ECE 417/598: Review Homework 4

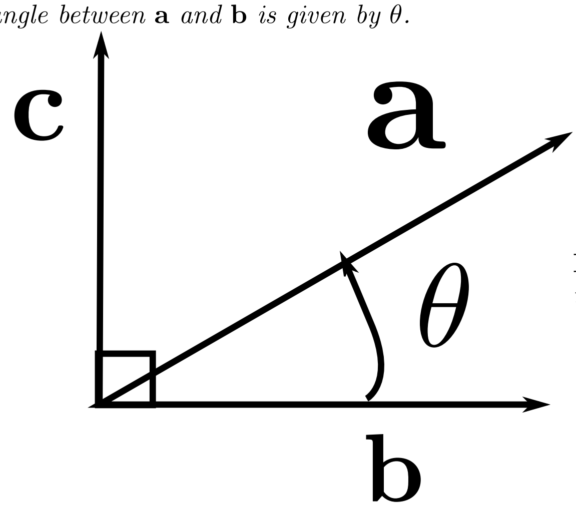
Max marks: 100 marks

Due on March 10th, 2021, midnight, 11:59 PM.

All notes so far are linked here.

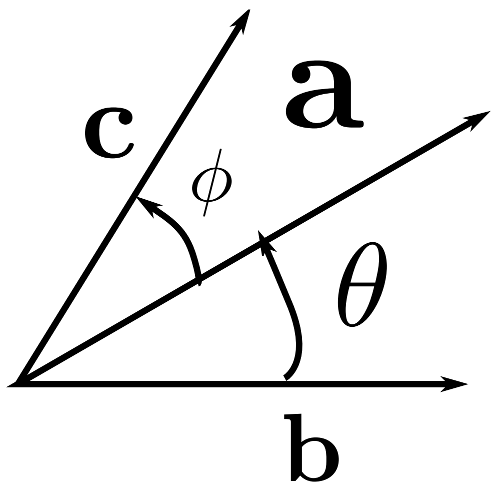
1 Trignometry and triangle laws of vector addition

Problem 1 *The magnitude of vector* $\mathbf{a} \in \mathbb{R}^n$ *is given to be* $\|\mathbf{a}\| = \alpha$. *Using the following figure, write* $\mathbf{a}$ *in terms of* α, θ, *vector* $\mathbf{b} \in \mathbb{R}^n$ *and* $\mathbf{c} \in \mathbb{R}^n$. *All three vectors lie in the same plane.* $\mathbf{b}$ *and* $\mathbf{c}$ *are perpendicular to each other. The angle between* $\mathbf{a}$ *and* $\mathbf{b}$ *is given by* θ.



(5 min, 5 marks)

Problem 2 *The magnitude of vector* $\mathbf{a} \in \mathbb{R}^n$ *is given to be* $\|\mathbf{a}\| = \alpha$. *Using the following figure, write* $\mathbf{a}$ *in terms of* α, θ, ϕ *vector* $\mathbf{b} \in \mathbb{R}^n$ *and* $\mathbf{c} \in \mathbb{R}^n$. *All three vectors lie in the same plane. The angle between* $\mathbf{a}$ *and* $\mathbf{b}$ *is given by* θ. *The angle between* $\mathbf{a}$ *and* $\mathbf{c}$ *is given by* ϕ. *Assume* $\theta+\phi \neq 0$. *When* $\theta+\phi = \frac{\pi}{/}2$, *is the solution is same as Problem 1? (Hint: You can convert this to Problem 1, by drawing a unit-vector perpendicular to* $\mathbf{b}$. *Call it* $\hat{\mathbf{d}}$. *First write* $\hat{\mathbf{d}}$ *in terms of* $\mathbf{c}$ *and others knowns and then write* $\mathbf{a}$ *in terms of* $\hat{\mathbf{d}}$ *and other knowns. You might want to use trignometric identities. The simplest form is not required.).*



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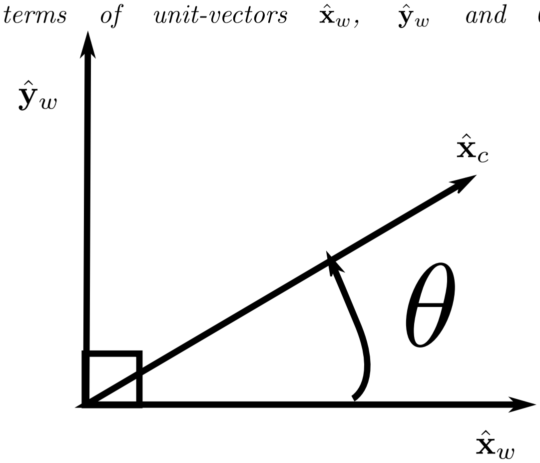
Problem 3 *Find unit-vector* $\hat{\mathbf{x}}_c$ *in terms of unit-vectors* $\hat{\mathbf{x}}_w$, $\hat{\mathbf{y}}_w$ *and* θ.



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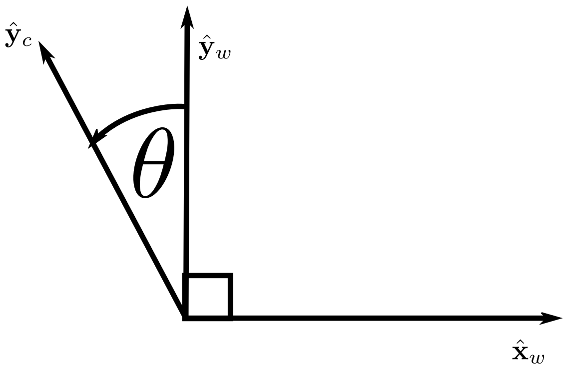
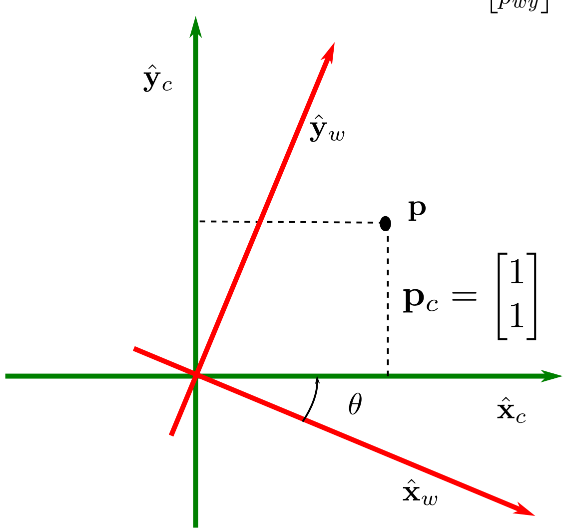
Problem 4 *Find unit-vector* $\hat{\mathbf{y}}_c$ *in terms of unit-vectors* $\hat{\mathbf{x}}_w$, $\hat{\mathbf{y}}_w$ *and* θ.



(5 min, 5 marks)

Problem 5 *Let the coordinates of a vector* $\mathbf{p}$ *in terms of* $\hat{\mathbf{x}}_c$ *and* $\hat{\mathbf{y}}_c$ *be* $\mathbf{p}_c = \begin{bmatrix} p_{cx} \\ p_{cy} \end{bmatrix}$, *so that:* $\mathbf{p} = p_{cx}\hat{\mathbf{x}}_c + p_{cy}\hat{\mathbf{y}}_c$. *Using the results from Prob 3 and Prob 4, write* $\mathbf{p}$ *in terms of* $\hat{\mathbf{x}}_w$ *and* $\hat{\mathbf{y}}_w$. *Thus derive the formula for rotation matrix* $R(\theta)$ *that converts coordinates from* $\mathbf{p}_c$ *to* $\mathbf{p}_w = \begin{bmatrix} p_{wx} \\ p_{wy} \end{bmatrix}$.



(10 min, 10 marks)

Problem 6 *We know that* $\|\mathbf{v}_{\perp,rot}\| = \|\mathbf{v}_\perp\|$. *Write* $\mathbf{v}_{\perp,rot}$ *in terms of* $\mathbf{v}_\perp$, $\mathbf{w}$ *and* θ. $\mathbf{v}_\perp$ *and* $\mathbf{w}$ *are known to be orthogonal to each other.*

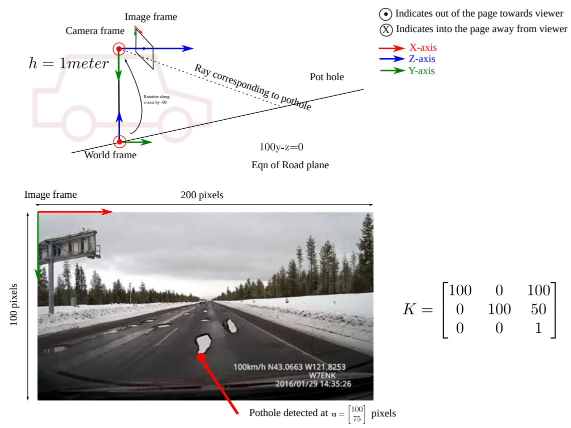
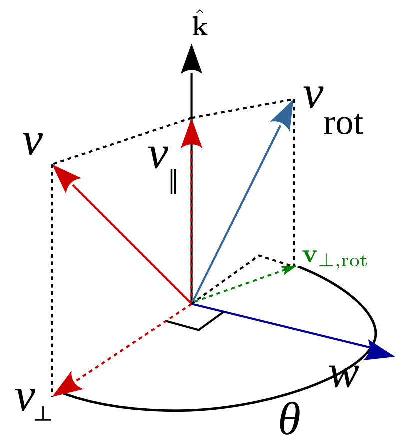


Figure 1: Point-plane triangulation



(5 min, 5 marks)

Problem 7 *In figure 1 find the 3D position of the pothole the World coordinate frame, in terms of* $h = 1$ *(the height of the camera), image-coordinates of the pothole* $\mathbf{u}$ *(provided in figure), camera matrix* K *(provided in figure). The Camera is mounted directly on top of the world frame, both of which are aligned to the gravity vector. The road is a perfect plane with a slope such that the equation of road plane in world-coordinate frame is given by* $100Y_w - Z_w = 0$ *and the pothole lies on the road plane. Provide the formula or pseudo-code for computing the pothole coordinates, and also substitute in the values. (20 min, 20 marks)*

Problem 8 *In figure 2 find the 3D representation of the lane the World coordinate frame, in terms of h (the height of the camera), image-representation of the line $\mathbf{l}$ (provided in figure), camera matrix K (provided in figure). Assume the lane to be a straight line. The Camera is mounted directly on top of the world frame, both of which are aligned to the gravity vector. The road is a perfect plane with a slope such that the equation of road plane in world-coordinate frame is given by $100Y_w - Z_w = 0$ and the pothole lies on the road plane. Provide the formula or pseudo-code for computing the pothole coordinates, and also substitute in the values. (20 min, 20 marks)*

*Hint 0: **Equation of plane in 3D.** Equation of a plane in 3D is given by $p_1X + p_2Y + p_3Z + p_4 = 0$. In matrix notation, you can write the equation plane as $\mathbf{p}_{1:3}^\top \mathbf{X} + p_4 = 0$, where $\mathbf{p}_{1:3} = [p_1, p_2, p_3]^\top$.*

*Hint 1: **3D Plane corresponding to the line in image-coordinates.** Let the equation of line in image-coordinates be $\mathbf{l}^\top \underline{\mathbf{u}} = 0$, where $\underline{\mathbf{u}} = \begin{bmatrix} \mathbf{u} \\ 1 \end{bmatrix} \in \mathbb{P}^2$ are all the points on the line. By pinhole camera model, if $\mathbf{X}_c \in \mathbb{R}^3$ are the corresponding points in 3D, then the equation of corresponding plane is given by $\mathbf{l}^\top (K\mathbf{X}_c) = 0$ which can also be written as $(K^\top \mathbf{l})^\top \mathbf{X}_c = 0$. If we compare it to the equation of plane $\mathbf{p}_{1:3}^\top \mathbf{X} + p_4 = 0$, then $\mathbf{p}_{1:3} = K^\top \mathbf{l}$ and $p_4 = 0$.*

*Hint 2: **Intersection of two planes in 3D is a line.** Equation of a plane in 3D is given by $p_1X_w + p_2Y_w + p_3Z_w + p_4 = 0$. In matrix notation, you can write the equation of the plane as $\mathbf{p}_{1:3}^\top \mathbf{X}_w + p_4 = 0$, where $\mathbf{p}_{1:3} = [p_1, p_2, p_3]^\top$. Let's say you have two planes $\mathbf{p}_{1:3}^\top \mathbf{X}_w + p_4 = 0$ and $\mathbf{q}_{1:3}^\top \mathbf{X}_w + q_4 = 0$. Their intersection is a line whose parameteric form is given by (why ? you have all the knowledge required to derive this):*

$$\mathbf{X}_w = \lambda(\mathbf{p}_{1:3} \times \mathbf{q}_{1:3}) + \begin{bmatrix} \mathbf{p}_{1:3}^\top \\ \mathbf{q}_{1:3}^\top \end{bmatrix}^\dagger \begin{bmatrix} -p_4 \\ -q_4 \end{bmatrix}, \tag{1}$$

where $A^\dagger$ denotes the pseudo-inverse of a matrix (a fat matrix in this case) and $\lambda \in \mathbb{R}$ is the free parameter and $\times$ denotes the vector cross-product.

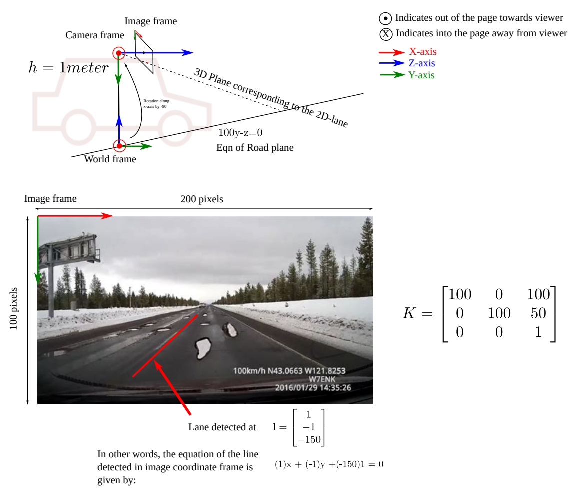


Figure 2: Line-plane triangulation

Problem 9 *You are a part of Tesla self-driving team. Team 1 provides you with lane-detection algorithms and their output. Team 2 provides you with detailed maps of road conditions. Your task is to write a function that solves problem 8 for arbitrary lanes detected by team 1 and for arbitrary plane provided by team 2. (Hint: Equation of a plane 3D is very similar to equation of line in 2D). What input representations of lane and plane would you ask for? Write a general algorithm or pseudo-code that solves problem 8. (30 min, 10 marks)*