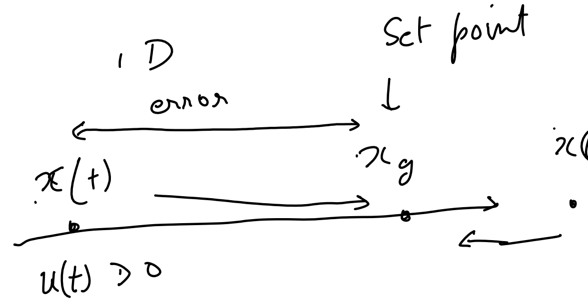
Controllers

① PID : Proportional Integral Derivative controller

② LQR: Linear quadratic regulator



PID

$$e(t) = x_g - x(t)$$

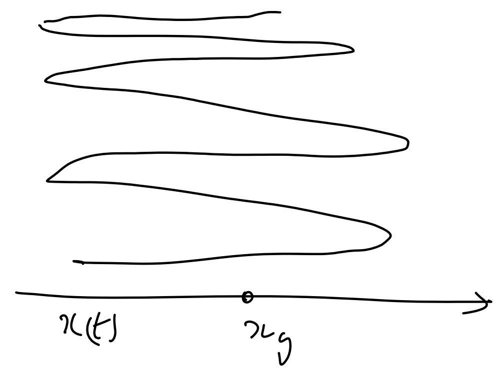
Proportional

$$u(t) = k_p e(t)$$

Proportional gain factor

How to find k_p?

PID tuning → Manual tuning



Integral term

$$\boxed{u(t) = k_p e(t) + k_I \int_0^{\tau} e(t)\,dt}$$

Integral gain

PI

↓

80%

Derivative term

$$\boxed{u(t) = \underbrace{k_p e(t)}_{\text{determines the direction + magnitude}} + \underbrace{k_I \int_0^{\tau} e(t)\,dt}_{\text{remove oscillations}} + \underbrace{k_D \frac{d}{dt} e(t)}_{\text{Remove shaky behaviour}}}$$

$$\begin{bmatrix} x \\ y \\ \theta \end{bmatrix} \qquad \begin{bmatrix} v \\ \omega \end{bmatrix}$$

In Jetbot

$$\underline{e}(t) = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} - \begin{bmatrix} x_g \\ y_g \\ \theta_g \end{bmatrix}$$

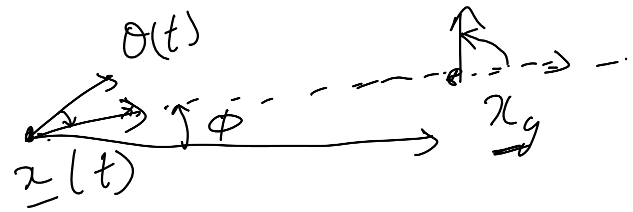
$$\underline{u}(t) = \begin{bmatrix} v \\ \omega \end{bmatrix}$$

v — linear vel

ω — angular vel

$$\underline{u}(t) = \underbrace{\begin{bmatrix} & & \\ & & \end{bmatrix}_{2\times3}}_{K_p} \begin{bmatrix} \underline{e}(t) \end{bmatrix} + \underbrace{\begin{bmatrix} & & \\ & & \end{bmatrix}_{2\times3}}_{K_I} \int_0^{\tau} \underline{e}(\tau)\,dt$$

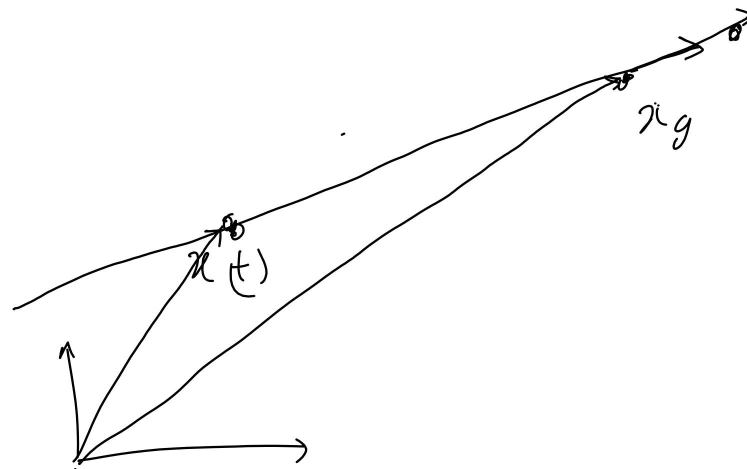
$$\left[\begin{array}{l} \underline{e_1(t)} = \theta(t) - \underbrace{\tan^{-1}\left(y_g - y(t) \,/\, x_g - x(t) \right)}_{\phi} \\ \omega(t) = k_p e_1(t) + k_I \displaystyle\int_0^{\tau} e_1(\tau)\,dt \end{array}\right.$$



$e_2(t) = \,?$

initial time

$$\underset{\text{direction}}{\hat{\underline{d}}} = \frac{\underline{x}_g - \underline{x}(0)}{\|\underline{x}_g - \underline{x}(0)\|}$$



$$\underline{e}(t) = \underline{x}_g - \underline{x}(t)$$

$$e_2(t) = \underbrace{\left(\underline{x}_g - \underline{x}(t)\right)^T \hat{\underline{d}}}_{\text{dot product}}$$

= projection of $(\underline{x}_g - \underline{x}(t))$ on $\hat{\underline{d}}$

$$v(t) = k_p e_2(t) + k_I \int_0^{\tau} e_2(\tau)\,d\tau + \ldots$$

Optimal control

$$u^*(t) = \underset{u(t)}{\text{minimize}} \text{ Cost function}$$

Assume a cost function $J_t(\underline{x}_t, \underline{u}_t)$

current state

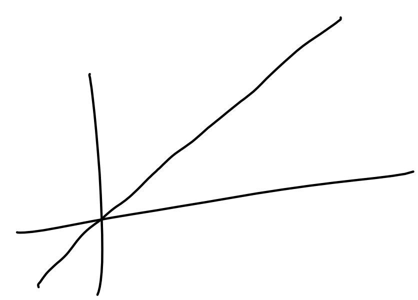
control signal

$$\boxed{u_0^*, u_1^*, \dots, u_T^* = \underset{u_0, u_1, \dots u_T}{\arg\min} \sum_{t=0}^{T} J_t(x_t, u_t) \quad \text{s.t.} \quad \underbrace{\underline{x}_{t+1} = f(\underline{x}_t, \underline{u}_t)}_{\text{system dynamics}}}$$

optimal control problem

LQR is the solution to the optimal control problem when the cost function J_t is QUADRATIC and the system dynamics f is LINEAR

What is a linear function?



Technical definition

A function f is linear if

$$\boxed{f(\alpha \underline{x} + \beta \underline{y}) = \alpha f(\underline{x}) + \beta f(\underline{y})} \quad \underline{x} \in \mathbb{R}^n, \ y \in \mathbb{R}^n$$

$\mathbb{R}$

Elementary $\boxed{+, \cdot, x^2, x^3, xy^2, yx^2}$, $\boxed{\exp(x)}$

$\boxed{\log(x), \cos(x), \sin(x)}$

Polynomial $= 3 + x + x^2 + 5x^2y + 6x^3y$

Linear functions are polynomials of degree 1 without the constant part

$$f(x) = 4x$$

Quadratic functions are polynomials of degree 2

In vector form

Linear functions are of the form:

$$f(\underline{x}) = A\underline{x}$$

scalar valued Quadratic functions are of the form:

$$f(\underline{x}) = \underline{\underline{x}^T Q \underline{x}} + \underline{p\underline{x}} + \underline{r}$$

System dynamics is linear

$$\underline{x}_{t+1} = f(\underline{x}_t, \underline{u}_t) = A\underline{x}_t + B\underline{u}_t$$

$$\underline{x}_{t+1} = A\underline{x}_t + B\underline{u}_t \qquad \underline{x}_t \in \mathbb{R}^{n\times 1}$$

What is the $\underbrace{\text{dimensionality}}_{\text{size}}$ of $\underbrace{A}_{n\times n}$ and $\underbrace{B}_{n\times m}$? $\quad \underline{u}_t \in \mathbb{R}^{m\times 1}$

Cost function is quadratic

$$J_t(\underline{x}_t, \underline{u}_t) = \underline{x}_t^T Q_t \underline{x}_t + \underline{u}_t^T R \underline{u}_t$$

LQR

$$\boxed{\underline{u}_{0\ldots t} = \arg\min_{\underline{u}_{0\ldots t}} \underline{x}_t^T Q_t \underline{x}_t + \underline{u}_t^T R \underline{u}_t \quad \text{s.t.} \quad \underline{x}_{t+1} = A\underline{x}_t + B\underline{u}_t}$$