ECE 417/598: Review Homework 4

Max marks: 100 marks

Due on March 10th, 2021, midnight, 11:59 PM.

All notes so far are linked here.

1 Trignometry and triangle laws of vector addition

Problem 1 *The magnitude of vector $\mathbf{a} \in \mathbb{R}^n$ is given to be $\|\mathbf{a}\| = \alpha$. Using the following figure, write $\mathbf{a}$ in terms of α, θ, vector $\mathbf{b} \in \mathbb{R}^n$ and $\mathbf{c} \in \mathbb{R}^n$. All three vectors lie in the same plane. $\mathbf{b}$ and $\mathbf{c}$ are perpendicular to each other. The angle between $\mathbf{a}$ and $\mathbf{b}$ is given by θ.*
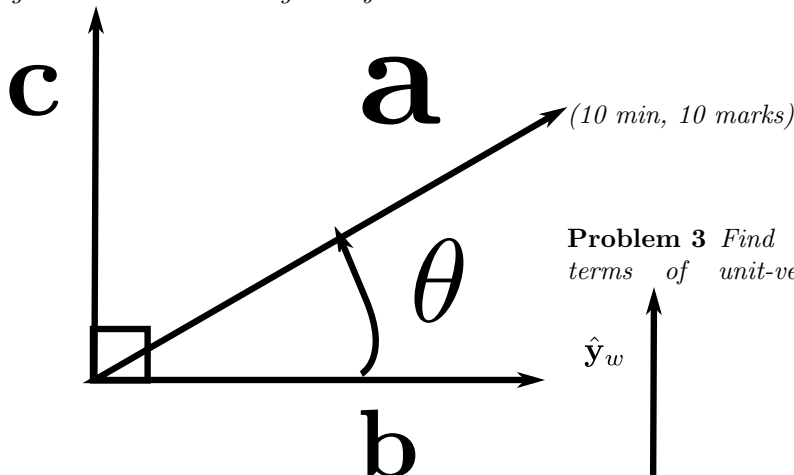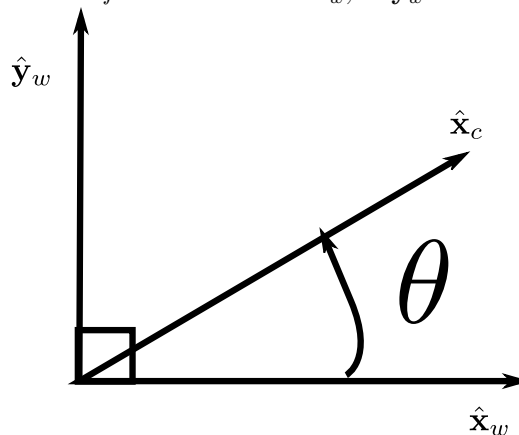
(5 min, 5 marks)

Problem 2 *The magnitude of vector $\mathbf{a} \in \mathbb{R}^n$ is given to be $\|\mathbf{a}\| = \alpha$. Using the following figure, write $\mathbf{a}$ in terms of α, θ, ϕ vector $\mathbf{b} \in \mathbb{R}^n$ and $\mathbf{c} \in \mathbb{R}^n$. All three vectors lie in the same plane. The angle between $\mathbf{a}$ and $\mathbf{b}$ is given by θ. The angle between $\mathbf{a}$ and $\mathbf{c}$ is given by ϕ. Assume $\theta + \phi \neq 0$. When $\theta + \phi = \frac{\pi}{7}2$, is the solution is same as Problem 1? (Hint: You can convert this to Problem 1, by drawing a unit-vector perpendicular to $\mathbf{b}$. Call it $\hat{\mathbf{d}}$. First write $\hat{\mathbf{d}}$ in terms of $\mathbf{c}$ and others knowns and then write $\mathbf{a}$ in terms of $\hat{\mathbf{d}}$ and other knowns. You might want to use trigonometric identities. The simplest form is not required.).*
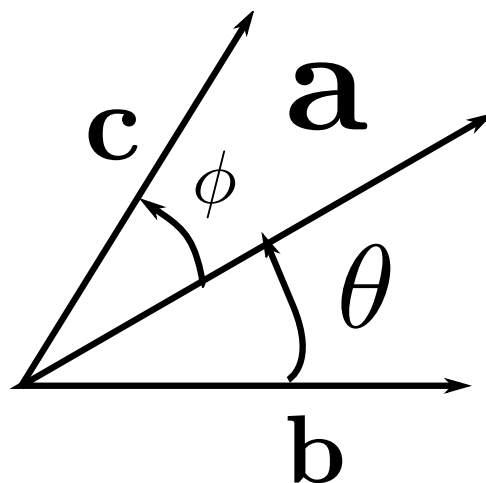
Problem 3 *Find unit-vector $\hat{\mathbf{x}}_c$ in terms of unit-vectors $\hat{\mathbf{x}}_w$, $\hat{\mathbf{y}}_w$ and θ.*

(10 min, 10 marks)

(5 min, 5 marks)

Problem 4 *Find unit-vector $\hat{\mathbf{y}}_c$ in terms of unit-vectors $\hat{\mathbf{x}}_w$, $\hat{\mathbf{y}}_w$ and θ.*

$\hat{\mathbf{y}}_c$ $\hat{\mathbf{y}}_w$

θ
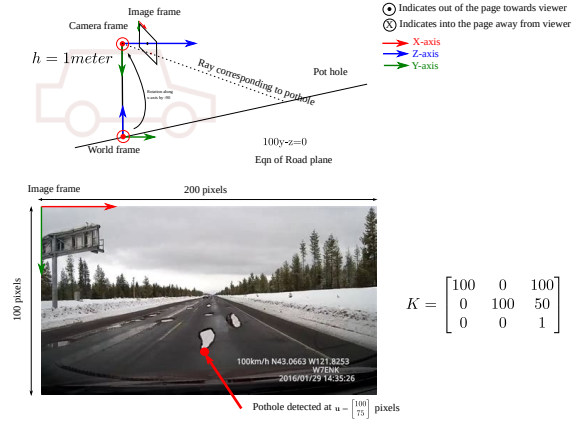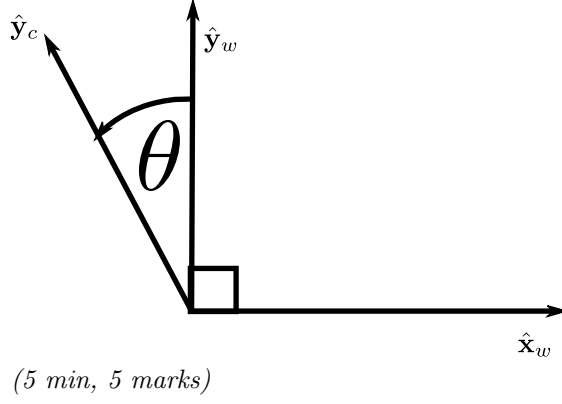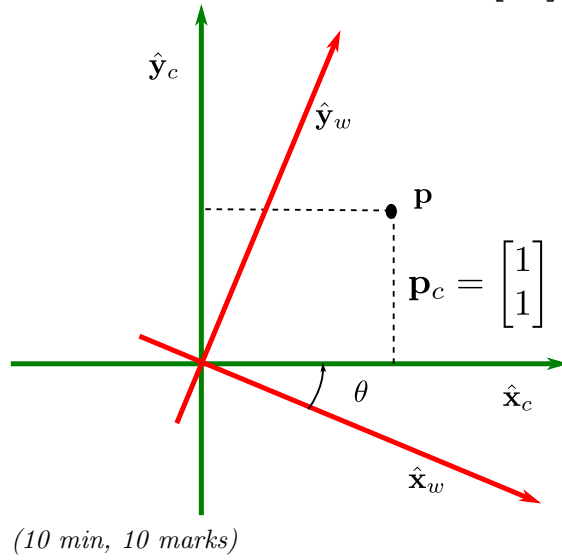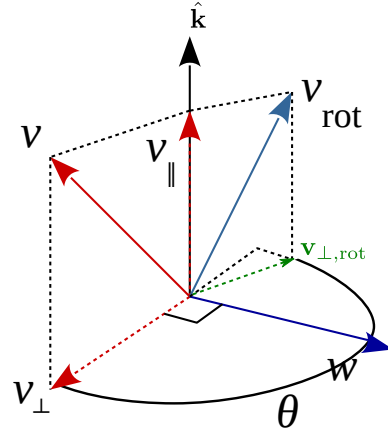
$\hat{\mathbf{x}}_w$

(5 min, 5 marks)

Figure 1: Point-plane triangulation

Problem 5 *Let the coordinates of a vector* $\mathbf{p}$ *in terms of* $\hat{\mathbf{x}}_c$ *and* $\hat{\mathbf{y}}_c$ *be* $\mathbf{p}_c = \begin{bmatrix} p_{cx} \\ p_{cy} \end{bmatrix}$, *so that:* $\mathbf{p} = p_{cx}\hat{\mathbf{x}}_c + p_{cy}\hat{\mathbf{y}}_c$. *Using the results from Prob 3 and Prob 4, write* $\mathbf{p}$ *in terms of* $\hat{\mathbf{x}}_w$ *and* $\hat{\mathbf{y}}_w$. *Thus derive the formula for rotation matrix* $R(\theta)$ *that converts coordinates from* $\mathbf{p}_c$ *to* $\mathbf{p}_w = \begin{bmatrix} p_{wx} \\ p_{wy} \end{bmatrix}$.

$\hat{\mathbf{y}}_c$

$\hat{\mathbf{y}}_w$

$\mathbf{p}$

$\mathbf{p}_c = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

θ $\hat{\mathbf{x}}_c$

$\hat{\mathbf{x}}_w$

(10 min, 10 marks)

$\hat{\mathbf{k}}$

$\mathbf{v}$ $\mathbf{v}_{rot}$

$\mathbf{v}_\parallel$

$\mathbf{v}_{\perp,rot}$

$\mathbf{w}$

$\mathbf{v}_\perp$ θ

(5 min, 5 marks)

Problem 7 *In figure 1 find the 3D position of the pothole the World coordinate frame, in terms of* $h = 1$ *(the height of the camera), image-coordinates of the pothole* $\mathbf{u}$ *(provided in figure), camera matrix* K *(provided in figure). The Camera is mounted directly on top of the world frame, both of which are aligned to the gravity vector. The road is a perfect plane with a slope such that the equation of road plane in world-coordinate frame is given by* $100Y_w - Z_w = 0$ *and the pothole lies on the road plane. Provide the formula or pseudo-code for computing the pothole coordinates, and also substitute in the values. (20 min, 20 marks)*

Problem 6 *We know that* $\|\mathbf{v}_{\perp,rot}\| = \|\mathbf{v}_\perp\|$. *Write* $\mathbf{v}_{\perp,rot}$ *in terms of* $\mathbf{v}_\perp$, $\mathbf{w}$ *and* θ. $\mathbf{v}_\perp$ *and* $\mathbf{w}$ *are known to be orthogonal to each other.*

2

Problem 8 *In figure 2 find the 3D representation of the lane the World coordinate frame, in terms of h (the height of the camera), image-representation of the line $\mathbf{l}$ (provided in figure), camera matrix K (provided in figure). Assume the lane to be a straight line. The Camera is mounted directly on top of the world frame, both of which are aligned to the gravity vector. The road is a perfect plane with a slope such that the equation of road plane in world-coordinate frame is given by $100Y_w - Z_w = 0$ and the lane lies on the road plane. Provide the formula or pseudocode for computing the 3D representation of the lane, and also substitute in the values. (20 min, 20 marks)*

*Hint 0: **Equation of plane in 3D.** Equation of a plane in 3D is given by $p_1 X + p_2 Y + p_3 Z + p_4 = 0$. In matrix notation, you can write the equation plane as $\mathbf{p}_{1:3}^\top \mathbf{X} + p_4 = 0$, where $\mathbf{p}_{1:3} = [p_1, p_2, p_3]^\top$.*

*Hint 1: **3D Plane corresponding to the line in image-coordinates.** Let the equation of line in image-coordinates be $\mathbf{l}^\top \underline{\mathbf{u}} = 0$, where $\underline{\mathbf{u}} = \begin{bmatrix} \mathbf{u} \\ 1 \end{bmatrix} \in \mathbb{P}^2$ are all the points on the line. By pinhole camera model, if $\mathbf{X}_c \in \mathbb{R}^3$ are the corresponding points in 3D, then the equation of corresponding plane is given by $\mathbf{l}^\top (K\mathbf{X}_c) = 0$ which can also be written as $(K^\top \mathbf{l})^\top \mathbf{X}_c = 0$. If we compare it to the equation of plane $\mathbf{p}_{1:3}^\top \mathbf{X} + p_4 = 0$, then $\mathbf{p}_{1:3} = K^\top \mathbf{l}$ and $p_4 = 0$.*

*Hint 2: **Intersection of two planes in 3D is a line.** Equation of a plane in 3D is given by $p_1 X_w + p_2 Y_w + p_3 Z_w + p_4 = 0$. In matrix notation, you can write the equation of the plane as $\mathbf{p}_{1:3}^\top \mathbf{X}_w + p_4 = 0$, where $\mathbf{p}_{1:3} = [p_1, p_2, p_3]^\top$. Let's say you have two planes $\mathbf{p}_{1:3}^\top \mathbf{X}_w + p_4 = 0$ and $\mathbf{q}_{1:3}^\top \mathbf{X}_w + q_4 = 0$. Their intersection is a line whose parameteric form is given by (why ? you have all the knowledge required to derive this):*

$$\mathbf{X}_w = \lambda(\mathbf{p}_{1:3} \times \mathbf{q}_{1:3}) + \begin{bmatrix} \mathbf{p}_{1:3}^\top \\ \mathbf{q}_{1:3}^\top \end{bmatrix}^\dagger \begin{bmatrix} -p_4 \\ -q_4 \end{bmatrix}, \quad (1)$$

where $A^\dagger$ denotes the pseudo-inverse of a matrix (a fat matrix in this case) and $\lambda \in \mathbb{R}$ is the free parameter and $\times$ denotes the vector cross-product.

Problem 9 *You are a part of Tesla self-driving team. Team 1 provides you with lane-detection algorithms and their output. Team 2 provides*
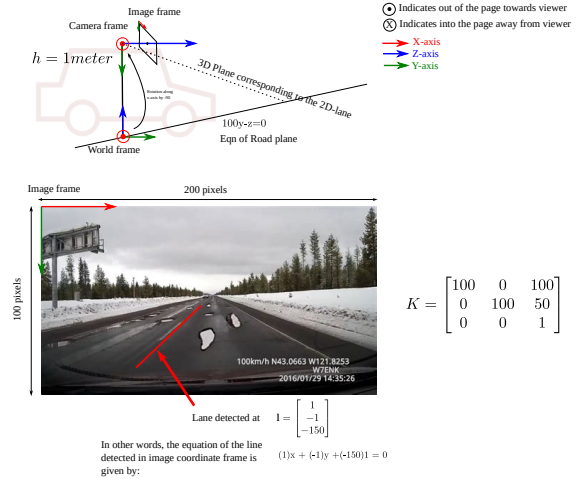

Figure 2: Line-plane triangulation

you with detailed maps of road conditions. Your task is to write a function that solves problem 8 for arbitrary lanes detected by team 1 and for arbitrary plane provided by team 2. (Hint: Equation of a plane 3D is very similar to equation of line in 2D). What input representations of lane and plane would you ask for? Write a general algorithm or pseudo-code that solves problem 8. (30 min, 10 marks)

3