Linear algebra drills

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1 Valid matrix multiplication and addition

- Notation: A matrix A is described as $A \in \mathbb{R}^{m \times n}$ (read as: A is in the set of real matrices of size m times n) if A is table of numbers of the form:

$$A = \begin{bmatrix} a_{11} & \ldots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \ldots & a_{mn,} \end{bmatrix}$$

and each a_{ij} is a real number also denoted as $a_{ij} \in \mathbb{R}$ (read as: a_{ij} is in the set of real numbers); for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, n\}$.

- Rule 1: Transpose of matrix swaps rows and columns. If $A \in \mathbb{R}^{m \times n}$, then $A^\top \in \mathbb{R}^{n \times m}$.

- Notation: A column vector is a matrix with only one column. Vectors are usually denoted by a boldface small letter $\mathbf{x} \in \mathbb{R}^{m \times 1}$ and sometimes the second dimension is omitted: $\mathbf{x} \in \mathbb{R}^m$.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$$

A row vector is a matrix with only one row. Row vectors are often denoted as transpose of column vector $\mathbf{x}^\top \in \mathbb{R}^{1 \times n}$. The 1 dimension of row vector is never omitted,

$$\mathbf{x}^\top = \begin{bmatrix} x_1 & \ldots & x_m \end{bmatrix}$$

- Rule 2: Matrix multiplication between $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{p \times q}$ is only valid when $n = p$. When matrix multiplication is valid, resulting matrix has size $C = AB \in \mathbb{R}^{m \times q}$.

- Role 3: Scalar-Matrix multiplication is not a type of matrix multiplication. It is a multiplication of a different type. Scalar-matrix multiplication is always valid irrespective of the size of the matrix.

- Rule 4: Matrix addition between $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{p \times q}$ is only valid when $m = p$ and $n = q$.

- Rule 5: Matrix division and matrix fractions are not defined. $\frac{A}{B}$ or A/B are not defined. (Matrix inverse (when defined) is just another matrix $C = A^{-1}$ that can be multiplied or added with other matrices. If $A \in \mathbb{R}^{m \times n}$ then $C = A^{-1} \in \mathbb{R}^{n \times m}$.)

Problem 1. *Are the following matrix operations valid? If not, which dimensions are unequal? If yes, what is the size of final matrix?*

1. *AB, where $A \in \mathbb{R}^{2 \times 3}$ and $B \in \mathbb{R}^{3 \times 4}$.*

2. *PQ, where $P \in \mathbb{R}^{2 \times 4}$ and $B \in \mathbb{R}^{2 \times 4}$.*

3. *$A^\top B$, where $A \in \mathbb{R}^{3 \times 5}$ and $B \in \mathbb{R}^{3 \times 4}$.*

4. *$AB^\top$, where $A \in \mathbb{R}^{2 \times 3}$ and $B \in \mathbb{R}^{3 \times 4}$.*

5. *$A^\top B^\top$, where $A \in \mathbb{R}^{2 \times 3}$ and $B \in \mathbb{R}^{3 \times 4}$.*

6. *$(BA)^\top$, where $A \in \mathbb{R}^{2 \times 3}$ and $B \in \mathbb{R}^{3 \times 4}$.*

7. *$A\mathbf{x}$, where $A \in \mathbb{R}^{2 \times 3}$ and $\mathbf{x} \in \mathbb{R}^3$.*

8. *$A^\top \mathbf{x}$, where $A \in \mathbb{R}^{2 \times 3}$ and $\mathbf{x} \in \mathbb{R}^{3 \times 1}$.*

9. *$\mathbf{x}A$, where $A \in \mathbb{R}^{2 \times 3}$ and $\mathbf{x} \in \mathbb{R}^3$.*

10. *$\mathbf{x}^\top B$, where $\mathbf{x} \in \mathbb{R}^3$ and $B \in \mathbb{R}^{3 \times 4}$.*

11. *$\mathbf{x}B$, where $\mathbf{x} \in \mathbb{R}^3$ and $B \in \mathbb{R}^{3 \times 4}$.*

12. $\mathbf{x}B^\top$, where $\mathbf{x} \in \mathbb{R}^3$ and $B \in \mathbb{R}^{3\times4}$.

13. $\mathbf{x}^\top B^\top$, where $\mathbf{x} \in \mathbb{R}^3$ and $B \in \mathbb{R}^{3\times4}$.

14. $\mathbf{x}^\top \mathbf{y}$, where $\mathbf{x} \in \mathbb{R}^3$ and $\mathbf{y} \in \mathbb{R}^3$.

15. $\mathbf{y}^\top \mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^3$ and $\mathbf{y} \in \mathbb{R}^3$.

16. $\mathbf{y}\mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^3$ and $\mathbf{y} \in \mathbb{R}^3$.

17. $\mathbf{y}^\top \mathbf{x}^\top$, where $\mathbf{x} \in \mathbb{R}^3$ and $\mathbf{y} \in \mathbb{R}^3$.

18. $\mathbf{y}\mathbf{x}^\top$, where $\mathbf{x} \in \mathbb{R}^3$ and $\mathbf{y} \in \mathbb{R}^3$.

19. $\mathbf{y}A\mathbf{x}^\top$, where $\mathbf{x} \in \mathbb{R}^3$, $A \in \mathbb{R}^{3\times4}$ and $\mathbf{y} \in \mathbb{R}^4$.

20. $\mathbf{y}^\top A^\top \mathbf{x}^\top$, where $\mathbf{x} \in \mathbb{R}^3$, $A \in \mathbb{R}^{3\times4}$ and $\mathbf{y} \in \mathbb{R}^4$.

21. $\mathbf{x}^\top A\mathbf{y}$, where $\mathbf{x} \in \mathbb{R}^3$, $A \in \mathbb{R}^{3\times4}$ and $\mathbf{y} \in \mathbb{R}^4$.

22. $\mathbf{y}^\top A^\top \mathbf{x}^\top$, where $\mathbf{x} \in \mathbb{R}^3$, $A \in \mathbb{R}^{3\times4}$ and $\mathbf{y} \in \mathbb{R}^4$.

23. $\mathbf{y}^\top A^\top \mathbf{x}^\top$, where $\mathbf{x} \in \mathbb{R}^3$, $A \in \mathbb{R}^{3\times4}$ and $\mathbf{y} \in \mathbb{R}^4$.

24. $\mathbf{x}^\top A^\top \mathbf{y}$, where $\mathbf{x} \in \mathbb{R}^3$, $A \in \mathbb{R}^{3\times4}$ and $\mathbf{y} \in \mathbb{R}^4$.

25. AB, where $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$.

26. PQ, where $P \in \mathbb{R}^{m\times p}$ and $B \in \mathbb{R}^{m\times p}$.

27. $A^\top B$, where $A \in \mathbb{R}^{n\times5}$ and $B \in \mathbb{R}^{n\times p}$.

28. $AB^\top$, where $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$.

29. $A^\top B^\top$, where $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$.

30. $(BA)^\top$, where $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$.

31. $A\mathbf{x}$, where $A \in \mathbb{R}^{m\times n}$ and $\mathbf{x} \in \mathbb{R}^n$.

32. $A^\top \mathbf{x}$, where $A \in \mathbb{R}^{m\times n}$ and $\mathbf{x} \in \mathbb{R}^{n\times1}$.

33. $\mathbf{x}A$, where $A \in \mathbb{R}^{m\times n}$ and $\mathbf{x} \in \mathbb{R}^n$.

34. $\mathbf{x}^\top B$, where $\mathbf{x} \in \mathbb{R}^n$ and $B \in \mathbb{R}^{n\times p}$.

35. $\mathbf{x}B$, where $\mathbf{x} \in \mathbb{R}^n$ and $B \in \mathbb{R}^{n\times p}$.

36. $\mathbf{x}B^\top$, where $\mathbf{x} \in \mathbb{R}^n$ and $B \in \mathbb{R}^{n\times p}$.

37. $\mathbf{x}^\top B^\top$, where $\mathbf{x} \in \mathbb{R}^n$ and $B \in \mathbb{R}^{n\times p}$.

38. $\mathbf{x}^\top \mathbf{y}$, where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n$.

39. $\mathbf{y}^\top \mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n$.

40. $\mathbf{y}\mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n$.

41. $\mathbf{y}^\top \mathbf{x}^\top$, where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n$.

42. $\mathbf{y}\mathbf{x}^\top$, where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n$.

43. $\mathbf{y}A\mathbf{x}^\top$, where $\mathbf{x} \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times p}$ and $\mathbf{y} \in \mathbb{R}^p$.

44. $\mathbf{y}^\top A^\top \mathbf{x}^\top$, where $\mathbf{x} \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times p}$ and $\mathbf{y} \in \mathbb{R}^p$.

45. $\mathbf{x}^\top A\mathbf{y}$, where $\mathbf{x} \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times p}$ and $\mathbf{y} \in \mathbb{R}^p$.

46. $\mathbf{y}^\top A^\top \mathbf{x}^\top$, where $\mathbf{x} \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times p}$ and $\mathbf{y} \in \mathbb{R}^p$.

47. $\mathbf{y}^\top A^\top \mathbf{x}^\top$, where $\mathbf{x} \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times p}$ and $\mathbf{y} \in \mathbb{R}^p$.

48. $\mathbf{x}^\top A^\top \mathbf{y}$, where $\mathbf{x} \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times p}$ and $\mathbf{y} \in \mathbb{R}^p$.

2 Solution

1. AB, where $A \in \mathbb{R}^{2\times 3}$ and $B \in \mathbb{R}^{3\times 4}$.
   Valid. $AB \in \mathbb{R}^{2\times 4}$.

2. PQ, where $P \in \mathbb{R}^{2\times 4}$ and $B \in \mathbb{R}^{2\times 4}$.
   Invalid, $4 \neq 2$.

3. $A^\top B$, where $A \in \mathbb{R}^{3\times 5}$ and $B \in \mathbb{R}^{3\times 4}$.
   Valid, $A^\top B \in \mathbb{R}^{5\times 4}$.

4. $AB^\top$, where $A \in \mathbb{R}^{2\times 3}$ and $B \in \mathbb{R}^{3\times 4}$.
   Invalid, because $3 \neq 4$.

5. $A^\top B^\top$, where $A \in \mathbb{R}^{2\times 3}$ and $B \in \mathbb{R}^{3\times 4}$.
   Invalid, because $2 \neq 4$.

6. $(BA)^\top$, where $A \in \mathbb{R}^{2\times 3}$ and $B \in \mathbb{R}^{3\times 4}$.
   Invalid, because $4 \neq 2$.

7. $A\mathbf{x}$, where $A \in \mathbb{R}^{2\times 3}$ and $\mathbf{x} \in \mathbb{R}^3$.
   Valid, $A\mathbf{x} \in \mathbb{R}^2$.

8. $A^\top \mathbf{x}$, where $A \in \mathbb{R}^{2\times 3}$ and $\mathbf{x} \in \mathbb{R}^{3\times 1}$.
   Invalid, because $2 \neq 3$.

9. $\mathbf{x}A$, where $A \in \mathbb{R}^{2\times 3}$ and $\mathbf{x} \in \mathbb{R}^3$.
   Invalid, because $1 \neq 2$.

10. $\mathbf{x}^\top B$, where $\mathbf{x} \in \mathbb{R}^3$ and $B \in \mathbb{R}^{3\times 4}$.
    Valid, $\mathbf{x}^\top B \in \mathbb{R}^{1\times 4}$.

11. $\mathbf{x}B$, where $\mathbf{x} \in \mathbb{R}^3$ and $B \in \mathbb{R}^{3\times 4}$.
    Invalid, because $1 \neq 3$.

12. $\mathbf{x}B^\top$, where $\mathbf{x} \in \mathbb{R}^3$ and $B \in \mathbb{R}^{3\times 4}$.
    Invalid, because $1 \neq 4$.

13. $\mathbf{x}^\top B^\top$, where $\mathbf{x} \in \mathbb{R}^3$ and $B \in \mathbb{R}^{3\times 4}$.
    Invalid, because $3 \neq 4$.

14. $\mathbf{x}^\top \mathbf{y}$, where $\mathbf{x} \in \mathbb{R}^3$ and $\mathbf{y} \in \mathbb{R}^3$.
    Valid, $\mathbf{x}^\top \mathbf{y} \in \mathbb{R}$.

15. $\mathbf{y}^\top \mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^3$ and $\mathbf{y} \in \mathbb{R}^3$.
    Valid, $\mathbf{y}^\top \mathbf{x} \in \mathbb{R}$.

16. $\mathbf{y}\mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^3$ and $\mathbf{y} \in \mathbb{R}^3$. Invalid, because $1 \neq 3$.

17. $\mathbf{y}^\top \mathbf{x}^\top$, where $\mathbf{x} \in \mathbb{R}^3$ and $\mathbf{y} \in \mathbb{R}^3$.
    Invalid, because $3 \neq 1$.

18. $\mathbf{y}\mathbf{x}^\top$, where $\mathbf{x} \in \mathbb{R}^3$ and $\mathbf{y} \in \mathbb{R}^3$.
    Valid, $\mathbf{y}\mathbf{x}^\top \in \mathbb{R}^{3\times 3}$.

19. $\mathbf{y}A\mathbf{x}^\top$, where $\mathbf{x} \in \mathbb{R}^3$, $A \in \mathbb{R}^{3\times 4}$ and $\mathbf{y} \in \mathbb{R}^4$.
    Invalid, because $1 \neq 3$ and $4 \neq 1$.

20. $\mathbf{y}^\top A^\top \mathbf{x}^\top$, where $\mathbf{x} \in \mathbb{R}^3$, $A \in \mathbb{R}^{3\times 4}$ and $\mathbf{y} \in \mathbb{R}^4$.
    Invalid, because $3 \neq 1$.

21. $\mathbf{x}^\top A\mathbf{y}$, where $\mathbf{x} \in \mathbb{R}^3$, $A \in \mathbb{R}^{3\times 4}$ and $\mathbf{y} \in \mathbb{R}^4$.
    Valid, $\mathbf{x}^\top A\mathbf{y} \in \mathbb{R}$.

22. $\mathbf{y}^\top A^\top \mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^3$, $A \in \mathbb{R}^{3\times 4}$ and $\mathbf{y} \in \mathbb{R}^4$.
    Valid, $\mathbf{y}^\top A^\top \mathbf{x} \in \mathbb{R}$.

23. $\mathbf{y}^\top A^\top \mathbf{x}^\top$, where $\mathbf{x} \in \mathbb{R}^3$, $A \in \mathbb{R}^{3\times 4}$ and $\mathbf{y} \in \mathbb{R}^4$.
    Invalid, because $3 \neq 1$.

24. $\mathbf{x}^\top A^\top \mathbf{y}$, where $\mathbf{x} \in \mathbb{R}^3$, $A \in \mathbb{R}^{3\times 4}$ and $\mathbf{y} \in \mathbb{R}^4$.
    Invalid, because $3 \neq 4$ and $3 \neq 4$.

25. AB, where $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$.
    Valid. $AB \in \mathbb{R}^{m\times p}$.

26. PQ, where $P \in \mathbb{R}^{m\times p}$ and $B \in \mathbb{R}^{m\times p}$.
    Invalid, $p \neq m$.

27. $A^\top B$, where $A \in \mathbb{R}^{n\times 5}$ and $B \in \mathbb{R}^{n\times p}$.
    Valid, $A^\top B \in \mathbb{R}^{5\times p}$.

28. $AB^\top$, where $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$.
    Invalid, because $n \neq p$.

29. $A^\top B^\top$, where $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$.
    Invalid, because $m \neq p$.

30. $(BA)^\top$, where $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$.
    Invalid, because $p \neq m$.

31. $A\mathbf{x}$, where $A \in \mathbb{R}^{m\times n}$ and $\mathbf{x} \in \mathbb{R}^n$.
    Valid, $A\mathbf{x} \in \mathbb{R}^m$.

32. $A^\top \mathbf{x}$, where $A \in \mathbb{R}^{m\times n}$ and $\mathbf{x} \in \mathbb{R}^{n\times 1}$.
    Invalid, because $m \neq n$.

33. $\mathbf{x}A$, where $A \in \mathbb{R}^{m\times n}$ and $\mathbf{x} \in \mathbb{R}^n$.
    Invalid, because $1 \neq m$.

34. $\mathbf{x}^\top B$, where $\mathbf{x} \in \mathbb{R}^n$ and $B \in \mathbb{R}^{n\times p}$.
    Valid, $\mathbf{x}^\top B \in \mathbb{R}^{1\times p}$.

35. $\mathbf{x}B$, where $\mathbf{x} \in \mathbb{R}^n$ and $B \in \mathbb{R}^{n \times p}$.
    Invalid, because $1 \neq n$.

36. $\mathbf{x}B^\top$, where $\mathbf{x} \in \mathbb{R}^n$ and $B \in \mathbb{R}^{n \times p}$.
    Invalid, because $1 \neq p$.

37. $\mathbf{x}^\top B^\top$, where $\mathbf{x} \in \mathbb{R}^n$ and $B \in \mathbb{R}^{n \times p}$.
    Invalid, because $n \neq p$.

38. $\mathbf{x}^\top \mathbf{y}$, where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n$.
    Valid, $\mathbf{x}^\top \mathbf{y} \in \mathbb{R}$.

39. $\mathbf{y}^\top \mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n$.
    Valid, $\mathbf{y}^\top \mathbf{x} \in \mathbb{R}$.

40. $\mathbf{y}\mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n$. Invalid, because
    $1 \neq n$.

41. $\mathbf{y}^\top \mathbf{x}^\top$, where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n$.
    Invalid, because $n \neq 1$.

42. $\mathbf{y}\mathbf{x}^\top$, where $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} \in \mathbb{R}^n$.
    Valid, $\mathbf{y}\mathbf{x}^\top \in \mathbb{R}^{n \times n}$.

43. $\mathbf{y}A\mathbf{x}^\top$, where $\mathbf{x} \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times p}$ and $\mathbf{y} \in \mathbb{R}^p$.
    Invalid, because $1 \neq n$ and $p \neq 1$.

44. $\mathbf{y}^\top A^\top \mathbf{x}^\top$, where $\mathbf{x} \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times p}$ and $\mathbf{y} \in \mathbb{R}^p$.
    Invalid, because $n \neq 1$.

45. $\mathbf{x}^\top A\mathbf{y}$, where $\mathbf{x} \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times p}$ and $\mathbf{y} \in \mathbb{R}^p$.
    Valid, $\mathbf{x}^\top A\mathbf{y} \in \mathbb{R}$.

46. $\mathbf{y}^\top A^\top \mathbf{x}$, where $\mathbf{x} \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times p}$ and $\mathbf{y} \in \mathbb{R}^p$.
    Valid, $\mathbf{y}^\top A^\top \mathbf{x} \in \mathbb{R}$.

47. $\mathbf{y}^\top A^\top \mathbf{x}^\top$, where $\mathbf{x} \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times p}$ and $\mathbf{y} \in \mathbb{R}^p$.
    Invalid, because $n \neq 1$.

48. $\mathbf{x}^\top A^\top \mathbf{y}$, where $\mathbf{x} \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times p}$ and $\mathbf{y} \in \mathbb{R}^p$.
    Invalid, because $n \neq p$ and $n \neq p$.