In [1]:
```python
# Refs:
# 1. https://github.com/karpathy/micrograd/tree/master/micrograd
# 2. https://github.com/mattjj/autodidact
# 3. https://github.com/mattjj/autodidact/blob/master/autograd/numpy/numpy_v
from collections import namedtuple
import numpy as np


def unbroadcast(target, g, axis=0):
    """Remove broadcasted dimensions by summing along them.
    When computing gradients of a broadcasted value, this is the right thing
    do when computing the total derivative and accounting for cloning.
    """
    while np.ndim(g) > np.ndim(target):
        g = g.sum(axis=axis)
    for axis, size in enumerate(target.shape):
        if size == 1:
            g = g.sum(axis=axis, keepdims=True)
    if np.iscomplexobj(g) and not np.iscomplex(target):
        g = g.real()
    return g

Op = namedtuple('Op', ['apply',
                   'vjp',
                   'name',
                   'nargs'])
```

Vector Jacobian Product for addition

$$\mathrm{f}(\mathrm{a}, \mathrm{b}) = \mathrm{a} + \mathrm{b}$$

where $\mathrm{a}, \mathrm{b}, \mathrm{f} \in \mathbb{R}^n$

Let $l(\mathrm{f}(\mathrm{a}, \mathrm{b})) \in \mathbb{R}$ be the eventual scalar output. We find $\frac{\partial l}{\partial \mathrm{a}}$ and $\frac{\partial l}{\partial \mathrm{b}}$ for Vector Jacobian product.

$$\frac{\partial}{\partial \mathrm{a}} l(\mathrm{f}(\mathrm{a}, \mathrm{b})) = \frac{\partial l}{\partial \mathrm{f}} \frac{\partial}{\partial \mathrm{a}}(\mathrm{a} + \mathrm{b}) = \frac{\partial l}{\partial \mathrm{f}}(\mathrm{I}_{n \times n} + 0_{n \times n}) = \frac{\partial l}{\partial \mathrm{f}}$$

Similarly,

$$\frac{\partial}{\partial \mathrm{b}} l(\mathrm{f}(\mathrm{a}, \mathrm{b})) = \frac{\partial l}{\partial \mathrm{f}}$$

In [2]:
```python
def add_vjp(dldf, a, b):
    dlda = unbroadcast(a, dldf)
    dldb = unbroadcast(b, dldf)
    return dlda, dldb
```

```
add = Op(
    apply=np.add,
    vjp=add_vjp,
    name='+',
    nargs=2)
```

VJP for element-wise multiplication

$$f(\alpha, \beta) = \alpha\beta$$

where $\alpha, \beta, f \in \mathbb{R}$

Let $l(f(\alpha, \beta)) \in \mathbb{R}$ be the eventual scalar output. We find $\frac{\partial l}{\partial \alpha}$ and $\frac{\partial l}{\partial \beta}$ for Vector Jacobian product.

$$\frac{\partial}{\partial \alpha} l(f(\alpha, \beta)) = \frac{\partial l}{\partial f} \frac{\partial}{\partial \alpha}(\alpha\beta) = \frac{\partial l}{\partial f}\beta$$

$$\frac{\partial}{\partial \beta} l(f(\alpha, \beta)) = \frac{\partial l}{\partial f} \frac{\partial}{\partial \beta}(\alpha\beta) = \frac{\partial l}{\partial f}\alpha$$

In [3]:
```
def mul_vjp(dldf, a, b):
    dlda = unbroadcast(a, dldf * b)
    dldb = unbroadcast(b, dldf * a)
    return dlda, dldb

mul = Op(
    apply=np.multiply,
    vjp=mul_vjp,
    name='*',
    nargs=2)
```

VJP for matrix-matrix, matrix-vector and vector-vector multiplication

Case 1: VJP for vector-vector multiplication

$$f(\mathrm{a}, \mathrm{b}) = \mathrm{a}^\top \mathrm{b}$$

where $f \in \mathbb{R}$, and $\mathrm{b}, \mathrm{a} \in \mathbb{R}^n$

Let $l(f(\mathrm{a}, \mathrm{b})) \in \mathbb{R}$ be the eventual scalar output. We find $\frac{\partial l}{\partial \mathrm{a}}$ and $\frac{\partial l}{\partial \mathrm{b}}$ for Vector Jacobian product.

$$\frac{\partial}{\partial \mathrm{a}} l(f(\mathrm{a}, \mathrm{b})) = \frac{\partial l}{\partial f} \frac{\partial}{\partial \mathrm{a}}(\mathrm{a}^\top \mathrm{b}) = \frac{\partial l}{\partial f}\mathrm{b}^\top$$

Similarly,

$$\frac{\partial}{\partial \mathrm{b}} l(f(\mathrm{a}, \mathrm{b})) = \frac{\partial l}{\partial f} \mathrm{a}^\top$$

Case 2: VJP for matrix-vector multiplication

Let

$$\mathrm{f}(\mathrm{A}, \mathrm{b}) = \mathrm{A}\mathrm{b}$$

where $\mathrm{f} \in \mathbb{R}^m$, $\mathrm{b} \in \mathbb{R}^n$, and $\mathrm{A} \in \mathbb{R}^{m \times n}$

Let $l(\mathrm{f}(\mathrm{A}, \mathrm{b})) \in \mathbb{R}$ be the eventual scalar output. We want to findfind $\frac{\partial l}{\partial \mathrm{A}}$ and $\frac{\partial l}{\partial \mathrm{b}}$ for Vector Jacobian product.

Let

$$\mathrm{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} \mathrm{a}_1^\top \\ \mathrm{a}_2^\top \\ \vdots \\ \mathrm{a}_m^\top \end{bmatrix}$$

, where each $\mathrm{a}_i^\top \in \mathbb{R}^{1 \times n}$ and $a_{ij} \in \mathbb{R}$.

Define matrix derivative of scalar to be:

$$\frac{\partial l}{\partial \mathrm{A}} = \begin{bmatrix} \frac{\partial l}{\partial a_{11}} & \frac{\partial l}{\partial a_{12}} & \cdots & \frac{\partial l}{\partial a_{1n}} \\ \frac{\partial l}{\partial a_{21}} & \frac{\partial l}{\partial a_{22}} & \cdots & \frac{\partial l}{\partial a_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial l}{\partial a_{m1}} & \frac{\partial l}{\partial a_{m2}} & \cdots & \frac{\partial l}{\partial a_{mn}} \end{bmatrix} = \begin{bmatrix} \frac{\partial l}{\partial \mathrm{a}_1} \\ \frac{\partial l}{\partial \mathrm{a}_2} \\ \vdots \\ \frac{\partial l}{\partial \mathrm{a}_m} \end{bmatrix}$$

$$\frac{\partial}{\partial \mathrm{A}} l(\mathrm{f}(\mathrm{a}, \mathrm{b})) = \frac{\partial l}{\partial \mathrm{f}} \frac{\partial}{\partial \mathrm{A}} (\mathrm{A}\mathrm{b})$$

.

Note that

$$\mathrm{A}\mathrm{b} = \begin{bmatrix} \mathrm{a}_1^\top \\ \mathrm{a}_2^\top \\ \vdots \\ \mathrm{a}_m^\top \end{bmatrix} \mathrm{b} = \begin{bmatrix} \mathrm{a}_1^\top \mathrm{b} \\ \mathrm{a}_2^\top \mathrm{b} \\ \vdots \\ \mathrm{a}_m^\top \mathrm{b} \end{bmatrix}$$

Since $\mathrm{a}_i^\top \mathrm{b}$ is a scalar, it is easier to find its derivative with respect to the matrix A.

$$\frac{\partial}{\partial \mathrm{A}} \mathrm{a}_i^\top \mathrm{b} = \begin{bmatrix} \frac{\partial \mathrm{a}_i^\top \mathrm{b}}{\partial \mathrm{a}_1} \\ \frac{\partial \mathrm{a}_i^\top \mathrm{b}}{\partial \mathrm{a}_2} \\ \vdots \\ \frac{\partial \mathrm{a}_i^\top \mathrm{b}}{\partial \mathrm{a}_i} \\ \vdots \\ \frac{\partial \mathrm{a}_i^\top \mathrm{b}}{\partial \mathrm{a}_m} \end{bmatrix} = \begin{bmatrix} 0_n^\top \\ 0_n^\top \\ \vdots \\ \mathrm{b}^\top \\ \vdots \\ 0_n^\top \end{bmatrix} \in \mathbb{R}^{m \times n}$$

Let

$$\frac{\partial l}{\partial \mathrm{f}} = \begin{bmatrix} \frac{\partial l}{\partial f_1} & \frac{\partial l}{\partial f_2} & \cdots & \frac{\partial l}{\partial f_m} \end{bmatrix}$$

Then

$$\frac{\partial l}{\partial \mathrm{f}} \frac{\partial}{\partial \mathrm{A}} \mathrm{a}_i^\top \mathrm{b} = \begin{bmatrix} \frac{\partial l}{\partial f_1} & \frac{\partial l}{\partial f_2} & \cdots & \frac{\partial l}{\partial f_m} \end{bmatrix} \begin{bmatrix} 0_n^\top \\ 0_n^\top \\ \vdots \\ \mathrm{b}^\top \\ \vdots \\ 0_n^\top \end{bmatrix} = \frac{\partial l}{\partial f_i} \mathrm{b}^\top \in \mathbb{R}^{1 \times n}$$

Returning to our original quest for

$$\frac{\partial}{\partial \mathrm{A}} l(\mathrm{f}(\mathrm{A}, \mathrm{b})) = \frac{\partial l}{\partial \mathrm{f}} \frac{\partial}{\partial \mathrm{A}} \mathrm{A}\mathrm{b} = \frac{\partial l}{\partial \mathrm{f}} \frac{\partial}{\partial \mathrm{A}} \begin{bmatrix} \mathrm{a}_1^\top \mathrm{b} \\ \mathrm{a}_2^\top \mathrm{b} \\ \vdots \\ \mathrm{a}_m^\top \mathrm{b} \end{bmatrix} = \begin{bmatrix} \frac{\partial l}{\partial \mathrm{f}} \frac{\partial}{\partial \mathrm{A}} \mathrm{a}_1^\top \mathrm{b} \\ \frac{\partial l}{\partial \mathrm{f}} \frac{\partial}{\partial \mathrm{A}} \mathrm{a}_2^\top \mathrm{b} \\ \vdots \\ \frac{\partial l}{\partial \mathrm{f}} \frac{\partial}{\partial \mathrm{A}} \mathrm{a}_m^\top \mathrm{b} \end{bmatrix} = \begin{bmatrix} \frac{\partial l}{\partial f_1} \mathrm{b}^\top \\ \frac{\partial l}{\partial f_2} \mathrm{b}^\top \\ \vdots \\ \frac{\partial l}{\partial f_m} \mathrm{b}^\top \end{bmatrix}$$

Note that

$$\begin{bmatrix} \frac{\partial l}{\partial f_1} \mathrm{b}^\top \\ \frac{\partial l}{\partial f_2} \mathrm{b}^\top \\ \vdots \\ \frac{\partial l}{\partial f_m} \mathrm{b}^\top \end{bmatrix} = \begin{bmatrix} \frac{\partial l}{\partial f_1} \\ \frac{\partial l}{\partial f_2} \\ \cdots \\ \frac{\partial l}{\partial f_m} \end{bmatrix} \mathrm{b}^\top = \left(\frac{\partial l}{\partial \mathrm{f}} \right)^\top \mathrm{b}^\top$$

We can group the terms inside a single transpose.

Which results in

$$\frac{\partial}{\partial \mathrm{A}} l(\mathrm{f}(\mathrm{A}, \mathrm{b})) = \left(\mathrm{b} \frac{\partial l}{\partial \mathrm{f}}\right)^\top$$

The derivative with respect to b is simpler:

$$\frac{\partial}{\partial \mathrm{b}} l(\mathrm{f}(\mathrm{A}, \mathrm{b})) = \frac{\partial l}{\partial \mathrm{f}} \frac{\partial}{\partial \mathrm{b}} (\mathrm{A}\mathrm{b}) = \frac{\partial l}{\partial \mathrm{f}} \mathrm{A}$$

Case 3: VJP for matrix-matrix multiplication

Let

$$\mathrm{F}(\mathrm{A}, \mathrm{B}) = \mathrm{A}\mathrm{B}$$

where $\mathrm{F} \in \mathbb{R}^{m \times p}$, $\mathrm{B} \in \mathbb{R}^{n \times p}$, and $\mathrm{A} \in \mathbb{R}^{m \times n}$

Let $l(\mathrm{F}(\mathrm{A}, \mathrm{B})) \in \mathbb{R}$ be the eventual scalar output. We want to find $\frac{\partial l}{\partial \mathrm{A}}$ and $\frac{\partial l}{\partial \mathrm{B}}$ for Vector Jacobian product.

Note that a matrix-matrix multiplication can be written in terms horizontal stacking of matrix-vector multiplications. Specifically, write F and B in terms of their column vectors:

$$\mathrm{B} = \begin{bmatrix} \mathrm{b}_1 & \mathrm{b}_2 & \dots & \mathrm{b}_p \end{bmatrix}$$

$$\mathrm{F} = \begin{bmatrix} \mathrm{f}_1 & \mathrm{f}_2 & \dots & \mathrm{f}_p \end{bmatrix}.$$

Then for all i

$$\mathrm{f}_i = \mathrm{A}\mathrm{b}_i$$

From the VJP of matrix-vector multiplication, we can write

$$\frac{\partial l}{\partial \mathrm{f}_i} \frac{\partial}{\partial \mathrm{A}} \mathrm{f}_i = \frac{\partial l}{\partial \mathrm{f}_i} \frac{\partial}{\partial \mathrm{A}} (\mathrm{A}\mathrm{b}_i) = \left(\mathrm{b}_i \frac{\partial l}{\partial \mathrm{f}_i}\right)^\top \in \mathbb{R}^{m \times n}$$

and for all $i \neq j$

$$\frac{\partial l}{\partial \mathrm{f}_j} \frac{\partial}{\partial \mathrm{A}} (\mathrm{A}\mathrm{b}_i) = 0_{m \times n}$$

Instead of writing $l(\mathrm{F})$, we can also write $l(\mathrm{f}_1, \mathrm{f}_2, \dots, \mathrm{f}_p)$, then by chain rule of functions with multiple arguments, we have,

$$\frac{\partial}{\partial \mathrm{A}} l(\mathrm{F}(\mathrm{A}, \mathrm{B})) = \frac{\partial}{\partial \mathrm{A}} l(\mathrm{f}_1, \mathrm{f}_2, \dots, \mathrm{f}_p) = \frac{\partial l}{\partial \mathrm{f}_1} \frac{\partial \mathrm{f}_1}{\partial \mathrm{A}} + \frac{\partial l}{\partial \mathrm{f}_2} \frac{\partial \mathrm{f}_2}{\partial \mathrm{A}} + \dots + \frac{\partial l}{\partial \mathrm{f}_p} \frac{\partial \mathrm{f}_p}{\partial \mathrm{A}}$$

$$\frac{\partial}{\partial \mathrm{A}} l(\mathrm{F}(\mathrm{A}, \mathrm{B})) = \left(\mathrm{b}_1 \frac{\partial l}{\partial \mathrm{f}_1}\right)^\top + \left(\mathrm{b}_2 \frac{\partial l}{\partial \mathrm{f}_2}\right)^\top + \dots + \left(\mathrm{b}_p \frac{\partial l}{\partial \mathrm{f}_p}\right)^\top$$
$$= \left(\mathrm{b}_1 \frac{\partial l}{\partial \mathrm{f}_1} + \mathrm{b}_2 \frac{\partial l}{\partial \mathrm{f}_2} + \dots + \mathrm{b}_p \frac{\partial l}{\partial \mathrm{f}_p}\right)^\top$$

It turns out that some of outer products can be compactly written as matrix-matrix multiplication: $$ \bfb_1\frac{\p l}{\p \bff_1}

- \bfb_2\frac{\p l}{\p \bff_2}
- \dots
- \bfb_p\frac{\p l}{\p \bff_p} =

$$\begin{bmatrix} \mathrm{b}_1 & \mathrm{b}_2 & \dots & \mathrm{b}_p \end{bmatrix}$$

$$\begin{bmatrix} \frac{\partial l}{\partial \mathrm{f}_1} \\ \frac{\partial l}{\partial \mathrm{f}_2} \\ \vdots \\ \frac{\partial l}{\partial \mathrm{f}_p} \end{bmatrix}$$

= \bfB \left(\frac{\p l}{\p \bfF}\right)^\top$$

Hence,

$$\frac{\partial}{\partial \mathrm{A}} l(\mathrm{F}(\mathrm{A}, \mathrm{B})) = \frac{\partial l}{\partial \mathrm{F}} \mathrm{B}^\top$$

The vector Jacobian product for B can be found by applying the above rule to $\mathrm{F}_2(\mathrm{A}, \mathrm{C}) = \mathrm{F}^\top(\mathrm{A}, \mathrm{B}) = \mathrm{B}^\top \mathrm{A}^\top = \mathrm{C}\mathrm{A}^\top$ where $\mathrm{C} = \mathrm{B}^\top$ and $\mathrm{F}_2 = \mathrm{F}^\top$.

$$\frac{\partial}{\partial \mathrm{C}} l(\mathrm{F}_2(\mathrm{A}, \mathrm{C})) = \frac{\partial l}{\partial \mathrm{F}_2} \mathrm{A}$$

Take transpose of both sides

$$\frac{\partial}{\partial \mathrm{C}^\top} l(\mathrm{F}_2^\top(\mathrm{A}, \mathrm{C})) = \mathrm{A}^\top \frac{\partial l}{\partial \mathrm{F}_2^\top}$$

Put back, $\mathrm{C} = \mathrm{B}^\top$ and $\mathrm{F}_2 = \mathrm{F}^\top$,

$$\frac{\partial}{\partial \mathrm{B}} l(\mathrm{F}(\mathrm{A}, \mathrm{B})) = \mathrm{A}^\top \frac{\partial l}{\partial \mathrm{F}}$$

In [4]:
```
def matmul_vjp(dldF, A, B):
    G = dldF
    if G.ndim == 0:
        # Case 1: vector-vector multiplication
        assert A.ndim == 1 and B.ndim == 1
```

```
        dldA = G*B
        dldB = G*A
        return (unbroadcast(A, dldA),
                unbroadcast(B, dldB))

    assert not (A.ndim == 1 and B.ndim == 1)

    # 1. If both arguments are 2-D they are multiplied like conventional mat
    # 2. If either argument is N-D, N > 2, it is treated as a stack of matri
    # residing in the last two indexes and broadcast accordingly.
    if A.ndim >= 2 and B.ndim >= 2:
        dldA = G @ B.swapaxes(-2, -1)
        dldB = A.swapaxes(-2, -1) @ G
    if A.ndim == 1:
        # 3. If the first argument is 1-D, it is promoted to a matrix by pre
        #    1 to its dimensions. After matrix multiplication the prepended
        A_ = A[np.newaxis, :]
        G_ = G[np.newaxis, :]
        dldA = G @ B.swapaxes(-2, -1)
        dldB = A_.swapaxes(-2, -1) @ G_ # outer product
    elif B.ndim == 1:
        # 4. If the second argument is 1-D, it is promoted to a matrix by ap
        #    a 1 to its dimensions. After matrix multiplication the appendec
        B_ = B[:, np.newaxis]
        G_ = G[:, np.newaxis]
        dldA = G_ @ B_.swapaxes(-2, -1) # outer product
        dldB = A.swapaxes(-2, -1) @ G
    return (unbroadcast(A, dldA),
            unbroadcast(B, dldB))


matmul = Op(
    apply=np.matmul,
    vjp=matmul_vjp,
    name='@',
    nargs=2)
```

In [5]:

```
def exp_vjp(dldf, x):
    dldx = dldf * np.exp(x)
    return (unbroadcast(x, dldx),)
exp = Op(
    apply=np.exp,
    vjp=exp_vjp,
    name='exp',
    nargs=1)
```

In [6]:

```
def log_vjp(dldf, x):
    dldx = dldf / x
    return (unbroadcast(x, dldx),)
log = Op(
    apply=np.log,
    vjp=log_vjp,
    name='log',
    nargs=1)
```

In [7]:
```
def sum_vjp(dldf, x, axis=None, **kwargs):
    if axis is not None:
        dldx = np.expand_dims(dldf, axis=axis) * np.ones_like(x)
    else:
        dldx = dldf * np.ones_like(x)
    return (unbroadcast(x, dldx),)

sum_ = Op(
    apply=np.sum,
    vjp=sum_vjp,
    name='sum',
    nargs=1)
```

In [18]:
```
def maximum_vjp(dldf, a, b):
    dlda = dldf * np.where(a > b, 1, 0)
    dldb = dldf * np.where(a > b, 0, 1)
    return unbroadcast(a, dlda), unbroadcast(b, dldb)

maximum = Op(
    apply=np.maximum,
    vjp=maximum_vjp,
    name='maximum',
    nargs=2)
```

In [19]:
```
NoOp = Op(apply=None, name='', vjp=None, nargs=0)
class Tensor:
    __array_priority__ = 100
    def __init__(self, value, grad=None, parents=(), op=NoOp, kwargs={}, rec
        self.value = np.asarray(value)
        self.grad = grad
        self.parents = parents
        self.op = op
        self.kwargs = kwargs
        self.requires_grad = requires_grad

    shape = property(lambda self: self.value.shape)
    ndim  = property(lambda self: self.value.ndim)
    size  = property(lambda self: self.value.size)
    dtype = property(lambda self: self.value.dtype)

    def __add__(self, other):
        cls = type(self)
        other = other if isinstance(other, cls) else cls(other)
        return cls(add.apply(self.value, other.value),
                   parents=(self, other),
                   op=add)
    __radd__ = __add__

    def __mul__(self, other):
        cls = type(self)
        other = other if isinstance(other, cls) else cls(other)
        return cls(mul.apply(self.value, other.value),
                   parents=(self, other),
                   op=mul)
    __rmul__ = __mul__
```

```
    def __matmul__(self, other):
        cls = type(self)
        other = other if isinstance(other, cls) else cls(other)
        return cls(matmul.apply(self.value, other.value),
                   parents=(self, other),
                   op=matmul)

    def exp(self):
        cls = type(self)
        return cls(exp.apply(self.value),
                 parents=(self,),
                 op=exp)

    def log(self):
        cls = type(self)
        return cls(log.apply(self.value),
                 parents=(self, ),
                 op=log)

    def __pow__(self, other):
        cls = type(self)
        other = other if isinstance(other, cls) else cls(other)
        return (self.log() * other).exp()

    def __div__(self, other):
        return self * (other**(-1))

    def __sub__(self, other):
        return self + (other * (-1))

    def __neg__(self):
        return self*(-1)

    def sum(self, axis=None):
        cls = type(self)
        return cls(sum_.apply(self.value, axis=axis),
                    parents=(self,),
                    op=sum_,
                    kwargs=dict(axis=axis))

    def maximum(self, other):
        cls = type(self)
        other = other if isinstance(other, cls) else cls(other)
        return cls(maximum.apply(self.value, other.value),
                    parents=(self, other),
                    op=maximum)

    def __repr__(self):
        cls = type(self)
        return f"{cls.__name__}(value={self.value}, op={self.op.name})" if s
        #return f"{cls.__name__}(value={self.value}, parents={self.parents},

    def backward(self, grad):
        self.grad = grad if self.grad is None else (self.grad+grad)
        if self.requires_grad and self.parents:
```

```
            p_vals = [p.value for p in self.parents]
            assert len(p_vals) == self.op.nargs
            p_grads = self.op.vjp(grad, *p_vals, **self.kwargs)
            for p, g in zip(self.parents, p_grads):
                p.backward(g)
```

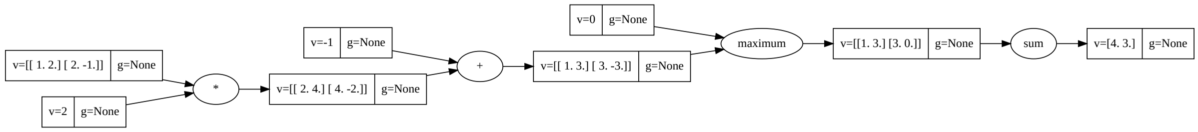
In [20]:
```
Tensor([1, 2]).sum()
```

Out[20]:
```
Tensor(value=3, op=sum)
```

In [68]:
```
try:
    from graphviz import Digraph
except ImportError as e:
    import subprocess
    subprocess.call("pip install --user graphviz".split())

def trace(root):
    nodes, edges = set(), set()
    def build(v):
        if v not in nodes:
            nodes.add(v)
            for p in v.parents:
                edges.add((p, v))
                build(p)
    build(root)
    return nodes, edges

def draw_dot(root, format='svg', rankdir='LR'):
    """
    format: png | svg | ...
    rankdir: TB (top to bottom graph) | LR (left to right)
    """
    assert rankdir in ['LR', 'TB']
    nodes, edges = trace(root)
    dot = Digraph(format=format, graph_attr={'rankdir': rankdir}) #, node_at

    for n in nodes:
        vstr = np.array2string(np.asarray(n.value), precision=4)
        gradstr= np.array2string(np.asarray(n.grad), precision=4)
        dot.node(name=str(id(n)), label = f"{{v={vstr} | g={gradstr}}}", sha
        if n.parents:
            dot.node(name=str(id(n)) + n.op.name, label=n.op.name)
            dot.edge(str(id(n)) + n.op.name, str(id(n)))

    for n1, n2 in edges:
        dot.edge(str(id(n1)), str(id(n2)) + n2.op.name)

    return dot
```

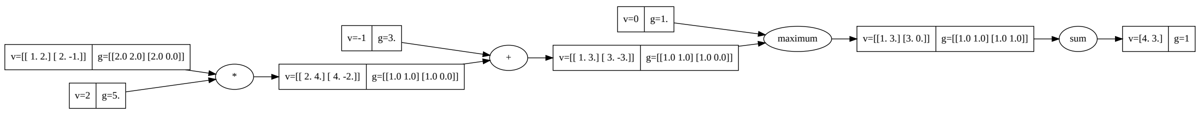
In [69]:
```
# a very simple example
x = Tensor([[1.0, 2.0],
            [2.0, -1.0]])
y = (x * 2 - 1).maximum(0).sum(axis=-1)
draw_dot(y)
```

Out[69]:



In [70]:
```
y.backward(np.ones_like(y))
draw_dot(y)
```

Out[70]:

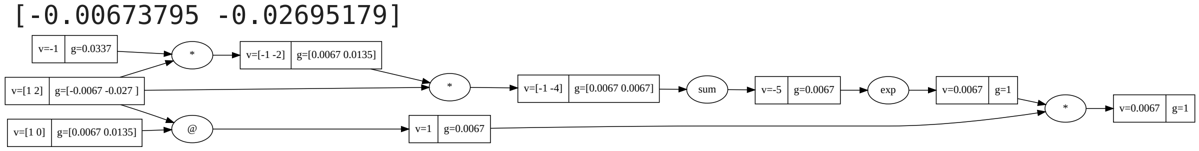


In [73]:
```
def f_np(x):
    b = [1, 0]
    return (x @ b)*np.exp((-x*x).sum(axis=-1))

def f_T(x):
    b = [1, 0]
    return (x @ b)*(-x*x).sum(axis=-1).exp()

def grad_f(x):
    xT = Tensor(x)
    y = f_T(xT)
    y.backward(np.ones_like(y.value))
    return xT.grad
```

In [74]:
```
xT = Tensor([1, 2])
out = f_T(xT)
out.backward(1)
print(xT.grad)
draw_dot(out)
```

```
[-0.00673795 -0.02695179]
```

Out[74]:



In [57]:
```
def numerical_jacobian(f, x, h=1e-10):
    n = x.shape[-1]
    eye = np.eye(n)
    x_plus_dx = x + h * eye # n x n
    num_jac = (f(x_plus_dx) - f(x)) / h # limit definition of the formula #
    if num_jac.ndim >= 2:
        num_jac = num_jac.swapaxes(-1, -2) # m x n
    return num_jac

# Compare our grad_f with numerical gradient
def check_numerical_jacobian(f, jac_f,  nD=2, **kwargs):
    x = np.random.rand(nD)
    print(x)
    num_jac = numerical_jacobian(f, x, **kwargs)
    print(num_jac)
    print(jac_f(x))
    return np.allclose(num_jac, jac_f(x), atol=1e-06, rtol=1e-4) # m x n
```

```python
## Throw error if grad_f is wrong
assert check_numerical_jacobian(f_np, grad_f)
```

```
[0.4717993  0.90549333]
[ 0.19560853 -0.30124125]
[ 0.19560835 -0.30124165]
```

In [ ]:

```python

```

In [ ]:

```python

```