ECE 498/598 Midterm 2 2023

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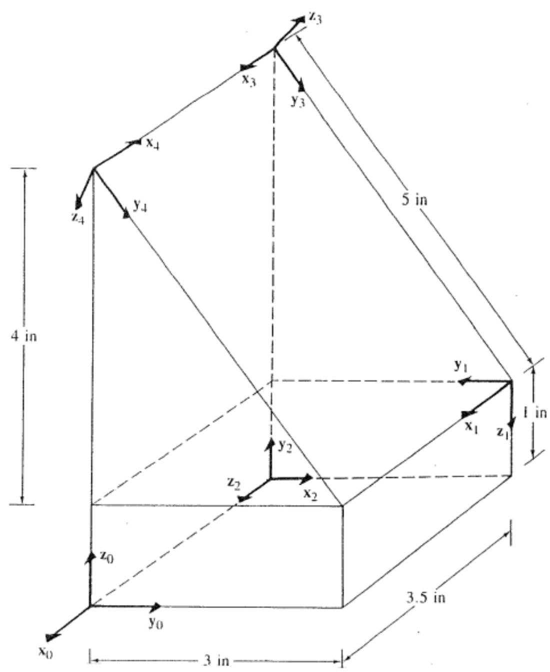
(1) Student name: Student email:

About the exam

1. There are total 4 problems. You must attempt all 4.
2. Maximum marks: 50.
3. Maximum time allotted: 50 min
4. Calculators are allowed.
5. One US Letter size or A4 size cheat sheet (both-sides) is allowed.

Problem 1 *Find the 4x4 transformation matrix* $^1T_0$ *that transforms coordinates from coordinate frame 1 to coordinate frame 0 (5 marks).*



Solution: I am going to write transform from frame 1 to frame 0. $^0T_1$

$$^0T_1 = \begin{bmatrix} 1 & 0 & 0 & -3.5 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}$$

The rotation matrix is obtained by writing the basis vectors of the destination coordinate system in the source coordinate system.

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Problem 2 *Consider a coordinate system OUVW whose ordered set of basis vectors given by* $\mathbf{u} = [3/7, 2/7, 6/7]^\top$, $\mathbf{v} = [2/7, 6/7, 3/7]^\top$ *and* $\mathbf{w} = [4, 5, 6]^\top$. *Another coordinate system OXYZ whose order set of basis vectors is,* $\mathbf{x} = [2/7, 6/7, -3/7]^\top$, $\mathbf{y} = [-6/7, 3/7, 2/7]^\top$ *and* $\mathbf{z} = [3/7, 2/7, 6/7]^\top$. *Find the rotation matrix* ${}^{ouvw}R_{oxyz}$ *that converts coordinates from frame OXYZ to frame OUVW. (10 marks)*

<u>Solution:</u>

Let OPQR be the coordinate system in which the coordinates of basis vectors are specified.

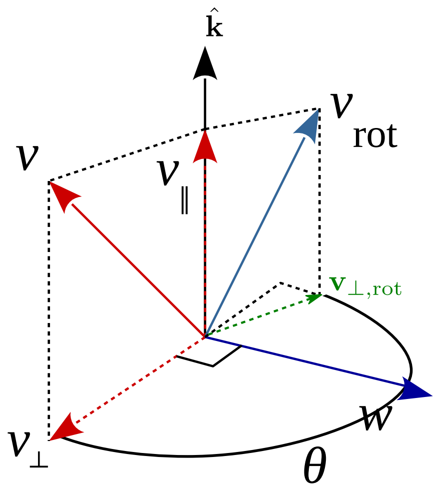
$$ {}^{OPQR}R_{OUVW} = \begin{bmatrix} | & | & | \\ \mathbf{u} & \mathbf{v} & \mathbf{w} \\ | & | & | \end{bmatrix} = \begin{bmatrix} 3/7 & 2/7 & 4 \\ 2/7 & 6/7 & 5 \\ 6/7 & 3/7 & 6 \end{bmatrix} \tag{2} $$

$$ {}^{OPQR}R_{OXYZ} = \begin{bmatrix} | & | & | \\ \mathbf{x} & \mathbf{y} & \mathbf{z} \\ | & | & | \end{bmatrix} = \begin{bmatrix} 2/7 & -6/7 & 3/7 \\ 6/7 & 3/7 & 2/7 \\ -3/7 & 2/7 & 6/7 \end{bmatrix} \tag{3} $$

$$ {}^{OUVW}R_{OXYZ} = {}^{OUVW}R_{OPQR}{}^{OPQR}R_{OXYZ} = (({}^{OPQR}R_{OUVW})^\top){}^{OPQR}R_{OXYZ} \tag{4} $$

$$ = \begin{bmatrix} 3/7 & 2/7 & 4 \\ 2/7 & 6/7 & 5 \\ 6/7 & 3/7 & 6 \end{bmatrix}^\top \begin{bmatrix} 2/7 & -6/7 & 3/7 \\ 6/7 & 3/7 & 2/7 \\ -3/7 & 2/7 & 6/7 \end{bmatrix} \tag{5} $$

Problem 3 *(Rodrigues formula) In the figure below, we are rotating point* $\mathbf{v}$ *around axis unit-vector* $\hat{\mathbf{k}}$ *by an angle* θ*. A unit vector* $\hat{\mathbf{w}}$ *is perpendicular to the both* $\mathbf{v}$ *and* $\hat{\mathbf{k}}$*. Another vector* $\mathbf{v}_\perp$ *is the projection of* $\mathbf{v}$ *onto a plane that is perpendicular to* $\hat{\mathbf{k}}$*. Note that* $\mathbf{v}_\perp$ *is perpendicular to both* $\hat{\mathbf{w}}$ *and* $\hat{\mathbf{k}}$*. First, (a) write the unit-vector* $\hat{\mathbf{w}}$ *in terms of* $\mathbf{v}$ *and* $\hat{\mathbf{k}}$*. (b) Then write the vector (including the correct magnitude)* $\mathbf{v}_\perp$ *in terms of* $\mathbf{v}$ *and* $\hat{\mathbf{k}}$*. (c) A vector* $\mathbf{v}_{\perp,rot}$ *is obtained by rotating* $\mathbf{v}_\perp$ *by an angle* θ*. Write the vector* $\mathbf{v}_{\perp,rot}$ *in terms of* $\mathbf{v}_\perp$*,* $\hat{\mathbf{w}}$ *and* θ*. (15 marks)*



Solution:

$$\hat{\mathbf{w}} = \frac{\hat{\mathbf{k}} \times \mathbf{v}}{\|\hat{\mathbf{k}} \times \mathbf{v}\|} \tag{6}$$

Note that $\hat{\mathbf{k}} \times \mathbf{v}$ has the magnitude $\|\hat{\mathbf{k}} \times \mathbf{v}\| = |\hat{\mathbf{k}}||\mathbf{v}|\sin(\alpha) = |\mathbf{v}|\sin(\alpha)$ where α is the angle between $\mathbf{v}$ and $\hat{\mathbf{k}}$.

$$\mathbf{v}_\perp = -\hat{\mathbf{k}} \times (\hat{\mathbf{k}} \times \mathbf{v}) \tag{7}$$

The magnitude of $\mathbf{v}_\perp$ is same as RHS because both $|\mathbf{v}_\perp| = |\mathbf{v}|\sin(\alpha)$ and $|-\hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\mathbf{v})| = |\mathbf{v}||\hat{\mathbf{k}}|^2\sin(\alpha)\sin(90^\circ) = |\mathbf{v}|\sin(\alpha)$ where α is the angle between $\mathbf{v}$ and $\hat{\mathbf{k}}$.

$$\mathbf{v}_{\perp,rot} = \mathbf{v}_\perp\cos(\theta) + \hat{\mathbf{w}}|\hat{\mathbf{k}} \times \mathbf{v}|\sin(\theta) \tag{8}$$

Problem 4 *The Euler angles of rotation YZX are given as θ, ϕ and ψ. Derive the rotation matrix corresponding to the Euler angle representation $R = R_x(\psi)R_z(\phi)R_y(\theta)$. Also derive an expression to convert the rotation matrix back to Euler angles. (20 marks).*

<u>Solution:</u>

$$R = R_x(\psi)R_z(\phi)R_y(\theta) \tag{9}$$

$$\implies \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\psi & -s_\psi \\ 0 & s_\psi & c_\psi \end{bmatrix} \begin{bmatrix} c_\phi & -s_\phi & 0 \\ s_\phi & c_\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\theta & 0 & s_\theta \\ 0 & 1 & 0 \\ -s_\theta & 0 & c_\theta \end{bmatrix} \tag{10}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\psi & -s_\psi \\ 0 & s_\psi & c_\psi \end{bmatrix} \begin{bmatrix} c_\phi c_\theta & -s_\phi & c_\phi s_\theta \\ s_\phi c_\theta & c_\phi & s_\phi s_\theta \\ -s_\theta & 0 & c_\theta \end{bmatrix} \tag{11}$$

$$\implies \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} c_\phi c_\theta & -s_\phi & c_\phi s_\theta \\ c_\psi s_\phi c_\theta + s_\psi s_\theta & c_\psi c_\phi & c_\psi s_\phi s_\theta - s_\psi c_\theta \\ s_\psi s_\phi c_\theta - c_\psi s_\theta & s_\psi c_\phi & s_\psi s_\phi s_\theta + c_\psi c_\theta \end{bmatrix} \tag{12}$$

$$\phi = \sin^{-1}(-r_{12}) \in [-\pi/2, \pi/2] \tag{13}$$
$$\theta = \text{arctan2}(r_{13}, r_{11}) \tag{14}$$
$$\psi = \text{arctan2}(r_{32}, r_{22}) \tag{15}$$