Linear algebra drills

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1 Valid matrix multiplication and addition

- Notation: A matrix A is described as $A \in \mathbb{R}^{m\times n}$ (read as: A is in the set of real matrices of size m times n) if A is table of numbers of the form:

$$A = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \dots & a_{mn}, \end{bmatrix}$$

  and each a_{ij} is a real number also denoted as $a_{ij} \in \mathbb{R}$ (read as: a_{ij} is in the set of real numbers); for all $i \in \{1, \dots, m\}$ and $j \in \{1, \dots, n\}$.

- Rule 1: Transpose of matrix swaps rows and columns. If $A \in \mathbb{R}^{m\times n}$, then $A^\top \in \mathbb{R}^{n\times m}$.

- Notation: A column vector is a matrix with only one column. Vectors are usually denoted by a boldface small letter $\mathbf{x} \in \mathbb{R}^{m\times 1}$ and sometimes the second dimension is omitted: $\mathbf{x} \in \mathbb{R}^m$.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$$

  A row vector is a matrix with only one row. Row vectors are often denoted as transpose of column vector $\mathbf{x}^\top \in \mathbb{R}^{1\times n}$. The 1 dimension of row vector is never omitted,

$$\mathbf{x}^\top = \begin{bmatrix} x_1 & \dots & x_m \end{bmatrix}$$

- Rule 2: Matrix multiplication between $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ is only valid when $n = p$. When matrix multiplication is valid, resulting matrix has size $C = AB \in \mathbb{R}^{m\times q}$.

- Role 3: Scalar-Matrix multiplication is not a type of matrix multiplication. It is a multiplication of a different type. Scalar-matrix multiplication is always valid irrespective of the size of the matrix.

- Rule 4: Matrix addition between $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{p\times q}$ is only valid when $m = p$ and $n = q$.

- Rule 5: Matrix division and matrix fractions are not defined. $\frac{A}{B}$ or A/B are not defined. (Matrix inverse (when defined) is just another matrix $C = A^{-1}$ that can be multiplied or added with other matrices. If $A \in \mathbb{R}^{m\times n}$ then $C = A^{-1} \in \mathbb{R}^{n\times m}$.)

Problem 1. *Are the following matrix operations valid? If not, which dimensions are unequal? If yes, what is the size of final matrix?*

1. AB, *where* $A \in \mathbb{R}^{2\times 3}$ *and* $B \in \mathbb{R}^{3\times 4}$.
2. PQ, *where* $P \in \mathbb{R}^{2\times 4}$ *and* $B \in \mathbb{R}^{2\times 4}$.
3. $A^\top B$, *where* $A \in \mathbb{R}^{3\times 5}$ *and* $B \in \mathbb{R}^{3\times 4}$.
4. $AB^\top$, *where* $A \in \mathbb{R}^{2\times 3}$ *and* $B \in \mathbb{R}^{3\times 4}$.
5. $A^\top B^\top$, *where* $A \in \mathbb{R}^{2\times 3}$ *and* $B \in \mathbb{R}^{3\times 4}$.
6. $(BA)^\top$, *where* $A \in \mathbb{R}^{2\times 3}$ *and* $B \in \mathbb{R}^{3\times 4}$.
7. $A\mathbf{x}$, *where* $A \in \mathbb{R}^{2\times 3}$ *and* $\mathbf{x} \in \mathbb{R}^3$.
8. $A^\top \mathbf{x}$, *where* $A \in \mathbb{R}^{2\times 3}$ *and* $\mathbf{x} \in \mathbb{R}^{3\times 1}$.
9. $\mathbf{x}A$, *where* $A \in \mathbb{R}^{2\times 3}$ *and* $\mathbf{x} \in \mathbb{R}^3$.
10. $\mathbf{x}^\top B$, *where* $\mathbf{x} \in \mathbb{R}^3$ *and* $B \in \mathbb{R}^{3\times 4}$.
11. $\mathbf{x}B$, *where* $\mathbf{x} \in \mathbb{R}^3$ *and* $B \in \mathbb{R}^{3\times 4}$.

12. $\mathbf{x}B^\top$, *where* $\mathbf{x} \in \mathbb{R}^3$ *and* $B \in \mathbb{R}^{3\times 4}$.

13. $\mathbf{x}^\top B^\top$, *where* $\mathbf{x} \in \mathbb{R}^3$ *and* $B \in \mathbb{R}^{3\times 4}$.

14. $\mathbf{x}^\top \mathbf{y}$, *where* $\mathbf{x} \in \mathbb{R}^3$ *and* $\mathbf{y} \in \mathbb{R}^3$.

15. $\mathbf{y}^\top \mathbf{x}$, *where* $\mathbf{x} \in \mathbb{R}^3$ *and* $\mathbf{y} \in \mathbb{R}^3$.

16. $\mathbf{y}\mathbf{x}$, *where* $\mathbf{x} \in \mathbb{R}^3$ *and* $\mathbf{y} \in \mathbb{R}^3$.

17. $\mathbf{y}^\top \mathbf{x}^\top$, *where* $\mathbf{x} \in \mathbb{R}^3$ *and* $\mathbf{y} \in \mathbb{R}^3$.

18. $\mathbf{y}\mathbf{x}^\top$, *where* $\mathbf{x} \in \mathbb{R}^3$ *and* $\mathbf{y} \in \mathbb{R}^3$.

19. $\mathbf{y}A\mathbf{x}^\top$, *where* $\mathbf{x} \in \mathbb{R}^3$, $A \in \mathbb{R}^{3\times 4}$ *and* $\mathbf{y} \in \mathbb{R}^4$.

20. $\mathbf{y}^\top A^\top \mathbf{x}^\top$, *where* $\mathbf{x} \in \mathbb{R}^3$, $A \in \mathbb{R}^{3\times 4}$ *and* $\mathbf{y} \in \mathbb{R}^4$.

21. $\mathbf{x}^\top A\mathbf{y}$, *where* $\mathbf{x} \in \mathbb{R}^3$, $A \in \mathbb{R}^{3\times 4}$ *and* $\mathbf{y} \in \mathbb{R}^4$.

22. $\mathbf{y}^\top A^\top \mathbf{x}^\top$, *where* $\mathbf{x} \in \mathbb{R}^3$, $A \in \mathbb{R}^{3\times 4}$ *and* $\mathbf{y} \in \mathbb{R}^4$.

23. $\mathbf{y}^\top A^\top \mathbf{x}^\top$, *where* $\mathbf{x} \in \mathbb{R}^3$, $A \in \mathbb{R}^{3\times 4}$ *and* $\mathbf{y} \in \mathbb{R}^4$.

24. $\mathbf{x}^\top A^\top \mathbf{y}$, *where* $\mathbf{x} \in \mathbb{R}^3$, $A \in \mathbb{R}^{3\times 4}$ *and* $\mathbf{y} \in \mathbb{R}^4$.

25. AB, *where* $A \in \mathbb{R}^{m\times n}$ *and* $B \in \mathbb{R}^{n\times p}$.

26. PQ, *where* $P \in \mathbb{R}^{m\times p}$ *and* $B \in \mathbb{R}^{m\times p}$.

27. $A^\top B$, *where* $A \in \mathbb{R}^{n\times 5}$ *and* $B \in \mathbb{R}^{n\times p}$.

28. $AB^\top$, *where* $A \in \mathbb{R}^{m\times n}$ *and* $B \in \mathbb{R}^{n\times p}$.

29. $A^\top B^\top$, *where* $A \in \mathbb{R}^{m\times n}$ *and* $B \in \mathbb{R}^{n\times p}$.

30. $(BA)^\top$, *where* $A \in \mathbb{R}^{m\times n}$ *and* $B \in \mathbb{R}^{n\times p}$.

31. $A\mathbf{x}$, *where* $A \in \mathbb{R}^{m\times n}$ *and* $\mathbf{x} \in \mathbb{R}^n$.

32. $A^\top \mathbf{x}$, *where* $A \in \mathbb{R}^{m\times n}$ *and* $\mathbf{x} \in \mathbb{R}^{n\times 1}$.

33. $\mathbf{x}A$, *where* $A \in \mathbb{R}^{m\times n}$ *and* $\mathbf{x} \in \mathbb{R}^n$.

34. $\mathbf{x}^\top B$, *where* $\mathbf{x} \in \mathbb{R}^n$ *and* $B \in \mathbb{R}^{n\times p}$.

35. $\mathbf{x}B$, *where* $\mathbf{x} \in \mathbb{R}^n$ *and* $B \in \mathbb{R}^{n\times p}$.

36. $\mathbf{x}B^\top$, *where* $\mathbf{x} \in \mathbb{R}^n$ *and* $B \in \mathbb{R}^{n\times p}$.

37. $\mathbf{x}^\top B^\top$, *where* $\mathbf{x} \in \mathbb{R}^n$ *and* $B \in \mathbb{R}^{n\times p}$.

38. $\mathbf{x}^\top \mathbf{y}$, *where* $\mathbf{x} \in \mathbb{R}^n$ *and* $\mathbf{y} \in \mathbb{R}^n$.

39. $\mathbf{y}^\top \mathbf{x}$, *where* $\mathbf{x} \in \mathbb{R}^n$ *and* $\mathbf{y} \in \mathbb{R}^n$.

40. $\mathbf{y}\mathbf{x}$, *where* $\mathbf{x} \in \mathbb{R}^n$ *and* $\mathbf{y} \in \mathbb{R}^n$.

41. $\mathbf{y}^\top \mathbf{x}^\top$, *where* $\mathbf{x} \in \mathbb{R}^n$ *and* $\mathbf{y} \in \mathbb{R}^n$.

42. $\mathbf{y}\mathbf{x}^\top$, *where* $\mathbf{x} \in \mathbb{R}^n$ *and* $\mathbf{y} \in \mathbb{R}^n$.

43. $\mathbf{y}A\mathbf{x}^\top$, *where* $\mathbf{x} \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times p}$ *and* $\mathbf{y} \in \mathbb{R}^p$.

44. $\mathbf{y}^\top A^\top \mathbf{x}^\top$, *where* $\mathbf{x} \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times p}$ *and* $\mathbf{y} \in \mathbb{R}^p$.

45. $\mathbf{x}^\top A\mathbf{y}$, *where* $\mathbf{x} \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times p}$ *and* $\mathbf{y} \in \mathbb{R}^p$.

46. $\mathbf{y}^\top A^\top \mathbf{x}^\top$, *where* $\mathbf{x} \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times p}$ *and* $\mathbf{y} \in \mathbb{R}^p$.

47. $\mathbf{y}^\top A^\top \mathbf{x}^\top$, *where* $\mathbf{x} \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times p}$ *and* $\mathbf{y} \in \mathbb{R}^p$.

48. $\mathbf{x}^\top A^\top \mathbf{y}$, *where* $\mathbf{x} \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times p}$ *and* $\mathbf{y} \in \mathbb{R}^p$.