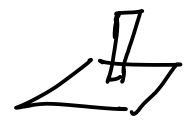
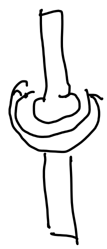
Denavit Hartenberg parameters (convention) Standardize the choice and description of robotic arms

Kinematic chains

https://www.youtube.com/watch?v=rA9tm0gTln8



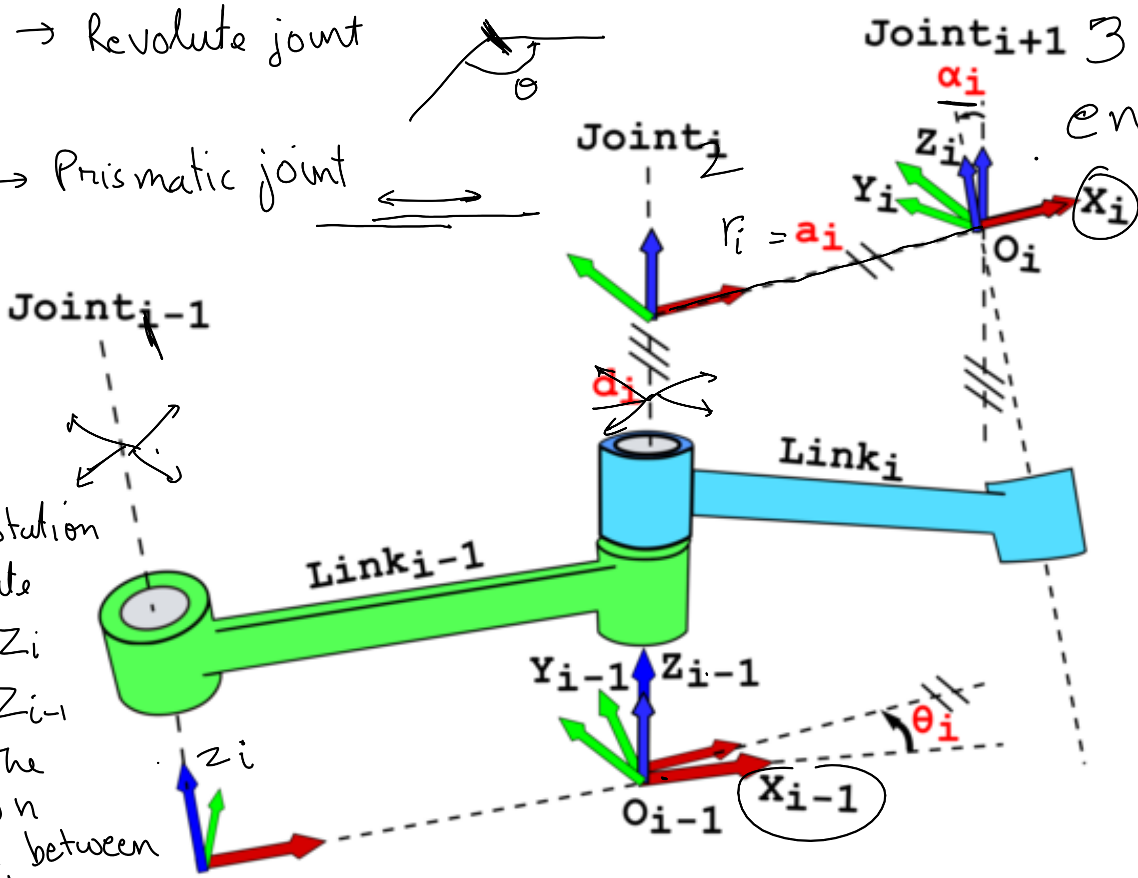
Robotic arm joints are mostly of two types



→ Revolute joint

→ Prismatic joint

Ball and socket joint



end effector

(1) Axis of rotation of revolute joints $= Z_i$ $= Z_{i-1}$

(2) Find the common normal between Z_i and Z_{i-1} $= x_i$

(3) x_{i-1} depends upon the previous link or is arbitrary

$$\underline{y}_{i-1} = \underline{Z}_{i-1} \times \underline{x}_{i-1} \quad , \quad \underline{y}_i = Z_i - \underline{x}_i$$

(5) Rotation (1) and translation (2) along the x-axis $= \alpha_i, r_i$ (x_i-axis)

" (3) " " (4) " " Z-axis $= \theta_i, d_i$ (z_{i-1} axis)

$$ {}^{i-1}_{i}R_{x_i}(\alpha_i) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha_i) & -\sin(\alpha_i) \\ 0 & \sin(\alpha_i) & \cos(\alpha_i) \end{bmatrix}_{3\times3} \rightarrow {}^{i-1}_{i}T^{R}_{x_i}(\alpha_i) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\alpha_i & -s\alpha_i & 0 \\ 0 & s\alpha_i & c\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= {}^{i-1}_{i}Rot_{x_i}(\alpha_i)$$

$$ {}^{i-1}_{i}T^{t}_{x_i}(r_i) = \begin{bmatrix} 1 & 0 & 0 & r_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^{i-1}_{i}Trans_{x_i}(r_i) $$

$$ {}^{i-1}_{i}T_x(\alpha_i, r_i) = \underbrace{{}^{i-1}_{i}Trans_{x_i}(r_i)}\ \underbrace{{}^{i-1}_{i}Rot_{x_i}(\alpha_i)} $$

$$ = \begin{bmatrix} 1 & 0 & 0 & r_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\alpha_i & -s\alpha_i & 0 \\ 0 & s\alpha_i & c\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ = \begin{bmatrix} 1 & 0 & 0 & r_i \\ 0 & c\alpha_i & -s\alpha_i & 0 \\ 0 & s\alpha_i & c\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{i-1}_{i}T_z(\theta_i; d_i) = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & 0 \\ s\theta_i & c\theta_i & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{i-1}_{i}T(\theta_i, d_i, \alpha_i, r_i) = {}^{i-1}_{i}T_z(\theta_i, d_i)\ {}^{i-1}_{x}T(\alpha_i, r_i) $$

Classical $\underbrace{\text{DH}}_{\text{Denavit Hartenberg}}$ parameters

For a robotic arm with n-links, a D-H table is typically provided

n-1 row

	θ_i	d_i	α_i	r_i
1			var	
2	var			
3				
.		var		
.				
n-1				

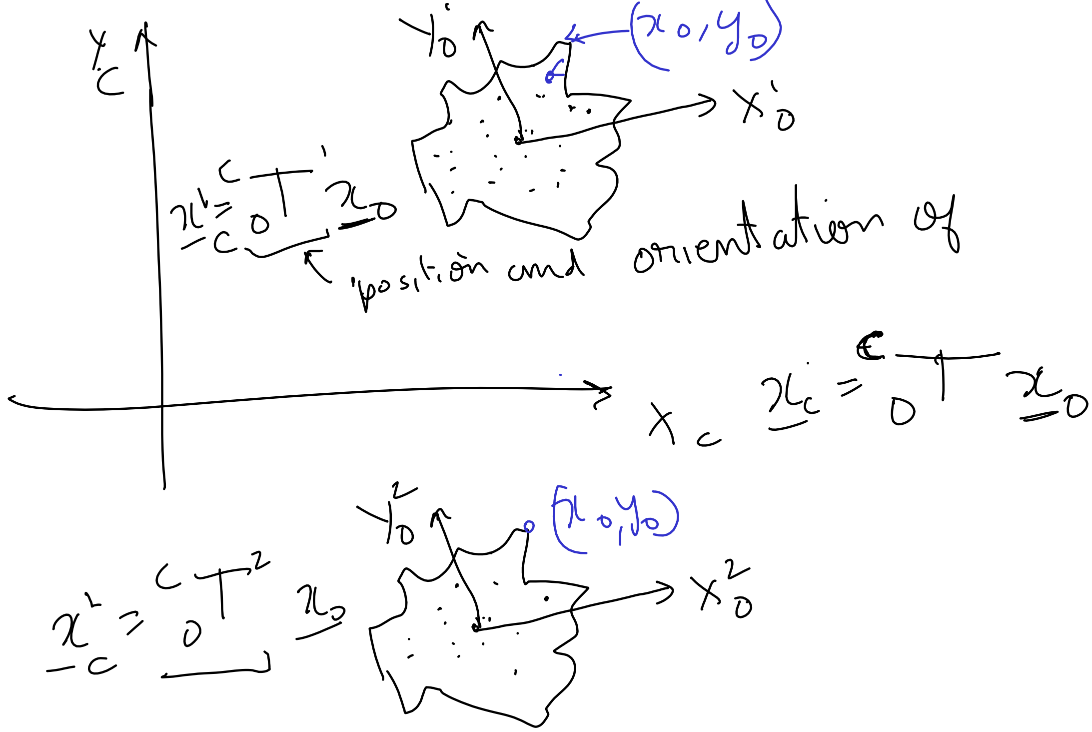
revolute joints

$\leftarrow$ prismatic joints

$$ {}^{0}_{n}T = {}^{0}_{1}T(\theta_1)\; {}^{1}_{2}T(\theta_2) \; \ldots \; {}^{n-1}_{n}T(\theta_n) $$

Forward Kinematics

Why Transformation matrices also describe position + orientation



321 Kinematic Structure

rotation orientation

$$\boxed{{}^0_3T = {}^0_1T(\theta_1)\;{}^1_2T(\theta_2)\;{}^2_3T(\theta_3)}$$

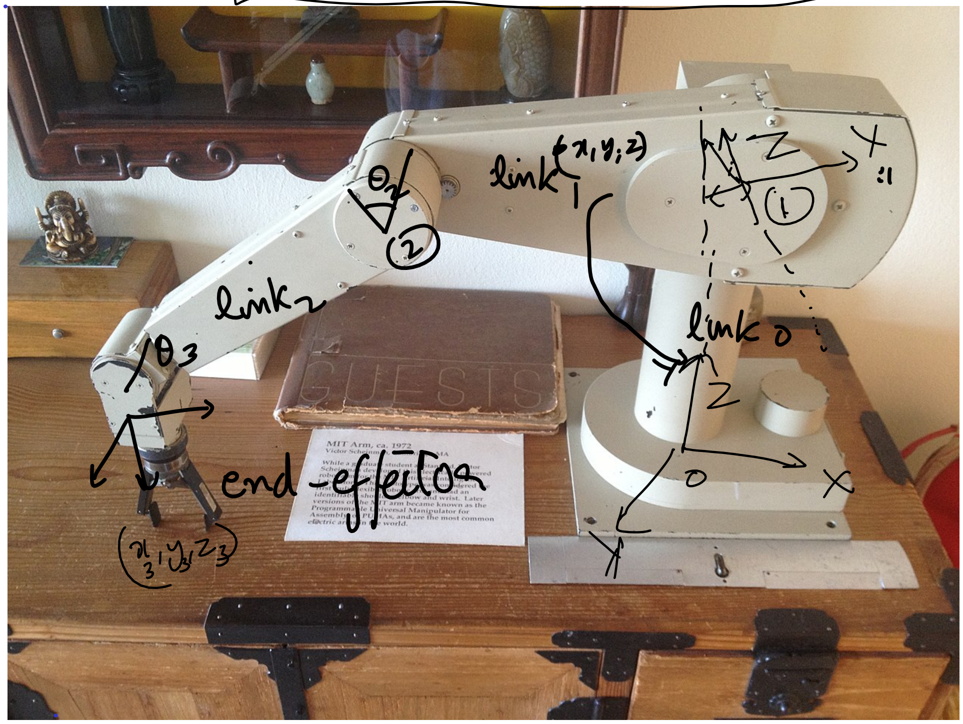
$$\boxed{x_0 = {}^0_3T\; x_3}$$

src

9Rot 3trans

12

4x4



description of robotic arm

dst

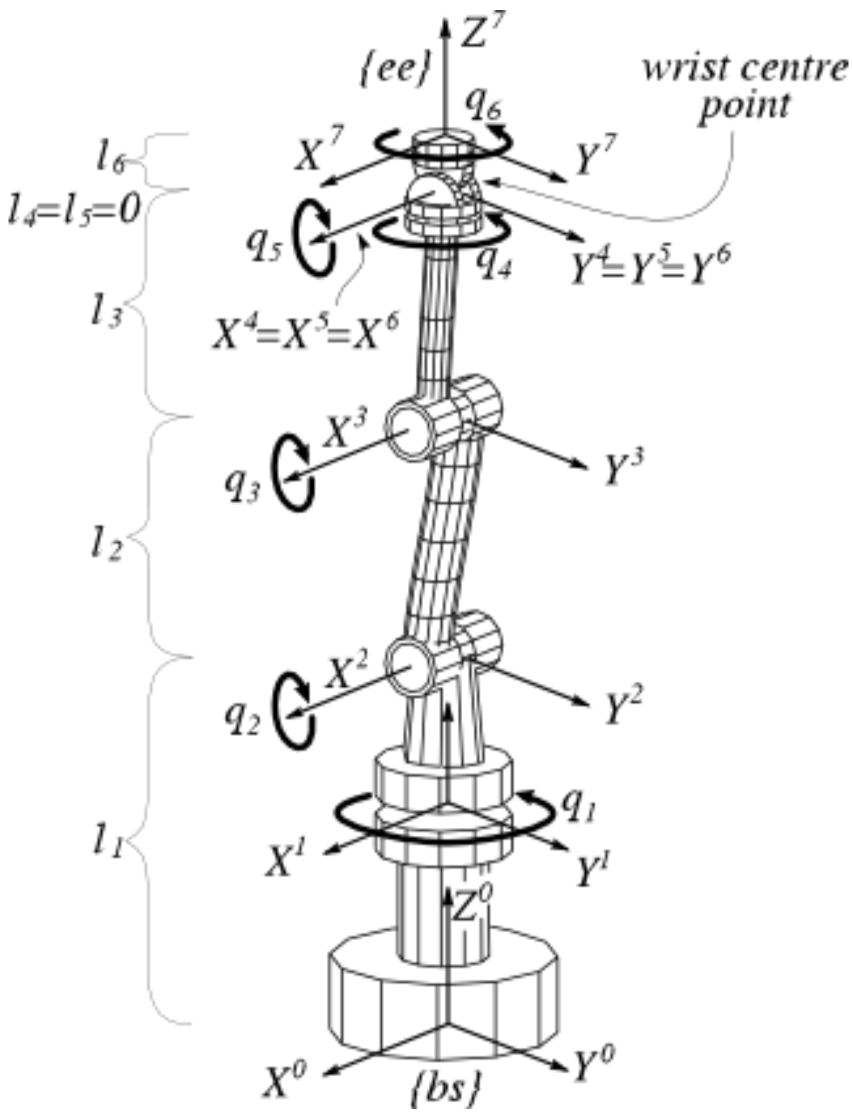
$$\boxed{x_0 = \boxed{{}^0_1T(\theta_1)\;{}^1_2T(\theta_2)\;{}^2_3T(\theta_3)}\; x_3}$$

src

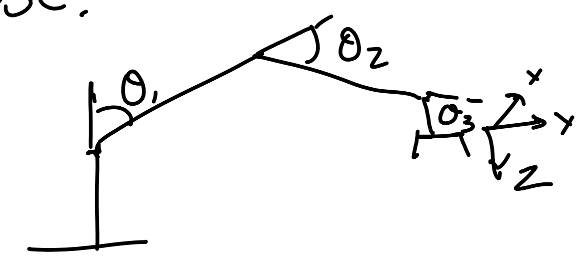
Forward Kinematics

is the problem of finding the end-effector pose in the base coordinate system when the joint angles (joint states) are given.

pos + orientat



Inverse Kinematics: The problem of finding motor angle (joint states) so that the end effector achieves a given pose.



Skip ① Closed form solutions for simple arms (2-DOF, 6-DOF)

Do FK analytically → Solve systems of eqns

② Numerical or iterative solutions

→ Small changes to motor angles (joint states) that move the end-effector towards desired pose

Given: Position of end effector $\underline{p} \in \mathbb{R}^{3\times1}$

Find: motor angle/joint states $\underline{\theta} \in \mathbb{R}^{n\times1}$

$$\underline{p} = {}^{0}_{n}T(\underline{\theta})\begin{bmatrix}0\\0\\0\\1\end{bmatrix} \Rightarrow \underline{p}(\underline{\theta}) \leftarrow \text{position of end-effector as a function of } \underline{\theta}$$

origin of end-effector

Taylor series approximation

Scalar valued functions $f(x)$

$$f(x+\Delta x) = f(x) + \underbrace{\frac{\Delta x}{1!} f'(x)} + \frac{1}{2!}\Delta x^2 f''(x) + \dots$$

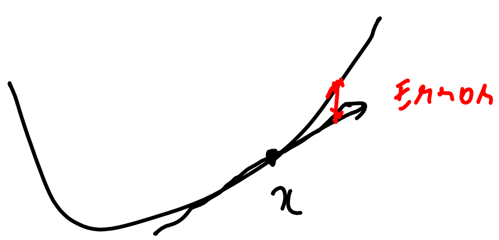
Vector-valued vector functions

$\underline{f}(\underline{x}) \in \mathbb{R}^{m\times 1}$ $\quad \underline{x} \in \mathbb{R}^{n\times 1}$

$$\underline{f}(\underline{x}+\Delta\underline{x}) = \underline{f}(\underline{x}) + \underbrace{J_x \underline{f}(\underline{x}) \Delta \underline{x}} + \dots O(\Delta x^2)$$

$$\underline{f}(\underline{x}+\Delta\underline{x}) \approx \underline{f}(\underline{x}) + J_x f(x) \Delta \underline{x}$$

for $\|\Delta \underline{x}\| \ll 1$



Inverse Kinematics

$\underline{p}(\underline{\theta}) \in \mathbb{R}^{3\times 1}$

$\underline{\theta} \in \mathbb{R}^{n\times 1}$

$J_\theta p(\underline{\theta}) \in \mathbb{R}^?$

$$\underline{p}(\underline{\theta}+\Delta\underline{\theta}) \approx \underbrace{\underline{p}(\underline{\theta})}_{3\times 1} + \underbrace{\underbrace{\boxed{J_\theta p(\underline{\theta})}}_{3\times n} \underbrace{\Delta\underline{\theta}}_{n\times 1}}_{3\times 1}$$

$$J_\theta \underline{p}(\underline{\theta}) = \begin{array}{c} \text{rows} \\ \downarrow \end{array} \begin{array}{c} \xrightarrow{\text{cols}} \\ \begin{bmatrix} \frac{\partial p_1(\underline{\theta})}{\partial \theta_1} & \frac{\partial p_1(\underline{\theta})}{\partial \theta_2} & \cdots & \frac{\partial p_1(\underline{\theta})}{\partial \theta_n} \\ \frac{\partial p_2(\underline{\theta})}{\partial \theta_1} & & & \\ \vdots & & & \\ \frac{\partial p_3(\underline{\theta})}{\partial \theta_1} & & & \frac{\partial p_3(\underline{\theta})}{\partial \theta_n} \end{bmatrix} \end{array} \in \mathbb{R}^{3\times n}$$

$$\underline{p}(\underline{\theta}) = \begin{bmatrix} p_1(\underline{\theta}) \\ p_2(\underline{\theta}) \\ p_3(\underline{\theta}) \end{bmatrix} \qquad \underline{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{bmatrix}$$

$$\underline{p}(\underline{\theta} + \underline{\Delta\theta}) \approx \underline{p}(\underline{\theta}) + J_\theta \underline{p}(\underline{\theta}) \underline{\Delta\theta}.$$

$$J_\theta \underline{p}(\underline{\theta}) \underline{\Delta\theta} = \underline{p}(\underline{\theta} + \underline{\Delta\theta}) - \underline{p}(\underline{\theta})$$

$$\underline{\Delta\theta} = \left[J_\theta \underline{p}(\underline{\theta})\right]^{\dagger} \left(\underline{p}(\underline{\theta} + \underline{\Delta\theta}) - \underline{p}(\underline{\theta})\right)$$

dagger symbol $\dagger$

Pseudo inverse

Inverse of a matrix is only defined for square matrices

Pseudo inverse of a matrix $A_{m\times n}$ is $A^{\dagger}_{n\times m}$ iff

$$A^{+}_{n\times m}\,\underbrace{\boxed{A_{m\times n}\,A^{+}_{n\times m}}}_{I} = A^{+}_{n\times m}$$

$$A_{m\times n}\,\underbrace{\boxed{A^{+}_{n\times m}\,A_{m\times n}}}_{I} = A_{m\times n}$$

$$A\underline{x} = \underline{b}$$

$$\Rightarrow \underline{x} = A^{+}\underline{b}$$

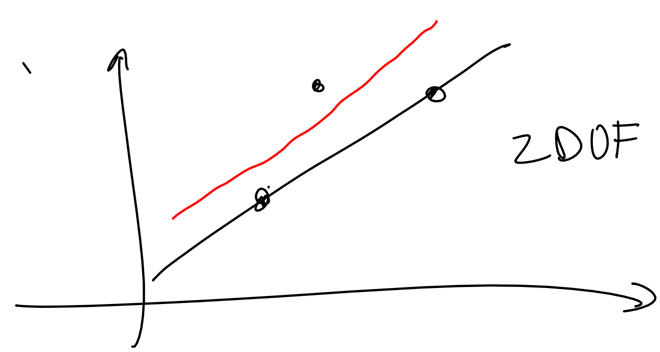
Solution to a system of Linear equations

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ a_{m1} & \cdots & & a_{mn} \end{bmatrix} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

If the number of equations $m > n$

the number of unknowns

how many exact solution $= 0$



$y = mx + c$