ECE 417/598: Homework 1

Max marks: 100

Due on Jan 28, 2021, 11:59 PM

You are allowed to use any matrix or linear algebra library (Eigen or xtensor), but no library that implements rotation matrices. You are not allowed to use Eigen/Geometry. You can use the following code for generating random rotation matrices: random_rotation.cpp.

1 Jan 24 Lecture: 3D transformations

Problem 1 *Degrees of Freedom of a quantity is the number independent scalar variables needed to represent that quantity. What is degrees of freedom required to*

1. *Position and orientation in 1-D*
2. *Position and orientation in 2-D*
3. *Position and orientation in 3-D*
4. *Position and orientation in 4-D*

(10 marks. Estimated time: 15 min) Justify your answer.

Solution

1. 1D: position requires 1-DoF, orientation requires a binary flag.
2. 2D: position requires 2-DoF, orientation requires a single angle: 1-DoF.
3. 3D: position requires 3-DoF, orientation requires 3-DoF because there are three pairs of axis X-Y, Y-Z, Z-X.
4. 4D: position requires 4-DoF, orientation requires 6-DoF because there are 6 pairs of axis: ${}^4C_2 = 6$.

Problem 2 *Write a program in C++ that checks if a given 3x3 matrix is a valid Rotation matrix is a valid Rotation matrix (check for orthonormality i.e. orthogonality and determinant = 1). You may use Eigen's matrix multiplication and determinant() function. (10 marks. Used in the following problems. Estimated time: 15 min).*

Solution Please look at the function `is_valid_rot` in file `hw1.cpp`.

Problem 3 *In class, we proved the expression to convert roll (θ), pitch (ϕ), yaw (ψ) from Euler Angles to Rotation matrix,*

$$R(\theta, \phi, \psi) = R_z(\psi)R_y(\phi)R_x(\theta). \tag{1}$$

What if we want to do the inverse? Prove that given a proper 3x3 rotation matrix $R = [[r_{ij}]_{i=1}^3]_{j=1}^3$ ($R^\top R = I$ and $det(R) = 1$) , the Euler angles are given by

$$\begin{bmatrix} \theta(R) \\ \phi(R) \\ \psi(R) \end{bmatrix} = \begin{bmatrix} \text{arctan2}(r_{32}, r_{33}) \\ -\arcsin(r_{31}) \\ \text{arctan2}(r_{21}, r_{11}) \end{bmatrix} \tag{2}$$

where r_{ij} is the element in ith row and jth column of the rotation matrix R. (10 marks. Used in the following problems. Estimated time: 15 min).

Solution

$$
\begin{aligned}
R(\theta,\phi,\psi) &= \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\phi & 0 & s_\phi \\ 0 & 1 & 0 \\ -s_\phi & 0 & c_\phi \end{bmatrix} \\
&\quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & c_\theta & -s_\theta \\ 0 & s_\theta & c_\theta \end{bmatrix} \\
&= \begin{bmatrix} c_\psi & -s_\psi & 0 \\ s_\psi & c_\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c_\phi & s_\phi s_\theta & s_\phi c_\theta \\ 0 & c_\theta & -s_\theta \\ -s_\phi & c_\phi s_\theta & c_\phi c_\theta \end{bmatrix} \\
&= \begin{bmatrix} c_\psi c_\phi & c_\psi s_\phi s_\theta - s_\psi c_\theta & c_\psi s_\phi c_\theta + s_\psi s_\theta \\ s_\psi c_\phi & s_\psi s_\phi s_\theta + c_\psi c_\theta & s_\psi s_\phi c_\theta - c_\psi s_\theta \\ -s_\phi & c_\phi s_\theta & c_\phi c_\theta \end{bmatrix}
\end{aligned} \tag{3}
$$

Problem 4 *Write a pair of functions in C++ that converts rotation matrix from XYZ Euler angles (roll, pitch, yaw) and vice versa. Test the pair of functions with randomly generated Euler angles. And check if the converted rotation matrix is orthonormal. What happens when pitch = $\pi/2$, are you able to convert from rotation matrix to Euler angle? Why or why not? (50 marks. Estimated time: 30 min)*

Problem 5 *Write a function in C++ that generates a 4x4 transformation matrix given XYZ Euler angles (roll, pitch, yaw) and translation. You can use the function that you wrote in Prob 4(20 marks. Estimated time: 15 min).*