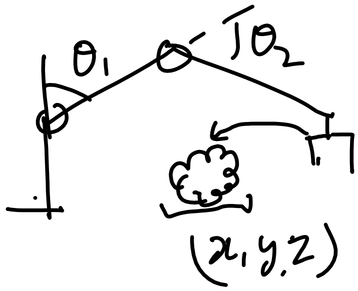
Forward and $\boxed{\text{inverse kinematics}}$

What should the joint ~~angles~~ state/conf. of the robot be so that the end-effector reaches a desired pose?



[How to move the end-effector to a desired pose [position + orientation]]

← end-effector

gripper or suction cup

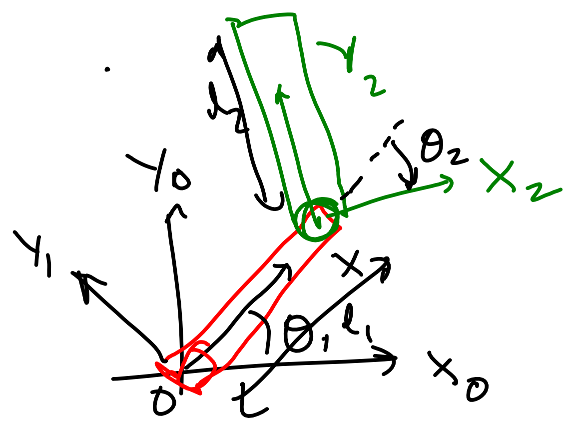
Forward kine.

If my joint ~~angles~~ state/conf. are given what would the pose of end-effector be?

Forward kinematics

$$ {}^{0}T_{2} = {}^{0}T_{1}(\theta_1, l_1)\, {}^{1}T_{2}(\theta_2, l_2) $$

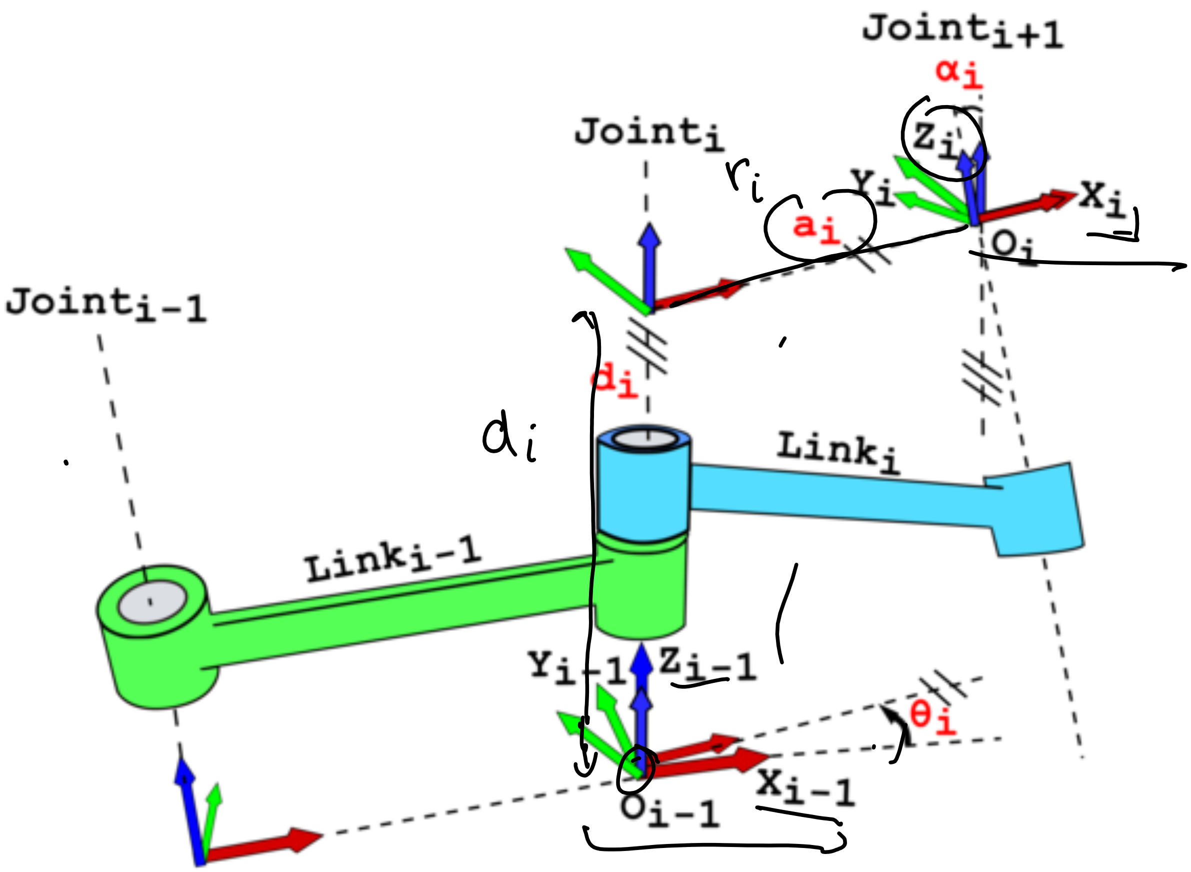
in terms of $\underbrace{\theta_1 \text{ and } \theta_2}_{\text{Given}}$



[Denavit Hartenberg Parameters/Convention]

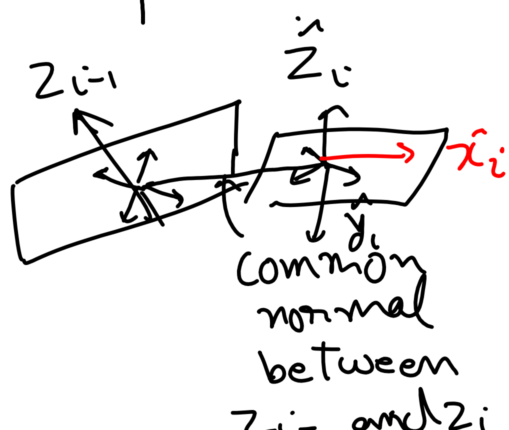
Denavit Hartenberg parameters

https://www.youtube.com/watch?v=rA9tm0gTln8



① $\hat{z}_{i-1}, \hat{z}_i$ aligned along the axis of rotation

② Choose $\hat{x}_i$ along the common normal between $\hat{z}_{i-1}, \hat{z}_i$



③ $\hat{y}_i = \hat{z}_i \times \hat{x}_i$

a) θ_i = Rotation along z_{i-1} (to align x_{i-1} with x_i)

b) d_i = translation along z_{i-1} (to align the origins)

c) α_i = Rotation along x_i (to align z_{i-1} with z_i)

d) r_i/a_i = translation along x_i (to align the origin)

(a) and (b) can be swapped
(c) and (d)

But Tranformation along z goes first
followed by " " x

$$ {}^{i-1}T_i = {}^{i-1}T_{zi} \; {}^{i-1}T_{xi} $$

$i-1$ target, i source

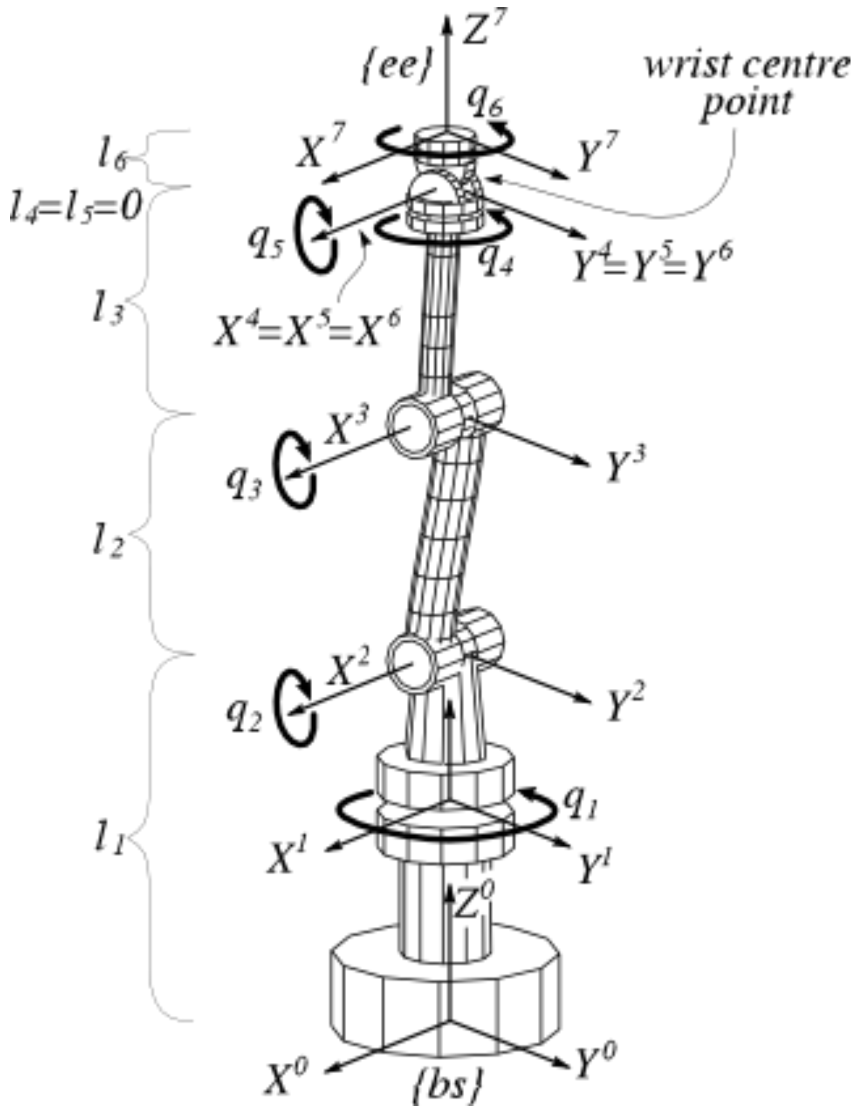
Tranformations are applied right to left

$$ {}^{i-1}T_{xi} = \begin{bmatrix} 1 & 0 & 0 & r_i \\ 0 & \cos\alpha_i & -\sin\alpha_i & 0 \\ 0 & \sin\alpha_i & \cos\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times 1} $$

3x3   3x1

$$ {}^{i-1}T_{zi} = \begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 & 0 \\ \sin\theta_i & \cos\theta_i & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}_{4\times 1} $$

θ_i, d_i



Z^7
$\{ee\}$
wrist centre point
q_6
X^7
Y^7
l_6
$l_4=l_5=0$
q_5
q_4
$Y^4=Y^5=Y^6$
l_3
$X^4=X^5=X^6$
X^3
q_3
Y^3
l_2
X^2
q_2
Y^2
q_1
l_1
X^1
Y^1
Z^0
X^0
Y^0
$\{bs\}$

Numerical solutions to IK problems: Jacobian inverse technique

Forward and inverse kinematics

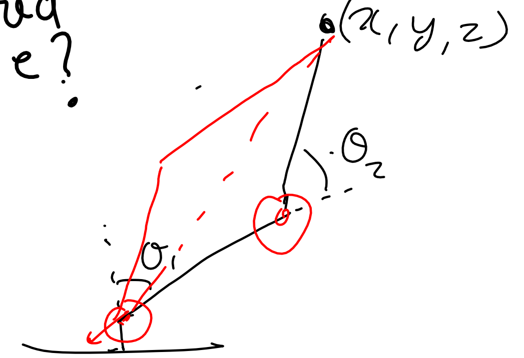
Inverse kinematics

↳ closed form solution

↳ Numerical/Iterative solutions

What should the joint ~~angles~~ state/conf. of the robot be so that the end-effector reaches a desired pose?

only polynomials of degree ≤ 5 have closed form solutions



$$\cancel{\theta_1 = x_1^2 + y_2 + \cos^{-1}\left(\frac{z}{y}\right)}$$

$\cos(\theta_1)$ $\underbrace{\sin(\theta_1)}_{x_1}$

$\sqrt{1 - x_1^2}$

Newton-Raphson method (Gradient descent) (optimization solution)

Suppose a function $y = f(x)$

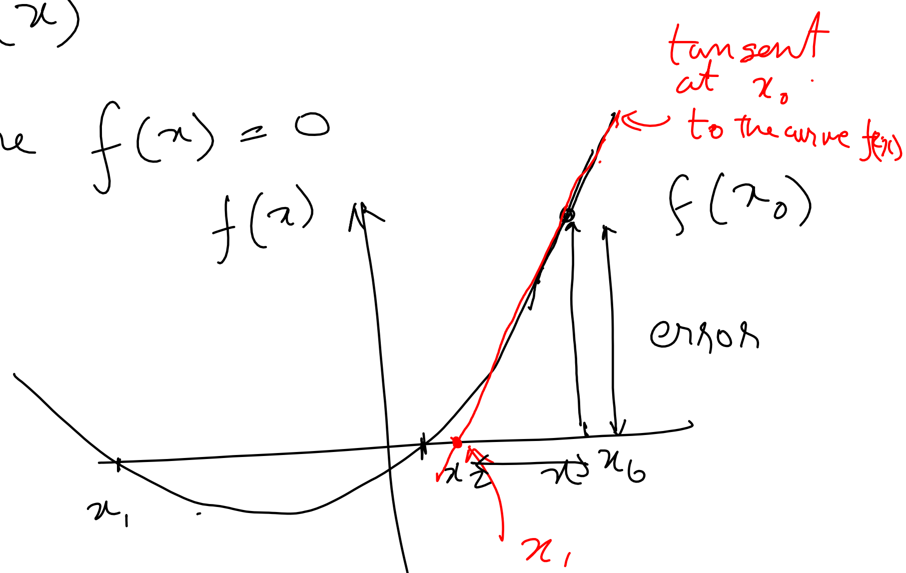
we want to find ~~all~~ any x where $f(x) = 0$

① Initial guess

x_0

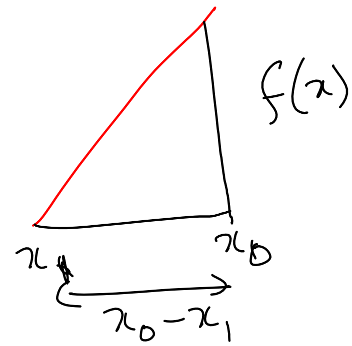
↑

iteration



(2) Improve the initial guess

$$f'(x)\Big|_{x_0} = \frac{f(x_0)}{x_0 - x_1}$$



$$\Rightarrow \quad x_0 - x_1 = \left[f'(x_0)\right]^{-1} f(x_0)$$

$$x_1 = x_0 - \left[f'(x_0)\right]^{-1} f(x_0)$$

(2) Repeat

$$x_2 = x_1 - \left[f'(x_1)\right]^{-1} f(x_1)$$

$$x_n = x_{n-1} - \left[f'(x_{n-1})\right]^{-1} f(x_{n-1})$$

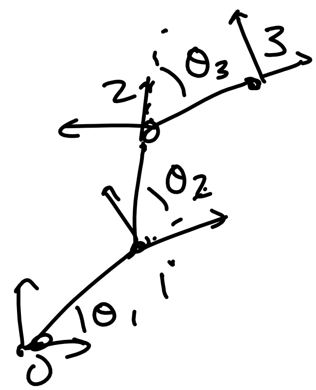
---

find the square root of 2

$$f(z) = z^2 - 2 = 0$$

$$z_n = z_{n-1} - \frac{(z_{n-1}^2 - 2)}{2z_{n-1}}$$

Forward Kinematics



$$\underbrace{{}^0T_3}_{\text{Given}} = {}^0T_1(\theta_1)\;{}^1T_2(\theta_2)\;{}^2T_3(\theta_3)$$

Find

$$\underbrace{\left({}^0T_3\right)^{-1}\,{}^0T_1(\theta_1)\;{}^1T_2(\theta_2)\;{}^2T_3(\theta_3) - I_{4\times4}} = 0_{4\times4}$$

$$\underbrace{F_{4\times4}(\theta_1, \theta_2, \theta_3)} = 0_{4\times4}$$

$$\underline{f}_{16\times1}\left(\underbrace{\begin{bmatrix}\theta_1\\ \theta_2\\ \theta_3\end{bmatrix}_{3\times1}}_{\text{vector}}\right) = \begin{bmatrix}0\\ \\ 0\end{bmatrix}_{16\times1}$$

$f(\underline{\theta})$ is a vector function

$$\underline{f}_{16\times1}(\underline{\theta}) = \left.\begin{bmatrix}f_1(\underline{\theta})\\ f_2(\underline{\theta})\\ \vdots\\ f_{16}(\underline{\theta})\end{bmatrix}\right\}\ \text{vector-valued vector function}$$

What's a Jacobian matrix

is ~~a~~ many derivative of vector-valued vector function

$$J\left[\underline{f}_{16\times1}(\underline{\theta}_{3\times1})\right] = \begin{array}{cc} & \text{inputs} \\ \text{outputs} & \begin{bmatrix} \frac{\partial f_1}{\partial\theta_1} & \frac{\partial f_1}{\partial\theta_2} & \cdots & \frac{\partial f_1}{\partial\theta_3} \\ \frac{\partial f_2}{\partial\theta_1} & \ddots & & \\ \vdots & & \ddots & \\ \frac{\partial f_{16}}{\partial\theta_1} & & & \ddots \end{bmatrix} \end{array}$$

Hand-waviness

scalar Newton Raphson generalizes to (vector Newton - Raphson)

Gauss Newton method

(1) Initial guess $\underline{\theta}_0$

Psuedo-inverse dagger

(2) $\underline{\theta}_1 = \underline{\theta}_0 - J\left[\underline{f}_{16\times1}(\underline{\theta}_{3\times1})\right]^{\dagger} \underline{f}_{16\times1}(\underline{\theta}_{3\times1})$

$$x_1 = x_0 - \left[f'(x_0)\right]^{-1} f(x_0)$$

(3) $$\underline{\theta}_n = \underline{\theta}_{n-1} - J\left[\underline{f}(\underline{\theta}_{n-1})\right]^{+} \underline{f}(\underline{\theta}_{n-1})$$

(1) What is a Psudo inverse?

(2) How can we compute the Jacobian?

(3) What is the relationship (similarities and differences) b/w Newton-Raphson, Gauss Newton, Gradient descents

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Problem 4 of Midterm helps in computing derivative of rotation matrices

$$K^3 = -K$$

$$R(\theta, \hat{k}) = I_{3\times 3} + \sin\theta\, K + (1-\cos\theta) K^2$$

$$\frac{\partial}{\partial \theta} R(\theta, \hat{k}) = 0 + \cos\theta\, \underset{\sim}{K} + (0 + \sin\theta) K^2$$

$$= -\cos\theta\, K^3 + \sin\theta\, K^2$$

$$= K\left(-\cos\theta\, K^2 + \sin\theta\, K\right)$$

$$= K\left(I - I - \cos\theta\, K^2 + \sin\theta\, K\right)$$

$$= K\left(I - \cos\theta K^2 + \sin\theta K\right) - K$$

$$= K\left(I + K^2 - K^2 - \cos\theta K^2 + \sin\theta K\right) - K$$

$$= K\left(I + (1-\cos\theta)K^2 + \sin\theta K\right) - K - K^3$$

$$= K R(\theta, \hat{k}) - \cancel{K} + \cancel{K}$$

$$\boxed{\frac{\partial}{\partial\theta} R(\theta, \hat{k}) = K R(\theta, \hat{k})}$$

$$\boxed{\frac{d}{dx} f(x) = a f(x)}$$

$\downarrow \uparrow$

scalar matrix

$$R(\theta, \hat{k}) = \exp(\theta K) = \frac{I}{1} + \frac{\theta K^1}{1!} + \frac{\theta^2 K^2}{2!} + \frac{\theta^3 K^3}{3!} + \dots \infty$$

matrix exponention

$$\frac{\partial}{\partial\theta} R(\theta, \hat{k}) = 0 + \frac{K^1}{1!} + \frac{2\theta K^2}{2!} + \frac{3\theta^2 K^3}{3!} + \dots$$

$$= K\left(\frac{I}{1!} + \frac{\theta K}{1!} + \frac{\theta^2 K^2}{2!} + \dots\right)$$

$$T = \begin{bmatrix} R(\theta, \hat{k})_{3\times 3} & \underline{t}_{3\times 1} \\ \underline{0}^T & 1 \end{bmatrix}$$

$$\frac{\partial T}{\partial\theta} = \begin{bmatrix} K R(\theta, \hat{k}) & 0 \\ 0 & 0 \end{bmatrix}$$

$$K = [\hat{\underline{k}}]_\times$$

Jacobian of forward kinematics

$$ {}^{0}T_3 = \underbrace{{}^{0}T_1(\theta_1)\ {}^{1}T_2(\theta_2)\ {}^{2}T_3(\theta_3)} $$

$$ \boxed{\frac{\partial\, {}^{0}T_3}{\partial \theta_1} = \left[\frac{\partial\, {}^{0}T_1(\theta_1)}{\partial \theta_1}\right] {}^{1}T_2(\theta_2)\ {}^{2}T_3(\theta_3)} $$

$$ \frac{\partial}{\partial \theta} A(\theta)\, B(\theta) = \left[\frac{\partial A(\theta)}{\partial \theta}\right] B(\theta) + A(\theta) \left[\frac{\partial B(\theta)}{\partial \theta}\right] $$

scalar

$$ C(\theta) = A(\theta)\, B(\theta) $$

$$ \begin{bmatrix} c_{11} & \cdots & \\ \vdots & & \end{bmatrix} = \begin{bmatrix} a_{11}(\theta) & \cdots & a_{1n}(\theta) \\ & & \end{bmatrix} \begin{bmatrix} b_{11}(\theta) & \\ \vdots & \\ b_{n1}(\theta) & \end{bmatrix} $$

$$ c_{11} = a_{11}(\theta)\, b_{11}(\theta) + a_{12}(\theta)\, b_{21}(\theta) + \cdots + a_{1n}(\theta)\, b_{n1}(\theta) $$

$$ \frac{\partial c_{11}}{\partial \theta} = \left\{ \left[\frac{\partial a_{11}(\theta)}{\partial \theta}\right] b_{11}(\theta) + a_{11}(\theta) \left[\frac{\partial}{\partial \theta} b_{11}(\theta)\right] \right\} $$

$$ + \cdots $$

$$\left\{\frac{\partial({}^0T_3)}{\partial\theta_2}\right\}_{4\times4} = {}^0T_1(\theta_1)\left[\frac{\partial}{\partial\theta_2}{}^1T_2(\theta_2)\right]{}^2T_3(\theta_3)$$

$$\mathrm{vec}({}^0T_3)_{4\times4} = [\;]_{16\times1} = \text{row-wise vectorization operator}$$

$$J_{\underline{\theta}}\left[\mathrm{vec}({}^0T_3)\right] = \begin{bmatrix} | & | & | \\ \frac{\partial\,\mathrm{vec}({}^0T_3)}{\partial\theta_1} & \frac{\partial\,\mathrm{vec}({}^0T_3)}{\partial\theta_2} & \frac{\partial\,\mathrm{vec}({}^0T_3)}{\partial\theta_3} \\ | & | & | \end{bmatrix}_{16\times3}$$

12×3

$$\frac{\partial}{\partial\theta_1}{}^0T_1(\theta_1) = \left[\frac{\partial T_z(\theta,d)}{\partial\theta_1}\right]\left[T_x(r,\alpha)\right]$$

$$= \begin{bmatrix} K_z R_z(\theta) & \underline{0}'_{3\times} \\ \boxed{\underline{0}^T_{1\times3} \quad 0'} \end{bmatrix} T_x(r,\alpha)$$

$$ {}^0T_3 = \begin{bmatrix} & \\ 0^T_{1\times3} & 1 \end{bmatrix}$$

we know how to compute Jacobian ✓

$$\underline{\theta}_n = \underline{\theta}_{n-1} - J\left[\underline{f}(\underline{\theta}_{n-1})\right]^{\dagger} \underbrace{\underline{f}(\theta_{n-1})}_{\text{error}}$$

$$\underline{f}(\underline{\theta}_{n-1}) = vec\left({}^{0}\underline{T}_3(\underline{\theta}_{n-1}) - \boxed{{}^{0}T_3^{*}}\right)$$

Desired end-effector pose

equivalent in terms of roots

$$\underline{f}(\underline{\theta}_{n-1}) = vec\left(\left[{}^{0}T_3^{*}\right]^{-1}\, {}^{0}T_3(\underline{\theta}_{n-1}) - \underline{I}_{4\times 4}\right)$$

$$J\left[\underline{f}(\underline{\theta}_{n-1})\right] = J\left[vec\left[{}^{0}T_3(\underline{\theta}_{n-1})\right]\right]$$

$$\underline{\theta}_n = \underline{\theta}_{n-1} - J\left[vec\left({}^{0}T_3(\theta_{n-1})\right)\right]^{\dagger} vec\left({}^{0}\underline{T}_3(\underline{\theta}_{n-1}) - \boxed{{}^{0}T_3^{*}}\right)$$

Pseudo Inverse

systems of eqns in multiple variable

① $$\boxed{\begin{aligned} 2x + 3y &= 4 \\ 5x + 7y &= 9 \end{aligned}} \quad \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \end{bmatrix} \quad A\underline{x} = \underline{b}$$

1 solution

$\underline{x} = A^{-1}\underline{b}$

No. of eqns = No of unknowns

② No of eqns > No of unknowns } no solution

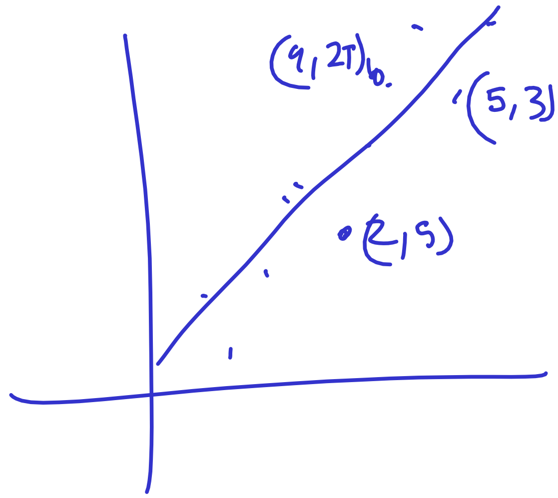
③ " < " } multiple solution

(2) No of eqns > No of unknowns

$$\begin{bmatrix} 2x + 3y \\ 5x + 7y \\ 6x + 13y \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 13 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} 2 & 3 \\ 5 & 7 \\ 6 & 13 \end{bmatrix}}_{A_{3\times 2}} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 9 \\ 13 \end{bmatrix}$$

$$A_{3\times 2}\ \underline{x}_{2\times 1} = \underline{b}_{3\times 1}$$



$$\left.\begin{aligned} 5 &= 2m + c \\ 27 &= 9m + c \\ 3 &= 5m + c \end{aligned}\right] \longrightarrow \text{Least square linear regression}$$