \\ - \underline{m}_n^T \underline{a} - \end{bmatrix}$$

$$J_{a_T}[\underline{a}_T^T M \underline{a}_T] = 2\underline{a}_T^T M$$

$$J_{\underline{a}_T}[\underline{m}^T \underline{a}_T] = \underline{m}^T \Big/ J_{a_T}[M\underline{a}_T] = M \qquad \begin{cases} Q = Q^T \\ R = R^T \end{cases}$$

$$J_{\underline{a}_T}\left\{(A \underline{s}_T + B a_T)^T Q (A \underline{s}_T + B \underline{a}_T) + \underline{a}_T^T R a_T\right\} = \underline{0}^T$$

$$2(A \underline{s}_T + B \underline{a}_T)^T Q J_{\underline{a}_T}\{(A \underline{s}_T + B \underline{a}_T)\} + 2 a_T^T R = \underline{0}^T$$

$$2(A \underline{s}_T + B \underline{a}_T)^T Q B + 2 a_T^T R = \underline{0}^T$$

$$\underline{a}_T^T = \quad - \underline{s}_T^T A^T Q B (R + B^T Q B)^{-1}$$

$$\underline{a}_T = -\left(R + B^T Q B\right)^{-T} B^T Q A \underline{s}_T$$

$$\underline{a}_T = -\underbrace{\left(R + B_T^T Q B_T\right)^{-1} B_T^T Q A_T}_{K_T} \underline{s}_T$$

$$\min_{\{a_{T-1}, a_T\}} \sum_{t=T-1}^{T} \left(\underline{s}_t^T Q \underline{s}_t + a_t^T R \underline{a}_t\right) + s_{T+1}^T Q s_{T+1}$$

$$\text{s.t.} \quad \underline{s}_{t+1} = A \underline{s}_t + B \underline{a}_t$$

$$\min_{a_{T-1}} s_{T-1}^T Q s_{T-1} + a_{T-1}^T R s_{T-1} + \underbrace{V(s_T)}_{\min_{a_T}}$$

$$\text{s.t.} \quad \underline{s}_T = A \underline{s}_{T-1} + B \underline{a}_{T-1}$$

Is $V(s_T)$ quadratic in s_T? $\qquad \underline{P_{T+1} = Q}$

$$V(s_{T+1}) = \underline{s}_{T+1}^T Q \underline{s}_{T+1} = \underline{s}_{T+1}^T P_{T+1} \underline{s}_{T+1}$$

$$V(s_T) = \min_{a_T} \underline{s}_T^T Q \underline{s}_T + \underline{a}_T^T R \underline{a}_T + \underbrace{s_{T+1}^T Q s_{T+1}}$$

$$= \min_{a_T} \quad '' \quad + \quad '' \quad + s_{T+1}^T P_{T+1} s_{T+1}$$

$$\underline{a}_T = -K_T \underline{s}_T$$

$$V(\underline{s}_T) = s_T^T Q s_T + \underline{s}_T^T K_T^T R K_T \underline{s}_T + s_{T+1}^T P_{T+1} s_{T+1}$$

$$\underline{s}_T^T P_T \underline{s}_T = \underbrace{\underline{s}_T^T \left(Q + K_T^T R K_T \right) \underline{s}_T} + \underbrace{(A\underline{s}_T + B\underline{a}_T)^T P_{T+1} (A\underline{s}_T + B\underline{a}_T)}$$

$$= \underline{\hspace{4cm}} + (A\underline{s}_T - BK_T s_T)^T P_{T+1} (A\underline{s}_T - BK_T s_T)$$

$V(\underline{s}_T)$

$$= \underline{s}_T^T \left(Q + K_T^T R K_T \right) \underline{s}_T + \underline{s}_T^T \left\{ (A - BK_T)^T P_{T+1} (A - BK_T) \right\} \underline{s}_T$$

This is Quadratic in $\underline{s}_T$

$$V(\underline{s}_T) = \underline{s}_T^T P_T \underline{s}_T =$$

$$\underline{P_T = Q + K_T^T R K_T + (A - BK_T)^T P_{T+1} (A - BK_T)}$$

$K_{T-1} = ?$

$P_{T-1} = ?$

$\vdots$

$$\min_{\underline{a}_T} (A_T s_T + B_T \underline{a}_T)^T Q (A_T \underline{s}_T + B_T a_T) + \underline{a}_T^T R \underline{a}_T$$

$$\min_{a_{T-1}} \underbrace{\underline{s}_{T-1}^T Q \underline{s}_{T-1}}_{\times} + \underline{a}_{T-1}^T R a_{T-1} + \underbrace{V(s_T)}_{\underline{s}_T^T P_T s_T}$$

$$\min_{a_{T-1}} \underline{a}_{T-1}^T R a_{T-1} + (A_{T-1} \underline{s}_{T-1} + B_{T-1} a_{T-1})^T P_T (A_{T-1} \underline{s}_{T-1} + B_{T-1} a_{T-1})$$

$$\underline{a}_{T-1} = -K_{T-1}\underline{s}_{T-1} =$$

$$K_{T-1} = \left(R + B_{T-1}^{T} P_T B_{T-1}\right)^{-1} B_{T-1} P_T A_{T-1}$$

Discrete Algebraic Riccati Equations for LQR

$$\begin{cases} P_T = Q + K_T^{T} R K_T + (A - BK_T)^{T} P_{T+1} (A - BK_T) \\ K_T = \left(R + B_T^{T} P_{T+1} B_T\right)^{-1} B_T P_{T+1} A_T \end{cases}$$

① Dynamic programming

② Quadratic optimization can be solved by taking derivatives and equating them to zero

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Is the unicyle model Linear?

$$\underbrace{\underline{s}_{t+1}}_{s_{t+1}} = \begin{bmatrix} x_{t+1} \\ y_{t+1} \\ \theta_{t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} x_t + v_t \cos(\theta_t)\Delta t \\ y_t + v_t \sin(\theta_t)\Delta t \\ \theta_t + \omega_t \Delta t \end{bmatrix}}_{\underline{s}_t}$$

$$\underline{a}_t = \begin{bmatrix} v_t \\ \omega_t \end{bmatrix}$$

$$\underline{s}_{t+1} = A\underline{s}_t + B\underline{a}_t$$

Non-linear in $\underline{\theta}_t, \underline{v}_t$

Closest Linear model

single integrator)

$$\underline{a}_t = \begin{bmatrix} v_{x_t} \\ v_{y_t} \end{bmatrix}$$

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \\ \theta_{t+1} \end{bmatrix} = \begin{bmatrix} x_t + v_{x_t}\Delta t \\ y_t + v_{y_t}\Delta t \\ \theta_t + \omega_t \Delta t \end{bmatrix} \qquad \underline{a}_t = \begin{bmatrix} v_{x_t} \\ v_{y_t} \\ \omega_t \end{bmatrix}$$

$$\| \underline{s}_t - \underline{s}_g \|^2 = \underbrace{(\underline{s}_{T+1} - \overset{\text{constant}}{\underset{\text{zero}}{\underline{s}_g}})^T Q_{T+1} (\underline{s}_{T+1} - \underline{s}_g)}$$

$$\tilde{\underline{s}}_t = \begin{bmatrix} \underline{s}_t \\ 1 \end{bmatrix}$$

$$\tilde{\underline{s}}_t^T \tilde{Q}_t \tilde{\underline{s}} = [\underline{s}_t^T \; 1] \overbrace{\begin{bmatrix} Q_t & \underline{a} \\ \underline{a}^T & b \end{bmatrix}}^{\tilde{Q}_t} \begin{bmatrix} s_t \\ 1 \end{bmatrix}$$

$$= \underline{s}_t^T Q_t \underline{s}_t + \underset{\sim}{2\underline{a}^T} Q_t \underline{s}_t + \underset{\sim}{b}$$

$$(\underline{s}_t - s_g)^T Q (\underline{s}_t - \underline{s}_g)$$

$$= \underline{s}_t^T Q \underline{s}_T - \underset{\sim}{2 s_g^T} Q \underline{s}_t + \underset{\sim}{\underline{s}_g^T Q \underline{s}_g}$$

$$\tilde{Q}_t = \begin{bmatrix} Q_t & -s_g \\ \underline{s}_g^T & \underline{s}_g^T Q_t \underline{s}_g \end{bmatrix} \qquad \tilde{\underline{s}}_t = \begin{bmatrix} s_t \\ 1 \end{bmatrix}$$