ECE 417 Midterm 2 2022 practice problem set

Instructor: Vikas Dhiman

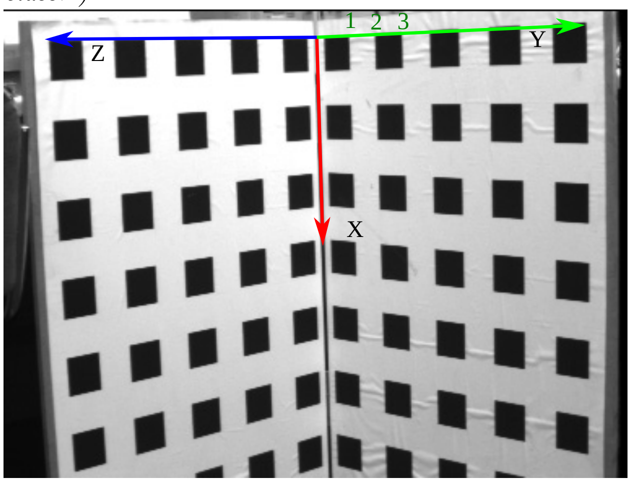
March 30th, 2021

(1) Student name: Student email:

About the exam

1. There are total 5 problems. You must attempt all 5.
2. Maximum marks: 50 (70 with bonus marks).
3. Maximum time allotted: 50 min
4. Calculators are allowed.
5. One US Letter size or A4 size cheat sheet (both-sides) is allowed.

Problem 1 *Given a set of $n \geq 6$ points $\underline{\mathbf{X}}_i \in \mathbb{P}^3$ for all $i \in \{1, \ldots, n\}$ in 3D projective space, and a set of corresponding points $\underline{\mathbf{u}}_i \in \mathbb{P}^2$ in an image, find the 3D to 2D projective $P \in \mathbb{R}^{3\times 4}$ matrix that converts $\mathbf{X}_i$ to $\underline{\mathbf{u}}_i = \lambda_i P \underline{\mathbf{X}}_i$. In other words, convert $\underline{\mathbf{u}}_i \times P\underline{\mathbf{X}}_i = 0$ into a familiar form $A\mathbf{y} = \mathbf{b}$ or $A\mathbf{y} = \mathbf{0}$ so that we can solve for P. For notation purposes, you can denote $\underline{\mathbf{u}}_i = [x_i, y_i, w_i]^\top$ and $P = \begin{bmatrix}\mathbf{p}_1^\top \\ \mathbf{p}_2^\top \\ \mathbf{p}_3^\top\end{bmatrix}$ where $\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3 \in \mathbb{R}^4$ are the rows of P represented as 4-D column vectors. (Practical motivation: We did camera calibration in lab using a single checker board. It is much easier to compute camera calibration using two mutually perpendicular checker boards so that all points do not lie on a single plane (hence linearly independent). One can make a coordinate system attached to the double checker and compute the 3D coordinates of each corner point in that system. Let $\underline{\mathbf{X}}_i \in \mathbb{P}^3$ be such points in 3D on the checker-board. Let $\underline{\mathbf{u}}_i \in \mathbb{P}^2$ be a point detected in the image so that we have one-to-one correspondence between $\underline{\mathbf{X}}_i$ and $\underline{\mathbf{u}}_i$. Finding the projection matrix $P \in \mathbb{R}^{3\times 4}$ then reduces to the above problem. We will cover the breakdown of P matrix into $P = K[R, t]$ in class.)*



u_i = K X_i

X_i = 3x1

u_i = K [R X_i + t]

u_i = K [R   t] [X_i]
                [1]

X_i = 4x1

P = K [R , t]

u_i = \lambda P X_i

u_i x P X_i = 0

P is 3x4 matrix

P = [p_1^T]
    [p_2^T]
    [p_3^T]

p_1 is 4x1 vector
p_2
p_3

u_i = [[0   -w_i   y_i]
       [w_i   0   -x_i]
       [-y_i   x_i   0]]

A y = b

P X_i = [p_1^T]   [p_1^T X_i]
        [p_2^T] X_i   = [p_2^T X_i]
        [p_3^T]   [p_3^T X_i]

$$P X_i = \begin{bmatrix} X_i^T p_1 \\ X_i^T p_2 \\ X_i^T p_3 \end{bmatrix}$$

p_1 is a 4x1
X_i is a 4x1
p_1^T is a 1x4

$p_1^T X_i$ is 1x1 scalar

$u_i \times P X_i = 0$

$p_1^T X_i = (p_1^T X_i)^T = X_i^T p_1 = p$

$$\begin{bmatrix} 0 & -w_i & y_i \\ w_i & 0 & -x_i \\ -y_i & x_i & 0 \end{bmatrix} \begin{bmatrix} X_i^T p_1 \\ X_i^T p_2 \\ X_i^T p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 - w_i X_i^T p_2 + y_i X_i^T p_3 \\ w_i X_i^T p_1 + 0 - x_i X_i^T p_3 \\ -y_i X_i^T p_1 + x_i X_i^T p_2 + 0 \end{bmatrix} = 0$$

$$\begin{bmatrix} 0^T & -w_i X_i^T & y_i X_i^T \\ w_i X_i^T & 0^T & -x_i X_i^T \\ -y_i X_i^T & x_i X_i^T & 0^T \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = 0$$

$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}$ is a 12x1 vector

Solution

1. Write cross product as a matrix operation

$$[\underline{\mathbf{u}}_i]_\times = \begin{bmatrix} 0 & -w_i & y_i \\ w_i & 0 & -x_i \\ -y_i & x_i & 0 \end{bmatrix}$$

2. Write $P\underline{\mathbf{X}}_i$ in terms of row vectors.

$$P\underline{\mathbf{X}}_i = \begin{bmatrix} \mathbf{p}_1^\top \\ \mathbf{p}_2^\top \\ \mathbf{p}_3^\top \end{bmatrix} \underline{\mathbf{X}}_i = \begin{bmatrix} \mathbf{p}_1^\top \underline{\mathbf{X}}_i \\ \mathbf{p}_2^\top \underline{\mathbf{X}}_i \\ \mathbf{p}_3^\top \underline{\mathbf{X}}_i \end{bmatrix}$$

3. Note that all the three terms like $\mathbf{p}_1^\top \underline{\mathbf{X}}_i$ are scalars hence they are symmetric. Hence $\mathbf{p}_1^\top \underline{\mathbf{X}}_i = \underline{\mathbf{X}}_i^\top \mathbf{p}_1$.

$$P\underline{\mathbf{X}}_i = \begin{bmatrix} \underline{\mathbf{X}}_i^\top \mathbf{p}_1 \\ \underline{\mathbf{X}}_i^\top \mathbf{p}_2 \\ \underline{\mathbf{X}}_i^\top \mathbf{p}_3 \end{bmatrix}$$

4. Substitute these values in the original equation $\underline{\mathbf{u}}_i \times P\underline{\mathbf{X}}_i = \mathbf{0}_{3\times 1}$.

$$\begin{bmatrix} 0 & -w_i & y_i \\ w_i & 0 & -x_i \\ -y_i & x_i & 0 \end{bmatrix} \begin{bmatrix} \underline{\mathbf{X}}_i^\top \mathbf{p}_1 \\ \underline{\mathbf{X}}_i^\top \mathbf{p}_2 \\ \underline{\mathbf{X}}_i^\top \mathbf{p}_3 \end{bmatrix} = \mathbf{0}_{3\times 1}$$

5. Matrix multiply

$$\begin{bmatrix} 0 - w_i \underline{\mathbf{X}}_i^\top \mathbf{p}_2 + y_i \underline{\mathbf{X}}_i^\top \mathbf{p}_3 \\ w_i \underline{\mathbf{X}}_i^\top \mathbf{p}_1 + 0 - x_i \underline{\mathbf{X}}_i^\top \mathbf{p}_3 \\ -y_i \underline{\mathbf{X}}_i^\top \mathbf{p}_1 + x_i \underline{\mathbf{X}}_i^\top \mathbf{p}_2 + 0 \end{bmatrix} = \mathbf{0}_{3\times 1}$$

6. Write the unknowns as a single vector, and the knowns as a matrix multiplication with the unknowns

$$\begin{bmatrix} \mathbf{0}^\top & -w_i \underline{\mathbf{X}}_i^\top & y_i \underline{\mathbf{X}}_i \\ w_i \underline{\mathbf{X}}_i^\top & \mathbf{0}^\top & -x_i \underline{\mathbf{X}}_i^\top \\ -y_i \underline{\mathbf{X}}_i^\top & x_i \underline{\mathbf{X}}_i^\top & \mathbf{0}^\top \end{bmatrix}_{3\times 12} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}_{12\times 1} = \mathbf{0}_{3\times 1}$$

7. Pick only two of the equations as only two are linearly independent.

$$\begin{bmatrix} \mathbf{0}^\top & -w_i \underline{\mathbf{X}}_i^\top & y_i \underline{\mathbf{X}}_i \\ w_i \underline{\mathbf{X}}_i^\top & \mathbf{0}^\top & -x_i \underline{\mathbf{X}}_i^\top \end{bmatrix}_{2\times 12} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}_{12\times 1} = \mathbf{0}_{2\times 1}$$

8. Collect all the equations from n pairs of corresponding points $\underline{\mathbf{u}}_1, \ldots, \underline{\mathbf{u}}_n$ and $\underline{\mathbf{X}}_1, \ldots, \underline{\mathbf{X}}_n$.

$$\begin{bmatrix} \mathbf{0}^\top & -w_1 \underline{\mathbf{X}}_1^\top & y_1 \underline{\mathbf{X}}_1 \\ w_1 \underline{\mathbf{X}}_1^\top & \mathbf{0}^\top & -x_1 \underline{\mathbf{X}}_1^\top \\ \vdots & \vdots & \vdots \\ \mathbf{0}^\top & -w_n \underline{\mathbf{X}}_n^\top & y_n \underline{\mathbf{X}}_n \\ w_n \underline{\mathbf{X}}_n^\top & \mathbf{0}^\top & -x_n \underline{\mathbf{X}}_n^\top \end{bmatrix}_{2n\times 12} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix}_{12\times 1} = \mathbf{0}_{2n\times 1}$$

9. P matrix has rank rank$(P) = 11$ because it has 12 elements and equivalence upto a scale factor. So the solution of the above equation can be computed from SVD by choosing the right singular vector corresponding to the smallest singular value.

$$A = \begin{bmatrix} \mathbf{0}^\top & -w_1\underline{\mathbf{X}}_1^\top & y_1\underline{\mathbf{X}}_1 \\ w_1\underline{\mathbf{X}}_1^\top & \mathbf{0}^\top & -x_1\underline{\mathbf{X}}_1^\top \\ \vdots & \vdots & \vdots \\ \mathbf{0}^\top & -w_n\underline{\mathbf{X}}_n^\top & y_n\underline{\mathbf{X}}_n \\ w_n\underline{\mathbf{X}}_n^\top & \mathbf{0}^\top & -x_n\underline{\mathbf{X}}_n^\top \end{bmatrix} = U\Sigma V^T$$

Let $V = [\mathbf{v}_1, \ldots, \mathbf{v}_n]$, then

$$\begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix} = \mathbf{v}_n$$

Now we can write the P matrix as

$$P = \begin{bmatrix} \mathbf{p}_1^\top \\ \mathbf{p}_2^\top \\ \mathbf{p}_3^\top \end{bmatrix}$$

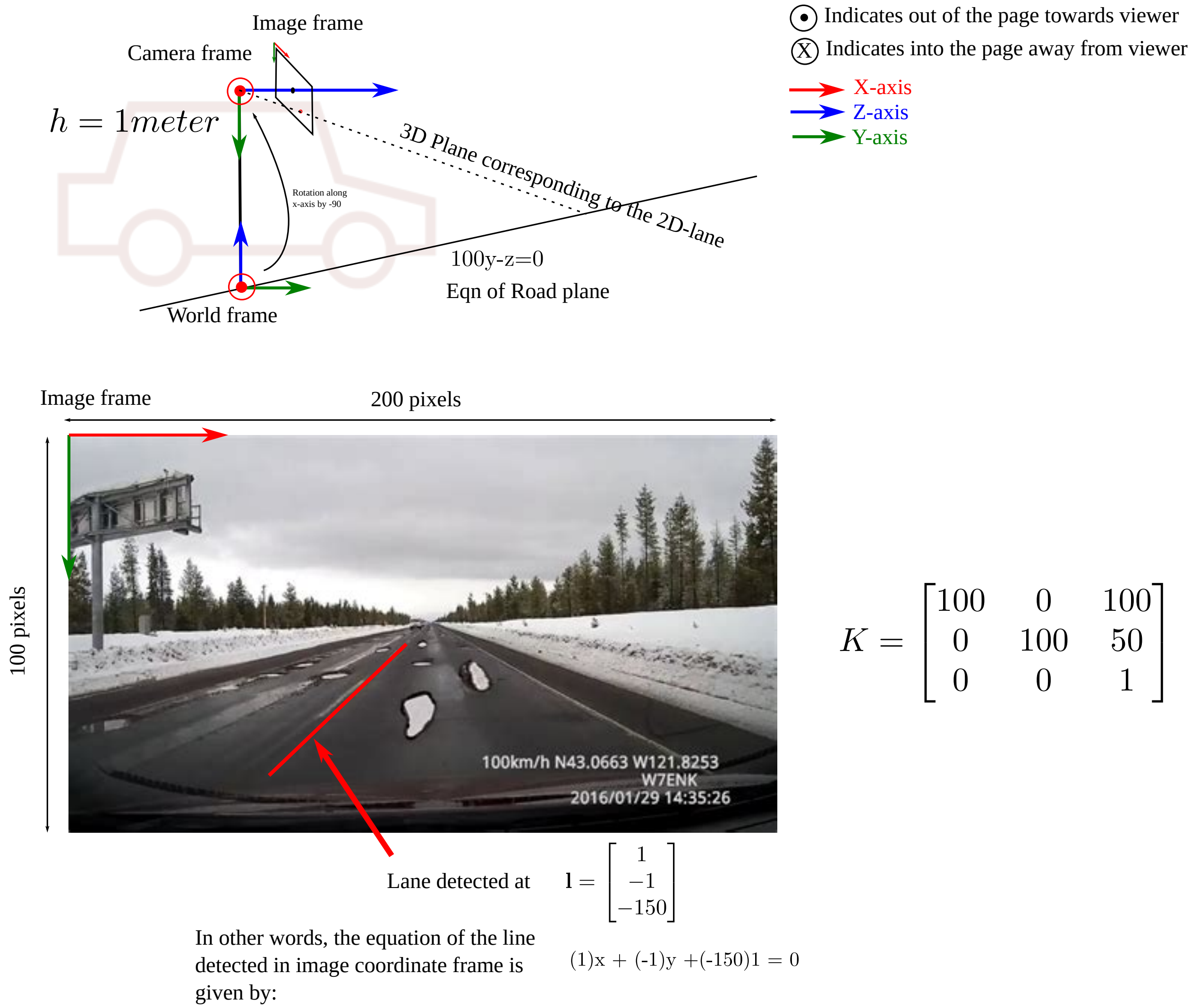


Figure 1: Line-plane triangulation

Problem 2 *In figure 1 find the 3D representation of the lane the World coordinate frame, in terms of h (the height of the camera), image-representation of the line $\mathbf{l}$ (provided in figure), camera matrix K (provided in figure). Assume the lane to be a straight line. The Camera is mounted directly on top of the world frame, both of which are aligned to the gravity vector. The road is a perfect plane with a slope such that the equation of road plane in world-coordinate frame is given by $100Y_w - Z_w = 0$ and the lane lies on the road plane. Provide the formula or pseudo-code for computing the 3D representation of the lane, and also substitute in the values. (20 min, 20 marks)*

Solution See homework 4 solution.

.

Problem 3 *Find the minimum point of the function,* $f(\mathbf{u}) = 2\mathbf{u}^\top A^\top A\mathbf{u} - 3\mathbf{u}^\top \mathbf{b} + 4\mathbf{c}^\top \mathbf{u} + d$. *Let* $\mathbf{u} \in \mathbb{R}^{n\times 1}$ *be a n-dimensional vector and sizes of* $A, \mathbf{b}, \mathbf{c}, d$ *be such that matrix multiplication and addition is valid. Also assume that* $A^\top A$ *is full rank, hence invertible.*

Problem 4 *Let matrix $A \in \mathbb{R}^{m\times n}$ be a $m \times n$ matrix. We are given that $B = A^\top A$ has n orthonormal eigen vectors $\mathbf{e}_1, \dots \mathbf{e}_n$ with corresponding eigen values as $\lambda_1 \dots \lambda_n$ such that $B\mathbf{e}_i = \lambda_i \mathbf{e}_i$ for all $i \in \{1, \dots n\}$. Let the rank of matrix A be r. Write the thin singular value decomposition of $A = U_{m\times r}\Sigma_{r\times r}V_{n\times r}^\top$ in terms of eigen values and eigen vectors of matrix $B = A^\top A$.*

Solution The matrix of right singular vectors of A is same as the eigen vector matrix of $B = A^\top A$.

$$V = \begin{bmatrix}\mathbf{e}_1 \dots \mathbf{e}_r\end{bmatrix} \in \mathbb{R}^{n\times r} \tag{1}$$

The matrix of singular values are the square root of eigen values of B.

$$\Sigma = \begin{bmatrix} \sqrt{\lambda_1} & \dots & 0 \\ 0 & \ddots & 0 \\ 0 & \dots & \sqrt{\lambda_r} \end{bmatrix} \in bbR^{r\times r} \tag{2}$$

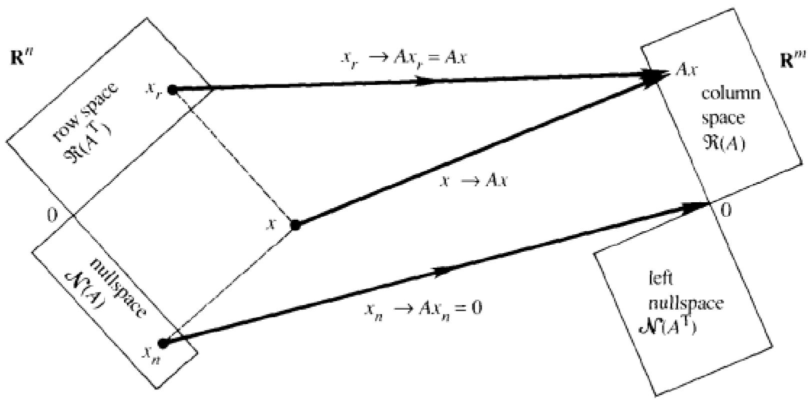
$$U = \begin{bmatrix}\mathbf{u}_1 \dots \mathbf{u}_r\end{bmatrix} \in \mathbb{R}^{m\times r} \tag{3}$$

$$\text{where } \mathbf{u}_i = \frac{A\mathbf{e}_i}{\sqrt{\lambda_i}} \tag{4}$$

Problem 5 *Let matrix $A \in \mathbb{R}^{m\times n}$ has the singular value decomposition (SVD) as $A = U\Sigma V^\top$ and rank of the matrix be $r = rank(A)$. Write the basis vectors of the four fundamental subspaces of matrix A in terms of SVD,*

1. *Null space of A ($\mathcal{N}(A) =?$).*
2. *Column space or range space ($\mathcal{R}(A) =?$).*
3. *Row space ($\mathcal{R}(A^\top) =?$).*
4. *Left null space ($\mathcal{N}(A^\top) =?$).*

You can denote the first r column vectors of U has $U_{1:r} \in \mathbb{R}^{m\times r}$ and the renaming $m-r$ vectors as $U_{r+1:m} \in \mathbb{R}^{m\times(m-r)}$. Similarly for V, first r column vectors of $V_{1:r} \in \mathbb{R}^{n\times r}$ and $V_{r+1:n} \in \mathbb{R}^{n\times(n-r)}$.



Solution

1. Null space of A ($\mathcal{N}(A) = V_{r+1:n}$).
2. Column space or range space ($\mathcal{R}(A) = U_{1:r}$).
3. Row space ($\mathcal{R}(A^\top) = V_{1:r}$).
4. Left null space ($\mathcal{N}(A^\top) = U_{r+1:m}$).

Extra practice problems

Problem 6 *Find a line passing through the following points*

$$\mathbf{u}_1 = [101, 203]^\top, \mathbf{u}_2 = [49, 102]^\top, \mathbf{u}_3 = [27, 51]^\top, \mathbf{u}_4 = [201, 403]^\top, \mathbf{u}_5 = [74, 151]^\top.$$

You can leave the output in terms of SVD.

Problem 7 *Find a plane passing through the following points*

$$\mathbf{x}_1 = [9.99, 101, 203]^\top, \mathbf{x}_2 = [5.1, 49, 102]^\top, \mathbf{x}_3 = [2.5, 27, 51]^\top, \mathbf{x}_4 = [21, 201, 403]^\top, \mathbf{x}_5 = [7.6, 74, 151]^\top.$$

You can leave the output in terms of SVD.

Problem 8 *Find the 3D line in parameteric representation that is formed by the intersection of two planes* $\mathbf{p}^\top \underline{\mathbf{x}} = 0$ *(with* $\mathbf{p} = [1, 2, 3, 4]^\top$*) and* $\mathbf{q}^\top \underline{\mathbf{x}} = 0$ *where* $\mathbf{q} = [-3, 2, 1, 4]^\top$.

Problem 9 *Find the point on the intersection of following 3D lines* $\mathbf{x} = \lambda_1 \mathbf{d}_1 + \mathbf{y}$ *and* $\mathbf{x} = \lambda_2 \mathbf{d}_2 + \mathbf{z}$. *Here* $\lambda_1 \in \mathbb{R}$ *and* $\lambda_2 \in \mathbb{R}$ *are the free parameters. The rest of the parameters have the following values*

$$\mathbf{d}_1 = [1, 2, 0]^\top, \mathbf{d}_2 = [-2, 1, 0]^\top, \mathbf{y} = [1, 2, 0]^\top, \mathbf{z} = [4, 5, 0]^\top$$

Problem 10 *Find the point of intersection of the 3D line* $\mathbf{x} = \lambda \mathbf{d} + \mathbf{x}_0$ *with the 3D plane* $\mathbf{p}^\top \underline{\mathbf{x}} = 0$. *The parameters have the following*

$$\mathbf{d} = [1, 2, 0]^\top, \mathbf{x}_0 = [3, 4, 5]^\top, \mathbf{p} = [1, 2, 0, 7]^\top$$

Practice problem solutions

Solution 6 Let $\mathbf{l} \in \mathbb{P}^2$ be the parameters of the line, so that $\underline{\mathbf{u}}^\top \mathbf{l} = 0$.

$$A = \begin{bmatrix} \mathbf{u}_1^\top & 1 \\ \mathbf{u}_2^\top & 1 \\ \mathbf{u}_3^\top & 1 \\ \mathbf{u}_4^\top & 1 \\ \mathbf{u}_5^\top & 1 \end{bmatrix}_{5\times 3}$$

We are looking for the solution of $A\mathbf{l} = 0$. Let the SVD of $A = U\Sigma V^\top$. Let $V = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3]$, then the representation of the line $\mathbf{l} = \mathbf{v}_3$.

Solution 7 Let $\mathbf{p} \in \mathbb{P}^3$ be the parameters of the plane, so that ${}^\top \mathbf{p} = 0$.

$$A = \begin{bmatrix} \mathbf{x}_1^\top & 1 \\ \mathbf{x}_2^\top & 1 \\ \mathbf{x}_3^\top & 1 \\ \mathbf{x}_4^\top & 1 \\ \mathbf{x}_5^\top & 1 \end{bmatrix}_{5\times 4}$$

We are looking for the solution of $A\mathbf{p} = 0$. Let the SVD of $A = U\Sigma V^\top$. Let $V = [\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4]$, then the representation of the line $\mathbf{p} = \mathbf{v}_4$.

Solution 8 $\mathbf{x} = \lambda(\mathbf{p}_{1:3} \times \mathbf{q}_{1:3}) + \begin{bmatrix} \mathbf{p}_{1:3}^\top \\ \mathbf{q}_{1:3} \end{bmatrix}^\dagger \begin{bmatrix} -p_4 \\ q_4 \end{bmatrix}$

Solution 9

$$\begin{bmatrix} \mathbf{d}_1 & -\mathbf{d}_2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \mathbf{z} - \mathbf{y}$$

or

$$\begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \mathbf{d}_1 & -\mathbf{d}_2 \end{bmatrix}^\dagger \mathbf{z} - \mathbf{y}$$

The point of intersection is given by

$$\mathbf{x} = \lambda_1 \mathbf{d}_1 + \mathbf{y}$$

Solution 10

$$\lambda \mathbf{p}_{1:3}^\top \mathbf{d} + \mathbf{p}_{1:3}^\top \mathbf{x}_0 + p_4 = 0$$

Solve for λ.

$$\lambda = -\frac{\mathbf{p}_{1:3}^\top \mathbf{x}_0 + p_4}{\mathbf{p}_{1:3}^\top \mathbf{d}}$$

Point of intersection is

$$\mathbf{x} = \lambda \mathbf{d} + \mathbf{x}_0$$