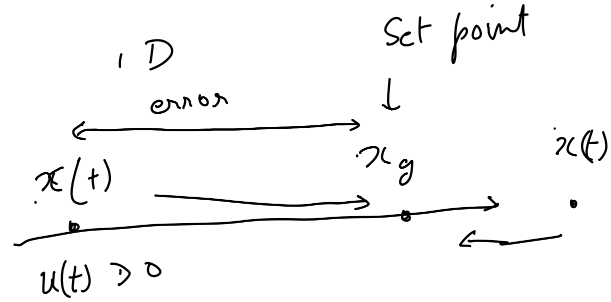
Controllers

① PID : Proportional Integral Derivative controller

② LQR : Linear quadratic regulator

PID



$$e(t) = x_g - x(t)$$

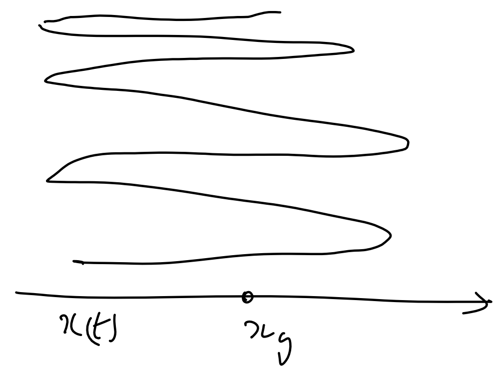
Proportional

$$u(t) = k_p e(t)$$

Proportional gain factor

How to find k_p?

PID tuning → Manual tuning



Integral term

PI

80%

$$\boxed{u(t) = k_p e(t) + k_I \int_0^\tau e(t)\,dt}$$

Integral gain

Derivative term

$$\begin{bmatrix} x \\ y \\ \theta \end{bmatrix} \qquad \begin{bmatrix} v \\ \omega \end{bmatrix}$$

$$\boxed{u(t) = \underbrace{k_p e(t)}_{\text{determines the direction + magnitude}} + \underbrace{k_I \int_0^\tau e(t)\,dt}_{\text{remove oscillations}} + \underbrace{k_D \frac{d}{dt} e(t)}_{\text{Remove shaky behaviour}}}$$

In Jetbot

$$\underline{e}(t) = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix} - \begin{bmatrix} x_g \\ y_g \\ \theta_g \end{bmatrix}$$

$$\underline{u}(t) = \begin{bmatrix} v \\ \omega \end{bmatrix}$$

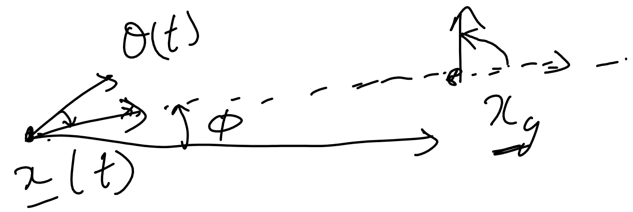
v — linear vel

ω — angular vel

$$\underline{u}(t) = \underbrace{\begin{bmatrix} & & \\ & & \end{bmatrix}_{2\times 3}}_{K_p} \begin{bmatrix} \underline{e}(t) \end{bmatrix} + \underbrace{\begin{bmatrix} & & \\ & & \end{bmatrix}_{2\times 3}}_{K_I} \int_0^\tau \underline{e}(\tau)\,dt$$

$$\underline{e_1(t)} = \theta(t) - \underbrace{\tan^{-1}\left(y_g - y(t) / x_g - x(t) \right)}_{\phi}$$

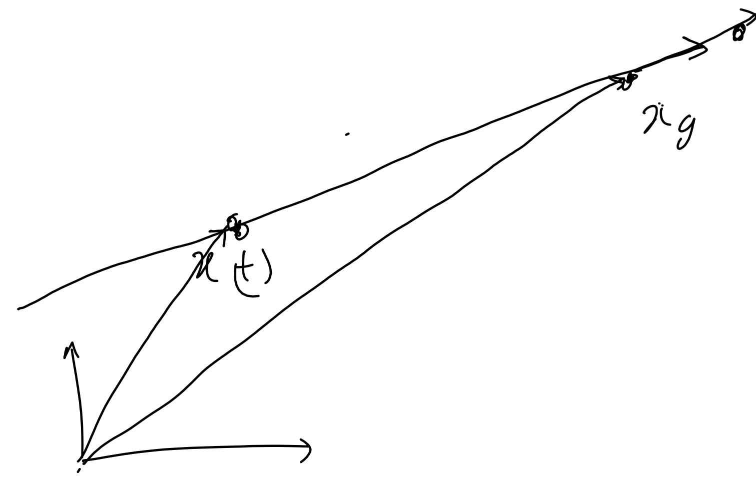
$$\omega(t) = k_p e_1(t) + k_I \int_0^\tau e_1(\tau)\, dt$$



$$e_2(t) = ?$$

initial time

$$\underset{\text{direction}}{\hat{\underline{d}}} = \frac{\underline{x}_g - \underline{x}(0)}{\|\underline{x}_g - \underline{x}(0)\|}$$



$$\underline{e}(t) = \underline{x}_g - \underline{x}(t)$$

$$e_2(t) = \underbrace{\left(\underline{x}_g - \underline{x}(t)\right)^T \hat{\underline{d}}}_{\text{dot product}}$$

= projection of $\left(\underline{x}_g - \underline{x}(t)\right)$ on $\hat{\underline{d}}$

$$v(t) = k_p e_2(t) + k_I \int_0^\tau e_2(\tau)\, d\tau + \dots$$

Optimal control

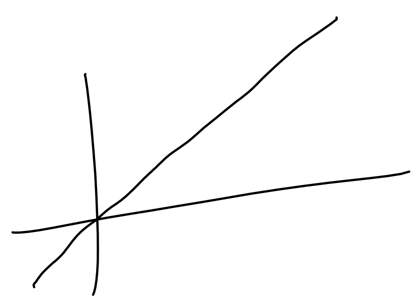
$$u^*(t) = \underset{u(t)}{\text{minimize}} \text{ Cost function}$$

Assume a cost function $J_t(\underset{\text{current state}}{\underline{x}_t}, \underset{\text{control signal}}{\underline{u}_t})$

$$\boxed{u_0^*, u_1^*, \dots, u_T^* = \underset{u_0, u_1, \dots u_T}{\arg\min} \sum_{t=0}^{T} J_t(x_t, u_t) \quad \text{s.t.} \quad \underbrace{\underline{x}_{t+1} = f(\underline{x}_t, \underline{u}_t)}_{\text{system dynamics}}}$$

optimal control problem

LQR is the solution to the optimal control problem when the cost function J_t is QUADRATIC and the system dynamics f is LINEAR



What is a linear function?

Technical definition

A function f is linear if

$$\boxed{f(\underset{\mathbb{R}}{\alpha}\, \underline{x} + \beta\, \underline{y}) = \alpha f(\underline{x}) + \beta f(\underline{y})} \quad \underline{x} \in \mathbb{R}^n,\ y \in \mathbb{R}^n$$

Elementary $\boxed{+, \cdot, x^2, x^3, xy^2, yx^2}$, $\boxed{\exp(x)}$

$\boxed{\log(x), \cos(x), \sin(x)}$

Polynomial $= 3 + x + x^2 + 5x^2y + 6x^3y$

Linear functions are polynomials of degree 1 without the constant part

$$f(x) = 4x$$

Quadratic functions are polynomials of degree 2

In vector form

Linear functions are of the form:

$$f(\underline{x}) = A\underline{x}$$

scalar valued Quadratic functions are of the form:

$$f(\underline{x}) = \underline{\underline{x}^T Q \underline{x}} + \underline{P\underline{x}} + \underline{r}$$

· System dynamics is linear

$$x_{t+1} = f(x_t, u_t) = A x_t + B u_t$$

$$\cdot x_{t+1} = A x_t + B u_t \qquad x_t \in \mathbb{R}^{n \times 1}$$

$$u_t \in \mathbb{R}^{m \times 1}$$

What is the dimensionality of $\underset{n \times n}{A}$ and $\underset{n \times m}{B}$?
size

Cost function is quadratic

$$J_t(x_t, u_t) = x_t^T Q_t x_t + u_t^T R u_t$$

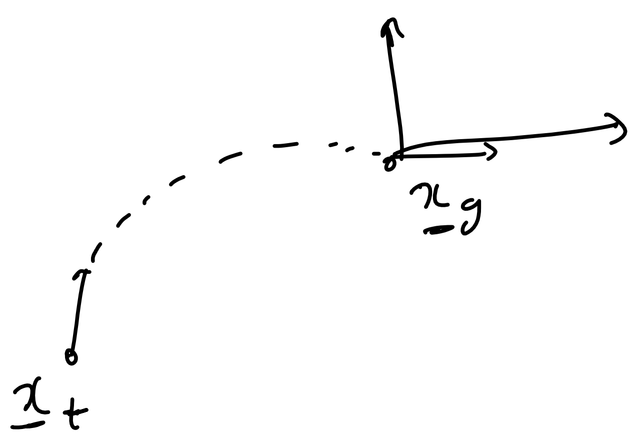
LQR

$$\boxed{u_{0\ldots t} = \underset{u_{0\ldots t}}{\arg\min}\ \underbrace{x_t^T Q_t x_t} + u_t^T R u_t \qquad \text{s.t.} \quad x_{t+1} = A x_t + B u_t}$$

Example

$$x_t = \begin{bmatrix} x \\ y \\ \theta \end{bmatrix}$$

$$x_g = \begin{bmatrix} x_g \\ y_g \\ \theta_g \end{bmatrix}$$



$$x'_t = x_t - x_g$$

$$\underbrace{\|\underline{x}_t - \underline{x}_g\|_2^2}_{\text{a quadratic cost}}$$

$$= (\underline{x}_t - \underline{x}_g)^T (\underline{x}_t - \underline{x}_g)$$

$$= (\underline{x}_t - \underline{x}_g)^T \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} (\underline{x}_t - \underline{x}_g)$$

$$= \underbrace{\underline{x}_t^T I_{3\times 3}\, \underline{x}_t}_{\text{Quadratic}} \underbrace{-2\, \underline{x}_g^T I_{3\times 3}\, \underline{x}_t}_{\text{Linear term}} + \underline{x}_g^T \underline{x}_g$$

$\underline{x}_g \rightarrow 0$

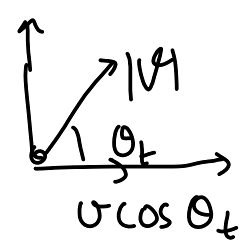
$\underline{x}'_t = \underline{x}_t - x_g$

$\underline{x}_t'^T \underline{x}'_t$

$= \underline{x}_t'^T \underset{\underset{Q}{\uparrow}}{\underline{I}_3} \underline{x}_t$

System dynamics

Unicycle model



$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \\ \theta_{t+1} \end{bmatrix} = \begin{bmatrix} x_t + \underline{v_t} \cos\theta_t \, dt \\ y_t + \underline{v_t} \sin\theta_t \, dt \\ \theta_t + \underline{\omega_t} \, dt \end{bmatrix}$$

$$\underline{x}_{t+1} = A\underline{x}_t + B\underline{u}_t$$

$$\begin{bmatrix} x_{t+1} \\ y_{t+1} \\ \theta_{t+1} \end{bmatrix} = \underbrace{\begin{bmatrix} x_t \\ y_t \\ \theta_t \end{bmatrix}}_{\underline{x}_t} + \underbrace{\begin{bmatrix} \cos\theta_t dt & 0 \\ \sin\theta_t dt & 0 \\ 0 & dt \end{bmatrix}_{3\times 2}}_{B} \underbrace{\begin{bmatrix} v_t \\ \omega_t \end{bmatrix}}_{\underline{u}_t}$$

$$A = I_{3\times 3}$$

$v_t^2 \propto KE$

$v_t = 0$
$\omega_t = 0$

$$\text{Cost function} = 0.99 \underbrace{(\underline{x}_t - \underline{x}_g)^T (\underline{x}_t - \underline{x}_g)}_{\text{we want to be close to goal}} + 0.01 \underbrace{\underline{u}_t^T \underline{u}_t}_{\text{lower energy}}$$

$$(x_t - x_g)^T \underbrace{\begin{bmatrix} 0.99 & 0 & 0 \\ 0 & 0.99 & 0 \\ 0 & 0 & 0.99 \end{bmatrix}}_{Q} (x_t - x_g) + u_t^T \underbrace{\begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}}_{R} u_t$$

How to solve LQR problem?

<u>Naive solution</u>

x_0

$$x_1 = A x_0 + B u_0$$

$$x_2 = A x_1 + B u_1$$

$$= A(Ax_0 + Bu_0) + Bu_1$$

$$= A^2 x_0 + ABu_0 + Bu_1$$

$$x_3 = Ax_2 + Bu_2$$

$$= A(A^2 x_0 + ABu_0 + Bu_1) + Bu_2$$

$$= A^3 x_0 + A^2 B u_0 + ABu_1 + Bu_2$$

$$\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_N \end{bmatrix}_{nN\times 1}}_{X} = \underbrace{\begin{bmatrix} B & 0 & 0 & \cdots & 0 \\ AB & B & 0 & \cdots & 0 \\ A^2B & AB & B & \cdots & 0 \\ & & & & \\ A^N B & A^{N-1}B & \cdots & \cdots & B \end{bmatrix}}_{G}{}_{nN\times mN} \underbrace{\begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{N-1} \end{bmatrix}_{mN\times 1}}_{U} + \underbrace{\begin{bmatrix} A \\ A^2 \\ A^3 \\ \vdots \\ A^N \end{bmatrix}_{nN\times n}}_{H} x_{0\;n\times 1}$$

$u_t \in \mathbb{R}^m$

$x_t \in \mathbb{R}^n$

known

$$X = G\,\boxed{U} + H x_0$$

unknown

$$\text{cost function} = \sum_{t=0}^{N} \left(\underbrace{x_t^T Q_t x_t} + u_t^T R_t u_t \right)$$

$$= \underbrace{\begin{bmatrix} x_0^T & x_1^T & x_2^T & \cdots & x_N^T \end{bmatrix}}_{X^T} \underbrace{\begin{bmatrix} Q_1 & 0 & 0 & \cdots & 0 \\ 0 & Q_2 & 0 & & 0 \\ 0 & 0 & Q_3 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & & Q_N \end{bmatrix}}_{P = \mathrm{diag}(Q_1, \dots Q_N)} \underbrace{\begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}}_{X}$$

$$= \begin{bmatrix} x_0^T Q_0 & x_1^T Q_1 & x_2^T Q_2 & \cdots & x_N^T Q_N \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}$$

$$\text{cost function} = X^T P X + U^T W U$$

$W = \mathrm{diag}(R_1, \dots R_N)$

$$= (GU + Hx_0)^T P (GU + Hx_0) + U^T W U$$

$$= U^T G^T P G U + 2 x_0^T H^T P G U + x_0^T H^T P H x_0 + U^T W U$$

$$= \underset{mN}{U}^T (G^T P G + W) \underset{mN}{U} + 2 x_0^T H^T P G U + x_0^T H^T P H x_0$$

$$\boxed{U^* = (\underbrace{G^T P G + W}_{mN \times mN})^{-1} G^T P^T H x_0}$$

Naive

Matrix inversion $= O(m^3 N^3)$

$m \approx$ small = control vector size

$N \approx$ number of time steps

Dynamic programming to solve LQR

(mathematical induction)

Assume that you have a solution for timestep $\underline{N}$

find a solution for " $\underline{N-1}$

Optimal control problem

$$\underset{\underline{u}_{1:T}}{\arg\min} \sum_{t=1}^{T} J_t(\underline{x}_t, \underline{u}_t)$$

such that $\quad \underline{x}_{t+1} = f(\underline{x}_t, u_t) \quad \forall t \in \{0, \dots T\}$

LQR

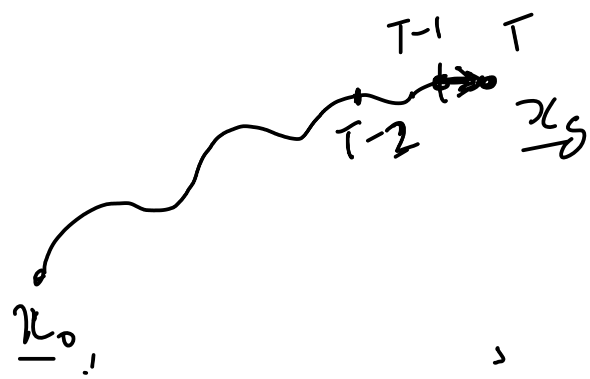
$$\underset{u_{1:T}}{\arg\min} \sum_{t=1}^{T} \underline{x}_t^T Q_t \underline{x}_t + \underline{u}_t^T R_t \underline{u}_t$$

such that $\quad \underline{x}_{t+1} = A \underline{x}_t + B \underline{u}_t \quad \forall t \in \{0, \dots, T\}$

LQR via dynamic programming

Cost to go $= V(\underline{x}_t)$

$$V_t(\underline{x}_t) = \sum_{k=t}^{T} J_t(x_t, u_t)$$

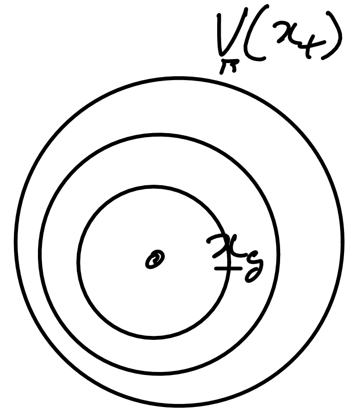


Policy $= \pi(\underline{x}_t) \mapsto \underline{u}_t$

is a function that returns control signal for every state cost to go depends upon policy

$$V_\pi(x_t) = \sum_{k=t}^{T} J_t(\underline{x}_t, \pi(\underline{x}_t))$$

$$J_t(\underline{x}_t, \underline{u}_t) = \|\underline{x}_t - \underline{x}_g\|^2$$



Optimal control as dynamic programming

$$\arg\min_{\underline{u}_{0:T}} \sum_{t=0}^{T} J_t(\underline{x}_t, \underline{u}_t)$$

$$= \underbrace{\arg\min_{\underline{u}_0} J_0(\underline{x}_0, \underline{u}_0)}_{\text{first time step}} + \underbrace{\min_{\underline{u}_{1:T}} \sum_{t=1}^{T} J_1(\underline{x}_t, \underline{u}_t)}_{\text{remaining}}$$

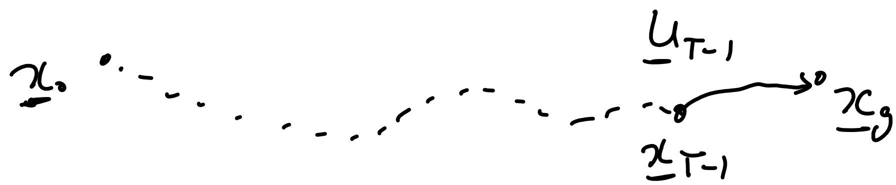
$$\boxed{\begin{aligned} V_*(\underline{x}_0) &= \min_{\underline{u}_0} J_0(x_0, u_0) + \underbrace{V_*(\underline{x}_1)}_{\text{cost to go}} \\ &\text{s.t. } x_1 = f(\underline{x}_0, u_0) \end{aligned}}$$

Dynamic programming for optimal control

This equation is called Bellman equation.
It appears in Q-learning (RL)

DQN

LQR



✓ $$\underset{u_{T-1}}{\arg\min} \quad x_g^T Q_T x_g + u_{T-1}^T R u_{T-1}$$

$$x_g = f(x_{T-1}, u_{T-1})$$

$$V_T(x_g) = x_g^T Q_T x_g$$

$$V_{T-1}(x_{T-1}) = \min_{u_{T-1}} V_T(x_g) + x_{T-1}^T Q_{T-1} x_{T-1} + u_{T-1}^T R u_{T-1}$$

$$x_T = f(x_{T-1}, u_{T-1})$$

$$V_t(x_t) = x_t^T P_t x_t$$

Assumption that $V_t(\,)$ is a quadratic function of x_t

$$V_{T-1}(\underline{x}_{T-1}) = \min_{\underline{u}_{T-1}} \underline{x}_T^T P_T \underline{x}_T + x_{T-1}^T Q_{T-1} x_{T-1} + \underline{u}_{T-1}^T R \underline{u}_{T-1}$$

$$x_T = A x_{T-1} + B u_{T-1}$$

$\underline{P_T = Q_T}$ for the last time step

$$V_{T-1}(x_{T-1}) = \min_{u_{T-1}} (Ax_{T-1} + Bu_{T-1})^T P_T (Ax_{T-1} + Bu_{T-1}) + \underline{u}_{T-1}^T R \underline{u}_{T-1} + \underbrace{\underline{x}_{T-1}^T Q_{T-1} \underline{x}_{T-1}}_{\text{independent of } u_{T-1}}$$

$$= \min_{u_{T-1}} \underline{u}_{T-1}^T B^T P_T B \underline{u}_{T-1} + \underline{u}_{T-1}^T R \underline{u}_{T-1} + 2\underline{x}_{T-1}^T A^T P_T B \underline{u}_{T-1} + \underline{x}_{T-1}^T A^T P_T A \underline{x}_{T-1} + \underline{x}_{T-1}^T Q_{T-1} \underline{x}_{T-1}$$

$$u_{T-1}^* = -\left(B^T P_T B + R\right)^{-1} B^T P_T A x_{T-1}$$

$$V_{T-1}(x_{T-1}) = x_{T-1}^T A^T P_T B \left(B^T P_T B + R\right)^{-T} \cancel{\left(B^T P_T B + R\right)}$$
$$\cancel{\left(B^T P_T B + R\right)^{-1}} \left(B^T P_T A x_{T-1}\right)$$
$$- 2 x_{T-1}^T A^T P_T B \left(B^T P_T B + R\right)^{-1} B^T P_T A x_{T-1}$$
$$+ x_{T-1}^T A^T P_T A x_{T-1} + x_{T-1}^T Q_{T-1} x_{T-1}$$

$$V_{T-1}(x_{T-1}) = x_{T-1}^T \Big(Q_{T-1} + A^T P_T A$$
$$- A^T P_T B \left(B^T P_T B + R\right)^{-1} B^T P_T A \Big) x_{T-1}$$

$$V_{T-1}(x_{T-1}) = x_{T-1}^T P_{T-1} x_{T-1}$$

$$\boxed{P_{T-1} = Q_{T-1} + A^T P_T A - A^T P_T B \left(B^T P_T B + R\right)^{-1} B^T P_T A}$$

$$\boxed{P_T = Q_T}$$

$$\boxed{u_{T-1}^* = -\left(B^T P_T B + R\right)^{-1} B^T P_T A x_{T-1}}$$