2D coordinate transforms
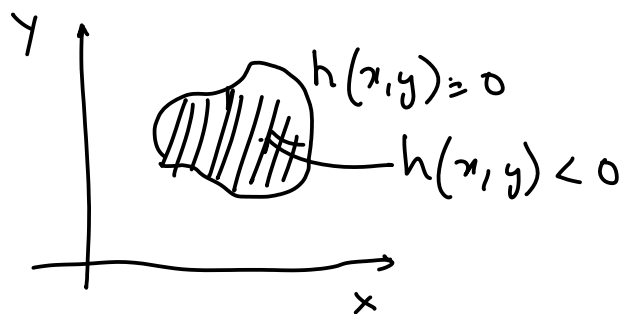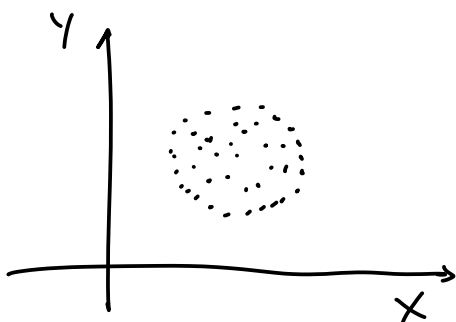
Rotations + translations
Orientation + position

$h(x,y) > 0$
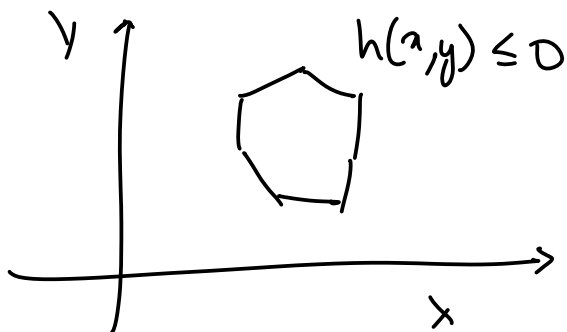
$h(x,y) = 0$

$h(x,y) < 0$

$$\text{Object} = \{(x,y) \in \mathbb{R}^2 : h(x,y) \le 0\}$$

Set of all points $(x,y) \in \mathbb{R}^2$

← set of all real numbers

such that

$$h(x,y) \le 0$$

$h(x,y) \le 0$

$h'(x,y) \le 0$

$h(x,y) \le 0$
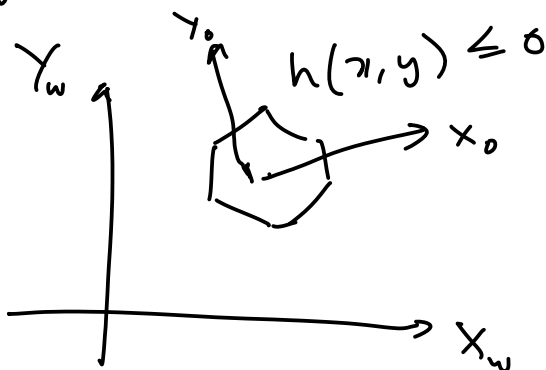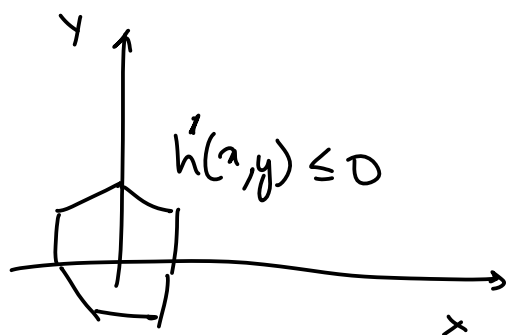
2D Rotations

① Rotation
② Translation

<u>Problem</u> (x_0, y_0) is given
in (X_0, Y_0) coordinate
frame. $(X_0 Y_0)$ has been rotated by angle Θ w.r.t. (X_w, Y_w)

$x_0 = x_w \cos\Theta + y_w \sin\Theta$

$y_w \sin\Theta$ x_0 $x_w \cos\Theta$

(x_0, y_0)
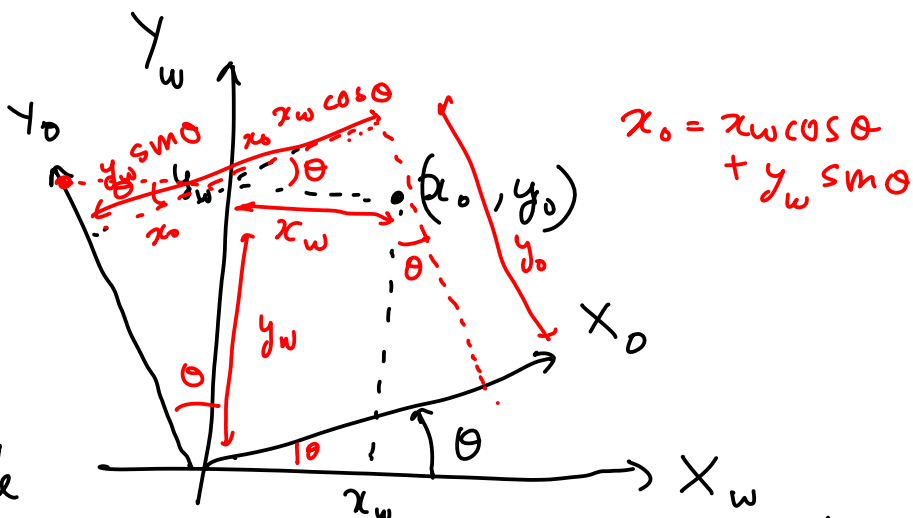
Find (x_w, y_w) in world coordinate frame

Proof using **Basis vectors**

In Linear algebra, **Basis vectors** are set of orthonormal unit vectors that span the entire space

Span is the set of all vectors that can be obtained by linear combinations of a given set of vectors

$$\text{Span} \left\{ \underbrace{\underline{a}, \underline{b}}_{\in \mathbb{R}^n} \right\} = \left\{ \underbrace{\alpha \underline{a} + \beta \underline{b}}_{\in \mathbb{R}^n}, \begin{array}{l} \alpha \in \mathbb{R} \\ \beta \in \mathbb{R} \end{array} \right\}$$

Standard Basis vector.

For example, in $\mathbb{R}^2$ $\quad \hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\hat{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

in $\mathbb{R}^3$ $\quad \hat{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\hat{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

in $\mathbb{R}^n$ $\quad \hat{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n \times 1} \cdots \hat{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}_{n \times}$

Basis vectors for $\mathbb{R}^n$

↳① All vectors must be perpendicular/orthogonal to each other

↳② They must be unit vectors

↳③ They must span the entire space $\mathbb{R}^n$

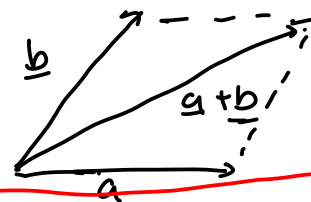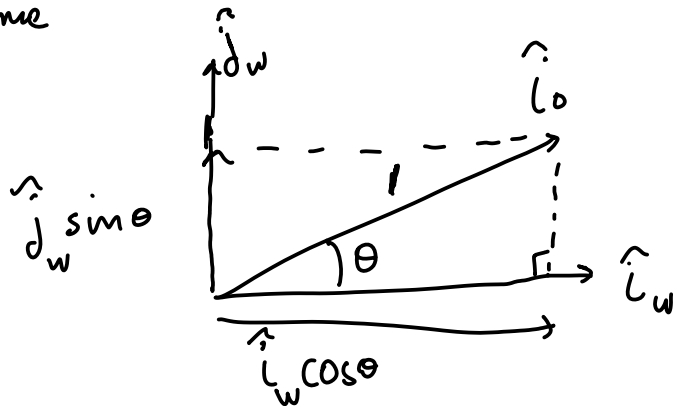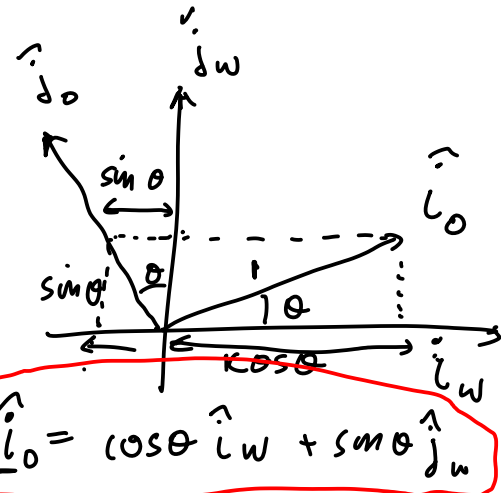Let Basis vector for (X_w, Y_w) be standard basis vector

$$\hat{i}_w = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad , \hat{j}_w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Any point in world coordinate frame

$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = x_w \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\hat{i}_w} + y_w \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\hat{j}_w}$$

Any point in the object coordinate frame

$$\begin{bmatrix} x_o \\ y_o \end{bmatrix} = x_o \hat{i}_o + y_o \hat{j}_o$$

$$\boxed{\hat{i}_o = \cos\theta \, \hat{i}_w + \sin\theta \, \hat{j}_w}$$

$$\boxed{\hat{j}_o = -\hat{i}_w \sin\theta + \hat{j}_w \cos\theta}$$

world	object

$$x_w \hat{i}_w + y_w \hat{j}_w = x_o \hat{i}_o + y_o \hat{j}_o$$

$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = x_o \left[\cos\theta \, \hat{i}_w + \sin\theta \, \hat{j}_w \right] + y_o \left\{ -\hat{i}_w \sin\theta + \hat{j}_w \cos\theta \right\}$$

Because we are using standard basis for world coordinate frame

$$= \left[x_o \cos\theta - y_o \sin\theta \right] \hat{i}_w + \left[x_o \sin\theta + y_o \cos\theta \right] \hat{j}_w$$

$$\downarrow \qquad\qquad \downarrow$$

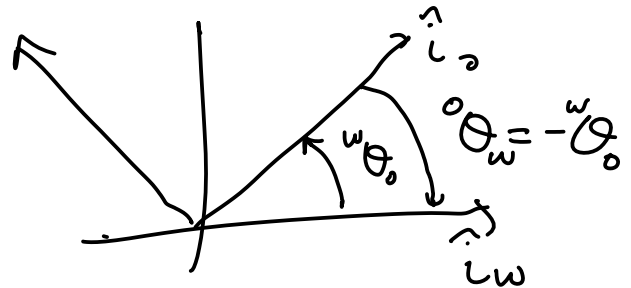$$= \qquad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad\qquad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} x_o \cos\theta - y_o \sin\theta \\ x_o \sin\theta + y_o \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_o \\ y_o \end{bmatrix}$$

$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \begin{bmatrix} {}^wR_0(\theta) \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$${}^wR_0(\theta) := \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}}_{{}^0R_w(\dot{\theta}) = {}^wR_0(-\theta)} \begin{bmatrix} x_w \\ y_w \end{bmatrix} = {}^wR_0(\theta)$$

$${}^0\theta_w = -{}^w\theta_0$$

$$= \begin{bmatrix} \cos\theta & +\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x_w \\ y_w \end{bmatrix}$$

$$\begin{bmatrix} {}^wR_0^T \end{bmatrix} {}^wR_0 = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\theta + \sin^2\theta & -c(\theta)s(\theta) + s(\theta)c(\theta) \\ -s(\theta)c(\theta) + c(\theta)s(\theta) & s^2(\theta) + c^2(\theta) \end{bmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_{2 \times 2}$$

$$\underbrace{R^T R = I}_{(A^{-1})A = I} \quad \longrightarrow \quad R^{-1} = R^T$$