ECE490 HW5

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HW5 Problem 1 (20 marks)

By following the Example 9.13 given below, solve the following problem (equivalent to Problem 20 from Bersekas):

A random variable X is characterized by a normal PDF with mean $\mu_0 = 20$, and a variance that is either $\sigma_0 = 16$ (negative class $Y = 0$) or $\sigma_1 = 25$ (positive $Y = 1$). We want to develop a likelihood test to distinguish between two classes, using three sample values x_1, x_2, x_3. Show that the likelihood ratio test is equivalent to picking class,

$$\hat{Y}(x) = \begin{cases} 1 & \text{if } x_1 + x_2 + x_3 > \gamma \\ 0 & \text{otherwise} \end{cases}$$

for some scalar γ. Determine the value of γ so that the probability of False positive rate is 0.05. What is the corresponding True positive rate?

Terminology translation

1. Read Section 9.3 from Bertsekas (Intro to Probability) (ungraded)

Bertsekas book uses different terminology than what was covered in the class.

1. The two Hypothesis are equivalent to two classes. H_1 is the positive class ($Y = 1$) and H_0 is the negative class ($Y = 0$).
2. The Rejection region contains all the x for which the Likelihood ratio test predicts the class of x to be H_0.

$$R = \text{ all the } x \text{ such that } L(x) \leq \eta$$

where $L(x) = \frac{f_x(x|Y=1)}{f_x(x|Y=0)}$ is the likelihood ratio.

Similarities between Problem 20 and Example 9.13

1. Both the problems deal with multiple Gaussian random variables. Example 9.13 deals with two Gaussian random variables X_1 and X_2. Problem 20 tdeals with three Gaussian random variables x_1, x_2, x_3. The samples of a random variable have the same distribution.
2. Only the parameters (mean and variance) of a Gaussian distributions in Example 9.13 and Problem 20 are different.
   a) Example 9.13 has mean = 0 and standard deviation = 1 (unit variance) under $Y = 0$.
   b) Example 9.13 has mean = 2 and standard deviation = 1 (unit variance) under $Y = 1$.
   c) Problem 20 has mean = 20 and standard deviation = $\sigma_0 = 16$ under $Y = 0$.
   d) Problem 20 has mean = 20 and standard deviation = $\sigma_1 = 25$ under $Y = 1$.
3. Both problems first find γ from false rejection probability $P(\hat{Y}(x) = 1|Y = 0)$ aka false positive rate. Both in Example 9.13 and Problem 20, the false rejection probability is 0.05.
4. Both problems use γ to find false acceptance probability aka false negative rate $P(\hat{Y}(x) = 0|Y = 1)$.

Example 9.13 (from Bertsekas Section 9.3) We observe two i.i.d. (independent identically distributed) random variables X_1 and X_2, with unit variance. Under negative class $Y = 0$ the random variables have common mean as 0; under positive class $Y = 1$ their common mean is 2. We fix the false positive rate to $\alpha = 0.05$. Show that the likelihood ratio test is equivalent to picking class,

$$\hat{Y}(x) = \begin{cases} 1 & \text{if } x_1 + x_2 > \gamma \\ 0 & \text{otherwise} \end{cases}$$

for some scalar γ. Determine the value of γ. What is the corresponding True positive rate?

Solution to 9.13

1. Definition 1: Gaussian PDF with mean μ and standard deviation σ is given by $f_X(X = x) = \frac{1}{\sigma\sqrt{2\pi}} \exp(-\frac{(x-\mu)^2}{2\sigma^2})$.
2. Theorem 1: Joint Gaussian PDF for two **independent** random variables X_1 and X_2 is the product of the two distributions,

$$f_{X_1,X_2}(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2)$$
$$f_{X_1,X_2}(x_1, x_2) = \frac{1}{\sigma_1\sqrt{2\pi}} \exp\left(-\frac{(x_1-\mu_1)^2}{2\sigma_1^2}\right) \frac{1}{\sigma_2\sqrt{2\pi}} \exp\left(-\frac{(x_2-\mu_2)^2}{2\sigma_2^2}\right)$$

   Multiplication of two exponents causes addition in the radix $\exp(a)\exp(b) = \exp(a + b)$.

$$f_{X_1,X_2}(x_1, x_2) = \frac{1}{\sigma_1\sigma_2 2\pi} \exp\left(-\frac{(x_1-\mu_1)^2}{2\sigma_1^2} - \frac{(x_2-\mu_2)^2}{2\sigma_2^2}\right)$$

3. Let the mean and standard deviation of i.i.d (independent and identically distributed) random variables X_1 X_2 under positive class $Y = 1$ be μ_1 and σ_1 respectively. Same be μ_0 and σ_0 under negative class $Y = 0$. The the Likelihood ratio is given by:

$$L(x) = \frac{f_{X_1,X_2}(x_1, x_2|Y = 1)}{f_{X_1,X_2}(x_1, x_2|Y = 0)}$$

   Under positive class $Y = 1$ the joint distribution is,

$$f_{X_1,X_2}(x_1, x_2|Y = 1) = f_{X_1}(x_1|Y = 1) f_{X_2}(x_2|Y = 1) = \frac{1}{\sigma_1^2(2\pi)} \exp\left(-\frac{(x_1-\mu_1)^2}{2\sigma_1^2} - \frac{(x_2-\mu_1)^2}{2\sigma_1^2}\right)$$

   Under Hypothesis $Y = 0$ the joint distribution is,

$$f_{X_1,X_2}(x_1, x_2|Y = 0) = f_{X_1}(x_1|Y = 0) f_{X_2}(x_2|Y = 0) = \frac{1}{\sigma_0^2(2\pi)} \exp\left(-\frac{(x_1-\mu_0)^2}{2\sigma_0^2} - \frac{(x_2-\mu_0)^2}{2\sigma_0^2}\right)$$

   The Likelihood ratio is (again using $\frac{\exp(a)}{\exp(b)} = \exp(a - b)$),

$$L(x) = \frac{f_{X_1,X_2}(x_1, x_2|Y = 1)}{f_{X_1,X_2}(x_1, x_2|Y = 0)} = \frac{\sigma_0^2}{\sigma_1^2} \exp\left(-\frac{(x_1-\mu_1)^2}{2\sigma_1^2} - \frac{(x_2-\mu_1)^2}{2\sigma_1^2} + \frac{(x_1-\mu_0)^2}{2\sigma_0^2} + \frac{(x_2-\mu_0)^2}{2\sigma_0^2}\right)$$

4. Likelihood ratio test is the comparison of the Likelihood ration $L(x)$ with some scalar η. We pick the positive class if $L(x) > \eta$. Equivalently, we pick positive class $\hat{Y}(x) = 1$ if,

$$L(x) = \frac{\sigma_0^2}{\sigma_1^2} \exp\left(-\frac{(x_1-\mu_1)^2}{2\sigma_1^2} + \frac{(x_1-\mu_0)^2}{2\sigma_0^2} - \frac{(x_2-\mu_1)^2}{2\sigma_1^2} + \frac{(x_2-\mu_0)^2}{2\sigma_0^2}\right) > \eta.$$

Since $\frac{\sigma_0^2}{\sigma^1}$ is positive, we can move this to other side of inequality,

$$\exp\left(-\frac{(x_1-\mu_1)^2}{2\sigma_1^2} - \frac{(x_2-\mu_1)^2}{2\sigma_1^2} + \frac{(x_1-\mu_0)^2}{2\sigma_0^2} + \frac{(x_2-\mu_0)^2}{2\sigma_0^2}\right) > \eta\frac{\sigma_1^2}{\sigma_0^2}.$$

Because $\log(z)$ is an increasing function, we can take log on both sides,

$$-\frac{(x_1-\mu_1)^2}{2\sigma_1^2} - \frac{(x_2-\mu_1)^2}{2\sigma_1^2} + \frac{(x_1-\mu_0)^2}{2\sigma_0^2} + \frac{(x_2-\mu_0)^2}{2\sigma_0^2} > \log(\eta) + \log(\sigma_1^2) - \log(\sigma_0^2)$$