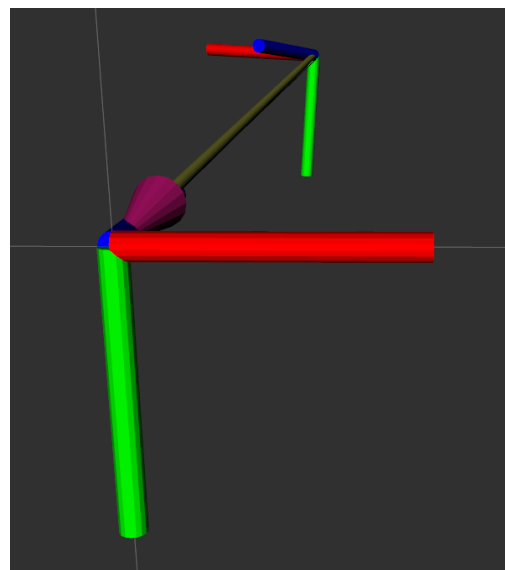
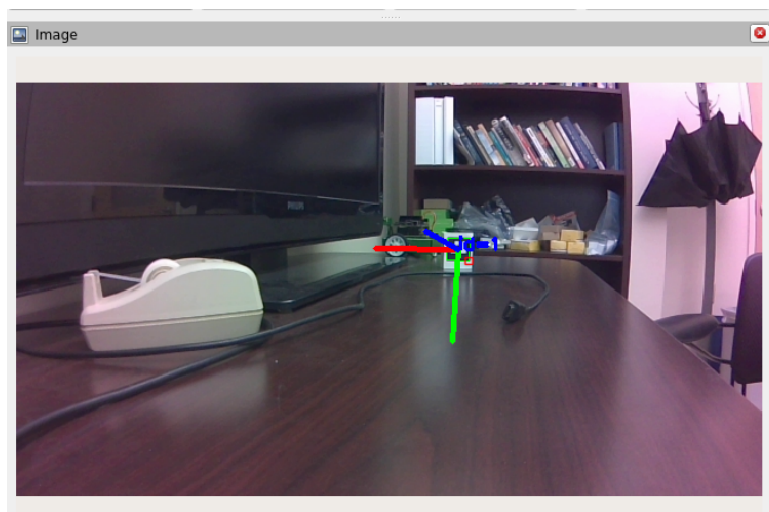


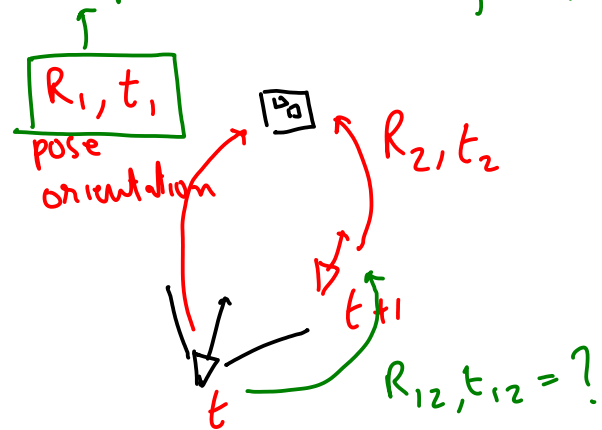
Rotations and translations/Coordinate transformations



```
root@nano-4gb-jp45:/home/jetbot/ece417# ros2 topic info /
aruco_detections
Type: aruco_opencv_msgs/msg/ArucoDetection
Publisher count: 1
Subscription count: 0

root@nano-4gb-jp45:/home/jetbot/ece417# ros2 topic echo --once /
aruco_detections
header:
  stamp:
    sec: 1727998810
    nanosec: 924374790
  frame_id: /v4l_frame
markers:
- marker_id: 1
  pose:
    position:
      x: 0.08918172498053901
      y: -0.10849999597426438
      z: 0.980432215194246
    orientation:
      x: -0.02973468393320003
      y: 0.9811997541144667
      z: -0.03227342049856023
      w: -0.18793966432455353
boards: []
---
```

joint representation = Transformation matrix

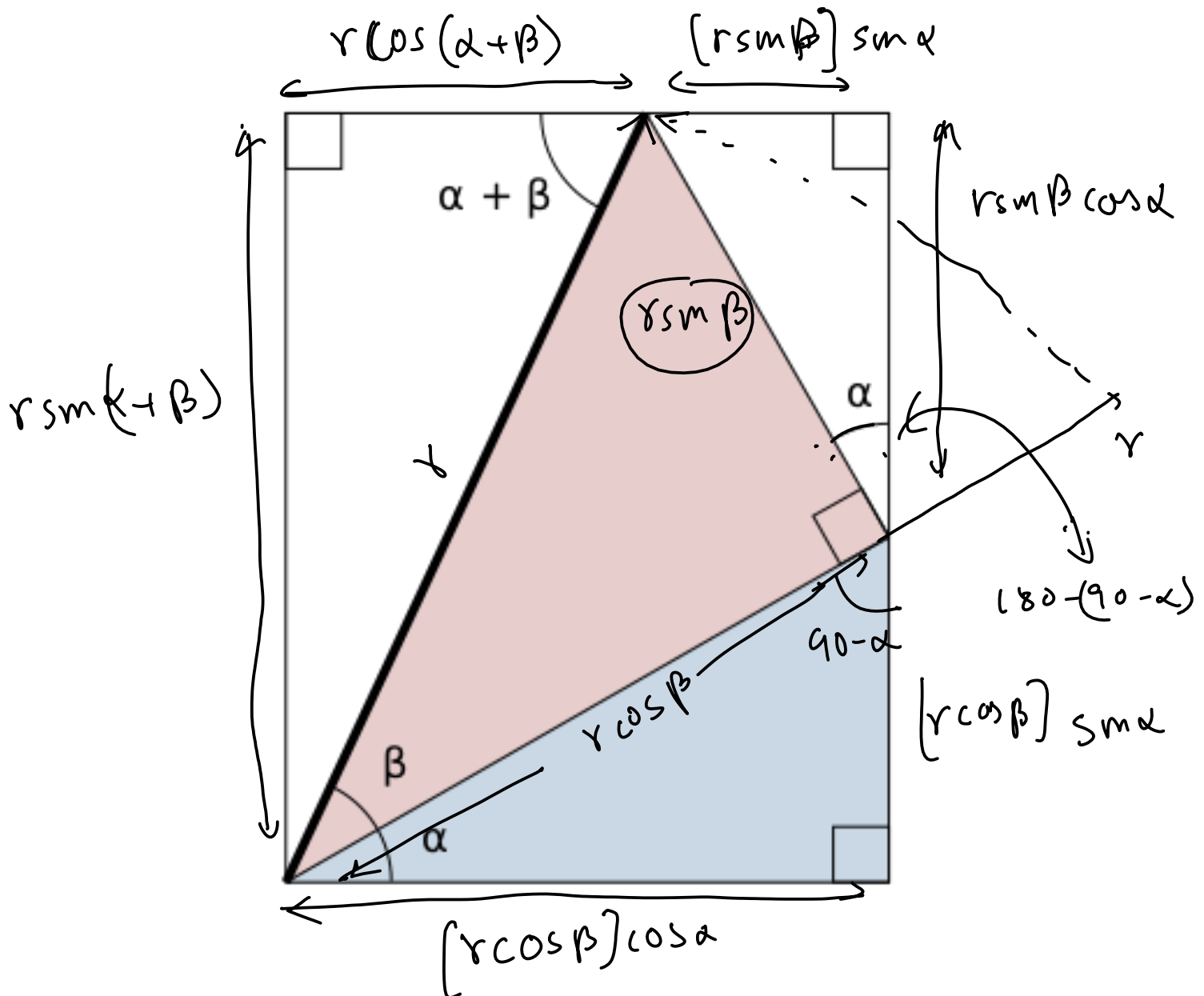
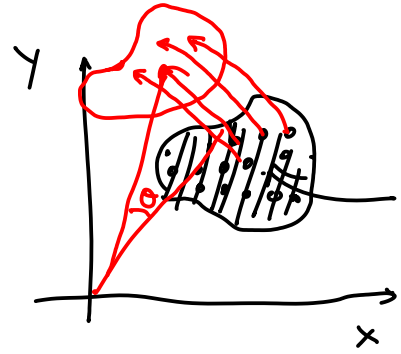


① Rigid bodies are sets of points

↳ Rotate

↳ Translate

All points in the rigid body
Rotate and translate by the
same amount



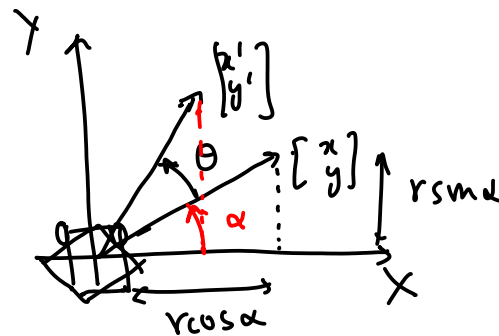
2D Rotation

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = ? \quad \text{in terms of } \theta \text{ and } \begin{bmatrix} x \\ y \end{bmatrix}$$

$$x = r \cos \alpha$$

$$y = r \sin \alpha$$

$$r = \sqrt{x^2 + y^2}$$

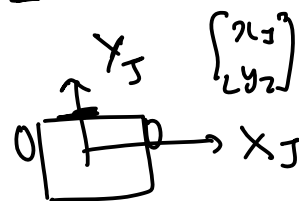


$$\begin{aligned} x' &= r \cos(\theta + \alpha) \\ &= \underline{r \cos \theta \cos \alpha} - \underline{r \sin \theta \sin \alpha} \\ &= x \cos \theta - y \sin \theta \end{aligned}$$

$$y' = r \sin(\theta + \alpha) = r \sin \theta \cos \alpha + r \cos \theta \sin \alpha = x \sin \theta + y \cos \theta$$

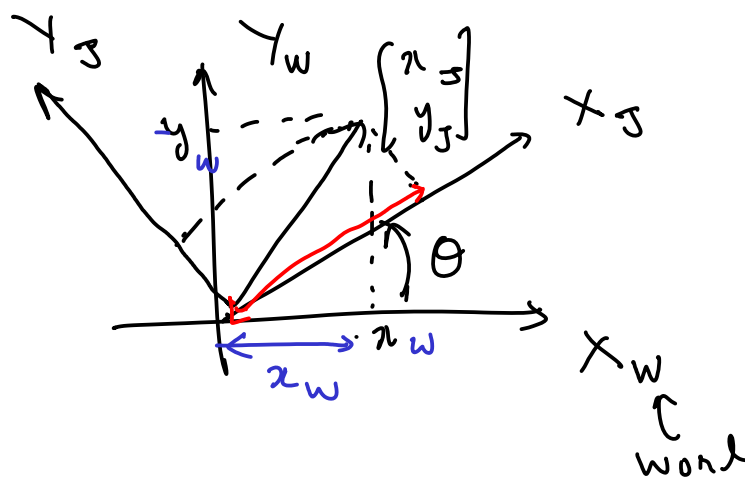
$$\underbrace{\begin{bmatrix} x' \\ y' \end{bmatrix}}_{\underline{p'}} = \underbrace{\begin{bmatrix} x \cos \theta - y \sin \theta \\ x \sin \theta + y \cos \theta \end{bmatrix}}_{\substack{2 \times 1 \\ R(\theta)}} = \underbrace{\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}}_{\substack{2 \times 2 \\ R(\theta)}} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\substack{2 \times 1 \\ \underline{p}}}$$

$$\underline{p'} = R(\theta) \underline{p}$$



$$\underbrace{\begin{bmatrix} x_w \\ y_w \end{bmatrix}}_{\underline{p}_w} = \underbrace{\begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}}_{\substack{2 \times 2 \\ {}^w R_J(\theta)}} \underbrace{\begin{bmatrix} x_J \\ y_J \end{bmatrix}}_{\substack{2 \times 1 \\ \underline{p}_J}}$$

SOURCE
↑
DESTINATION



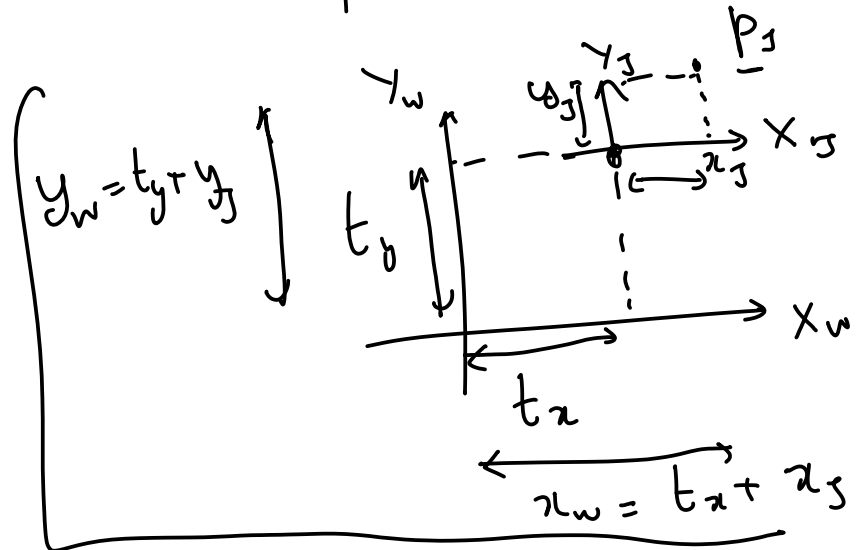
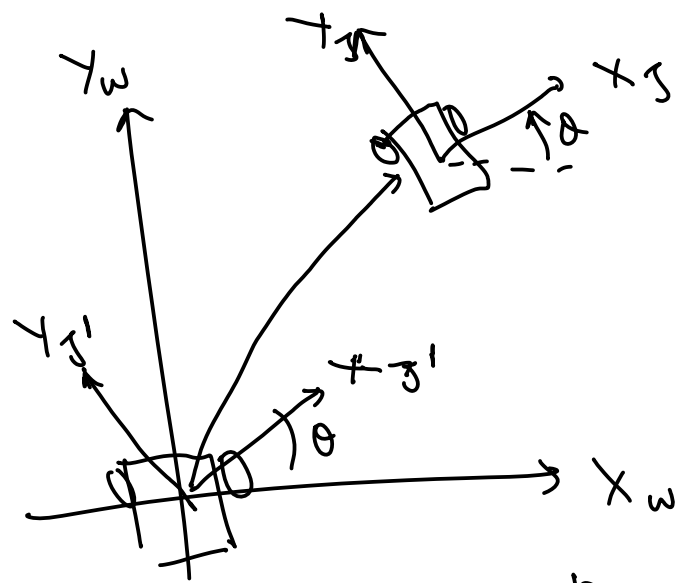
$$\underline{p}_w = {}^w R_J(\theta) \underline{p}_J$$

$$\underline{p}_{j'} = {}^{j'}R_j \underline{p}_j$$

$$\underline{p}_w = \underline{p}_{j'} + {}^w\underline{t}_{j'}$$

$$\underline{p}_w = {}^{j'}R_j \underline{p}_j + {}^w\underline{t}_{j'}$$

$$\underline{p}_w = {}^wR_j \underline{p}_j + {}^w\underline{t}_j$$



$$\begin{aligned} \begin{bmatrix} x_w \\ y_w \end{bmatrix} &= \begin{bmatrix} x_j + t_x \\ y_j + t_y \end{bmatrix} \\ &= \begin{bmatrix} x_j \\ y_j \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \end{bmatrix} \end{aligned}$$

$$\underline{p}_w = \underline{p}_j + {}^w\underline{t}_j$$

Matrices = capital letters

Vectors = small letters
with underscore

Scalars = small letters
without
underscore

Transformation matrix

$$\underline{p}_w = {}^w R_s \underline{p}_s + {}^w \underline{t}_s$$

\nwarrow orientation \swarrow position

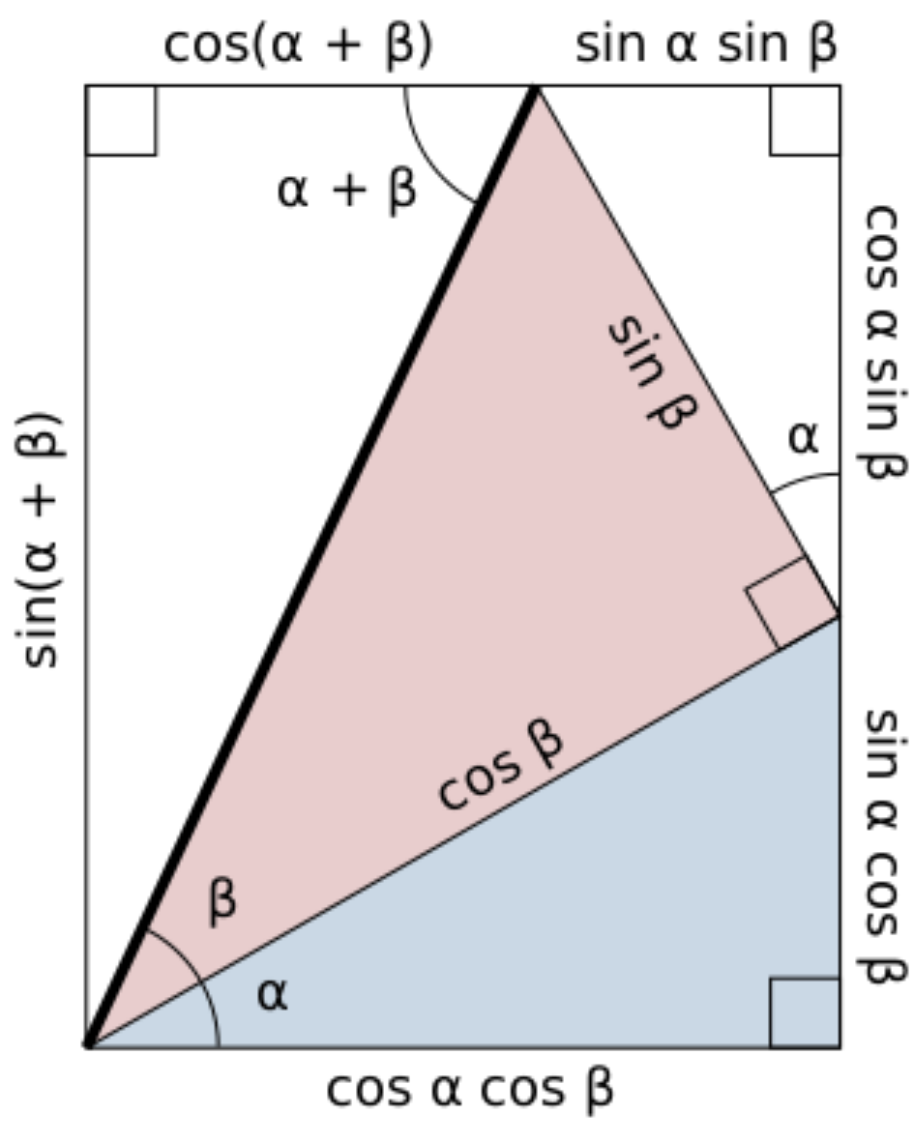
$\underbrace{{}^w R_s, {}^w \underline{t}_s}$

$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}_{2 \times 2} \begin{bmatrix} x_s \\ y_s \end{bmatrix}_{2 \times 1} + \begin{bmatrix} t_1 \\ t_2 \end{bmatrix}_{2 \times 1}$$

$$\begin{bmatrix} x_w \\ y_w \\ 1 \end{bmatrix}_{3 \times 1} = \underbrace{\begin{bmatrix} r_{11} & r_{12} & | & t_1 \\ r_{21} & r_{22} & | & t_2 \\ \hline 0 & 0 & | & 1 \end{bmatrix}}_{\substack{{}^w T_s \\ 3 \times 3}} \begin{bmatrix} x_s \\ y_s \\ 1 \end{bmatrix}_{3 \times 1}$$

\nwarrow homogeneous coordinate

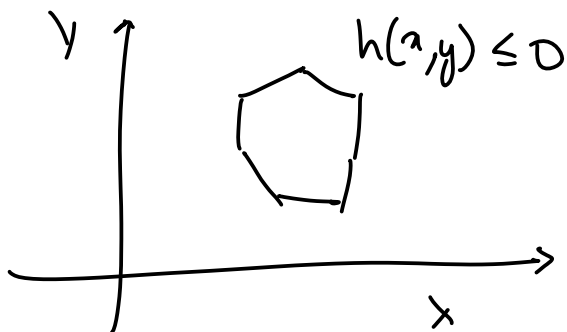
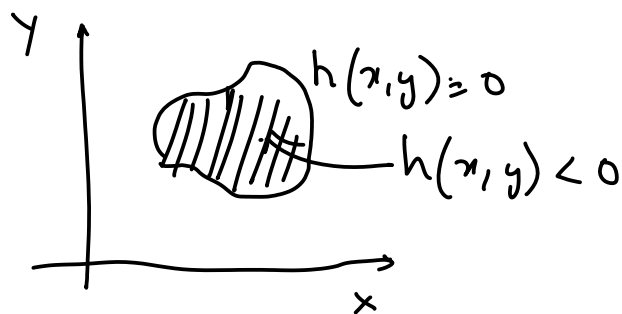
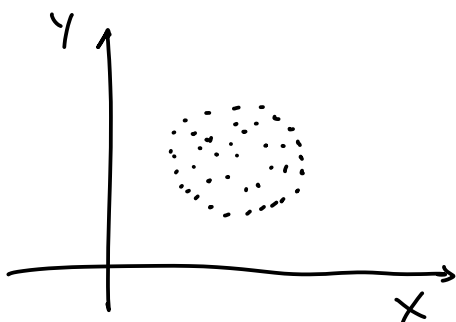
$$\begin{bmatrix} \underline{p}_w \\ 1 \end{bmatrix} = \begin{bmatrix} {}^w R_s & | & {}^w \underline{t}_s \\ \hline \underline{0}_{2 \times 1} & | & 1 \end{bmatrix} \begin{bmatrix} \underline{p}_s \\ 1 \end{bmatrix}$$



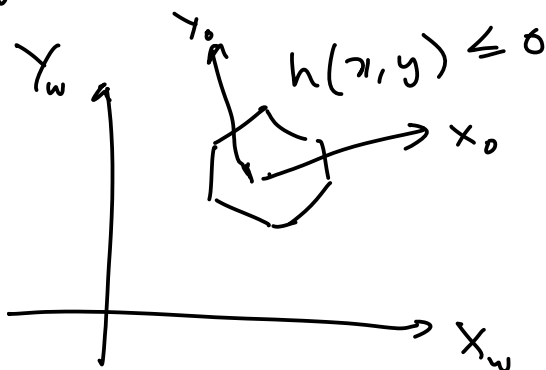
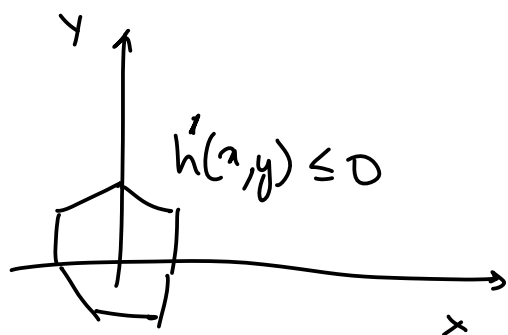
2D coordinate transforms

Rotations + translations
Orientation + position

$$h(x,y) > 0$$



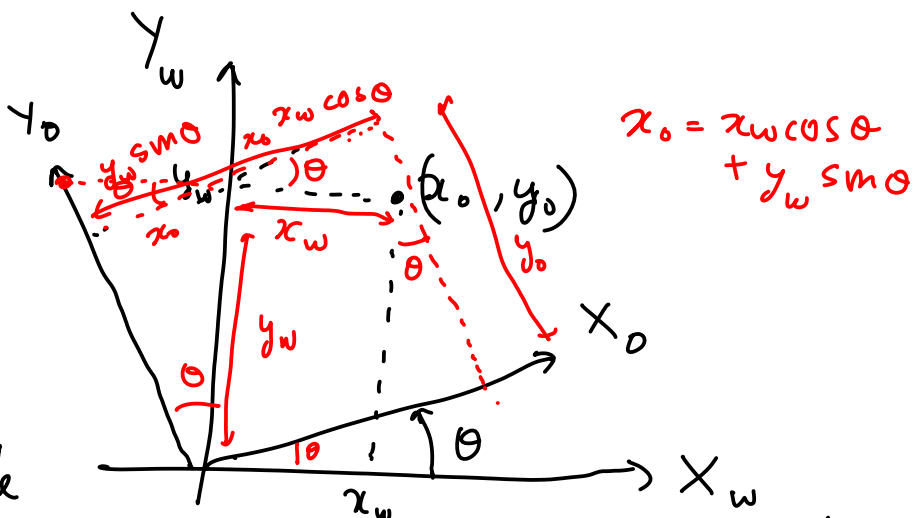
Object = $\{(x,y) \in \mathbb{R}^2 : h(x,y) \leq 0\}$
Set of all points $(x,y) \in \mathbb{R}^2$
such that $h(x,y) \leq 0$
set of all real numbers



2D Rotations

- ① Rotation
- ② Translation

Problem (x_0, y_0) is given
in (x_0, y_0) coordinate
frame. (x_0, y_0) has been rotated by angle θ w.r.t. (x_w, y_w)



Find (x_w, y_w) in world coordinate frame

Proof using Basis vectors

In Linear algebra, Basis vectors are set of orthonormal unit vectors that span the entire space

Span is the set of all vectors that can be obtained by linear combinations of a given set of vectors

$$\text{Span} \{ \underset{\substack{\uparrow \\ \in \mathbb{R}^n}}{\underline{a}}, \underset{\substack{\uparrow \\ \in \mathbb{R}^n}}{\underline{b}} \} = \{ \underset{\substack{\uparrow \\ \in \mathbb{R}^n}}{\underline{\alpha a + \beta b}}, \underset{\substack{\uparrow \\ \in \mathbb{R}^n}}{\beta \in \mathbb{R}} \}$$

Standard Basis vector.

For example, in \mathbb{R}^2
in \mathbb{R}^3

$$\hat{i} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \hat{j} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$
$$\hat{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \hat{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \hat{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\text{in } \mathbb{R}^n \quad \hat{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}_{n \times 1} \dots \hat{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}_{n \times 1}$$

Basis vectors for \mathbb{R}^n

↳ ① All vectors must be perpendicular/orthogonal to each other

↳ ② They must be unit vectors

↳ ③ They must span the entire space \mathbb{R}^n

Let Basis vector for (x_w, y_w) be standard basis vector

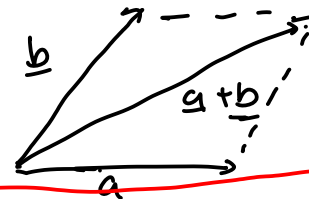
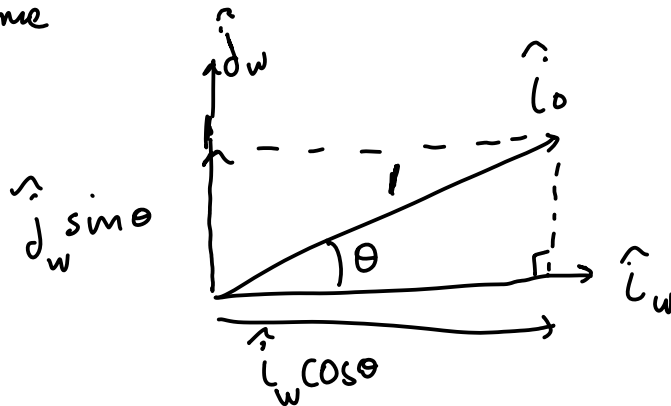
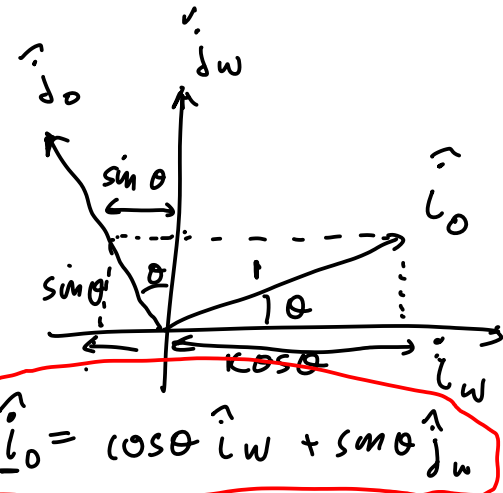
$$\hat{i}_w = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \hat{j}_w = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Any point
in world
coordinate
frame

$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = x_w \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\hat{i}_w} + y_w \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{\hat{j}_w}$$

Any point
in the object
coordinate
frame

$$\begin{bmatrix} x_o \\ y_o \end{bmatrix} = x_o \hat{i}_o + y_o \hat{j}_o$$



$$\hat{j}_o = -\hat{i}_w \sin \theta + \hat{j}_w \cos \theta$$

world

object

$$x_w \hat{i}_w + y_w \hat{j}_w = x_o \hat{i}_o + y_o \hat{j}_o$$

$$= x_o [\cos \theta \hat{i}_w + \sin \theta \hat{j}_w] + y_o [-\hat{i}_w \sin \theta + \hat{j}_w \cos \theta]$$

$$= [x_o \cos \theta - y_o \sin \theta] \hat{i}_w + [x_o \sin \theta + y_o \cos \theta] \hat{j}_w$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

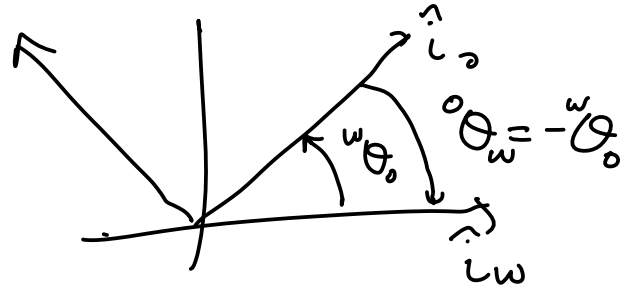
$$= \begin{bmatrix} x_o \cos \theta - y_o \sin \theta \\ x_o \sin \theta + y_o \cos \theta \end{bmatrix}$$

$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x_o \\ y_o \end{bmatrix}$$

Because
we are
using
standard
basis
for world
coordinate
frame

$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \begin{bmatrix} {}^w R_0(\theta) \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$$

$${}^w R_0(\theta) := \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$



$$\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x_w \\ y_w \end{bmatrix}$$

$${}^0 R_w(\theta) = {}^w R_0(-\theta) = {}^w R_0^T(\theta)$$

$$= \begin{bmatrix} \cos \theta & +\sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x_w \\ y_w \end{bmatrix}$$

$$\begin{bmatrix} {}^w R_0^T \end{bmatrix} {}^w R_0 = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2 \theta + \sin^2 \theta & -c(\theta)s(\theta) + s(\theta)c(\theta) \\ -s(\theta)c(\theta) + c(\theta)s(\theta) & s^2(\theta) + c^2(\theta) \end{bmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \underline{I}_{2 \times 2}$$

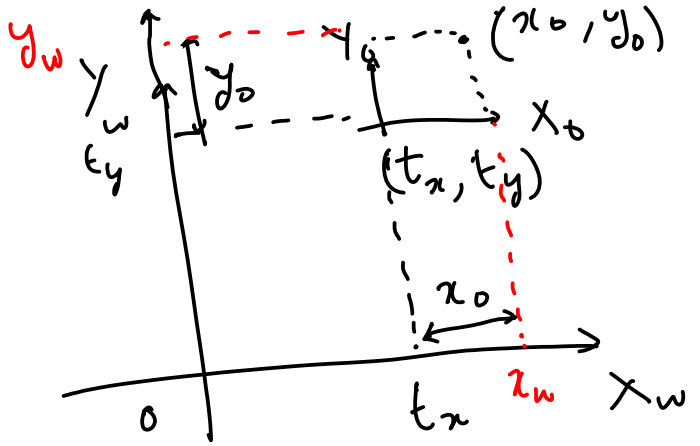
$$\underline{R^T R = I}$$

$$(A^{-1})A = I$$

$$R^{-1} = R^T$$

2D Translation

$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \underbrace{\begin{bmatrix} t_x \\ t_y \end{bmatrix}}_{\text{translation vector}} + \begin{bmatrix} x_o \\ y_o \end{bmatrix}$$

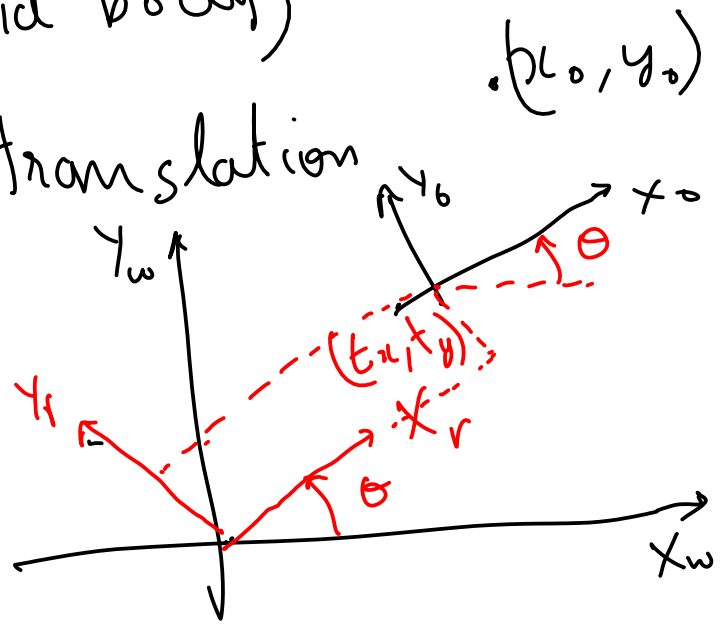


2D transformation (Rigid body)

Rotation followed by translation

$$\begin{bmatrix} x_r \\ y_r \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x_o \\ y_o \end{bmatrix}$$

$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \begin{bmatrix} t_x \\ t_y \end{bmatrix} + \begin{bmatrix} x_r \\ y_r \end{bmatrix}$$



$$\begin{bmatrix} x_w \\ y_w \end{bmatrix} = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} \begin{bmatrix} x_o \\ y_o \end{bmatrix} + \underbrace{\begin{bmatrix} t_x \\ t_y \end{bmatrix}}_{\underline{t}}$$

$$\underline{x}_w = {}^w_0 R(\theta) \underline{x}_o + {}^w_0 \underline{t}$$

$$\underline{x}_0 = ? (\underline{x}_w)$$

$$\rightarrow \underline{x}_w - {}^w_0 \underline{t} = {}^w_0 R \underline{x}_0$$

$${}^w_0 R = {}^w_0 R(0)$$

Multiply on the left of both sides by ${}^w_0 R^T$

$$\rightarrow {}^w_0 R^T (\underline{x}_w - {}^w_0 \underline{t}) = ({}^w_0 R^T {}^w_0 R) \underline{x}_0$$

$$\frac{\underline{x}_w - {}^w_0 \underline{t}}{{}^w_0 R} = \underline{x}_0$$

NEVER EVER
DO THIS

$$\underline{x}_0 = {}^w_0 R^T \underline{x}_w - {}^w_0 R^T {}^w_0 \underline{t} \quad \text{--- (1)}$$

$$\underline{x}_0 = {}^0_w R \cdot \underline{x}_w + {}^0_w \underline{t} \quad \text{--- (2)}$$

Compare (1) and (2)

$${}^0_w R = {}^w_0 R^T$$

and

$${}^0_w \underline{t} = - {}^w_0 R^T {}^w_0 \underline{t}$$

$$\underline{x}_w = {}^w_0 R \underline{x}_0 + {}^w_0 \underline{t}$$

$$\begin{bmatrix} \underline{x}_w \\ 1 \end{bmatrix} = \begin{bmatrix} x_w \\ y_w \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} {}^wR_{2 \times 2} & {}^w\vec{t}_{2 \times 1} \\ O_{1 \times 2}^T & 1 \end{bmatrix}}_{\substack{{}^wT_o \\ 3 \times 3}} \begin{bmatrix} \underline{x}_o_{2 \times 1} \\ 1 \end{bmatrix}$$

Block matrix

Block Matrix multiplication

$$\checkmark \quad \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} E & F \\ G & H \end{bmatrix} = \begin{bmatrix} AE + BG & AF + BH \\ CE + DG & CF + DH \end{bmatrix}$$

$$\begin{bmatrix} {}^wR_{2 \times 2} & {}^w\vec{t}_{2 \times 1} \\ O_{1 \times 2}^T & 1 \end{bmatrix} \begin{bmatrix} \underline{x}_o_{2 \times 1} \\ 1 \end{bmatrix} = \begin{bmatrix} {}^wR \underline{x}_o + {}^w\vec{t}_o \\ O^T \underline{x}_o + 1 \end{bmatrix} = \begin{bmatrix} {}^wR \underline{x}_o + {}^w\vec{t}_o \\ 1 \end{bmatrix} = \begin{bmatrix} \underline{x}_w \\ 1 \end{bmatrix}$$

$$\underline{x}_w = {}^wT_o \underline{x}_o$$

$${}^wT_o = \begin{bmatrix} {}^wR_{2 \times 2} & {}^w\vec{t}_{2 \times 1} \\ O_{1 \times 2}^T & 1 \end{bmatrix}$$

Transformation matrix

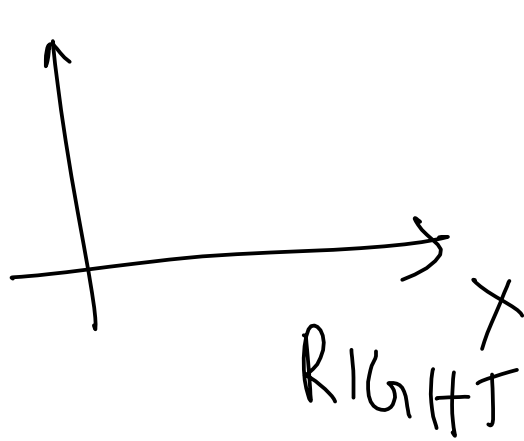
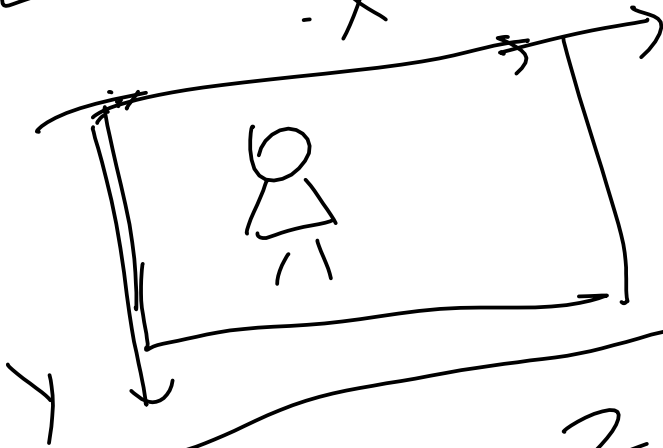
In 2D
and
3D

Right hand ✓

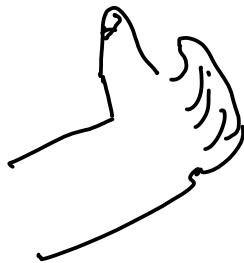
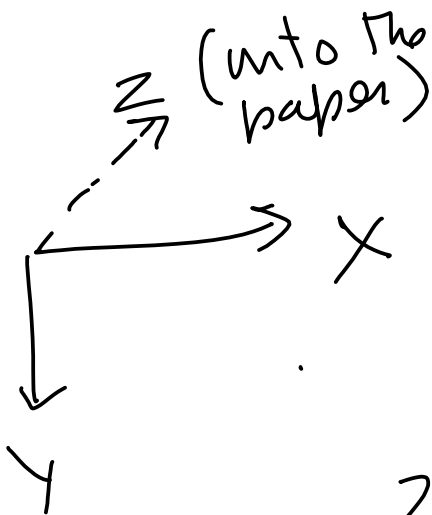
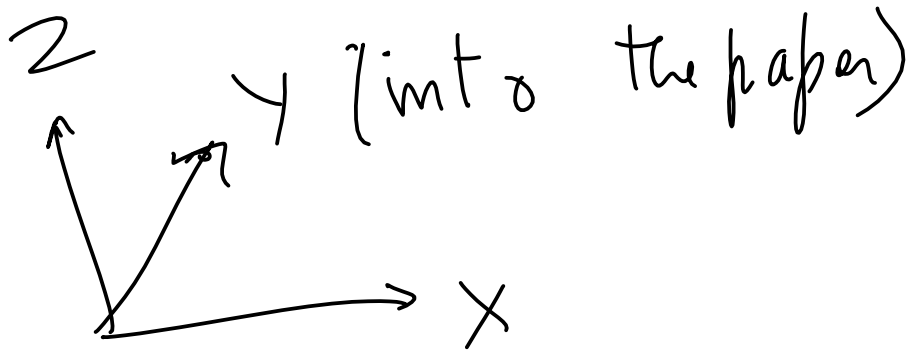
Left hand ✗

LEFT

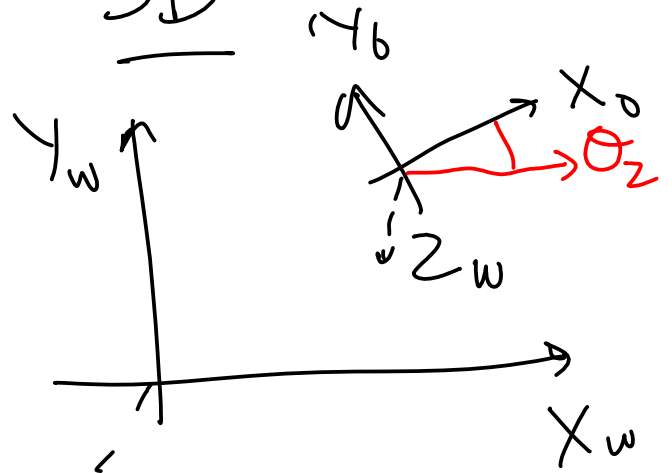
-X



In 3D



Extending 2D to 3D



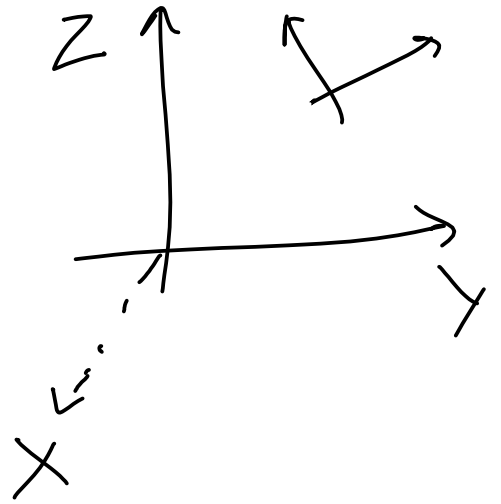
\hat{z}_w (out of the paper)

$$\begin{bmatrix} x_w \\ y_w \\ z_w = 0 \end{bmatrix} = \begin{bmatrix} \cos \theta_2 & -\sin \theta_2 & 0 \\ \sin \theta_2 & \cos \theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_o \\ y_o \\ z_o \end{bmatrix} + \begin{bmatrix} t_x \\ t_y \\ 0 \end{bmatrix}$$

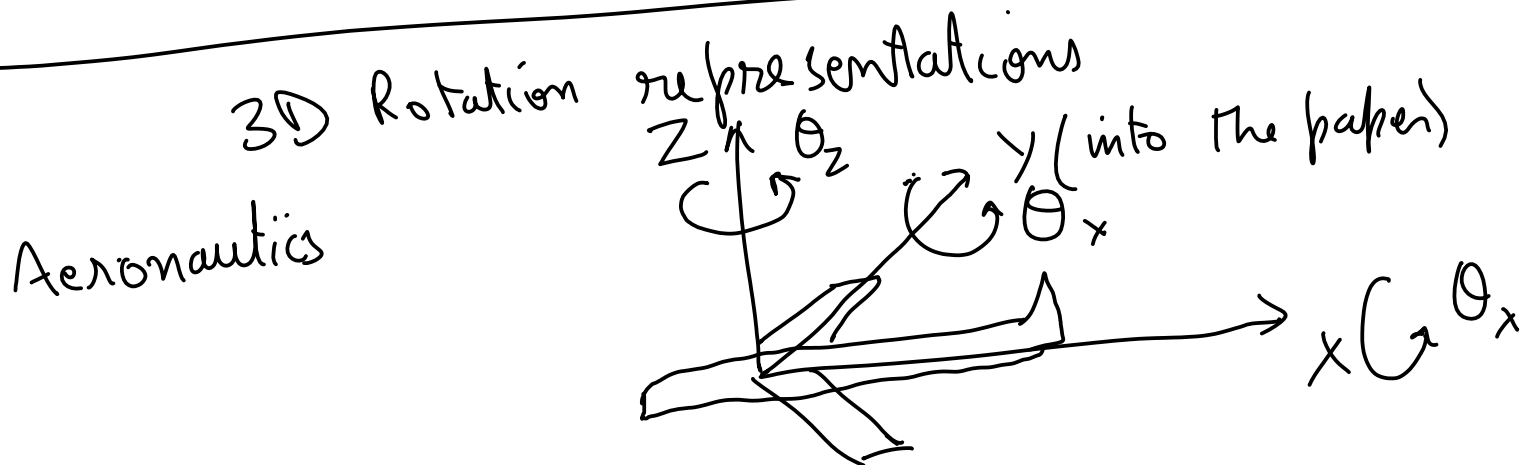
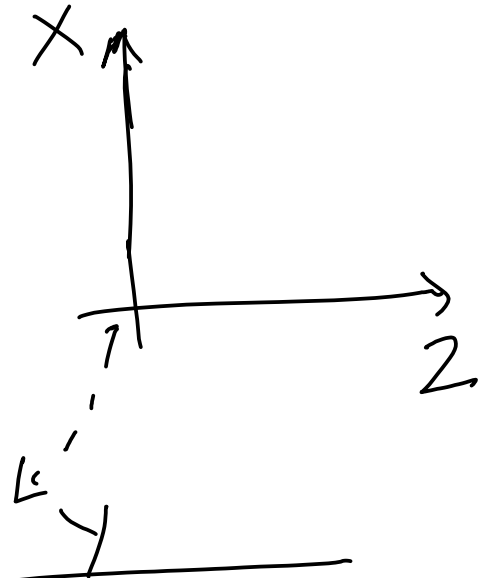
Rotation along Z-axis changes
only X-Y coordinates

$$R(\theta_2) = \begin{bmatrix} \cos \theta_2 & -\sin \theta_2 & 0 \\ \sin \theta_2 & \cos \theta_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R(\theta_x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & -\sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix}$$



$$R(\theta_y) = \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix}$$

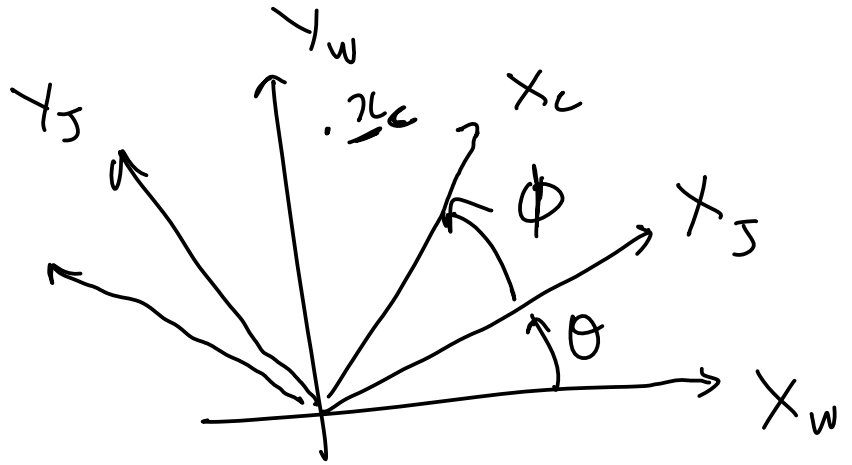


$\theta_x = \text{roll}$
 $\theta_y = \text{pitch}$
 $\theta_z = \text{yaw}$

$$R = R(\theta_z) R(\theta_y) R(\theta_x)$$

↑
Yaw
↑
Pitch
↑
Roll

Chain rotation, translation, transformations



$$\underline{x}_j = {}^j_c R(\phi) \underline{x}_c$$

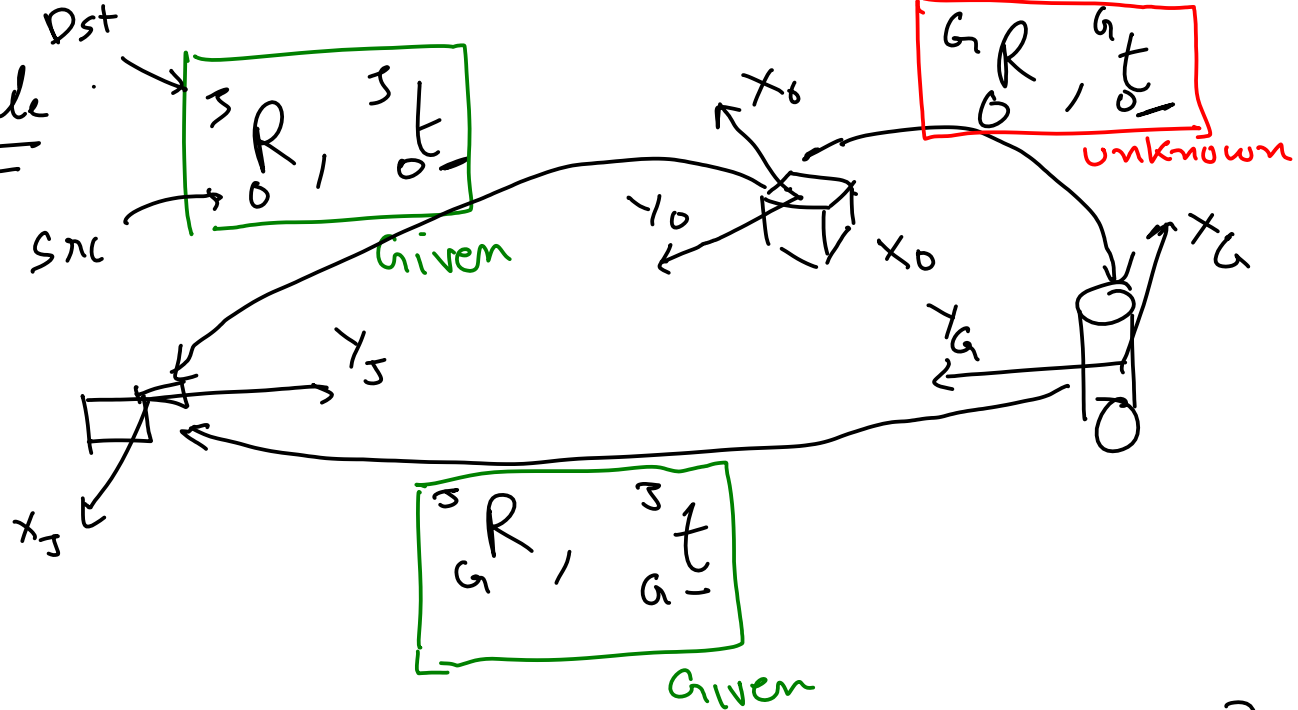
$$\underline{x}_w = {}^w_j R(\theta) \underline{x}_j$$

$$= R(\theta) \left[R(\phi) \underline{x}_c \right]$$

$$= \underbrace{\left(R(\theta) R(\phi) \right)}_{{}^w_c R} \underline{x}_c$$

$${}^w_c R = {}^w_j R(\theta) {}^j_c R(\phi)$$

$${}^w_c T = {}^w_j T {}^j_c T$$

Example

$${}^0T_1 = \begin{bmatrix} {}^0R_1 & {}^0t_1 \\ 0^T & 1 \end{bmatrix}, \quad {}^1T_2 = \begin{bmatrix} {}^1R_2 & {}^1t_2 \\ 0^T & 1 \end{bmatrix}$$

$${}^0T_2 = \begin{bmatrix} {}^0R_2 & {}^0t_2 \\ 0^T & 1 \end{bmatrix}, \quad {}^0T_3 = \left({}^0T_2^{-1} \right) {}^2T_3$$

$$= {}^0T_2 {}^2T_3 \dots$$

$${}^0T_3 \underline{x}_3 = \left({}^0T_2 \left({}^2T_3 \underline{x}_3 \right) \right)$$

\underline{x}_2

\underline{x}_3

\underline{x}_2

① $R_{3D} = R(\theta_z) R(\theta_y) R(\theta_x)$

$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ \text{yaw} & \text{pitch} & \text{roll} \end{array}$

This sequence

$\begin{array}{c} XYZ \\ \hline ZYX \\ \vdots \end{array}$

6 possible
= 3!

$\dot{\theta}_x \xrightarrow{\text{then}} \theta_y \downarrow \text{then} \theta_z$

Euler angle representation of 3D rotation is a sequence of rotation around standard axis

Euler representation with XYZ then

$R_{3D} = R(\theta_z) R(\theta_y) R(\theta_x)$

Conversion from Euler angles to Rotation matrix

How to do The opposite?

convert from Rotation matrix to Euler angles?

$$R(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix} = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}$$

$$\theta = ? = \tan^{-1} \left(\frac{r_{21}}{r_{11}} \right) \in \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$$

$$\theta = \arctan 2(r_{21}, r_{11}) \in (-\pi, \pi)$$

$$\begin{array}{c}
 \text{3D} \quad \downarrow^z \quad \downarrow^y \quad \downarrow^x \\
 R = R(\psi) R(\phi) R(\theta) \\
 = \begin{bmatrix} c(\psi) & -s(\psi) & 0 \\ s(\psi) & c(\psi) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} c(\phi) & 0 & s(\phi) \\ 0 & 1 & 0 \\ -s(\phi) & 0 & c(\phi) \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & c(\theta) & -s(\theta) \\ 0 & s(\theta) & c(\theta) \end{bmatrix}
 \end{array}$$

$$= \begin{bmatrix} \boxed{c(\psi)c(\phi)} & c(\psi)c(\phi)s(\theta) & c(\psi)s(\phi)c(\theta) + s(\psi)s(\theta) \\ \boxed{s(\psi)c(\phi)} & s(\psi)s(\phi)s(\theta) + c(\psi)c(\theta) & s(\psi)s(\phi)c(\theta) - c(\psi)s(\theta) \\ \boxed{-s(\phi)} & \boxed{c(\phi)s(\theta)} & \boxed{c(\phi)c(\theta)} \end{bmatrix}$$

$$= \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

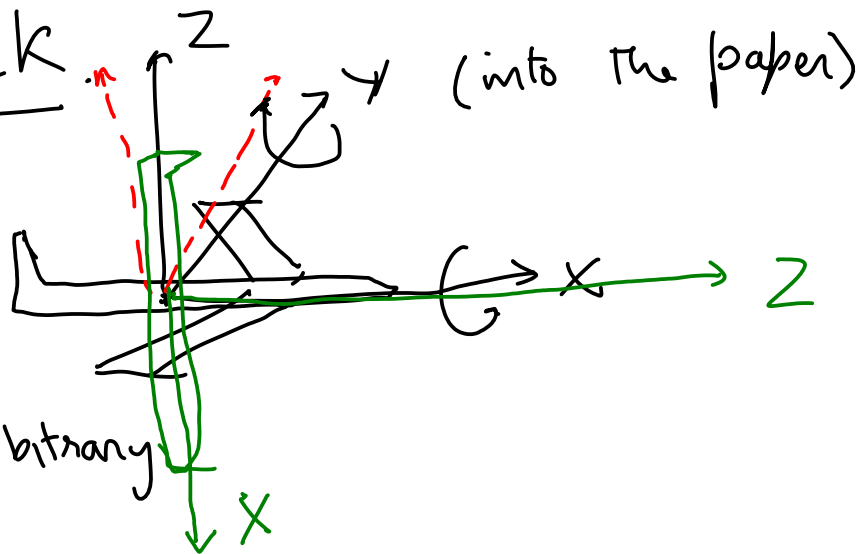
$$\phi = -\sin^{-1}(r_{31}) \in [0, \pi]$$

$$\frac{r_{21}}{r_{11}} = \frac{\sin(\psi) \cancel{c(\phi)}}{\cos(\psi) \cancel{c(\phi)}} \Rightarrow \psi = \arctan 2(r_{21}, r_{11})$$

$$\theta = \arctan 2(r_{32}, r_{33})$$

conversion from Rotation matrix to Euler angles

Gimbal lock



$$\theta_x = 30^\circ \leftarrow \text{arbitrary}$$

$$[\theta_y = \underline{90^\circ}]$$

$$\theta_z = 45^\circ \leftarrow \text{arbitrary}$$

Euler angles $\xrightarrow{\text{deterministically}}$ Rot mat
 $\xleftarrow{\text{multiple solutions}}$

Other representations. It is impossible to unambiguously represent 3D rotation with only 3 numbers

Degree of freedom

but needs 4 numbers
+ 1 constraint
to represent it

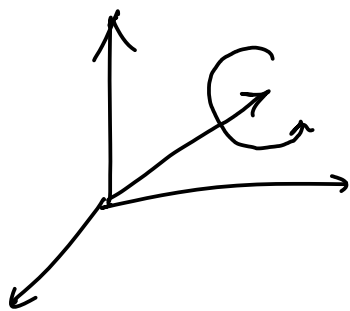
3D rot = 3 DOF

② Axis-angle representation ← (3) Quaternions

↓
Rot mats
↓
Algebra

Quaternions
[quaternion algebra]
≡ [complex numbers]

② Axis-angle representation



Any 3D rotation can be represented
as a unit vector (axis) and rotation angle
around it.

$$\text{Axis} = \underline{a} = [a_x, a_y, a_z]$$

$$\text{angle} = \theta$$

$$\text{constraint: } \|a\|_2 = \sqrt{a_x^2 + a_y^2 + a_z^2} = 1$$

$$\left. \begin{array}{l} \text{Free scalars} = 4 \\ \text{Constraint} = 1 \end{array} \right\} \text{Degree of freedom} = 3$$

DOF of 2D Rot matrix = ?

$$R = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}$$

Free scalars = 4

Two vector constraints :

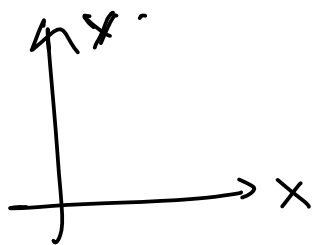
$$\left[\begin{array}{l} R^T R = I = R R^T \\ \det(R) = +1 \end{array} \right]$$

2 scalar constraints

1 scalar constraint

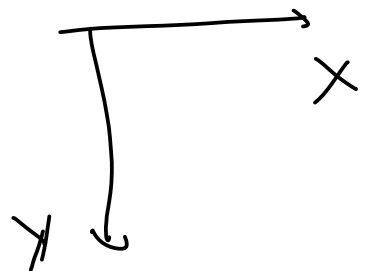
Reflection matrices also satisfy

Check for valid rotation



not Rotation

$$\begin{array}{c} \xrightarrow{\text{reflection}} \\ \det(\text{Reflection}) \\ = -1 \end{array}$$



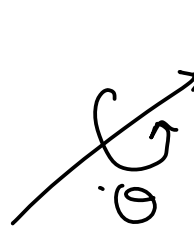
Rodrigues rotation formula

Axis angle \rightarrow Rot matrix
 (θ, \hat{k})

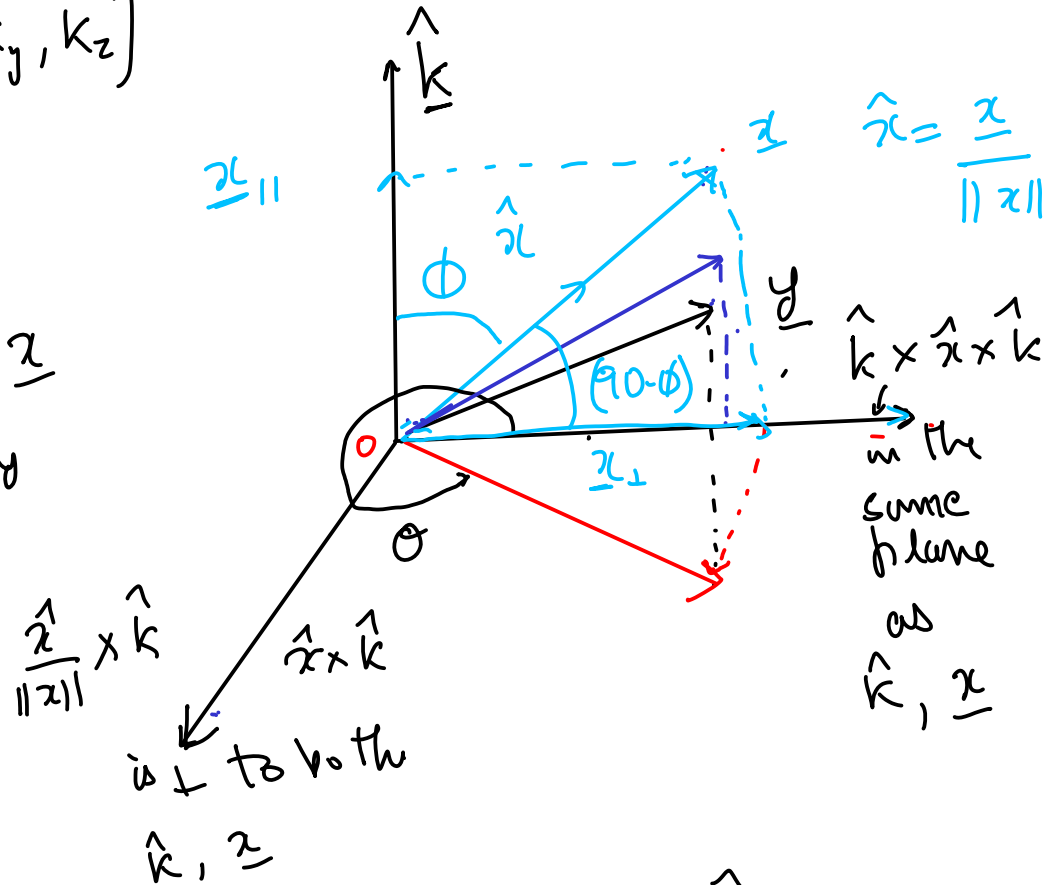
$$R = I + \sin \theta [k_x] + (1 - \cos \theta) [k_x]^2$$

axis angle representation \longrightarrow Rotation matrix (3D)
 Rodrigues rotation formula

$\hat{k} = [k_x, k_y, k_z]$



$\underline{y} =$ by rotating \underline{x} around \hat{k} by angle θ



In the plane of \hat{k} and $\hat{k} \times \hat{x} \times \hat{k}$, \underline{x} can be projected into two component

$$\underline{x} = \underline{x}_{\parallel} + \underline{x}_{\perp}$$

$$\underline{x}_{\parallel} = (\hat{k} \cdot \underline{x}) \hat{k}$$

$$\underline{x}_{\perp} = \left[\left(\hat{k} \times (\hat{x} \times \hat{k}) \right) \cdot \underline{x} \right] (\hat{k} \times \hat{x} \times \hat{k})$$

$$\underline{x}_{\perp} = \hat{k} \times (\underline{x} \times \hat{k}) = \hat{k} \times (-\hat{k} \times \underline{x}) = \underline{\hat{k} \times (\hat{k} \times \underline{x})}$$

$\hat{k} \cdot \underline{x} = |\hat{k}| |\underline{x}| \cos \phi$

$$\rightarrow |\underline{x} \times \hat{k}| = |\underline{x}| |\hat{k}| \sin \phi$$

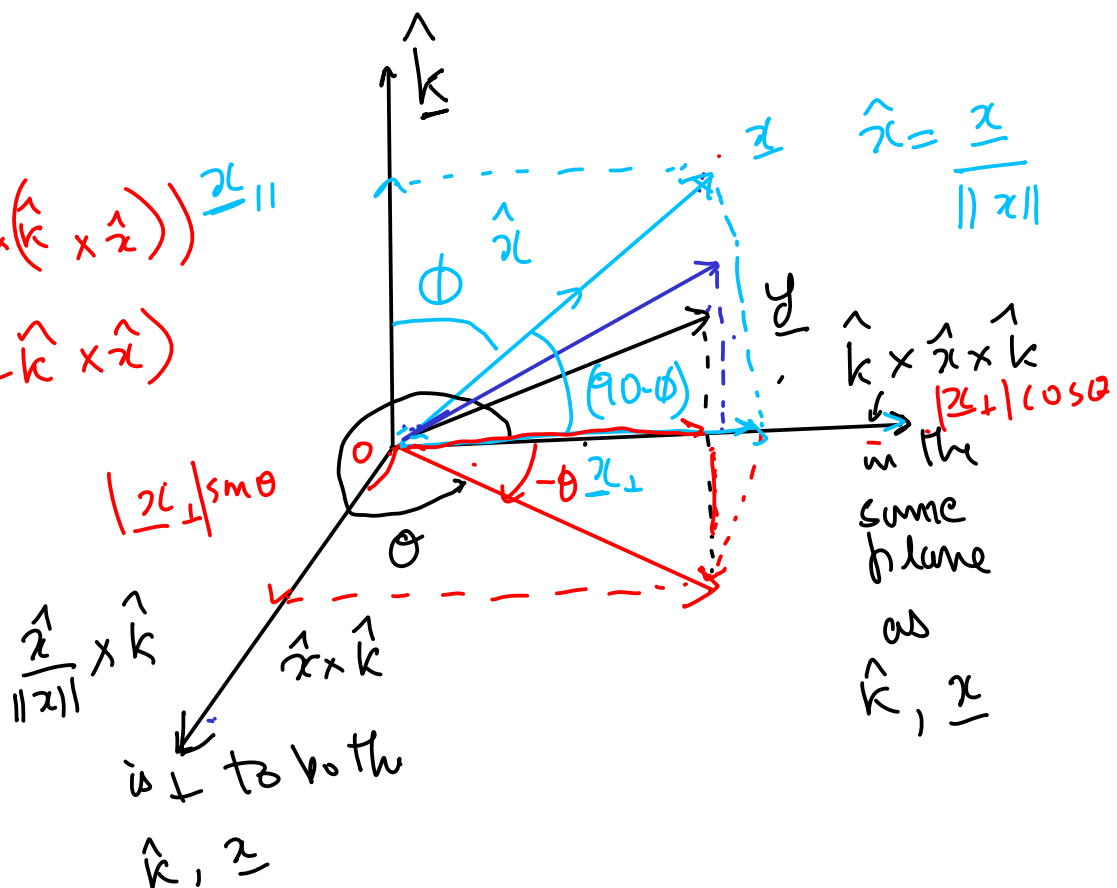
$$\rightarrow \underline{x} \times \hat{k} = (\hat{x} \times \hat{k}) (|\underline{x}| \sin \phi)$$

$$= |\underline{x}| \cos \phi$$

$$|\underline{x}| \sin \phi = |\underline{x}| \sqrt{1 - \cos^2 \phi}$$

$$\underline{y} = \underline{x}_{||} + \underline{x}_{\perp \text{rot}}$$

$$\underline{x}_{\perp \text{rot}} = |\underline{x}_{\perp}| \cos(\theta) (\hat{k} \times (\hat{k} \times \hat{x})) + |\underline{x}_{\perp}| \sin(\theta) (-\hat{k} \times \hat{x})$$



$$\underline{x}_{\perp \text{rot}} = \underline{x}_{\perp} \cos \theta (-\hat{k} \times (\hat{k} \times \hat{x})) - \underline{x}_{\perp} \sin \theta (-\hat{k} \times \hat{x})$$

$$= \cos \theta (-\hat{k} \times (\hat{k} \times \underline{x}))$$

$$- \sin \theta (-\hat{k} \times \underline{x})$$

$$= \sin \theta (\hat{k} \times \underline{x}) - \cos \theta (\hat{k} \times (\hat{k} \times \underline{x}))$$

$$(\hat{k} \times \hat{x}) |\underline{x}_{\perp}| = (\hat{k} \times \hat{x}) |\underline{x}| \sin \theta = \hat{k} \times \underline{x}$$

$$|\underline{a} \times \underline{b}| = |\underline{a}| |\underline{b}| \sin \phi$$

$$\underline{y} = \underline{x}_{||} + \underline{x}_{\perp \text{rot}}$$

Rodrigues' formula

$$\underline{y} = (\hat{k} \cdot \underline{x}) \hat{k} + \sin \theta (\hat{k} \times \underline{x}) - \cos \theta (\hat{k} \times (\hat{k} \times \underline{x}))$$

$$x_{||} = (\hat{k} \cdot \underline{x}) \hat{k}$$

$$\underline{x} = \underline{x}_{||} + \underline{x}_{\perp}$$

$$= \underline{x} - \underline{x}_{\perp}$$

$$= \underline{x} - \left(-\hat{k} \times (\hat{k} \times \underline{x}) \right)$$

$$= \underline{x} + \hat{k} \times (\hat{k} \times \underline{x})$$

$$\underline{y} = \underline{x} + \hat{k} \times (\hat{k} \times \underline{x}) + \sin \theta (\hat{k} \times \underline{x}) - \cos \theta (\hat{k} \times (\hat{k} \times \underline{x}))$$

$$\underline{y} = \underline{x} + \sin \theta (\hat{k} \times \underline{x}) + (1 - \cos \theta) (\hat{k} \times (\hat{k} \times \underline{x}))$$

$$\underline{y} = \underbrace{R(\theta, \hat{k})}_{\text{matrix}} \underline{x}$$

Writing cross product using matrix notation

$$\underline{a} \times \underline{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$$= \hat{i}(a_y b_z - b_y a_z) + \hat{j}(a_z b_x - b_z a_x) + \hat{k}(a_x b_y - b_x a_y)$$

$$\underline{a} = \begin{bmatrix} a_x \\ a_y \\ a_z \end{bmatrix}$$

$$\underline{b} = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

$$= \begin{bmatrix} a_y b_z - b_y a_z \\ -a_x b_z + b_x a_z \\ a_x b_y - b_x a_y \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} a_y b_z - b_y a_z \\ -a_x b_z + b_x a_z \\ a_x b_y + b_x a_y \end{bmatrix}}_{\underline{a} \times \underline{b}} = \underbrace{\begin{bmatrix} 0 & -a_z & a_y \\ a_z & 0 & -a_x \\ a_y & a_x & 0 \end{bmatrix}}_{[\underline{a}_x]} \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$$

Cross product matrix
of vector

$$\hat{k} \times \underline{x} = \underbrace{[\underline{k}_x]}_K \underline{x} = K \underline{x}$$

$$\hat{k} \times (\hat{k} \times \underline{x}) = [\underline{k}_x] ([\underline{k}_x] \underline{x}) = K(K \underline{x}) = K^2 \underline{x}$$

$$\begin{aligned} \underline{y} &= \underline{x} + \sin \theta (K \underline{x}) + (1 - \cos \theta) K^2 \underline{x} \\ &= \underbrace{\left[\underline{I}_{3 \times 3} + \sin \theta K + (1 - \cos \theta) K^2 \right]}_{R(\theta, \hat{k})} \underline{x} \end{aligned}$$

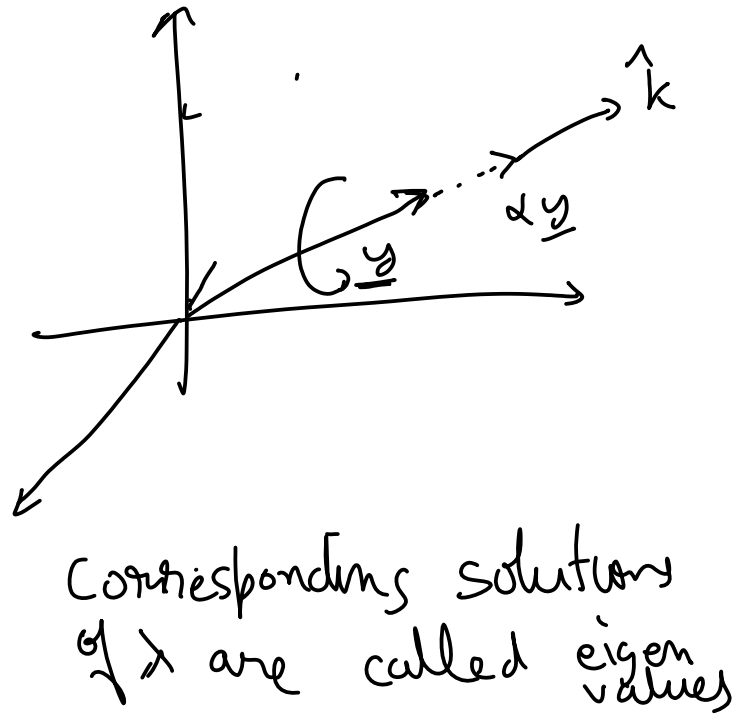
Oct 25

Convert. from Rot matrix to axis angle representation

$$\underline{\alpha} \underline{y} = R \underline{y}$$

R is 3×3 matrix
 \underline{y} is 3×1 vector
 α is scalar

$\rightarrow \underline{A} \underline{v} = \lambda \underline{v}$
Eigen vectors
of a matrix A
are all the solution
for \underline{v} from the
above equation



$$\underline{A} \underline{v} - \lambda \underline{v} = 0$$
$$\Rightarrow \underbrace{(\underline{A} - \lambda \underline{I})}_{\text{matrix}} \underbrace{\underline{v}}_{\text{vector}} = 0$$

$\det(\underline{A} - \lambda \underline{I}) = 0 \} \rightarrow$ solve for eigen value

The axis of rotation is an eigen vector of
the rotation matrix.

$$\underline{y} = R \underline{x} \Rightarrow \|\underline{y}\| = \|\underline{x}\|$$

$$\|\underline{y}\| = \sqrt{y_1^2 + y_2^2 + \dots + y_n^2}$$

$$= \sqrt{\begin{bmatrix} y_1 & y_2 & \dots & y_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}}$$

$$= \sqrt{\underline{y}^T \underline{y}}$$

$$\underline{y}^T \underline{y} = (\underline{R} \underline{x})^T (\underline{R} \underline{x})$$

$$= (\underline{x}^T \underline{R}^T) (\underline{R} \underline{x})$$

$$= \underline{x}^T (\underline{R}^T \underline{R}) \underline{x}$$

$$= \underline{x}^T \underline{x}$$

eigen value \downarrow eigen vector \downarrow

$$\underline{1} \underline{k} = \underline{R} \underline{k}$$

$\underline{y} \quad \underline{k}$ is along the axis of rotation for \underline{R}

The axis of rotation is the eigen vector of the rotation matrix corresponding to eigen value 1.

$$\boxed{\det(\underline{R} - \underline{I}) = 0}$$

Use `numpy.linalg.eig()` to find eigen value and eigen vector

For 3×3 Rotation matrix

if $\theta = 0^\circ$ or 180°

special case

else ($\theta \neq 0, \theta \neq 180^\circ$)

$$R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

$$\hat{k} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} = \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} \frac{1}{(2\sin\theta)}$$

How to compute angle θ in axis-angle

$$R = I + K \sin\theta + (1 - \cos\theta) K^2$$

$$K^2 = \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix} \begin{bmatrix} 0 & -k_z & k_y \\ k_z & 0 & -k_x \\ -k_y & k_x & 0 \end{bmatrix}$$

A is a Symmetric matrix if $A^T = A$ $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$

A is a skew-symmetric matrix if $A^T = -A$

$$\begin{bmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & 0 \\ -a_{13} & 0 & 0 \end{bmatrix}$$

$$K^2 = \begin{bmatrix} -(k_x^2 + k_y^2) & k_y k_x & k_z k_x \\ k_y k_x & -(k_x^2 + k_z^2) & k_z k_y \\ k_z k_x & k_z k_y & -(k_x^2 + k_y^2) \end{bmatrix}$$

$$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \sin \theta +$$

$$\begin{bmatrix} -(k_x^2 + k_y^2) & k_y k_x & k_z k_x \\ k_y k_x & -(k_x^2 + k_z^2) & k_z k_y \\ k_z k_x & k_z k_y & -(k_x^2 + k_y^2) \end{bmatrix} (1 - \cos \theta)$$

$$\hat{k} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} \text{ is a unit vector}$$

$$k_x^2 + k_y^2 + k_z^2 = 1$$

$$R = \begin{bmatrix} r_{11} & & \\ & r_{22} & \\ & & r_{33} \end{bmatrix}$$

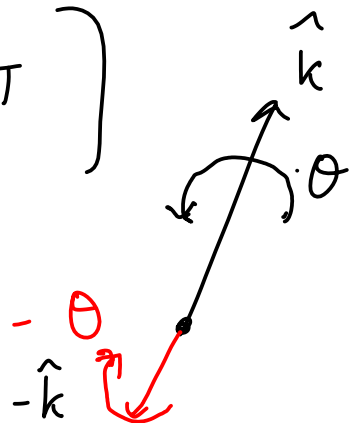
$$\text{trace}(R) = r_{11} + r_{22} + r_{33} = 1 + 1 + 1 - 2 \underbrace{(k_x^2 + k_y^2 + k_z^2)}_1 (1 - \cos \theta)$$

$$\Rightarrow \text{tr}(R) = r_{11} + r_{22} + r_{33} = 3 - 2 + 2 \cos \theta$$

$$\Rightarrow \text{tr}(R) = 1 + 2 \cos \theta$$

$$\Rightarrow \boxed{\theta = \cos^{-1} \left(\frac{\text{tr}(R) - 1}{2} \right)} \in [0, \pi]$$

If $\theta = \frac{3\pi}{4} = 270^\circ$ rotated around \hat{k}
 this is same as $\theta = \frac{\pi}{4} = 90^\circ$ " " $-\hat{k}$



$$\therefore \theta = 180^\circ$$

$$k_x = \pm \sqrt{(r_{11}+1)/2}$$

$$k_y = \pm \sqrt{(r_{22}+1)/2}$$

$$k_z = \pm \sqrt{(r_{33}+1)/2}$$

else ($\theta \neq 0, \theta \neq 180^\circ$)

$$\hat{k} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} = \begin{bmatrix} r_{32} - r_{23} \\ r_{13} - r_{31} \\ r_{21} - r_{12} \end{bmatrix} \frac{1}{(2\sin\theta)}$$

$$R = \begin{bmatrix} r_{11} & r_{12} & \\ r_{21} & & \\ & & r_{33} \end{bmatrix}$$

Quaternions

$$q = w + ix + jy + kz$$

$$\hat{k} = \begin{bmatrix} k_x \\ k_y \\ k_z \end{bmatrix} \theta$$

Axis angle to quat

$$\boxed{\begin{aligned} i^2 &= -1, j^2 = -1, k^2 = -1 \\ ijk &= -1 \end{aligned}}$$

Complex numbers

$$q = \left[\begin{array}{c} \cancel{\sin} \left(\frac{\theta}{2} \right) \\ \cos \left(\frac{\theta}{2} \right) \end{array} \right], \quad k_x \cancel{\cos} \left(\frac{\theta}{2} \right), \quad k_y \cancel{\cos} \left(\frac{\theta}{2} \right), \quad k_z \cancel{\cos} \left(\frac{\theta}{2} \right) \\ \begin{array}{cccc} w & x & y & z \end{array}$$

$$q = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

OR

$$q = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

$$q = \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$$

$$\hat{k} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} / \frac{\cancel{\cos}(\theta/2)}{\cancel{\sin}(\theta/2)}$$

$$\sqrt{x^2 + y^2 + z^2} = \cancel{\cos}^{\sin} \left(\frac{\theta}{2} \right)$$

$$w \quad z \quad \cancel{\sin}^{\cos} \frac{\theta}{2}$$

$$\theta = 2 \arctan 2 \left(w, \sqrt{x^2 + y^2 + z^2} \right)$$

$$\theta = 2 \arctan 2 \left(\sqrt{x^2 + y^2 + z^2}, w \right)$$