$$f(\underline{x}) = (x_1^2 - 2x_1 x_2)\exp(-x_2^2 - x_3^2)$$

$$\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$= a_{11} x_1^2 + a_{22} x_2^2 + a_{33} x_3^2 + (a_{21} + a_{12}) x_1 x_2 + (a_{32} + a_{23}) x_2 x_3 + (a_{31} + a_{13}) x_1 x_3$$

$$x_1^2 - 2x_1 x_2 = \underline{x}^T A \underline{x}$$

where $A = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$$-x_2^2 - x_3 = -\underline{x}^T B \underline{x}$$

where $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Rules

1. $$\frac{\partial}{\partial \underline{x}} \underline{b}^T \underline{x} = \underline{b}^T$$

2. $$\frac{\partial}{\partial \underline{x}} \underline{x}^T A \underline{x} = 2x^T A$$ if A is symmetric

   $$\frac{\partial}{\partial \underline{x}} \underline{x}^T A \underline{x} = x^T (A + A^T)$$ if A is not symmetric

Chain rule

3. $$\frac{\partial}{\partial \underline{x}} f(\underline{g}(\underline{x})) = \frac{\partial f}{\partial \underline{g}} \underbrace{\boxed{\frac{\partial \underline{g}}{\partial \underline{x}}}}_{J_{xg}}$$

Product rule

4. $$\frac{\partial}{\partial \underline{x}} f(\underline{x}) g(\underline{x}) = \frac{\partial f(\underline{x})}{\partial \underline{x}} g(x) + f(x) \frac{\partial g(\underline{x})}{\partial x}$$

4.a $$\frac{\partial}{\partial \underline{x}} \underline{g}(\underline{x}) f(\underline{x}) = \underline{g}(\underline{x}) \frac{\partial f(\underline{x})}{\partial \underline{x}} + f(\underline{x}) \frac{\partial}{\partial \underline{x}} \underline{g}(\underline{x})$$

(4.b) $$\frac{\partial}{\partial \underline{x}} \underline{f}(\underline{x})^T \underline{g}(\underline{x}) = \underline{f}(\underline{x})^T \underbrace{\boxed{\frac{\partial \underline{g}(\underline{x})}{\partial \underline{x}}}}_{J_{xg}} + \underline{g}(\underline{x})^T \underbrace{\boxed{\frac{\partial \underline{f}(\underline{x})}{\partial \underline{g}}}}_{J_{xf}}$$

$$f(\underline{x}) = \underbrace{\underline{x}^T A \underline{x}}_{①} \underbrace{\exp(-\underline{x}^T B \underline{x})}_{②}$$

scalar

$$\frac{\partial}{\partial \underline{x}} f(\underline{x}) = \underbrace{(2\underbrace{\underline{x}^T}_{1\times3}\underbrace{A}_{3\times3})}_{1\times3} \underbrace{\exp(-\underline{x}^T B \underline{x})}_{\text{scalar}} + (\underline{x}^T A \underline{x}) \frac{\partial}{\partial \underline{x}} \left[\exp(-\underline{x}^T B \underline{x})\right]$$

$$\frac{\partial}{\partial \underline{x}}\left[\exp(-\underline{x}^T B \underline{x})\right] = \underbrace{\exp(\underbrace{-\underline{x}^T B \underline{x}}_{\text{Scalar}})}_{\text{Scalar}} \underbrace{\left(-2\,\underbrace{\underline{x}^T}_{1\times 3}\,\underbrace{B}_{3\times 3}\right)}_{1\times 3} \in \mathbb{R}^{1\times 3}$$

$$\frac{\partial}{\partial \underline{x}} f(\underline{x}) = (2\underline{x}^T A)\,\underbrace{\exp(-\underline{x}^T B \underline{x})} + (\underline{x}^T A \underline{x})\,\underbrace{\exp(-\underline{x}^T B \underline{x})}\,(-2\underline{x}^T B)$$

$$\nabla_x^T f(\underline{x}) = 2\exp(-\underline{x}^T B \underline{x})\left[\underline{x}^T A - (\underline{x}^T A \underline{x})\,\underline{x}^T B\right] \in \mathbb{R}^{1\times 3}$$

$$\nabla_x f(\underline{x}) \in \mathbb{R}^{3\times 1}$$

$$H_{xx} f(x) = \frac{\partial}{\partial \underline{x}}\left[\nabla_x f(x)\right] \in \mathbb{R}^{3\times 3}$$

$$= \begin{bmatrix} \frac{\partial}{\partial \underline{x}}[\nabla_x f(x)]_1 & \in \mathbb{R}^{1\times 3} \\ \frac{\partial}{\partial \underline{x}}[\nabla_x f(x)]_2 & \in \mathbb{R}^{1\times 3} \\ \frac{\partial}{\partial \underline{x}}[\nabla_x f(x)]_3 & \in \mathbb{R}^{1\times 3} \end{bmatrix}$$

$$A^T = A$$

$$\nabla_x^T f(\underline{x}) = 2\exp(-\underline{x}^T B \underline{x})\left[\underbrace{\underline{x}^T A}_{1\times 3} - \underbrace{(\underline{x}^T A \underline{x})}_{\text{scalar}}\,\underbrace{\underline{x}^T B}_{1\times 3}\right]$$

$$\nabla_x f(\underline{x}) = \underbrace{2\exp(-\underline{x}^T B \underline{x})}_{\text{Scalar}}\,\underbrace{\left[\underbrace{A\underline{x}}_{3\times 1} - (\underline{x}^T A \underline{x})\,\underbrace{B\underline{x}}_{3\times 1}\right]} \in \mathbb{R}^{3\times}$$

New product rule

$$f(\underline{x}) \qquad\qquad g(\underline{x})$$

$$\boxed{\frac{\partial}{\partial \underline{x}} f(\underline{x})\,g(\underline{x}) = \underbrace{g(\underline{x})}_{m\times 1}\,\underbrace{\frac{\partial f(\underline{x})}{\partial \underline{x}}}_{1\times n}{}^{m\times n} + \underbrace{\left[\frac{\partial}{\partial \underline{x}} g(\underline{x})\right]}_{m\times n} f(\underline{x})}$$

$H_{xx} f(\underline{x})$

$$\frac{\partial}{\partial \underline{x}} \nabla_x f(\underline{x}) = \underbrace{\left[A\underline{x} - (\underline{x}^T A \underline{x}) B\underline{x}\right]}_{3\times 1} \underbrace{\left(2\exp(-\underline{x}^T B\underline{x})(-2x^T B)\right)}_{1\times 3}$$

$$+ \frac{\partial}{\partial \underline{x}}\left[A\underline{x} - (\underline{x}^T A \underline{x}) B\underline{x}\right]\left(2\exp(-\underline{x}^T B \underline{x})\right)$$

$$\frac{\partial}{\partial \underline{x}}(A\underline{x}) = \underbrace{\begin{bmatrix} a_1^T \\ a_2^T \\ a_3^T \end{bmatrix}}_{A} \underline{x} = \begin{bmatrix} a_1^T x \\ a_2^T x \\ a_3^T x \end{bmatrix} = \begin{bmatrix} \frac{\partial a_1^T x}{\partial x} \\ \frac{\partial}{\partial x} a_2^T x \\ \frac{\partial}{\partial x} a_3^T x \end{bmatrix} = \begin{bmatrix} a_1^T \\ a_2^T \\ a_3^T \end{bmatrix} = A$$

New rule
$$\boxed{\frac{\partial}{\partial x} A\underline{x} = A}$$

$$\frac{\partial}{\partial \underline{x}} \nabla_x f(\underline{x}) = \left[A\underline{x} - \underbrace{(\underline{x}^T A \underline{x})}_{\text{scaler}} \underbrace{B\underline{x}}_{3\times 1}\right]\left(2\exp(-\underline{x}^T B\underline{x})(-2x^T B)\right)$$

$$+ \left[A - (B\underline{x})(2\underline{x}^T A) - B(\underline{x}^T A \underline{x})\right]\left(2\exp(-\underline{x}^T B \underline{x})\right)$$

$$= 2\exp(-\underline{x}^T B \underline{x})\left[-2A\underline{x}\,\underline{x}^T B + 2(\underline{x}^T A \underline{x})(B\underline{x}\,\underline{x}^T B) + A - 2B\underline{x}\,x^T A - B(\underline{x}^T A \underline{x})\right]$$

$$A\underline{x}\,\underline{x}^T B = B^T \underline{x}\,\underline{x}^T A^T = B\underline{x}\,\underline{x}^T A$$

$$H_{xx} f(\underline{x}) = 2\exp(-\underline{x}^T B\underline{x})\left[-4A\underline{x}\,\underline{x}^T B + 2(\underline{x}^T A \underline{x})(B\underline{x}\,\underline{x}^T B - B) + A\right]$$