ECE 417/598: Eigen Value Decomposition, Singular Value Decompsition

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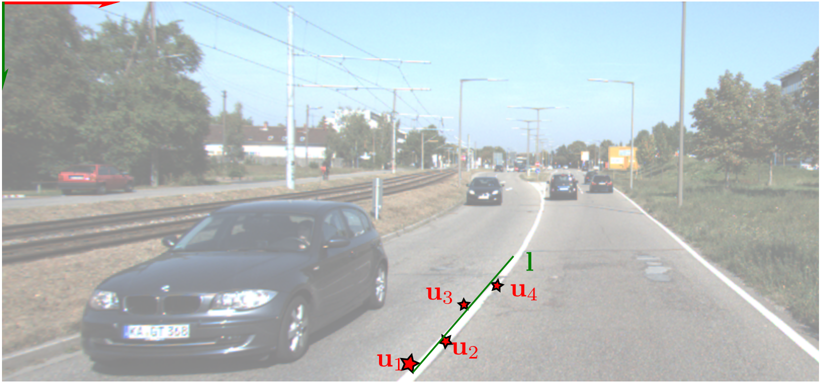


$$\underline{\mathbf{u}}_1 = [100, 98, 1]^\top$$
$$\underline{\mathbf{u}}_2 = [105, 95, 1]^\top$$
$$\underline{\mathbf{u}}_3 = [107, 90, 1]^\top$$
$$\underline{\mathbf{u}}_4 = [110, 85, 1]^\top$$

Find the line $\mathbf{l}$ such that it is the "closest line" passing through $\mathbf{u}_1, \dots, \mathbf{u}_4$.

$$U = \begin{bmatrix} \mathbf{u}_1^\top \\ \mathbf{u}_2^\top \\ \mathbf{u}_3^\top \\ \mathbf{u}_4^\top \end{bmatrix}$$

We want to solve for $\mathbf{l}$ such that

$$U\mathbf{l} = 0$$

Eigenvalues and Eigenvectors

For a square matrix A, the λ_i and $\mathbf{x}_i$ that satisfy the following equation are called eigenvalues and eigenvectors respectively.

$$A\mathbf{x} = \lambda\mathbf{x} \text{ or } (A - \lambda I)\mathbf{x} = 0 \tag{1}$$

λ is chosen to ensure that $A - \lambda I$ has null space, hence, characteristic equation

$$\det(A - \lambda I) = 0 \tag{2}$$

For symmetrix matrix $A = A^\top$, eigenvalues are real, and eigenvectors are orthonormal,

$$A[\mathbf{x}_1, \dots, \mathbf{x}_n] = [\mathbf{x}_1, \dots, \mathbf{x}_n] \begin{bmatrix} \lambda_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \lambda_n \end{bmatrix} \tag{3}$$

$$AS = S\Lambda \tag{4}$$

$$\text{if } A = A^\top \text{ then } A = S\Lambda S^\top \tag{5}$$

Numerical example

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 4 & 6 \end{bmatrix}$$

Find eigen values and eigen vectors Eigen library.

`https://github.com/wecacuee/ECE417-Mobile-Robots/blob/master/docs/slides/03-04-linear-algebra_files/findeig.cpp`

Not all matrices possess n linearly independent eigenvectors, and therefore not all matrices are diagonalizable. The standard example of a ”defective matrix” is

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

If the eigenvectors $x_1, \ldots, x_k$ correspond to different eigenvalues $\lambda_1, \ldots, \lambda_k$ then those eigenvectors are linearly independent.

Find the eigen values and vectors of rotation matrix

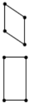
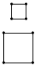
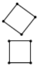
$$R(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Find the eigen values and vectors of rotation matrix

$$R(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{bmatrix}$$

Compute the exponential of matrix $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$ using the series expansion of $\exp(A)$ and the fact that $A^n = S\Lambda^n S^{-1}$.

Hierarchy of transforms

Group	Matrix	Distortion	Invariant properties
Projective 8 dof	$\begin{bmatrix} h_{11} & h_{12} & h_{13} \\ h_{21} & h_{22} & h_{23} \\ h_{31} & h_{32} & h_{33} \end{bmatrix}$		Concurrency, collinearity, **order of contact**: intersection (1 pt contact); tangency (2 pt contact); inflections (3 pt contact with line); tangent discontinuities and cusps. cross ratio (ratio of ratio of lengths).
Affine 6 dof	$\begin{bmatrix} a_{11} & a_{12} & t_x \\ a_{21} & a_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Parallelism, ratio of areas, ratio of lengths on collinear or parallel lines (e.g. midpoints), linear combinations of vectors (e.g. centroids). The line at infinity, $\mathbf{l}_\infty$.
Similarity 4 dof	$\begin{bmatrix} sr_{11} & sr_{12} & t_x \\ sr_{21} & sr_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Ratio of lengths, angle. The circular points, **I**, **J** (see section 2.7.3).
Euclidean 3 dof	$\begin{bmatrix} r_{11} & r_{12} & t_x \\ r_{21} & r_{22} & t_y \\ 0 & 0 & 1 \end{bmatrix}$		Length, area

Singular Value Decomposition (SVD)

$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^\top \tag{6}$$

$$A^\top A = V \Sigma^2 V^{-1} \tag{7}$$

$$A^\top A \mathbf{v}_i = \lambda_i \mathbf{v}_i \qquad \lambda_i = \sigma_i^2 \tag{8}$$

$$AV = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} \tag{9}$$

$$U^+ = \Sigma^{-1} A V^+ \tag{10}$$

Numerical example

Find singular value decomposition

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 2 & 4 & 6 \end{bmatrix}$$