# ECE 417 Midterm 2 2022 practice problem set

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**About the exam**

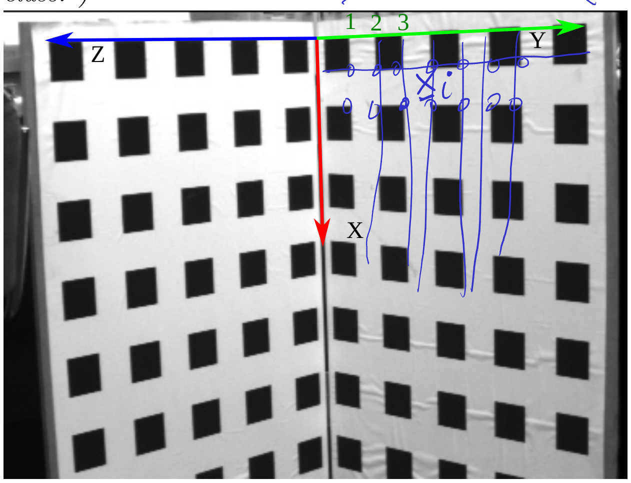
1. There are total 5 problems. You must attempt all 5.
2. Maximum marks: 50 (70 with bonus marks).
3. Maximum time allotted: 50 min
4. Calculators are allowed.
5. One US Letter size or A4 size cheat sheet (both-sides) is allowed.

$n \geq 6$   $6, 7, 9 \ldots 9 \ldots$   $\underline{X}_i = \begin{bmatrix} x_i \\ y_i \\ z_i \\ 1 \end{bmatrix}$

$n$ points   $\underline{X}_i \in \mathbb{P}^3$

**Problem 1** *Given a set of $n \geq 6$ points $\underline{\mathbf{X}}_i \in \mathbb{P}^3$ for all $i \in \{1, \ldots, n\}$ in 3D projective space, and a set of corresponding points $\underline{\mathbf{u}}_i \in \mathbb{P}^2$ in an image, find the 3D to 2D projective $P \in \mathbb{R}^{3\times4}$ matrix that converts $\mathbf{X}_i$ to $\underline{\mathbf{u}}_i = \lambda_i P \underline{\mathbf{X}}_i$. In other words, convert $\underline{\mathbf{u}}_i \times P\underline{\mathbf{X}}_i = 0$ into a familiar form $A\mathbf{y} = \mathbf{b}$ or $A\mathbf{y} = \mathbf{0}$ so that we can solve for $P$. For notation purposes, you can denote $\underline{\mathbf{u}}_i = [x_i, y_i, w_i]^\top$ and $P = \begin{bmatrix} \mathbf{p}_1^\top \\ \mathbf{p}_2^\top \\ \mathbf{p}_3^\top \end{bmatrix}$ where $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \in \mathbb{R}^4$ are the rows of $P$ represented as 4-D column vectors. (Practical motivation: We did camera calibration in lab using a single checker board. It is much easier to compute camera calibration using two mutually perpendicular checker boards so that all points do not lie on a single plane (hence linearly independent). One can make a coordinate system attached to the double checker* board *and compute the 3D coordinates of each corner point in that system. Let $\underline{\mathbf{X}}_i \in \mathbb{P}^3$ be such points in 3D on the checker-board. Let $\underline{\mathbf{u}}_i \in \mathbb{P}^2$ be a point detected in the image so that we have one-to-one correspondence between $\underline{\mathbf{X}}_i$ and $\underline{\mathbf{u}}_i$. Finding the projection matrix $P \in \mathbb{R}^{3\times4}$ then reduces to the above problem. We will cover the breakdown of $P$ matrix into $P = K[R, t]$ in class. )*

$R, t$   $\rightarrow K$




$\underline{u}_i \in \mathbb{P}^2 = \begin{bmatrix} x_i \\ y_i \\ w_i \end{bmatrix}$

3x4 matrix

$P \in \mathbb{R}^{3\times4}$

$\underset{3\times1}{\underline{u}_i} = \lambda_i P \underline{X}_i \leftarrow 4\times1$ vector

$\lambda_i \in \mathbb{R}$

known

$\underline{u}_i \times P \underline{X}_i = 0$

unknown

known

$Ay = b$

or $Ay = 0$

unknown

$$P = \underbrace{\begin{bmatrix} p_{11} & \ldots & p_{14} \\ p_{21} & \ldots & p_{24} \\ p_{31} & \ldots & p_{34} \end{bmatrix}}_{}$$

4 colums, 3 rows

$$P = \begin{bmatrix} \underline{p}_1^T \\ \underline{p}_2^T \\ \underline{p}_3^T \end{bmatrix}$$

4 cols, 3 rows

$\underline{p}_1, \underline{p}_2, \underline{p}_3 \in \mathbb{R}^4$   $4 \times 1$ vectors

$\underline{u}_i = \lambda_i K(R X_i + t)$

$\underline{u}_i = \lambda K \overset{4}{\overbrace{[R_{3\times3} \quad t]}} \underbrace{\begin{bmatrix} X_i \\ 1 \end{bmatrix}}_{\underline{X}_i}$

$\underline{u}_i = \lambda_i P \underline{X}_i$

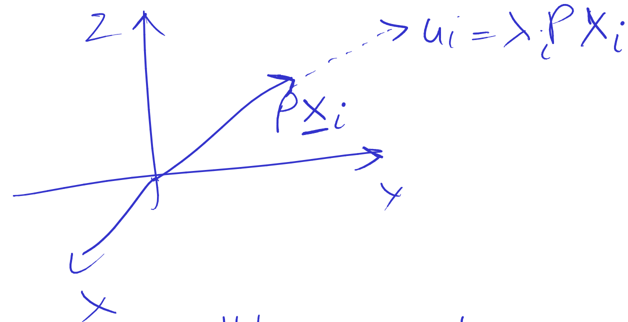


$\underline{u}_1 = \lambda_1 P \underline{X}_1, \quad \underline{u}_2 = \lambda_2 P \underline{X}_2 \quad \ldots \ldots, \quad u_n = \lambda_n P X_n$

$\underline{u}_i = \lambda_i P \underbrace{\underline{X}_i}$

$\underbrace{P\,\underline{X}_i}_{3\times 1 \text{ vector}}$ $\lambda_i \in \mathbb{R}$

$\underset{3\times 1}{\underline{u}_i}$ vecto




$\underline{a} \parallel \underline{b} \Rightarrow \underline{a} \times \underline{b} = 0$

$\|\underline{a} \times \underline{b}\| = \|a\|\,\|b\| \sin\theta$
$= 0$

$\underline{u}_i \times P\underline{X}_i = 0$

$$\underline{u}_i \times = \begin{bmatrix} 0 & -w_i & y_i \\ w_i & 0 & -x_i \\ -y_i & x_i & 0 \end{bmatrix}$$

$$\underset{3\times 4}{P}\,\underset{4\times 1}{\underline{X}_i} = \begin{bmatrix} \underline{p}_1^T \\ \underline{p}_2^T \\ \underline{p}_3^T \end{bmatrix} \underline{X}_i$$

$$= \begin{bmatrix} \underline{p}_1^T \underline{X}_i \\ p_2^T X_i \\ p_3^T X_i \end{bmatrix}_{3\times 1}$$

$$= \begin{bmatrix} \underline{X}_i^T \underline{p}_1 \\ \underline{X}_i^T \underline{p}_2 \\ \underline{X}_i^T \underline{p}_3 \end{bmatrix}$$

$\underline{p}_1^T \underline{X}_i$ is a scalar

$\underline{p}_1$ is a $4\times 1$ vector

$X_i$ is a $4\times 1$ vector

$\underset{1\times 4}{p_1^T}\,\underset{4\times 1}{X_i}$ is a $1\times 1$ matrix
or a scalar

$$\underline{p}_1^T \underline{X}_i = (\underline{p}_1^T \underline{X}_i)^T = \underline{X}_i^T \underline{p}_1$$

$$(\underline{u}_i)_\times P \underline{X}_i = 0_{3\times1}$$

$$\begin{bmatrix} 0 & -w_i & y_i \\ w_i & 0 & -x_i \\ -y_i & x_i & 0 \end{bmatrix}_{3\times3} \begin{bmatrix} \underline{X}_i^T \underline{p}_1 \\ \underline{X}_i^T \underline{p}_2 \\ \underline{X}_i^T \underline{p}_3 \end{bmatrix}_{3\times1} = 0_{3\times1}$$

$$\begin{bmatrix} 0 - w_i \underline{X}_i^T \underline{p}_2 + y_i \underline{X}_i^T \underline{p}_3 \\ w_i \underline{X}_i^T \underline{p}_1 + 0 - x_i \underline{X}_i^T \underline{p}_3 \\ -y_i \underline{X}_i^T \underline{p}_1 + x_i \underline{X}_i^T \underline{p}_2 + 0 \end{bmatrix}_{3\times1} = 0_{3\times1}$$

$$\begin{array}{ccc} \overset{4}{\longleftrightarrow} & \overset{4}{\longleftrightarrow} & \overset{4}{\longleftrightarrow} \end{array}$$

$$\begin{bmatrix} 0_{4\times1}^T & -w_i \underline{X}_i^T & y_i \underline{X}_i^T \\ w_i \underline{X}_i^T & 0_{4\times1}^T & -x_i \underline{X}_i^T \\ -y_i \underline{X}_i^T & x_i \underline{X}_i^T & 0_{4\times1}^T \end{bmatrix}_{3\times12} \begin{bmatrix} \underline{p}_1 \\ \underline{p}_2 \\ \underline{p}_3 \end{bmatrix}_{12\times1} = 0_{3\times1}$$

$\underline{p}_1$ is $4\times1$

knowns — unknowns

$$\underline{u}_i = \lambda_i P \underline{X}_i$$

$$\begin{bmatrix} 0_{4\times1}^T & -w_i \underline{X}_i^T & y_i \underline{X}_i^T \\ w_i \underline{X}_i^T & 0_{4\times1}^T & -x_i \underline{X}_i^T \end{bmatrix} \begin{bmatrix} \underline{p}_1 \\ \underline{p}_2 \\ \underline{p}_3 \end{bmatrix}_{12\times1} = 0_{2\times1}$$

$$\underbrace{\begin{bmatrix} 0_{4\times1}^T & -w_1 \underline{X}_1^T & y_1 \underline{X}_1^T \\ w_1 \underline{X}_1^T & 0_{4\times1}^T & -x_1 \underline{X}_1^T \\ & \vdots & \\ 0_{4\times1}^T & -w_n \underline{X}_n^T & y_n \underline{X}_n^T \\ w_n \underline{X}_n^T & 0_{4\times1}^T & -x_n \underline{X}_n^T \end{bmatrix}}_{A}{}_{2n\times12} \underbrace{\begin{bmatrix} \underline{p}_1 \\ \underline{p}_2 \\ \underline{p}_3 \end{bmatrix}_{12\times1}}_{\underline{y}} = 0_{2n\times1} = 0$$

$$A\underline{y} = 0$$

$$\Rightarrow \underline{y} \in N(A)$$

$$\underset{2n\times 2n}{A = U}\ \overset{\overset{2n\times 12}{\times}}{\Sigma}\ V^T_{12\times 12}$$

$P$ has 12 elements

$$\text{rank}(A) = 11$$

$$\underline{u}_i = \underline{\lambda}\ P\ \underline{X}_i$$

$$1P \quad 2P \quad 3P$$

$$DOF(P) = 11$$

$$V = [\underline{v}_1 \ \ldots\ldots \ \underline{v}_{12}]$$

$$\underline{y} = \begin{bmatrix} \underline{p}_1 \\ \underline{p}_2 \\ \underline{p}_3 \end{bmatrix} \in N(A) = \underline{v}_{12}$$

$$P = \begin{bmatrix} \underline{p}_1^T \\ \underline{p}_2^T \\ \underline{p}_3^T \end{bmatrix}$$

**Solution** Watch lecture `https://drive.google.com/file/d/1cY02DTagpckbYl5gS0PYBu569vlZUNN6/view?usp=sharing`

1. Write cross product as a matrix operation

$$[\underline{\mathbf{u}}_i]_\times = \begin{bmatrix} 0 & -w_i & y_i \\ w_i & 0 & -x_i \\ -y_i & x_i & 0 \end{bmatrix}$$

2. Write $P\underline{\mathbf{X}}_i$ in terms of row vectors.

$$P\underline{\mathbf{X}}_i = \begin{bmatrix} \mathbf{p}_1^\top \\ \mathbf{p}_2^\top \\ \mathbf{p}_3^\top \end{bmatrix} \underline{\mathbf{X}}_i = \begin{bmatrix} \mathbf{p}_1^\top \underline{\mathbf{X}}_i \\ \mathbf{p}_2^\top \underline{\mathbf{X}}_i \\ \mathbf{p}_3^\top \underline{\mathbf{X}}_i \end{bmatrix}$$

3. Note that all the three terms like $\mathbf{p}_1^\top \underline{\mathbf{X}}_i$ are scalars hence they are symmetric. Hence $\mathbf{p}_1^\top \underline{\mathbf{X}}_i = \underline{\mathbf{X}}_i^\top \mathbf{p}_1$ .

$$P\underline{\mathbf{X}}_i = \begin{bmatrix} \underline{\mathbf{X}}_i^\top \mathbf{p}_1 \\ \underline{\mathbf{X}}_i^\top \mathbf{p}_2 \\ \underline{\mathbf{X}}_i^\top \mathbf{p}_3 \end{bmatrix}$$

4. Substitute these values in the original equation $\underline{\mathbf{u}}_i \times P\underline{\mathbf{X}}_i = \mathbf{0}_{3\times 1}$.

$$\begin{bmatrix} 0 & -w_i & y_i \\ w_i & 0 & -x_i \\ -y_i & x_i & 0 \end{bmatrix} \begin{bmatrix} \underline{\mathbf{X}}_i^\top \mathbf{p}_1 \\ \underline{\mathbf{X}}_i^\top \mathbf{p}_2 \\ \underline{\mathbf{X}}_i^\top \mathbf{p}_3 \end{bmatrix} = \mathbf{0}_{3\times 1}$$

5. Matrix multiply

$$\begin{bmatrix} 0 - w_i \underline{\mathbf{X}}_i^\top \mathbf{p}_2 + y_i \underline{\mathbf{X}}_i^\top \mathbf{p}_3 \\ w_i \underline{\mathbf{X}}_i^\top \mathbf{p}_1 + 0 - x_i \underline{\mathbf{X}}_i^\top \mathbf{p}_3 \\ -y_i \underline{\mathbf{X}}_i^\top \mathbf{p}_1 + x_i \underline{\mathbf{X}}_i^\top \mathbf{p}_2 + 0 \end{bmatrix} = \mathbf{0}_{3\times 1}$$

6. Write the unknowns as a single vector, and the knowns as a matrix multiplication with the unknowns

$$\begin{bmatrix} \mathbf{0}^\top & -w_i \underline{\mathbf{X}}_i^\top & y_i \underline{\mathbf{X}}_i \\ w_i \underline{\mathbf{X}}_i^\top & \mathbf{0}^\top & -x_i \underline{\mathbf{X}}_i^\top \\ -y_i \underline{\mathbf{X}}_i^\top & x_i \underline{\mathbf{X}}_i^\top & \mathbf{0}^\top \end{bmatrix}_{3\times 12} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}_{12\times 1} = \mathbf{0}_{3\times 1}$$

7. Pick only two of the equations as only two are linearly independent.

$$\begin{bmatrix} \mathbf{0}^\top & -w_i \underline{\mathbf{X}}_i^\top & y_i \underline{\mathbf{X}}_i \\ w_i \underline{\mathbf{X}}_i^\top & \mathbf{0}^\top & -x_i \underline{\mathbf{X}}_i^\top \end{bmatrix}_{2\times 12} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}_{12\times 1} = \mathbf{0}_{2\times 1}$$

8. Collect all the equations from $n$ pairs of corresponding points $\underline{\mathbf{u}}_1, \ldots, \underline{\mathbf{u}}_n$ and $\underline{\mathbf{X}}_1, \ldots, \underline{\mathbf{X}}_n$.

$$\begin{bmatrix} \mathbf{0}^\top & -w_1 \underline{\mathbf{X}}_1^\top & y_1 \underline{\mathbf{X}}_1 \\ w_1 \underline{\mathbf{X}}_1^\top & \mathbf{0}^\top & -x_1 \underline{\mathbf{X}}_1^\top \\ \vdots & \vdots & \vdots \\ \mathbf{0}^\top & -w_n \underline{\mathbf{X}}_n^\top & y_n \underline{\mathbf{X}}_n \\ w_n \underline{\mathbf{X}}_n^\top & \mathbf{0}^\top & -x_n \underline{\mathbf{X}}_n^\top \end{bmatrix}_{2n\times 12} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}_{12\times 1} = \mathbf{0}_{2n\times 1}$$

9. $P$ matrix has rank $\text{rank}(P) = 11$ because it has 12 elements and equivalence upto a scale factor. So the solution of the above equation can be computed from SVD by choosing the right singular vector corresponding to the smallest singular value.

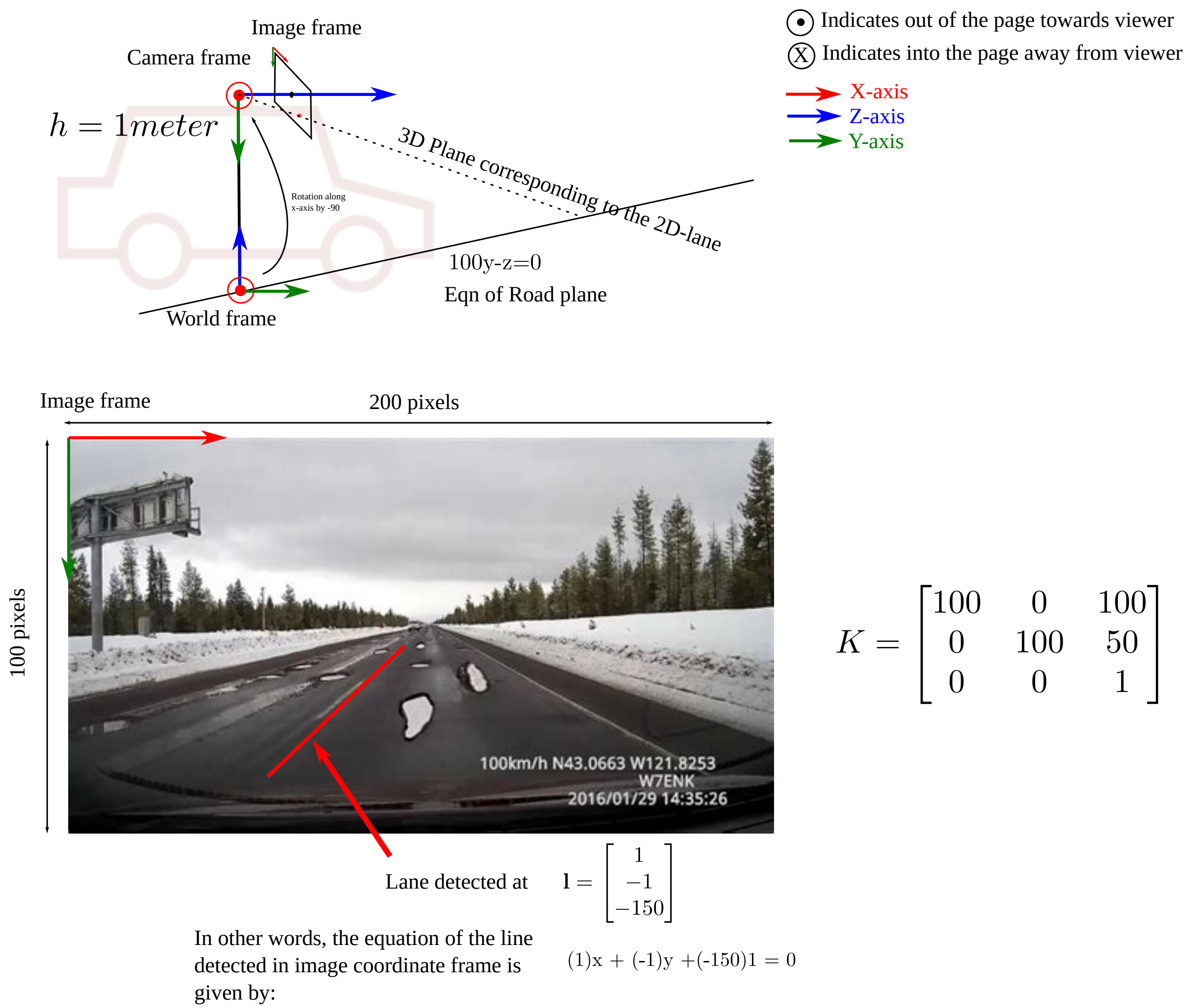
$$A = \begin{bmatrix} \mathbf{0}^\top & -w_1\underline{\mathbf{X}}_1^\top & y_1\underline{\mathbf{X}}_1 \\ w_1\underline{\mathbf{X}}_1^\top & \mathbf{0}^\top & -x_1\underline{\mathbf{X}}_1^\top \\ \vdots & \vdots & \vdots \\ \mathbf{0}^\top & -w_n\underline{\mathbf{X}}_n^\top & y_n\underline{\mathbf{X}}_n \\ w_n\underline{\mathbf{X}}_n^\top & \mathbf{0}^\top & -x_n\underline{\mathbf{X}}_n^\top \end{bmatrix} = U\Sigma V^T$$

Let $V = [\mathbf{v}_1, \ldots, \mathbf{v}_{\cancel{n}}]$, then
12

$$\begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix} = \mathbf{v}_{\cancel{n}} \; 12$$

Now we can write the $P$ matrix as

$$P = \begin{bmatrix} \mathbf{p}_1^\top \\ \mathbf{p}_2^\top \\ \mathbf{p}_3^\top \end{bmatrix}$$




$$K = \begin{bmatrix} 100 & 0 & 100 \\ 0 & 100 & 50 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{l} = \begin{bmatrix} 1 \\ -1 \\ -150 \end{bmatrix}$$

Figure 1: Line-plane triangulation

**Problem 2** *In figure 1 find the 3D representation of the lane the World coordinate frame, in terms of $h$ (the height of the camera), image-representation of the line $\mathbf{l}$ (provided in figure), camera matrix $K$ (provided in figure). Assume the lane to be a straight line. The Camera is mounted directly on top of the world frame, both of which are aligned to the gravity vector. The road is a perfect plane with a slope such that the equation of road plane in world-coordinate frame is given by $100Y_w - Z_w = 0$ and the lane lies on the road plane. Provide the formula or pseudo-code for computing the 3D representation of the lane, and also substitute in the values. (20 min, 20 marks)*

**Solution** Watch lecture `https://drive.google.com/file/d/1JaEwLxQ2BvT30sVxshmglBz_v5FjC1hW/view?usp=sharing` See homework 4 solution.

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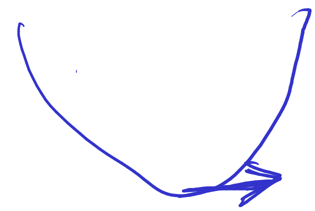
$\mathbb{R}^{m\times n}$ $\quad$ $\mathbb{R}^n$ $\quad$ $\in\mathbb{R}$

$$f(\underline{u}) = 2\,\underline{u}^T \underbrace{A^T A}_{\mathbb{R}^{n\times n}} \underline{u} - 3\,\underline{u}^T \underline{b} + 4\,\underline{c}^T \underline{u} + d \quad \in \mathbb{R}$$

$$\frac{\partial f(\underline{u})}{\partial \underline{u}} = 0$$



$$\frac{\partial f(\underline{u})}{\partial \underline{u}} = 0$$

$$\frac{\partial}{\partial \underline{u}} \underline{u}^T Q \underline{u} = 2 Q \underline{u}$$

$$\frac{\partial}{\partial \underline{u}} \underline{q}^T \underline{u} = \underline{q}$$

$$\frac{\partial}{\partial u} \underline{u}^T \underline{q} = \frac{\partial}{\partial u} \underline{q}^T u = \underline{q}$$

$$\frac{\partial f(\underline{u})}{\partial \underline{u}} = 0$$

$$\frac{\partial}{\partial \underline{u}} 2\underline{u}^T A^T A \underline{u} = 2(2A^T A) u$$

$$= 4 A^T A \underline{u}$$

$$\frac{\partial}{\partial \underline{u}} (-3\underline{u}^T \underline{b}) = -3\underline{b}$$

$$\frac{\partial}{\partial \underline{u}} 4\underline{c}^T \underline{u} = 4\underline{c}$$

$$\frac{\partial}{\partial \underline{u}} d = 0$$

$$\frac{\partial f(\underline{u})}{\partial \underline{u}} = 4A^TA\underline{u} - 3\underline{b} + 4\underline{c} = 0$$

$$\underline{u} = (A^TA)^{-1}\left(\frac{3}{4}\underline{b} - \underline{c}\right)$$

**Problem 3** *Find the minimum point of the function,* $f(\mathbf{u}) = 2\mathbf{u}^\top A^\top A\mathbf{u} - 3\mathbf{u}^\top \mathbf{b} + 4\mathbf{c}^\top \mathbf{u} + d$. *Let* $\mathbf{u} \in \mathbb{R}^{n\times 1}$ *be a n-dimensional vector and sizes of* $A, \mathbf{b}, \mathbf{c}, d$ *be such that matrix multiplication and addition is valid. Also assume that* $A^\top A$ *is full rank, hence invertible.*

**Solution** Watch this lecture `https://drive.google.com/file/d/1wgY2LAw7LQnh_IyHY0XDAr2yorXta93Z/view?usp=sharing`

**Problem 4** *Let matrix* $A \in \mathbb{R}^{m\times n}$ *be a* $m \times n$ *matrix. We are given that* $B = A^\top A$ *has* $n$ *orthonormal eigen vectors* $\mathbf{e}_1, \dots \mathbf{e}_n$ *with corresponding eigen values as* $\lambda_1 \dots \lambda_n$ *such that* $B\mathbf{e}_i = \lambda_i \mathbf{e}_i$ *for all* $i \in \{1, \dots n\}$. *Let the rank of matrix* $A$ *be* $r$. *Write the thin singular value decomposition of* $A = U_{m\times r}\Sigma_{r\times r}V_{n\times r}^\top$ *in terms of eigen values and eigen vectors of matrix* $B = A^\top A$.

**Solution** Go through these slides. Watch this lecture `https://drive.google.com/file/d/13aO_XI7kykQNOs5RJOfhqtL5f` `view?usp=sharing`

The matrix of right singular vectors of $A$ is same as the eigen vector matrix of $B = A^\top A$.

$$V = \left[\mathbf{e}_1 \dots \mathbf{e}_r\right] \in \mathbb{R}^{n\times r} \tag{1}$$

The matrix of singular values are the square root of eigen values of $B$.

$$\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & \dots & 0 \\ 0 & \ddots & 0 \\ 0 & \dots & \sqrt{\lambda_r} \end{bmatrix} \in bbR^{r\times r} \tag{2}$$

$$U = \left[\mathbf{u}_1 \dots \mathbf{u}_r\right] \in \mathbb{R}^{m\times r} \tag{3}$$

$$\text{where } \mathbf{u}_i = \frac{A\mathbf{e}_i}{\sqrt{\lambda_i}} \tag{4}$$

$A$

$\underbrace{B}_{\in \mathbb{R}^{n\times n}} = A^T A \Rightarrow \quad B\underline{e}_i = \lambda_i \underline{e}_i$

$$B\underbrace{\left[\underline{e}_1 \dots \underline{e}_n\right]}_{E} = \left[\underline{e}_1 \dots \underline{e}_n\right]\underbrace{\begin{bmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{bmatrix}}_{\Lambda}$$

$$BE = E\Lambda$$

$$B = E\Lambda E^{-1} \leftarrow \text{Eigen value decomposition}$$

$$A^T A = E \Lambda E^{-1} \qquad E^{-1} = E^T$$

$$= E \Lambda E^T \qquad \lambda_i \geq 0$$

$$\underset{m \times n}{A} = U \Sigma V^T \qquad V = E$$

$$V = [\underline{e}_1 \; \ldots \; \underline{e}_n] \qquad \underline{\text{defn}}$$

$$\Sigma = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix}_{m \times n} \qquad \sigma_i \equiv \sqrt{\lambda_i} \qquad \underline{\underline{\text{defn}}}$$

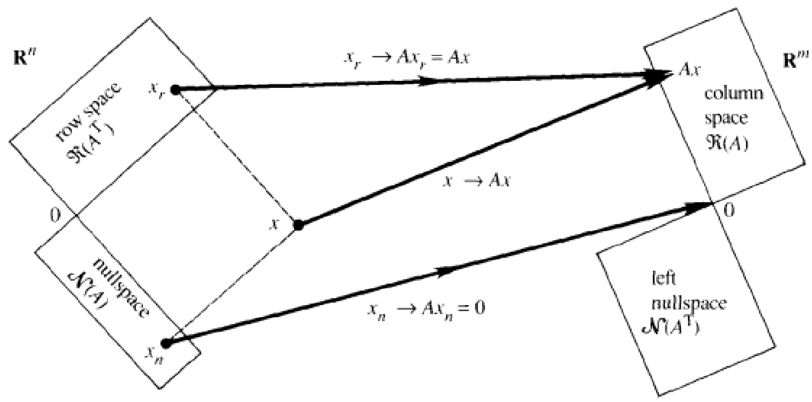
$$U = [\underline{u}_1, \ldots \; \underline{u}_m]_{m \times m}$$

$$\underline{u}_i = \frac{A \underline{e}_i}{\sigma_i} = \frac{A e_i}{\sqrt{\lambda_i}} \qquad \underline{\underline{\text{defn}}}$$

**Problem 5** *Let matrix $A \in \mathbb{R}^{m \times n}$ has the singular value decomposition (SVD) as $A = U\Sigma V^\top$ and rank of the matrix be $r = rank(A)$. Write the basis vectors of the four fundamental subspaces of matrix $A$ in terms of SVD,*

1. *Null space of $A$ ($\mathcal{N}(A) =?$).*
2. *Column space or range space ($\mathcal{R}(A) =?$).*
3. *Row space ($\mathcal{R}(A^\top) =?$).*
4. *Left null space ($\mathcal{N}(A^\top) =?$).*

*You can denote the first $r$ column vectors of $U$ has $U_{1:r} \in \mathbb{R}^{m\times r}$ and the renaming $m - r$ vectors as $U_{r+1:m} \in \mathbb{R}^{m\times(m-r)}$. Similarly for $V$, first $r$ column vectors of $V_{1:r} \in \mathbb{R}^{n\times r}$ and $V_{r+1:n} \in \mathbb{R}^{n\times(n-r)}$.*




$A^T = V\Sigma^T U^T$

$A = U\Sigma V^T \qquad r = rank(A)$

$$A = \begin{bmatrix} U_{1:r} & U_{r+1:m} \end{bmatrix} \begin{bmatrix} \sigma_1 & & & & \\ & \sigma_2 & & 0 & \\ & & \ddots & & 0 \\ & 0 & & \sigma_r & \\ & & & & 0 \\ & 0 & & & \ddots & 0 \end{bmatrix}_{m\times n} \begin{bmatrix} V_{1:r}^T \\ V_{r+1:n}^T \end{bmatrix}$$

$$\Updownarrow \quad \begin{bmatrix} V_{1:r} & V_{r+1:n} \end{bmatrix}^T$$

$$= U_{1:r}\Sigma V_{1:r}^T + U_{r+1:m}\, 0\, V_{r+1:n}^T$$

$\mathcal{N}(A) = V_{r+1:n} \qquad \mathcal{N}(A^T) = U_{r+1:m}$

$R(A) = U_{1:r} \qquad R(A^T) = V_{1:r}$

**Solution** Watch lecture `https://drive.google.com/file/d/17Cr0rMr567gfRNrrj9ri9vtsFaNLqW7t/view?usp=sharing`

1. Null space of $A$ ($\mathcal{N}(A) = V_{r+1:n}$).
2. Column space or range space ($\mathcal{R}(A) = U_{1:r}$).
3. Row space ($\mathcal{R}(A^\top) = V_{1:r}$).
4. Left null space ($\mathcal{N}(A^\top) = U_{r+1:m}$).

## Extra practice problems

**Problem 6** *Find a line passing through the following points*

$$\mathbf{u}_1 = [101, 203]^\top, \mathbf{u}_2 = [49, 102]^\top, \mathbf{u}_3 = [27, 51]^\top, \mathbf{u}_4 = [201, 403]^\top, \mathbf{u}_5 = [74, 151]^\top.$$

*You can leave the output in terms of SVD.*

**Problem 7** *Find a plane passing through the following points*

$$\mathbf{x}_1 = [9.99, 101, 203]^\top, \mathbf{x}_2 = [5.1, 49, 102]^\top, \mathbf{x}_3 = [2.5, 27, 51]^\top, \mathbf{x}_4 = [21, 201, 403]^\top, \mathbf{x}_5 = [7.6, 74, 151]^\top.$$

*You can leave the output in terms of SVD.*

**Problem 8** *Find the 3D line in parameteric representation that is formed by the intersection of two planes* $\mathbf{p}^\top \underline{\mathbf{x}} = 0$ *(with* $\mathbf{p} = [1, 2, 3, 4]^\top$*) and* $\mathbf{q}^\top \underline{\mathbf{x}} = 0$ *where* $\mathbf{q} = [-3, 2, 1, 4]^\top$.

**Problem 9** *Find the point on the intersection of following 3D lines* $\mathbf{x} = \lambda_1 \mathbf{d}_1 + \mathbf{y}$ *and* $\mathbf{x} = \lambda_2 \mathbf{d}_2 + \mathbf{z}$*. Here* $\lambda_1 \in \mathbb{R}$ *and* $\lambda_2 \in \mathbb{R}$ *are the free parameters. The rest of the parameters have the following values*

$$\mathbf{d}_1 = [1, 2, 0]^\top, \mathbf{d}_2 = [-2, 1, 0]^\top, \mathbf{y} = [1, 2, 0]^\top, \mathbf{z} = [4, 5, 0]^\top$$

**Problem 10** *Find the point of intersection of the 3D line* $\mathbf{x} = \lambda \mathbf{d} + \mathbf{x}_0$ *with the 3D plane* $\mathbf{p}^\top \underline{\mathbf{x}} = 0$*. The parameters have the following*

$$\mathbf{d} = [1, 2, 0]^\top, \mathbf{x}_0 = [3, 4, 5]^\top, \mathbf{p} = [1, 2, 0, 7]^\top$$

Soln 6 :

$$\underline{l}^T \underline{u} = 0 = \underline{u}^T \underline{l}$$

$$l = \begin{bmatrix} l_1 \\ l_2 \\ l_3 \end{bmatrix}, \quad \underline{u} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

$$ax + by + c = 0$$

$$l_1 x + l_2 y + l_3 = 0$$

$$\underbrace{\begin{bmatrix} l_1 & l_2 & l_3 \end{bmatrix}}_{\underline{l}^T} \underbrace{\begin{bmatrix} x \\ y \\ 1 \end{bmatrix}}_{\underline{u}} = 0$$

$$\underline{u}_1 = \begin{bmatrix} 101 \\ 203 \\ 1 \end{bmatrix} \qquad \underline{u}_2 = \begin{bmatrix} 49 \\ 102 \\ 1 \end{bmatrix} \quad \cdots$$

$$\underline{u}_1^T \underline{l} = 0$$

$$\underline{u}_2^T \underline{l} = 0$$

$$\vdots$$

$$\underline{u}_5^T \underline{l} = 0$$

$$\underbrace{\begin{bmatrix} \underline{u}_1^T \\ \underline{u}_2^T \\ \vdots \\ \underline{u}_5^T \end{bmatrix}}_{A} \underline{l} = 0$$

$$\underline{l} \in N(A) \quad \ni \qquad A_{5\times 3} = U \Sigma V^T_{3\times 3}$$

$$\underline{l} = \underline{v}_3 \qquad V = [\underline{v}_1, \underline{v}_2, \underline{v}_3]$$

## Soln 7

$$ax + by + cz + d = 0$$

$$\underbrace{\begin{bmatrix} a & b & c & d \end{bmatrix}}_{\underline{p}^T} \underbrace{\begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}}_{\underline{x}} = 0$$

$$\underline{p}^T \underline{x} = 0$$

$$\underline{x}^T \underline{p} = 0$$

$$\underline{x}_1 = \begin{bmatrix} 9.99 \\ 108 \\ 203 \\ 1 \end{bmatrix}, \quad \underline{x}_2 = \begin{bmatrix} 5.1 \\ 49 \\ 102 \\ 1 \end{bmatrix}, \quad \ldots \quad \underline{x}_5 = \begin{bmatrix} 7.6 \\ 74 \\ 151 \\ 1 \end{bmatrix}$$

$$\underline{x}_1^T \underline{p} = 0$$

$$\underline{x}_2^T \underline{p} = 0$$

$$\vdots$$

$$x_5^T \underline{p} = 0$$

$$\underbrace{\begin{bmatrix} \underline{x}_1^T \\ \underline{x}_2^T \\ \vdots \\ \underline{x}_5^T \end{bmatrix}_{5\times 4}}_{A} \underline{p} = 0$$

$\underline{p} \in \mathcal{N}(A)$

$A = U \Sigma V^T_{4\times4}$

$V = [\underline{v}_1, \dots, \underline{v}_4]$

$\text{rank}(A) = 3$

$p = \underline{v}_4$

HW4 + HW5 way

$$\begin{bmatrix} p_{1:3}^T \\ q_{1:3}^T \end{bmatrix} \underline{x} = \begin{bmatrix} -p_4 \\ -q_4 \end{bmatrix}$$

$$\underline{x} = \lambda (p_{1:3} \times q_{1:3}) + \begin{bmatrix} p_{1:3}^T \\ q_{1:3}^T \end{bmatrix}^{\#} \begin{bmatrix} -p_4 \\ -q_4 \end{bmatrix}$$

Soln 8

$\underline{p}^T \underline{x} = 0$

$q^T \underline{x} = 0$

$$\underbrace{\begin{bmatrix} p^T \\ q^T \end{bmatrix}_{2\times4}}_{A} \underline{x} = 0$$

$A \underline{x} = 0$

$\underline{x} \in \mathcal{N}(A)$ $\quad A = U \Sigma V^T_{4\times4}$

$V = [\underline{v}_1 \ \dots \ \underline{v}_4]$

$\underline{x} = \lambda_3 \underline{v}_3 + \lambda_4 \underline{v}_4$

$\underline{x} = \underline{v}_3 + \frac{\lambda_4}{\lambda_3} \underline{v}_4 = \underline{v}_3 + t\, \underline{v}_4$ parametric eqn of line

Soln 7

$$\underline{x} = \lambda_1 \underline{d}_1 + \underline{y}$$

$$\underline{x} = \lambda_2 \underline{d}_2 + \underline{z}$$

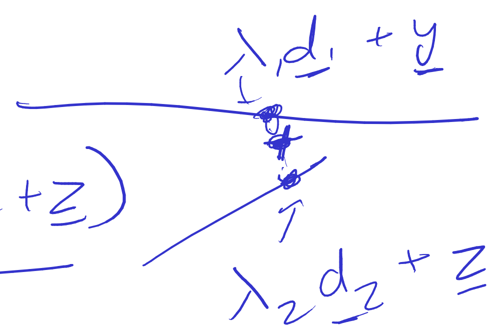
$$\lambda_1 \underline{d}_1 + \underline{y} = \lambda_2 \underline{d}_2 + \underline{z}$$

$$\lambda_1 \underline{d}_1 - \lambda_2 \underline{d}_2 = \underline{z} - \underline{y}$$

$$\underbrace{\begin{bmatrix} \underline{d}_1 & -\underline{d}_2 \end{bmatrix}_{3\times 2}}_{A} \underbrace{\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}_{2\times 1}} = \underbrace{\underline{z} - \underline{y}}_{\underline{b}} \quad 3\times 1$$

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = A^{+} \underline{b} = (A^T A)^{-1} A^T (\underline{z} - \underline{y})$$

$$\underline{x} = \frac{(\lambda_1 \underline{d}_1 + \underline{y}) + (\lambda_2 \underline{d}_2 + \underline{z})}{2}$$



Soln 10

$$\vec{x} = \lambda \vec{d} + \vec{x}_0$$ — 3D line

$$\underline{p}^T \underline{x} = 0$$ — 3D plane

$$p_{1:3}^T \vec{x} + p_4 = 0$$

$$p_{1:3}^T (\lambda \vec{d} + \vec{x}_0) + p_4 = 0$$

$$\lambda (p_{1:3}^T \vec{d}) + p_{1:3}^T \vec{x}_0 + p_4 = 0$$

$$\lambda = \left[ \frac{-p_{1:3}^T \vec{x}_0 - p_4}{p_{1:3}^T \vec{d}} \right]$$

$$\left. \vec{x} = \lambda \vec{d} + \vec{x}_0 \right\} \text{Point of intersection}$$

## Practice problem solutions

**Solution 6** Watch lecture `https://drive.google.com/file/d/13aO_XI7kykQNOs5RJ0fhqtL5fgEzVfkN/view?usp=sharing` Let $\mathbf{l} \in \mathbb{P}^2$ be the parameters of the line, so that $\underline{\mathbf{u}}^\top \mathbf{l} = 0$.

$$A = \begin{bmatrix} \mathbf{u}_1^\top & 1 \\ \mathbf{u}_2^\top & 1 \\ \mathbf{u}_3^\top & 1 \\ \mathbf{u}_4^\top & 1 \\ \mathbf{u}_5^\top & 1 \end{bmatrix}_{5\times 3}$$

We are looking for the solution of $A\mathbf{l} = 0$. Let the SVD of $A = U\Sigma V^\top$. Let $V = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$, then the representation of the line $\mathbf{l} = \mathbf{v}_3$.

**Solution 7** Watch the lecture `https://drive.google.com/file/d/1PEdgHCf6UdOWYMVtCRlsjfctDcnBm8E9/view?usp=sharing`

Let $\mathbf{p} \in \mathbb{P}^3$ be the parameters of the plane, so that $\underline{\mathbf{x}}^\top \mathbf{p} = 0$.

$$A = \begin{bmatrix} \mathbf{x}_1^\top & 1 \\ \mathbf{x}_2^\top & 1 \\ \mathbf{x}_3^\top & 1 \\ \mathbf{x}_4^\top & 1 \\ \mathbf{x}_5^\top & 1 \end{bmatrix}_{5\times 4}$$

We are looking for the solution of $A\mathbf{p} = 0$. Let the SVD of $A = U\Sigma V^\top$. Let $V = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4]$, then the representation of the line $\mathbf{p} = \mathbf{v}_4$.

**Solution 8** Watch the lecture `https://drive.google.com/file/d/1JaEwLxQ2BvT30sVxshmglBz_v5FjC1hW/view?usp=sharing` $\mathbf{x} = \lambda(\mathbf{p}_{1:3} \times \mathbf{q}_{1:3}) + \begin{bmatrix} \mathbf{p}_{1:3}^\top \\ \mathbf{q}_{1:3} \end{bmatrix}^\dagger \begin{bmatrix} -p_4 \\ q_4 \end{bmatrix}$

**Solution 9** Watch the lecture `https://drive.google.com/file/d/1foVVQBCOkrljjJ3f-UP4zsHn4l-c611e/view?usp=sharing`

$$\begin{bmatrix} \mathbf{d}_1 & -\mathbf{d}_2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \mathbf{z} - \mathbf{y}$$

or

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \mathbf{d}_1 & -\mathbf{d}_2 \end{bmatrix}^\dagger \mathbf{z} - \mathbf{y}$$

The point of intersection is given by

$$\mathbf{x} = \lambda_1 \mathbf{d}_1 + \mathbf{y}$$

**Solution 10** Watch the lecture `https://drive.google.com/file/d/1wgY2LAw7LQnh_IyHYOXDAr2yorXta93Z/view?usp=sharing`

$$\lambda \mathbf{p}_{1:3}^\top \mathbf{d} + \mathbf{p}_{1:3}^\top \mathbf{x}_0 + p_4 = 0$$

Solve for $\lambda$.

$$\lambda = -\frac{\mathbf{p}_{1:3}^\top \mathbf{x}_0 + p_4}{\mathbf{p}_{1:3}^\top \mathbf{d}}$$

Point of intersection is

$$\mathbf{x} = \lambda \mathbf{d} + \mathbf{x}_0$$