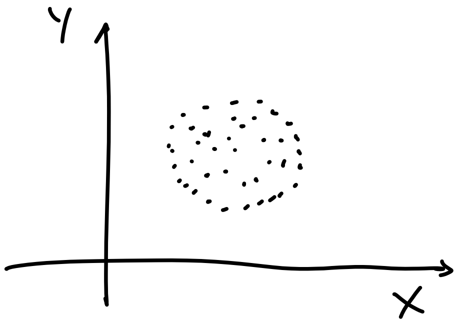
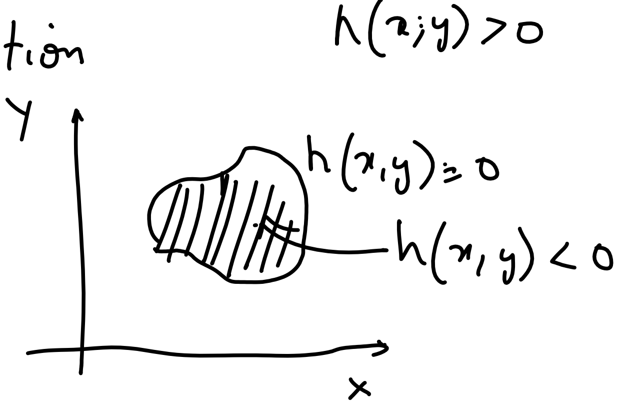
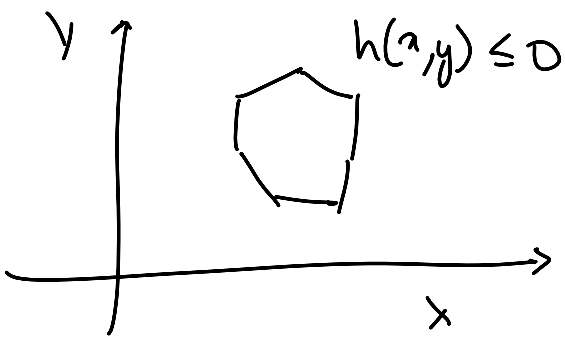
# 2D coordinate transforms

Rotations + translations

Orientation + position



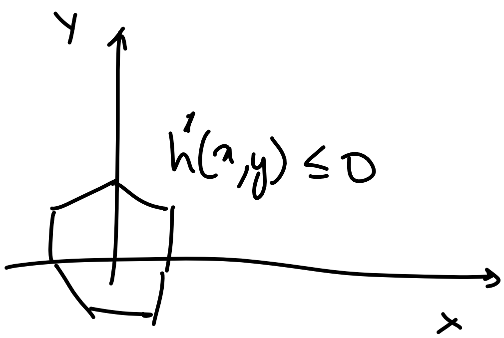



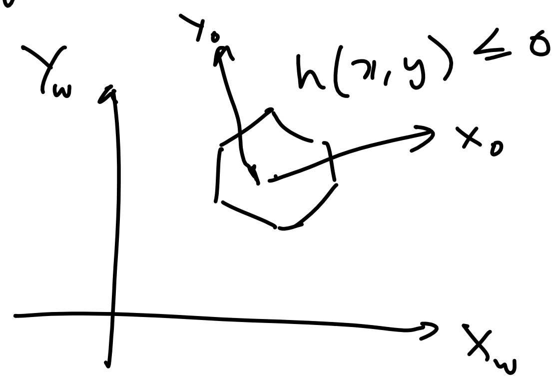

$$\text{Object} = \{(x,y) \in \mathbb{R}^2 : h(x,y) \le 0\}$$

Set of all points $(x,y) \in \mathbb{R}^2$ ← set of all real numbers

such that

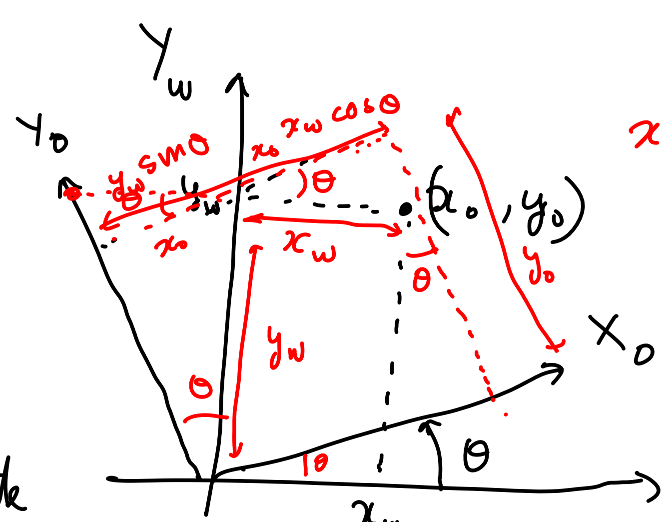
$h(x,y) \le 0$




## 2D Rotations

1. Rotation
2. Translation




$x_o = x_w \cos\theta + y_w \sin\theta$

Problem: $(x_o, y_o)$ is given in $(X_o, Y_o)$ coordinate frame. $(X_o\,Y_o)$ has been rotated by angle $\theta$ w.r.t. $(X_w, Y_w)$

Find $(x_w, y_w)$ in world coordinate frame

Proof using Basis vectors

In Linear algebra, Basis vectors are set of orthonormal unit vectors that span the entire space

Span is the set of all vectors that can be obtained by linear combinations of a given set of vectors

$$\text{Span}\{\underbrace{\underline{a}, \underline{b}}_{\in \mathbb{R}^n}\} = \{\underbrace{\alpha \underline{a} + \beta \underline{b}}_{\in \mathbb{R}^n}, \begin{matrix} \alpha \in \mathbb{R} \\ \beta \in \mathbb{R} \end{matrix}\}$$

Standard Basis vector.

For example, in $\mathbb{R}^2$ $\quad \hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \hat{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

in $\mathbb{R}^3$ $\quad \hat{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \hat{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

in $\mathbb{R}^n$ $\quad \hat{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n \times 1} \quad \cdots \quad \hat{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}_{n \times 1}$

Basis vectors for $\mathbb{R}^n$

↳ ① All vectors must be perpendicular/orthogonal to each other

↳ ② They must be unit vectors

↳ ③ They must span the entire space $\mathbb{R}^n$

Let Basis vector for $(X_w, Y_w)$ be standard basis vector

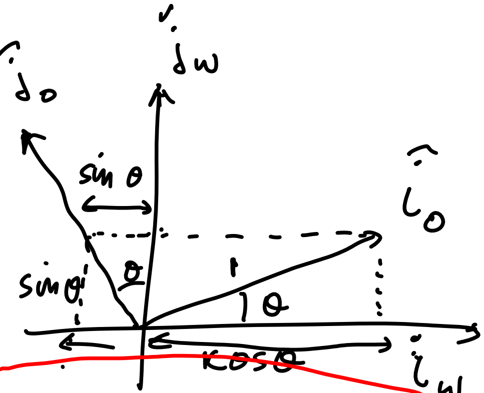
$\hat{i}_w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \hat{j}_w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Any point in world coordinate frame

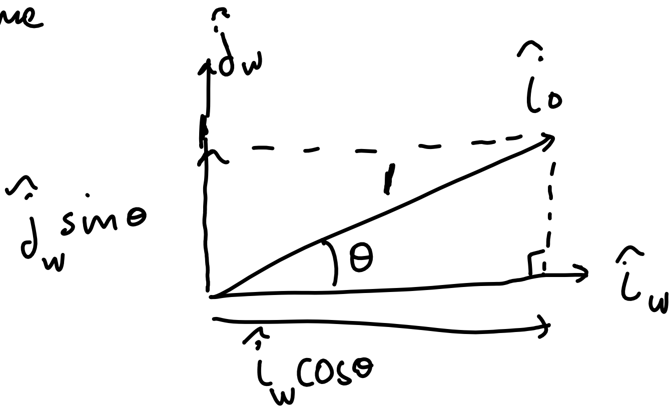
$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = x_w \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\hat{i}_w} + y_w \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\hat{j}_w}$$

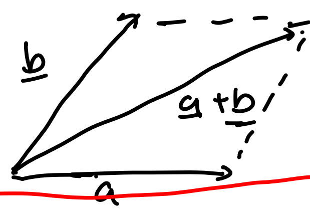
Any point in the object coordinate frame

$$\begin{bmatrix} x_o \\ y_o \end{bmatrix} = x_o \hat{i}_o + y_o \hat{j}_o$$




$$\boxed{\hat{i}_o = \cos\theta\, \hat{i}_w + \sin\theta\, \hat{j}_w}$$




$$\boxed{\hat{j}_o = -\hat{i}_w \sin\theta + \hat{j}_w \cos\theta}$$

$$\overbrace{x_w \hat{i}_w + y_w \hat{j}_w}^{\text{world}} = \overbrace{x_o \hat{i}_o + y_o \hat{j}_o}^{\text{object}}$$

$$\underbrace{\begin{bmatrix} x_w \\ y_w \end{bmatrix}}_{} = x_o \left[\cos\theta\, \hat{i}_w + \sin\theta\, \hat{j}_w\right] + y_o \left[-\hat{i}_w \sin\theta + \hat{j}_w \cos\theta\right]$$

Because we are using standard basis for world coordinate frame

$$= \left[x_o \cos\theta - y_o \sin\theta\right] \hat{i}_w + \left[x_o \sin\theta + y_o \cos\theta\right] \hat{j}_w$$

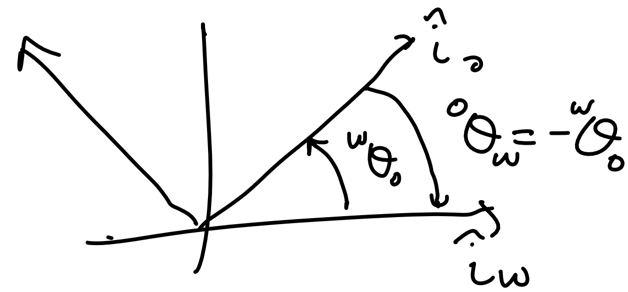
$$= \qquad \hat{i}_w \to \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \hat{j}_w \to \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} x_o \cos\theta - y_o \sin\theta \\ x_o \sin\theta + y_o \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_o \\ y_o \end{bmatrix}$$

$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \begin{bmatrix} {}^{w}R_{o}(\theta) \end{bmatrix} \begin{bmatrix} x_o \\ y_o \end{bmatrix}$$

$${}^{w}R_{o}(\theta) := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$




$$\begin{bmatrix} x_o \\ y_o \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}}_{{}^{o}R_{w}(\theta) = {}^{w}R_{o}(-\theta) = {}^{w}R_{o}^{T}(\theta)} \begin{bmatrix} x_w \\ y_w \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & +\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_w \\ y_w \end{bmatrix}$$

$$\begin{bmatrix} {}^{w}R_{o}^{T} \end{bmatrix} {}^{w}R_{o} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -c(\theta)s(\theta) + s(\theta)c(\theta) \\ -s(\theta)c(\theta) + c(\theta)s(\theta) & s^2(\theta) + c^2(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_{2\times2}$$

$$\underbrace{R^T R = I}$$

$$(A^{-1})A = I$$

$$R^{-1} = R^T$$

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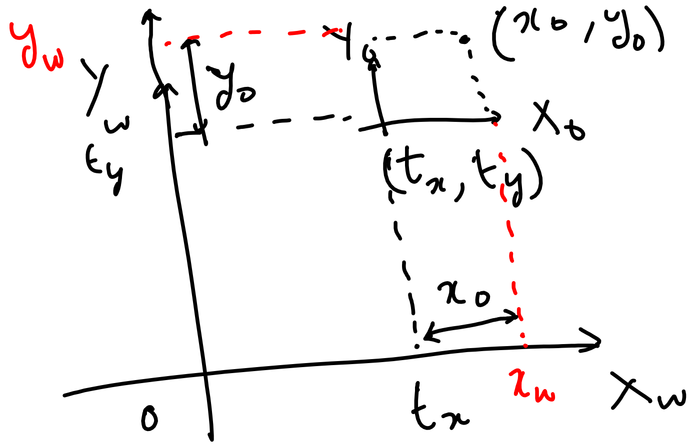
## 2D Translation

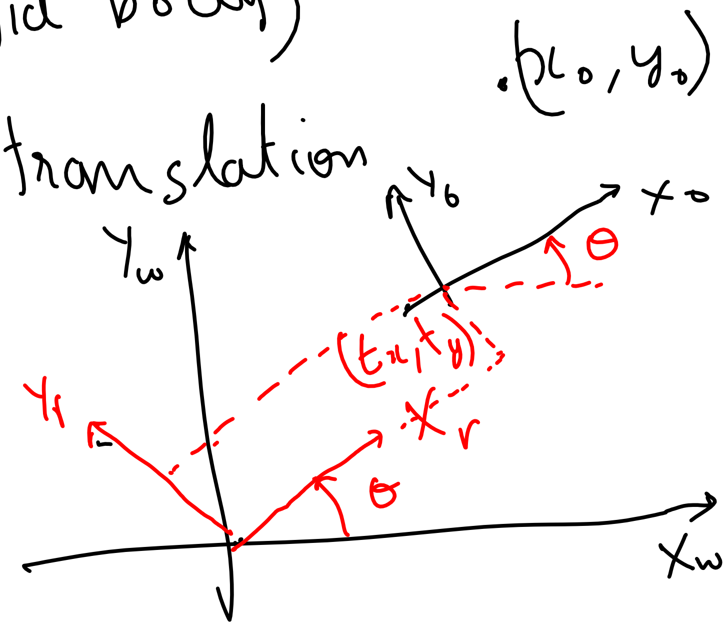
$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \underbrace{\begin{bmatrix} t_x \\ t_y \end{bmatrix}}_{\text{translation vector}} + \begin{bmatrix} x_o \\ y_o \end{bmatrix}$$




## 2D transformation (Rigid body)

Rotation followed by translation

$$\begin{bmatrix} x_r \\ y_r \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_o \\ y_o \end{bmatrix}$$




$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \begin{bmatrix} t_x \\ t_y \end{bmatrix} + \begin{bmatrix} x_r \\ y_r \end{bmatrix}$$

$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}\begin{bmatrix} x_o \\ y_o \end{bmatrix} + \underbrace{\begin{bmatrix} t_x \\ t_y \end{bmatrix}}_{\underline{t}}$$

$$\boxed{\underline{x}_w = {}^{w}_{o}R(\theta)\, \underline{x}_o + {}^{w}_{o}\underline{t}}$$

$$\underline{x}_o = ?(\underline{x}_w)$$

$${}^{w}_{o}R = {}^{w}_{o}R(\theta)$$

$$\underline{x}_w - {}^{w}_{o}\underline{t} = {}^{w}_{o}R\,\underline{x}_o$$

Multiply on the left of both sides by ${}^{w}_{o}R^T$

$$\frac{\underline{x}_w - {}^{w}_{o}\underline{t}}{{}^{w}_{o}R} = \underline{x}_o$$

NEVER EVER DO THIS

$${}^{w}_{o}R^T(\underline{x}_w - {}^{w}_{o}\underline{t}) = ({}^{w}_{o}R^T\,{}^{w}_{o}R)\,\underline{x}_o$$

$$\underline{x}_o = {}^{w}_{o}R^T\underline{x}_w - {}^{w}_{o}R^T\,{}^{w}_{o}\underline{t} \quad \text{--- (1)}$$

$$\underline{x}_o = {}^{o}_{w}R\cdot\underline{x}_w + {}^{o}_{w}\underline{t} \quad \text{--- (2)}$$

Compare (1) and (2)

$$\boxed{{}^{o}_{w}R = {}^{w}_{o}R^T} \quad \text{and} \quad \boxed{{}^{o}_{w}\underline{t} = -{}^{w}_{o}R^T\,{}^{w}_{o}\underline{t}}$$

$$\underline{x}_w = {}^{w}_{o}R\,\underline{x}_o + {}^{w}_{o}\underline{t}$$

$$\begin{bmatrix}\underline{x}_w\\1\end{bmatrix} = \begin{bmatrix}x_w\\y_w\\1\end{bmatrix} = \underbrace{\begin{bmatrix}{}^{w}_{o}R_{2\times2} & {}^{w}_{o}\underline{t}_{2\times1}\\ 0^T_{1\times2} & 1\end{bmatrix}}_{{}^{w}_{o}T_{3\times3}} \begin{bmatrix}\underline{x}_{o\,2\times1}\\1\end{bmatrix}$$

Block matrix

Block Matrix multiplication

$$\begin{bmatrix}A & B\\C & D\end{bmatrix}\begin{bmatrix}E & F\\G & H\end{bmatrix} = \begin{bmatrix}AE+BG & AF+BH\\CE+DG & CF+DH\end{bmatrix}$$

$$\begin{bmatrix}{}^{w}_{o}R_{2\times2} & {}^{w}_{o}\underline{t}_{2\times1}\\ 0^T_{1\times2} & 1\end{bmatrix}\begin{bmatrix}\underline{x}_{o\,2\times1}\\1\end{bmatrix} = \begin{bmatrix}{}^{w}_{o}R\,\underline{x}_o + {}^{w}\underline{t}_o\\ \underline{0}^T\underline{x}_o + 1\end{bmatrix}$$

$$= \begin{bmatrix}{}^{w}_{o}R\,\underline{x}_o + {}^{w}\underline{t}_o\\ 1\end{bmatrix} = \begin{bmatrix}\underline{x}_w\\1\end{bmatrix}$$

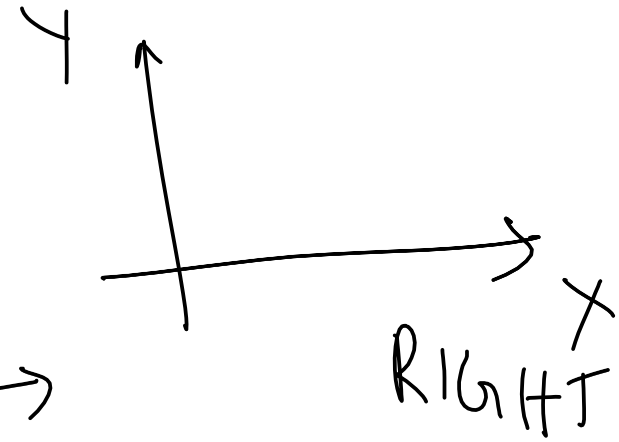
$$\underline{x}_w = {}^{w}_{o}T\,\underline{x}_o$$

Transformation matrix

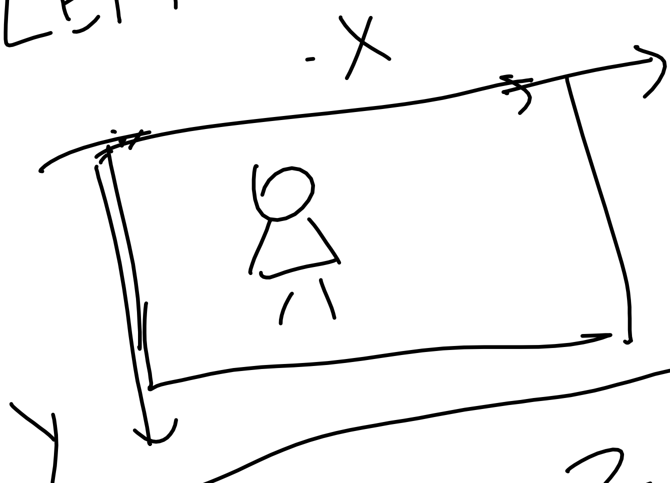
$${}^{w}T_o = \begin{bmatrix}{}^{w}_{o}R_{2\times2} & {}^{w}_{o}\underline{t}_{2\times1}\\ 0^T_{1\times2} & 1\end{bmatrix}$$

In 2D and 3D → $\boxed{\text{Right hand}}$ ✓

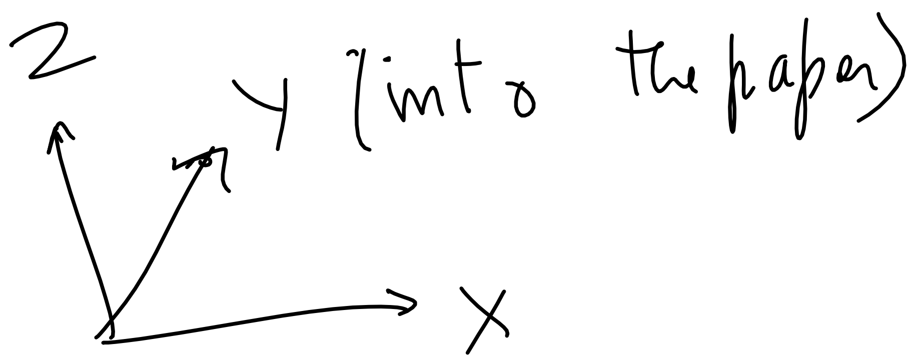
→ Left hand ✗

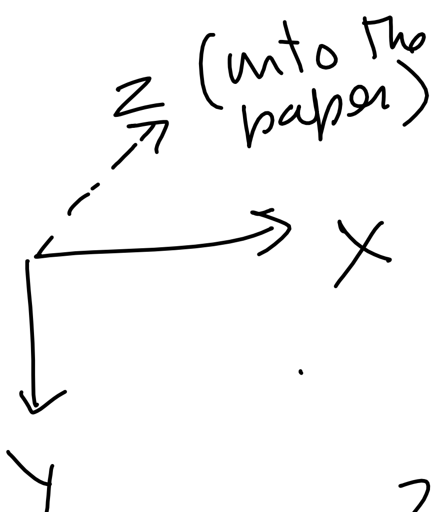



LEFT

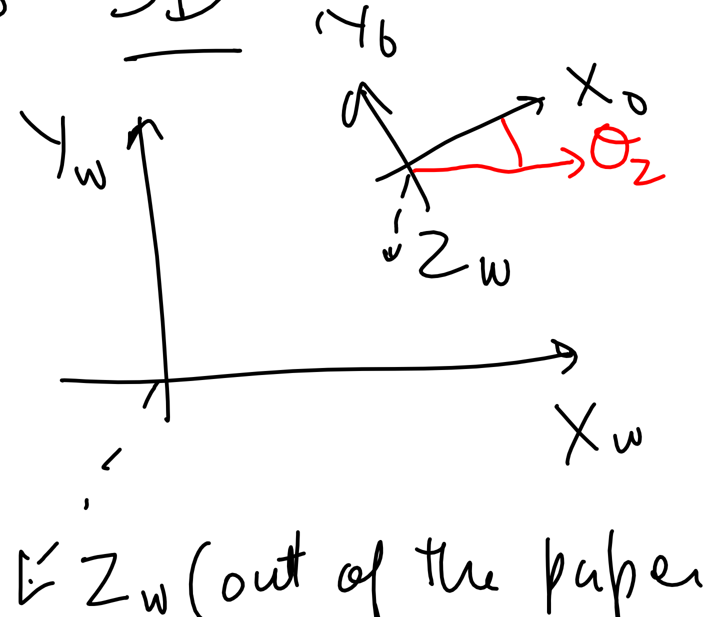



In 3D




# Extending 2D to 3D

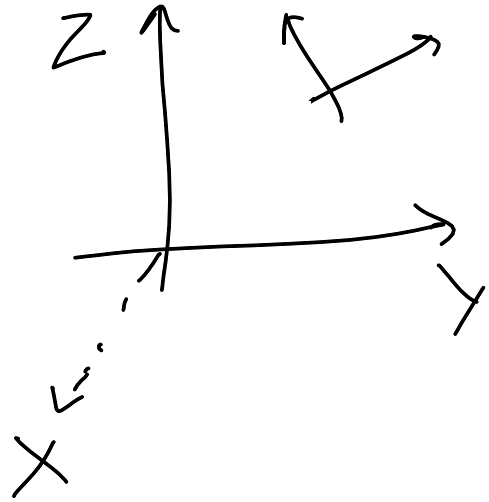



$$\begin{bmatrix} x_w \\ y_w \\ z_w=0 \end{bmatrix} = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ 0 \end{bmatrix}$$

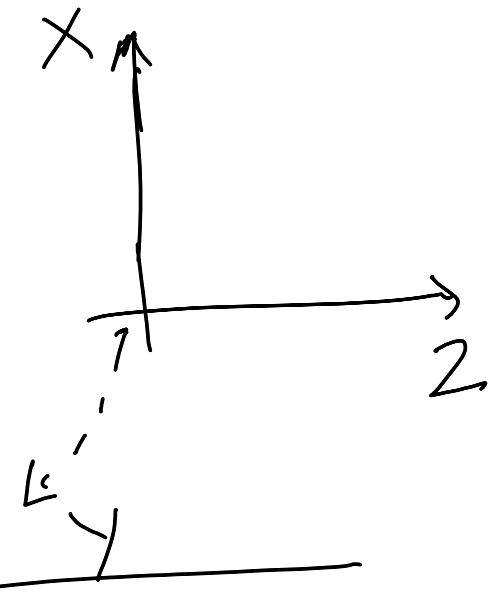
Rotation along Z-axis changes only X-Y coordinates

$$R(\theta_z) = \begin{bmatrix} \cos\theta_z & -\sin\theta_z & 0 \\ \sin\theta_z & \cos\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R(\theta_x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \underline{-\sin\theta} \\ 0 & \underline{\sin\theta} & \cos\theta \end{bmatrix}$$




$$R(\theta_y) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix}$$



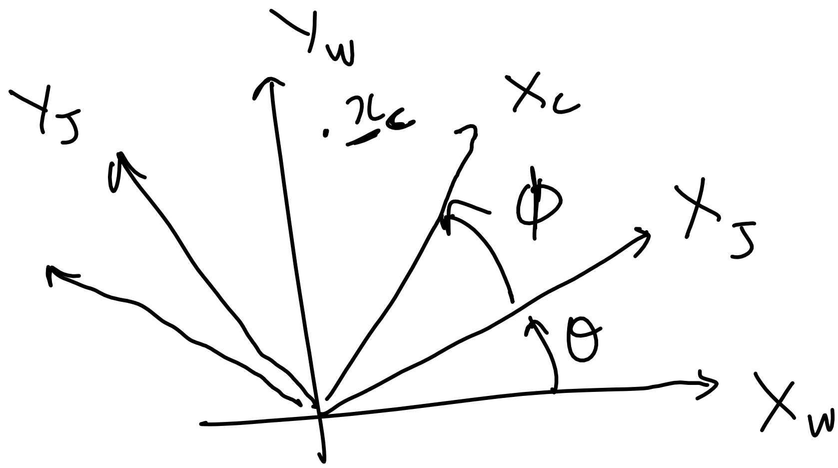

3D Rotation representations

Aeronautics


$\theta_x$ = roll
$\theta_y$ = pitch
$\theta_z$ = yaw

$$\boxed{R = \underset{\text{Yaw}}{R(\theta_z)}\,\underset{\text{Pitch}}{R(\theta_y)}\,\underset{\text{Roll}}{R(\theta_x)}}$$

Chain rotation, translation, transformations




$$\underline{x}_J = {}^J_c R(\phi)\, \underline{x}_c$$

$$\underline{x}_w = {}^w_J R(\theta)\, \underline{x}_J$$

$$= R(\theta)\left[R(\phi)\,\underline{x}_c\right]$$

$$= \underbrace{\left[R(\theta)\; R(\phi)\right]}_{{}^w_c R} \underline{x}_c$$

$$\boxed{{}^w_c R = {}^w_{\cancel{J}} R(\theta)\; {}^{\cancel{J}}_c R(\phi)}$$

$$\boxed{{}^w_c T = {}^w_J T \;\; {}^J_c T}$$