ECE490 HW5

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HW5 Problem 1 (20 marks)

By following the Example 9.13 given below, solve the following problem (equivalent to Problem 20 from Bersekas):

A random variable X is characterized by a normal PDF with mean $\mu_0 = 20$, and a variance that is either $\sigma_0 = 16$ (negative class $Y = 0$) or $\sigma_1 = 25$ (positive $Y = 1$). We want to develop a likelihood test to distinguish between two classes, using three sample values x_1, x_2, x_3. Show that the likelihood ratio test is equivalent to picking class,

$$\hat{Y}(x) = \begin{cases} 1 & \text{if } x_1 + x_2 + x_3 > \gamma \\ 0 & \text{otherwise} \end{cases}$$

for some scalar γ. Determine the value of γ so that the probability of False positive rate is 0.05. What is the corresponding True positive rate?

Terminology translation

1. Read Section 9.3 from Bertsekas (Intro to Probability) (ungraded)

Bertsekas book uses different terminology than what was covered in the class.

1. The two Hypothesis are equivalent to two classes. H_1 is the positive class ($Y = 1$) and H_0 is the negative class ($Y = 0$).
2. The Rejection region contains all the x for which the Likelihood ratio test predicts the class of x to be H_0.

$$R = \text{ all the } x \text{ such that } L(x) \leq \eta$$

where $L(x) = \frac{f_x(x|Y=1)}{f_x(x|Y=0)}$ is the likelihood ratio.

Example 9.13 (from Bertsekas Section 9.3) We observe two i.i.d. (independent identically distributed) random variables X_1 and X_2, with unit variance. Under negative class $Y = 0$ the random variables have common mean as 0; under positive class $Y = 1$ their common mean is 2. We fix the false positive rate to $\alpha = 0.05$. Show that the likelihood ratio test is equivalent to picking class,

$$\hat{Y}(x) = \begin{cases} 1 & \text{if } x_1 + x_2 > \gamma \\ 0 & \text{otherwise} \end{cases}$$

for some scalar γ. Determine the value of γ. What is the corresponding True positive rate?

Solution to 9.13

1. Definition 1: Gaussian PDF with mean μ and standard deviation σ is given by $f_X(X = x) = \frac{1}{\sigma\sqrt{2\pi}} \exp(-\frac{(x-\mu)^2}{2\sigma^2})$.

2. Theorem 1: Joint Gaussian PDF for two **independent** random variables X_1 and X_2 is the product of the two distributions,

$$f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1) f_{X_2}(x_2)$$
$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{\sigma_1\sqrt{2\pi}} \exp\left(-\frac{(x_1-\mu_1)^2}{2\sigma_1^2}\right) \frac{1}{\sigma_2\sqrt{2\pi}} \exp\left(-\frac{(x_2-\mu_2)^2}{2\sigma_2^2}\right)$$

Multiplication of two exponents causes addition in the radix $\exp(a)\exp(b) = \exp(a+b)$.

$$f_{X_1,X_2}(x_1,x_2) = \frac{1}{\sigma_1\sigma_2 2\pi} \exp\left(-\frac{(x_1-\mu_1)^2}{2\sigma_1^2} - \frac{(x_2-\mu_2)^2}{2\sigma_2^2}\right)$$

3. Theorem 2: Addition of two i.i.d Gaussian random variables X_1 and X_2 with same mean and variance μ_X and σ_X^2.

   The probability distribution of $Z = X_1 + X_2$ is a Gaussian (proof omitted here) with mean $\mu_Z = E[Z] = E[X_1 + X_2] = E[X_1] + E[X_2] = 2\mu_X$ and the standard deviation (use $\sigma_Y^2 = \text{Var}[Y] = E[Y^2] - E[Y]^2$)

$$\sigma_Z^2 = E[(X_1+X_2)^2] - \mu_Z^2 = E[X_1^2 + X_2^2 + 2X_1X_2] + \mu_Z^2 = E[X_1^2] + E[X_2^2] + 2E[X_1X_2] - \mu_Z^2$$

   Note that for independent random variables X_1 and X_2, $E[X_1X_2] = E[X_1]E[X_2] = \mu_X^2$. Also, $E[X_1^2] = \sigma_X^2 + \mu_X^2 = E[X_2^2]$. We get

$$\sigma_Z = \sqrt{2\sigma_X^2 + 2\mu_X^2 + 2\mu_X^2 - \mu_Z^2} = \sqrt{2\sigma_X^2 + 2\mu_X^2 + 2\mu_X^2 - 4\mu_X^2} = \sqrt{2}\sigma_X$$

   In general when we add n i.i.d. Gaussian random variables, the mean of the sum $Z_n = X_1 + \cdots + X_n$ gets multiplied by n ($\mu(Z_n) = n\mu_X$) and standard deviation gets multiplied by $\sqrt{n}$ ($\sigma(Z_n) = \sqrt{n}\sigma_X$).

4. Let the mean and standard deviation of i.i.d (independent and identically distributed) random variables X_1 X_2 under positive class $Y = 1$ be μ_1 and σ_1 respectively. Same be μ_0 and σ_0 under negative class $Y = 0$. The the Likelihood ratio is given by:

$$L(x) = \frac{f_{X_1,X_2}(x_1,x_2|Y=1)}{f_{X_1,X_2}(x_1,x_2|Y=0)}$$

   Under positive class $Y = 1$ the joint distribution is,

$$f_{X_1,X_2}(x_1,x_2|Y=1) = f_{X_1}(x_1|Y=1) f_{X_2}(x_2|Y=1) = \frac{1}{\sigma_1^2(2\pi)} \exp\left(-\frac{(x_1-\mu_1)^2}{2\sigma_1^2} - \frac{(x_2-\mu_1)^2}{2\sigma_1^2}\right)$$

   Under Hypothesis $Y = 0$ the joint distribution is,

$$f_{X_1,X_2}(x_1,x_2|Y=0) = f_{X_1}(x_1|Y=0) f_{X_2}(x_2|Y=0) = \frac{1}{\sigma_0^2(2\pi)} \exp\left(-\frac{(x_1-\mu_0)^2}{2\sigma_0^2} - \frac{(x_2-\mu_0)^2}{2\sigma_0^2}\right)$$

   The Likelihood ratio is (again using $\frac{\exp(a)}{\exp(b)} = \exp(a-b)$),

$$L(x) = \frac{f_{X_1,X_2}(x_1,x_2|Y=1)}{f_{X_1,X_2}(x_1,x_2|Y=0)} = \frac{\sigma_0^2}{\sigma_1^2} \exp\left(-\frac{(x_1-\mu_1)^2}{2\sigma_1^2} - \frac{(x_2-\mu_1)^2}{2\sigma_1^2} + \frac{(x_1-\mu_0)^2}{2\sigma_0^2} + \frac{(x_2-\mu_0)^2}{2\sigma_0^2}\right)$$

5. Likelihood ratio test is the comparison of the Likelihood ration $L(x)$ with some scalar η. We pick the positive class if $L(x) > \eta$. Equivalently, we pick positive class $\hat{Y}(x) = 1$ if,

$$L(x) = \frac{\sigma_0^2}{\sigma_1^2} \exp\left(-\frac{(x_1-\mu_1)^2}{2\sigma_1^2} + \frac{(x_1-\mu_0)^2}{2\sigma_0^2} - \frac{(x_2-\mu_1)^2}{2\sigma_1^2} + \frac{(x_2-\mu_0)^2}{2\sigma_0^2}\right) > \eta.$$

Since $\frac{\sigma_0^2}{\sigma^1}$ is positive, we can move this to other side of inequality,

$$\exp\left(-\frac{(x_1-\mu_1)^2}{2\sigma_1^2} - \frac{(x_2-\mu_1)^2}{2\sigma_1^2} + \frac{(x_1-\mu_0)^2}{2\sigma_0^2} + \frac{(x_2-\mu_0)^2}{2\sigma_0^2}\right) > \eta\frac{\sigma_1^2}{\sigma_0^2}.$$

Because $\log(z)$ is an increasing function, we can take log on both sides,

$$-\frac{(x_1-\mu_1)^2}{2\sigma_1^2} - \frac{(x_2-\mu_1)^2}{2\sigma_1^2} + \frac{(x_1-\mu_0)^2}{2\sigma_0^2} + \frac{(x_2-\mu_0)^2}{2\sigma_0^2} > \log(\eta\frac{\sigma_1^2}{\sigma_0^2})$$

6. We can substitute the given values for this Example 9.13 problem, where $\mu_0 = 0$, $\mu_1 = 2$, $\sigma_0 = \sigma_1 = 1$ (unit variance). We pick the positive class $\hat{Y}(x) = 1$ if,

$$-\frac{(x_1-2)^2}{2} - \frac{(x_2-2)^2}{2} + \frac{(x_1)^2}{2} + \frac{(x_2)^2}{2} > \log(\eta\frac{\sigma_1^2}{\sigma_0^2})$$

$$-(x_1^2 - 2x_1 + 4) - (x_2^2 - 2x_2 + 4) + (x_1)^2 + (x_2)^2 > 2\log(\eta\frac{\sigma_1^2}{\sigma_0^2})$$

$$+2x_1 - 4 + 2x_2 - 4 > 2\log(\eta\frac{\sigma_1^2}{\sigma_0^2})$$

$$+2x_1 - 4 + 2x_2 - 4 > 2\log(\eta\frac{\sigma_1^2}{\sigma_0^2})$$

$$x_1 + x_2 > \log(\eta\frac{\sigma_1^2}{\sigma_0^2}) + 4$$

Let $\gamma = \log(\eta\frac{\sigma_1^2}{\sigma_0^2}) + 4$.

In summary,

$$\hat{Y}(x) = \begin{cases} 1 & \text{if } L(x) > \eta \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 1 & \text{if } x_1 + x_2 > \gamma \\ 0 & \text{otherwise} \end{cases}$$

7. We are given the false positive rate:

$$P(\hat{Y}(x) = 1 | Y = 0) = 0.05$$

By liklihood ratio test, $\hat{Y}(x) = 1$ when $x_1 + x_2 > \gamma$.

$$P(X_1 + X_2 > \gamma | Y = 0) = 0.05$$

We swapped x_1 (and x_2) with X_1 (and X_2) to highlight that X_1 (and X_2) is a random variable. Use Theorem 2 to get $\mu_Z = 2\mu_X$ and $\sigma_Z = \sqrt{2}\sigma_X$. Normal tables allow us to look values for zero mean unit normal distributions.

$$P\left(\frac{X_1 + X_2 - \mu_Z}{\sigma_Z} > \frac{\gamma - \mu_Z}{\sigma_Z}\Big| Y = 0\right) = 0.05$$

$$1 - P\left(\frac{X_1 + X_2 - \mu_Z}{\sigma_Z} \le \frac{\gamma - \mu_Z}{\sigma_Z}\Big| Y = 0\right) = 0.05$$

$$\Phi\left(\frac{\gamma - \mu_Z}{\sigma_Z}\right) = 1 - 0.05 = 0.95$$

Looking at the normal tables, $\Phi(1.65) = 0.95$, hence $\frac{\gamma-\mu_Z}{\sigma_Z} = 1.65$ or $\gamma = 1.65\sigma_Z = 1.65\sqrt{2}\sigma_X = 1.65\sqrt{2}\sigma_0 = 2.331$.

In summary, when $\gamma = 2.331$, the false positive rate is 0.05.

8. To find True positive rate, use $\gamma = 2.331$

$$P(\hat{Y} = 1|Y = 1) = P(X_1 + X_2 > \gamma|Y = 1) = 1 - P\left(\frac{X_1 + X_2 - \mu_Z}{\sigma_Z} \le \frac{\gamma - \mu_Z}{\sigma_Z}|Y = 1\right)$$

When $Y = 1$, the $\mu_X = \mu_1 = 2$ and $\sigma_X = \sigma_1 = 1$. Hence $\mu_Z = 2\mu_X = 4$ and $\sigma_Z = \sqrt{2}\sigma_X = \sqrt{2}$. With these values, the True positive rate is,

$$P(\hat{Y} = 1|Y = 1) = 1 - \Phi\left(\frac{\gamma - \mu_Z}{\sigma_Z}\right).$$

or

$$P(\hat{Y} = 1|Y = 1) = 1 - \Phi\left(\frac{2.331 - 4}{\sqrt{2}}\right) = 1 - \Phi(-1.180) = 1 - (1 - \Phi(1.180)) = \Phi(1.180) = 0.964$$

In summary, when γ is chosen so that false positive rate is 0.05, then $\gamma = 2.331$ and true positive rate is 0.964.