SYDNEY GRAMMAR SCHOOL

CANDIDATE NUMBER

2020 **Trial HSC Examination**

Form VI Mathematics Extension 2

Wednesday 12th August 2020

General Instructions

- Reading time — 10 minutes
- Working time — 3 hours
- Attempt all questions.
- Write using black pen.
- Calculators approved by NESA may be used.
- A loose reference sheet is provided separate to this paper.

Total Marks: 100

Section I (10 marks) Questions 1–10

- This section is multiple-choice. Each question is worth 1 mark.
- Record your answers on the provided answer sheet.

Section II (90 marks) Questions 11–16

- Relevant mathematical reasoning and calculations are required.
- Start each question in a new booklet.

Collection

- If you use multiple booklets for a question, place them inside the first booklet for the question.
- Arrange your solutions in order.
- Write your candidate number on this page and on the multiple choice sheet.
- Place everything inside this question booklet.

Checklist

- Reference sheet
- Multiple-choice answer sheet
- 6 booklets per boy
- Candidature: 78 pupils

Writer: WJM

Section I

Questions in this section are multiple-choice.
Choose a single best answer for each question and record it on the provided answer sheet.

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1. Which of the following is the converse of $\sim P \Rightarrow Q$?

   (A) $\sim Q \Rightarrow P$

   (B) $Q \Rightarrow \sim P$

   (C) $P \Rightarrow Q$

   (D) $\sim P \Rightarrow \sim Q$

2. Which of the following is a primitive of $\tan^4 2x \sec^2 2x$?

   (A) $\tan^5 2x$

   (B) $\frac{1}{2}\tan^5 2x$

   (C) $\frac{1}{5}\tan^5 2x$

   (D) $\frac{1}{10}\tan^5 2x$

3. What is the smallest positive value of θ such that $e^{i\theta} \times e^{2i\theta} = i$?

   (A) $\frac{\pi}{12}$

   (B) $\frac{\pi}{6}$

   (C) $\frac{\pi}{3}$

   (D) $\frac{5\pi}{6}$

4. What is the approximate size of the angle between the vectors $\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$?

   (A) $57°$

   (B) $93°$

   (C) $123°$

   (D) $158°$

5. What are the zeros of the polynomial $P(x) = x^3 - 3x^2 + x + 5$?

   (A) $1,\ -2+i,\ -2-i$

   (B) $1,\ 2+i,\ 2-i$

   (C) $-1,\ -2+i,\ -2-i$

   (D) $-1,\ 2+i,\ 2-i$

6. Which expression is equivalent to $\displaystyle\int (\ln x)^2\, dx$?

   (A) $x\,(\ln x)^2 - 2\displaystyle\int \ln x\, dx$

   (B) $(\ln x)^2 - 2\displaystyle\int \ln x\, dx$

   (C) $x\,(\ln x)^2 - 2\displaystyle\int x\ln x\, dx$

   (D) $2\displaystyle\int \ln x\, dx$

7. The displacement x of a particle in metres after t seconds is given by $x = 2 + 4\sin^2 t$. How far will the particle travel in the first 2π seconds?

   (A) 0 metres

   (B) 2 metres

   (C) 8 metres

   (D) 16 metres

8. The polynomial $P(z)$ has real coefficients. The complex number α is of the form $a + ib$, where a and b are both real, non-zero and distinct.

   If $P(a)$, $P'(a)$, $P(b)$, $P'(b)$ and $P(\alpha)$ are all zero, what is the minimum degree of $P(z)$?

   (A) 4

   (B) 5

   (C) 6

   (D) 7

9. Without evaluating the integrals, which of the following integrals has the largest value?

   (A) $\displaystyle\int_{-\pi}^{\pi} x\cos x\,dx$

   (B) $\displaystyle\int_{-1}^{1} \ln(x^2+1)\,dx$

   (C) $\displaystyle\int_{0}^{1} (2^{-x}-1)\,dx$

   (D) $\displaystyle\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \left(\sin^{-1} x\right)^3\,dx$

10. A complex number z is defined such that $|z-1| = |z+2-i\sqrt{3}|$.

    What is the value of $\operatorname{Arg}(z)$ when $|z|$ is a minimum?

    (A) $\dfrac{\pi}{6}$

    (B) $\dfrac{\pi}{3}$

    (C) $\dfrac{2\pi}{3}$

    (D) $\dfrac{5\pi}{6}$

End of Section I

The paper continues in the next section

Section II

This section consists of long-answer questions.
Marks may be awarded for reasoning and calculations.
Marks may be lost for poor setting out or poor logic.
Start each question in a new booklet.

QUESTION ELEVEN (15 marks) Start a new answer booklet. **Marks**

(a) Express $\dfrac{1-8i}{2-i}$ in the form $a+ib$, where a and b are real. **2**

(b) Find:

(i) $\displaystyle\int x\cos x\ dx$ **2**

(ii) $\displaystyle\int \frac{dx}{x^2+4x+8}$ **2**

(c) Find any values of λ for which the vectors $\begin{bmatrix} -2 \\ \lambda \\ 2\lambda \end{bmatrix}$ and $\begin{bmatrix} 4 \\ \lambda \\ -1 \end{bmatrix}$ are perpendicular. **2**

(d) (i) Find the constants A, B and C such that **2**

$$\frac{5x^2-x+5}{(x^2+2)(x-1)} = \frac{Ax+B}{x^2+2} + \frac{C}{x-1}\,.$$

(ii) Hence find $\displaystyle\int \frac{5x^2-x+5}{(x^2+2)(x-1)}\,dx\,.$ **2**

(e) Three lines have equations: **3**

$$y = px + b_1$$
$$y = qx + b_2$$
$$y = rx + b_3$$

where p, q, r, b_1, b_2 and b_3 are real constants and p, q and r are distinct.

Use proof by contradiction to show algebraically that these lines cannot be perpendicular to one another.

QUESTION TWELVE (15 marks) Start a new answer booklet. **Marks**

(a) Sketch the region in the complex plane which simultaneously satisfies **3**

$$|z| < \sqrt{2} \text{ and } 0 \le \arg(z) \le \frac{\pi}{4}.$$

Clearly label the coordinates of any corners of the region, indicating if they are included in the region.

(b) Using the substitution $t = \tan\frac{x}{2}$, or otherwise, evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{2 + \sin x + 2\cos x}$. **3**

(c) Use proof by contraposition to show that for $x \in \mathbf{Z}$, if $x^2 - 6x + 5$ is even, then x is odd. **2**

(d) (i) By solving the equation $z^3 + 1 = 0$, find the three cube roots of -1. **2**

(ii) Let ω be a non-real cube root of -1. Show that $\omega^2 = \omega - 1$. **2**

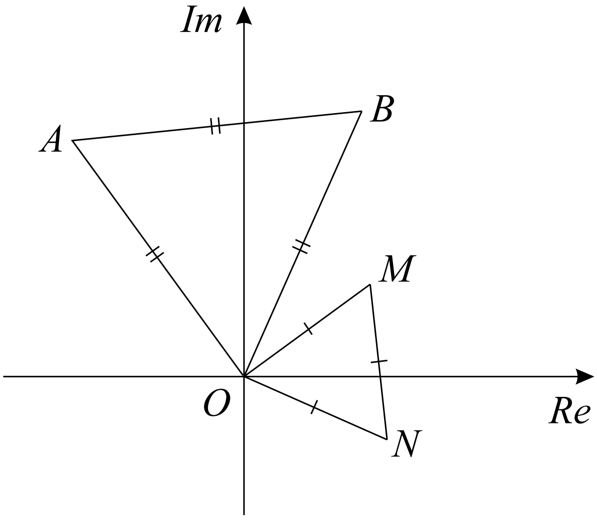
(iii) Hence simplify $(1 - \omega)^6$. **1**

(e) If x and y are positive real numbers, then $x + y \ge 2\sqrt{xy}$. (Do NOT prove this.) **2**

If a and b are positive real numbers, show that $(a + b)\left(\frac{1}{a} + \frac{1}{b}\right) \ge 4$.

QUESTION THIRTEEN (16 marks) Start a new answer booklet. **Marks**

(a)



The diagram above shows the points O, A, B, M and N on the complex plane. These points correspond to the complex numbers 0, a, b, m and n respectively. The triangles OAB and OMN are equilateral. Let $\alpha = e^{\frac{i\pi}{3}}$.

(i) Explain why $m = \alpha n$. $\boxed{1}$

(ii) Use complex numbers to show that $AM = BN$. $\boxed{2}$

(b) Two of the zeros of $P(x) = x^4 - 12x^3 + 54x^2 - 108x + 85$ are $a + ib$ and $2a + ib$, where a and b are real and $b > 0$.

(i) Find the values of a and b. $\boxed{3}$

(ii) Hence or otherwise express $P(x)$ as the product of quadratic factors with real coefficients. $\boxed{1}$

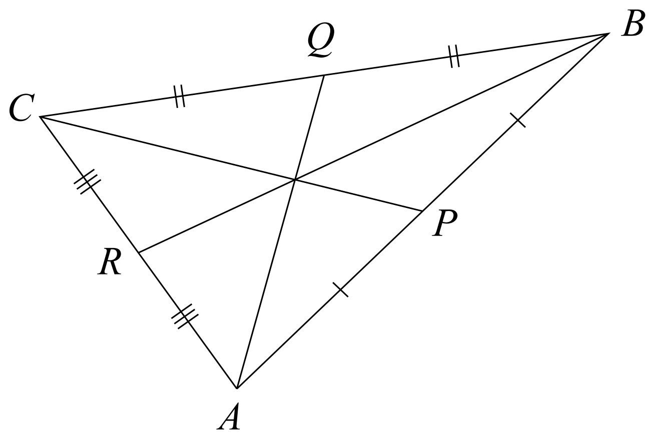
(c) Two lines are defined by $\underset{\sim}{v} = \begin{bmatrix} 2 \\ -1 \\ -5 \end{bmatrix} + \lambda \begin{bmatrix} 4 \\ -2 \\ -5 \end{bmatrix}$ and $\underset{\sim}{u} = \begin{bmatrix} 4 \\ -3 \\ 3 \end{bmatrix} + \mu \begin{bmatrix} -5 \\ 3 \\ 1 \end{bmatrix}$, where $\lambda, \mu \in \mathbf{R}$. $\boxed{3}$

Show that the two lines intersect at a single point.

Question Thirteen continues over the page

QUESTION THIRTEEN (Continued)

(d)



The diagram above shows $\triangle ABC$, where A, B and C have position vectors $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ respectively. The points P, Q and R bisect the intervals AB, BC and CA respectively.

(i) Show that $\overrightarrow{AQ} = \frac{1}{2}(\underset{\sim}{c} + \underset{\sim}{b}) - \underset{\sim}{a}$. $\boxed{1}$

(ii) Show that $\overrightarrow{AQ} + \overrightarrow{BR} + \overrightarrow{CP} = \underset{\sim}{0}$. $\boxed{2}$

(e) A sequence a_n is defined recursively by $a_n = a_{n-1} + 3n^2$, where $a_0 = 0$. Use mathematical induction to show that $a_n = \dfrac{n(n+1)(2n+1)}{2}$ for all integers $n \geq 0$. $\boxed{3}$

QUESTION FOURTEEN (14 marks) Start a new answer booklet. **Marks**

(a) The polynomial $P(x) = x^5 + px^4 + qx^3 + (2q-1)x^2 + 4px + r$, where p, q, $r \in \mathbf{R}$, has a zero of $x = -1$ with multiplicity 3.

(i) Find the values of p, q and r. $\boxed{3}$

(ii) Hence find the other zeros of $P(x)$. $\boxed{2}$

(b) Let $I_n = \displaystyle\int_0^{\frac{\pi}{2}} x^n \sin x \, dx$, for integers $n \geq 0$.

(i) Show that $I_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}$ for $n \geq 2$. $\boxed{2}$

(ii) Hence evaluate $\displaystyle\int_0^{\frac{\pi}{2}} x^2 \sin x \, dx$. $\boxed{2}$

(c) Let $z = e^{i\theta}$.

(i) Show that $z^n - \dfrac{1}{z^n} = 2i\sin(n\theta)$. $\boxed{1}$

(ii) Show that $\left(z - \dfrac{1}{z}\right)^5 = \left(z^5 - \dfrac{1}{z^5}\right) - 5\left(z^3 - \dfrac{1}{z^3}\right) + 10\left(z - \dfrac{1}{z}\right)$. $\boxed{1}$

(iii) Hence find $\displaystyle\int \sin^5\theta \, d\theta$. $\boxed{3}$

QUESTION FIFTEEN (16 marks) Start a new answer booklet. **Marks**

(a) Find $\displaystyle\int \frac{\sqrt{x}}{1+x}\,dx\,.$ $\boxed{2}$

(b) Use mathematical induction to show that for all **odd** integers $n \geq 1$, $4^n + 5^n + 6^n$ is divisible by 15. $\boxed{3}$

(c) A package with mass m kg is dropped from a stationary hovering helicopter. As the package falls vertically it experiences a force due to gravity of $10m$ Newtons. When a parachute on the package is deployed, it experiences a resistive force of magnitude mkv Newtons, where v is the velocity of the package in metres per second and k is a positive constant.

The vertical displacement of the package y metres from the position where the parachute is deployed satisfies

$$m\ddot{y} = 10m - mkv\,,$$

where the downwards direction is taken as positive.

(i) Let v_T be the terminal velocity of the package with the parachute deployed. Find v_T in terms of k. $\boxed{1}$

(ii) The parachute on the package is deployed when its velocity reaches $\dfrac{20}{k}\,\text{ms}^{-1}$.

(α) Show that $y = \dfrac{1}{k^2}\left(20 - kv + 10\ln\left|\dfrac{10}{10-kv}\right|\right).$ $\boxed{3}$

(β) In the time that it takes the package to fall 50 m after the parachute is deployed, its velocity decreases by 25%. Find the value of k, giving your answer correct to two decimal places. $\boxed{2}$

(d) Two lines $\underset{\sim}{r_1}$ and $\underset{\sim}{r_2}$ have equations

$$\underset{\sim}{r_1} = \begin{bmatrix} 0 \\ 5 \\ 4 \end{bmatrix} + \lambda \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} \quad \text{and} \quad \underset{\sim}{r_2} = \begin{bmatrix} -2 \\ 4 \\ 1 \end{bmatrix} + \mu \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}, \text{ where } \lambda,\ \mu \in \mathbf{R}.$$

The point A lies on the first line with parameter $\lambda = p$, and the point B lies on the second line with parameter $\mu = q$.

(i) Write $\overrightarrow{AB}$ as a column vector, writing the components in terms of p and q. $\boxed{1}$

(ii) Calculate the value of $\left|\overrightarrow{AB}\right|$ when $\overrightarrow{AB}$ is perpendicular to both $\underset{\sim}{r_1}$ and $\underset{\sim}{r_2}$. $\boxed{3}$

(iii) State the range of values that $\left|\overrightarrow{AB}\right|$ can take as p and q vary. $\boxed{1}$

QUESTION SIXTEEN (14 marks) Start a new answer booklet. **Marks**

(a) (i) The function $f(x)$ is continuous for all $x \in \mathbf{R}$. **2**

Use the substitution $x = \pi - u$ to show that

$$\int_0^\pi x f(\sin x)\, dx = \frac{\pi}{2} \int_0^\pi f(\sin x)\, dx\,.$$

(ii) Hence evaluate $\displaystyle\int_0^\pi (1+2x)\frac{\sin^3 x}{1+\cos^2 x}\, dx\,.$ **3**

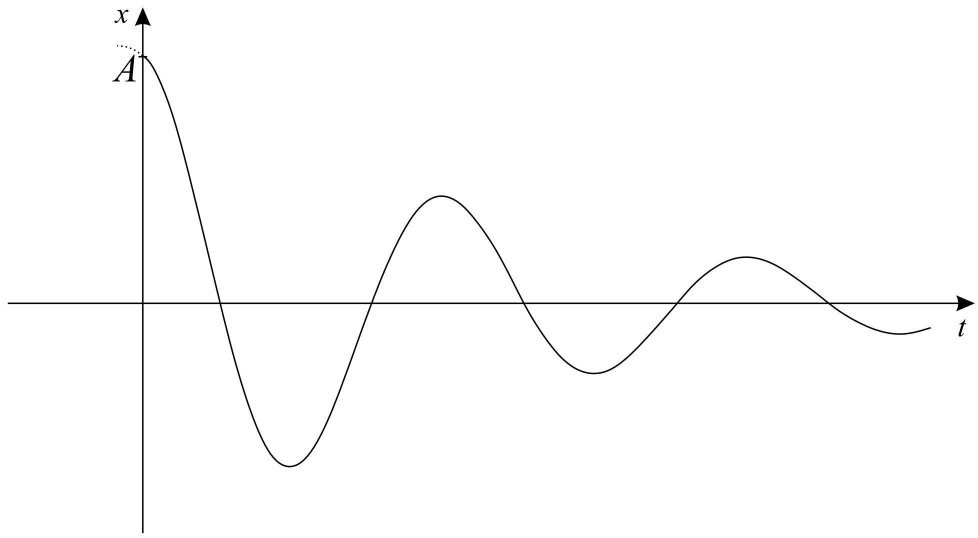
(b) An object of unit mass is attached to a spring. When the object is pulled and released, it experiences a force proportional to its displacement x metres, where $x = 0$ is taken as the centre of motion. The object moves in simple harmonic motion and the acceleration of the object is given by $\ddot{x} = -P^2 x$ for some constant $P > 0$.

When the spring and object are submerged in a liquid, the object also experiences a resistive force proportional to its velocity. Thus, the acceleration of the object is given by

$$\ddot{x} = -P^2 x - Q\dot{x} \qquad (*)$$

for some constant $Q > 0$.

The spring is stretched and the object is released. A timer is started once the object reaches $x = A$, where $A > 0$. That is, $x = A$ when $t = 0$. A graph of the displacement of the object submerged in liquid after t seconds is shown as follows:



The following questions relate to the motion of the object while it is submerged in liquid and $t \geq 0$.

(i) Show that $x = Ae^{-kt}\cos nt$ is a solution to the differential equation $(*)$ if $k = \frac{1}{2}Q$ and $n = \frac{1}{2}\sqrt{4P^2 - Q^2}$. You may assume that $4P^2 - Q^2 > 0$. **3**

(ii) Let x_r be the displacement of the object the rth time that it is instantaneously at rest. **2**

Show that $x_1 = -Ae^{\frac{k\alpha}{n}}\cos\alpha \times e^{-\frac{k\pi}{n}}$, where $\alpha = \tan^{-1}\left(\dfrac{k}{n}\right)$.

(iii) The value of the coefficient P relates to the stiffness of the spring, while the value of the coefficient Q relates to the viscosity of the liquid. **4**

Show that the total distance that the object will move while submerged in a liquid for $t \geq 0$ is dependent only on the value of the ratio $\dfrac{P}{Q}$.

END OF PAPER

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Sydney Grammar School

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$\boxed{2020}$ **Trial HSC Examination**

Form VI Mathematics Extension 2

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- Fill in the circle completely.
- Each question has only one correct answer.

Question One

A ◯ B ◯ C ◯ D ◯

Question Two

A ◯ B ◯ C ◯ D ◯

Question Three

A ◯ B ◯ C ◯ D ◯

Question Four

A ◯ B ◯ C ◯ D ◯

Question Five

A ◯ B ◯ C ◯ D ◯

Question Six

A ◯ B ◯ C ◯ D ◯

Question Seven

A ◯ B ◯ C ◯ D ◯

Question Eight

A ◯ B ◯ C ◯ D ◯

Question Nine

A ◯ B ◯ C ◯ D ◯

Question Ten

A ◯ B ◯ C ◯ D ◯

Maths Ext. 2 Trial - Solutions

MC

1. $Q \Rightarrow \sim P$ is the converse of $\sim P \Rightarrow Q$

$\therefore$ B ✓

2. $\int \tan^4 2x \sec^2 2x\,dx = \frac{1}{2}\int \tan^4 2x \times \frac{d}{dx}(\tan 2x)\,dx$

$= \frac{1}{10}\tan^5 2x + c$

$\therefore$ D ✓

3. $e^{i\theta} \times e^{2i\theta} = e^{3i\theta}$ $\qquad i = e^{i\frac{\pi}{2}}$

$\Rightarrow 3i\theta = i\frac{\pi}{2}$

$\theta = \frac{\pi}{6}$

$\therefore$ B ✓

4. Let $\underset{\sim}{a} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$, $\underset{\sim}{b} = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$.

$\underset{\sim}{a} \cdot \underset{\sim}{b} = 2 - 6 - 1$

$= -5$

$|\underset{\sim}{a}| = \sqrt{4+9+1}$

$= \sqrt{14}$

$|\underset{\sim}{b}| = \sqrt{1+4+1}$

$= \sqrt{6}$

$\cos\theta = \frac{\underset{\sim}{a} \cdot \underset{\sim}{b}}{|\underset{\sim}{a}| \times |\underset{\sim}{b}|}$

$= \frac{-5}{\sqrt{14} \times \sqrt{6}}$

$\theta \doteqdot 123°$

$\therefore$ C ✓

5. Sum of zeros must be 3, ruling out A & C

Product of zeros must be -5

$-1 \times (2+i)(2-i) = 4 - i^2$

$= -5$

$\therefore$ D ✓

(6) $\int (\ln x)^2 dx = \int (\ln x)^2 \times \frac{d}{dx}(x)\, dx$

$= x(\ln x)^2 - \int 2\ln x \times \frac{1}{x} \times x\, dx$

$= x(\ln x)^2 - 2\int \ln x\, dx$

$\therefore$ (A)

(7) $x = 2 + 4\sin^2 t$

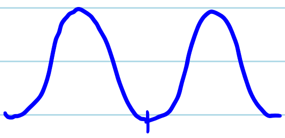
$= 2 + 4 \times \frac{1}{2}(1 - \cos 2t)$

$= 4 - 2\cos 2t$

Period $= \frac{2\pi}{2}$

$= \pi$

$\therefore$ In 2π seconds, particle travels 2 full cycles with amplitude 2 metres.



$2 \times 8 = 16$

$\therefore$ (D)

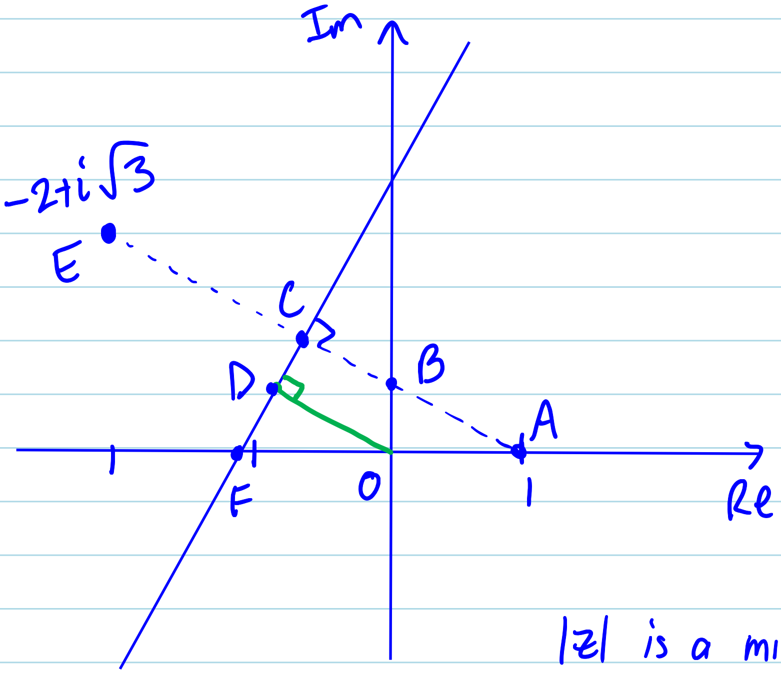
(8) $P(a) = P'(a) = 0$, so a is a double zero
$P(b) = P'(b) = 0$, so b is a double zero.
Since $P(z)$ has real coefficients and $P(\alpha) = 0$, $P(\bar{\alpha}) = 0$.
$\therefore$ α and $\bar{\alpha}$ are zeros

$\therefore$ minimum degree is 6 $\therefore$ (C)

(9) $x\cos x \rightarrow$ odd
$(\sin^{-1} x)^3 \rightarrow$ odd
$2^{-x} - 1 \leq 0$ for $0 \leq x \leq 4$
$\ln(x^2+1) \geq 0$ for $-1 \leq x \leq 1$

$\therefore$ (B)

(10) z lies on the perpendicular bisector between 1 and $-2+i\sqrt{3}$



$m_{AE} = -\frac{\sqrt{3}}{3}$

$= -\frac{1}{\sqrt{3}}$

$\therefore m_{CD} = \sqrt{3}$

$\Rightarrow \angle CFO = \frac{\pi}{3}$ and

$\angle OAB = \frac{\pi}{6}$

$|z|$ is a minimum when z can be represented by D, where $OD \perp FC$.

$\therefore OD \parallel AB$ and $\arg(z) = \frac{5\pi}{6}$ when $|z|$ is a minimum.

$\therefore$ (D)

11 (a) $\frac{1-8i}{2-i} \times \frac{2+i}{2+i} = \frac{2+i-16i+8}{4+1}$ ✓

$= 2-3i$ ✓

(b)(i) $\int x\cos x \, dx = \int x . \frac{d}{dx}(\sin x) dx$

$= x\sin x - \int \sin x \, dx$ ✓

$= x\sin x + \cos x + c$ ✓

(ii) $\int \frac{dx}{x^2+4x+8} = \int \frac{dx}{(x+2)^2+4}$ ✓

$= \frac{1}{2}\tan^{-1}\left(\frac{x+2}{2}\right) + c$ ✓

(c) $\begin{bmatrix} -2 \\ \lambda \\ 2\lambda \end{bmatrix} \cdot \begin{bmatrix} 4 \\ \lambda \\ -1 \end{bmatrix} = 0$ ✓

$\therefore -8 + \lambda^2 - 2\lambda = 0$

$\lambda^2 - 2\lambda - 8 = 0$

$(\lambda-4)(\lambda+2) = 0$

$\therefore \lambda = 4 \text{ or } -2$ ✓

(d) (i) $\frac{5x^2-x+5}{(x^2+2)(x-1)} = \frac{Ax+B}{x^2+2} + \frac{C}{x-1}$

$5x^2-x+5 = (Ax+B)(x-1) + C(x^2+2)$

Let $x=1$: $5-1+5 = 3C$

$C = 3$ ✓ (any 1 value)

Equate coef. of x^2: $5 = A+C$

$\therefore A = 2$

Let $x=0$: $5 = -B + 2C$

$B = 1$ ✓ (all values)

(ii) $\int \frac{5x^2-x+5}{(x^2+2)(x-1)} dx = \int \left(\frac{2x+1}{x^2+2} + \frac{3}{x-1}\right) dx$

$= \int \left(\frac{2x}{x^2+2} + \frac{1}{x^2+2} + \frac{3}{x-1}\right) dx$

✓ (any one correct integral)

$= \ln(x^2+2) + \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + 3\ln|x-1| + c$ ✓

11 (e) Assume that the lines $y = px + b_1$, $y = qx + b_2$, $y = rx + b_3$ are perpendicular.

Then
$$pq = -1 \quad (1)$$
$$pr = -1 \quad (2)$$
$$qr = -1 \quad (3)$$

(1) × (2): $p^2 qr = 1 \quad (4)$

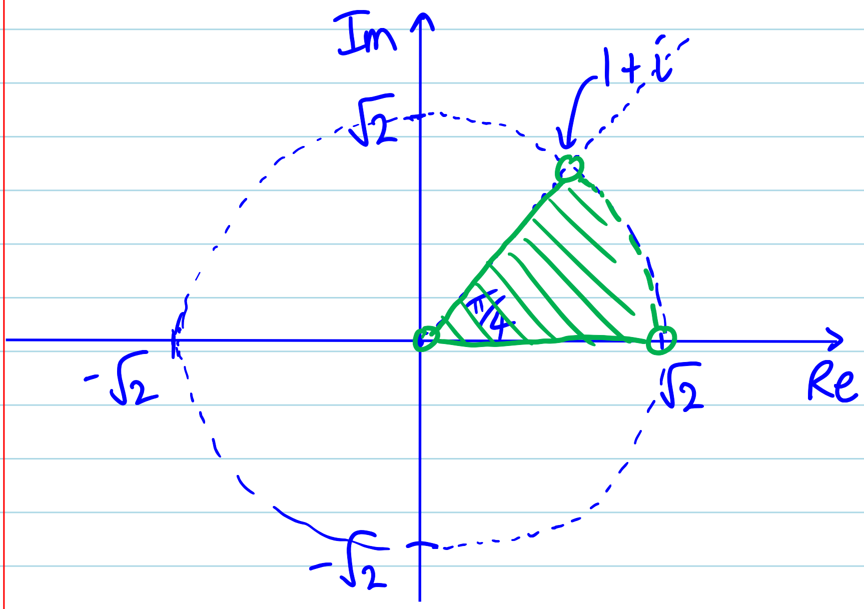
Sub. (3) into (4): $p^2 \times -1 = 1$

$p^2 = -1$

$\Rightarrow$ contradiction, since $p \in \mathbb{R}$

$\therefore$ 3 lines of the form $y = mx + b$ can't be perpendicular.

(12) (a) $|z| < \sqrt{2}$, $0 \le \arg(z) \le \frac{\pi}{4}$



✓ ✓] one mark for correctly identifying each region

✓ Correct interaction, including boundaries.

(b) $$\int_0^{\frac{\pi}{2}} \frac{dx}{2 + \sin x + 2\cos x}$$

Let $t = \tan \frac{x}{2}$

$$\frac{dt}{dx} = \frac{1}{2} \sec^2 \frac{x}{2}$$

$$= \frac{1}{2}\left(1 + \tan^2 \frac{x}{2}\right)$$

$$\therefore dx = \frac{2dt}{1+t^2}$$

When $x = 0$, $t = 0$

$x = \frac{\pi}{2}$, $t = 1$

$$= \int_0^1 \frac{\frac{2dt}{1+t^2}}{2 + \frac{2t}{1+t^2} + \frac{2(1-t^2)}{1+t^2}}$$ ✓

$$= \int_0^1 \frac{2dt}{2+2t^2+2t+2-2t^2}$$

$$= \int_0^1 \frac{2dt}{2t+4}$$

$$= \int_0^1 \frac{dt}{t+2}$$ ✓

$$= \Big[\ln(t+2)\Big]_0^1$$

$$= \ln 3 - \ln 2$$ ✓

(c) Proof by contraposition: If x is even, then x^2-6x+5 is odd.

Let $x=2n$, where $n \in \mathbb{Z}^+$ so that x is even.

correctly identifying contrapositive

$$\begin{aligned} x^2-6x+5 &= (2n)^2-6(2n)+5 \\ &= 4n^2-12n+5 \\ &= 2(2n^2-6n+2)+1 \\ &= 2M+1, \quad \text{where } M \text{ is an integer.} \end{aligned}$$

which is odd.

$\therefore$ by contraposition, if x^2-6x+5 is even, x is odd.

(d)(i) $z^3+1=0$

Let $z=\text{cis}\,\theta$

then by De Moivre's theorem:

$$\begin{aligned} \text{cis}\,3\theta &= -1 \\ &= \text{cis}\,\pi \end{aligned}$$

$$\therefore 3\theta = \pi, 3\pi, 5\pi$$

$$\theta = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$$

So the cube roots of -1 are $\text{cis}\frac{\pi}{3}$, $\text{cis}\,\pi=-1$, $\text{cis}\frac{5\pi}{3}$

(ii) $z^3+1=0$

$(z+1)(z^2-z+1)=0$

As -1 is a zero of $z+1$, if $w=\text{cis}\frac{\pi}{3}$ or $\text{cis}\frac{5\pi}{3}$,

$$\begin{aligned} w^2-w+1 &= 0 \\ w^2 &= w-1 \end{aligned}$$

(iii)
$$\begin{aligned} (1-w)^6 &= (-(w-1))^6 \\ &= (-w^2)^6 \\ &= w^{12} \\ &= (w^3)^4 \\ &= (-1)^4 \\ &= 1 \end{aligned}$$

(e) $x + y \geq 2\sqrt{xy}$

Let $x = a$, $y = b$: $a + b \geq 2\sqrt{ab}$ ①

Let $x = \frac{1}{a}$, $y = \frac{1}{b}$: $\frac{1}{a} + \frac{1}{b} \geq 2\sqrt{\frac{1}{a} \times \frac{1}{b}}$

$\frac{1}{a} + \frac{1}{b} \geq \frac{2}{\sqrt{ab}}$ ②

① × ②: $(a+b)\left(\frac{1}{a} + \frac{1}{b}\right) \geq 2\sqrt{ab} \times \frac{2}{\sqrt{ab}}$

≥ 4

13 (a)(i) As ΔOMN is equilateral, $\angle MON = \frac{\pi}{3}$ and $|m| = |n|$.

So m can be obtained by rotating n anti-clockwise about the origin by $\frac{\pi}{3}$.

i.e. $m = n \times \text{cis}\frac{\pi}{3}$
$= \alpha n$ ✓

must be specific about nature of rotation

(ii) $m = \alpha n$

Similarly, $a = \alpha b$.

$$AM = |a - m|$$ ✓
$$= |\alpha b - \alpha n|$$
$$= |\alpha| \times |b - n|$$
$$= 1 \times BN$$

$\therefore AM = BN$ ✓

(b) $P(x) = x^4 - 12x^3 + 54x^2 - 108x + 85$

(i) As the coefficients of $P(z)$ are real, $a - ib$ and $2a - ib$ are also zeros of $P(z)$.

Sum of zeros: $(a+ib) + (a-ib) + (2a+ib) + (2a-ib) = -\frac{-12}{1}$
$$6a = 12$$
$$a = 2$$ ✓

Product of zeros: $(a+ib)(a-ib)(2a+ib)(2a-ib) = \frac{85}{1}$ ✓
$$(a^2+b^2)(4a^2+b^2) = 85$$
$$4a^4 + 5a^2b^2 + b^4 = 85$$
$$b^4 + 20b^2 + 64 = 85$$
$$b^4 + 20b^2 - 21 = 0$$
$$(b^2+21)(b^2-1) = 0$$

Since b is real and $b > 0$, $b = 1$.

$\therefore a = 2,\ b = 1$ ✓

(ii) $P(x) = (x - (2+i))(x - (2-i))(x - (4+i))(x - (4-i))$
$= (x^2 - 4x + 5)(x^2 - 8x + 17)$ ✓

13 (c) $\underset{\sim}{v} = \begin{bmatrix} 2+4\lambda \\ -1-2\lambda \\ -5-5\lambda \end{bmatrix}$, $\underset{\sim}{u} = \begin{bmatrix} 4-5\mu \\ -3+3\mu \\ 3+\mu \end{bmatrix}$

Lines intersect if a solution to the system of equations exist:

$$2+4\lambda = 4-5\mu$$
$$4\lambda + 5\mu - 2 = 0 \quad (1)$$

$$-1-2\lambda = -3+3\mu$$
$$-2\lambda - 3\mu + 2 = 0 \quad (2)$$

$$-5-5\lambda = 3+\mu$$
$$5\lambda + \mu + 8 = 0 \quad (3)$$

(1) + 2×(2):
$$-\mu + 2 = 0$$
$$\mu = 2$$

sub. into (1):
$$4\lambda + 10 - 2 = 0$$
$$\lambda = -2$$

Check (3):
$$LHS = 5\times -2 + 2 + 8$$
$$= 0$$

As the solution to (1) and (2) satisfies (3), the lines intersect.

13 (d)(i)

$$\vec{AQ} = \vec{AB} + \vec{BQ}$$
$$= (\underset{\sim}{b} - \underset{\sim}{a}) + \tfrac{1}{2}(\underset{\sim}{c} - \underset{\sim}{b})$$
$$= \underset{\sim}{b} - \underset{\sim}{a} + \tfrac{1}{2}\underset{\sim}{c} - \tfrac{1}{2}\underset{\sim}{b}$$
$$= \tfrac{1}{2}(\underset{\sim}{b} + \underset{\sim}{c}) - \underset{\sim}{a}$$

(ii) Similarly,
$$\vec{BR} = \tfrac{1}{2}(\underset{\sim}{a} + \underset{\sim}{c}) - \underset{\sim}{b}$$
$$\vec{CP} = \tfrac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b}) - \underset{\sim}{c}$$

$$\vec{AQ} + \vec{BR} + \vec{CP} = \tfrac{1}{2}(\underset{\sim}{b} + \underset{\sim}{c}) - \underset{\sim}{a} + \tfrac{1}{2}(\underset{\sim}{a} + \underset{\sim}{c}) - \underset{\sim}{b} + \tfrac{1}{2}(\underset{\sim}{a} + \underset{\sim}{b}) - \underset{\sim}{c}$$
$$= \underset{\sim}{0}$$

(e) $a_n = a_{n-1} + 3n^2, \quad a_0 = 0.$ $\quad a_n = \dfrac{n(n+1)(2n+1)}{2}$

Prove true for $n=0$: $\quad a_0 = 0$ (given)

Using formula: $\quad a_0 = \dfrac{0(0+1)(2\times0+1)}{2}$
$$= 0$$

$\therefore$ true for $n = 0$

Assume true for some $n=k$, i.e. $a_k = \dfrac{k(k+1)(2k+1)}{2}$

Prove true for $n=k+1$:

RTP: $\quad a_{k+1} = \dfrac{(k+1)(k+2)(2(k+1)+1)}{2}$
$$= \dfrac{(k+1)(k+2)(2k+3)}{2}$$

$$\text{LHS} = a_{k+1}$$
$$= a_k + 3(k+1)^2 \quad \text{(from definition)}$$
$$= \dfrac{k(k+1)(2k+1)}{2} + 3(k+1)^2 \quad \text{(from assumption)}$$

13(e) contd.

$$= \frac{k(k+1)(2k+1) + 6(k+1)^2}{2}$$

$$= \frac{(k+1)[k(2k+1) + 6(k+1)]}{2}$$

$$= \frac{(k+1)(2k^2 + 7k + 6)}{2}$$

$$= \frac{(k+1)(k+2)(2k+3)}{2}$$ ✓

$$= RHS$$

$\therefore \quad a_n = \frac{n(n+1)(2n+1)}{2}$ for integers $n \geq 1$ by mathematical induction.

14 (a)(i) $P(x) = x^5 + px^4 + qx^3 + (2q-1)x^2 + 4px + r$

$$P'(x) = 5x^4 + 4px^3 + 3qx^2 + 2(2q-1)x + 4p$$
$$P''(x) = 20x^3 + 12px^2 + 6qx + 2(2q-1)$$

$P''(-1) = 0$:
$$-20 + 12p - 6q + 4q - 2 = 0$$
$$12p - 2q = 22$$
$$6p - q = 11 \quad (1)$$

$P'(-1) = 0$:
$$5 - \cancel{4p} + 3q - 2(2q-1) + \cancel{4p} = 0$$
$$5 + 3q - 4q + 2 = 0$$
$$q = 7$$

sub. into (1):
$$6p - 7 = 11$$
$$6p = 18$$
$$p = 3$$

$P(-1) = 0$:
$$-1 + p - q + (2q-1) - 4p + r = 0$$
$$-1 + 3 - 7 + 2 \times 7 - 1 - 4 \times 3 + r = 0$$
$$r = 4$$

(ii) The coefficients of $P(x)$ are real, so let the zeros be $-1, -1, -1, a+ib, a-ib$, $a, b \in \mathbb{R}$.

Sum of zeros:
$$-3 + 2a = -3$$
$$a = 0$$

Product of zeros:
$$(-1)^3 \times (a+ib)(a-ib) = -4$$
$$-(a^2+b^2) = -4$$
$$b^2 = 4 \quad (a=0)$$
$$b = \pm 2$$

$\therefore$ the other zeros are $2i$ and $-2i$

(14) (a) Alternate solution

$P(x) = x^5 + px^4 + qx^3 + (2q-1)x^2 + 4px + r$

Let the zeros be $\alpha, \beta, -1, -1, -1$.

Correctly obtaining 4 equations ✓

Product of zeros: $-\alpha\beta = -r$

$\alpha\beta = r$ (1)

Zeros four at a time:

$\alpha\beta + \alpha\beta + \alpha\beta - \alpha - \beta = 4p$

$3\alpha\beta - \alpha - \beta = 4p$ (2)

Zeros three at a time:

$-\alpha\beta - \alpha\beta - \alpha\beta + \alpha + \alpha + \alpha + \beta + \beta + \beta - 1 = -(2q-1)$

$-3\alpha\beta + 3\alpha + 3\beta - 1 = -(2q-1)$

$3\alpha\beta - 3\alpha - 3\beta + 1 = 2q - 1$ (3)

Zeros two at a time:

$\alpha\beta - \alpha - \alpha - \alpha - \beta - \beta - \beta + 1 + 1 + 1 = q$

$\alpha\beta - 3\alpha - 3\beta + 3 = q$ (4)

Sum of zeros:

$\alpha + \beta - 3 = -p$ (5)

(2) + 4×(5): $3\alpha\beta - \alpha - \beta + 4(\alpha + \beta - 3) = 0$

$\alpha\beta + \alpha + \beta - 4 = 0$ (6)

sub. (4) into (3): $3\alpha\beta - 3\alpha - 3\beta + 1 = 2(\alpha\beta - 3\alpha - 3\beta + 3) - 1$

$\alpha\beta + 3\alpha + 3\beta - 4 = 0$ (7)

(7) − (6): $2\alpha + 2\beta = 0$

$\therefore \alpha + \beta = 0$ (8) ✓

sub. into (6): $\alpha\beta - 4 = 0$

$\alpha\beta = 4$ (9)

Substituting (8) and (9) into (1), (4), and (5) gives:

$r = 4$

$q = 4 - 3 \times 0 + 3 = 7$ ✓

$p = 3 - 0 = 3$

(ii) From (8): $\beta = -\alpha$

sub. into (9): $-\alpha^2 = 4$ ✓

$\alpha^2 = -4$

$\alpha = \pm 2i$

$\therefore$ the other zeros are $2i$ and $-2i$ ✓

(14) (b)(i) $I_n = \int_0^{\frac{\pi}{2}} x^n \sin x \, dx$

$= \int_0^{\frac{\pi}{2}} x^n \cdot \frac{d}{dx}(-\cos x)\, dx$

$= \left[-x^n \cos x\right]_0^{\frac{\pi}{2}} + n\int_0^{\frac{\pi}{2}} x^{n-1}\cos x \, dx$ ✓

$= 0 + n\int_0^{\frac{\pi}{2}} x^{n-1} \times \frac{d}{dx}(\sin x)\, dx$

$= n\left(\left[x^{n-1}\sin x\right]_0^{\frac{\pi}{2}} - (n-1)\int_0^{\frac{\pi}{2}} x^{n-2}\sin x \, dx\right)$

$\therefore I_n = n\left(\left(\frac{\pi}{2}\right)^{n-1} - (n-1)I_{n-2}\right)$

$= n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)I_{n-2}$ ✓

(ii) $I_0 = \int_0^{\frac{\pi}{2}} \sin x \, dx$

$= \left[-\cos x\right]_0^{\frac{\pi}{2}}$

$= 0 - (-1)$

$= 1$ ✓

$\int_0^{\frac{\pi}{2}} x^2 \sin x \, dx = I_2$

$= 2\left(\frac{\pi}{2}\right)^{2-1} - 2 \times 1 \times 1$

$= \pi - 2$ ✓

(c)(i) $z^n - \frac{1}{z^n} = e^{in\theta} - \frac{1}{e^{in\theta}}$ (by De Moivre's theorem)

$= e^{in\theta} - e^{-in\theta}$

$= \cos(n\theta) + i\sin(n\theta) - (\cos(-n\theta) + i\sin(-n\theta))$

$= \cos(n\theta) + i\sin(n\theta) - \cos(n\theta) + i\sin(n\theta)$ as cosine even, sine odd.

$= 2i\sin(n\theta)$ ✓

(relatively lenient here)

14 (c) (ii) $\left(z-\frac{1}{z}\right)^5 = \binom{5}{0}z^5 + \binom{5}{1}z^4\left(-\frac{1}{z}\right)^1 + \binom{5}{2}z^3\left(-\frac{1}{z}\right)^2 + \binom{5}{3}z^2\left(-\frac{1}{z}\right)^3 + \binom{5}{4}z\left(-\frac{1}{z}\right)^4 + \binom{5}{5}\left(-\frac{1}{z}\right)^5$

$$= z^5 - 5z^3 + 10z - 10\times\frac{1}{z} + 5\times\frac{1}{z^3} - \frac{1}{z^5}$$

$$= \left(z^5 - \frac{1}{z^5}\right) - 5\left(z^3 - \frac{1}{z^3}\right) + 10\left(z - \frac{1}{z}\right) \checkmark$$

(iii) Substituting result from part (i) into identity in part (ii):

$$(2i\sin\theta)^5 = 2i\sin 5\theta - 5(2i\sin 3\theta) + 10(2i\sin\theta) \checkmark$$

$$32i\sin^5\theta = 2i\sin 5\theta - 10i\sin 3\theta + 20i\sin\theta$$

$$\therefore \sin^5\theta = \frac{1}{16}\left(\sin 5\theta - 5\sin 3\theta + 10\sin\theta\right) \checkmark$$

So $\int \sin^5\theta\, d\theta = \frac{1}{16}\left(-\frac{1}{5}\cos 5\theta + \frac{5}{3}\cos 3\theta - 10\cos\theta\right) + C \checkmark$

$$= -\frac{1}{80}\cos 5\theta + \frac{5}{48}\cos 3\theta - \frac{5}{8}\cos\theta + C$$

(15) (a) $\displaystyle\int \frac{\sqrt{x}\,dx}{1+x}$ Let $u=\sqrt{x}$

$u^2 = x$

$2u\,du = dx$

$$= \int \frac{u \times 2u\,du}{1+u^2}$$

$$= 2\int \frac{u^2\,du}{1+u^2}$$

$$= 2\int \left(\frac{u^2+1}{1+u^2} - \frac{1}{1+u^2}\right) du$$

$$= 2\int \left(1 - \frac{1}{1+u^2}\right) du$$

$$= 2u - 2\tan^{-1}(u) + c$$

$$= 2\sqrt{x} - 2\tan^{-1}(\sqrt{x}) + c$$

(b) * Rewrite problem as:

Show that $4^{2n+1} + 5^{2n+1} + 6^{2n+1}$ is a multiple of 15 for all $n \geq 0$.

Prove true for $n=0$:

$4^1 + 5^1 + 6^1 = 15$

$\therefore$ true when $n=0$.

Assume true for some $n=k$, where $k \geq 0$.

i.e. $4^{2k+1} + 5^{2k+1} + 6^{2k+1} = 15M$, where $M \in \mathbb{Z}$.

Prove true for $n=k+1$:

RTP: $4^{2k+3} + 5^{2k+3} + 6^{2k+3}$ is a multiple of 15.

$$4^{2k+3} + 5^{2k+3} + 6^{2k+3} = 4^2 \times 4^{2k+1} + 5^2 \times 5^{2k+1} + 6^2 \times 6^{2k+1}$$

$$= 16(15M - 5^{2k+1} - 6^{2k+1}) + 25 \times 5^{2k+1} + 36 \times 6^{2k+1}$$

(from assumption)

$$= 16 \times 15M + 9 \times 5^{2k+1} + 20 \times 6^{2k+1}$$

$$= 16 \times 15M + 45 \times 5^{2k} + 120 \times 6^{2k}$$

$$= 15(16M + 3 \times 5^{2k} + 8 \times 6^{2k})$$

$= 15N$, where $N \in \mathbb{Z}$ since $k \geq 0$

$\therefore$ $4^n + 5^n + 6^n$ is a multiple of 15 for all odd $n \geq 1$.

(15)(c)(i) $\ddot{y} = 0$ when travelling at terminal velocity.

$$\therefore \quad 10m - mkv_T = 0$$

$$v_T = \frac{10}{k} \text{ ms}^{-1}$$

(ii)(α)

$$m\ddot{y} = 10m - mkv$$

$$\ddot{y} = 10 - kv$$

$$v \cdot \frac{dv}{dy} = 10 - kv$$

$$\frac{dy}{dv} = \frac{v}{10 - kv}$$

$$= -\frac{1}{k} \times \frac{-kv + 10 - 10}{10 - kv}$$

$$\int -k\,dy = \int \left(1 - \frac{10}{10 - kv}\right) dv$$

(correctly rearranging as separable DE or equivalent)

$$-ky = \left(v + \frac{10}{k} \ln|10 - kv|\right) + c$$

When $y = 0$, $v = \frac{20}{k}$

$$\therefore \quad c = -\frac{20}{k} - \frac{10}{k} \ln\left|10 - k \times \frac{20}{k}\right|$$

$$= -\frac{20}{k} - \frac{10}{k} \ln 10$$

$$\therefore \quad -ky = v + \frac{10}{k} \ln|10 - kv| - \frac{20}{k} - \frac{10}{k} \ln 10$$

$$k^2 y = -kv - 10\ln|10 - kv| + 20 + 10\ln 10$$

$$= 20 - kv + 10\ln\left|\frac{10}{10 - kv}\right|$$

$$\therefore \quad y = \frac{1}{k^2}\left(20 - kv + 10\ln\left|\frac{10}{10 - kv}\right|\right)$$

(β) When $y = 50$, $v = \frac{3}{4} \times \frac{20}{k}$

$$= \frac{15}{k}$$

$$50 = \frac{1}{k^2}\left(20 - k \times \frac{15}{k} + 10\ln\left|\frac{10}{10 - k \times \frac{15}{k}}\right|\right)$$

$$50 = \frac{1}{k^2}\left(20 - 15 + 10\ln 2\right)$$

$$k^2 = \frac{5 + 10\ln 2}{50}, \quad \therefore \quad k = 0.49 \text{ (2 d.p.)}$$

(15)(d)(i)

$$\underset{\sim}{r_1} = \begin{bmatrix} -\lambda \\ 5+4\lambda \\ 4+3\lambda \end{bmatrix}, \quad \therefore \quad \vec{OA} = \begin{bmatrix} -p \\ 5+4p \\ 4+3p \end{bmatrix}$$

$$\underset{\sim}{r_2} = \begin{bmatrix} -2-\mu \\ 4+2\mu \\ 1+2\mu \end{bmatrix}, \quad \therefore \quad \vec{OB} = \begin{bmatrix} -2-q \\ 4+2q \\ 1+2q \end{bmatrix}$$

$$\vec{AB} = \vec{OB} - \vec{OA}$$
$$= \begin{bmatrix} -2-q+p \\ -1+2q-4p \\ -3+2q-3p \end{bmatrix} \quad ✓$$

(ii) $|\vec{AB}|$ is a minimum when $\vec{AB}$ is perpendicular to both lines.

$$\vec{AB} \cdot \begin{bmatrix} -1 \\ 4 \\ 3 \end{bmatrix} = 0: \quad 2+q-p-4+8q-16p-9+6q-9p = 0$$
$$-11+15q-26p = 0 \quad (1)$$

$$\vec{AB} \cdot \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix} = 0: \quad 2+q-p-2+4q-8p-6+4q-6p = 0$$
$$-6+9q-15p = 0$$
$$3q = 5p+2 \quad (2)$$

✓ ✓

sub. into (1): $-11+5(5p+2)-26p = 0$
$$-11+25p+10-26p = 0$$
$$p = -1$$

sub. into (2): $3q = -5+2$
$$= -3$$
$$q = -1$$

When $p=-1, q=-1$:
$$\vec{AB} = \begin{bmatrix} -2-(-1)-1 \\ -1+2(-1)-4(-1) \\ -3+2(-1)-3(-1) \end{bmatrix}$$
$$= \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$$

and $|\vec{AB}| = \sqrt{(-2)^2+1^2+(-2)^2}$
$$= \sqrt{9}$$
$$= 3 \text{ units} \quad ✓$$

(15) (d)(iii) Note that $|\overrightarrow{AB}|$ is a minimum when $\overrightarrow{AB}$ is perpendicular to both $\underset{\sim}{r}_1$ and $\underset{\sim}{r}_2$.

As such, $|\overrightarrow{AB}| \geq 3$ ✓ (with equality when $p=-1$ and $q=-1$)

16(a)(i)

$$\int_0^{\pi} x\, f(\sin x)\,dx$$

Let $x = \pi - u$

$dx = -du$

When $x = 0,\ u = \pi$

$x = \pi,\ u = 0$

$$= \int_{\pi}^{0} (\pi - u) f(\sin(\pi - u)) \times -1\, du$$

$$= \int_0^{\pi} (\pi - u) f(\sin u)\, du \quad \checkmark$$ since $\sin(\pi - u) = \sin u$

$$= \pi \int_0^{\pi} f(\sin u)\,du - \int_0^{\pi} u f(\sin u)\,du$$

$$= \pi \int_0^{\pi} f(\sin x)\,dx - \int_0^{\pi} x f(\sin x)\,dx$$ (dummy variables)

$$\therefore 2\int_0^{\pi} x f(\sin x)\,dx = \pi \int_0^{\pi} f(\sin x)\,dx$$

$$\int_0^{\pi} x f(\sin x)\,dx = \frac{\pi}{2}\int_0^{\pi} f(\sin x)\,dx \quad \checkmark$$

(ii)

$$\int_0^{\pi} (1+2x)\frac{\sin^3 x}{1+\cos^2 x}\,dx = \int_0^{\pi} \frac{\sin^3 x\,dx}{1+\cos^2 x} + 2\int_0^{\pi} x\,\frac{\sin^3 x\,dx}{1+\cos^2 x}$$

as $\dfrac{\sin^3 x}{1+\cos^2 x} = \dfrac{\sin^3 x}{1+(1-\sin^2 x)}$, using result from part (i): $\checkmark$

$$= \int_0^{\pi} \frac{\sin^3 x\,dx}{1+\cos^2 x} + 2\times\frac{\pi}{2}\int_0^{\pi} \frac{\sin^3 x\,dx}{1+\cos^2 x}$$

$$= (1+\pi)\int_0^{\pi} \frac{\sin x.(1-\cos^2 x)\,dx}{1+\cos^2 x}$$

Let $u = \cos x$

$du = -\sin x\,dx$

$x = 0,\ u = 1$

$x = \pi,\ u = -1$

$$= (1+\pi)\int_1^{-1} \frac{1-u^2}{1+u^2} \times -du \quad \checkmark$$

$$= (1+\pi)\int_{-1}^{1} \frac{1-u^2}{1+u^2}\,du$$

$$= (1+\pi)\int_{-1}^{1} \left(\frac{-(1+u^2)+2}{1+u^2}\right)du$$

$$= (1+\pi)\int_{-1}^{1} \left(-1 + \frac{2}{1+u^2}\right)du$$

$$= (1+\pi)\left[-u + 2\tan^{-1}(u)\right]_{-1}^{1}$$
$$= (1+\pi)\left(-1+\frac{\pi}{2}-\left(1-\frac{\pi}{2}\right)\right)$$
$$= (1+\pi)(\pi-2)$$

16 (b)(i)

$$x = Ae^{-kt}\cos nt$$
$$\dot{x} = -kAe^{-kt}\cos nt - nAe^{-kt}\sin nt$$
$$\ddot{x} = k^2Ae^{-kt}\cos nt + nkAe^{-kt}\sin nt + nkAe^{-kt}\sin nt - n^2Ae^{-kt}\cos nt$$
$$= (Ak^2 - An^2)e^{-kt}\cos nt + (2Ank)e^{-kt}\sin nt \quad ①$$

Also, $\ddot{x} = -p^2x - Q\dot{x}$
$$= -p^2Ae^{-kt}\cos nt - Q(-kAe^{-kt}\cos nt - nAe^{-kt}\sin nt)$$
$$= (-Ap^2 + AkQ)e^{-kt}\cos nt + AnQ.e^{-kt}\sin nt \quad ②$$

As the expressions in the RHS of ① and ② must be equivalent, equate coefficients of $e^{-kt}\cos nt$ and $e^{-kt}\sin nt$:

$$2\cancel{An}k = \cancel{An}Q$$
$$\therefore k = \tfrac{1}{2}Q$$

$$\cancel{A}k^2 - \cancel{A}n^2 = -\cancel{A}p^2 + \cancel{A}kQ$$
$$\left(\tfrac{1}{2}Q\right)^2 - n^2 = -p^2 + \tfrac{1}{2}Q^2$$
$$n^2 = p^2 - \tfrac{1}{4}Q^2$$
$$= \tfrac{1}{4}(4p^2 - Q^2)$$
$$\therefore n = \tfrac{1}{2}\sqrt{4p^2 - Q^2}$$

Also note that when $t=0$, $x = A \times e^0 \times \cos 0$
$= A$

So the initial displacement is also satisfied.

(16) contd.

(b)(ii) Particle is at rest when $\dot{x}=0$:

$$-Ake^{-kt}\cos nt - Ane^{-kt}\sin nt = 0$$
$$n\sin nt = -k\cos nt$$
$$\tan nt = -\frac{k}{n}$$ ✓

As $k, n > 0$, $-\frac{k}{n} < 0$. $\therefore$ for positive t:

$$nt = \pi - \tan^{-1}\left(\frac{k}{n}\right),\ 2\pi - \tan^{-1}\left(\frac{k}{n}\right),\ 3\pi - \tan^{-1}\left(\frac{k}{n}\right), \ldots$$

$$t = \frac{1}{n}(\pi-\alpha),\ \frac{1}{n}(2\pi-\alpha),\ \frac{1}{n}(3\pi-\alpha), \ldots, \text{ where } \alpha = \tan^{-1}\left(\frac{k}{n}\right)$$

Particle first comes to rest when $t = \frac{1}{n}(\pi-\alpha)$

$$\therefore\ x_1 = Ae^{-k\times\frac{1}{n}(\pi-\alpha)}\cos\left(n\times\frac{1}{n}(\pi-\alpha)\right)$$
$$= Ae^{-\frac{k\pi}{n}+\frac{k\alpha}{n}}\cos(\pi-\alpha)$$
$$= -Ae^{\frac{k\alpha}{n}}\cos\alpha \times e^{-\frac{k\pi}{n}}$$ ✓, since $\cos(\pi-\alpha) = -\cos\alpha$

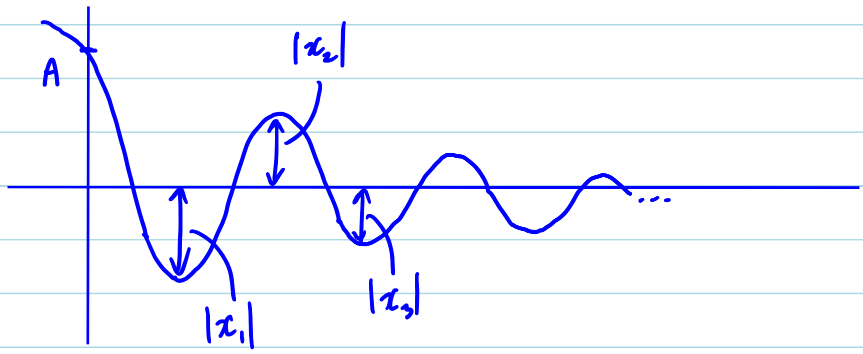
(iii) When $t = \frac{1}{n}(2\pi-\alpha)$:

$$x_2 = Ae^{-k\times\frac{1}{n}(2\pi-\alpha)}\cos\left(n\times\frac{1}{n}(2\pi-\alpha)\right)$$
$$= Ae^{\frac{k\alpha}{n}}\cos(2\pi-\alpha)\times e^{-\frac{2k\pi}{n}}$$ ✓ (some correct exploration of x_2)
$$= Ae^{\frac{k\alpha}{n}}\cos\alpha \times e^{-\frac{2k\pi}{n}}$$ since $\cos(2\pi-\alpha) = \cos\alpha$

Note that for successive values of t, $\cos nt$ will alternate between $\cos\alpha$ and $-\cos\alpha$. Each successive value of x_r can be found by multiplying the previous by $-e^{-\frac{k\pi}{n}}$.

$$\therefore\ |x_r| = |x_{r-1}| \times e^{-\frac{k\pi}{n}}$$, forming a G.P.

✓ (recognising geometric progression)



Let total distance travelled by D metres.

Then $D = A + 2|x_1| + 2|x_2| + 2|x_3| + \ldots$

$$= A + 2\left(|x_1| + |x_2| + |x_3| + \ldots\right)$$

$$= A + 2 \times \frac{A e^{\frac{k\alpha}{n}} \cos\alpha \, e^{-\frac{k\pi}{n}}}{1 - e^{-\frac{k\pi}{n}}}$$

$$= A\left(1 + \frac{2Ae^{\frac{k\alpha}{n}}\cos\alpha}{e^{\frac{k\pi}{n}} - 1}\right)$$

where $\alpha = \tan^{-1}\left(\frac{k}{n}\right)$

Note that for a given A, D is dependent on $\frac{k}{n}$.

Now $\frac{k}{n} = \frac{\frac{1}{2}Q}{\frac{1}{2}\sqrt{4P^2 - Q^2}}$

$$= \frac{1}{\sqrt{\frac{4P^2 - Q^2}{Q^2}}}$$

$$= \frac{1}{\sqrt{4\left(\frac{P}{Q}\right)^2 - 1}}$$

$\therefore$ For a given A, D can be expressed as a function of $\frac{k}{n}$, and $\frac{k}{n}$ can be expressed as a function of $\frac{P}{Q}$.

$\therefore$ The total distance travelled depends only on $\frac{P}{Q}$.