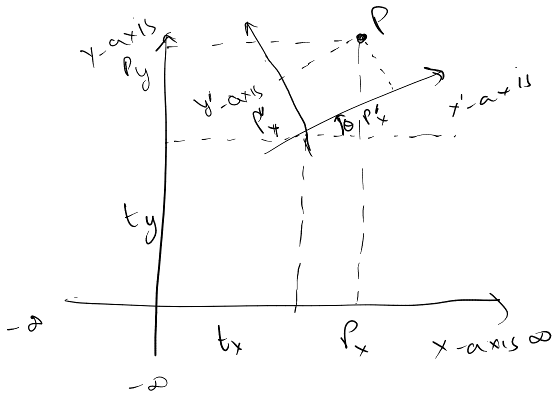
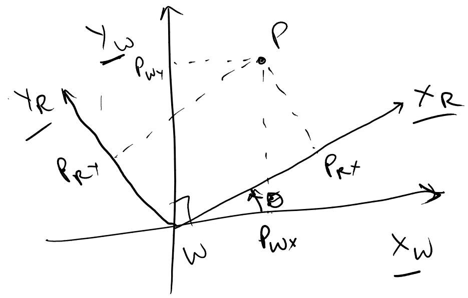
Coordinate transformations



Rotation in 2D

W = World



$$\left.\begin{array}{l} \underline{X_W} = \begin{bmatrix}1\\0\end{bmatrix} \\ \underline{Y_W} = \begin{bmatrix}0\\1\end{bmatrix}\end{array}\right\}$$ Basis vector → a set of vectors

(1) linearly independent

(2) span the space

$Y_W^T Y_W = X_W^T X_W = 1$ (3) unit vector

$X_W^T Y_W = 0$ (4) Mutually orthogonal

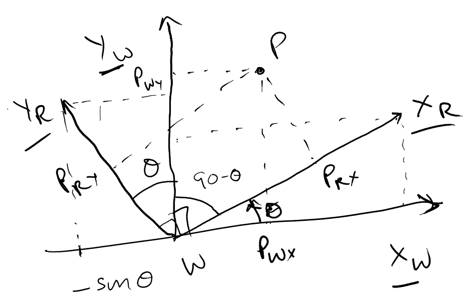
$$P = P_{Wx} \underline{X_W} + P_{Wy} \underline{Y_W} = \begin{bmatrix} \underline{X_W} & \underline{Y_W} \end{bmatrix} \begin{bmatrix} P_{Wx} \\ P_{Wy} \end{bmatrix} \rightarrow \underline{P_W}$$

$$= P_{Rx} \underline{X_R} + P_{Ry} \underline{Y_R} = \underbrace{\begin{bmatrix} \underline{X_R} & \underline{Y_R} \end{bmatrix}}_{B_R} \underbrace{\begin{bmatrix} P_{Rx} \\ P_{Ry} \end{bmatrix}}_{\underline{P_R}}$$

$$B_W = \begin{bmatrix} \underline{X_W} & \underline{Y_W} \end{bmatrix}$$

$$B_W \underline{P_W} = B_R \underline{P_R}$$

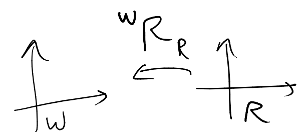
$$\boxed{\underline{P_W} = B_W^{-1} B_R \underline{P_R}}$$



$$\|X_R\| = 1$$

$$\underline{X_R} = \cos(\theta)\, \underline{X_W} + \sin\theta\, \underline{Y_W}$$
$$= \begin{bmatrix} \underline{X_W} & \underline{Y_W} \end{bmatrix} \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} = B_W \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$$

$$\underline{Y_R} = -\sin(\theta)\, \underline{X_W} + \cos\theta\, \underline{Y_W}$$
$$= B_W \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$$



$$B_R = \begin{bmatrix} \underline{X_R} & \underline{Y_R} \end{bmatrix} = B_W \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{{}^W R_R(\theta)}$$

$$B_W = \begin{bmatrix} X_W & Y_W \end{bmatrix}$$

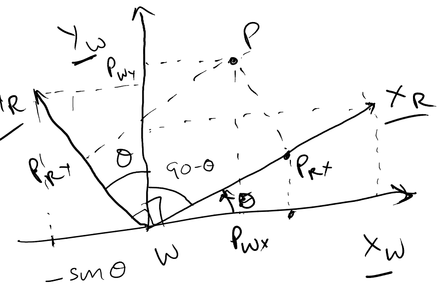
$$B_W^T B_W = \begin{bmatrix} \overset{1}{X_W^T X_W} & \overset{0}{X_W^T Y_W} \\ \underset{0}{Y_W^T X_W} & \underset{1}{Y_W^T Y_W} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \underline{I}$$

$$\underline{P_W} = B_W^{-1} B_R \underline{P_R} = B_W^{-1} \left(B_W\, {}^W R_R(\theta) \right) \underline{P_R}$$

$$= B_W^T B_W\, {}^W R_R(\theta)\, \underline{P_R} = {}^W R_R(\theta)\, \underline{P_R}$$

$$R \xleftarrow{\theta \ \text{clockwise}} W$$

$$\underline{P_W} = {}^{W}R_{R}(\theta)\ \underline{P_R}$$



$${}^{W}R_{R}(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

Exercise

$$\underline{P_R} = \underline{{}^{R}R_{W}(\theta)}\ \underline{P_W}$$

Properties of Rot Mat ${}^{W}R_{R} =$ Orthonormal Basis matrices

$${}^{W}R_{R} = B_W^{-1} B_R \Rightarrow ({}^{W}R_{R})^T ({}^{W}R_{R})$$ (2) orthogonality

$$= B_W^T B_R$$

$$= (B_R^T B_W)(B_W^T B_R)$$

$$= B_R^T \underbrace{(B_W B_W^T)}_{I} B_R = B_R^T B_R$$

$$= I$$

(1) $\det(R) = +1$

$$\left| \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \right| = \cos\theta\cos\theta - (\sin\theta(-\sin\theta))$$

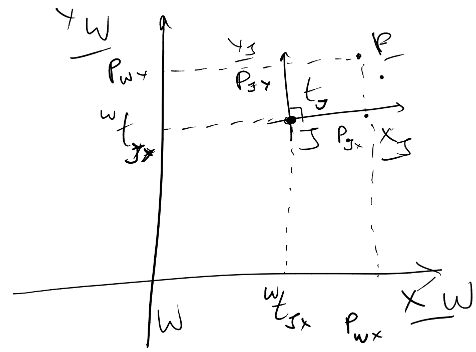
$$= \cos^2\theta + \sin^2\theta = 1$$

(2) $R^T R = R R^T = I$

necessary and sufficient

If (1) and (2) are satisfied for any matrix M then M represents a "proper" rotation

Translation only



J = Jetbot
W = World

$$\underline{x_J} \parallel x_W \qquad y_J \parallel y_W$$

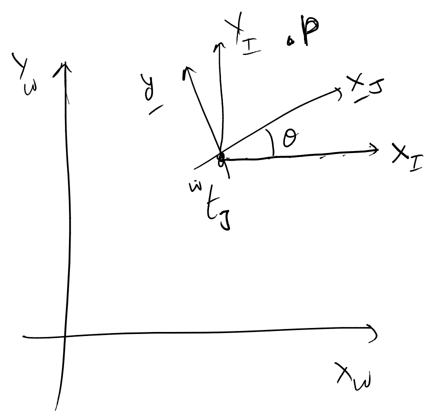
$$P_{Wx} = {}^{W}t_{Jx} + P_{Jx}$$

$$P_{WY} = {}^{W}t_{JY} + P_{JY}$$

$$\underline{P_W} = {}^{W}\underline{t_J} + \underline{P_J}$$

Trans + Rot

$$\underline{P_W} \xleftarrow{\text{trans}} \underline{P_I} \xleftarrow{\text{Rot}} \underline{P_J}$$



J = Jetbot
W = World
I = Inter

$$\underline{P_I} = {}^{W}R_J(\theta)\, \underline{P_J}$$

$$\underline{{}^{W}R_J(\theta)} = \begin{bmatrix} c\theta & -s\theta \\ s\theta & c\theta \end{bmatrix}$$

$$\underline{P_W} = {}^{W}t_J + \underline{P_I} = {}^{W}t_J + {}^{W}R_J(\theta)\,\underline{P_J}$$

$$= {}^{W}R_J(\theta)\,\underline{P_J} + {}^{W}t_J$$

2D

$$\underline{P_W}_{2\times1} = \underbrace{\begin{bmatrix} {}^{W}R_J(\theta)_{2\times2} & {}^{W}t_{J\,2\times1} \end{bmatrix}}_{\text{coordinate transformation}} \begin{bmatrix} \underline{P_J}_{2\times1} \\ 1 \end{bmatrix}_{3\times1}$$

Homogeneous coordinate

$$\begin{bmatrix} \underline{P_W}_{2\times1} \\ 1 \end{bmatrix}_{3\times1} = \begin{bmatrix} {}^{W}R_J(\theta)_{2\times2} & {}^{W}t_{J\,2\times1} \\ \underline{0}^T_{1\times2} & 1 \end{bmatrix} \begin{bmatrix} \underline{P_J} \\ 1 \end{bmatrix}_{3\times1}$$

$$\begin{bmatrix} \underline{p}_{W\,2\times1} \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} {}^{W}R_J(\theta)_{2\times2} & {}^{W}t_{J\,2\times1} \\ \underline{0}^T_{1\times2} & 1 \end{bmatrix}}_{{}^{W}T_J} \begin{bmatrix} \underline{p}_J \\ 1 \end{bmatrix}_{3\times1}$$

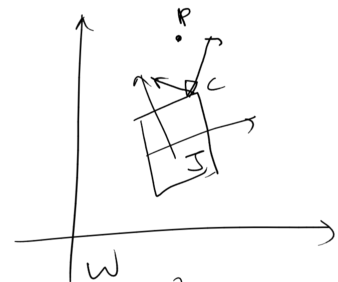
${}^{W}T_J$ = Transformation matrix

3×3 in 2D case

= Rot + trans b/w coordinate frames

$$\begin{bmatrix} \underline{p}_W \\ 1 \end{bmatrix} = {}^{W}T_J \begin{bmatrix} \underline{p}_J \\ 1 \end{bmatrix}$$

$$\underline{p}_W \xleftarrow{{}^{W}T_J} \underline{p}_J \xleftarrow{{}^{J}T_C} \underline{p}_C$$



$$\begin{bmatrix} \underline{p}_W \\ 1 \end{bmatrix} = {}^{W}T_J \begin{bmatrix} \underline{p}_J \\ 1 \end{bmatrix}, \quad \begin{bmatrix} \underline{p}_J \\ 1 \end{bmatrix} = \underbrace{{}^{J}T_C \begin{bmatrix} \underline{p}_C \\ 1 \end{bmatrix}}$$

$$\begin{bmatrix} \underline{p}_W \\ 1 \end{bmatrix} = \underbrace{{}^{W}T_J\, {}^{J}T_C}_{{}^{W}T_C} \begin{bmatrix} \underline{p}_C \\ 1 \end{bmatrix}$$

$${}^{W}T_C = {}^{W}T_J\, {}^{J}T_C$$