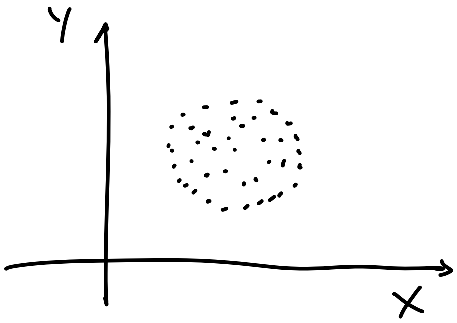
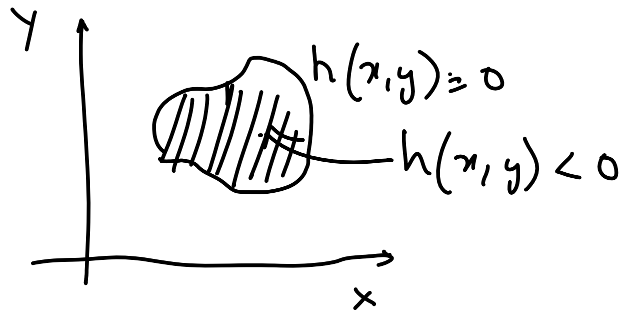
2D coordinate transforms

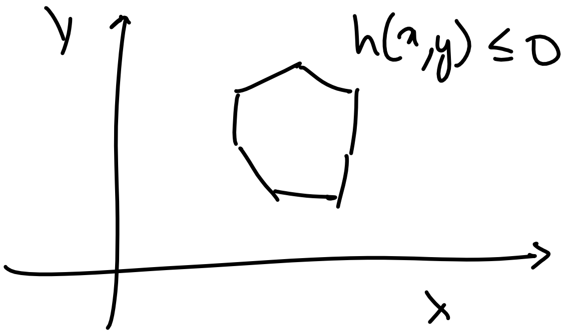
Rotations + translations

Orientation + position

$h(x,y) > 0$





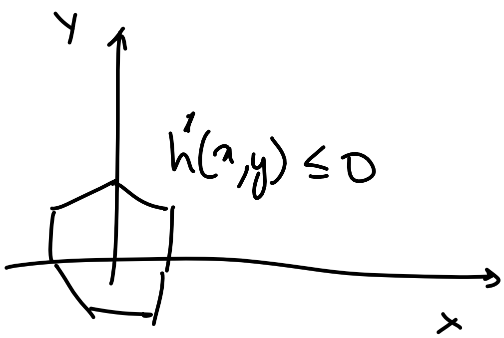


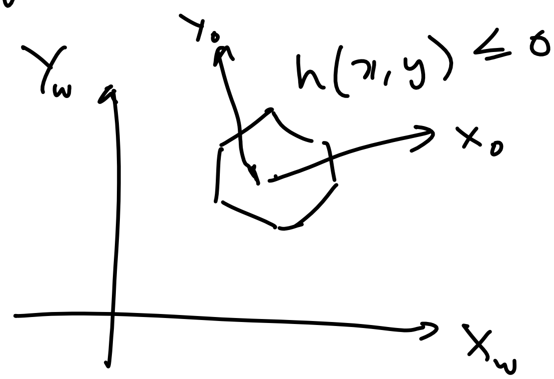
Object $= \{(x,y) \in \mathbb{R}^2 : h(x,y) \leq 0\}$

Set of all points $(x,y) \in \mathbb{R}^2$ ← set of all real numbers

such that

$h(x,y) \leq 0$

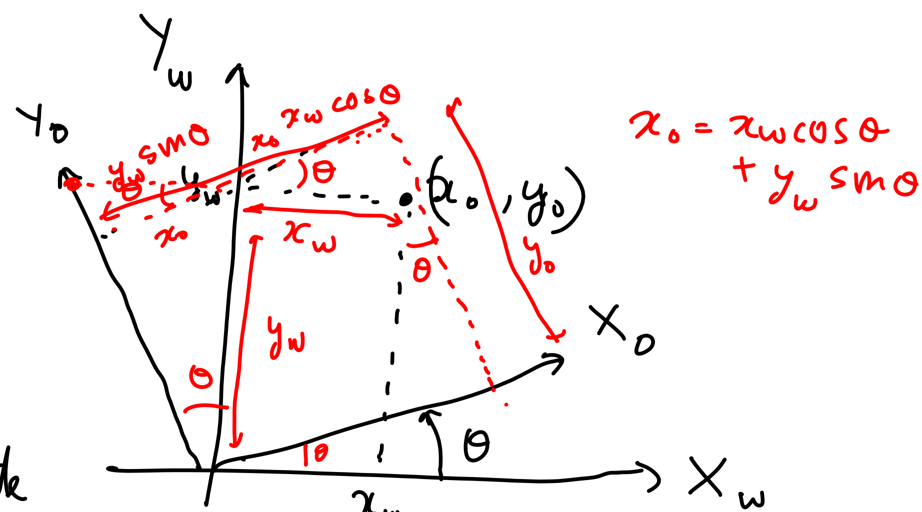




2D Rotations

① Rotation

② Translation



$x_o = x_w \cos\theta + y_w \sin\theta$

Problem: (x_o, y_o) is given in (X_o, Y_o) coordinate frame. $(X_o Y_o)$ has been rotated by angle θ w.r.t. (X_w, Y_w)

Find (x_w, y_w) in world coordinate frame

Proof using Basis vectors

In Linear algebra, Basis vectors are set of orthonormal unit vectors that span the entire space

Span is the set of all vectors that can be obtained by linear combinations of a given set of vectors

$$\text{Span}\{\underline{a}, \underline{b}\} = \left\{ \alpha \underline{a} + \beta \underline{b}, \; \begin{matrix} \alpha \in \mathbb{R} \\ \beta \in \mathbb{R} \end{matrix} \right\}$$

$\in \mathbb{R}^n$ $\qquad \in \mathbb{R}^n$

Standard Basis vector.

For example, in $\mathbb{R}^2$ $\quad \hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; \hat{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

in $\mathbb{R}^3$ $\quad \hat{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \; \hat{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \; \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

in $\mathbb{R}^n$ $\quad \hat{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n \times 1} \; \cdots \; \hat{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}_{n \times 1}$

Basis vectors for $\mathbb{R}^n$

- ① All vectors must be perpendicular/orthogonal to each other
- ② They must be unit vectors
- ③ They must span the entire space $\mathbb{R}^n$

Let Basis vector for (X_w, Y_w) be standard basis vector

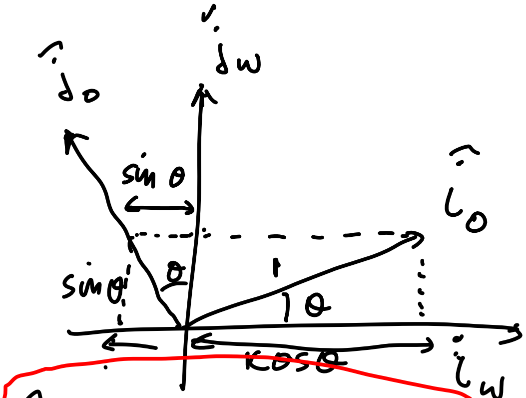
$\hat{i}_w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \; \hat{j}_w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Any point in world coordinate frame

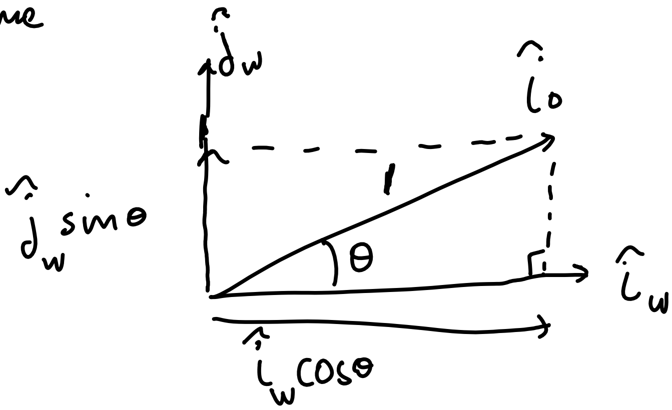
$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = x_w \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\hat{i}_w} + y_w \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\hat{j}_w}$$

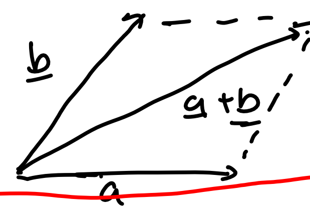
Any point in the object coordinate frame

$$\begin{bmatrix} x_o \\ y_o \end{bmatrix} = x_o \hat{i}_o + y_o \hat{j}_o$$



$$\boxed{\hat{i}_o = \cos\theta\, \hat{i}_w + \sin\theta\, \hat{j}_w}$$





$$\boxed{\hat{j}_o = -\hat{i}_w \sin\theta + \hat{j}_w \cos\theta}$$

$$\overbrace{x_w \hat{i}_w + y_w \hat{j}_w}^{\text{world}} = \overbrace{x_o \hat{i}_o + y_o \hat{j}_o}^{\text{object}}$$

$$\underbrace{\begin{bmatrix} x_w \\ y_w \end{bmatrix}}_{} = x_o \left[\cos\theta\, \hat{i}_w + \sin\theta\, \hat{j}_w\right] + y_o \left[-\hat{i}_w \sin\theta + \hat{j}_w \cos\theta\right]$$

$$= \left[x_o \cos\theta - y_o \sin\theta\right] \hat{i}_w + \left[x_o \sin\theta + y_o \cos\theta\right] \hat{j}_w$$

Because we are using standard basis for world coordinate frame

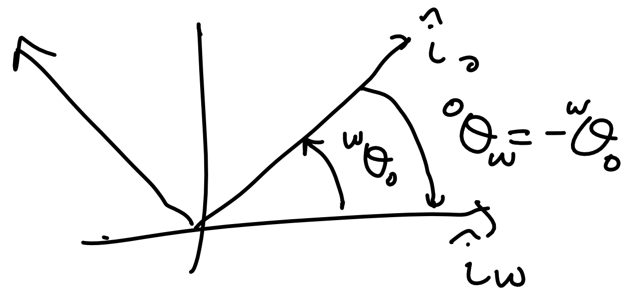
$$= \quad \hat{i}_w \to \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \hat{j}_w \to \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} x_o \cos\theta - y_o \sin\theta \\ x_o \sin\theta + y_o \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_o \\ y_o \end{bmatrix}$$

$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \begin{bmatrix} {}^wR_o(\theta) \end{bmatrix} \begin{bmatrix} x_o \\ y_o \end{bmatrix}$$

$${}^wR_o(\theta) := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



$$\begin{bmatrix} x_o \\ y_o \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}}_{{}^oR_w(\theta) = {}^wR_o(-\theta) = {}^wR_o^T(\theta)} \begin{bmatrix} x_w \\ y_w \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & +\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_w \\ y_w \end{bmatrix}$$

$$\begin{bmatrix} {}^wR_o^T \end{bmatrix} {}^wR_o = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -c(\theta)s(\theta) + s(\theta)c(\theta) \\ -s(\theta)c(\theta) + c(\theta)s(\theta) & s^2(\theta) + c^2(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_{2\times 2}$$

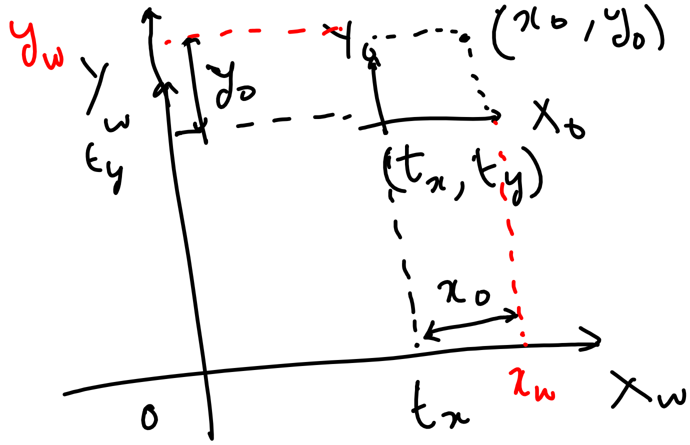
$$\underbrace{R^T R = I}$$

$$(A^{-1})A = I$$

$$R^{-1} = R^T$$

2D Translation

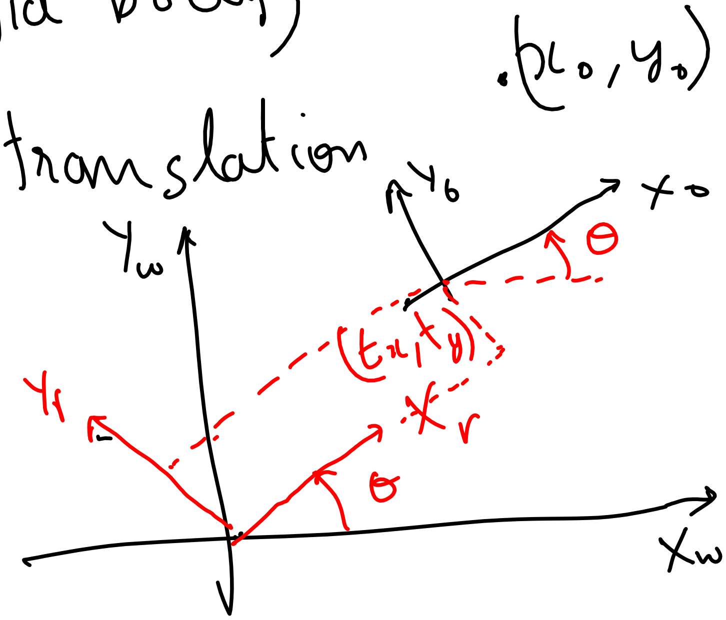
$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \underbrace{\begin{bmatrix} t_x \\ t_y \end{bmatrix}}_{\text{translation vector}} + \begin{bmatrix} x_o \\ y_o \end{bmatrix}$$



2D transformation (Rigid body)

Rotation followed by translation

$$\begin{bmatrix} x_r \\ y_r \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_o \\ y_o \end{bmatrix}$$



$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \begin{bmatrix} t_x \\ t_y \end{bmatrix} + \begin{bmatrix} x_r \\ y_r \end{bmatrix}$$

$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}\begin{bmatrix} x_o \\ y_o \end{bmatrix} + \underbrace{\begin{bmatrix} t_x \\ t_y \end{bmatrix}}_{\underline{t}}$$

$$\boxed{\underline{x}_w = {}^{w}_{o}R(\theta)\, \underline{x}_o + {}^{w}_{o}\underline{t}}$$

$$\underline{x}_0 = ?(\underline{x}_w)$$

$$\underline{x}_w - {}^{w}_{0}\underline{t} = {}^{w}_{0}R\,\underline{x}_0$$

$${}^{w}_{0}R = {}^{w}_{0}R(\theta)$$

Multiply on the left of both sides by ${}^{w}_{0}R^T$

$${}^{w}_{0}R^T\left(\underline{x}_w - {}^{w}_{0}\underline{t}\right) = \left({}^{w}_{0}R^T\,{}^{w}_{0}R\right)\underline{x}_0$$

$$\frac{\underline{x}_w - {}^{w}_{0}\underline{t}}{{}^{w}_{0}R} = \underline{x}_0$$

NEVER EVER DO THIS

$$\underline{x}_0 = {}^{w}_{0}R^T\,\underline{x}_w - {}^{w}_{0}R^T\,{}^{w}_{0}\underline{t} \quad \text{(1)}$$

$$\underline{x}_0 = {}^{0}_{w}R\cdot\underline{x}_w + {}^{0}_{w}\underline{t} \quad \text{(2)}$$

Compare (1) and (2)

$$\boxed{{}^{0}_{w}R = {}^{w}_{0}R^T} \quad \text{and} \quad \boxed{{}^{0}_{w}\underline{t} = -\,{}^{w}_{0}R^T\,{}^{w}_{0}\underline{t}}$$

$$\underline{x}_w = {}^{w}_{0}R\,\underline{x}_0 + {}^{w}_{0}\underline{t}$$

$$
\begin{bmatrix} \underline{x}_w \\ 1 \end{bmatrix} = \begin{bmatrix} x_w \\ y_w \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} {}^w_o R_{2\times 2} & {}^w_o \underline{t}_{2\times 1} \\ 0^T_{1\times 2} & 1 \end{bmatrix}}_{{}^w_o T_{3\times 3}} \begin{bmatrix} \underline{x}_{o\,2\times 1} \\ 1 \end{bmatrix}
$$

Block matrix

Block Matrix multiplication

✓
$$
\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} AE+BG & AF+BH \\ CE+DG & CF+DH \end{bmatrix}
$$

$$
\begin{bmatrix} {}^w_o R_{2\times 2} & {}^w_o \underline{t}_{2\times 1} \\ 0^T_{1\times 2} & 1 \end{bmatrix} \begin{bmatrix} \underline{x}_{o\,2\times 1} \\ 1 \end{bmatrix} = \begin{bmatrix} {}^w_o R\, \underline{x}_o + {}^w\underline{t}_o \\ \underline{0}^T \underline{x}_o + 1 \end{bmatrix}
$$

$$
= \begin{bmatrix} {}^w_o R\, \underline{x}_o + {}^w\underline{t}_o \\ 1 \end{bmatrix} = \begin{bmatrix} \underline{x}_w \\ 1 \end{bmatrix}
$$

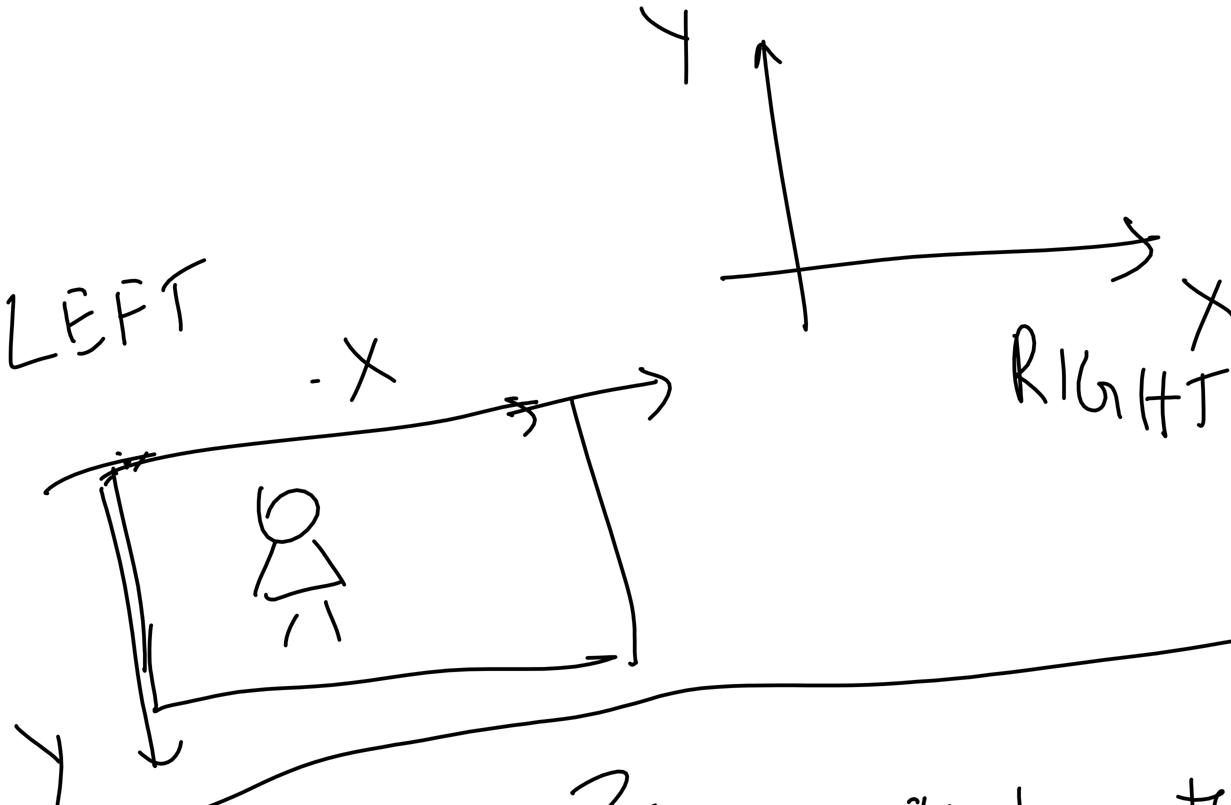
$$
\underline{x}_w = {}^w_o T\, \underline{x}_o
$$

Transformation matrix

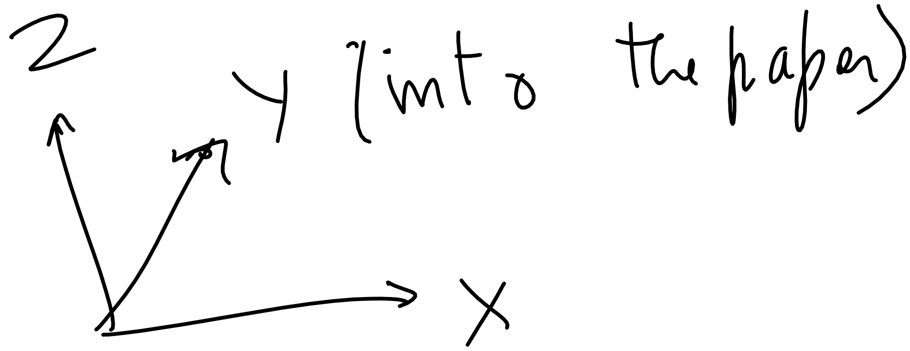
$$
{}^w T_o = \begin{bmatrix} {}^w_o R_{2\times 2} & {}^w_o \underline{t}_{2\times 1} \\ 0^T_{1\times 2} & 1 \end{bmatrix}
$$

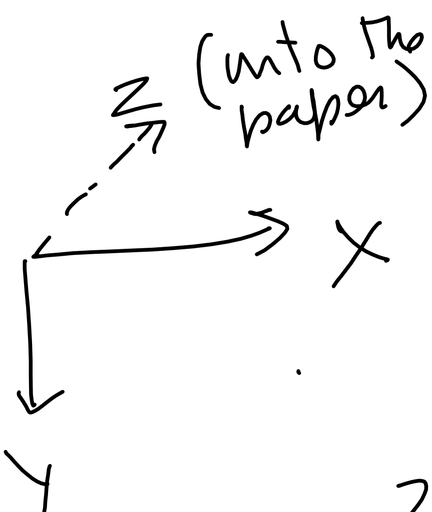
In 2D and 3D

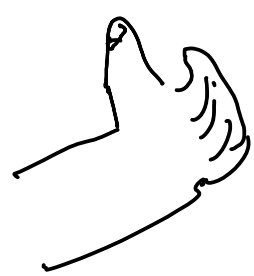
- Right hand ✓
- Left hand ✗



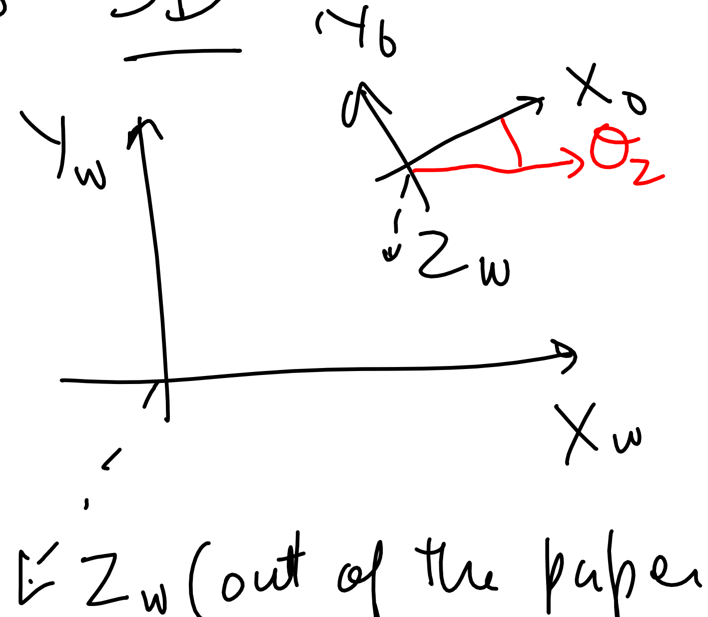
In 3D







Extending 2D to 3D

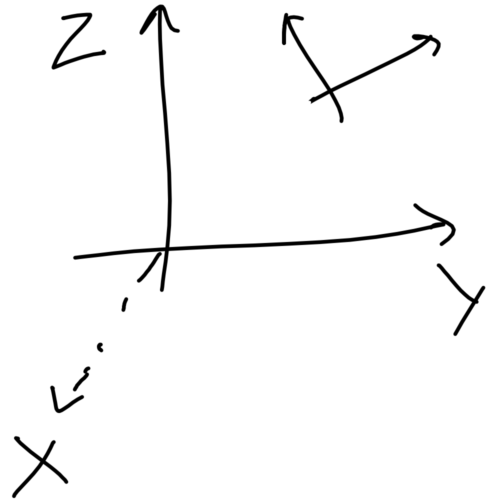


$$\begin{bmatrix} x_w \\ y_w \\ z_w = 0 \end{bmatrix} = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ 0 \end{bmatrix}$$

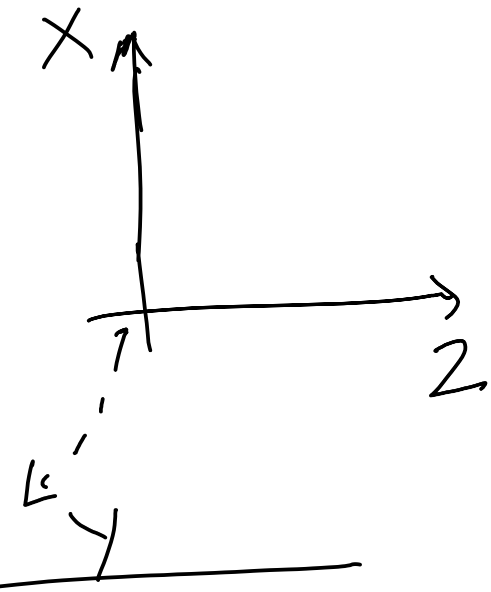
Rotation along Z-axis changes only X-Y coordinates

$$R(\theta_z) = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R(\theta_x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \underline{-\sin\theta} \\ 0 & \underline{\sin\theta} & \cos\theta \end{bmatrix}$$



$$R(\theta_y) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$



3D Rotation representations

Aeronautics

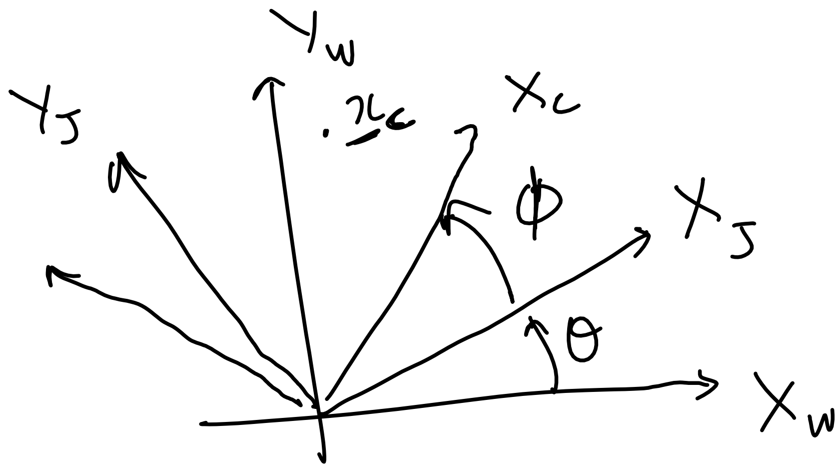
θ_x = roll

θ_y = pitch

θ_z = yaw

$$\boxed{R = \underbrace{R(\theta_z)}_{\text{Yaw}} \underbrace{R(\theta_y)}_{\text{Pitch}} \underbrace{R(\theta_x)}_{\text{Roll}}}$$

Chain rotation, translation, transformations



$$\underline{x}_J = {}^J_C R(\phi)\, \underline{x}_c$$

$$\underline{x}_w = {}^w_J R(\theta)\, \underline{x}_J$$

$$= R(\theta)\left[R(\phi)\,\underline{x}_c\right]$$

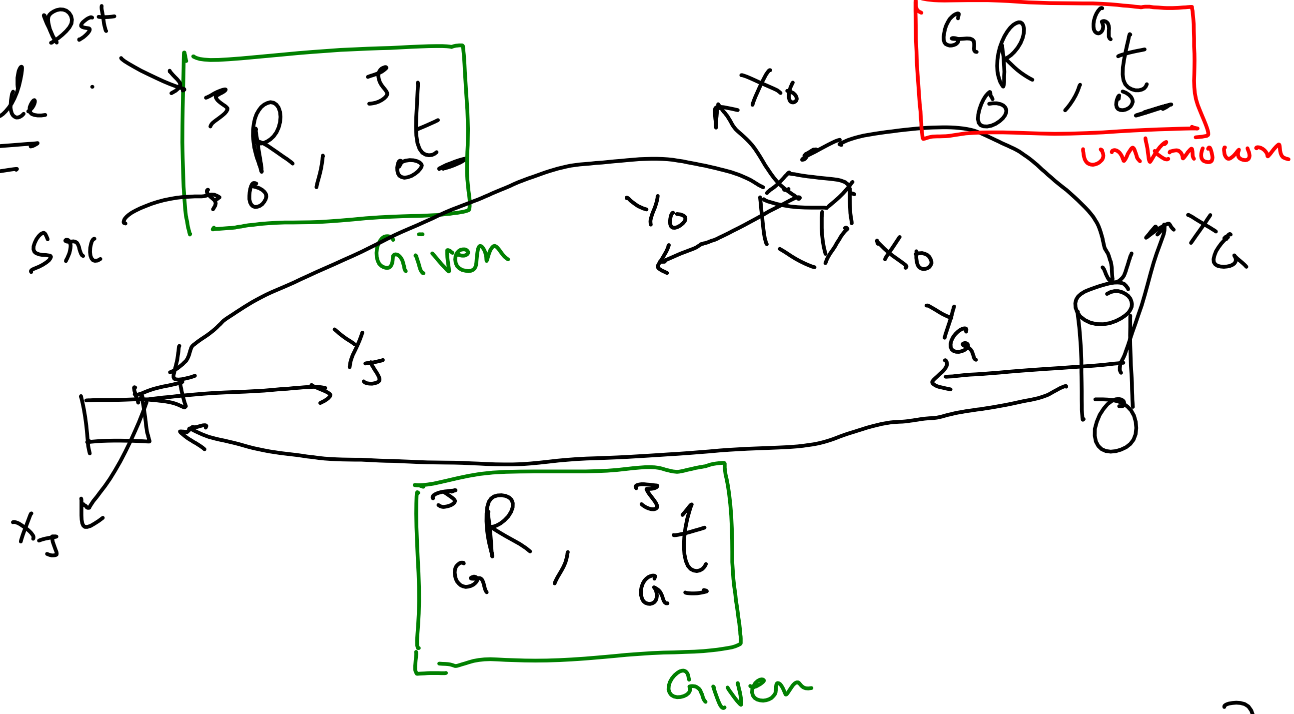
$$= \underbrace{\left(R(\theta)\, R(\phi)\right)}_{{}^w_c R}\, \underline{x}_c$$

$$\boxed{{}^w_c R = {}^w_{\cancel{J}} R(\theta)\; {}^{\cancel{J}}_c R(\phi)}$$

$$\boxed{{}^w_c T = {}^w_J T \; {}^J_c T}$$

10/20

Example:



$${}^J_O T = \begin{bmatrix} {}^J_O R & {}^J_O \underline{t} \\ 0^T & 1 \end{bmatrix}, \quad {}^J_G T = \begin{bmatrix} {}^J_G R & {}^J_G \underline{t} \\ 0^T & 1 \end{bmatrix}$$

$${}^G_O T = \begin{bmatrix} {}^G_O R & {}^G_O \underline{t} \\ 0^T & 1 \end{bmatrix} \qquad {}^G_O T = \left({}^J_G T\right)^{-1} {}^J_O T$$

$$= {}^G_J T \, {}^J_O T \ldots$$

$$\underbrace{{}^G_O T \, \underline{x}_O}_{\underline{x}_G} = \underbrace{\left({}^G_J T \underbrace{\left({}^J_O T \, \underline{x}_O\right)}_{\underline{x}_J}\right)}_{\underline{x}_G}$$

① $R_{3D} = \underset{\text{yaw}}{R(\theta_z)}\,\underset{\text{pitch}}{R(\theta_y)}\,\underset{\text{roll}}{R(\theta_x)}$

$\theta_x \xrightarrow{\text{then}} \theta_y \xrightarrow{\text{then}} \theta_z$

This sequences

$\overrightarrow{XYZ}$

ZYX

⋮

6 possible $= 3!$

Euler angle representation of 3D rotation is a sequence of rotation around standard axis

Euler representation with $\overrightarrow{XYZ}$ then

$R_{3D} = R(\theta_z)R(\theta_y)R(\theta_x)$

Conversion from Euler angles to Rotation matrix

How to do the opposite?

convert from Rotation matrix to Euler angles?

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}$$

$$\theta = ? = \tan^{-1}\left(\frac{r_{21}}{r_{11}}\right) \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$$

$$\underbrace{\theta = \operatorname{arctan2}(r_{21}, r_{11})} \in (-\pi, \pi)$$

3D

$$R = \overset{z}{\downarrow}R(\psi)\,\overset{y}{\downarrow}R(\phi)\,\overset{x}{\downarrow}R(\theta)$$

$$= \begin{bmatrix} c(\psi) & -s(\psi) & 0 \\ s(\psi) & c(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c(\phi) & 0 & s(\phi) \\ 0 & 1 & 0 \\ -s(\phi) & 0 & c(\phi) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c(\theta) & -s(\theta) \\ 0 & s(\theta) & c(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{c(\psi)c(\phi)} & c(\psi)c(\phi)s(\theta) - s(\psi)c(\theta) & c(\psi)s(\phi)c(\theta) + s(\psi)s(\theta) \\ \boxed{s(\psi)c(\phi)} & s(\psi)s(\phi)s(\theta) + c(\psi)c(\theta) & s(\psi)s(\phi)c(\theta) - c(\psi)s(\theta) \\ \boxed{-s(\phi)} & \boxed{c(\phi)s(\theta)} & \boxed{c(\phi)c(\theta)} \end{bmatrix}$$

$$= \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

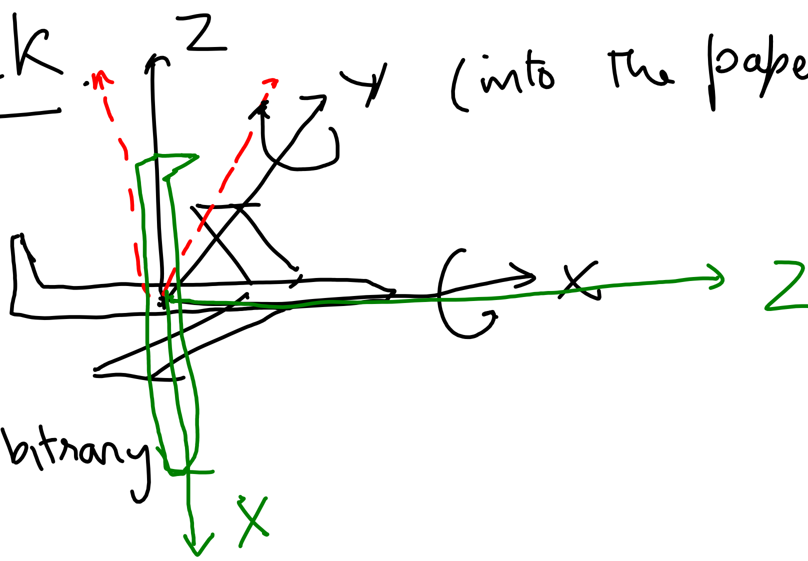
$$\phi = -\sin^{-1}(r_{31}) \in [0, \pi]$$

$$\frac{r_{21}}{r_{11}} = \frac{\sin(\psi)\,\cancel{c(\phi)}}{\cos(\psi)\,\cancel{c(\phi)}} \Rightarrow \psi = \arctan2(r_{21}, r_{11})$$

$$\theta = \arctan2(r_{32}, r_{33})$$

Conversion from Rotation matrix to Euler angles

Gimbal lock



$\theta_x = 30^\circ \leftarrow$ arbitrary

$[\theta_y = 90^\circ]$

$\theta_z = 45^\circ \leftarrow$ arbitrary

$$\text{Euler angles} \underset{\text{multiple solutions}}{\overset{\text{deterministically}}{\rightleftarrows}} \text{Rot mat}$$

Other representations. It is impossible to unambiguously represent 3D rotation with only 3 number

Degree of freedom

3D rot = 3 DOF

but needs 4 numbers + 1 constraint to represent it

② Axis-angle representation ← | ③ Quaternions

↓

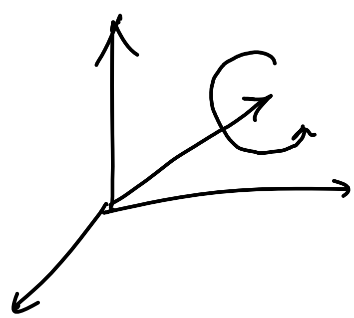
Rot mats

↓

Algebra

Quaternion algebra ≡ complex numbers

② Axis-angle representation



Any 3D rotation can be represented as a unit vector (axis) and rotation angle around it.

axis = $\underline{a} = [a_x, a_y, a_z]$

angle = θ

Constraint : $\|a\|_2 = \sqrt{a_x^2 + a_y^2 + a_z^2} = 1$

Free scalars $= 4$
Constraint $= 1$ } Degree of freedom $= 3$

DOF of 2D Rot matrix $= ?$

$$R = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Free scalars $= 4$

True for N-D rotations

Two vector constraints :

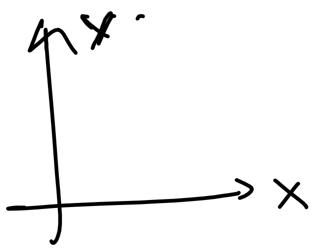
$$\boxed{\begin{aligned} R^T R = \underline{I} = R R^T \\ \det(R) = +1 \end{aligned}}$$

$R^T R = I = R R^T$ } 2 scalar constraints

$\det(R) = +1$ } 1 scalar constraint

Check for valid rotation

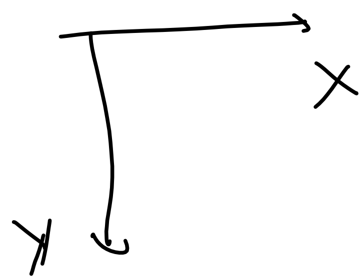
Reflection matrices also satisfy



not Rotation

reflection

det(Reflection) $= -1$



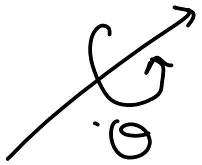
Rodrigues rotation formula

Axis angle $(\theta, \hat{k}) \longrightarrow$ Rot matrix

$$\underline{R = I + \sin\theta\,[k_\times] + (1-\cos\theta)[k_\times]^2}$$

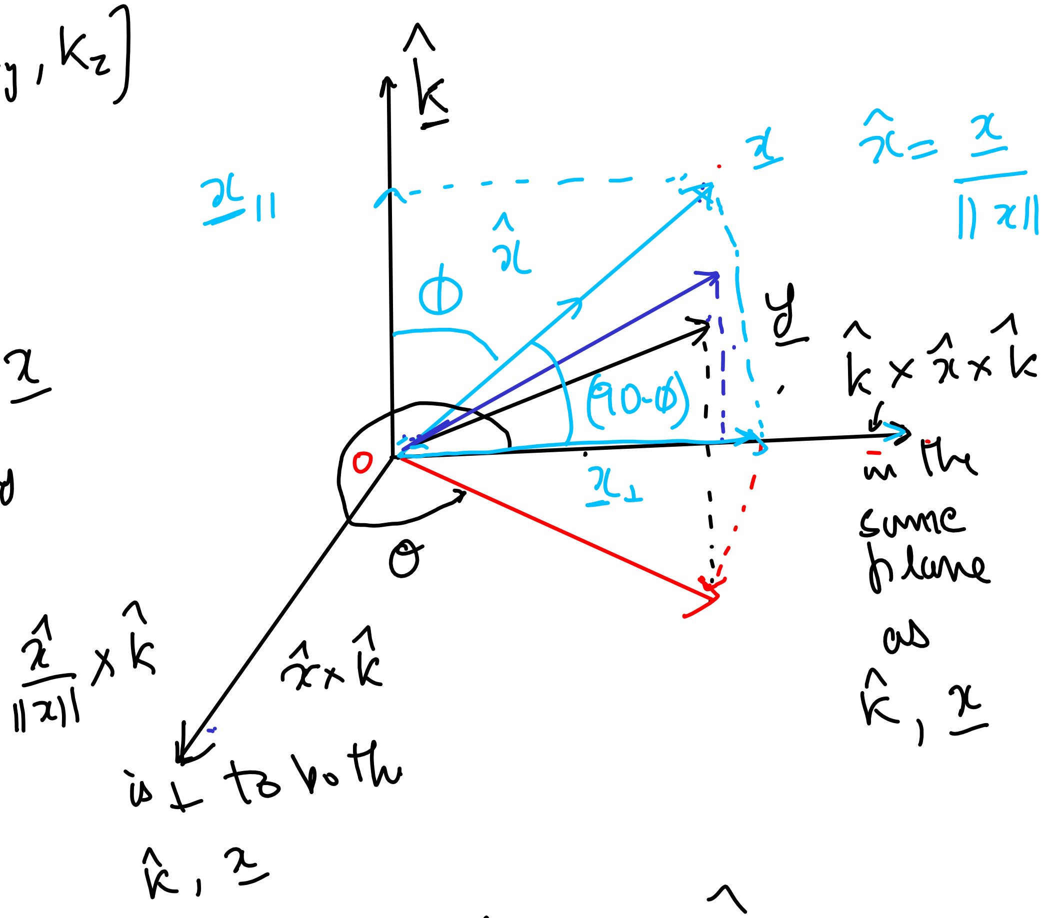
axis angle representation $\longrightarrow$ Rotation matrix (3D)

Rodrigues rotation formula



$\hat{k} = [k_x, k_y, k_z]$

y = by rotating x around $\hat{k}$ by angle θ



In the plane of $\hat{k}$ and $\hat{k} \times \hat{x} \times \hat{k}$,

x can be projected into two component

$$x = x_{\|} + x_{\perp}$$

$$x_{\|} = (\hat{k} \cdot x)\hat{k}$$

$$x_{\perp} = \left[\left(\hat{k} \times (\hat{x} \times \hat{k})\right) \cdot x\right] (\hat{k} \times \hat{x} \times \hat{k})$$

$$x_{\perp} = \hat{k} \times (x \times \hat{k}) = \hat{k} \times (-\hat{k} \times x) = -\hat{k} \times (\hat{k} \times x)$$

$$\hat{k} \cdot x = |\hat{k}||x|\cos\phi = |x|\cos\phi$$

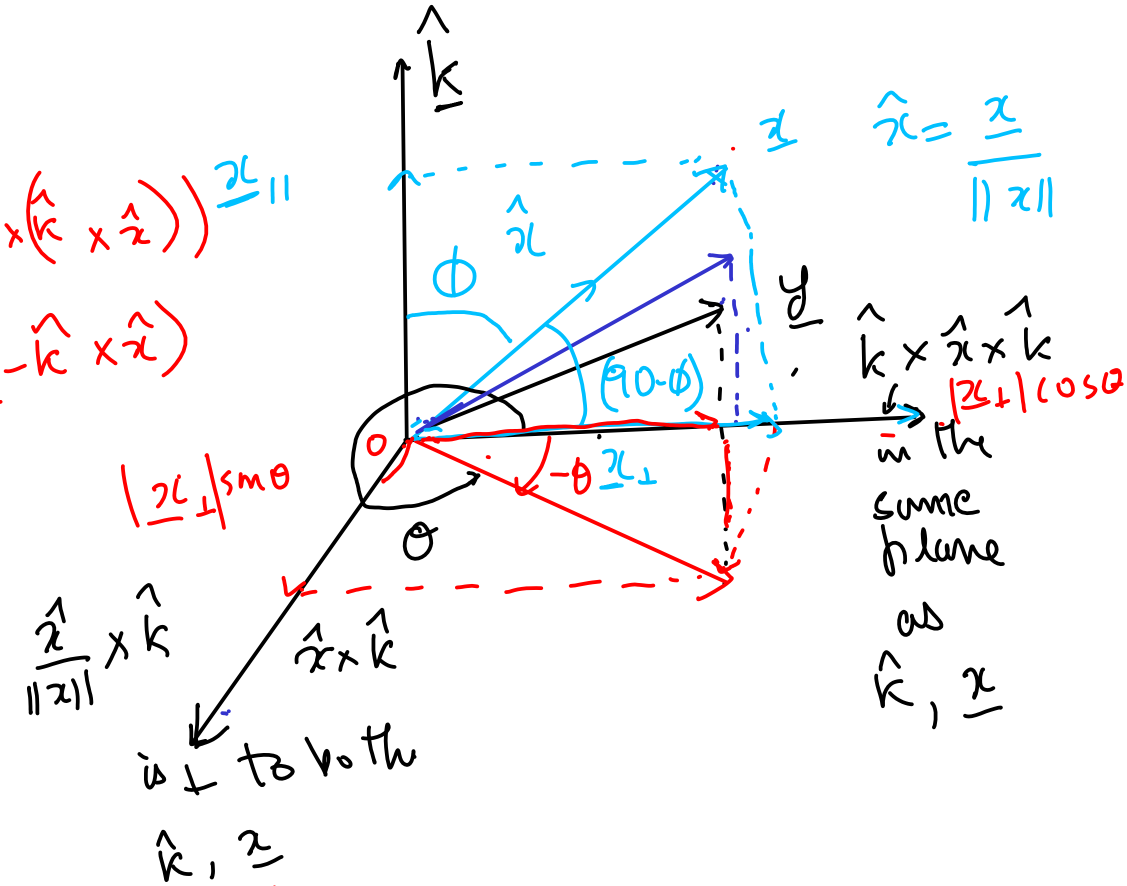
$$\to |x \times \hat{k}| = |x||k|\sin\phi$$

$$\to x \times \hat{k} = (\hat{x} \times \hat{k})(|x|\sin\phi)$$

$$\boxed{|x|\sin\phi} = |x|\sqrt{1-\cos^2\phi}$$

$$\underline{y} = \underline{x}_{\parallel} + \underline{x}_{\perp rot}$$

$$\underline{x}_{\perp rot} = |\underline{x}_{\perp}|\cos(-\theta)\left(-\hat{k} \times (\hat{k} \times \hat{x})\right)$$
$$+ |\underline{x}_{\perp}|\sin(-\theta)(-\hat{k} \times \hat{x})$$



$$\underline{x}_{\perp rot} = |\underline{x}_{\perp}|\cos\theta\left(-\hat{k} \times (\hat{k} \times \hat{x})\right) - |\underline{x}_{\perp}|\sin\theta(-\hat{k} \times \hat{x})$$

$$= \cos\theta\left(-\hat{k} \times (\hat{k} \times \underline{x})\right) - \sin\theta(-\hat{k} \times \underline{x})$$

$$= \sin\theta(\hat{k} \times \underline{x}) - \cos\theta\left(\hat{k} \times (\hat{k} \times \underline{x})\right)$$

$$(\hat{k} \times \hat{x})|\underline{x}_{\perp}| = (\hat{k} \times \hat{x})|x|\sin\phi = \hat{k} \times \underline{x}$$

$$|\underline{a} \times \underline{b}| = |a||b|\sin\phi$$

$$\underline{y} = \underline{x}_{\parallel} + \underline{x}_{\perp} rot$$

Rodrigues formula

$$\boxed{\underline{y} = (\hat{k} \cdot \underline{x})\hat{k} + \sin\theta(\hat{k} \times \underline{x}) - \cos\theta(\hat{k} \times \hat{k} \times \underline{x})}$$

$$x_{\parallel} = (\hat{k}\cdot\underline{x})\,\hat{k} \qquad \underline{x} = \underline{x}_{\parallel} + \underline{x}_{\perp}$$

$$= \underline{x} - x_{\perp}$$

$$= \underline{x} - \left(-\hat{k}\times\left(\hat{k}\times\underline{x}\right)\right)$$

$$= x + \hat{k}\times\left(\hat{k}\times\underline{x}\right)$$

$$\underline{y} = \underline{x} + \hat{k}\times\left(\hat{k}\times\underline{x}\right) + \sin\theta\left(\hat{k}\times\underline{x}\right) - \cos\theta\left(\hat{k}\times\left(\hat{k}\times\underline{x}\right)\right)$$

$$\boxed{\underline{y} = \underline{x} + \sin\theta\left(\underline{\hat{k}}\times\underline{x}\right) + \left(1-\cos(\theta)\right)\left(\hat{k}\times\left(\hat{k}\times\underline{x}\right)\right)}$$

$$\underline{y} = \underbrace{R(\theta,\hat{k})}_{\text{matrix}}\ \underline{x}$$

Writing cross product using matrix notation

$$\underline{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix} \qquad \underline{b} = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

$$\underline{a}\times\underline{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$= \hat{i}\,(a_y b_z - b_y a_z) + j\,(\qquad = \begin{bmatrix} a_y b_z - b_y a_z \\ -a_x b_z + b_x a_z \\ a_x b_y + b_x a_y \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} a_y b_z - b_y a_z \\ -a_x b_z + b_x a_z \\ a_x b_y + b_x a_y \end{bmatrix}}_{\underline{a} \times \underline{b}} = \boxed{\underbrace{\begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ a_y & a_x & 0 \end{bmatrix}}_{\lfloor a_\times \rfloor}} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

Cross product matrix of vector

$$\hat{k} \times \underline{x} = \underbrace{\lfloor k_\times \rfloor}_{K} \underline{x} = K \underline{x}$$

$$\hat{k} \times (\hat{k} \times \underline{x}) = \lfloor K_\times \rfloor (\lfloor K_\times \rfloor \underline{x}) = K(K \underline{x})$$

$$= K^2 \underline{x}$$

$$\underline{y} = \underline{x} + \sin\theta \, (K\underline{x}) + (1 - \cos\theta) K^2 \underline{x}$$

$$= \underbrace{\left[I_{3\times3} + \sin\theta \, K + (1 - \cos\theta) K^2 \right]}_{R(\theta, \hat{k})} \underline{x}$$