Before you turn this problem in, make sure everything runs as expected. First, **restart the kernel** (in the menubar, select Kernel→Restart) and then **run all cells** (in the menubar, select Cell→Run All).

Make sure you fill in any place that says `YOUR CODE HERE` or "YOUR ANSWER HERE", as well as your name and collaborators below:

```
In [ ]: NAME = ""
        COLLABORATORS = ""
```

Differentiation options

Numpy ⟺ Pytorch

+ GPU

+ Automatic differentiation

def f($\underline{x}$): n

return

1. Numerical differentiation

2. Symbolic differentiation

3. Automatic differentiation

   A. Forward mode differentiation

   B. Reverse mode differentiation

1. Numerical differentiation :

check_numerical_jacobian

$$\left.\frac{\partial f}{\partial x_1}\right|_{\underline{x}} = \frac{f\left(\underline{x} + \begin{bmatrix}\varepsilon\\0\\0\\0\end{bmatrix}\right) - f(\underline{x})}{\varepsilon} \quad \varepsilon = 1e-6$$

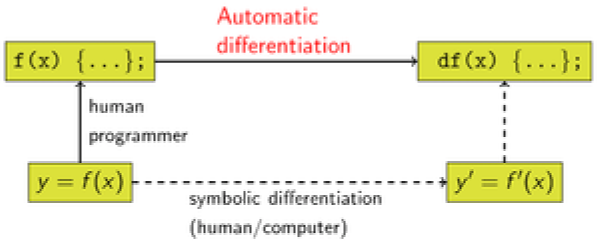
$$\frac{\partial f}{\partial x_2} = \frac{f\left(\underline{x} + \begin{bmatrix}0\\\varepsilon\\0\\0\end{bmatrix}\right) - f(\underline{x})}{\boxed{\varepsilon}}$$

For n-dim derivative : 2n calls to function f

O(n)

2. Symbolic differentiation

3. Automatic differentiation


Automatic differentiation
f(x) {...};
df(x) {...};
human programmer
y = f(x)
y' = f'(x)
symbolic differentiation (human/computer)

3.A Forward mode

Example:

$$z = f(x_1, x_2) = x_1 x_2 + \sin(x_1)$$

3.B Reverse mode

② Symbolic diff. [close Automatic diff]

Tensorflow
Pytorch
Jax

x = Symbol()

f = x**2 + 2*x

$\text{grad}(f, x) \longrightarrow \underbrace{\text{print a functional form of derivative}}_{\cancel{\sin}'(\sin(x))}$

more optimizations possible in symbolic derivatives

Diff wrt Auto diff

a)

b) Derivative computation is done for all $\underline{x}$ input

c) $\underline{f(x)}$ has to be computed as a separate process

Auto. Diff

① Create a library of atomic function

atomic function		derivative function
$(f(x) + g(\underline{x}))$	$\longleftrightarrow$	$\frac{\partial}{\partial x} f(x) + \frac{\partial}{\partial x} g(x)$
*		
**		
sin	$\longleftrightarrow$	cos
cos	$\longleftrightarrow$	-sin
exp	$\longleftrightarrow$	exp

② Write the function whose derivative you want in terms of the atomic functions

③ Then derivative of any program written in Terms of atomic functions, can be computed by chain rule.

How?

$$f(\underline{x}) = \exp(-\underline{x}**2)$$

$$f(x) = \underbrace{\exp}\Big(\underbrace{\_\_mul\_\_\big(\underbrace{\underbrace{\_\_pow\_\_(x,2)}}_{\text{deriv}}, -1\big)}_{y}\Big)$$

$$f(x) = \overbrace{\overbrace{\overbrace{\frac{\partial \exp(z)}{\partial \underline{z}}}^{1} \cdot \frac{\partial}{\partial \underline{y}}\_\_mul\_\_(y)}^{Z \quad 2} \cdot \frac{\partial}{\partial \underline{x}}\_\_pow\_\_(x,2}^{3}$$

$$\underbrace{\frac{\partial \exp(z)}{\partial z} \cdot \underbrace{\frac{\partial}{\partial y}\_\_mul\_\_(y) \cdot \underbrace{\frac{\partial}{\partial x}\_\_pow\_\_(x,2}_{1}}_{2}}_{3}$$

$$\text{Auto Diff} \begin{cases} \to \underline{\text{Forward diff}} \\ \to \underline{\text{Backward diff.}} \ (\text{Backpropagation}) \end{cases}$$

[compution complexity is diff.]

$x_1 = \text{ForwardDiff}(1, 1)$

$x_2 = \text{ForwardDiff}(2, 0)$

Example:

$x_1 + x_2$

$$z = f(x_1, x_2) = \underbrace{x_1 x_2}_{\text{ForwardDiff}(2,\ 2)} \boxed{+} \underbrace{\sin(x_1)}_{\text{ForwardDiff}(\sin(1),\ \cos(1))}$$

```
In [ ]: import numpy as np
        class ForwardDiff: ✓
            def __init__(self, value, grad=None):
                self.value = value ✓
                self.grad = np.zeros_like(value) if grad is None else grad ✓


            def __add__(self, other):
                cls = type(self)          ForwardDiff
                other = other if isinstance(other, cls) else cls(other)
                out = cls(self.value + other.value,
                          self.grad + other.grad)
                return out
            __radd__ = __add__   same fn

            def __repr__(self):
                return f"{self.__class__.__name__}(data={self.value}, grad={self.gra
        x = ForwardDiff(2, 1)
        y = ForwardDiff(3, 0)

        f = x + y
        f
```

printing this class

$$f(a, b) = a + b$$

$$\frac{d}{dy} f(a, b) = \frac{da}{dy} + \frac{db}{dy}$$

$$f(x, y) = x + y$$

$$\frac{d}{dx} f(x, y) = \underbrace{\boxed{\frac{dx}{dx}}}_{=1} + \boxed{\frac{dy}{dx}} = 0$$

$$f.grad = \frac{df}{dx}$$

```
In [ ]: oldFD = ForwardDiff # Bad practice: do not do it
        class ForwardDiff(oldFD):
            def __mul__(self, other):
                cls = type(self)
                other = other if isinstance(other, cls) else cls(other)
                out = cls(self.value * other.value,
                          other.value * self.grad +
                          self.value * other.grad)
                return out

            __rmul__ = __mul__

        x = ForwardDiff(2, 0)
        y = ForwardDiff(3, 1)

        f1 = x * y
        f2 = 2*x + 3*y + x*y
        f1, f2
```

$$f(x, y) = x * y$$

$$\frac{d}{dz} f(x, y) = x * \frac{dy}{dz} + y * \frac{dx}{dz}$$

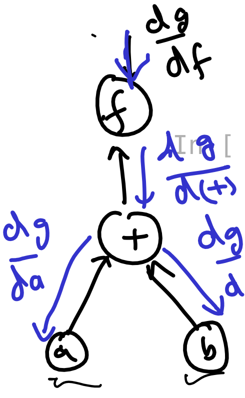
```
In [ ]: oldFD = ForwardDiff # Bad practice: do not do it
        class ForwardDiff(oldFD):
            def log(self):
                cls = type(self)
                return cls(np.log(self.value),
                           1/self.value * self.grad)
            def exp(self):
                cls = type(self)
```

$$\frac{d}{dz} \log(x) = \frac{1}{x} \cdot \frac{dx}{dz}$$

```
        out_val = np.exp(self.value)
        return cls(out_val,
                   out_val * self.grad)
```

$$\frac{d}{dz} exp(x) = exp(x) \frac{dx}{dz}$$

```
    def sin(self):
        cls = type(self)
        return cls(np.sin(self.value),
                   np.cos(self.value) * self.grad)

    def cos(self):
        cls = type(self)
        return cls(np.cos(self.value),
                  -np.sin(self.value) * self.grad)

    def __pow__(self, other):
        cls = type(self)
        other = other if isinstance(other, cls) else cls(other)
        return (self.log() * other).exp()

    def __neg__(self): # -self
        return self * -1

    def __sub__(self, other): # self - other
        return self + (-other)

    def __truediv__(self, other): # self / other
        return self * other**-1

    def __rtruediv__(self, other): # other / self
        return other * self**-1



x = ForwardDiff(2, 1)
y = ForwardDiff(3, 0)

f = x**y
f
```



In [ ]:

```
import numpy as np
def add_vjp(a, b, grad):
    return grad, grad


def no_parents_vjp(grad):
    return (grad,)

class ReverseDiff:
    def __init__(self, value, parents=(), op='', vjp=no_parents_vjp):
        self.value = value
        self.parents = parents
        self.op = op
        self.vjp = vjp
        self.grad = None

    def backward(self, grad):
```

$\frac{dg}{df}$

vjp = vector -jacobian -product

$\frac{dg}{db}$

$f(a,b) = a+b$

given $\frac{dg}{df}$

$$\frac{dg}{da} = \boxed{\frac{dg}{df} \cdot \frac{df}{da}} = \frac{dg}{df}$$

$g(f)$

$f(a,b) = a+b$

arguments to The operations

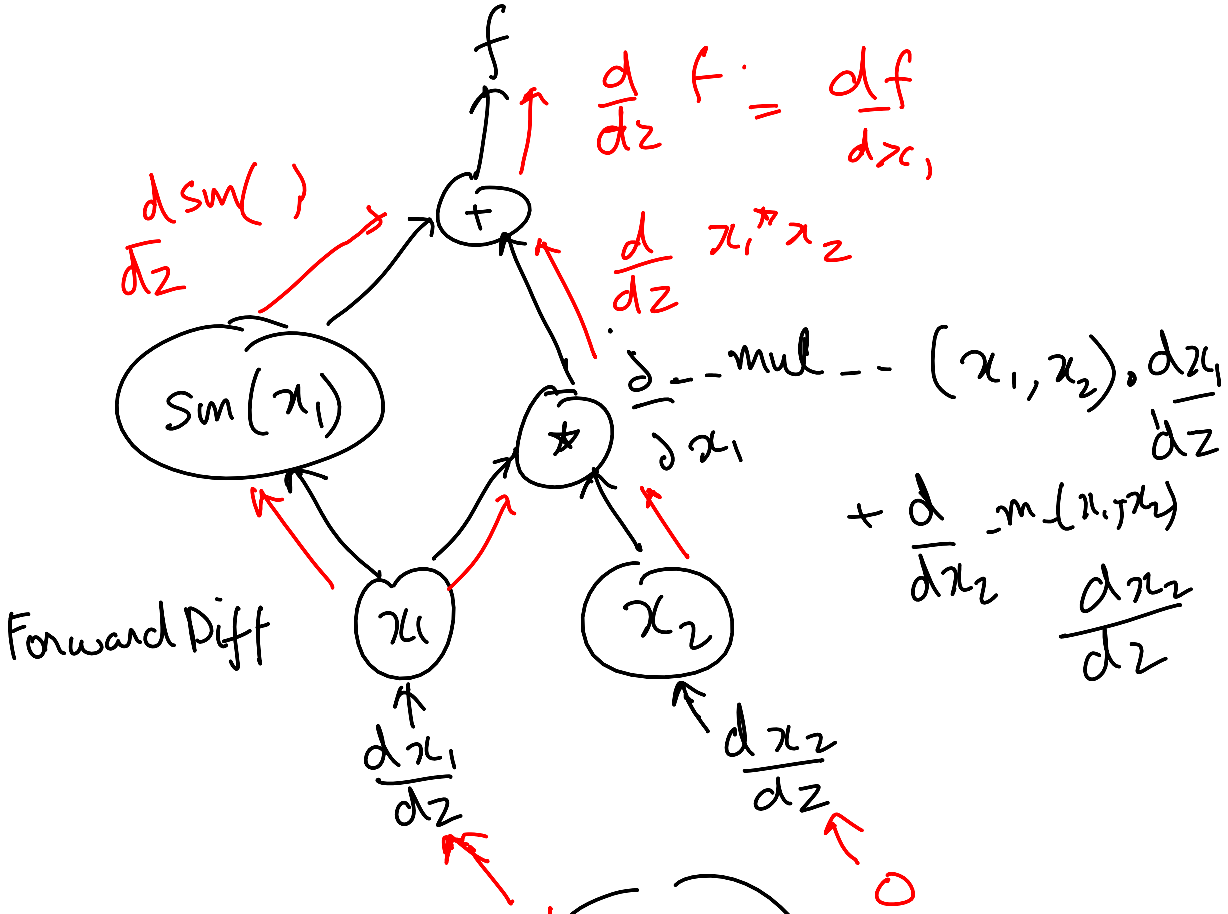
= operation

= deriv

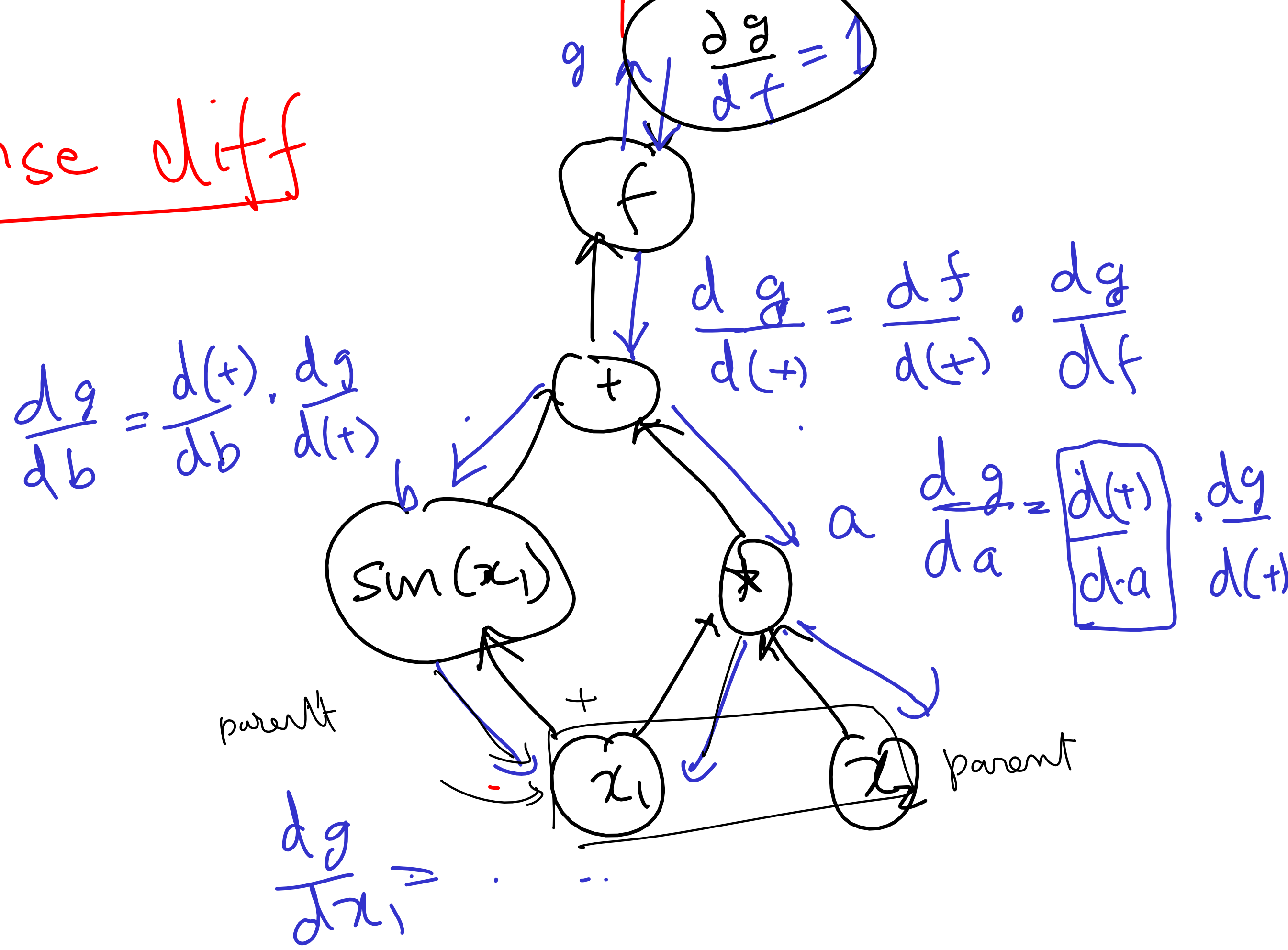
$$\frac{dg}{db} = \frac{dg}{df} \cdot \frac{df}{db} = \frac{dg}{df}$$

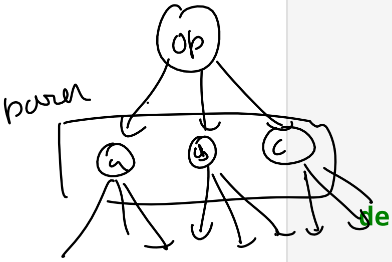
Forward Diff

$$f(x) = x_1 * x_2 + \sin(x_1)$$


f
$\frac{d}{dz} f = \frac{df}{dx_1}$
$\frac{d\sin(\;)}{dz}$
+
$\frac{d}{dz} x_1 * x_2$
$\frac{\partial}{\partial x_1}$ _mul_ $(x_1, x_2) \cdot \frac{dx_1}{dz}$
$+ \frac{d}{dx_2}$ _m_(x_1, x_2) $\frac{dx_2}{dz}$
$\sin(x_1)$
*
Forward Diff
x_1
x_2
$\frac{dx_1}{dz}$
$\frac{dx_2}{dz}$
0

Reverse diff


g
$\frac{dg}{df} = 1$
f
$\frac{dg}{d(+)} = \frac{df}{d(+)} \cdot \frac{dg}{df}$
+
$\frac{dg}{db} = \frac{d(+)}{db} \cdot \frac{dg}{d(+)}$
b
a
$\frac{dg}{da} = \boxed{\frac{d(+)}{da}} \cdot \frac{dg}{d(+)}$
$\sin(x_1)$
*
+
parent
x_1
x_2
parent
$\frac{dg}{dx_1} =$ . . .



```
        self.grad = grad
        op_args = [p.value for p in self.parents]
        grads = self.vjp(*op_args, grad)
        for g, p in zip(grads, self.parents):
            p.backward(g)

    def __add__(self, other):
        cls = type(self)
        other = other if isinstance(other, cls) else cls(other)
        out = cls(self.value + other.value,
                  parents=(self, other),
                  op='+',
                  vjp=add_vjp)
        return out

    __radd__ = __add__

    def __repr__(self):
        cls = type(self)
        return f"{cls.__name__}(value={self.value}, parents={self.parents},

x = ReverseDiff(2)
y = ReverseDiff(3)

f = x + y + 3
f.backward(1)
f
x.grad, y.grad
```

+ self.grad

if self.f.grad is not None: grad

$\frac{dg}{df}$

x_1 = ReverseDiff(2)

y = Rev

In [ ]:

```
oldRD = ReverseDiff # Bad practice: do not do it

def mul_vjp(a, b, grad):
    return grad * b, grad * a

class ReverseDiff(oldRD):
    def __mul__(self, other):
        cls = type(self)
        other = other if isinstance(other, cls) else cls(other)
        out = cls(self.value * other.value,
                 parents=(self, other),
                 op='*',
                 vjp=mul_vjp)
        return out

    __rmul__ = __mul__

x = ReverseDiff(2)
y = ReverseDiff(3)

f1 = 5*x + 7* y
f1.backward(1)
x.grad, y.grad
```

$g = f_1$

$\frac{dg}{df} = 1$

In [ ]:

```
f2 = x*y
f2.backward(1)
```

```
x.grad, y.grad
```