Continuous Optimization (Chapter 7: MML Book)

Latex macros

Recall geometry of a derivative

Definition (Directional derivative)

Directional dervivative of a function $f(\mathbf{x}) : \mathbb{R}^n \to \mathbb{R}$ with respect to a given vector $\mathbf{u} \in \mathbb{R}$ is defined as

$$D_{\mathbf{u}} f(\mathbf{x}) = \lim_{\epsilon \to 0} \frac{f(\mathbf{x} + \epsilon \mathbf{u}) - f(\mathbf{x})}{\epsilon}$$

Ref Khan Academy

Ref Libretexts/12%3A_Functions_of_Several_Variables/12.06%3A_Directional_Derivatives

Vector calculus chain rule (a theorem)

Given a function composition $\mathbf{f}(\mathbf{x}) = \mathbf{g}(\mathbf{h}(\mathbf{x})) = (\mathbf{g} \circ \mathbf{h})(\mathbf{x})$ where $\mathbf{h} : \mathbb{R}^n \to \mathbb{R}^m$, $\mathbf{g} : \mathbb{R}^m \to \mathbb{R}^p$ and $\mathbf{h} : \mathbb{R}^m \to \mathbb{R}^p$

$$\frac{\p\mathbf{f}}{\p\mathbf{x}} = \frac{\p\mathbf{f}}{\p\mathbf{h}} \frac{\p\mathbf{h}}{\p\mathbf{x}}$$

or denoting the derivatives as Jacobian matrices we have,

$$\mathcal{J}_{\mathbf{x}} \mathbf{f} = \mathcal{J}_{\mathbf{h}}[\mathbf{f}] \mathcal{J}_{\mathbf{x}}[\mathbf{h}]$$

Theorem (Directional derivative is gradient dot product with the direction)

Express the trajectory in the direction $\mathbf{u}$ as a function of time t as

$$\mathbf{g}(t) = \mathbf{x} + t\mathbf{u}$$

Note that the Jacobian of $\mathbf{g}(t)$ wrt t is simply $\mathbf{u}$,

$$\mathcal{J}_t \mathbf{g}(t) = \mathbf{u}$$

Recall the definition of directional derivative,

$$D_{\mathbf{u}} f(\mathbf{x}) = \lim_{\epsilon \to 0} \frac{f(\mathbf{x} + \epsilon \mathbf{u}) - f(\mathbf{x})}{\epsilon}.$$

Compare it with the derivative of $f(\mathbf{g}(t))$ with respect to t at $t = 0$

$$\frac{\color{red}{\backslash\mathrm{p}} f(\mathbf{g}(t))}{\color{red}{\backslash\mathrm{p}} t} = \lim_{\epsilon \to 0} \frac{f(\mathbf{x} + (t + \epsilon)\mathbf{u}) - f(\mathbf{x} + t\mathbf{u})}{\epsilon}\bigg|_{t=0}.$$

$$\frac{\color{red}{\backslash\mathrm{p}} f(\mathbf{g}(t))}{\color{red}{\backslash\mathrm{p}} t} = \lim_{\epsilon \to 0} \frac{f(\mathbf{x} + \epsilon \mathbf{u}) - f(\mathbf{x})}{\epsilon} = D_{\mathbf{u}} f(\mathbf{x}).$$

We can compute $\frac{\color{red}{\backslash\mathrm{p}} f(\mathbf{g}(t))}{\color{red}{\backslash\mathrm{p}} t}$ by chain rule,

$$D_{\mathbf{u}} f(\mathbf{x}) = \mathcal{J}_t f(\mathbf{g}(t)) = \mathcal{J}_{\mathbf{x}} f(\mathbf{x}) \mathcal{J}_t \mathbf{g} = \nabla_{\mathbf{x}}^{\top} f(\mathbf{x}) \mathbf{u}$$

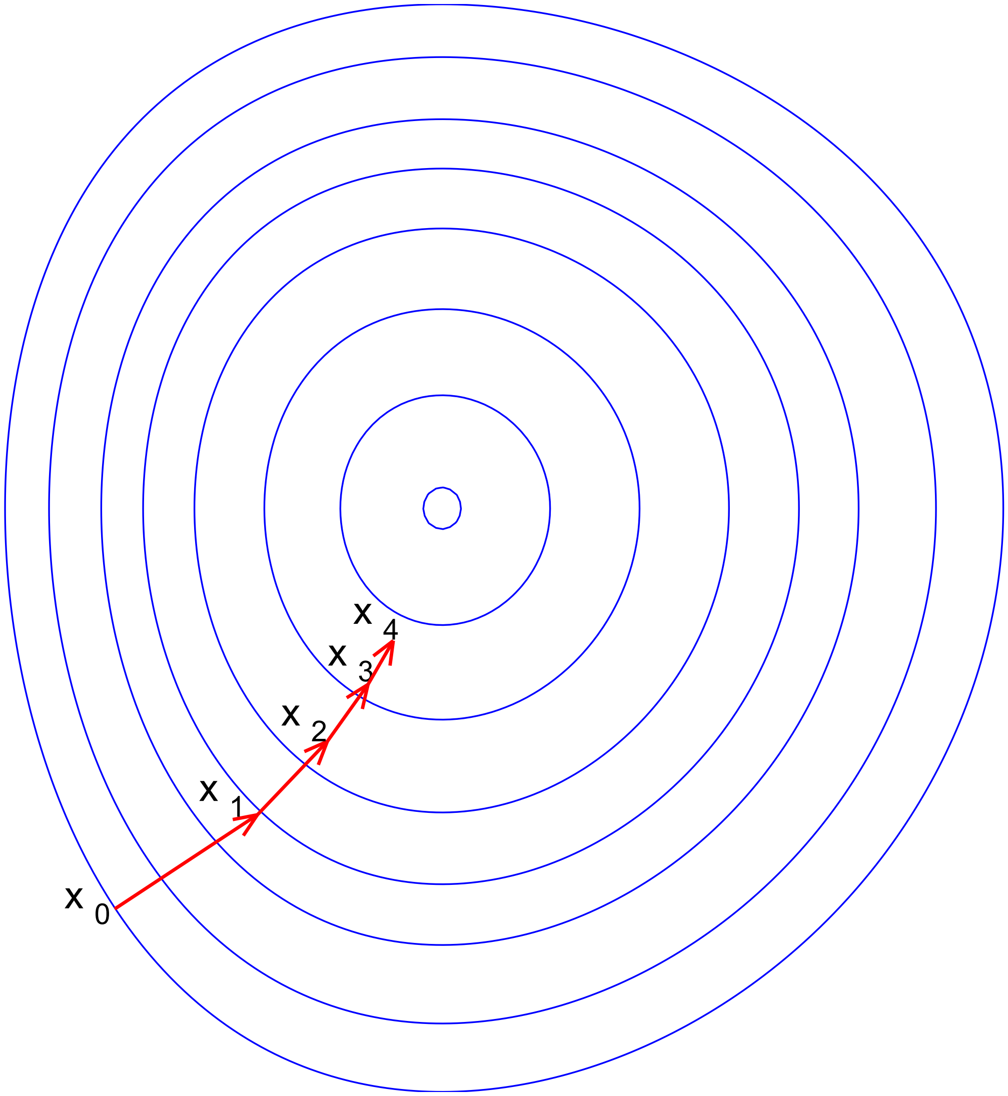
Theorem : The direction of stepest ascent and descent

Let $\hat{\mathbf{u}}$ be of unit magnitude. The directional derivative represents how the function changes in the direction $\hat{\mathbf{u}}$.

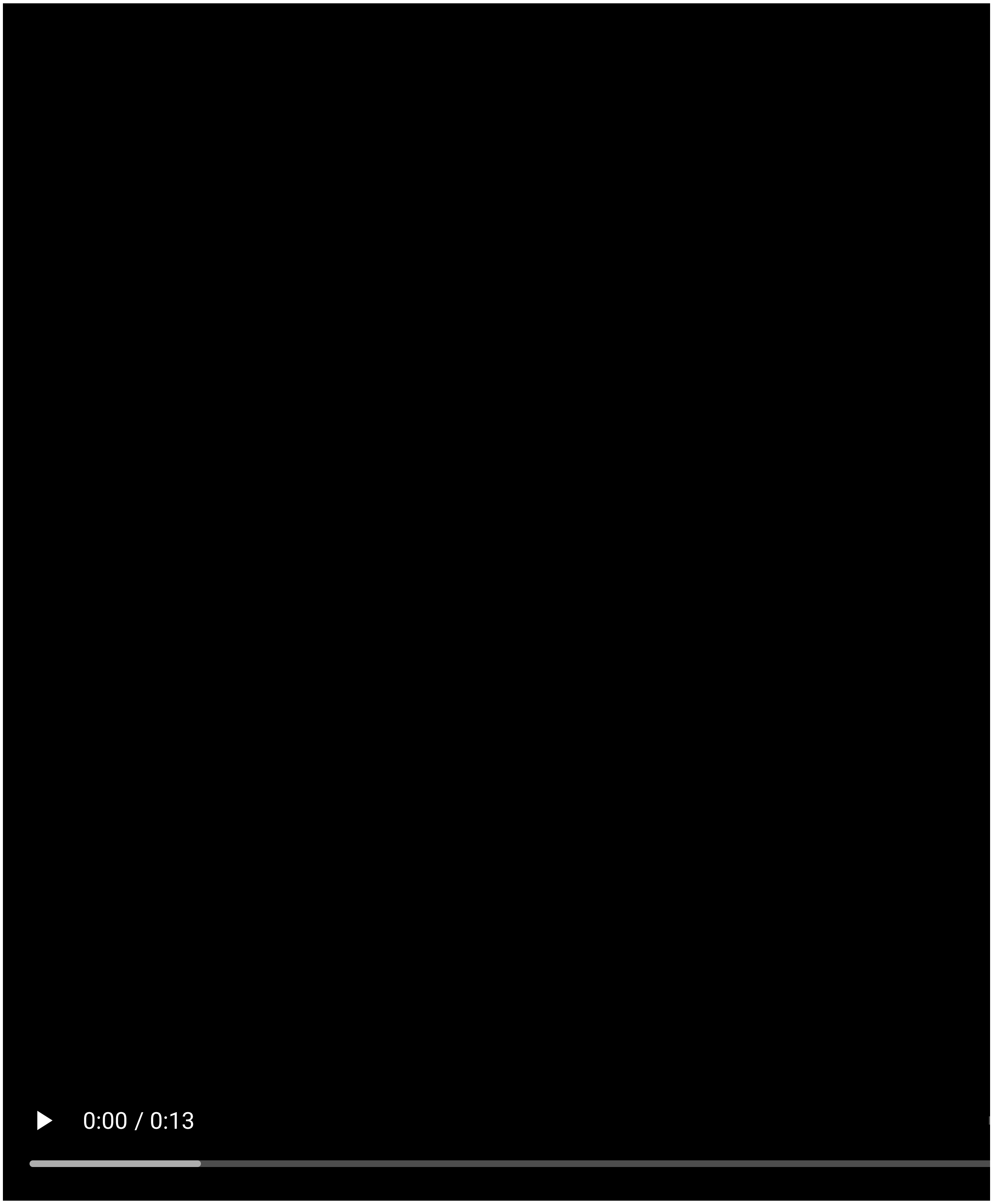
$$D_{\hat{\mathbf{u}}} f(\mathbf{x}) = \nabla_{\mathbf{x}}^{\top} f(\mathbf{x}) \hat{\mathbf{u}} = \|\nabla_{\mathbf{x}} f(\mathbf{x})\| \cos(\theta),$$

where θ is the angle between $\nabla_{\mathbf{x}} f(\mathbf{x})$ and $\hat{\mathbf{u}}$. The change is maximum when $\theta = 0$ and $\cos(\theta) = 1$ and the change is minimum when $\theta = 180^{\circ}$ and $\cos(\theta) = -1$.

In other words the function f increases the most (stepest ascent) when $\hat{\mathbf{u}} \propto \nabla_{\mathbf{x}} f(\mathbf{x})$ and decreases the most (steepest descent) when $\hat{\mathbf{u}} \propto -\nabla_{\mathbf{x}} f(\mathbf{x})$.



x 0
x 1
x 2
x 3
x 4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