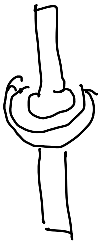
Denavit Hartenberg parameters (convention) Standardize the choice and description of robotic arms

Kinematic chains

https://www.youtube.com/watch?v=rA9tm0gTln8

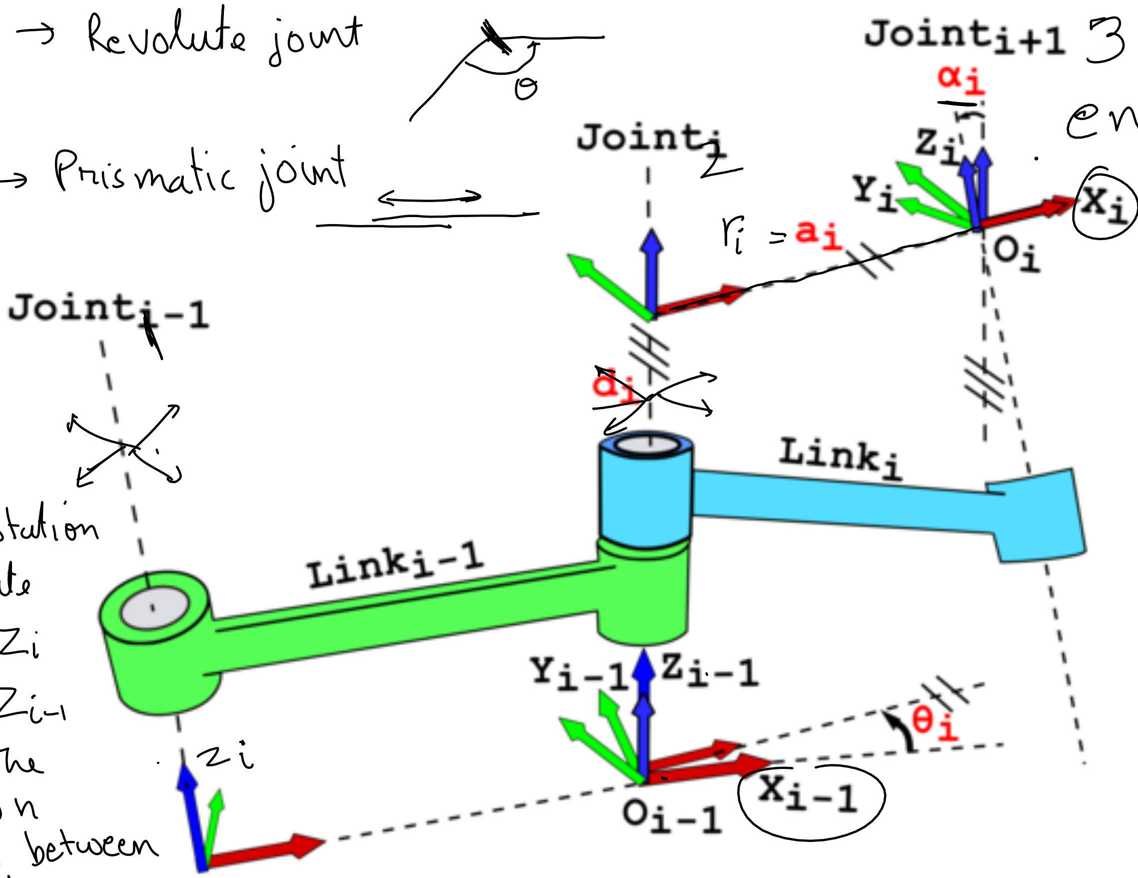
Robotic arm joints are mostly of two types



Ball and socket joint

→ Revolute joint

→ Prismatic joint



(1) Axis of rotation of revolute joints $= Z_i$ $= Z_{i-1}$

(2) Find the common normal between Z_i and Z_{i-1} $= x_i$

(3) x_{i-1} depends upon the previous link or is arbitrary

$\underline{y}_{i-1} = \underline{Z}_{i-1} \times \underline{x}_{i-1}$, $\underline{y}_i = Z_i - \underline{x}_i$

(5) Rotation (1) and translation (2) along the x-axis $= \alpha_i, r_i$ (x_i-axis)

" (3) " " (4) " " Z-axis $= \theta_i, d_i$ (Z_{i-1} axis)

$$ {}^{i-1}_{i}R_{x_i}(\alpha_i) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha_i) & -\sin(\alpha_i) \\ 0 & \sin(\alpha_i) & \cos(\alpha_i) \end{bmatrix}_{3\times3} \rightarrow {}^{i-1}_{i}T^{R}_{x_i}(\alpha_i) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\alpha_i & -s\alpha_i & 0 \\ 0 & s\alpha_i & c\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ = {}^{i-1}_{i}Rot_{x_i}(\alpha_i) $$

$$ {}^{i-1}_{i}T^{t}_{x_i}(r_i) = \begin{bmatrix} 1 & 0 & 0 & r_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^{i-1}_{i}\mathrm{Trans}_{x_i}(r_i) $$

$$ {}^{i-1}_{i}T_{x}(\alpha_i, r_i) = \underbrace{{}^{i-1}_{i}\mathrm{Trans}_{x_i}(r_i)}\ \underbrace{{}^{i-1}_{i}\mathrm{Rot}_{x_i}(\alpha_i)} $$

$$ = \begin{bmatrix} 1 & 0 & 0 & r_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & c\alpha_i & -s\alpha_i & 0 \\ 0 & s\alpha_i & c\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ = \begin{bmatrix} 1 & 0 & 0 & r_i \\ 0 & c\alpha_i & -s\alpha_i & 0 \\ 0 & s\alpha_i & c\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{i-1}_{i}T_{z}(\theta_i; d_i) = \begin{bmatrix} c\theta_i & -s\theta_i & 0 & 0 \\ s\theta_i & c\theta_i & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

$$ {}^{i-1}_{i}T(\theta_i, d_i, \alpha_i, r_i) = {}^{i-1}_{i}T_{z}(\theta_i, d_i)\ {}^{i-1}_{x}T(\alpha_i, r_i) $$

Classical $\underbrace{\text{DH}}_{\text{Denavit Hartenberg}}$ parameters

For a robotic arm with
n-links, a D-H table is typically provided

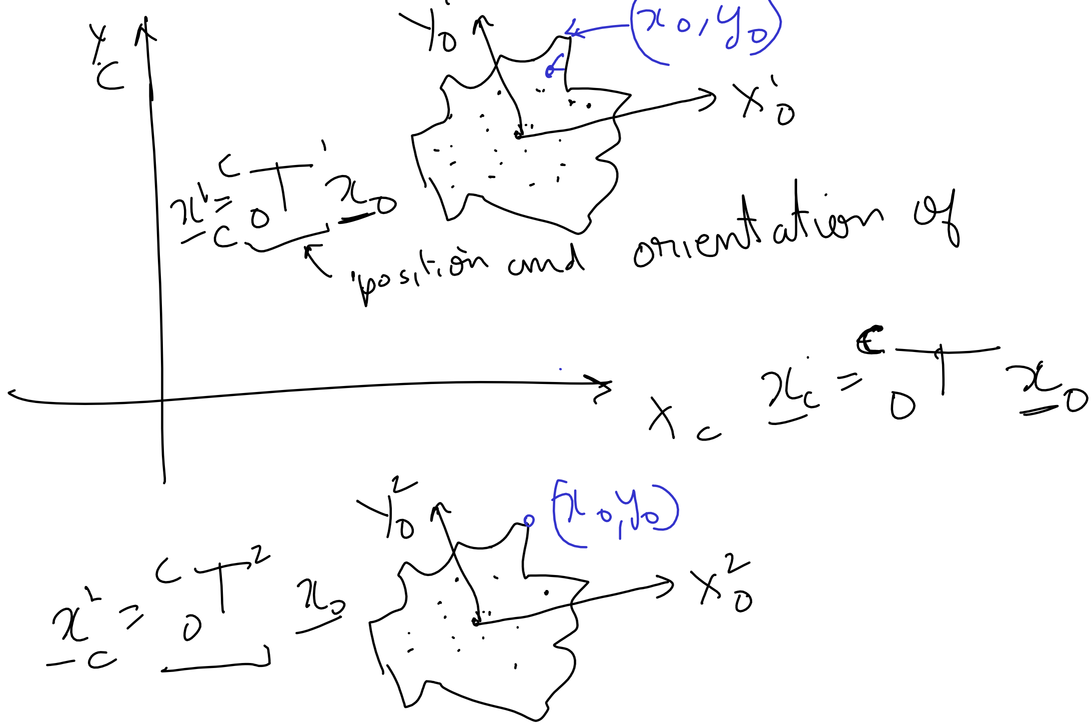
n-1 row

	θ_i	d_i	α_i	r_i	
1			var		revolute joints
2	var				
3					
.		var			← prismatic joints
.					
n-1					

$${}^{0}_{n}T = {}^{0}_{1}T(\theta_1)\,{}^{1}_{2}T(\theta_2)\ \ldots\ {}^{n-1}_{n}T(\theta_n)$$

Forward Kinematics

Why Transformation matrices also describe position + orientation



321 Kinematic Structure

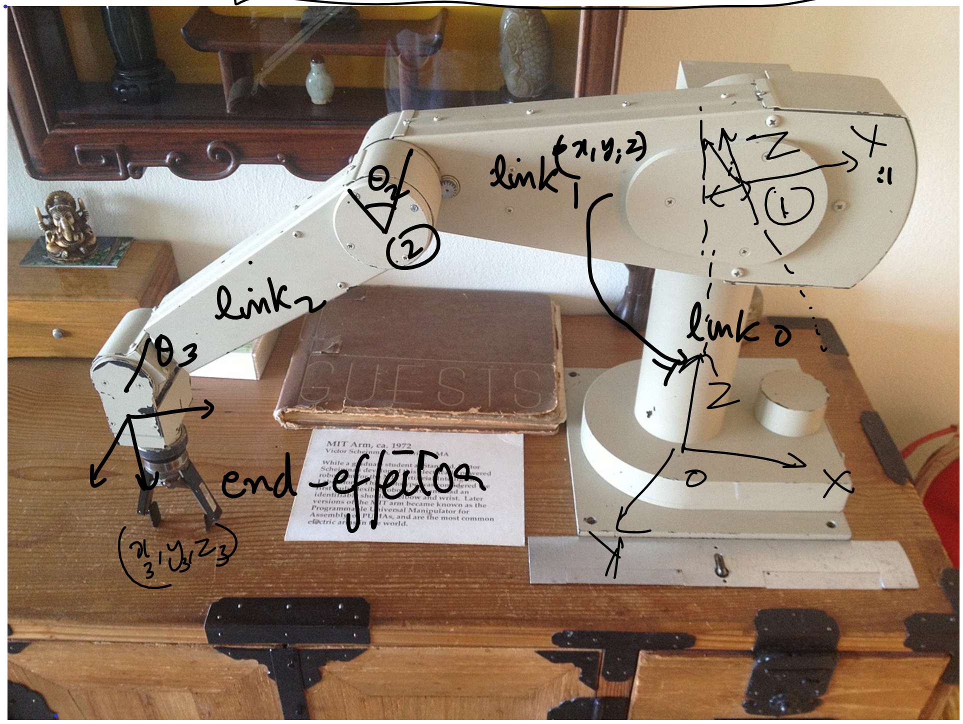
rotation
orientation

$$\boxed{{}^0_3T = {}^0_1T(\theta_1)\;{}^1_2T(\theta_2)\;{}^2_3T(\theta_3)}$$

$$\boxed{x_0 = {}^0_3T\; x_3}$$

src

9Rot 3trans
12
4x4



description of robotic arm

dst

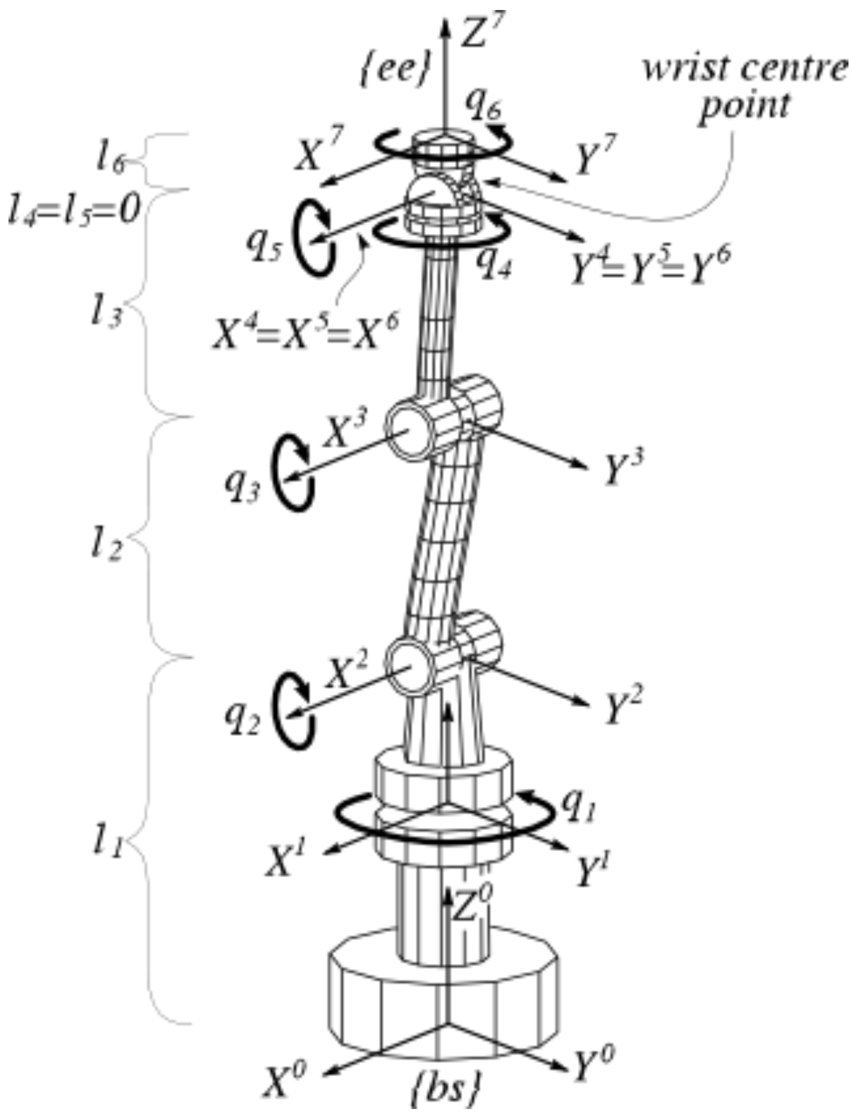
$$x_0 = {}^0_1T(\theta_1)\;{}^1_2T(\theta_2)\;{}^2_3T(\theta_3)\;x_3$$

src

Forward Kinematics

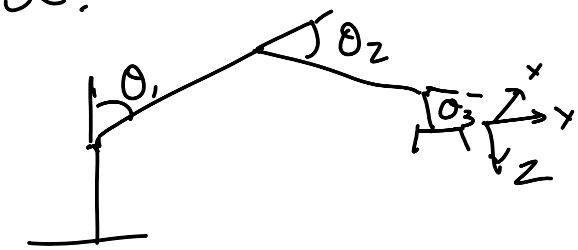
is the problem of finding the end-effector pose in the base coordinate system when the joint angles (joint states) are given.

pos + orientat



Numerical solutions to IK problems: Jacobian inverse technique

Inverse Kinematics: The problem of finding motor angle (joint states) so that the end effector achieves a given pose.



Skip ① Closed form solutions for simple arms (2-DOF, 6-DOF)

Do FK analytically → Solve systems of eqns

② Numerical or iterative solutions

→ Small changes to motor angles (joint states) that move the end-effector towards desired pose

Given: Position of end effector $\underline{p} \in \mathbb{R}^{3\times 1}$

Find: motor angle/joint states $\underline{\theta} \in \mathbb{R}^{n\times 1}$

$$\underline{p} = {}^{0}_{n}T(\underline{\theta}) \underbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{origin of end-effector}} \Rightarrow \underline{p}(\underline{\theta})$$

← position of end-effector as a function of $\underline{\theta}$

Taylor series approximation

Scalar valued functions $\quad f(x)$

$$f(x+\Delta x) = f(x) + \frac{\Delta x}{1!} f'(x) + \frac{1}{2!} \Delta x^2 f''(x) + \ldots$$

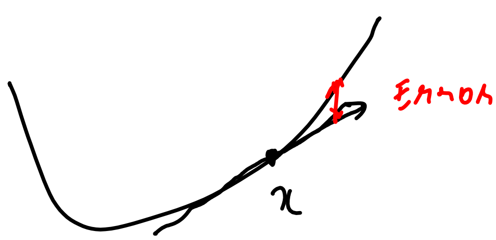
Vector-valued vector functions

$\underline{f}(\underline{x}) \in \mathbb{R}^{m\times 1} \qquad \underline{x} \in \mathbb{R}^{n\times 1}$

$$\underline{f}(\underline{x}+\Delta\underline{x}) = \underline{f}(\underline{x}) + \frac{J_x \underline{f}(\underline{x}) \Delta \underline{x}}{1!} + \ldots O(\Delta x^2)$$

$$\underline{f}(\underline{x}+\Delta\underline{x}) \approx \underline{f}(\underline{x}) + J_x f(x) \Delta \underline{x}$$

for $\|\Delta \underline{x}\| \ll 1$



Inverse Kinematics

$$\underline{p}(\underline{\theta}+\Delta\underline{\theta}) \approx \underbrace{\underline{p}(\underline{\theta})}_{3\times 1} + \underbrace{\underbrace{J_\theta p(\underline{\theta})}_{3\times n} \underbrace{\Delta\underline{\theta}}_{n\times 1}}_{3\times 1}$$

$\underline{p}(\underline{\theta}) \in \mathbb{R}^{3\times 1}$

$\underline{\theta} \in \mathbb{R}^{n\times 1}$

$J_\theta p(\underline{\theta}) \in \mathbb{R}^{?}$

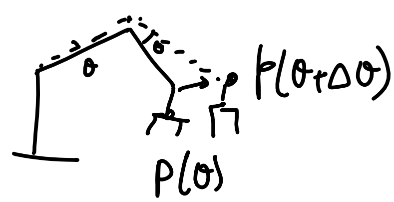
rows ↓, cols →

$$J_\theta \underline{p}(\underline{\theta}) = \begin{bmatrix} \frac{\partial p_1(\underline{\theta})}{\partial \theta_1} & \frac{\partial p_1(\underline{\theta})}{\partial \theta_2} & \cdots & \frac{\partial p_1(\underline{\theta})}{\partial \theta_n} \\ \frac{\partial p_2(\underline{\theta})}{\partial \theta_1} & & & \\ \vdots & & & \\ \frac{\partial p_3(\underline{\theta})}{\partial \theta_1} & & & \frac{\partial p_3(\underline{\theta})}{\partial \theta_n} \end{bmatrix} \in \mathbb{R}^{3\times n}$$

$$\underline{p}(\underline{\theta}) = \begin{bmatrix} p_1(\underline{\theta}) \\ p_2(\underline{\theta}) \\ p_3(\underline{\theta}) \end{bmatrix} \qquad \underline{\theta} = \begin{bmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_n \end{bmatrix}$$

$$\underline{p}(\underline{\theta} + \underline{\Delta\theta}) \approx \underline{p}(\underline{\theta}) + J_\theta \underline{p}(\underline{\theta}) \underline{\Delta\theta}$$

$$J_\theta \underline{p}(\underline{\theta}) \underline{\Delta\theta} = \underline{p}(\underline{\theta} + \underline{\Delta\theta}) - \underline{p}(\underline{\theta})$$



$$\boxed{\underline{\Delta\theta} = \left[J_\theta \underline{p}(\underline{\theta})\right]^{\dagger} \left(\underline{p}(\underline{\theta} + \underline{\Delta\theta}) - \underline{p}(\underline{\theta})\right)}$$

① How to find $J_\theta \underline{p}(\theta)$

② What is $[\ ]^{\dagger}$

dagger symbol †

Pseudo inverse

Inverse of a matrix is only defined for square matrices

Pseudo inverse of a matrix $A_{m\times n}$ is $A^{\dagger}_{n\times m}$ iff

$$A^{\dagger}_{n\times m}\,\underbrace{\boxed{A_{m\times n}\,A^{\dagger}_{n\times m}}}_{I} = A^{\dagger}_{n\times m}$$

$$A_{m\times n}\,\underbrace{\boxed{A^{\dagger}_{n\times m}\,A_{m\times n}}}_{I} = A_{m\times n}$$

Capital letter = matrices
small letters = vector or scalars

$$A\underline{x} = \underline{b}$$

$$\Rightarrow \underline{x} = \underbrace{A^{\dagger}}\underline{b}$$

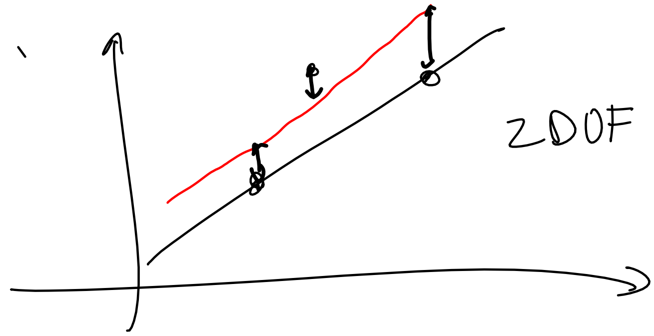
Solution to a system of Linear equations

$$\begin{array}{c} \\ m \updownarrow \end{array}\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ a_{m1} & \cdots & & a_{mn} \end{bmatrix}_{m\times n} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{n\times 1} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}_{m\times 1}$$

If the number of equations $\underline{m} > n$
↑
the number of unknowns

how many $\underbrace{\text{exact}}_{\times}$ solution = 0

if want approximate solution
then you can minimize an error



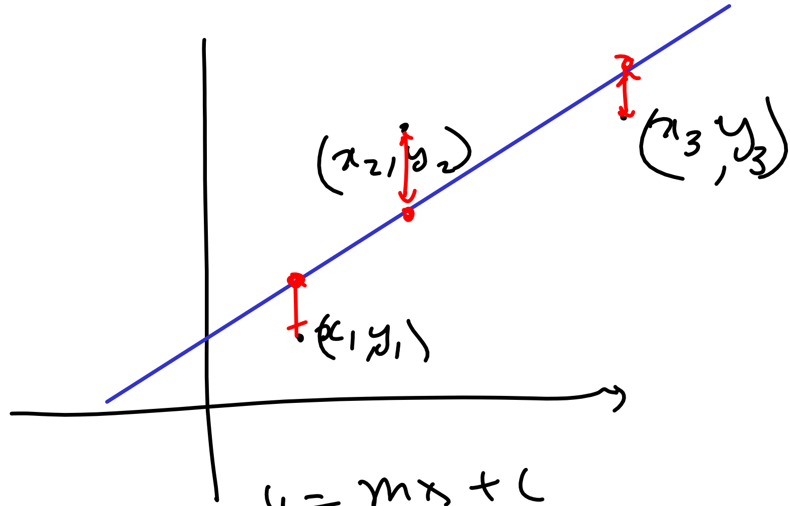
2DOF

$y = mx + c$

$$A\underline{x} = \underline{b}$$

$$\min_{\underline{x}} \|A\underline{x} - \underline{b}\|^2$$

$$y_1 = m x_1 + c$$
$$y_2 = m x_2 + c$$
$$y_3 = m x_3 + c$$



$y = mx + c$

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}}_{\underline{b}} = \underbrace{\begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ x_3 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} m \\ c \end{bmatrix}}_{\underline{x}}$$

$$\|x\|_2^2 = \underline{x}^T \underline{x}$$

$$\|\underline{x}\|_2^2 = x_1^2 + x_2^2 + \dots x_n^2$$

$$\underline{x}^T \underline{x} = [x_1, x_2 \dots x_n] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1^2 + x_2^2 + \dots x_n$$

$$\|A\underline{x} - \underline{b}\|^2 = (A\underline{x} - b)^T (A\underline{x} - b)$$

$$= (\underline{x}^T A^T - \underline{b}^T)(A\underline{x} - \underline{b})$$

$$= \underbrace{\underline{x}^T A^T A \underline{x}}_{\text{quadratic form in } \underline{x}} \underbrace{- \underline{b}^T A \underline{x}^{①} - \underline{x}^T A^T \underline{b}^{②}}_{\text{linear term in } x} + \underline{b}^T \underline{b}$$

completing the squares

$$\underbrace{(x - \underline{y})^T M (x - \underline{y})}_{\rightarrow \text{when } x \doteq y} + \underbrace{}_{\text{not containing } x}$$

$$\underbrace{\underline{b}^T A \underline{x}^{①}}_{1\times 1} = \underbrace{\underline{x}^T A^T \underline{b}^{②}}_{1\times 1}$$

$1\times m$ $m\times n$ $n\times 1$ $1\times n$ $n\times m$ $m\times 1$

why is this true

$$A\underline{x} = \underline{b}$$
$$A \in \mathbb{R}^{m\times n}$$
$$\underline{x} \in \mathbb{R}^{n\times 1}$$
$$\underline{b} \in \mathbb{R}^{m\times 1}$$

$$(\underline{b}^T A \underline{x})^T = (\underline{b}^T A \underline{x})$$

$\hookrightarrow \underline{x}^T A^T \underline{b} = \underline{b}^T A \underline{x}$

$(PQR)^T = R^T Q^T P^T$

$$\|A\underline{x}-\underline{b}\|^2 = \underline{x}^T A^T A \underline{x} - 2\underline{b}^T A \underline{x} + \underline{b}^T\underline{b}$$

$$= x^T \underbrace{(A^TA)(A^TA)^{-1}(A^TA)}_{I} x$$

$$-2\underline{b}^T A \underbrace{(A^TA)^{-1}(A^TA)}_{I} \underline{x}$$

$$+ \underline{b}^T A (A^TA)^{-1} A^T \underline{b}$$

$$- \underline{b}^T A (A^TA)^{-1} A^T \underline{b}$$

$$+ b b$$

$$\left(x^2 - 2\frac{b}{a}x + \frac{c}{a}\right)$$

$$= \left(x^2 - \frac{2b}{a}x + \boxed{\frac{b^2}{a^2}}\right) - \boxed{\frac{b^2}{a^2}} + \frac{c}{a}$$

$$= \left(x - \frac{b}{a}\right)\left(x - \frac{b}{a}\right) - \frac{b^2}{a^2} + \frac{c}{a}$$

inverse

$(PQR)^{-1} = R^{-1}Q^{-1}P^{-1}$

$$\cancel{\underline{x}^T \underbrace{(A^TA)}\,\underbrace{(A^TA)^{-1}(A^TA)}\,x} = \left(x^T (A^TA) \underbrace{A^{-1}}\right)\left(A^{-T}(A^TA)\underline{x}\right)$$

$\times$ not defined

$$\cancel{-2\underline{b}^T A (A^TA)^{-1}(A^TA)\underline{x}} = -2\underline{b}^T \underbrace{A A^{-1}} \left(A^{-T}(A^TA)\underline{x}\right)$$

cannot do this because A^{-1} is not defined

$$(A^TA\underline{x} - \underline{y})^T (A^TA)^{-1} (A^TA\underline{x} - \underline{y})$$

$$= (\underline{x}^T A^T A - \underline{y}^T)(A^TA)^{-1}(A^TA\underline{x} - \underline{y})$$

$$= \underline{x}^T (A^TA)(A^TA)^{-1}(A^TA)\underline{x} - 2\underbrace{\underline{y}^T}(A^TA)^{-1}(A^TA)\underline{x}$$

$$+ \underbrace{\underline{y}^T}(A^TA)^{-1}\underbrace{\underline{y}}$$

$$= x^T (A^TA)\underbrace{(A^TA)^{-1}(A^TA)}_{I}\underline{x}$$

$\underline{y} = A^T\underline{b}$

will complete the square

$$-2\underbrace{\underline{b}^T A}\underbrace{(A^TA)^{-1}(A^TA)}_{I}\underline{x}$$

$$+ \underbrace{\underline{b}^T A}(A^TA)^{-1}\underbrace{A^T\underline{b}}$$

$$\|A\underline{x} - \underline{b}\|^2 = \overbrace{(A^TA\underline{x} - A^T\underline{b})^T(A^TA)^{-1}(A^TA\underline{x} - A^T\underline{b})}^{\text{depends upon } x \geq 0}$$

$$- \underline{b}^TA(A^TA)^{-1}A^T\underline{b} + \underline{b}^T\underline{b}$$

If we make first term 0, then the whole thing is minimised

$$A^TA\underline{x} - A^Tb = 0$$

$$\Rightarrow \quad \underline{x} = \boxed{\underbrace{(A^TA)^{-1}A^T}_{A^+ \text{ for } m > n}\underline{b}}$$

$A^T_{n\times m}A_{m\times n}$

$(A^TA)_{n\times n}$

for $m < n$

$$\underline{x} = \underbrace{A^T (AA^T)^{-1}}_{A^+ \text{ for } m<n} \underline{b}$$