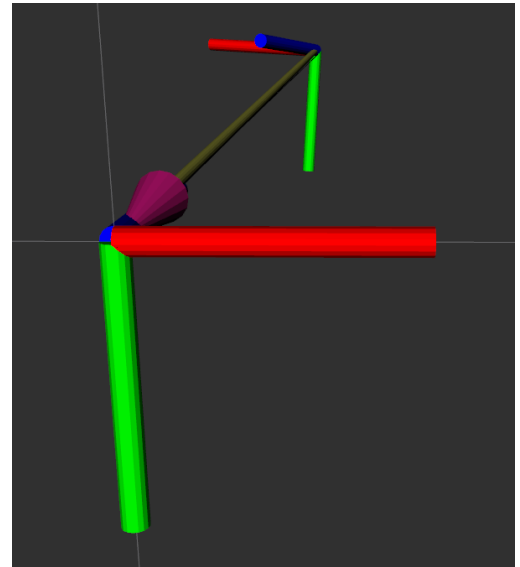
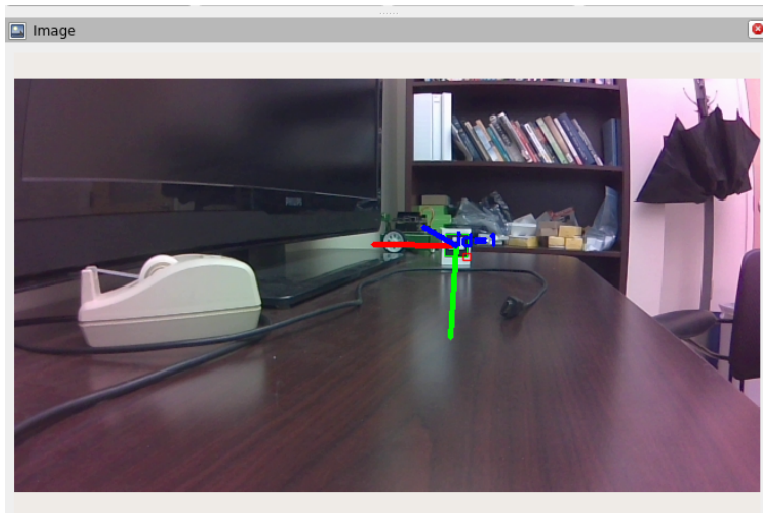


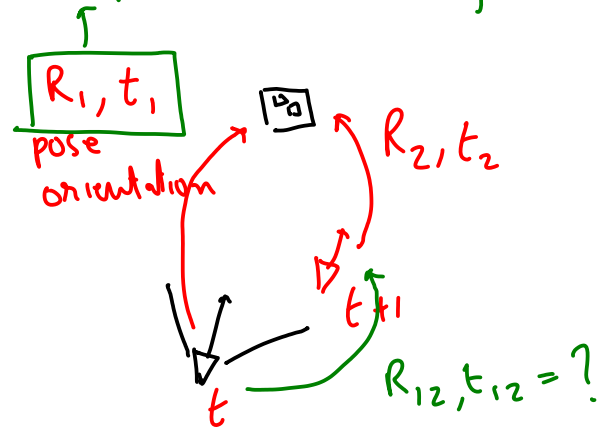
## Rotations and translations/Coordinate transformations



```
root@nano-4gb-jp45:/home/jetbot/ece417# ros2 topic info /
aruco_detections
Type: aruco_opencv_msgs/msg/ArucoDetection
Publisher count: 1
Subscription count: 0

root@nano-4gb-jp45:/home/jetbot/ece417# ros2 topic echo --once /
aruco_detections
header:
  stamp:
    sec: 1727998810
    nanosec: 924374790
  frame_id: /v4l_frame
markers:
- marker_id: 1
  pose:
    position:
      x: 0.08918172498053901
      y: -0.10849999597426438
      z: 0.980432215194246
    orientation:
      x: -0.02973468393320003
      y: 0.9811997541144667
      z: -0.03227342049856023
      w: -0.18793966432455353
boards: []
---
```

joint representation = Transformation matrix

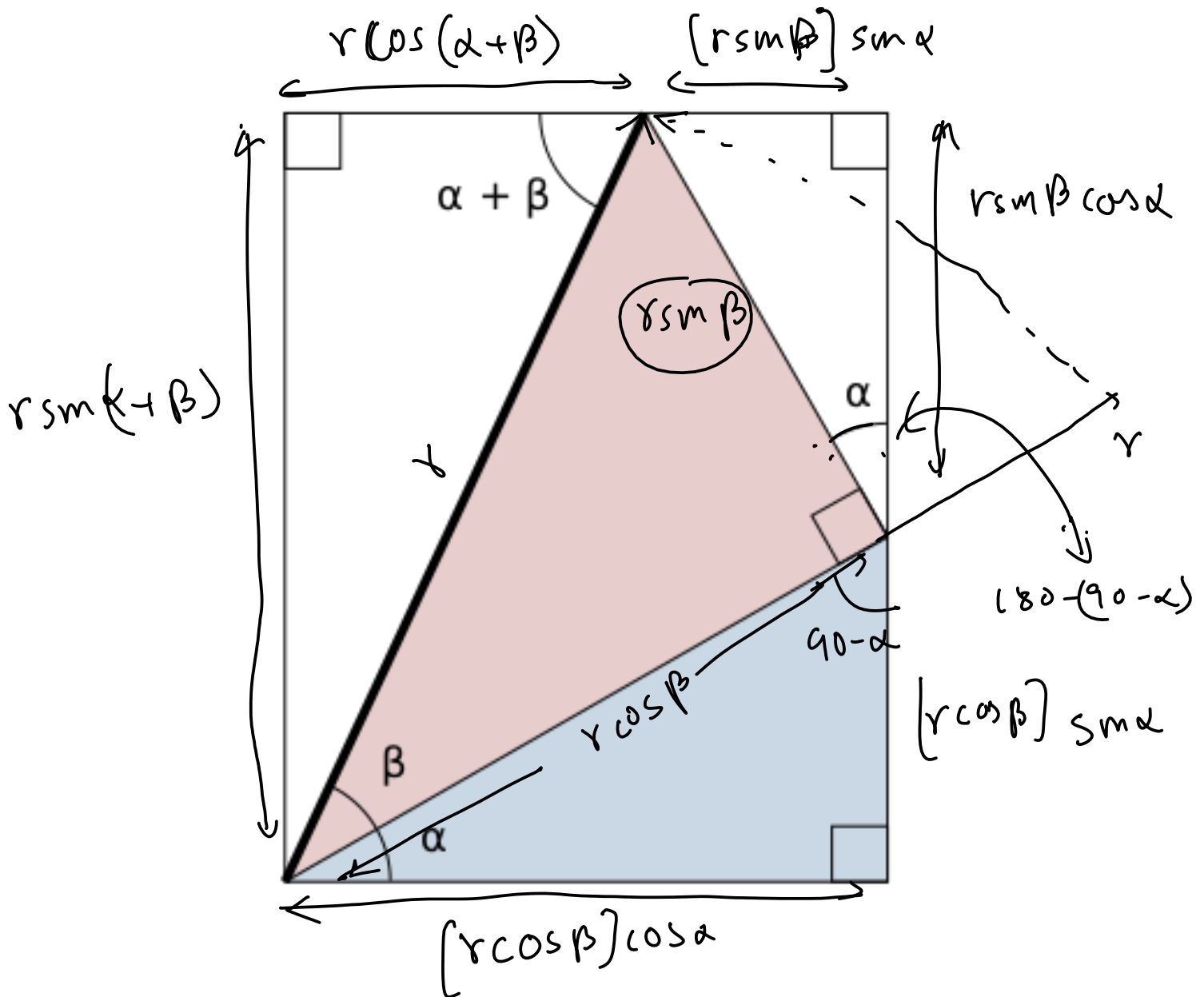
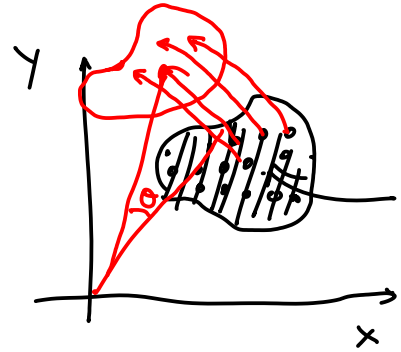


① Rigid bodies are sets of points

↳ Rotate

↳ Translate

All points in the rigid body  
Rotate and translate by the  
same amount



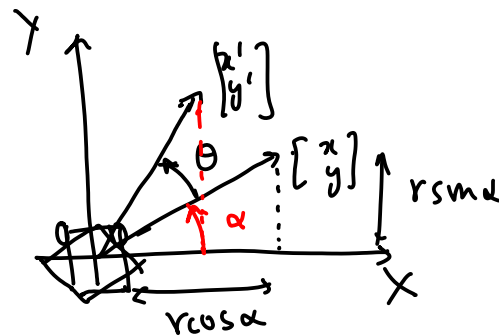
## 2D Rotation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = ? \quad \text{in terms of } \theta \text{ and } \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x = r \cos \alpha$$

$$y = r \sin \alpha$$

$$r = \sqrt{x^2 + y^2}$$

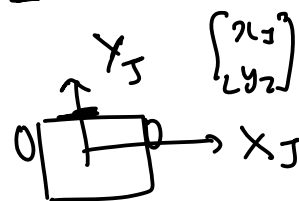


$$\begin{aligned} x' &= r \cos(\theta + \alpha) \\ &= r \cos \theta \cos \alpha - r \sin \theta \sin \alpha \\ &= x \cos \theta - y \sin \theta \end{aligned}$$

$$y' = r \sin(\theta + \alpha) = r \sin \theta \cos \alpha + r \cos \theta \sin \alpha = x \sin \theta + y \cos \theta$$

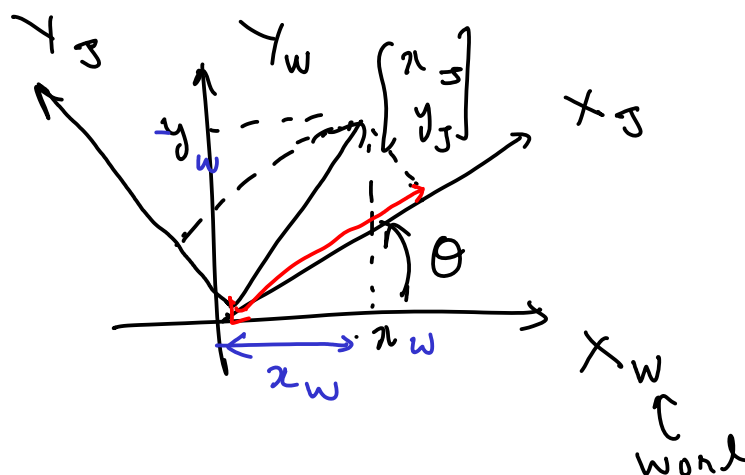
$$\underbrace{\begin{bmatrix} x' \\ y' \end{bmatrix}}_{\underline{p}'} = \underbrace{\begin{bmatrix} x \cos \theta - y \sin \theta \\ x \sin \theta + y \cos \theta \end{bmatrix}}_{\substack{2 \times 1 \\ R(\theta)}} = \underbrace{\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}}_{\substack{2 \times 2 \\ R(\theta)}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\substack{2 \times 1 \\ \underline{p}}}$$

$$\underline{p}' = R(\theta) \underline{p}$$



$$\underbrace{\begin{bmatrix} x_w \\ y_w \end{bmatrix}}_{\underline{p}_w} = \underbrace{\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}}_{\substack{2 \times 2 \\ {}^w R_J(\theta)}} \underbrace{\begin{bmatrix} x_J \\ y_J \end{bmatrix}}_{\substack{2 \times 1 \\ \underline{p}_J}}$$

SOURCE  
↑  
DESTINATION



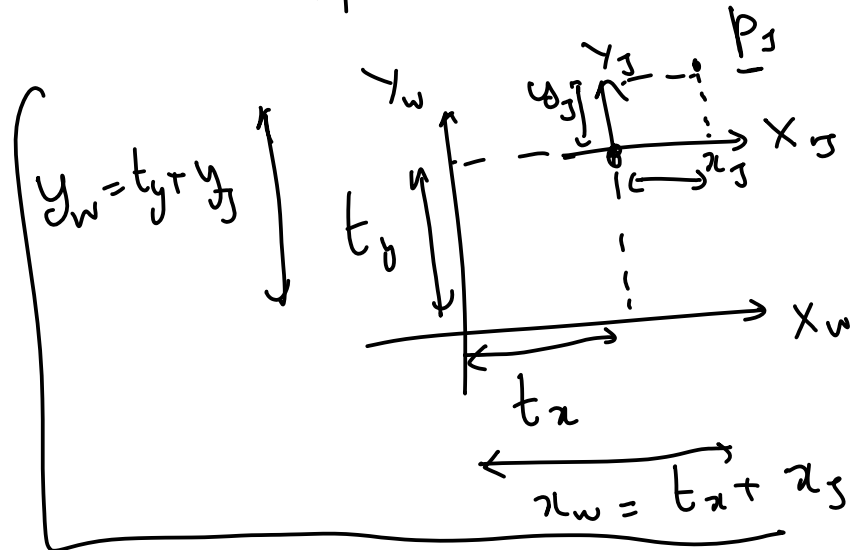
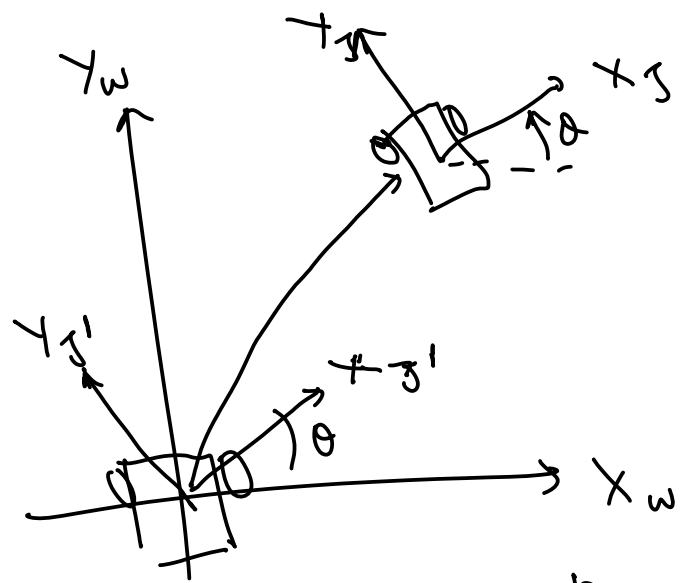
$$\underline{p}_w = {}^w R_J(\theta) \underline{p}_J$$

$$\underline{p}_{j'} = {}^{j'}R_j \underline{p}_j$$

$$\underline{p}_w = \underline{p}_{j'} + {}^w\underline{t}_{j'}$$

$$\underline{p}_w = {}^{j'}R_j \underline{p}_j + {}^w\underline{t}_{j'}$$

$$\underline{p}_w = {}^wR_j \underline{p}_j + {}^w\underline{t}_j$$



$$\begin{aligned} \begin{bmatrix} x_w \\ y_w \end{bmatrix} &= \begin{bmatrix} x_j + t_x \\ y_j + t_y \end{bmatrix} \\ &= \begin{bmatrix} x_j \\ y_j \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix} \end{aligned}$$

$$\underline{p}_w = \underline{p}_j + {}^w\underline{t}_j$$

Matrices = capital letters

Vectors = small letters  
with underscore

Scalars = small letters  
without  
underscore

# Transformation matrix

$$\underline{p}_w = {}^w R_s \underline{p}_s + {}^w \underline{t}_s$$

$\nwarrow$  orientation       $\nwarrow$  position

$\underbrace{{}^w R_s, {}^w \underline{t}_s}$

$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}_{2 \times 2} \begin{bmatrix} x_s \\ y_s \end{bmatrix}_{2 \times 1} + \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}_{2 \times 1}$$

$$\begin{bmatrix} x_w \\ y_w \\ 1 \end{bmatrix}_{3 \times 1} = \underbrace{\begin{bmatrix} r_{11} & r_{12} & | & t_1 \\ r_{21} & r_{22} & | & t_2 \\ \hline 0 & 0 & | & 1 \end{bmatrix}}_{\substack{{}^w T_s \\ 3 \times 3}} \begin{bmatrix} x_s \\ y_s \\ 1 \end{bmatrix}_{3 \times 1}$$

$\nwarrow$  homogeneous coordinate

$$\begin{bmatrix} \underline{p}_w \\ 1 \end{bmatrix} = \begin{bmatrix} {}^w R_s & | & {}^w \underline{t}_s \\ \hline \underline{0}_{2 \times 1} & | & 1 \end{bmatrix} \begin{bmatrix} \underline{p}_s \\ 1 \end{bmatrix}$$

# Properties of a Rotation matrix

① What is the inverse of a Rotation matrix

$$AA^{-1} = A^{-1}A = I$$

Prereq,

$$U = \begin{bmatrix} \underline{u}_1 & \dots & \underline{u}_n \\ 1 & & 1 \end{bmatrix}$$

orthonormal matrix

when

$$\underline{u}_i^T \underline{u}_i = 1$$

$$\underline{u}_i^T \underline{u}_j = 0 \quad i \neq j$$

$$R = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

is an orthonormal matrix

$$\begin{bmatrix} \cos \theta & \sin \theta \end{bmatrix} \begin{bmatrix} \cos \theta \\ \sin \theta \end{bmatrix} = \cos^2 \theta + \sin^2 \theta = 1$$

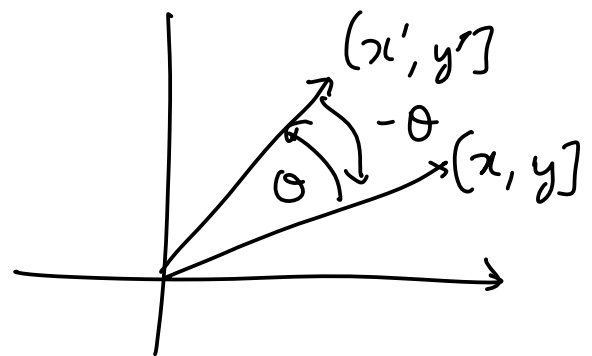
$$\begin{bmatrix} \cos \theta & \sin \theta \end{bmatrix} \begin{bmatrix} -\sin \theta \\ \cos \theta \end{bmatrix} = -\cos \theta \sin \theta + \sin \theta \cos \theta = 0$$

For all orthonormal matrices, their inverse is their transpose  $U^{-1} = U^T$

$$R^{-1} = R^T$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} R \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = R^{-1} \begin{bmatrix} x' \\ y' \end{bmatrix}$$



$$R^{-1} = \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix} = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} = R^T$$

① Rotation matrices are orthonormal  $R R^T = R^T R = I$  / Rotation matrix inverse is its transpose

Are all orthonormal matrices rotation matrices?

Answer : No



②  $\det(R) = 1$

In general an orthogonal matrix,  $U$

$$\det(U) \in \{-1, +1\}$$

Reflection  
matrix

Rotation matrix

$$\det \begin{pmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{pmatrix} = \cos^2 \theta - (-\sin^2 \theta) = 1$$

2D transformation matrix

$$\begin{bmatrix} p_w \\ i \end{bmatrix} = \begin{bmatrix} {}^w R_s & {}^w t_s \\ \underline{0}^T & 1 \end{bmatrix} \begin{bmatrix} p_s \\ 1 \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} {}^w R_s & {}^w t_s \\ \underline{0}^T & 1 \end{bmatrix}}_{{}^w T_s}$$

$${}_s T_w = ? = {}^w T_s^{-1}$$

$$\underline{p_w} = {}^w R_s \underline{p_s} + {}^w t_s$$

$$\underline{p_w} - {}^w t_s = {}^w R_s \underline{p_s}$$

$$\underline{p_s} = \frac{\underline{p_w} - {}^w t_s}{{}^w R_s}$$

BLASPHEMY

X

Left multiply by  ${}^w R_s^T$

$${}^w R_s^T (\underline{p}_w - {}^w t_s) = \underbrace{({}^w R_s^T) R_s}_{I} \underline{p}_s$$

$${}^w R_s^T \underline{p}_w - {}^w R_s^T {}^w t_s = \underline{p}_s$$

$$\underbrace{\begin{bmatrix} {}^w R_s^T & (-{}^w R_s^T {}^w t_s) \\ \mathbf{0}_{2 \times 1}^T & 1 \end{bmatrix}}_{{}^w T_s^{-1}} \begin{bmatrix} \underline{p}_w \\ 1 \end{bmatrix} = \begin{bmatrix} \underline{p}_s \\ 1 \end{bmatrix}$$

$${}^s T_w = \begin{bmatrix} {}^w R_s^T & -{}^w R_s^T {}^w t_s \\ \mathbf{0}_{2 \times 1}^T & 1 \end{bmatrix}$$

Intuitive interpretation

① Rotation → ② translation

← ←

$$\underbrace{\underline{p}' = R \underline{p} + \underline{t}}_{\text{Rotation first}} \neq \underbrace{\underline{p}' = R (\underline{p} + \underline{t})}_{\text{translation first}}$$

SO(2)

Special  
Orthogonal group

O(2)

Orthogonal group

$$\left\{ R_{2 \times 2} : R^T R = I, \det(R) = 1 \right\} \left\{ U : \begin{matrix} U^T & U \\ 2 \times 2 & 2 \times 2 \end{matrix} = \begin{matrix} I \\ 2 \times 2 \end{matrix} \right\}$$

SE(2)

Special Euclidean group

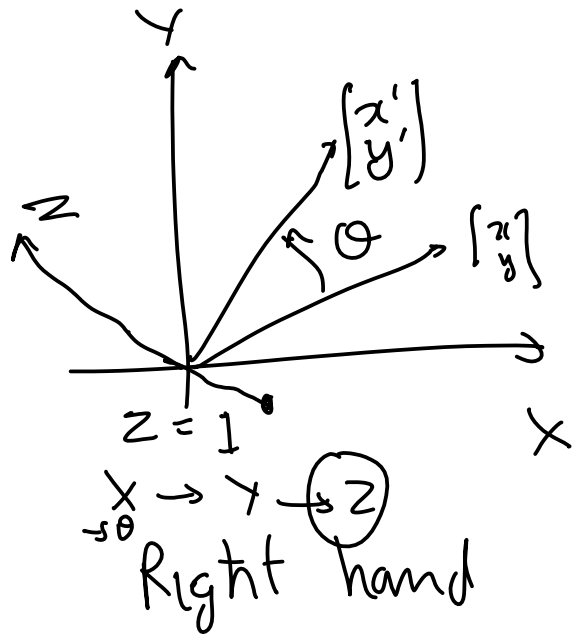
$$\left\{ R_{2 \times 2}, t_{2 \times 1} : R_{2 \times 2} \in \underline{SO(2)}, t \in \boxed{\mathbb{R}^2} \right\}$$

2D vectors  
in real space

# 3D Rotations

## 2D Rotation around Z-axis

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \underbrace{\begin{bmatrix} R(\theta_z) & 0 \\ 0 & 1 \end{bmatrix}}_{R_z(\theta_z)} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$



around X-axis  $R_z(\theta_z)$

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \begin{bmatrix} \cos\theta_x & -\sin\theta_x \\ \sin\theta_x & \cos\theta_x \end{bmatrix} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

around Y-axis  $R_x(\theta_x)$   $\odot \rightarrow \begin{bmatrix} x \rightarrow z \\ -y \end{bmatrix}$

coordinate frame

$X \rightarrow Y : Z$

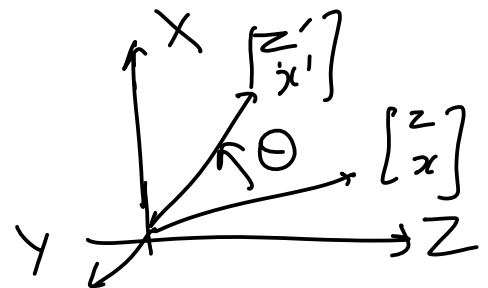
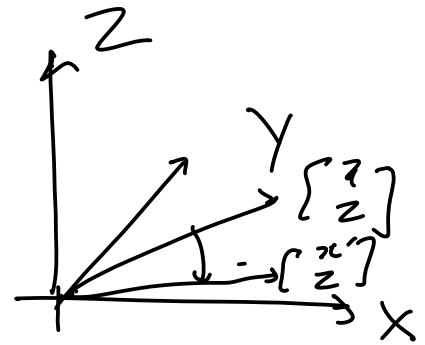
$Y \rightarrow Z : X$

$Z \rightarrow X : Y$

curl finger thumb

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$R_y(\theta_y) X \rightarrow \begin{bmatrix} x \rightarrow z \\ -y \end{bmatrix}$



$$R_p = \begin{bmatrix} R_z(\theta_z) R_y(\theta_y) R_x(\theta_x) p \end{bmatrix}$$

$\begin{matrix} 3 \times 3 & & 3 \times 3 & & 3 \times 3 & & 3 \times 1 \end{matrix}$

All 3D rotations can be decomposed into 3 axis aligned rotations

three angles of rotations  
are called Euler angles

12 possible  
combination

Roll - Pitch - Yaw  
(1) (2) (3)



$$R = R_z(\theta_z) R_y(\theta_y) R_x(\theta_x)$$

←

Application of  
Rotation matrices  
is from right  
to left

# Composition of Rotations / Transformations

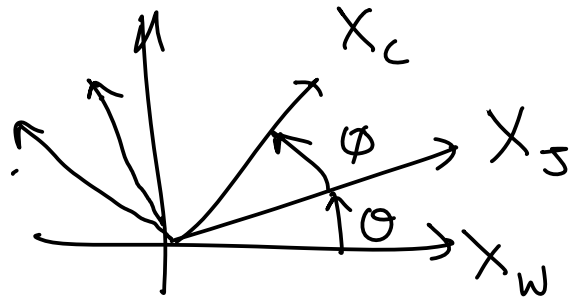
$$\begin{aligned}\underline{p}_s &= {}^s R_c \underline{p}_c \\ \underline{p}_w &= {}^w R_s \underline{p}_s\end{aligned}$$

$$\boxed{{}^w R_c = ?}$$

$$\underline{p}_w = {}^w R_s \underbrace{{}^s R_c \underline{p}_c}_{\underline{p}_s}$$

$$\underline{p}_w = \underbrace{{}^w R_s {}^s R_c}_{{}^w R_c} \underline{p}_c$$

$$\boxed{{}^w R_c = {}^w R_s {}^s R_c}$$



$$\begin{aligned}& \mathbb{I}_s \\ & (R_1 R_2) \\ & \text{orthonormal?} \\ & \det(R_1 R_2) \stackrel{?}{=} 1\end{aligned}$$

$$\underbrace{(R_1 R_2)^{-1}} = R_2^{-1} R_1^{-1}$$

$$\begin{aligned}\det(R_1 R_2) &= \det(R_1) \det(R_2) = R_2^T R_1^T \\ &= 1 \\ &= \underbrace{(R_1 R_2)^T}\end{aligned}$$

$${}^w T_c = {}^w T_J {}^J T_c$$

Why are there 12 possible Euler angles?

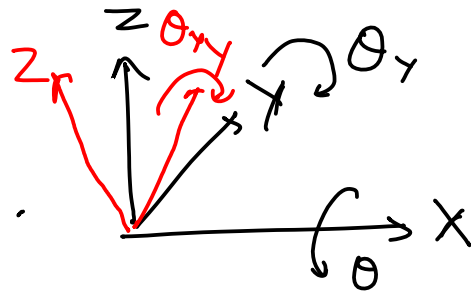
$$R = \begin{matrix} R_z & R_y & R_x \\ R_y & R_z & R_x \end{matrix}$$

$6 = 3!$  possibilities  
permutations

When chaining rotations there are two possibilities

① either rotate along the new axis after first rotation

② or rotate along the original axis



# 3D Rotation representation

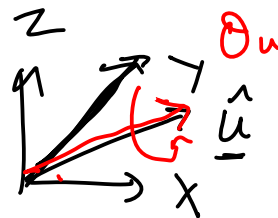
→  $3 \times 3$ ,  $\det(R)=1$ ,  $R^T R = I \equiv$  Rotation matrix

→ 3 Euler angles  $(\theta_x, \theta_y, \theta_z)$

→ Quaternions  $(x, y, z, w)$

→ Axis-angle representation  $(\hat{u}, \theta_u)$

(Rodrigues formula)



$$A = \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}_{3 \times 3} = 9 \text{ DOF}$$

$$A^T = A \quad \text{if } A \text{ is symmetric} = 6 \text{ DOF}$$

$U_{3 \times 3}$  is orthonormal

$$U = \begin{bmatrix} \underline{u}_1 & \underline{u}_2 & \underline{u}_3 \end{bmatrix}$$

$$\begin{aligned} \underline{u}_1^T \underline{u}_2 &= 0 \\ \underline{u}_2^T \underline{u}_3 &= 0 \\ \underline{u}_3^T \underline{u}_1 &= 0 \end{aligned}$$

$$\begin{aligned} \underline{u}_1^T \underline{u}_1 &= 1 \\ \underline{u}_2^T \underline{u}_2 &= 1 \\ \underline{u}_3^T \underline{u}_3 &= 1 \end{aligned}$$

→ 3 DOF



Conversions

Roll-pitch-Yaw  $\longrightarrow$  Rotation matrix

$$R = R_z(\theta_z) R_y(\theta_y) R_x(\theta_x)$$

$$= \begin{bmatrix} c\theta_z & -s\theta_z & 0 \\ s\theta_z & c\theta_z & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c\theta_y & 0 & s\theta_y \\ 0 & 1 & 0 \\ -s\theta_y & 0 & c\theta_y \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\theta_x & -s\theta_x \\ 0 & s\theta_x & c\theta_x \end{bmatrix}$$

Rotation matrix  $\longrightarrow$  Roll pitch Yaw angles

$$R = \begin{bmatrix} c(\theta_z) c(\theta_y) & \dots & \dots \\ s(\theta_z) c(\theta_y) & \dots & \dots \\ -s(\theta_y) & c(\theta_y) s(\theta_x) & c(\theta_y) c(\theta_x) \end{bmatrix}$$

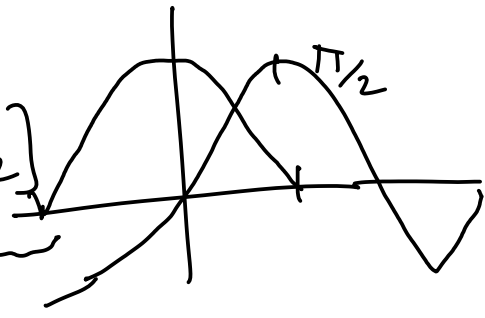
$$= \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\times \left[ \tan(\theta_x) = \frac{r_{32}}{r_{33}} \right] \Rightarrow \theta_x = \tan^{-1} \left( \frac{r_{32}}{r_{33}} \right) \in \left( -\pi/2, \pi/2 \right)$$

$$\rightarrow \theta_x = \arctan 2 (r_{33}, r_{32}) \in (-\pi, \pi]$$

$$-\sin(\theta_y) = r_{31}$$

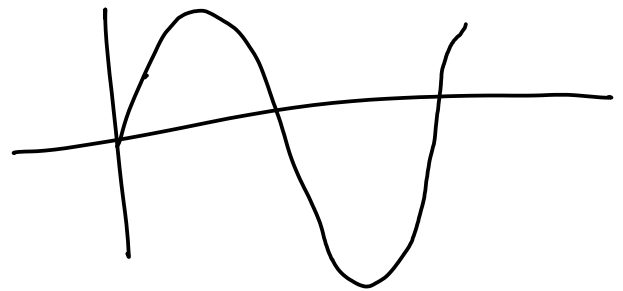
$$\theta_y = \sin^{-1}(-r_{31}) \in [-\pi/2, \pi/2]$$



$$\theta_x = \arctan 2 \left( \frac{\text{sign}(\cos(\theta_y)) r_{33}}{\text{sign}(\cos(\theta_y)) r_{32}} \right)$$

$$\theta_z = \arctan 2 \left( \frac{\text{sign}(\cos(\theta_y)) r_{21}}{\text{sign}(\cos(\theta_y)) r_{11}} \right)$$

$R$   $\rightarrow$  Euler angles  
 $\rightarrow$  Euler angles



$$\sin(\pi - \theta) = \sin(\theta)$$

$$\theta_y = \sin^{-1}(-r_{31}) \in [-\pi/2, \pi/2]$$

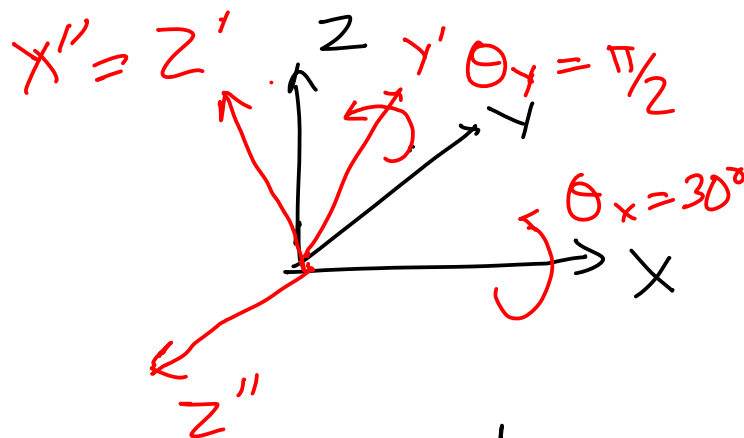
$$\theta_y' = \pi - \theta$$

# Gimbal Lock (Problem with Euler angles)

$$\theta_y = 90^\circ \neq \pi/2$$

$$\cos(\theta_y) = 0$$

Then  $\theta_x$  and  $\theta_z$  can be swapped

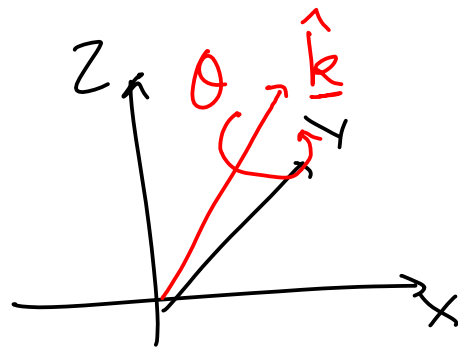


Other representations  $\left\{ \begin{array}{l} \text{Quaternions} \\ \text{Axis-angle representation} \end{array} \right.$

Axis-angle representation

$$\hat{k} = [k_x, k_y, k_z]$$

$$\|\hat{k}\| = \sqrt{k_x^2 + k_y^2 + k_z^2} = 1$$



$\theta$

3DOF, 4 numbers, 1 constraint

Axis angle  $\xrightarrow[\text{formula}]{\text{Rodrigues}}$  Rot Matrix

$$R = \underline{I} + \underbrace{\sin \theta [\hat{k}_x]} + (1 - \cos \theta) [\hat{k}_x]^2$$

$$[\hat{k}_x] = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix} : \text{Cross product matrix}$$

Cross product matrix

$$\underline{a} \times \underline{b} = \underbrace{[\underline{a}_x]}_{\substack{\text{Matrix} \\ \text{Cross product}}} \underline{b}$$

$$\begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{bmatrix} = \hat{i} (a_y b_z - b_y a_z) + \hat{j} (a_z b_x - a_x b_z) + \hat{k} (a_x b_y - b_x a_y)$$

$$= \begin{bmatrix} a_y b_z - b_y a_z \\ a_z b_x - a_x b_z \\ a_x b_y - b_x a_y \end{bmatrix}$$

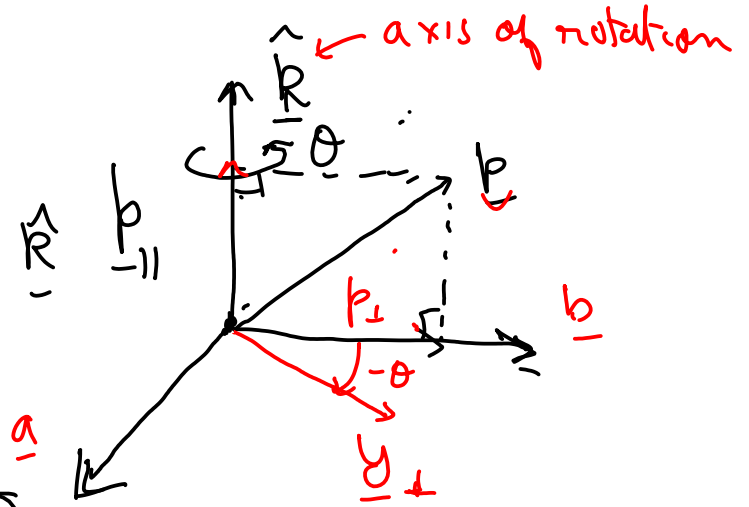
$$= \boxed{\begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ -a_y & a_x & 0 \end{bmatrix}}_{3 \times 3} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

$A = \underbrace{[\underline{a}_\times]}_{\text{cross-product matrix of } \underline{a}}$

$\underbrace{A^T = -A}_{\text{skew-symmetric matrices}}$

$$\begin{aligned} R \underline{p} &= \left( \underline{I} + \sin \theta \underbrace{[\hat{\underline{k}}_\times]}_{\text{skew-symmetric}} + (1 - \cos \theta) [\hat{\underline{k}}_\times]^2 \right) \underline{p} \\ &= \underline{p} + \sin \theta (\hat{\underline{k}} \times \underline{p}) + (1 - \cos \theta) \hat{\underline{k}} \times (\hat{\underline{k}} \times \underline{p}) \end{aligned}$$

$\underline{y} = ?$  after rotating  
point  $\underline{p}$  around  $\hat{\underline{r}}$   
by angle  $\theta$



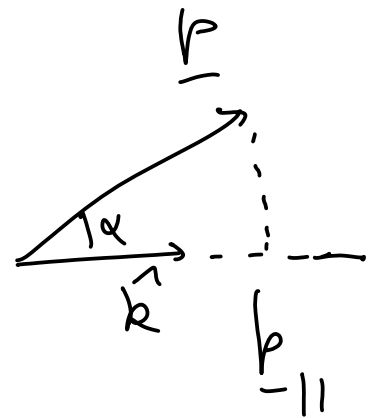
① Split  $\underline{p}$  into  $\underline{p}_{||} + \underline{p}_{\perp}$

$$\underline{y} = \underline{p}_{||} + \underline{y}_{\perp}$$

② Rotate  $\underline{p}_{\perp}$  by angle  $\theta$  to get  $\underline{y}_{\perp}$

③ Combine  $\underline{p}_{||}$  and  $\underline{y}_{\perp}$  to get  $\underline{y}$

$$\underline{p}_{||} = \underbrace{(\hat{\underline{r}}^T \underline{p})}_{|\underline{p}| \cos(\alpha)} \hat{\underline{r}}$$



Find  $\underline{a}$  that is  $\underline{a} \perp \hat{\underline{r}}$  and  $\underline{a} \perp \underline{p}$  } 3D

$$\underline{a} = \underline{p} \times \hat{\underline{r}} = -(\hat{\underline{r}} \times \underline{p}), \quad \underline{b} = ?$$

Find  $\underline{b}$  that is  $\underline{b} \perp \hat{\underline{r}}$  and  $\underline{b} \perp \underline{a}$

$$\underline{b} = \hat{\underline{r}} \times \underline{a} = \hat{\underline{r}} \times (\underline{p} \times \hat{\underline{r}}) = -\hat{\underline{r}} \times (\hat{\underline{r}} \times \underline{p})$$

$$\underline{a} \times (\underline{b} \times \underline{c}) \neq (\underline{a} \times \underline{b}) \times \underline{c}$$

Cross product is not associative

$$\hat{\underline{k}} \times \hat{\underline{k}} = ?$$

0

$$\underline{b} = -\hat{\underline{k}} \times (\hat{\underline{k}} \times \underline{p})$$

$$\underline{b} = -K (K \underline{p}) = -K^2 \underline{p}$$

$$[\hat{\underline{k}}_{\times}] = K$$

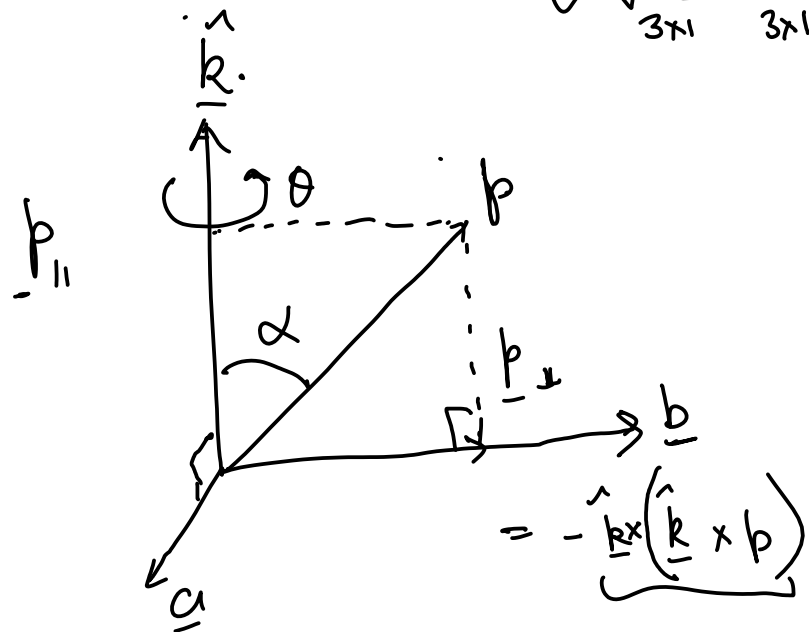
$$\begin{aligned} \underline{b} &= -(\hat{\underline{k}} \times \hat{\underline{k}}) \times \underline{p} \\ &= -(K \hat{\underline{p}}) \times \underline{p} \end{aligned}$$

$\underbrace{\quad}_{3 \times 3} \underbrace{\quad}_{3 \times 1} \underbrace{\quad}_{3 \times 1}$

Magnitude of  $\underline{p}_{\perp} = ?$   
in terms of  $\alpha$

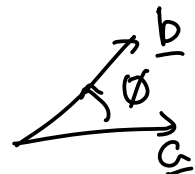
$$|\underline{p}_{\perp}| = |\underline{p}| \sin \alpha$$

$$|\underline{b}| = ?$$



$$\begin{aligned} |\underline{b}| &= |\hat{\underline{k}} \times \underline{p}| = |\hat{\underline{k}}| |\underline{p}| \sin \alpha \\ &= |\underline{p}| \sin \alpha \end{aligned}$$

$$|\underline{a} \times \underline{b}| = |\underline{a}| |\underline{b}| \sin \theta$$



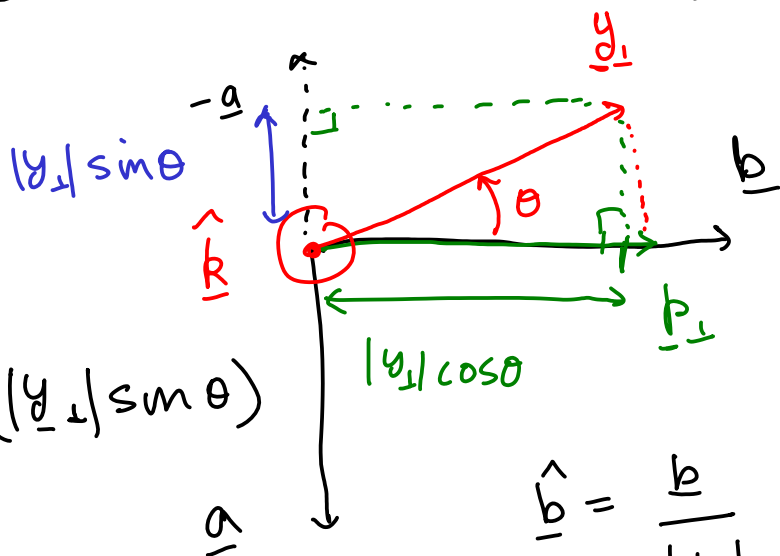
$$|\underline{b}| = |\hat{\underline{k}} \times \underline{a}| = |\hat{\underline{k}}| |\underline{a}| \sin(90^\circ) = |\underline{p}| \sin \alpha = |\underline{p}_{\perp}|$$

$$\underline{p}_\perp = \underline{b} = -\hat{\underline{k}} \times (\hat{\underline{k}} \times \underline{p})$$

$$\begin{aligned} \textcircled{1} \quad \underline{p} &= \underline{p}_{||} + \underline{p}_\perp \\ &= (\hat{\underline{k}}^T \underline{p}) \hat{\underline{k}} - \hat{\underline{k}} \times (\hat{\underline{k}} \times \underline{p}) \end{aligned}$$

② Rotate  $\underline{p}_\perp$  by  $\theta$  in the plane made by

$$|\underline{y}_\perp| = |\underline{p}_\perp|$$



$$\underline{y}_\perp = (|\underline{y}_\perp| \cos \theta) \hat{\underline{b}} + (-\hat{\underline{a}}) (|\underline{y}_\perp| \sin \theta)$$

$$\hat{\underline{b}} = \frac{\underline{b}}{|\underline{b}|}$$

$$\hat{\underline{a}} = \frac{\underline{a}}{|\underline{a}|}$$

$$= (|\underline{p}_\perp| \cos \theta) \hat{\underline{b}} + (-\hat{\underline{a}}) (|\underline{p}_\perp| \sin \theta)$$

$$= \underbrace{(|\underline{p}_\perp| \hat{\underline{b}})}_{\underline{b}} \cos \theta + (-\hat{\underline{a}}) (|\underline{p}_\perp| \sin \theta)$$

$$= \underline{b} \cos \theta - \underline{a} \sin \theta$$

$$= -\hat{\underline{k}} \times (\hat{\underline{k}} \times \underline{p}) \cos \theta + (\hat{\underline{k}} \times \underline{p}) \sin \theta$$

$$\begin{aligned} |\hat{\underline{a}}| |\underline{a}| &= \underline{a} \\ |\underline{p}_\perp| &= |\underline{p}| \sin \alpha \\ |\underline{a}| &= |\underline{p}| \sin \alpha \end{aligned}$$



$$\underline{y} = \underline{p}_{||} + \underline{p}_{\perp}$$

$$= \underbrace{(\hat{\underline{k}}^T \underline{p}) \hat{\underline{k}}} + (\hat{\underline{k}} \times \underline{p}) \sin \theta - \hat{\underline{k}} \times (\hat{\underline{k}} \times \underline{p}) \cos \theta$$

$$\underline{p} = \underline{p}_{||} + \underline{p}_{\perp} \Rightarrow \underline{p}_{||} = \underline{p} - \underline{p}_{\perp}$$

$$= \underline{p} + \hat{\underline{k}} \times (\hat{\underline{k}} \times \underline{p})$$

$$\underline{y} = \underline{p} + \underbrace{\hat{\underline{k}} \times (\hat{\underline{k}} \times \underline{p})} + (\hat{\underline{k}} \times \underline{p}) \sin \theta - \underbrace{\hat{\underline{k}} \times (\hat{\underline{k}} \times \underline{p})}_{\cos \theta}$$

$$= \underline{p} + (\hat{\underline{k}} \times \underline{p}) \sin \theta + (1 - \cos \theta) \hat{\underline{k}} \times (\hat{\underline{k}} \times \underline{p})$$

$$[\hat{\underline{k}} \times] = \underline{K} \quad \text{cross-product matrix}$$

$$= \underline{p} + \underline{K} \underline{p} \sin \theta + (1 - \cos \theta) \underline{K}^2 \underline{p}$$

$$= \underbrace{\left[ \underline{I}_{3 \times 3} + \underline{K} \sin \theta + (1 - \cos \theta) \underline{K}^2 \right]}_{\substack{R(\hat{\underline{k}}, \theta) \\ \text{axis} \quad \text{angle}}} \underline{p}$$

$A$  is angle representation

$R(\hat{k}, \theta)$

$(\hat{k}, \theta)$

?

?

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix} = \begin{bmatrix} I_{3 \times 3} + K \sin \theta + (1 - \cos \theta) K^2 \end{bmatrix}$$

$$\hat{k} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} \Rightarrow K = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix}$$

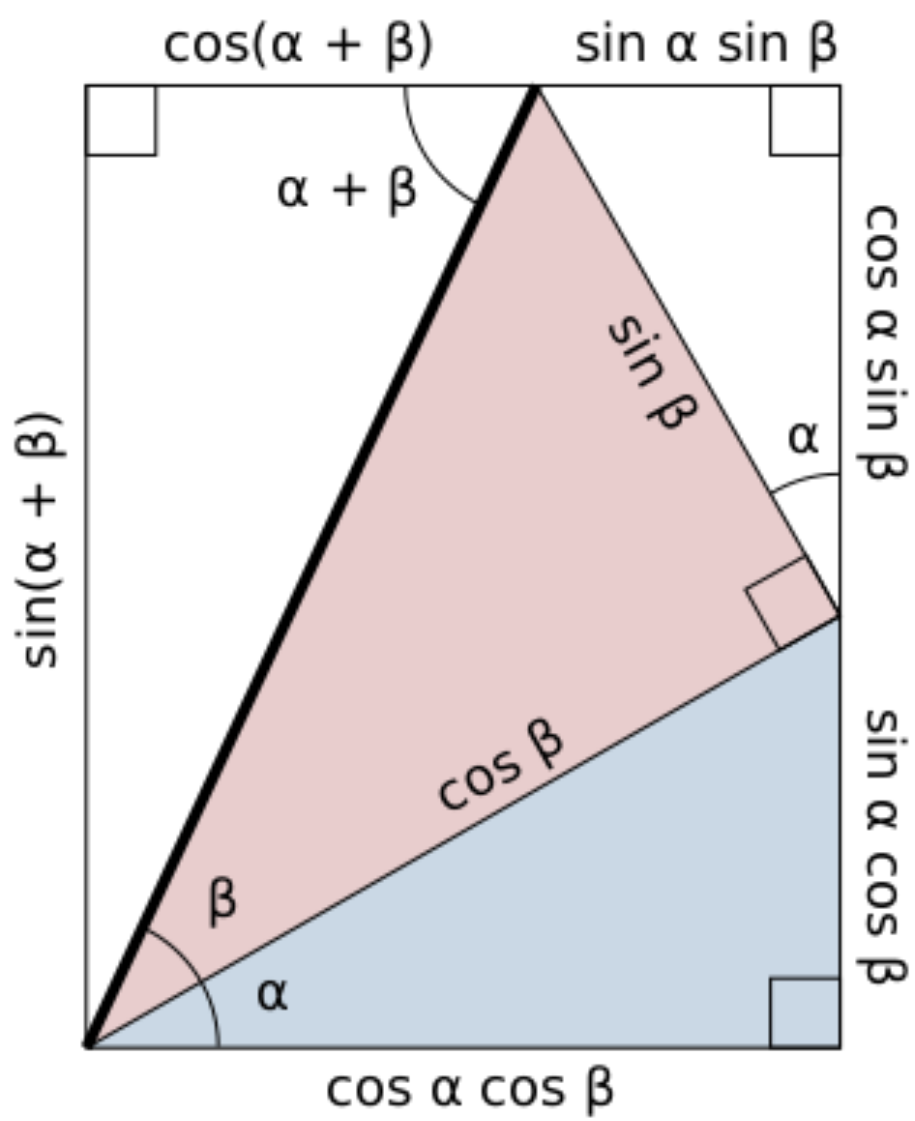
$$K^2 = \begin{bmatrix} -(k_z^2 + k_y^2) & k_y k_x & k_z k_x \\ k_y k_x & -(k_x^2 + k_z^2) & k_z k_y \\ k_z k_x & k_z k_y & -(k_x^2 + k_y^2) \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$I + K \sin \theta + (1 - \cos \theta) K^2$$

$$\begin{bmatrix} r_{11} & & \\ & r_{22} & \\ & & r_{33} \end{bmatrix} = \begin{bmatrix} 1 + 0 - (1 - \cos \theta)(k_z^2 + k_y^2) & & \\ & 1 - (1 - \cos \theta)(k_x^2 + k_z^2) & \\ & & 1 - (1 - \cos \theta)(k_x^2 + k_y^2) \end{bmatrix}$$

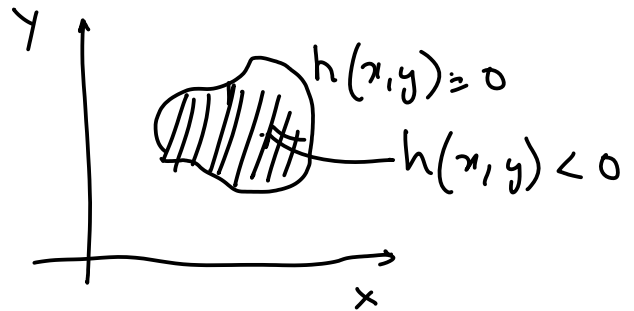
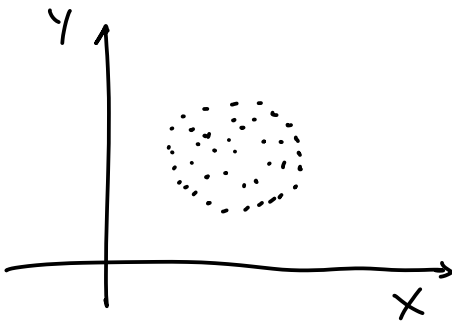
$$\text{trace}(R) = r_{11} + r_{22} + r_{33} = 2 \cos \theta (k_x^2 + k_y^2 + k_z^2) = 2 \omega \sin \theta$$



# 2D coordinate transforms

Rotations + translations  
Orientation + position

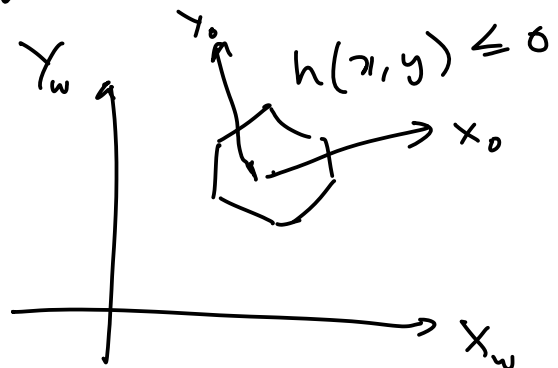
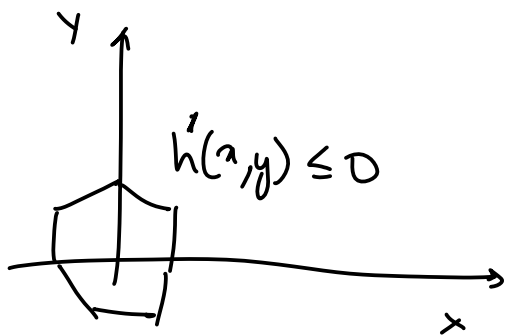
$$h(x,y) > 0$$



$$\text{Object} = \{ (x,y) \in \mathbb{R}^2 : h(x,y) \leq 0 \}$$

Set of all points  $(x,y) \in \mathbb{R}^2$   
such that  $h(x,y) \leq 0$

set of all real numbers

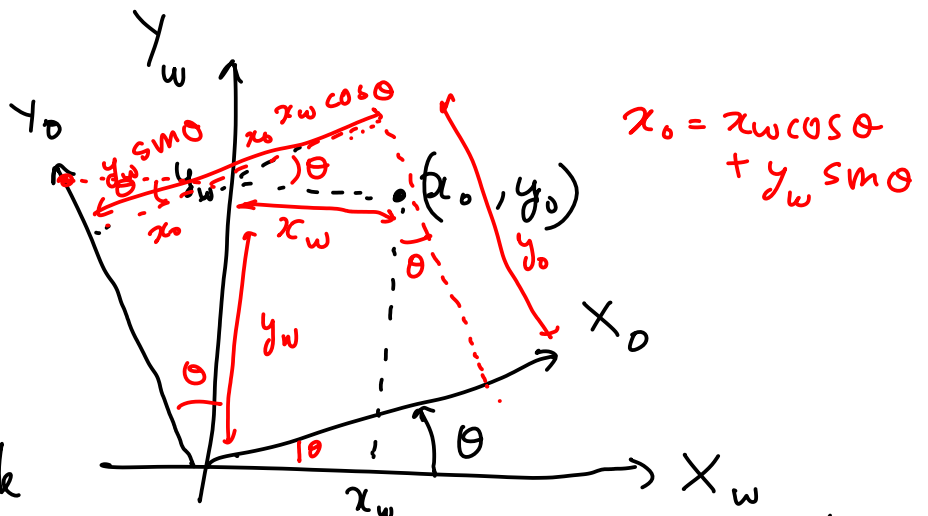


## 2D Rotations

- ① Rotation
- ② Translation

Problem

$(x_0, y_0)$  is given in  $(x_0, y_0)$  coordinate frame.  $(x_0, y_0)$  has been rotated by angle  $\theta$  w.r.t.  $(x_w, y_w)$



Find  $(x_w, y_w)$  in world coordinate frame

Proof using Basis vectors

In Linear algebra, Basis vectors are set of orthonormal unit vectors that span the entire space

Span is the set of all vectors that can be obtained by linear combinations of a given set of vectors

$$\text{Span} \{ \underset{\substack{\uparrow \\ \in \mathbb{R}^n}}{\underline{a}}, \underset{\substack{\uparrow \\ \in \mathbb{R}^n}}{\underline{b}} \} = \{ \underset{\substack{\uparrow \\ \in \mathbb{R}^n}}{\underline{\alpha a + \beta b}}, \underset{\substack{\uparrow \\ \in \mathbb{R}^n}}{\alpha \in \mathbb{R}, \beta \in \mathbb{R}} \}$$

Standard Basis vector.

For example, in  $\mathbb{R}^2$   
in  $\mathbb{R}^3$

$$\hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \hat{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$\hat{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \hat{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{in } \mathbb{R}^n \quad \hat{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n \times 1} \dots \hat{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}_{n \times 1}$$

Basis vectors for  $\mathbb{R}^n$

↳ ① All vectors must be perpendicular/orthogonal to each other

↳ ② They must be unit vectors

↳ ③ They must span the entire space  $\mathbb{R}^n$

Let Basis vector for  $(x_w, y_w)$  be standard basis vector

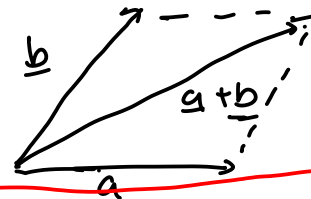
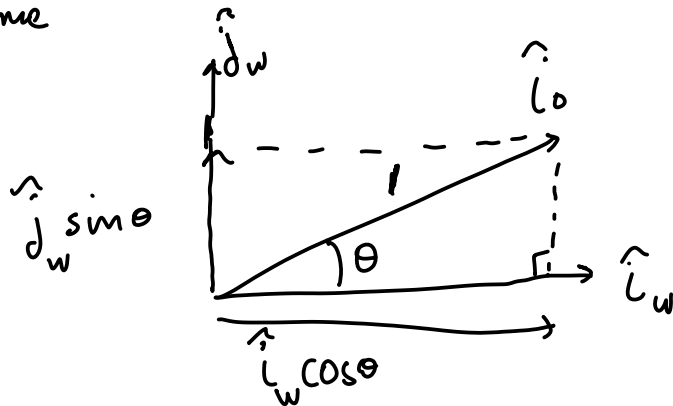
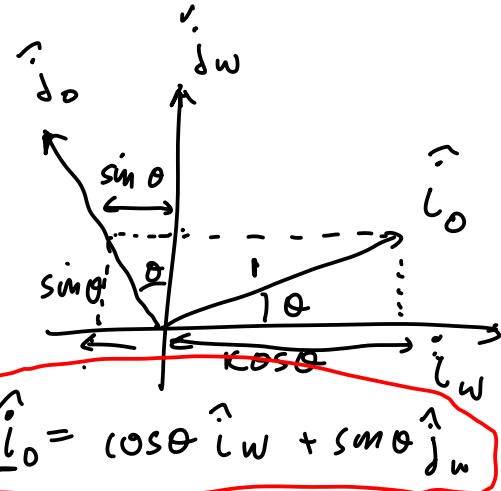
$$\hat{i}_w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \hat{j}_w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Any point  
in world  
coordinate  
frame

$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = x_w \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\hat{i}_w} + y_w \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\hat{j}_w}$$

Any point  
in the object  
coordinate  
frame

$$\begin{bmatrix} x_o \\ y_o \end{bmatrix} = x_o \hat{i}_o + y_o \hat{j}_o$$



$$\hat{j}_o = -\hat{i}_w \sin \theta + \hat{j}_w \cos \theta$$

world

object

$$x_w \hat{i}_w + y_w \hat{j}_w = x_o \hat{i}_o + y_o \hat{j}_o$$

$$= x_o [\cos \theta \hat{i}_w + \sin \theta \hat{j}_w] + y_o [-\hat{i}_w \sin \theta + \hat{j}_w \cos \theta]$$

$$= [x_o \cos \theta - y_o \sin \theta] \hat{i}_w + [x_o \sin \theta + y_o \cos \theta] \hat{j}_w$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

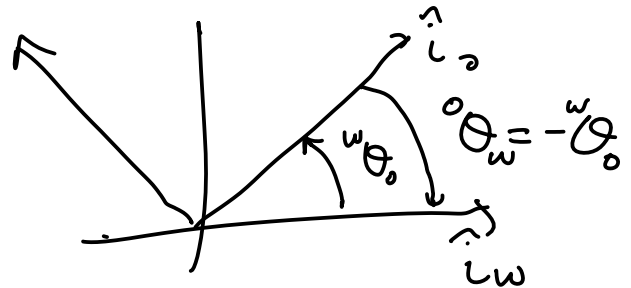
$$= \begin{bmatrix} x_o \cos \theta - y_o \sin \theta \\ x_o \sin \theta + y_o \cos \theta \end{bmatrix}$$

$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x_o \\ y_o \end{bmatrix}$$

Because  
we are  
using  
standard  
basis  
for world  
coordinate  
frame

$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \begin{bmatrix} {}^w R_0(\theta) \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$${}^w R_0(\theta) := \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$



$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_w \\ y_w \end{bmatrix}$$

$${}^0 R_w(\theta) = {}^w R_0(-\theta) = {}^w R_0^T(\theta)$$

$$= \begin{bmatrix} \cos \theta & +\sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x_w \\ y_w \end{bmatrix}$$

$$\begin{bmatrix} {}^w R_0^T \end{bmatrix} {}^w R_0 = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 \theta + \sin^2 \theta & -c(\theta)s(\theta) + s(\theta)c(\theta) \\ -s(\theta)c(\theta) + c(\theta)s(\theta) & s^2(\theta) + c^2(\theta) \end{bmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_{2 \times 2}$$

$$\underline{R^T R = I}$$

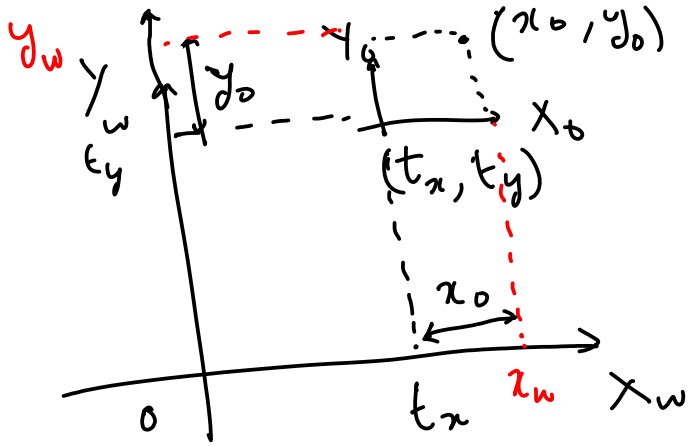
$$(A^{-1})A = I$$

$$R^{-1} = R^T$$



## 2D Translation

$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \underbrace{\begin{bmatrix} t_x \\ t_y \end{bmatrix}}_{\text{translation vector}} + \begin{bmatrix} x_o \\ y_o \end{bmatrix}$$

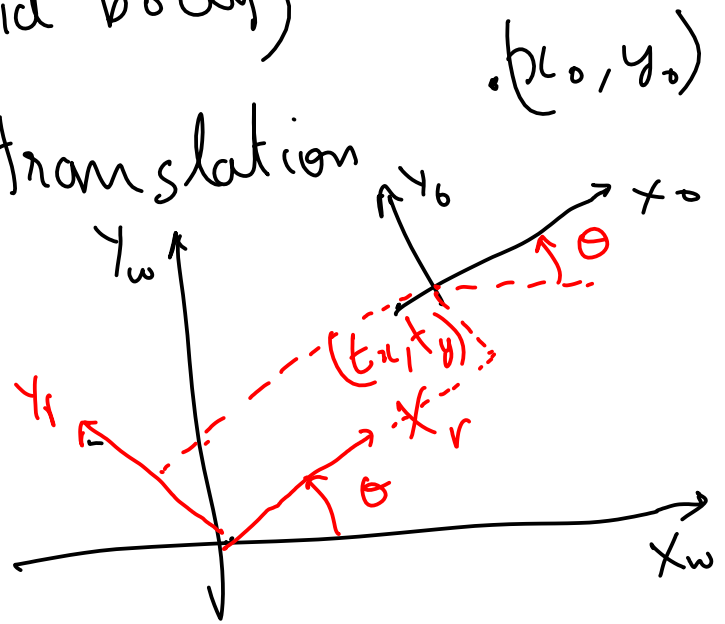


## 2D transformation (Rigid body)

Rotation followed by translation

$$\begin{bmatrix} x_r \\ y_r \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x_o \\ y_o \end{bmatrix}$$

$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \begin{bmatrix} t_x \\ t_y \end{bmatrix} + \begin{bmatrix} x_r \\ y_r \end{bmatrix}$$



$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x_o \\ y_o \end{bmatrix} + \underbrace{\begin{bmatrix} t_x \\ t_y \end{bmatrix}}_{\underline{t}}$$

$$\underline{x}_w = {}^w_0 R(\theta) \underline{x}_o + {}^w_0 \underline{t}$$

$$\underline{x}_0 = ? (\underline{x}_w)$$

$$\rightarrow \underline{x}_w - {}^w_0 \underline{t} = {}^w_0 R \underline{x}_0$$

$${}^w_0 R = {}^w_0 R(0)$$

Multiply on the left of both sides by  ${}^w_0 R^T$

$$\rightarrow {}^w_0 R^T (\underline{x}_w - {}^w_0 \underline{t}) = ({}^w_0 R^T {}^w_0 R) \underline{x}_0$$

$$\frac{\underline{x}_w - {}^w_0 \underline{t}}{{}^w_0 R} = \underline{x}_0$$

NEVER EVER  
DO THIS

$$\underline{x}_0 = {}^w_0 R^T \underline{x}_w - {}^w_0 R^T {}^w_0 \underline{t} \quad \text{--- (1)}$$

$$\underline{x}_0 = {}^0_w R \cdot \underline{x}_w + {}^0_w \underline{t} \quad \text{--- (2)}$$

Compare (1) and (2)

$${}^0_w R = {}^w_0 R^T$$

and

$${}^0_w \underline{t} = - {}^w_0 R^T {}^w_0 \underline{t}$$

$$\underline{x}_w = {}^w_0 R \underline{x}_0 + {}^w_0 \underline{t}$$

$$\begin{bmatrix} \underline{x}_w \\ 1 \end{bmatrix} = \begin{bmatrix} x_w \\ y_w \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} {}^wR_{02 \times 2} & {}^w\vec{t}_{0-2 \times 1} \\ O_{1 \times 2}^T & 1 \end{bmatrix}}_{\substack{{}^wT_0 \\ 3 \times 3}} \begin{bmatrix} \underline{x}_0_{2 \times 1} \\ 1 \end{bmatrix}$$

Block matrix

Block Matrix multiplication

$$\checkmark \quad \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{bmatrix}$$

$$\begin{bmatrix} {}^wR_{02 \times 2} & {}^w\vec{t}_{0-2 \times 1} \\ O_{1 \times 2}^T & 1 \end{bmatrix} \begin{bmatrix} \underline{x}_0_{2 \times 1} \\ 1 \end{bmatrix} = \begin{bmatrix} {}^wR_{02 \times 2} \underline{x}_0 + {}^w\vec{t}_0 \\ O_{1 \times 2}^T \underline{x}_0 + 1 \end{bmatrix}$$

$$= \begin{bmatrix} {}^wR_{02 \times 2} \underline{x}_0 + {}^w\vec{t}_0 \\ 1 \end{bmatrix} = \begin{bmatrix} \underline{x}_w \\ 1 \end{bmatrix}$$

$$\underline{x}_w = {}^wT_0 \underline{x}_0$$

$${}^wT_0 = \begin{bmatrix} {}^wR_{02 \times 2} & {}^w\vec{t}_{0-2 \times 1} \\ O_{1 \times 2}^T & 1 \end{bmatrix}$$

Transformation matrix

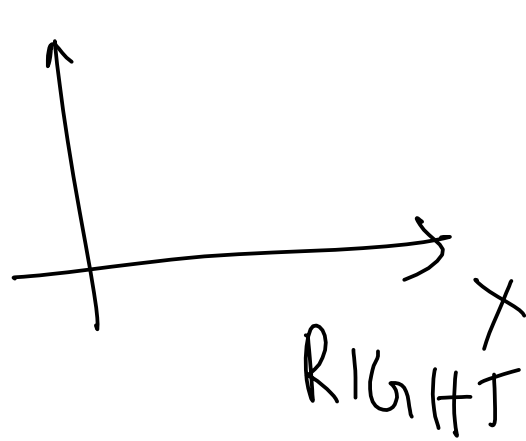
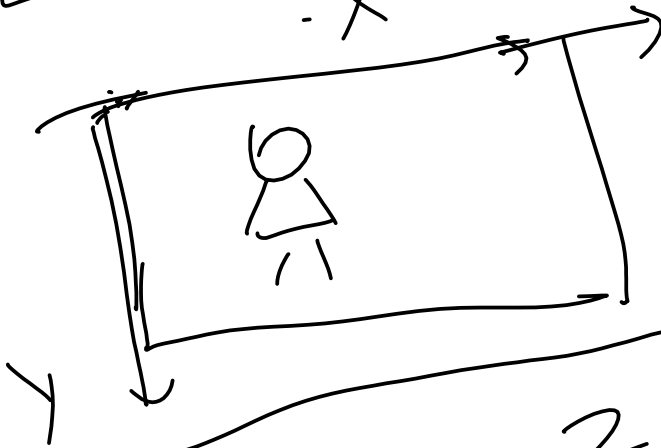
In 2D  
and  
3D

Right hand ✓

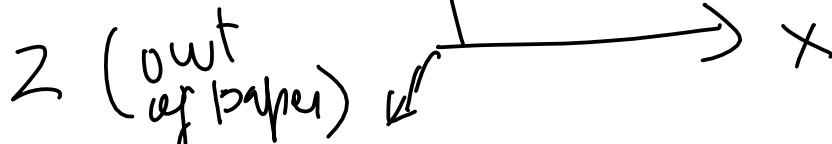
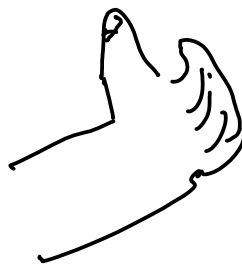
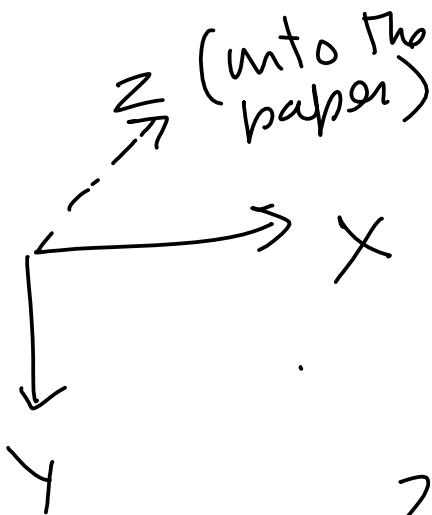
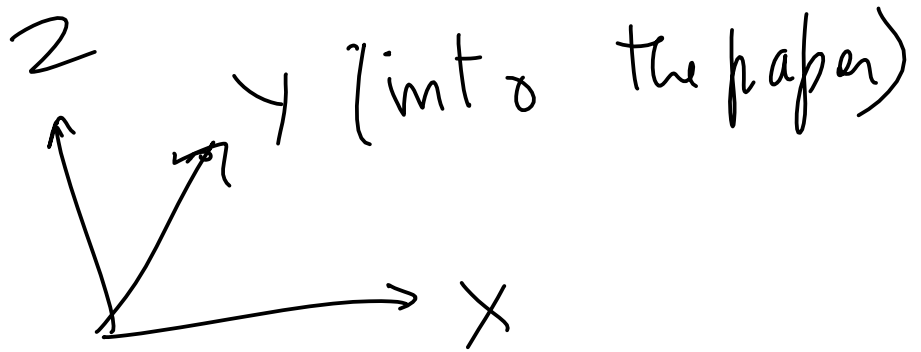
Left hand ✗

LEFT

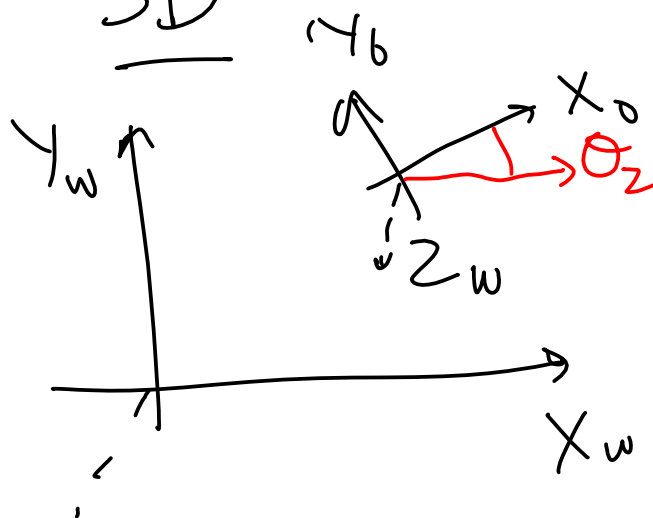
-X



In 3D



Extending 2D to 3D



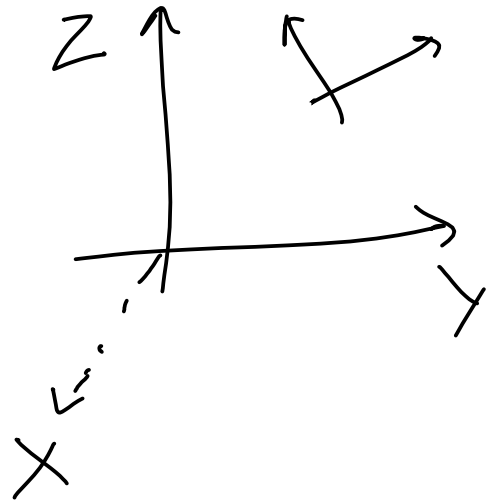
$\hat{z}_w$  (out of the paper)

$$\begin{bmatrix} x_w \\ y_w \\ z_w = 0 \end{bmatrix} = \begin{bmatrix} \cos \theta_2 & -\sin \theta_2 & 0 \\ \sin \theta_2 & \cos \theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ 0 \end{bmatrix}$$

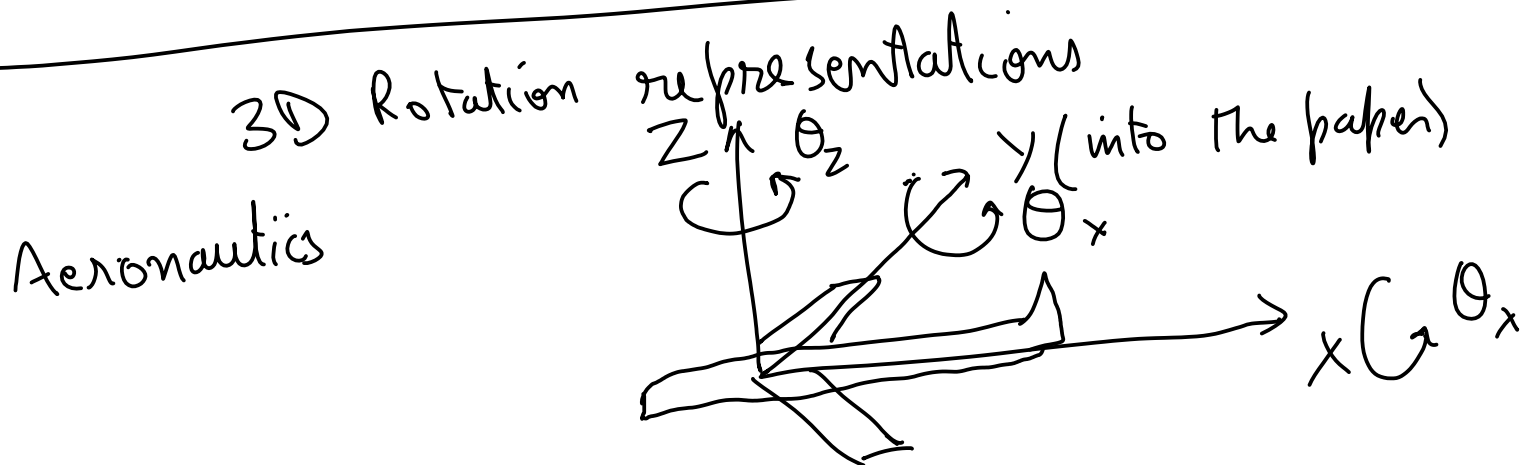
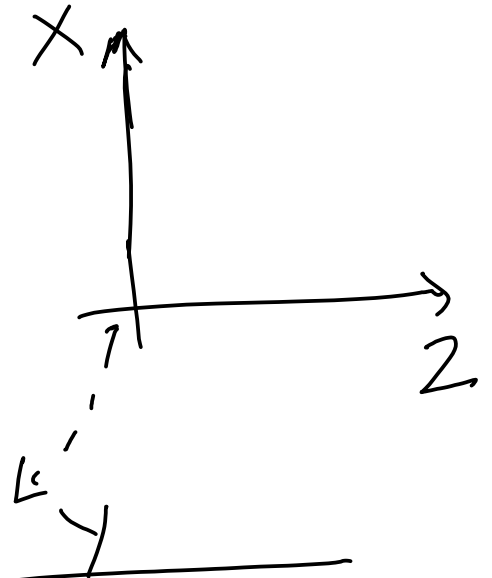
Rotation along Z-axis changes  
only X-Y coordinates

$$R(\theta_2) = \begin{bmatrix} \cos \theta_2 & -\sin \theta_2 & 0 \\ \sin \theta_2 & \cos \theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R(\theta_x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}$$



$$R(\theta_y) = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$$

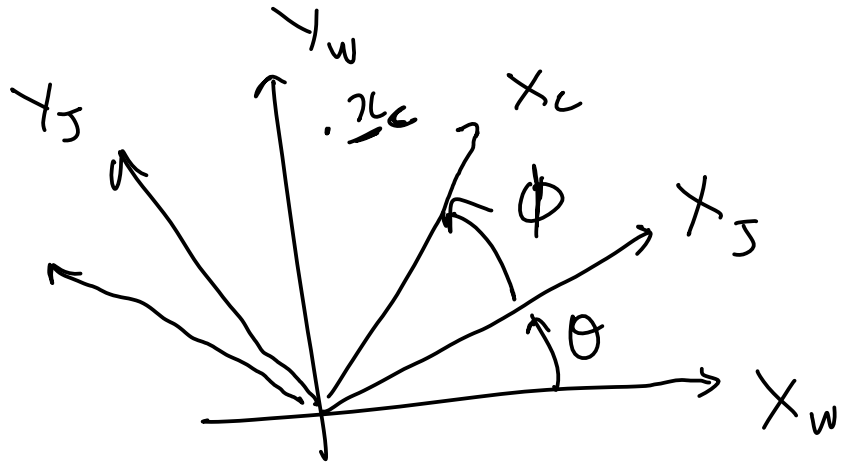


$\theta_x = \text{roll}$   
 $\theta_y = \text{pitch}$   
 $\theta_z = \text{yaw}$

$$R = R(\theta_z) R(\theta_y) R(\theta_x)$$

$\uparrow$  Yaw     $\uparrow$  Pitch     $\uparrow$  Roll

# Chain rotation, translation, transformations



$$\underline{x}_j = {}^j_c R(\phi) \underline{x}_c$$

$$\underline{x}_w = {}^w_j R(\theta) \underline{x}_j$$

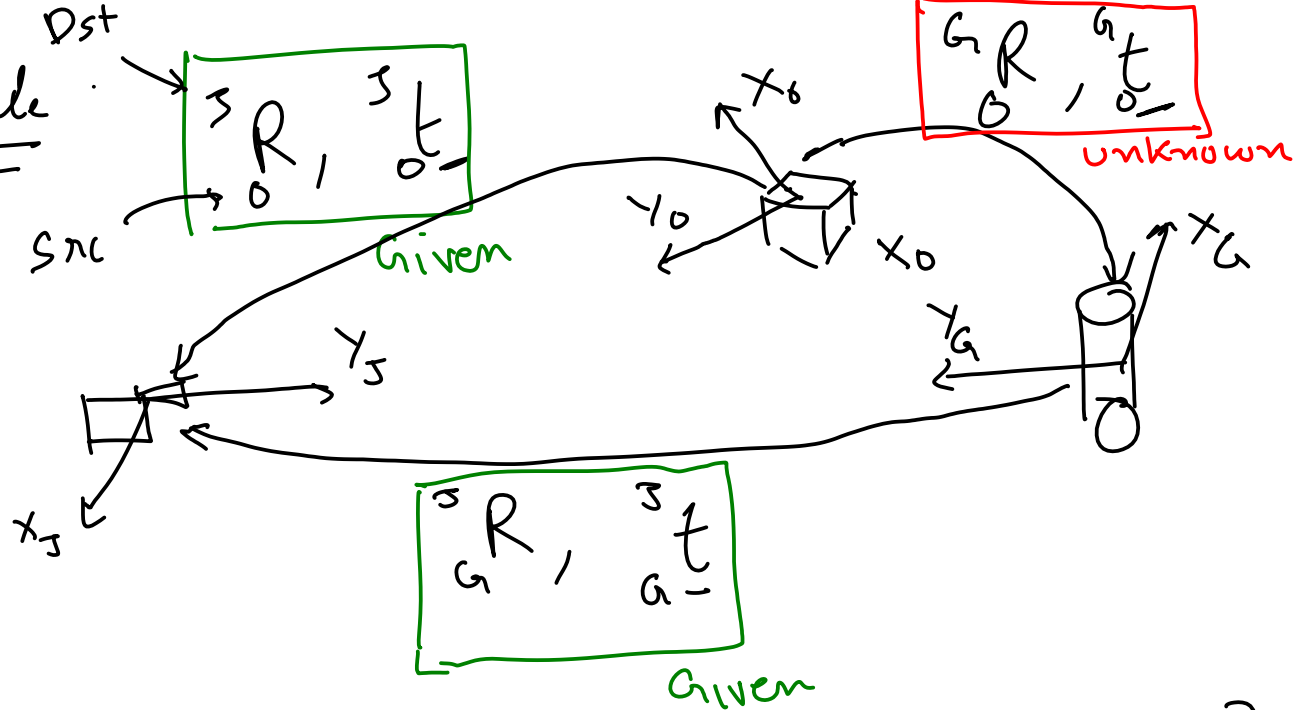
$$= R(\theta) \left[ R(\phi) \underline{x}_c \right]$$

$$= \underbrace{\left( R(\theta) R(\phi) \right)}_{{}^w_c R} \underline{x}_c$$

${}^w_c R$

$${}^w_c R = {}^w_j R(\theta) {}^j_c R(\phi)$$

$${}^w_c T = {}^w_j T {}^j_c T$$

Example

$${}^J_0T = \begin{bmatrix} {}^3R & {}^3t \\ 0^T & 1 \end{bmatrix}, \quad {}^S_A T = \begin{bmatrix} {}^3R & {}^3t \\ 0^T & 1 \end{bmatrix}$$

$${}^G_0T = \begin{bmatrix} {}^6R & {}^6t \\ 0^T & 1 \end{bmatrix}, \quad {}^G_0T = ({}^S_0T^{-1}) {}^S_0T$$

$$= {}^G_3T {}^J_0T \dots$$

$$\underbrace{{}^G_0T}_{\mathcal{L}_G} \underbrace{\chi_6}_{\mathcal{L}_G} = \left( {}^G_3T \left( {}^J_0T \chi_0 \right) \right)$$

$$\underbrace{\quad}_{\mathcal{L}_G}$$



①  $R_{3D} = R(\theta_z) R(\theta_y) R(\theta_x)$

$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{yaw} & \text{pitch} & \text{roll} \end{array}$

This sequence

$\begin{array}{c} XYZ \\ \hline ZYX \\ \vdots \end{array}$

6 possible

$= 3!$

$\dot{\theta}_x \xrightarrow{\text{then}} \theta_y$   
 $\downarrow \text{then}$   
 $\theta_z$

Euler angle representation of 3D rotation is a sequence of rotation around standard axis

Euler representation with XYZ then

$R_{3D} = R(\theta_z) R(\theta_y) R(\theta_x)$

Conversion from Euler angles to Rotation matrix

How to do The opposite?

convert from Rotation matrix to Euler angles?

$$R(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}$$

$$\theta = ? = \tan^{-1} \left( \frac{r_{21}}{r_{11}} \right) \in \left[ -\frac{\pi}{2}, \frac{\pi}{2} \right]$$

$$\theta = \arctan 2(r_{21}, r_{11}) \in (-\pi, \pi)$$

$$\begin{array}{c} 3D \\ \downarrow^z \quad \downarrow^y \quad \downarrow^x \\ R = R(\psi) R(\phi) R(\theta) \end{array}$$

$$= \begin{bmatrix} c(\psi) & -s(\psi) & 0 \\ s(\psi) & c(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c(\phi) & 0 & s(\phi) \\ 0 & 1 & 0 \\ -s(\phi) & 0 & c(\phi) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c(\theta) & -s(\theta) \\ 0 & s(\theta) & c(\theta) \end{bmatrix}$$

$$= \begin{bmatrix} \boxed{c(\psi)c(\phi)} & c(\psi)c(\phi)s(\theta) & c(\psi)s(\phi)c(\theta) + s(\psi)s(\theta) \\ \boxed{s(\psi)c(\phi)} & s(\psi)s(\phi)s(\theta) + c(\psi)c(\theta) & s(\psi)s(\phi)c(\theta) - c(\psi)s(\theta) \\ \boxed{-s(\phi)} & \boxed{c(\phi)s(\theta)} & \boxed{c(\phi)c(\theta)} \end{bmatrix}$$

$$= \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

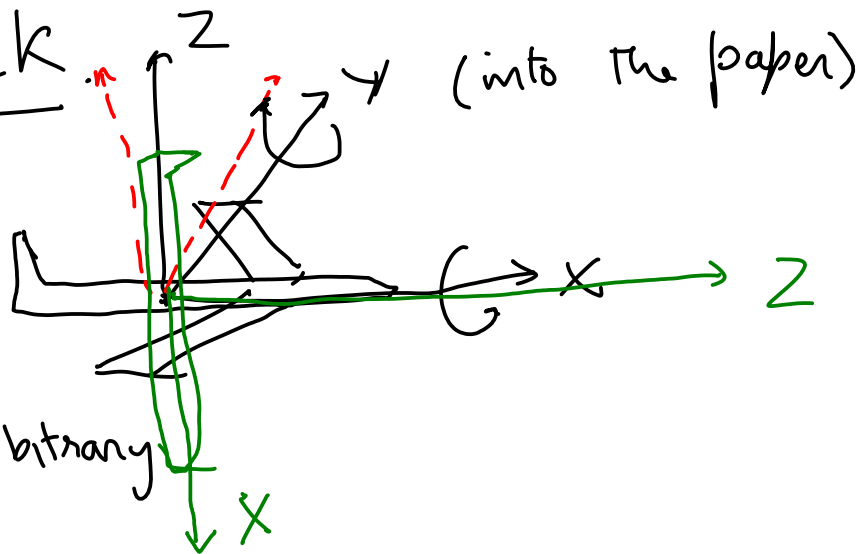
$$\phi = -\sin^{-1}(r_{31}) \in [0, \pi]$$

$$\frac{r_{21}}{r_{11}} = \frac{\sin(\psi) \cancel{c(\phi)}}{\cos(\psi) \cancel{c(\phi)}} \Rightarrow \psi = \arctan 2(r_{21}, r_{11})$$

$$\theta = \arctan 2(r_{32}, r_{33})$$

conversion from Rotation matrix to Euler angles

Gimbal lock



$$\theta_x = 30^\circ \leftarrow \text{arbitrary}$$

$$[\theta_y = \underline{90^\circ}]$$

$$\theta_z = 45^\circ \leftarrow \text{arbitrary}$$

Euler angles  $\xrightarrow{\text{deterministically}}$  Rot mat  
 $\nwarrow$  multiple solutions

Other representations. It is impossible to unambiguously represent 3D rotation with only 3 number

Degree of freedom

but needs 4 numbers  
+ 1 constraint  
to represent it

3D rot = 3 DOF

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② Axis-angle representation ← (3) Quaternions

↓

Rot mats

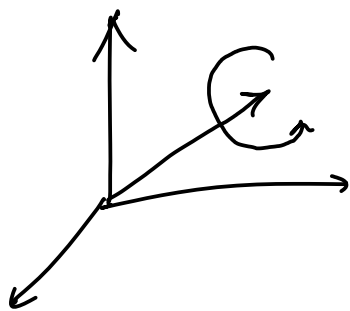
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Algebra

Quaternions algebra  
≡ complex numbers

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② Axis-angle representation



Any 3D rotation can be represented  
as a unit vector (axis) and rotation angle  
around it.

$$\text{Axis} = \underline{a} = [a_x, a_y, a_z]$$

$$\text{angle} = \theta$$

$$\text{constraint: } \|a\|_2 = \sqrt{a_x^2 + a_y^2 + a_z^2} = 1$$

$$\left. \begin{array}{l} \text{Free scalars} = 4 \\ \text{Constraint} = 1 \end{array} \right\} \text{Degree of freedom} = 3$$

DOF of 2D Rot matrix = ?

$$R = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

Free scalars = 4

Two vector constraints :

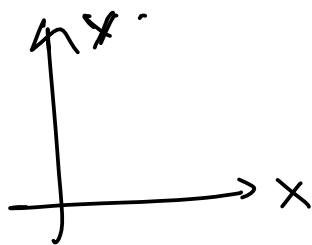
$$\left[ \begin{array}{l} R^T R = I = R R^T \\ \det(R) = +1 \end{array} \right]$$

2 scalar constraints

1 scalar constraint

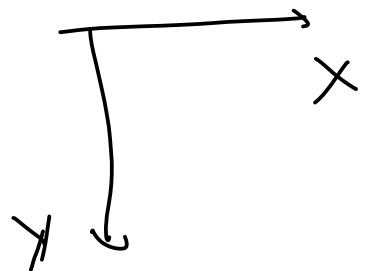
Reflection matrices also satisfy

Check for valid rotation



not Rotation

$$\begin{array}{c} \xrightarrow{\text{reflection}} \\ \det(\text{Reflection}) \\ = -1 \end{array}$$



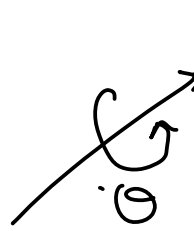
Rodrigues rotation formula

Axis angle  $\rightarrow$  Rot matrix  
 $(\theta, \hat{k})$

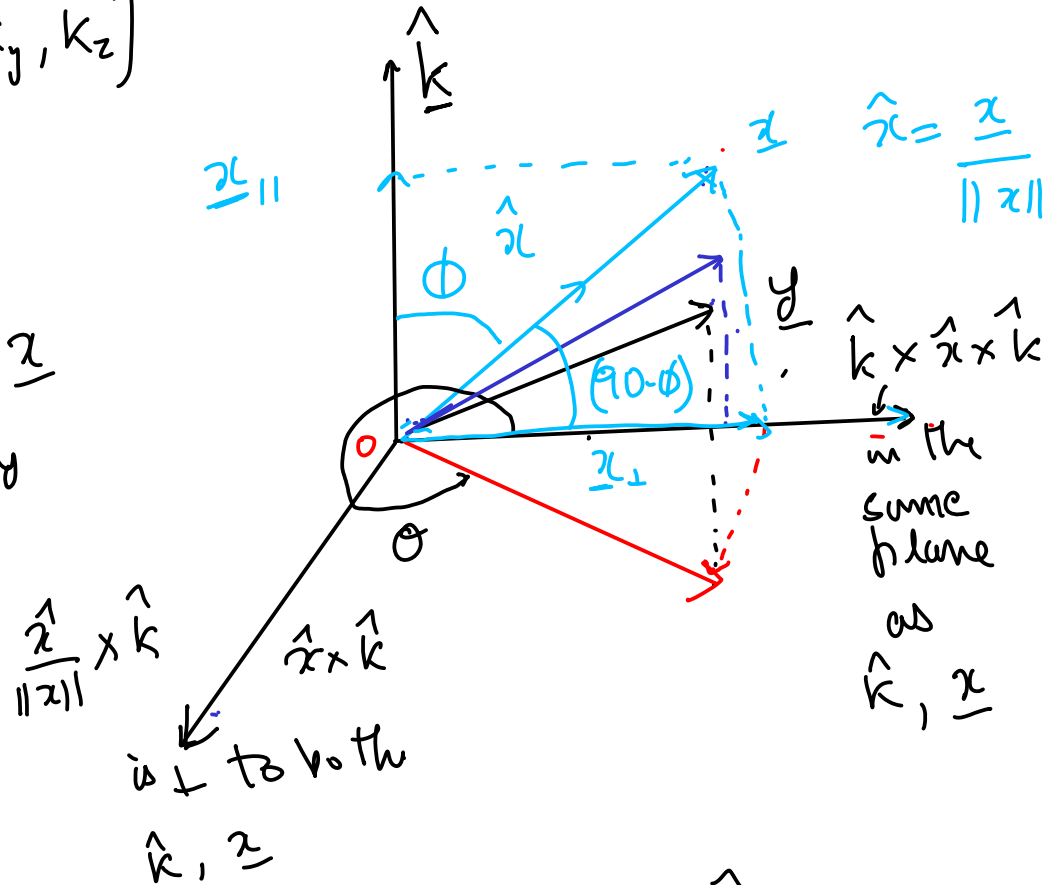
$$R = I + \sin \theta [k_x] + (1 - \cos \theta) [k_x]^2$$

axis angle representation  $\longrightarrow$  Rotation matrix (3D)  
 Rodrigues rotation formula

$\hat{k} = [k_x, k_y, k_z]$



$\underline{y} =$  by rotating  $\underline{x}$  around  $\hat{k}$  by angle  $\theta$



In the plane of  $\hat{k}$  and  $\hat{k} \times \hat{x} \times \hat{k}$ ,  $\underline{x}$  can be projected into two component

$$\underline{x} = \underline{x}_{||} + \underline{x}_{\perp}$$

$$\underline{x}_{||} = (\hat{k} \cdot \underline{x}) \hat{k}$$

$$\underline{x}_{\perp} = \left[ \left( \hat{k} \times (\hat{x} \times \hat{k}) \cdot \underline{x} \right) (\hat{k} \times \hat{x} \times \hat{k}) \right]$$

$$\underline{x}_{\perp} = \hat{k} \times (\underline{x} \times \hat{k}) = \hat{k} \times (-\hat{k} \times \underline{x}) = \underline{x} \times \hat{k}$$

$\hat{k} \cdot \underline{x} = |\hat{k}| |\underline{x}| \cos \phi$

$$\rightarrow |\underline{x} \times \hat{k}| = |\underline{x}| |\hat{k}| \sin \phi$$

$$\rightarrow \underline{x} \times \hat{k} = (\hat{x} \times \hat{k}) (|\underline{x}| \sin \phi)$$

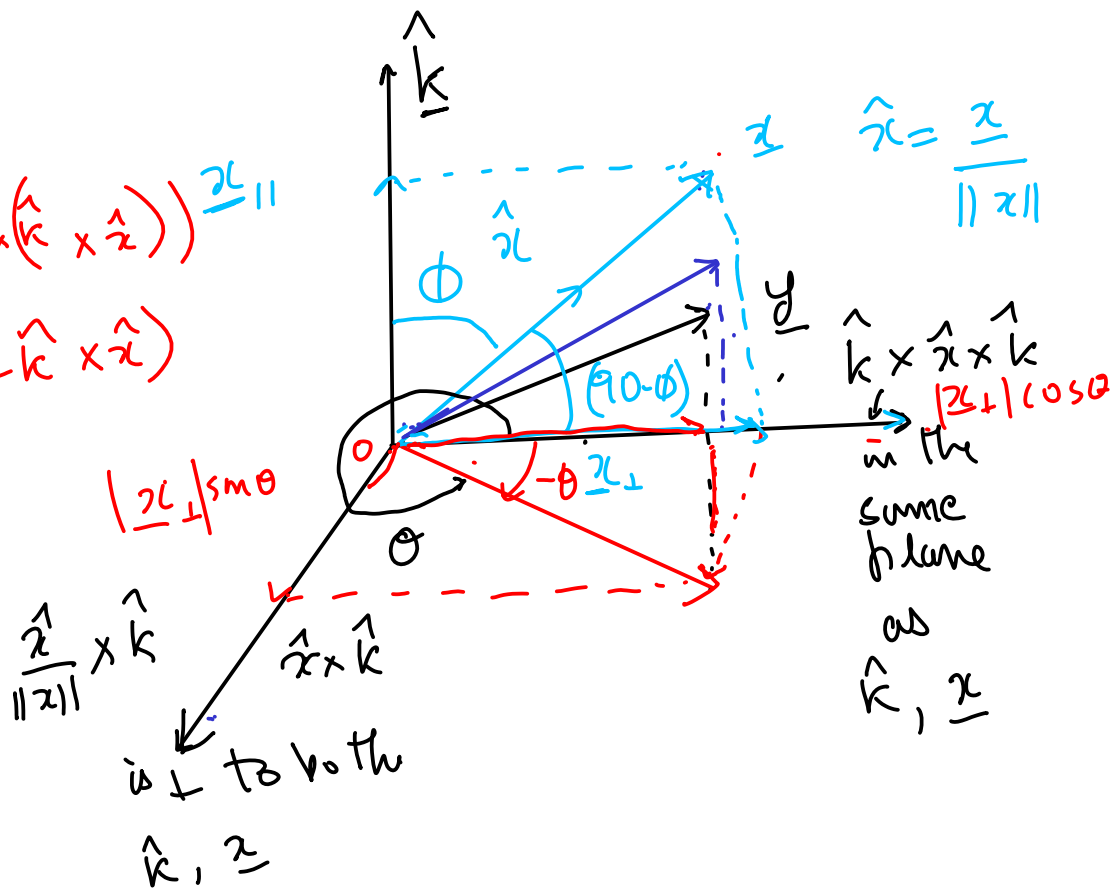
$$= |\underline{x}| \cos \phi$$

$$|\underline{x}| \sin \phi = |\underline{x}| \sqrt{1 - \cos^2 \phi}$$

$$\underline{y} = \underline{x}_{||} + \underline{x}_{\perp \text{rot}}$$

$$\underline{x}_{\perp \text{rot}} = |\underline{x}_{\perp}| \cos(\theta) (\hat{k} \times (\hat{k} \times \hat{x})) \quad \underline{x}_{||}$$

$$+ |\underline{x}_{\perp}| \sin(\theta) (-\hat{k} \times \hat{x})$$



$$\underline{x}_{\perp \text{rot}} = \underline{x}_{\perp} \cos \theta (-\hat{k} \times (\hat{k} \times \hat{x}))$$

$$- \underline{x}_{\perp} \sin \theta (-\hat{k} \times \hat{x})$$

$$= \cos \theta (-\hat{k} \times (\hat{k} \times \underline{x}))$$

$$- \sin \theta (-\hat{k} \times \underline{x})$$

$$= \sin \theta (\hat{k} \times \underline{x})$$

$$- \cos \theta (\hat{k} \times (\hat{k} \times \underline{x}))$$

$$(\hat{k} \times \hat{x}) |\underline{x}_{\perp}| =$$

$$= (\hat{k} \times \hat{x}) |\underline{x}| \sin \theta = \hat{k} \times \underline{x}$$

$$|\underline{a} \times \underline{b}| = |\underline{a}| |\underline{b}| \sin \phi$$

$$\underline{y} = \underline{x}_{||} + \underline{x}_{\perp \text{rot}}$$

Rodrigues' formula

$$\underline{y} = (\hat{k} \cdot \underline{x}) \hat{k} + \sin \theta (\hat{k} \times \underline{x}) - \cos \theta (\hat{k} \times \hat{k} \times \underline{x})$$

$$x_{||} = (\hat{k} \cdot \underline{x}) \hat{k}$$

$$\underline{x} = \underline{x}_{||} + \underline{x}_{\perp}$$

$$= \underline{x} - \underline{x}_{\perp}$$

$$= \underline{x} - \left( -\hat{k} \times (\hat{k} \times \underline{x}) \right)$$

$$= \underline{x} + \hat{k} \times (\hat{k} \times \underline{x})$$

$$\underline{y} = \underline{x} + \hat{k} \times (\hat{k} \times \underline{x}) + \sin \theta (\hat{k} \times \underline{x}) - \cos \theta (\hat{k} \times (\hat{k} \times \underline{x}))$$

$$\underline{y} = \underline{x} + \sin \theta (\hat{k} \times \underline{x}) + (1 - \cos \theta) (\hat{k} \times (\hat{k} \times \underline{x}))$$

$$\underline{y} = \underbrace{R(\theta, \hat{k})}_{\text{matrix}} \underline{x}$$

Writing cross product using matrix notation

$$\underline{a} \times \underline{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$= \hat{i}(a_y b_z - b_y a_z) + \hat{j}(a_z b_x - b_z a_x) + \hat{k}(a_x b_y - b_x a_y)$$

$$\underline{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

$$\underline{b} = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

$$= \begin{bmatrix} a_y b_z - b_y a_z \\ -a_x b_z + b_x a_z \\ a_x b_y - b_x a_y \end{bmatrix}$$



$$\underbrace{\begin{bmatrix} a_y b_z - b_y a_z \\ -a_x b_z + b_x a_z \\ a_x b_y + b_x a_y \end{bmatrix}}_{\underline{a} \times \underline{b}} = \underbrace{\begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ a_y & a_x & 0 \end{bmatrix}}_{[\underline{a}_x]} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

Cross product matrix  
of vector

$$\hat{k} \times \underline{x} = \underbrace{[\underline{k}_x]}_K \underline{x} = K \underline{x}$$

$$\hat{k} \times (\hat{k} \times \underline{x}) = [\underline{k}_x] ([\underline{k}_x] \underline{x}) = K(K \underline{x}) = K^2 \underline{x}$$

$$\begin{aligned} \underline{y} &= \underline{x} + \sin \theta (K \underline{x}) + (1 - \cos \theta) K^2 \underline{x} \\ &= \underbrace{\left[ \underline{I}_{3 \times 3} + \sin \theta K + (1 - \cos \theta) K^2 \right]}_{R(\theta, \hat{k})} \underline{x} \end{aligned}$$

Oct 25

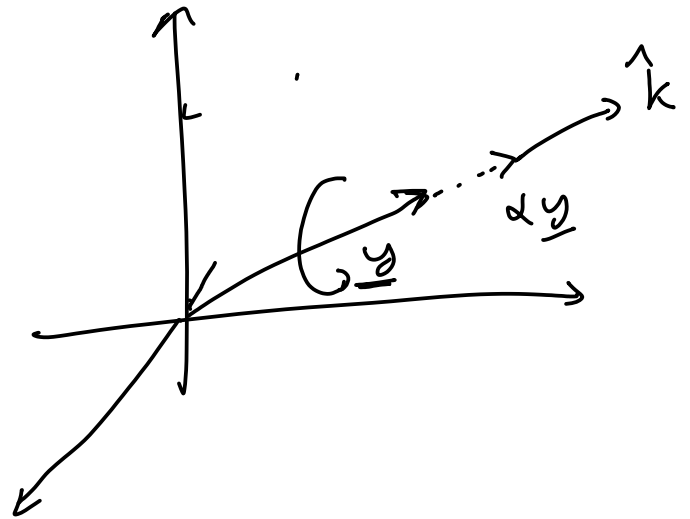
Convert. from Rot matrix to axis angle representation

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$$\underline{\alpha} \underline{y} = R \underline{y}$$

$R$  is  $3 \times 3$  matrix  
 $\underline{y}$  is  $3 \times 1$  vector  
 $\alpha$  is scalar

$\rightarrow \underline{A} \underline{v} = \lambda \underline{v}$   
Eigen vectors  
of a matrix  $A$   
are all the solution  
for  $\underline{v}$  from the  
above equation



Corresponding solutions  
of  $\lambda$  are called eigen  
values

$$\underline{A} \underline{v} - \lambda \underline{v} = 0$$

$$\Rightarrow \underbrace{(\underline{A} - \lambda \underline{I})}_{\text{matrix}} \underbrace{\underline{v}}_{\text{vector}} = 0$$

$\det(\underline{A} - \lambda \underline{I}) = 0 \} \rightarrow$  solve for eigen value

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The axis of rotation is an eigen vector of  
the rotation matrix.

$$\underline{y} = R \underline{x} \Rightarrow \|\underline{y}\| = \|\underline{x}\|$$

$$\|\underline{y}\| = \sqrt{y_1^2 + y_2^2 + \dots + y_n^2}$$

$$= \sqrt{\begin{bmatrix} y_1 & y_2 & \dots & y_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}$$

$$= \sqrt{\underline{y}^T \underline{y}}$$

$$\underline{y}^T \underline{y} = (\underline{R} \underline{x})^T (\underline{R} \underline{x})$$

$$= (\underline{x}^T \underline{R}^T) (\underline{R} \underline{x})$$

$$= \underline{x}^T (\underline{R}^T \underline{R}) \underline{x}$$

$$= \underline{x}^T \underline{x}$$

eigen value  $\downarrow$  eigen vector  $\downarrow$

$$\underline{1} \underline{k} = \underline{R} \underline{k}$$

$\underline{y} \quad \underline{k}$  is along the axis of rotation for  $\underline{R}$

The axis of rotation is the eigen vector of the rotation matrix corresponding to eigen value 1.

$$\boxed{\det(\underline{R} - \underline{I}) = 0}$$

Use `numpy.linalg.eig()` to find eigen value and eigen vector

For  $3 \times 3$  Rotation matrix

if  $\theta = 0^\circ$  or  $180^\circ$

special case

else ( $\theta \neq 0, \theta \neq 180^\circ$ )

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\hat{k} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} = \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} \frac{1}{(2\sin\theta)}$$

How to compute angle  $\theta$  in axis-angle

$$R = I + K \sin\theta + (1 - \cos\theta) K^2$$

$$K^2 = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix} \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix}$$

A is a Symmetric matrix if  $A^T = A$   $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$

A is a skew-symmetric matrix if  $A^T = -A$

$$\begin{bmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & 0 \\ -a_{13} & 0 & 0 \end{bmatrix}$$

$$K^2 = \begin{bmatrix} -(k_x^2 + k_y^2) & k_y k_x & k_z k_x \\ k_y k_x & -(k_x^2 + k_z^2) & k_z k_y \\ k_z k_x & k_z k_y & -(k_x^2 + k_y^2) \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sin \theta +$$

$$\begin{bmatrix} -(k_x^2 + k_y^2) & k_y k_x & k_z k_x \\ k_y k_x & -(k_x^2 + k_z^2) & k_z k_y \\ k_z k_x & k_z k_y & -(k_x^2 + k_y^2) \end{bmatrix} (1 - \cos \theta)$$

$$\hat{k} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} \text{ is a unit vector}$$

$$k_x^2 + k_y^2 + k_z^2 = 1$$

$$R = \begin{bmatrix} r_{11} & & \\ & r_{22} & \\ & & r_{33} \end{bmatrix}$$

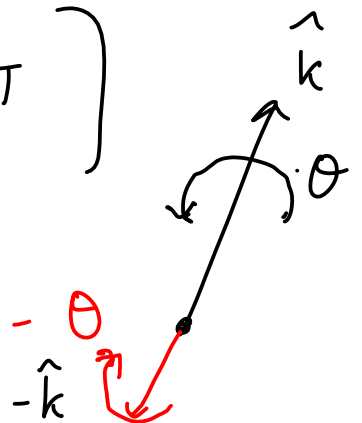
$$\text{trace}(R) = r_{11} + r_{22} + r_{33} = 1 + 1 + 1 - 2 \underbrace{(k_x^2 + k_y^2 + k_z^2)}_1 (1 - \cos \theta)$$

$$\Rightarrow \text{tr}(R) = r_{11} + r_{22} + r_{33} = 3 - 2 + 2 \cos \theta$$

$$\Rightarrow \text{tr}(R) = 1 + 2 \cos \theta$$

$$\Rightarrow \boxed{\theta = \cos^{-1} \left( \frac{\text{tr}(R) - 1}{2} \right)} \in [0, \pi]$$

If  $\theta = \frac{3\pi}{4} = 270^\circ$  rotated around  $\hat{k}$   
 this is same as  $\theta = \frac{\pi}{4} = 90^\circ$  " "  $-\hat{k}$



$$\therefore \theta = 180^\circ$$

$$k_x = \pm \sqrt{(r_{11}+1)/2}$$

$$k_y = \pm \sqrt{(r_{22}+1)/2}$$

$$k_z = \pm \sqrt{(r_{33}+1)/2}$$

else ( $\theta \neq 0, \theta \neq 180^\circ$ )

$$\hat{k} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} = \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} \frac{1}{(2\sin\theta)}$$

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

Quaternions

$$q = w + ix + jy + kz$$

$$\hat{k} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} \theta$$

Axis angle to quat

$$\boxed{\begin{aligned} i^2 &= -1, j^2 = -1, k^2 = -1 \\ ijk &= -1 \end{aligned}}$$

Complex numbers

$$q = \left[ \begin{array}{c} \sin(\frac{\theta}{2}) \\ \cos(\frac{\theta}{2}) \end{array} \right], \quad k_x \cos(\frac{\theta}{2}), \quad k_y \sin(\frac{\theta}{2}), \quad k_z \sin(\frac{\theta}{2})$$

$w$                        $x$                        $y$                        $z$

$$q = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

OR

$$q = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

$$q = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

$$\hat{k} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} / \sin(\frac{\theta}{2})$$

$$\sqrt{x^2 + y^2 + z^2} = \cancel{\cos}^{\sin} \left( \frac{\theta}{2} \right)$$

$$w \quad z \quad \cancel{\sin}^{\cos} \frac{\theta}{2}$$

$$\theta = 2 \arctan 2 \left( w, \sqrt{x^2 + y^2 + z^2} \right)$$

$$\theta = 2 \arctan 2 \left( \sqrt{x^2 + y^2 + z^2}, w \right)$$

