Problem set

Proofs

1. In your own words, prove using trignometry that the 2D rotation matrix is given by (10 marks)

$$R = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$$

2. Derive the expression for Euler angles roll, pitch yaw from a given 3D rotation matrix (10 marks)

3. Derive the rodrigues formula for a rotation matrix that rotates a point around a given unit vector k for an angle theta. (30 marks)

4. Derive the axis-angle representation from a given 3D rotation matrix (10 marks)

.

.

These are homework 1-6 from Prof Rick Eason's course: https://web.eece.maine.edu/eason/ece417/

You are allowed to use python to find these answers

For the rotation matrix:

$$ {}^{XYZ}\mathbf{R}_{UVW} = \begin{bmatrix} 2/7 & -6/7 & 3/7 \\ 6/7 & 3/7 & 2/7 \\ -3/7 & 2/7 & 6/7 \end{bmatrix} $$

1. Show that ${}^{XYZ}\mathbf{R}_{UVW}$ is a proper rotation matrix. (5 marks)
2. Show that $\mathbf{R}^{-1}$ is equal to $\mathbf{R}^{T}$ where ($\mathbf{R}$ is shorthand for ${}^{XYZ}\mathbf{R}_{UVW}$). HINT: taking the inverse is not required. (5 marks)
3. Compute $\mathbf{R}\,\mathbf{A}$ where matrix $\mathbf{A}$ is given by (5 marks)

$$ \mathbf{A} = \begin{bmatrix} 3/7 & 2/7 & 6/7 \\ 2/7 & 6/7 & -3/7 \\ -6/7 & 3/7 & 2/7 \end{bmatrix} $$

4. If $\mathbf{P}_{UVW} = (1,2,3)^T$, what is $\mathbf{P}_{XYZ}$ given (using ${}^{XYZ}R_{UVW}$ above) (5 marks)
5. If $\mathbf{P}_{XYZ} = (1,2,3)^T$, what is $\mathbf{P}_{UVW}$? given (using ${}^{XYZ}R_{UVW}$ above) (5 marks)
6. If the OUVW system has basis vectors $\mathbf{U} = (1/\sqrt{2}, 0, 1/\sqrt{2})^T$, $\mathbf{V} = (-1/\sqrt{2}, 0, 1/\sqrt{2})^T$, $\mathbf{W} = (0, -1, 0)^T$, and the OXYZ system has basis vectors $\mathbf{X} = (1, 0, 0)^T$, $\mathbf{Y} = (0, 1/\sqrt{2}, -1/\sqrt{2})^T$, $\mathbf{Z} = (0, 1/\sqrt{2}, 1/\sqrt{2})^T$, then what is the corresponding rotation matrix between the two systems? (5 marks)

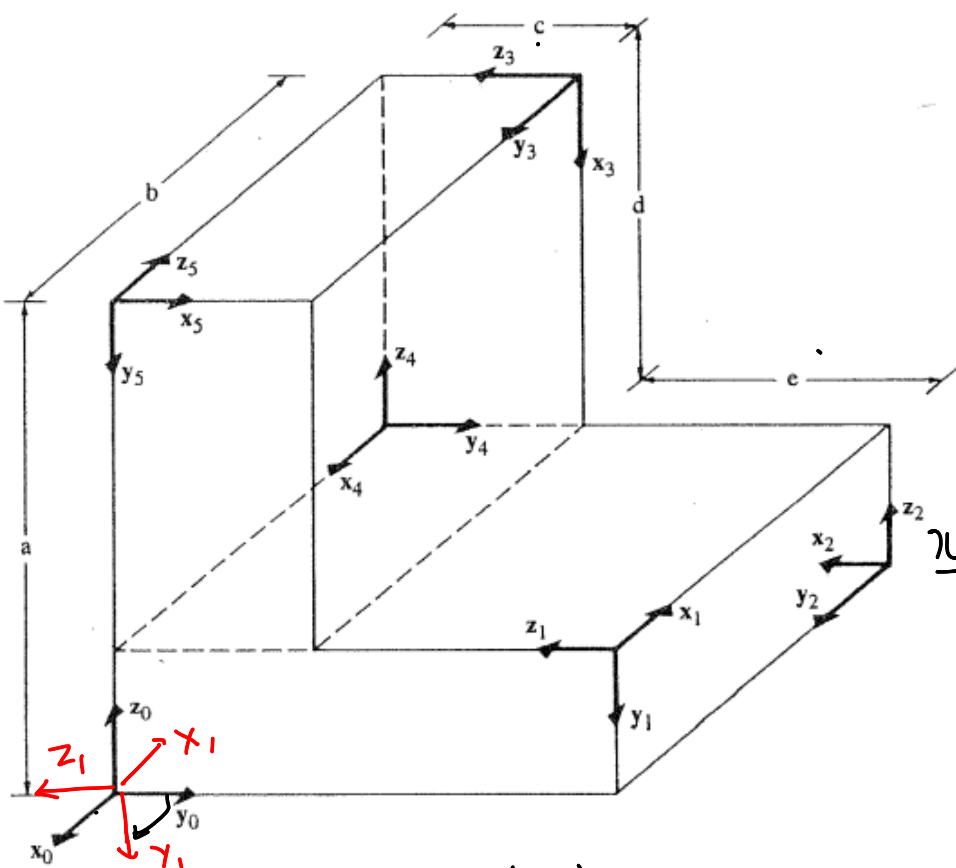
Given the following 4x4 homogeneous transformation matrices:

$$ {}^{B}\mathbf{T}_{A} = \begin{bmatrix} 2/7 & -6/7 & 3/7 & 1 \\ 6/7 & 3/7 & 2/7 & 2 \\ -3/7 & 2/7 & 6/7 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{C}\mathbf{T}_{B} = \begin{bmatrix} 3/7 & 2/7 & 6/7 & 4 \\ 2/7 & 6/7 & -3/7 & 5 \\ -6/7 & 3/7 & 2/7 & 6 \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

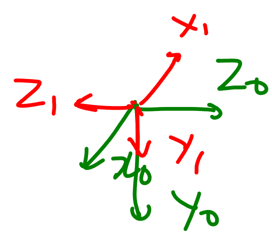
1. Give the inverse of Matrix ${}^{B}\mathbf{T}_{A}$. (5 marks)
2. What is the direction of the X-axis of system **A** w.r.t. system **B**? What is the direction of the Y-axis of system **A** w.r.t system **B**? Where is the origin of system **A** w.r.t. system **B**? (5 marks)
3. What is the direction of the X-axis of system **B** w.r.t. system **A**? What is the direction of the Y-axis of system **B** w.r.t system **A**? Where is the origin of system **B** w.r.t. system **A**? (5 marks)
4. What is ${}^{C}\mathbf{T}_{A}$? (5 marks)
5. For the point $(0, 1, 2)^T$ in system **A**, what are it's coordinates in system **B**? (5 marks)
6. For the point $(0, 1, 2)^T$ in system **B**, what are it's coordinates in system **A**? (5 marks)

2.6 For the figure shown below, find the 4 × 4 homogeneous transformation matrices ${}^{i-1}\mathbf{A}_i$ and ${}^{0}\mathbf{A}_i$ for $i = 1, 2, 3, 4, 5$.

${}^{0}A_1 \quad {}^{1}A_2$



$$\underline{x}_0 = {}^{0}R\,\underline{x}_1 + {}^{0}\underline{t}_1$$



$$\cancel{{}^{0}R_1 = R_{y_0}(180^\circ)\,R_{x_0}(-90^\circ)}$$

$$= \begin{bmatrix} \cos(180^\circ) & 0 & \sin(180^\circ) \\ 0 & 1 & 0 \\ -\sin(180^\circ) & 0 & \cos(180^\circ) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(-90) & -\sin(-90) \\ 0 & \sin(-90) & \cos(-90) \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$ {}^{0}\underline{t}_1 = \begin{bmatrix} 0 \\ +(a-d) \\ +2c \end{bmatrix} $$

$$ {}^{0}R_1 = R_{x_0}(-90^\circ)\, R_{y_0}(180^\circ) $$

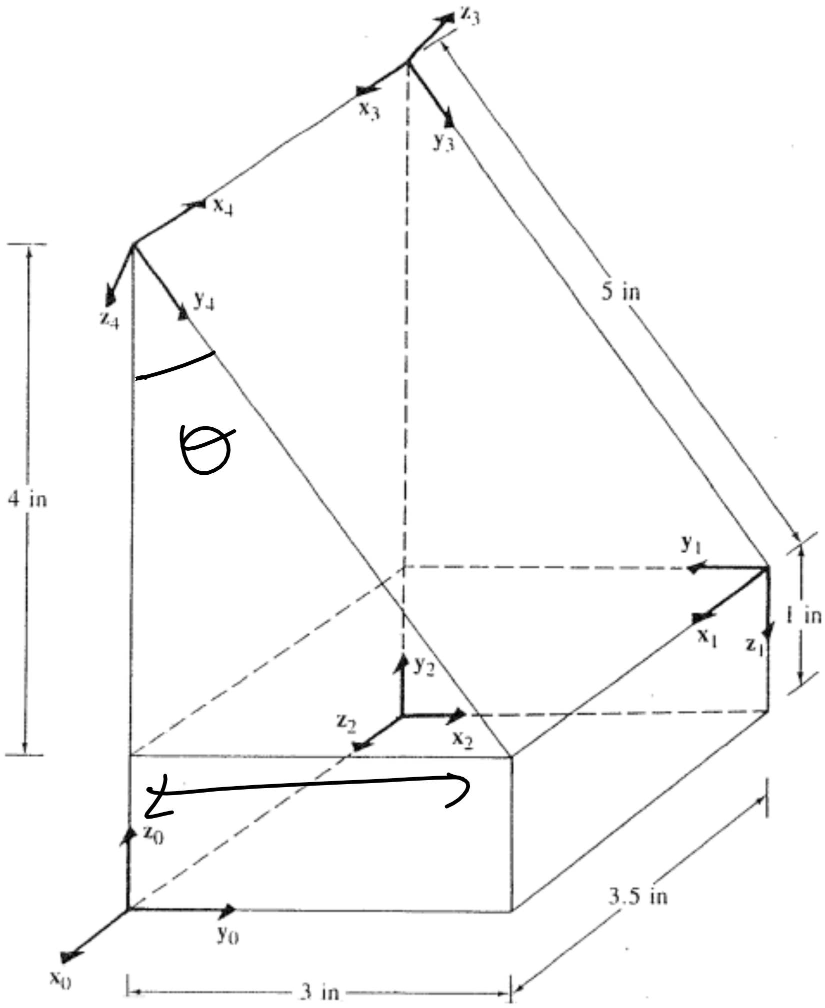
$$ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(-90) & -\sin(-90) \\ 0 & \sin(-90) & \cos(-90) \end{bmatrix} \begin{bmatrix} \cos(180^\circ) & 0 & \sin(180^\circ) \\ 0 & 1 & 0 \\ -\sin(180^\circ) & 0 & \cos(180^\circ) \end{bmatrix} $$

$$ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} $$

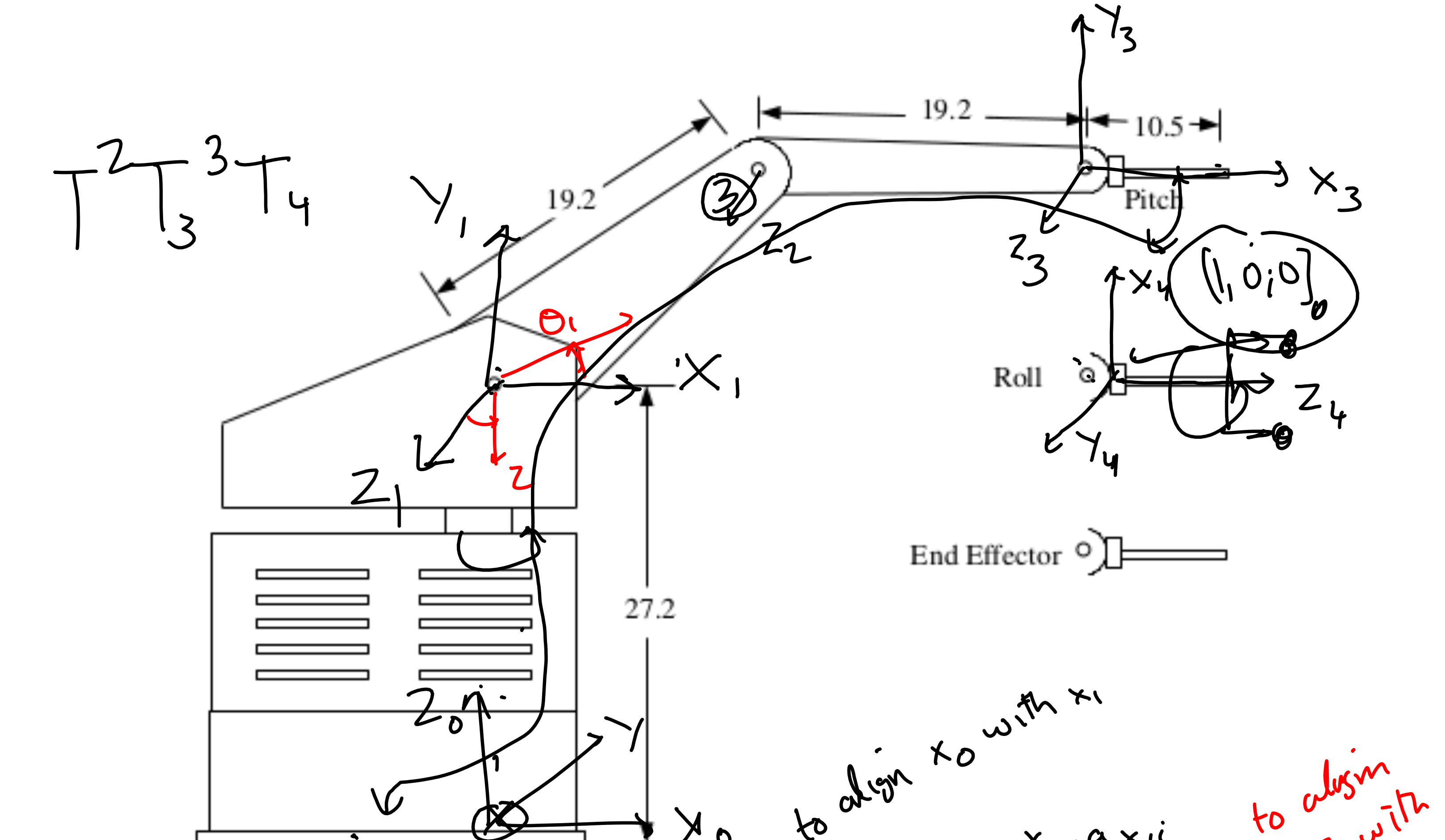
$$ = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} $$

$$ {}^{0}A_1 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & +(a-d) \\ 0 & -1 & 0 & +2c \\ 0 & 0 & 0 & 1 \end{bmatrix} $$

2.7 For the figure shown below, find the 4 × 4 homogeneous transformation matrices ${}^{i-1}\mathbf{A}_i$ and ${}^{0}\mathbf{A}_i$ for $i = 1, 2, 3, 4$.



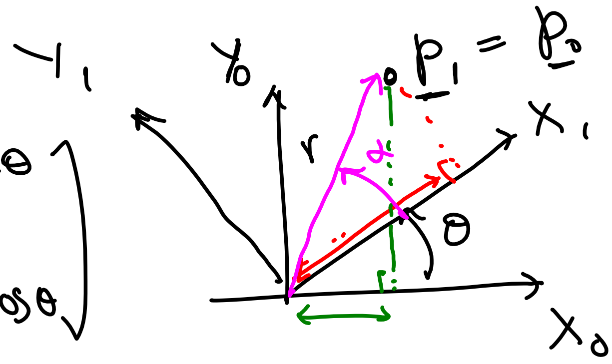
The following applies to our LabVolt robots. Draw the coordinate frame for each link (don't forget to include link 0). Label each axis and indicate how you define the joint angles. Then fill in the joint parameter table. Use a dot for an axis pointing out of the page and an X for an axis pointing in.



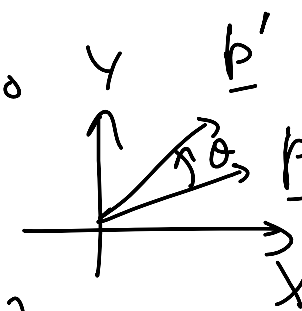
z-axis　　x-axis　　to align Z_0 with Z_1

	Axis	d_i	θ_i	a_i	α_i	
$Z_0 \to Z_1$	1	27.2	var	0	90°	x_1
$Z_1 \to Z_2$	2					
$Z_2 \to Z_3$	3					
$Z_3 \to Z_4$	4	0	90°	0		
	5					

$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



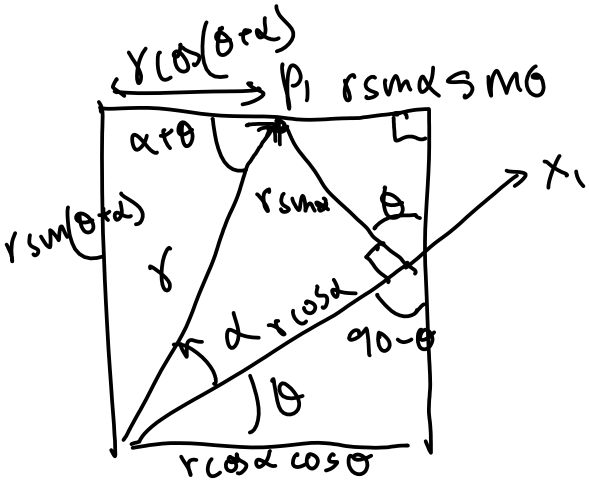
$$\boxed{\underline{p}' = R(\theta)\,\underline{p}}$$



$$R(\theta) \overset{?}{=} {}^1R_0 \quad \text{or} \quad {}^0R_1$$

$$\boxed{\underline{p}_0 = {}^0R_1(\theta)\,\underline{p}_1}$$

$$\underline{p}_1 = \begin{bmatrix} r\cos\alpha \\ r\sin\alpha \end{bmatrix}$$



$$\underline{p}_0 = \begin{bmatrix} r\cos(\theta+\alpha) \\ r\sin(\theta+\alpha) \end{bmatrix} = \begin{bmatrix} r\cos\alpha\cos\theta - r\sin\alpha\sin\theta \\ r\sin\theta\cos\alpha + r\sin\alpha\cos\theta \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{R(\theta)} \underbrace{\begin{bmatrix} r\cos\alpha \\ r\sin\alpha \end{bmatrix}}_{\underline{p}_1}$$

Position and orientation of coordinate frame 0 w.r.t 1

≡ Transformation that will take points from Frame 0 to Frame 1

✓ ARUCO marker detector

${}^{c}R_{m}\ {}^{c}\underline{t}_{m}$

quaternion → Rot mat