Problem set for Midterm 2

These are homework 1-6 from Prof Rick Eason's course:
https://web.eece.maine.edu/eason/ece417/

For the rotation matrix:

$$ {}^{XYZ}\mathbf{R}_{UVW} = \begin{bmatrix} 2/7 & -6/7 & 3/7 \\ 6/7 & 3/7 & 2/7 \\ -3/7 & 2/7 & 6/7 \end{bmatrix} $$

1. Show that ${}^{XYZ}\mathbf{R}_{UVW}$ is a proper rotation matrix. $\det(R) = 1 \quad R^T R = I$
2. Show that $\mathbf{R}^{-1}$ is equal to $\mathbf{R}^T$ where ($\mathbf{R}$ is shorthand for ${}^{XYZ}\mathbf{R}_{UVW}$). HINT: taking the inverse is not required.
3. Compute $\mathbf{R}\,\mathbf{A}$ where matrix $\mathbf{A}$ is given by

$$ \mathbf{A} = \begin{bmatrix} 3/7 & 2/7 & 6/7 \\ 2/7 & 6/7 & -3/7 \\ -6/7 & 3/7 & 2/7 \end{bmatrix} $$

4. If $\mathbf{P}_{UVW} = (1,2,3)^T$, what is $\mathbf{P}_{XYZ}$ given (using ${}^{XYZ}R_{UVW}$ above)
5. If $\mathbf{P}_{XYZ} = (1,2,3)^T$, what is $\mathbf{P}_{UVW}$? given (using ${}^{XYZ}R_{UVW}$ above)
6. If the OUVW system has standard basis vectors $\mathbf{U} = (1/\sqrt{2}, 0, 1/\sqrt{2})^T$, $\mathbf{V} = (-1/\sqrt{2}, 0, 1/\sqrt{2})^T$, $\mathbf{W} = (0, -1, 0)^T$, and the OXYZ system has basis vectors $\mathbf{X} = (1, 0, 0)^T$, $\mathbf{Y} = (0, 1/\sqrt{2}, -1/\sqrt{2})^T$, $\mathbf{Z} = (0, 1/\sqrt{2}, 1/\sqrt{2})^T$, then what is the corresponding rotation matrix between the two systems?

$$ \underline{P}_{xyz} = {}^{xyz}R_{uvw}\, \underline{P}_{uvw} $$

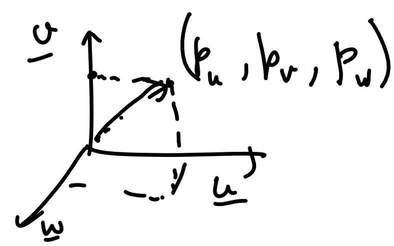
$$ \underline{P}_{uvw} = \left({}^{xyz}R_{uvw}\right)^T \underline{P}_{xyz} $$

unit vectors that are orthonormal + span the entire space

Basis

$$ \hat{\underline{u}} = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}, \quad \hat{\underline{v}} = \begin{bmatrix} \\ \\ \end{bmatrix}, \quad \hat{\underline{w}} = \begin{bmatrix} \\ \\ \end{bmatrix} $$

$$ \hat{\underline{x}} = \begin{bmatrix} \\ \\ \end{bmatrix} \quad \hat{\underline{y}} = \begin{bmatrix} \\ \\ \end{bmatrix} \quad \hat{\underline{z}} = \begin{bmatrix} \\ \\ \end{bmatrix} $$




$$ \underline{p} = p_u \hat{\underline{u}} + p_v \hat{\underline{v}} + p_w \hat{\underline{w}} $$

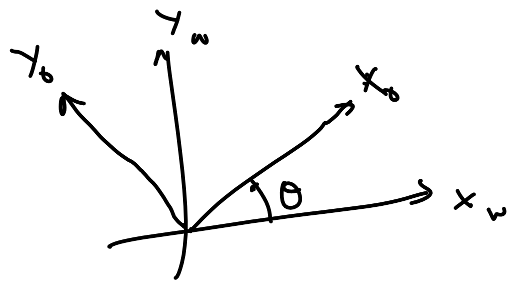
$$ \underline{p} = p_x \hat{\underline{x}} + p_y \hat{\underline{y}} + p_z \hat{\underline{z}} $$

$$\underline{p} = \underbrace{\begin{bmatrix} \underline{\hat{u}}_{3\times1} & \underline{\hat{v}}_{3\times1} & \underline{\hat{w}}_{3\times1} \end{bmatrix}_{3\times3}}_{\text{Basis matrix}} \underbrace{\begin{bmatrix} p_u \\ p_v \\ p_w \end{bmatrix}}_{\text{coordinates}} = \underbrace{\begin{bmatrix} \underline{\hat{x}} & \underline{\hat{y}} & \underline{\hat{z}} \end{bmatrix}_{3\times3}}_{\text{Basis matrix}} \underbrace{\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}}_{\text{coordinates}}$$

$$B_{uvw}\, \underline{p}_{uvw} = B_{xyz}\, \underline{p}_{xyz}$$

$$\underline{p}_{uvw} = \underbrace{B_{uvw}^{-1} B_{xyz}}_{{}^{uvw}}\, \underline{p}_{xyz} \qquad B_{uvw}^{-1} = B_{uvw}^{T}$$

$$\underline{p}_{uvw} = \underline{{}^{uvw}_{xyz}R}\; \underline{p}_{xyz}$$




$$\underline{X}_o = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \qquad \underline{X}_w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\underline{Y}_o = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} \qquad \underline{Y}_w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Given the following 4x4 homogeneous transformation matrices:

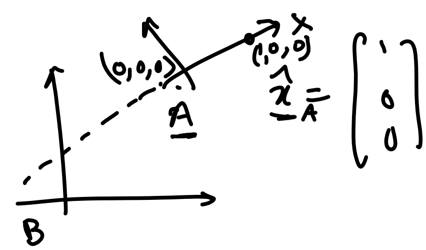
$$\boxed{{}^{B}T_{A}} = \begin{bmatrix} \overbrace{\begin{matrix} 2/7 & -6/7 & 3/7 \\ 6/7 & 3/7 & 2/7 \\ -3/7 & 2/7 & 6/7 \end{matrix}}^{R} & \overbrace{\begin{matrix} 1 \\ 2 \\ 3 \end{matrix}}^{t} \\ \begin{matrix} 0 & 0 & 0 \end{matrix} & 1 \end{bmatrix} \qquad \boxed{{}^{C}T_{B}} = \begin{bmatrix} 3/7 & 2/7 & 6/7 & 4 \\ 2/7 & 6/7 & -3/7 & 5 \\ -6/7 & 3/7 & 2/7 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$${}^{B}T_{A}^{-1} = \begin{bmatrix} R^T_{3\times3} & -R^T t \\ 0 & 1 \end{bmatrix}$$

$$\begin{pmatrix} y \\ 1 \end{pmatrix}_{4\times1} = \begin{pmatrix} R_{3\times3} & t \\ 0^T & 1 \end{pmatrix}_{4\times4} \begin{pmatrix} x \\ 1 \end{pmatrix}_{4\times1} \Rightarrow y = Rx + t \Rightarrow x = R^T y - R^T t$$

1. Give the inverse of Matrix ${}^{B}T_{A}$.
2. What is the direction of the X-axis of system **A** w.r.t. system **B**? What is the direction of the Y-axis of system **A** w.r.t system **B**? Where is the origin of system **A** w.r.t. system **B**?
3. What is the direction of the X-axis of system **B** w.r.t. system **A**? What is the direction of the Y-axis of system **B** w.r.t system **A**? Where is the origin of system **B** w.r.t. system **A**?
4. What is $\boxed{{}^{C}T_{A}}$?
5. For the point $(0, 1, 2)^T$ in system **A**, what are it's coordinates in system **B**?
6. For the point $(0, 1, 2)^T$ in system **B**, what are it's coordinates in system **A**?

$$\begin{bmatrix} \hat{x}_B \\ 1 \end{bmatrix} = {}^{B}T_{A} \begin{bmatrix} \hat{x}_A \\ 1 \end{bmatrix}$$



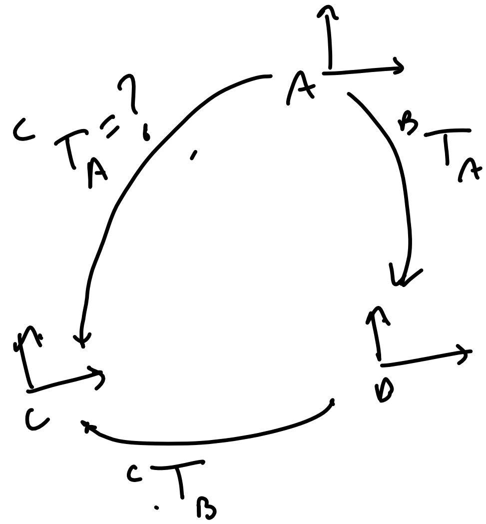

$$= {}^{B}T_{A} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} r_{1\,3\times1} & r_{2\,3\times1} & r_{3\,3\times1} & t_{3\times1} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= r_{1\,3\times1} + t_{3\times1}$$

$$O_B = \left[ {}^{B}T_{A} \right] \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = t_{3\times1} \leftarrow \text{origin of } A \text{ wrt } B$$

Direction of X-axis $= r_1 = \begin{bmatrix} 2/7 \\ 6/7 \\ -3/7 \end{bmatrix}$

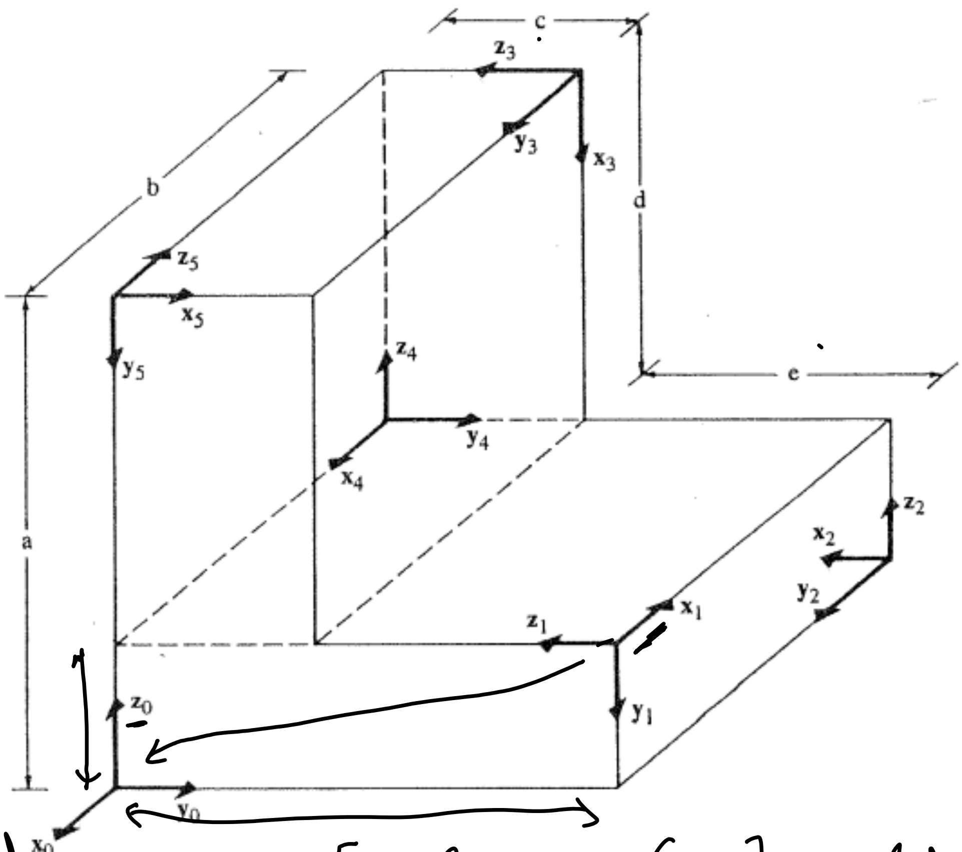
Y-axis $= r_2$




Composition of transformation

$$\boxed{{}^{C}T_A = {}^{C}T_B \, {}^{B}T_A}$$

**2.6** For the figure shown below, find the 4 × 4 homogeneous transformation matrices $^{i-1}\mathbf{A}_i$ and $^0\mathbf{A}_i$ for $i = 1, 2, 3, 4, 5$.




Not standard notation

$x$-axis of 1 wrt 0

$$\boxed{{}^0\underline{x}_1} = \begin{bmatrix}-1\\0\\0\end{bmatrix} \quad {}^0\underline{y}_1 = \begin{bmatrix}0\\0\\-1\end{bmatrix}, \quad {}^0\underline{z}_1 = \begin{bmatrix}0\\-1\\0\end{bmatrix}, \quad {}^0\underline{O}_1 = \begin{bmatrix}0\\c+e\\a-d\end{bmatrix}$$

$$\underline{p} = \underbrace{\begin{bmatrix}{}^0\underline{x}_1 & {}^0\underline{y}_1 & {}^0\underline{z}_1\end{bmatrix}_{3\times3}}_{\text{Basis matrix}} \underbrace{\begin{bmatrix}p_{1x}\\p_{1y}\\p_{1z}\end{bmatrix}}_{\text{coordinate}} + {}^0\underline{O}_1 = {}^0\underline{B}_1\,\underline{p}_1 + {}^0\underline{O}_1$$

$$\underline{p} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}_{3\times3}\begin{bmatrix}p_{0x}\\p_{0y}\\p_{0z}\end{bmatrix} + {}^0\underline{O}_0 = {}^0\underline{B}_0\,\underline{P}_0 + \begin{bmatrix}0\\0\\0\end{bmatrix}$$

$$\underline{p}_0 = \overset{I}{{}^0B_0^T}\left[\underbrace{{}^0B_1}_{{}^0R_1}\,\underline{p}_1 + {}^0\underline{O}_1\right]$$

$$= I^T\left[{}^0\underline{B}_1\,\underline{p}_1 + {}^0\underline{O}_1\right]$$

$$= {}^0\underline{B}_1\,\underline{p}_1 + \underset{\uparrow\,{}^0\underline{t}_1}{{}^0\underline{O}_1}$$

$${}^0\underline{B}_1 = {}^0R_1$$

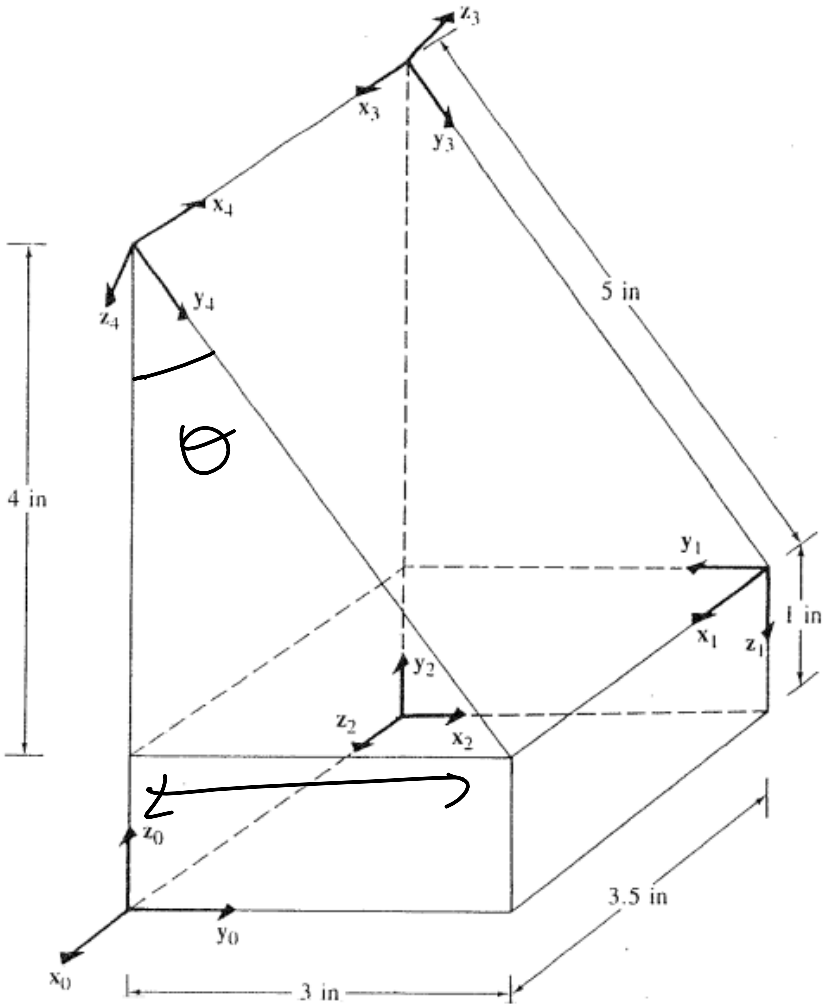
$$\underline{p}_0 = \overbrace{\underbrace{\begin{bmatrix} {}^0\underline{x}_1 & {}^0\underline{y}_1 & {}^0\underline{z}_1 \end{bmatrix}}_{\boxed{{}^0R_1}}}^{\text{Basis vectors of 1 wrt 0}} \underline{p}_1$$

${}^0R_1$ ← Rotation matrix that converts from $\underline{1}$ to $\underline{0}$

$$\underline{p}_0 = {}^0R_1\, \underline{p}_1 + {}^0\underline{O}_1$$

$$\begin{bmatrix} \underline{p}_0 \\ \hline 1 \end{bmatrix} = \begin{bmatrix} {}^0R_{1\ 3\times1} & {}^0\underline{O}_{1\ 3\times1} \\ 0\ 0\ 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{p}_1 \\ \hline 1 \end{bmatrix}$$

2.7 For the figure shown below, find the 4 × 4 homogeneous transformation matrices ${}^{i-1}\mathbf{A}_i$ and ${}^{0}\mathbf{A}_i$ for $i = 1, 2, 3, 4$.




$\theta = \tan^{-1}\left(\frac{3}{4}\right)$

Euler angle representation 3D rotations have 3 d.o.f.

Many different set are possible Three common ones are

System I $R_{z,\phi} \to R_{u,\theta} \to R_{w,\psi} \Rightarrow \overline{R}_{z,\phi}\,\overline{R}_{u,\theta}\,\overline{R}_{w,\psi}$ gyroscopic motion

System II $R_{z,\phi} \to R_{v,\theta} \to R_{w,\psi} \Rightarrow \overline{R}_{z,\phi}\,\overline{R}_{v,\theta}\,\overline{R}_{w,\psi}$ Robotics

System III $R_{x,\psi} \to R_{y,\theta} \to R_{z,\phi} \Rightarrow \overline{R}_{z,\phi}\,\overline{R}_{y,\theta}\,\overline{R}_{x,\psi}$ Roll pitch Yaw Aerospace

**Write the solution for each of the following in terms of atan2 before giving your final answer (e.g., $\theta$ = atan2(a,b) = ... )**

1. If $\sin\theta = -1/2$ and $\cos\theta = \sqrt{3}/2$ what is $\theta$?
2. If $-k \sin\theta = 3$ and $k \cos\theta = 3\sqrt{3}$ what is $\theta$?
3. If $\sin\phi \sin\theta = 0.2$ and $\sin\phi \cos\theta = -0.3$ and $\sin\phi > 0$ what is $\theta$? Repeat for $\sin\phi < 0$.
4. If $-\sin\phi \sin\theta = 0.2$ and $\sin\phi \cos\theta = 0.3$ and $\sin\phi > 0$ what is $\theta$? Repeat for $\sin\phi < 0$.
5. Using the System II matrix solve for $\phi + \psi$ in terms of the matrix elements $r_{ij}$ if $\theta = 0$ degrees. Also solve for $\phi - \psi$ if $\theta = 180$ degrees.
6. Using the System I matrix (given below) solve for $\phi$, $\theta$, and $\psi$ in terms of the matrix elements $r_{ij}$. Include solutions for the special cases for $\theta$.

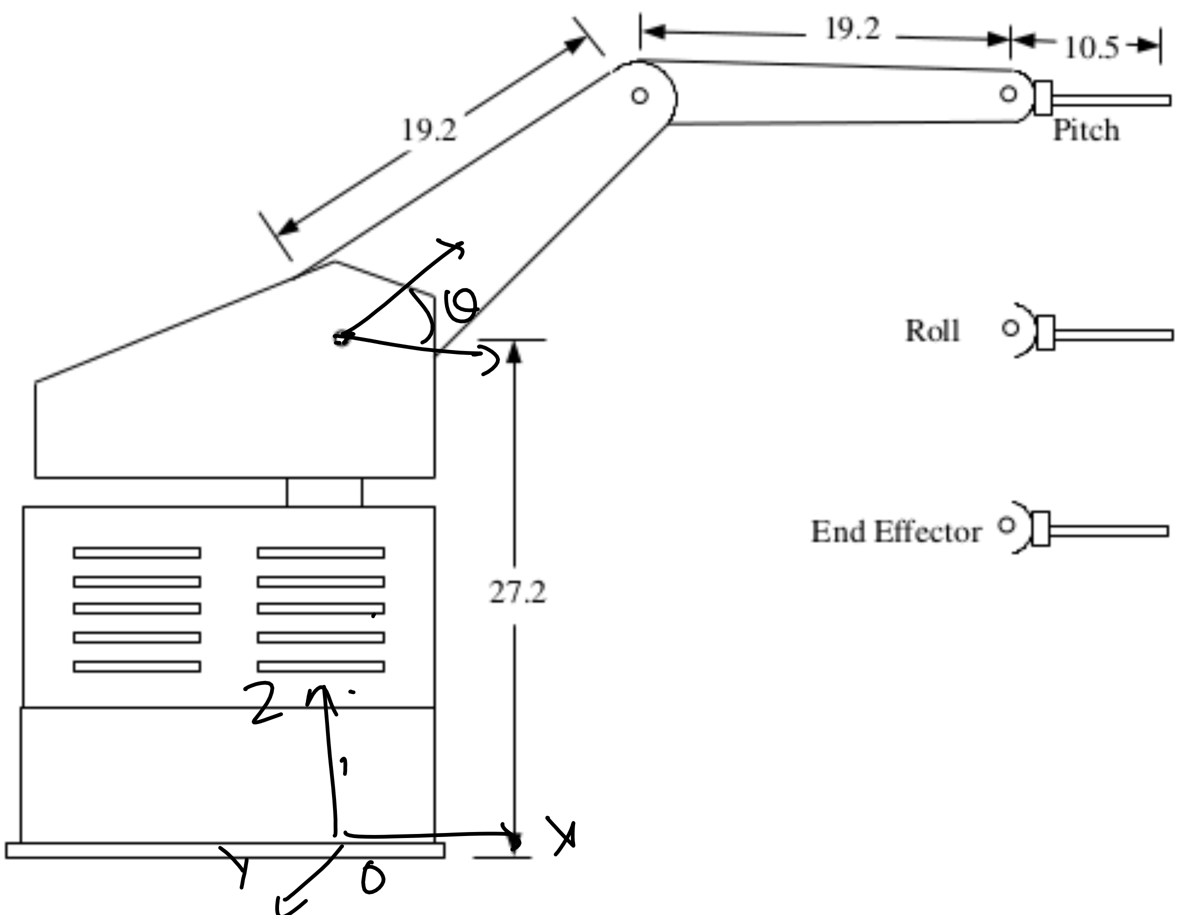
**System I**

$$\begin{bmatrix} \cos\phi\cos\psi - \sin\phi\cos\theta\sin\psi & -\cos\phi\sin\psi - \sin\phi\cos\theta\cos\psi & \sin\phi\sin\theta \\ \sin\phi\cos\psi + \cos\phi\cos\theta\sin\psi & -\sin\phi\sin\psi + \cos\phi\cos\theta\cos\psi & -\cos\phi\sin\theta \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{bmatrix}$$

Euler angles = $R_z(\psi)\, R_y(\phi)\, R_x(\theta)$

1. Convert the matrix $\mathbf{R}$ (aka ${}^{XYZ}\mathbf{R}_{UVW}$ ) of Homework 1 to System II Euler angles (2 solutions)
2. Convert the matrix $\mathbf{R}$ (aka ${}^{XYZ}\mathbf{R}_{UVW}$ ) of Homework 1 to axis-angle form (2 solutions)
3. Convert System II Euler angles $(90°, 90°, 90°)^T$ to a 3x3 rotation matrix
4. Convert System II Euler angles $(30°, 60°, 90°)^T$ to a 3x3 rotation matrix
5. Convert axis-angle given by a rotation of 120° about an axis through $(1, -1, 1)^T$ to a 3x3 rotation matrix. (Remember to normalize the vector first.)
6. For a rotation about the X axis by 90°, convert to
    a. System II Euler angles (2 solutions)
    b. Axis-angle form (2 solutions)
    c. A 3x3 rotation matrix

The following applies to our LabVolt robots. Draw the coordinate frame for each link (don't forget to include link 0). Label each axis and indicate how you define the joint angles. Then fill in the joint parameter table. Use a dot for an axis pointing out of the page and an X for an axis pointing in.




| Axis | $d_i$ | $\theta_i$ | $a_i$ | $\alpha_i$ |
|---|---|---|---|---|
| 1 | | var | | |
| 2 | | | | |
| 3 | | | | |
| 4 | | | | |
| 5 | | | | |

1. Convert the following quaternions to rotation matrix and axis/angle forms: $(1,0,0,0)^T$, $(0,1,0,0)^T$, $(0,0,0,-1)^T$, and $(1/2, 1/2, -1/2, -1/2)^T$

2. What **two** quaternions correspond to the inverse of $(1/2, -1/2, 1/2, -1/2)^T$ ?

3. What **two** quaternions correspond to each of the following (HINT: convert to unit vectors):
    1. A +90 degree rotation about the +X axis.
    2. A -90 degree rotation about the +X axis.
    3. A 90 degree rotation about the axis (1, 0, 1).
    4. A 180 degree rotation about the axis (1, 0, 1).
    5. A 180 degree rotation about the axis (1,1,-1).

4. Multiply the quaternions $(1/2, 1/2, -1/2, -1/2)^T$ and $(-1/7, 4/7, 4/7, -4/7)^T$ (**not** dot product).

5. What is the angle between two orientations given by the quaternions $(1/2, 1/2, -1/2, -1/2)^T$ and $(-1/7, 4/7, 4/7, -4/7)^T$ ?

6. Transform the vector $(1, 2, 3)^T$ by the orientation given by $(1/2, 1/2, -1/2, -1/2)^T$ (without converting to another representation of orientation).

7. Convert the orientation to quaternion form

$$\mathbf{R} = \begin{bmatrix} 4/9 & -7/9 & 4/9 \\ -1/9 & 4/9 & 8/9 \\ -8/9 & -4/9 & 1/9 \end{bmatrix}$$

8. Give the quaternion corresponding to System II Euler angles $(\phi, \theta, \psi) = (30, -45, 60)$.