ECE 417/598: Review Homework 4

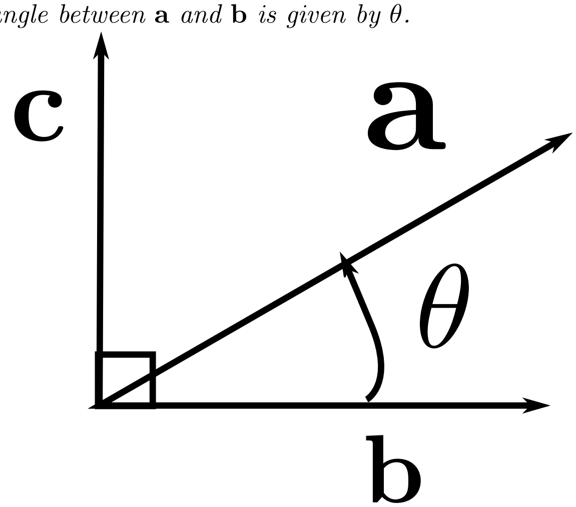
Max marks: 100 marks

Due on March 10th, 2021, midnight, 11:59 PM.

All notes so far are linked here.

1 Trignometry and triangle laws of vector addition

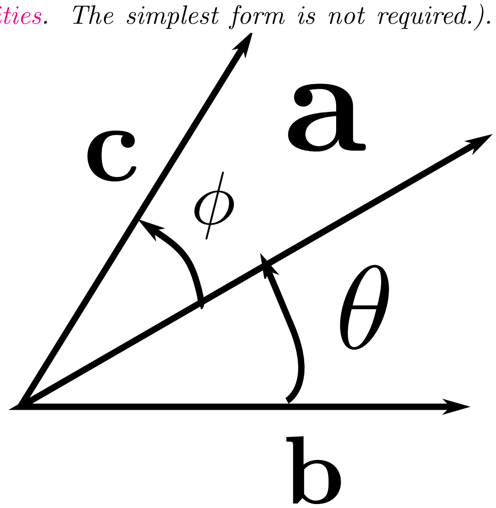
Problem 1 *The magnitude of vector* $\mathbf{a} \in \mathbb{R}^n$ *is given to be* $\|\mathbf{a}\| = \alpha$. *Using the following figure, write* $\mathbf{a}$ *in terms of* α, θ, *vector* $\mathbf{b} \in \mathbb{R}^n$ *and* $\mathbf{c} \in \mathbb{R}^n$. *All three vectors lie in the same plane.* $\mathbf{b}$ *and* $\mathbf{c}$ *are perpendicular to each other. The angle between* $\mathbf{a}$ *and* $\mathbf{b}$ *is given by* θ.



(5 min, 5 marks)

Solution Draw a triangle that has sides parallel to $\mathbf{b}$ and $\mathbf{c}$ and adds up to $\mathbf{a}$. Since $\mathbf{b}$ and $\mathbf{c}$ are perpendicular to each other, it is a right triangle. By trigonometry the side parallel to $\mathbf{b}$ must have length $\alpha\cos(\theta)$ and the side parallel to $\mathbf{c}$ must have the length $\alpha\sin(\theta)$. So by triangle law of vector addition $\mathbf{a} = \alpha\cos(\theta)\frac{\mathbf{b}}{\|\mathbf{b}\|} + \alpha\sin(\theta)\frac{\mathbf{c}}{\|\mathbf{c}\|}$.

Problem 2 *The magnitude of vector* $\mathbf{a} \in \mathbb{R}^n$ *is given to be* $\|\mathbf{a}\| = \alpha$. *Using the following figure, write* $\mathbf{a}$ *in terms of* α, θ, ϕ *vector* $\mathbf{b} \in \mathbb{R}^n$ *and* $\mathbf{c} \in \mathbb{R}^n$. *All three vectors lie in the same plane. The angle between* $\mathbf{a}$ *and* $\mathbf{b}$ *is given by* θ. *The angle between* $\mathbf{a}$ *and* $\mathbf{c}$ *is given by* ϕ. *Assume* $\theta+\phi \neq 0$. *When* $\theta+\phi = \frac{\pi}{/}2$, *is the solution is same as Problem 1? (Hint: You can convert this to Problem 1, by drawing a unit-vector perpendicular to* $\mathbf{b}$. *Call it* $\hat{\mathbf{d}}$. *First write* $\hat{\mathbf{d}}$ *in terms of* $\mathbf{c}$ *and others knowns and then write* $\mathbf{a}$ *in terms of* $\hat{\mathbf{d}}$ *and other knowns. You might want to use* trignometric identities. *The simplest form is not required.).*



(10 min, 10 marks)

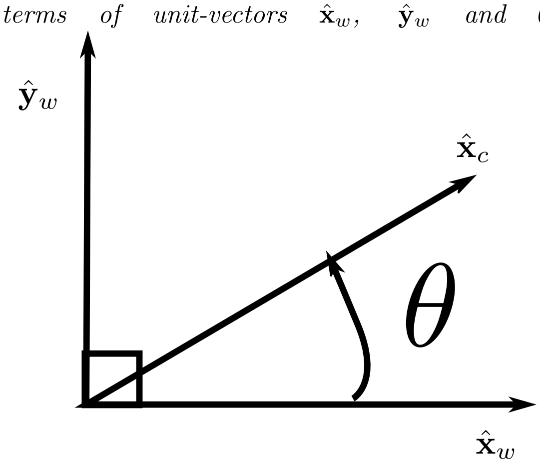
Solution You can convert this to Problem 1, by drawing a unit-vector perpendicular to $\mathbf{b}$. Call it $\hat{\mathbf{d}}$. Define unit vectors $\hat{\mathbf{b}} = \frac{\mathbf{b}}{\|\mathbf{b}\|}$ and $\hat{\mathbf{c}} = \frac{\mathbf{c}}{\|\mathbf{c}\|}$.

$$\hat{\mathbf{c}} = \cos(\theta+\phi)\hat{\mathbf{b}} + \sin(\theta+\phi)\hat{\mathbf{d}}$$
$$\implies \hat{\mathbf{d}} = \frac{\hat{\mathbf{c}} - \cos(\theta+\phi)\hat{\mathbf{b}}}{\sin(\theta+\phi)} \tag{1}$$

(Note that this procedure of finding orthogonal vectors from a set of non-orthogonal vectors has a name: Gram-Schmidt orthogonalization.)

$$
\begin{aligned}
\mathbf{a} &= \alpha\cos(\theta)\hat{\mathbf{b}} + \alpha\sin(\theta)\hat{\mathbf{d}} \\
&= \alpha\cos(\theta)\hat{\mathbf{b}} + \alpha\sin(\theta)\frac{\hat{\mathbf{c}} - \cos(\theta+\phi)\hat{\mathbf{b}}}{\sin(\theta+\phi)} \\
&= \frac{\alpha\hat{\mathbf{b}}}{\sin(\theta+\phi)}\left(\cos(\theta)\sin(\theta+\phi) - \cos(\theta+\phi)\sin(\theta)\right) \\
&\quad + \frac{\alpha\sin(\theta)}{\sin(\theta+\phi)}\hat{\mathbf{c}} \\
&= \frac{\alpha\sin(\phi)}{\sin(\theta+\phi)}\hat{\mathbf{b}} + \frac{\alpha\sin(\theta)}{\sin(\theta+\phi)}\hat{\mathbf{c}} \qquad (2)
\end{aligned}
$$

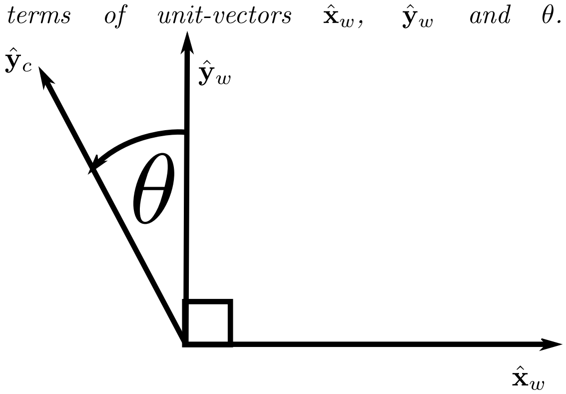
Problem 3 *Find unit-vector $\hat{\mathbf{x}}_c$ in terms of unit-vectors $\hat{\mathbf{x}}_w$, $\hat{\mathbf{y}}_w$ and θ.*



(5 min, 5 marks)

Solution $\hat{\mathbf{x}}_c = \cos(\theta)\hat{\mathbf{x}}_w + \sin(\theta)\hat{\mathbf{y}}_w$

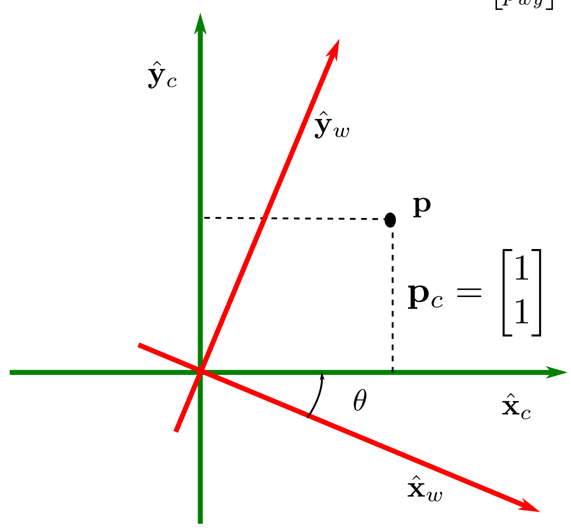
Problem 4 *Find unit-vector $\hat{\mathbf{y}}_c$ in terms of unit-vectors $\hat{\mathbf{x}}_w$, $\hat{\mathbf{y}}_w$ and θ.*



(5 min, 5 marks)

Solution $\hat{\mathbf{y}}_c = -\sin(\theta)\hat{\mathbf{x}}_w + \cos(\theta)\hat{\mathbf{y}}_w$

Problem 5 *Let the coordinates of a vector $\mathbf{p}$ in terms of $\hat{\mathbf{x}}_c$ and $\hat{\mathbf{y}}_c$ be $\mathbf{p}_c = \begin{bmatrix} p_{cx} \\ p_{cy} \end{bmatrix}$, so that: $\mathbf{p} = p_{cx}\hat{\mathbf{x}}_c + p_{cy}\hat{\mathbf{y}}_c$. Using the results from Prob 3 and Prob 4, write $\mathbf{p}$ in terms of $\hat{\mathbf{x}}_w$ and $\hat{\mathbf{y}}_w$. Thus derive the formula for rotation matrix $R(\theta)$ that converts coordinates from $\mathbf{p}_c$ to $\mathbf{p}_w = \begin{bmatrix} p_{wx} \\ p_{wy} \end{bmatrix}$.*



(10 min, 10 marks)

Solution

$$
\begin{aligned}
\mathbf{p} &= p_{cx}\hat{\mathbf{x}}_c + p_{cy}\hat{\mathbf{y}}_c \\
&= p_{cx}(\cos(\theta)\hat{\mathbf{x}}_w + \sin(\theta)\hat{\mathbf{y}}_w) \\
&\quad + p_{cy}(-\sin(\theta)\hat{\mathbf{x}}_w + \cos(\theta)\hat{\mathbf{y}}_w) \\
&= \underbrace{(p_{cx}\cos(\theta) - p_{cy}\sin(\theta))}_{p_{wx}}\hat{\mathbf{x}}_w \\
&\quad + \underbrace{(-p_{cx}\sin(\theta) + p_{cy}\cos(\theta))}_{p_{wy}}\hat{\mathbf{y}}_w \qquad (3)
\end{aligned}
$$

Or

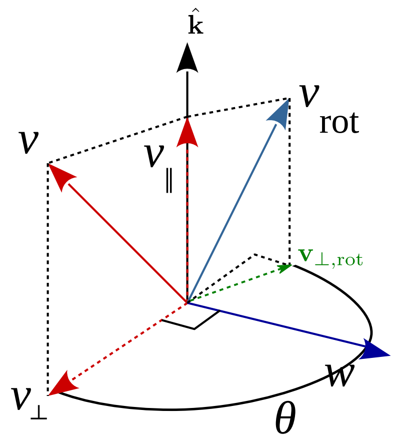
$$
\begin{aligned}
p_{wx} &= p_{cx}\cos(\theta) - p_{cy}\sin(\theta) \\
p_{wy} &= p_{cx}\sin(\theta) + p_{cy}\cos(\theta) \qquad (4)
\end{aligned}
$$

In a matrix representation,

$$
\begin{aligned}
\begin{bmatrix} p_{wx} \\ p_{wy} \end{bmatrix} &= \begin{bmatrix} p_{cx}\cos(\theta) - p_{cy}\sin(\theta) \\ p_{cx}\sin(\theta) + p_{cy}\cos(\theta) \end{bmatrix} \\
&= \underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}_{R(\theta)} \begin{bmatrix} p_{cx} \\ p_{cy} \end{bmatrix} \qquad (5)
\end{aligned}
$$

Problem 6 *We know that $\|\mathbf{v}_{\perp,rot}\| = \|\mathbf{v}_\perp\|$. Write $\mathbf{v}_{\perp,rot}$ in terms of $\mathbf{v}_\perp$, $\mathbf{w}$ and θ. $\mathbf{v}_\perp$*

and $\mathbf{w}$ *are known to be orthogonal to each other.*



(5 min, 5 marks)

Solution

$$\mathbf{v}_{\perp,rot} = \|\mathbf{v}_{\perp}\| \frac{\mathbf{v}_{\perp}}{\|\mathbf{v}_{\perp}\|}\cos(\theta) + \|\mathbf{v}_{\perp}\| \frac{\mathbf{w}}{\|\mathbf{w}\|}\sin(\theta)$$

Problem 7 *In figure 1 find the 3D position of the pothole the World coordinate frame, in terms of* $h = 1$ *(the height of the camera), image-coordinates of the pothole* $\mathbf{u}$ *(provided in figure), camera matrix* K *(provided in figure). The Camera is mounted directly on top of the world frame, both of which are aligned to the gravity vector. The road is a perfect plane with a slope such that the equation of road plane in world-coordinate frame is given by* $100Y_w - Z_w = 0$ *and the pothole lies on the road plane. Provide the formula or pseudo-code for computing the pothole coordinates, and also substitute in the values. (20 min, 20 marks)*

Solution Let $\mathbf{X}_c \in \mathbb{R}^3$ be the 3-D coordinates of the point camera-coordinate frame. Let $\lambda \neq 0 \in \mathbb{R}$ be unknown scalar that represents the depth of the point. Then the camera coordinates can be written in terms of image point using camera pinhole model,

$$\mathbf{X}_c = \lambda K^{-1}\underline{\mathbf{u}} \tag{6}$$

Since the camera-coordinate frame is obtained by rotation along X-axis by $-\frac{\pi}{2}$,

$$\begin{aligned} {}^wR_c &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(-\frac{\pi}{2}) & -\sin(-\frac{\pi}{2}) \\ 0 & \sin(-\frac{\pi}{2}) & \cos(-\frac{\pi}{2}) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \end{aligned} \tag{7}$$

$${}^w\mathbf{t}_c = \begin{bmatrix} 0 \\ 0 \\ h \end{bmatrix} \tag{8}$$

Then the equation of pothole in world coordinates is given by,

$$\mathbf{X}_w = {}^wR_c\mathbf{X}_c + {}^w\mathbf{t}_c = {}^wR_c(\lambda K^{-1}\underline{\mathbf{u}}) + {}^w\mathbf{t}_c \tag{9}$$

The equation of plane in the world-coordinates is given to be $100Y_w - Z_w = 0$. In general, let the equation of plane be given by,

$$\begin{aligned} 0X_w + 100Y_w - Z_w + 0 &= 0 \\ \text{or } \begin{bmatrix} 0 & 100 & -1 & 0 \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \\ 1 \end{bmatrix} &= 0 \\ \text{or } \mathbf{p}^\top \begin{bmatrix} \mathbf{X}_w \\ 1 \end{bmatrix} &= 0, \end{aligned} \tag{10}$$

where the $\mathbf{p} = [0, 100, -1, 0]^\top$. To find the intersection of plane and line we solve for λ in the following equation,

$$\mathbf{p}^\top \begin{bmatrix} {}^wR_c(\lambda K^{-1}\underline{\mathbf{u}}) + {}^w\mathbf{t}_c \\ 1 \end{bmatrix} = 0. \tag{11}$$

Let $\mathbf{p}_{1:3} = \begin{bmatrix} 0 & 100 & -1 \end{bmatrix}^\top$ be the first three coordinats of $\mathbf{p}$, then we have,

$$\mathbf{p}_{1:3}^\top({}^wR_c(\lambda K^{-1}\underline{\mathbf{u}}) + {}^w\mathbf{t}_c) = 0. \tag{12}$$

Rearranging the terms, we get,

$$\lambda = \frac{-\mathbf{p}_{1:3}^\top {}^w\mathbf{t}_c}{\mathbf{p}_{1:3}^\top {}^wR_cK^{-1}\underline{\mathbf{u}}} \tag{13}$$

Then the 3D coordinates of pothole is given by,

$$\mathbf{X}_w = {}^wR_c(\lambda K^{-1}\underline{\mathbf{u}}) + {}^w\mathbf{t}_c \tag{14}$$

$$\text{where } \lambda = \frac{-\mathbf{p}_{1:3}^\top {}^w\mathbf{t}_c}{\mathbf{p}_{1:3}^\top {}^wR_cK^{-1}\underline{\mathbf{u}}} \tag{15}$$

Substitute in the values,

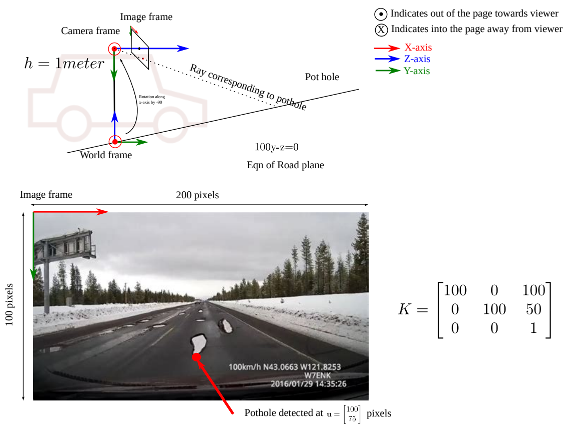


Figure 1: Point-plane triangulation

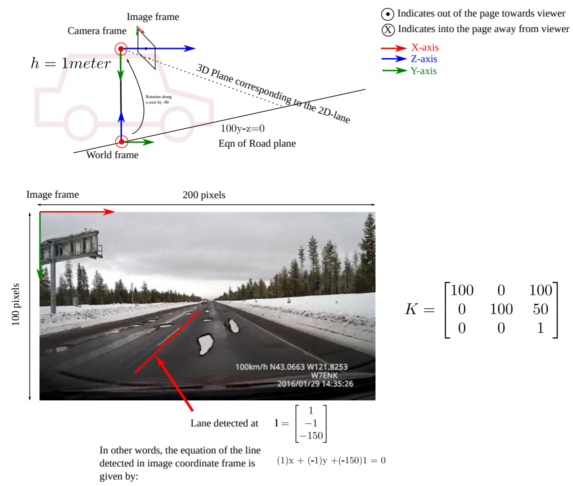


Figure 2: Line-plane triangulation

$$\lambda = 0.009975$$

$$\mathbf{X}_w = \begin{bmatrix} 0 \\ 0.009975 \\ 0.9975 \end{bmatrix}$$

Problem 8 *In figure 2 find the 3D representation of the lane the World coordinate frame, in terms of h (the height of the camera), image-representation of the line $\mathbf{l}$ (provided in figure), camera matrix K (provided in figure). Assume the lane to be a straight line. The Camera is mounted directly on top of the world frame, both of which are aligned to the gravity vector. The road is a perfect plane with a slope such that the equation of road plane in world-coordinate frame is given by $100Y_w - Z_w = 0$ and the lane lies on the road plane. Provide the formula or pseudo-code for computing the 3D representation of the lane, and also substitute in the values. (20 min, 20 marks)*

*Hint 0: **Equation of a plane in 3D.** Equation of a plane in 3D is given by $p_1X + p_2Y + p_3Z + p_4 = 0$. In matrix notation, you can write the equation plane as $\mathbf{p}_{1:3}^\top \mathbf{X} + p_4 = 0$, where $\mathbf{p}_{1:3} = [p_1, p_2, p_3]^\top$.*

*Hint 1: **3D Plane corresponding to the line in image-coordinates.** Let the equation of line in image-coordinates be $\mathbf{l}^\top \underline{\mathbf{u}} = 0$, where $\underline{\mathbf{u}} = \begin{bmatrix} \mathbf{u} \\ 1 \end{bmatrix} \in \mathbb{P}^2$ are all the points on the line. By pinhole camera model, if $\mathbf{X}_c \in \mathbb{R}^3$ are the corresponding points in 3D, then the equation of corresponding plane is given by $\mathbf{l}^\top (K\mathbf{X}_c) = 0$ which can also be written as $(K^\top \mathbf{l})^\top \mathbf{X}_c = 0$. If we compare it to the equation of plane $\mathbf{p}_{1:3}^\top \mathbf{X} + p_4 = 0$, then $\mathbf{p}_{1:3} = K^\top \mathbf{l}$ and $p_4 = 0$.*

*Hint 2: **Intersection of two planes in 3D is a line.** Equation of a plane in 3D is given by $p_1X_w + p_2Y_w + p_3Z_w + p_4 = 0$. In matrix notation, you can write the equation of the plane as $\mathbf{p}_{1:3}^\top \mathbf{X}_w + p_4 = 0$, where $\mathbf{p}_{1:3} = [p_1, p_2, p_3]^\top$. Let's say you have two planes $\mathbf{p}_{1:3}^\top \mathbf{X}_w + p_4 = 0$ and $\mathbf{q}_{1:3}^\top \mathbf{X}_w + q_4 = 0$. Their intersection is a line whose parameteric form is given by (why ? you have all the knowledge required to derive this):*

$$\mathbf{X}_w = \lambda(\mathbf{p}_{1:3} \times \mathbf{q}_{1:3}) + \begin{bmatrix} \mathbf{p}_{1:3}^\top \\ \mathbf{q}_{1:3}^\top \end{bmatrix}^\dagger \begin{bmatrix} -p_4 \\ -q_4 \end{bmatrix}, \quad (16)$$

where $A^\dagger$ denotes the pseudo-inverse of a matrix (a fat matrix in this case) and $\lambda \in \mathbb{R}$ is the free parameter and $\times$ denotes the vector cross-product.

Solution Since $\mathbf{l}$ is the representation of a line in perspective space, let $\underline{\mathbf{u}} \in \mathbb{P}^2$ be the coordinates of the space, the equation of the line is given by

$$\mathbf{l}^\top \underline{\mathbf{u}} = 0 \quad (17)$$

Let $\mathbf{X}_c \in \mathbb{R}^3$ be the 3-D coordinates of the point camera-coordinate frame. Let $\lambda \neq 0 \in \mathbb{R}$ be unknown scalar that represents the depth of the point. Following the pinhole camera model, $\underline{\mathbf{u}} =$

$K\mathbf{X}_c$, Then the equation of the plane in camera-coordinate frame is given by,

$$\mathbf{l}^\top(K\mathbf{X}_c) = 0 \tag{18}$$

Since the camera-coordinate frame is obtained by rotation along X-axis by $-\frac{\pi}{2}$,

$$\begin{aligned} {}^wR_c &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(-\frac{\pi}{2}) & -\sin(-\frac{\pi}{2}) \\ 0 & \sin(-\frac{\pi}{2}) & \cos(-\frac{\pi}{2}) \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \end{aligned} \tag{19}$$

$${}^w\mathbf{t}_c = \begin{bmatrix} 0 \\ 0 \\ h \end{bmatrix} \tag{20}$$

Then the equation of pothole in world coordinates is given by,

$$\mathbf{X}_w = {}^wR_c\mathbf{X}_c + {}^w\mathbf{t}_c \tag{21}$$

Rearranging the equation, we can get the equation of camera coordinates in terms of world coordinates,

$$\mathbf{X}_c = {}^wR_c^\top\mathbf{X}_w - {}^wR_c^\top {}^w\mathbf{t}_c \tag{22}$$

Now, we can write the equation of plane corresponding the line in the image in world coordinate frame,

$$\mathbf{l}^\top(K^wR_c^\top\mathbf{X}_w - K^wR_c^\top {}^w\mathbf{t}_c) = 0 \tag{23}$$

$$\text{or } \mathbf{l}^\top K^wR_c^\top\mathbf{X}_w - \mathbf{l}^\top K^wR_c^\top {}^w\mathbf{t}_c = 0 \tag{24}$$

The equation of plane in the world-coordinates is given to be $100Y_w - Z_w = 0$. In general, let the equation of plane be given by,

$$\begin{aligned} 0X_w + 100Y_w - Z_w + 0 &= 0 \\ \text{or } \begin{bmatrix} 0 & 100 & -1 \end{bmatrix} \begin{bmatrix} X_w \\ Y_w \\ Z_w \end{bmatrix} + 0 &= 0 \\ \text{or } \mathbf{p}^\top\mathbf{X}_w + p_4 &= 0, \end{aligned} \tag{25}$$

where the $\mathbf{p} = [0, 100, -1]^\top$ and $p_4 = 0$. Now, we have two equations that we have to find the intersection of. A line in 3D is given by intersection of two planes,

$$\mathbf{l}^\top K^wR_c^\top\mathbf{X}_w - \mathbf{l}^\top K^wR_c^\top {}^w\mathbf{t}_c = 0 \tag{26}$$

$$\mathbf{p}^\top\mathbf{X}_w + p_4 = 0. \tag{27}$$

Now, we have three unknowns and two equations, we can solve an under-determined system of equations of using psuedo-inverse,

$$\begin{aligned} \mathbf{l}^\top K^wR_c^\top\mathbf{X}_w &= \mathbf{l}^\top K^wR_c^\top {}^w\mathbf{t}_c \\ \mathbf{p}^\top\mathbf{X}_w &= -p_4. \end{aligned} \tag{28}$$

Or we can collect the equations in matrix form,

$$\underbrace{\begin{bmatrix} \mathbf{l}^\top K^wR_c^\top \\ \mathbf{p}^\top \end{bmatrix}}_{A} \mathbf{X}_w = \underbrace{\begin{bmatrix} \mathbf{l}^\top K^wR_c^\top {}^w\mathbf{t}_c \\ -p_4 \end{bmatrix}}_{\mathbf{b}}. \tag{29}$$

The pseudo-inverse solution gives us a minimum norm solution of the system of equations. That is we get a point that is closest to the origin.

$$\mathbf{X}_{w,0} = A^\dagger\mathbf{b} = A^\top(AA^\top)^{-1}\mathbf{b}. \tag{30}$$

Here, we have written the equation of plane in $\mathbf{p}^\top\mathbf{x} + p_4 = 0$ form. Another way of writing the same equation is $\mathbf{p}^\top(\mathbf{x} - \mathbf{x}_0) = 0$ where $\mathbf{x}_0$ is any point on the plane. Since, we have got the point on both the planes, we can write the equation of planes in the form $\mathbf{X}_w - \mathbf{X}_{w,0}$,

$$\mathbf{l}^\top K^wR_c^\top(\mathbf{X}_w - \mathbf{X}_{w,0}) = 0 \tag{31}$$

$$\mathbf{p}^\top(\mathbf{X}_w - \mathbf{X}_{w,0}) = 0 \tag{32}$$

Note that $\mathbf{l}^\top K^wR_c^\top$ is 1×3 row vector,

$$({}^wR_cK^\top\mathbf{l})^\top(\mathbf{X}_w - \mathbf{X}_{w,0}) = 0 \tag{33}$$

$$\mathbf{p}^\top(\mathbf{X}_w - \mathbf{X}_{w,0}) = 0 \tag{34}$$

Since $\mathbf{X}_w - \mathbf{X}_{w,0}$ is a vector perpendicular to both the vectors in 3D, we can compute the $\mathbf{X}_w - \mathbf{X}_{w,0}$ by cross product,

$$\mathbf{X}_w - \mathbf{X}_{w,0} = \lambda(\mathbf{p} \times {}^wR_cK^\top\mathbf{l}), \tag{35}$$

where $\lambda \in \mathbb{R}$ is an arbitrary scalar. The equation of line in 3D is given by,

$$\mathbf{X}_w = \lambda(\mathbf{p} \times {}^wR_cK^\top\mathbf{l}) + \mathbf{X}_{w,0}. \tag{36}$$

$$\mathbf{X}_w = \lambda \begin{bmatrix} -9900 \\ 100 \\ 10000 \end{bmatrix} + \begin{bmatrix} 0.50505 \\ 0.0049995 \\ 0.49995 \end{bmatrix}$$

Problem 9 *You are a part of Tesla self-driving team. Team 1 provides you with lane-detection algorithms and their output. Team 2 provides you with detailed maps of road conditions. Your* *task is to write a function that solves problem 8 for arbitrary lanes detected by team 1 and for arbitrary plane provided by team 2. (Hint: Equation of a plane 3D is very similar to equation of line in 2D). What input representations of lane and plane would you ask for? Write a general algorithm or pseudo-code that solves problem 8. (30 min, 10 marks)*

Solution

- I will ask Team 1 to provide the equation of lane detected in perspective representation $\mathbf{l}$ so that all the points $\underline{\mathbf{u}} \in \mathbb{P}^2$ on the line in image-coordinates satisfy $\mathbf{l}^\top \underline{\mathbf{u}} = 0$.
- I will ask Team 2 to provide the equation of road plane in world coordinates in perspective representation $\underline{\mathbf{p}}$ so that all the points $\mathbf{X}_w \in \mathbb{R}^3$ on the plane satisfy $\underline{\mathbf{p}}^\top \begin{bmatrix} \mathbf{X}_w \\ 1 \end{bmatrix} = 0$. Equivalently, we define $\underline{\mathbf{X}}_w = \begin{bmatrix} \mathbf{X}_w \\ 1 \end{bmatrix}$, then the representation of line in perspective space is $\underline{\mathbf{p}}^\top \underline{\mathbf{X}}_w = 0$.

Algorithm for solving Problem 8.

1. Find the equation of plane in 3D that corresponds to the equation of the line $\mathbf{l}^\top \underline{\mathbf{u}} = 0$ in the image. The equation of plane is given by $\mathbf{l}^\top K \mathbf{X}_c = 0$.
2. Transform the equation of plane from camera coordinates to world coordinate frame. The equation of plane in world coordinate frame is given by,
$$\mathbf{l}^\top K {}^w R_c^\top \mathbf{X}_w - \mathbf{l}^\top K {}^w R_c^\top {}^w\mathbf{t}_c = 0.$$
This equation is of the form,
$$\mathbf{q}_{1:3}^\top \mathbf{X}_w + q_4 = 0,$$
$\mathbf{q}_{1:3} = {}^w R_c K^\top \mathbf{l}$ and $q_4 = -\mathbf{l}^\top K {}^w R_c^\top {}^w\mathbf{t}_c$.
3. Let $\mathbf{p}_{1:3}$ be the first three coordinates of the vector $\underline{\mathbf{p}}$ and p_4 be the fourth coordinate. Then the equation of the plane is,
$$\mathbf{p}_{1:3}^\top \mathbf{X}_w + p_4 = 0$$
4. The equation of line in 3D given by intersection of two planes,
$$\mathbf{q}_{1:3}^\top \mathbf{X}_w + q_4 = 0 \tag{37}$$
$$\mathbf{p}_{1:3}^\top \mathbf{X}_w + p_4 = 0, \tag{38}$$
can be computed by constructing a matrix $A \in \mathbb{R}^{2\times 3}$ such that $\mathbf{b} \in \mathbb{R}^2$ such that,
$$\underbrace{\begin{bmatrix} \mathbf{q}_{1:3}^\top \\ \mathbf{p}_{1:3}^\top \end{bmatrix}}_{A} \mathbf{X}_w = \underbrace{\begin{bmatrix} -q_4 \\ -p_4 \end{bmatrix}}_{\mathbf{b}}. \tag{39}$$
Now, we can solve for a point $\mathbf{X}_{w,0}$ closest to the origin that lies on both the planes,
$$\mathbf{X}_{w,0} = A^\dagger \mathbf{b} = A^\top (AA^\top)^{-1} \mathbf{b}. \tag{40}$$
The equation of planes can then be re-written as,
$$A(\mathbf{X}_w - \mathbf{X}_{w,0}) = 0. \tag{41}$$
where
$$A = \begin{bmatrix} \mathbf{q}_{1:3}^\top \\ \mathbf{p}_{1:3}^\top \end{bmatrix}. \tag{42}$$
Since $\mathbf{X}_w - \mathbf{X}_{w,0}$ is a vector that is perpendicular to both $\mathbf{q}_{1:3}$ and $\mathbf{p}_{1:3}$ in 3D, the vector is given by cross product,
$$\mathbf{X}_w - \mathbf{X}_{w,0} = \lambda(\mathbf{q}_{1:3} \times \mathbf{q}_{1:3}), \tag{43}$$
where $\lambda \in \mathbb{R}$ is the free parameter. Thus the equation of line is given by,
$$\mathbf{X}_w = \lambda(\mathbf{q}_{1:3} \times \mathbf{p}_{1:3}) + \mathbf{X}_{w,0}. \tag{44}$$