

# ECE 417/598: Review Homework 4

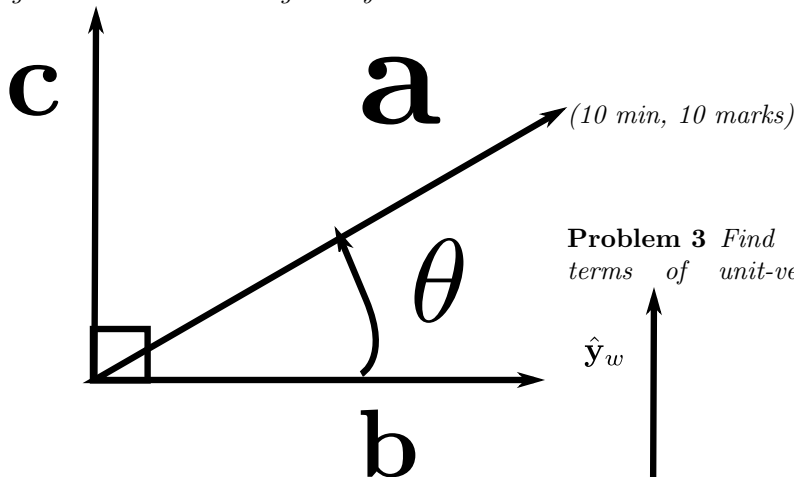
Max marks: 100 marks

Due on March 10th, 2021, midnight, 11:59 PM.

All notes so far are [linked here](#).

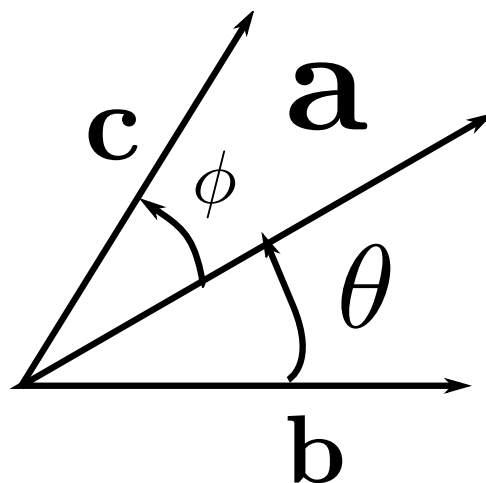
## 1 Trigonometry and triangle laws of vector addition

**Problem 1** The magnitude of vector  $\mathbf{a} \in \mathbb{R}^n$  is given to be  $\|\mathbf{a}\| = \alpha$ . Using the following figure, write  $\mathbf{a}$  in terms of  $\alpha$ ,  $\theta$ , vector  $\mathbf{b} \in \mathbb{R}^n$  and  $\mathbf{c} \in \mathbb{R}^n$ . All three vectors lie in the same plane.  $\mathbf{b}$  and  $\mathbf{c}$  are perpendicular to each other. The angle between  $\mathbf{a}$  and  $\mathbf{b}$  is given by  $\theta$ .

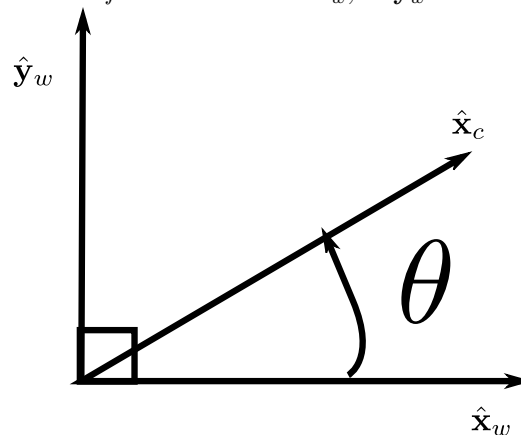


(5 min, 5 marks)

**Problem 2** The magnitude of vector  $\mathbf{a} \in \mathbb{R}^n$  is given to be  $\|\mathbf{a}\| = \alpha$ . Using the following figure, write  $\mathbf{a}$  in terms of  $\alpha$ ,  $\theta$ ,  $\phi$ , vector  $\mathbf{b} \in \mathbb{R}^n$  and  $\mathbf{c} \in \mathbb{R}^n$ . All three vectors lie in the same plane. The angle between  $\mathbf{a}$  and  $\mathbf{b}$  is given by  $\theta$ . The angle between  $\mathbf{a}$  and  $\mathbf{c}$  is given by  $\phi$ . Assume  $\theta + \phi \neq 0$ . When  $\theta + \phi = \frac{\pi}{2}$ , is the solution is same as Problem 1? (Hint: You can convert this to Problem 1, by drawing a unit-vector perpendicular to  $\mathbf{b}$ . Call it  $\hat{\mathbf{d}}$ . First write  $\hat{\mathbf{d}}$  in terms of  $\mathbf{c}$  and others knowns and then write  $\mathbf{a}$  in terms of  $\hat{\mathbf{d}}$  and other knowns. You might want to use [trigonometric identities](#). The simplest form is not required.).



**Problem 3** Find unit-vector  $\hat{\mathbf{x}}_c$  in terms of unit-vectors  $\hat{\mathbf{x}}_w$ ,  $\hat{\mathbf{y}}_w$  and  $\theta$ .



(5 min, 5 marks)

**Problem 4** Find unit-vector  $\hat{\mathbf{y}}_c$  in terms of unit-vectors  $\hat{\mathbf{x}}_w$ ,  $\hat{\mathbf{y}}_w$  and  $\theta$ .

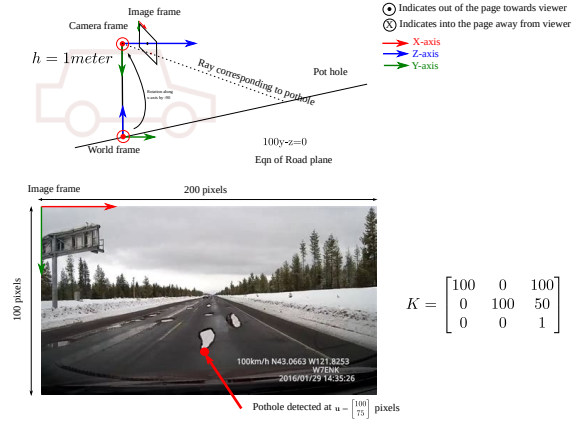
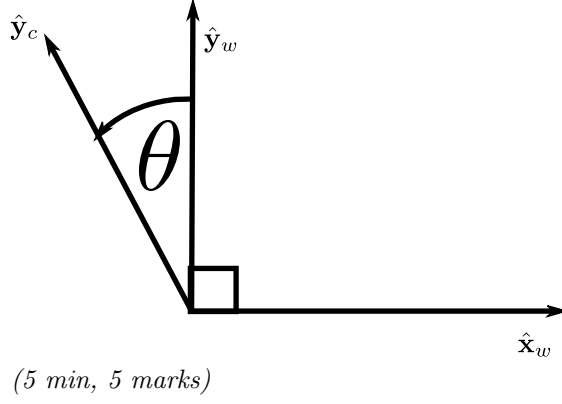
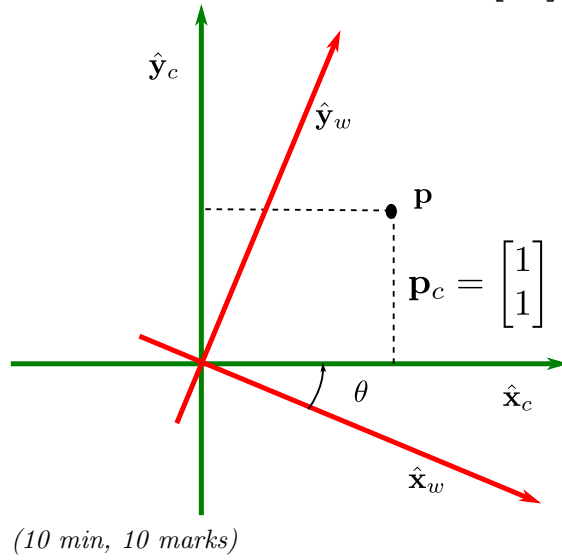
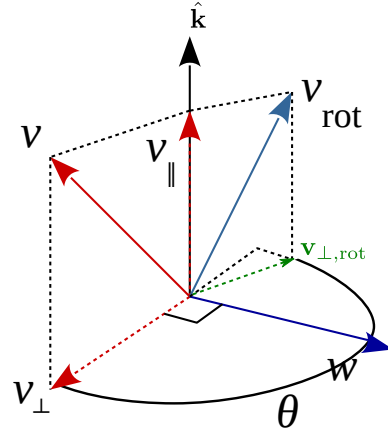


Figure 1: Point-plane triangulation

**Problem 5** Let the coordinates of a vector  $\mathbf{p}$  in terms of  $\hat{\mathbf{x}}_c$  and  $\hat{\mathbf{y}}_c$  be  $\mathbf{p}_c = \begin{bmatrix} p_{cx} \\ p_{cy} \end{bmatrix}$ , so that:  $\mathbf{p} = p_{cx}\hat{\mathbf{x}}_c + p_{cy}\hat{\mathbf{y}}_c$ . Using the results from Prob 3 and Prob 4, write  $\mathbf{p}$  in terms of  $\hat{\mathbf{x}}_w$  and  $\hat{\mathbf{y}}_w$ . Thus derive the formula for rotation matrix  $R(\theta)$  that converts coordinates from  $\mathbf{p}_c$  to  $\mathbf{p}_w = \begin{bmatrix} p_{wx} \\ p_{wy} \end{bmatrix}$ .



**Problem 6** We know that  $\|\mathbf{v}_{\perp, \text{rot}}\| = \|\mathbf{v}_{\perp}\|$ . Write  $\mathbf{v}_{\perp, \text{rot}}$  in terms of  $\mathbf{v}_{\perp}$ ,  $\mathbf{w}$  and  $\theta$ .  $\mathbf{v}_{\perp}$  and  $\mathbf{w}$  are known to be orthogonal to each other.



**Problem 7** In figure 1 find the 3D position of the pothole the World coordinate frame, in terms of  $h = 1$  (the height of the camera), image-coordinates of the pothole  $\mathbf{u}$  (provided in figure), camera matrix  $K$  (provided in figure). The Camera is mounted directly on top of the world frame, both of which are aligned to the gravity vector. The road is a perfect plane with a slope such that the equation of road plane in world-coordinate frame is given by  $100Y_w - Z_w = 0$  and the pothole lies on the road plane. Provide the formula or pseudo-code for computing the pothole coordinates, and also substitute in the values. (20 min, 20 marks)

**Problem 8** In figure 2 find the 3D representation of the lane the World coordinate frame, in terms of  $h$  (the height of the camera), image-representation of the line  $\mathbf{l}$  (provided in figure), camera matrix  $K$  (provided in figure). Assume the lane to be a straight line. The Camera is mounted directly on top of the world frame, both of which are aligned to the gravity vector. The road is a perfect plane with a slope such that the equation of road plane in world-coordinate frame is given by  $100Y_w - Z_w = 0$  and the pothole lies on the road plane. Provide the formula or pseudo-code for computing the pothole coordinates, and also substitute in the values. (20 min, 20 marks)

**Hint 0: Equation of plane in 3D.** Equation of a plane in 3D is given by  $p_1X + p_2Y + p_3Z + p_4 = 0$ . In matrix notation, you can write the equation plane as  $\mathbf{p}_{1:3}^\top \mathbf{X} + p_4 = 0$ , where  $\mathbf{p}_{1:3} = [p_1, p_2, p_3]^\top$ .

**Hint 1: 3D Plane corresponding to the line in image-coordinates.** Let the equation of line in image-coordinates be  $\mathbf{l}^\top \mathbf{u} = 0$ , where  $\mathbf{u} = \begin{bmatrix} u \\ 1 \end{bmatrix} \in \mathbb{P}^2$ . By pinhole camera model, if  $\mathbf{X}_c \in \mathbb{R}^3$  are the corresponding points in 3D, then the equation of corresponding plane is given by  $\mathbf{l}^\top (K\mathbf{X}_c) = 0$  which can also be written as  $(K^\top \mathbf{l})^\top \mathbf{X}_c = 0$ . If we compare it to the equation of plane  $\mathbf{p}_{1:3}^\top \mathbf{X} + p_4 = 0$ , then  $\mathbf{p}_{1:3} = K^\top \mathbf{l}$  and  $p_4 = 0$ .

**Hint 2: Intersection of two planes in 3D is a line.** Equation of a plane in 3D is given by  $p_1X_w + p_2Y_w + p_3Z_w + p_4 = 0$ . In matrix notation, you can write the equation plane as  $\mathbf{p}_{1:3}^\top \mathbf{X}_w + p_4 = 0$ , where  $\mathbf{p}_{1:3} = [p_1, p_2, p_3]^\top$ . Let's say you have two planes  $\mathbf{p}_{1:3}^\top \mathbf{X}_w + p_4 = 0$  and  $\mathbf{q}_{1:3}^\top \mathbf{X}_w + q_4 = 0$ . Their intersection is a line whose parameteric form is given by (why ? you have all the knowledge required to derive this):

$$\mathbf{X}_w = \lambda(\mathbf{p}_{1:3} \times \mathbf{q}_{1:3}) + \begin{bmatrix} \mathbf{p}_{1:3}^\top \\ \mathbf{q}_{1:3}^\top \end{bmatrix}^\dagger \begin{bmatrix} -p_4 \\ -q_4 \end{bmatrix}, \quad (1)$$

where  $A^\dagger$  denotes the pseudo-inverse of a matrix (a fat matrix in this case) and  $\lambda \in \mathbb{R}$  is the free parameter and  $\times$  denotes the vector cross-product.

**Problem 9** You are a part of Tesla self-driving team. Team 1 provides you with lane-detection algorithms and their output. Team 2 provides

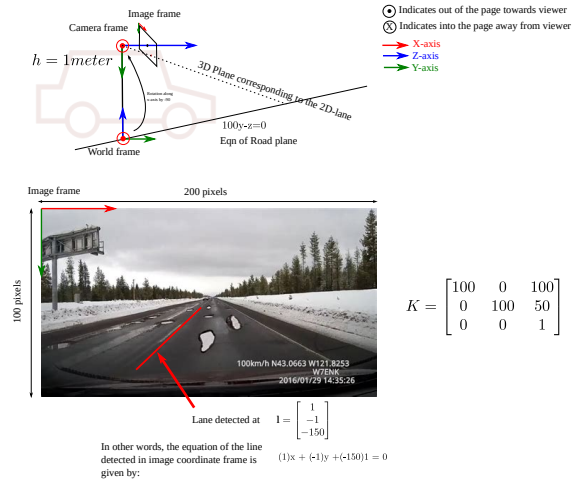


Figure 2: Line-plane triangulation

you with detailed maps of road conditions. Your task is to write a function that solves problem 8 for arbitrary lanes detected by team 1 and for arbitrary plane provided by team 2. (Hint: Equation of a plane 3D is very similar to equation of line in 2D). What input representations of lane and plane would you ask for? Write a general algorithm or pseudo-code that solves problem 8. (30 min, 10 marks)