Some Aspects of the Flow of Stratified Fluids

I. A Theoretical Investigation

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Abstract

The following paper is the first of a series of two relating to the problem of internal oscillations of a fluid in a gravity field with vertical gradients of density and velocity. The theoretical analysis in this paper will be supplemented by a report on an experimental investigation along the same lines to be published in a future issue of this journal.

The exact, steady-state equations of motion and continuity of a perfect liquid moving two-dimensionally, with an arbitrary vertical distribution of density and velocity, are integrated once to yield a second-order differential equation. This equation is examined with regard to uniqueness and stability of the motion. A criterion is developed giving a sufficient condition for the motion to be uniquely determined by the configuration of the topography over which the fluid moves. It appears, further, that the condition of uniqueness is also a condition that a certain integrated quantity, called the kinetic potential of the motion, be a maximum. The suggestion is offered that this may correspond to a form of fluid instability.

A detailed study is made in the special cases of a uniform basic velocity, and a certain type of shearing flow. In either case, it is shown that an internal Froude number of about 1/3 divides the motion into two states, one of which is called supercritical, the other subcritical. From several viewpoints, these regimes are analogous to the corresponding states of flow of water in a channel. In the subcritical state the flow is in the form of standing wave patterns. When flowing supercritically, conditions may be favorable for the formation of internal "hydraulic jumps".

1. Introduction

From a meteorological viewpoint, considerable interest is attached to the flow of a fluid with a vertical gradient of density. The atmosphere itself is an example of such a fluid, and the existence of stable, unstable, and neutral internal gravity waves in a stratified medium of this kind must be of great meteorological importance. A striking example is the "Bishop Wave", a phenomenon now under investigation (COLSON, 1952).

TEPPER (1950) and FREEMAN (1948) have developed models of waves on an interface of two fluids of different density which appear to have application to atmospheric phenomena. In their investigations primary importance is attached to the instability of such waves with respect to the ultimate development of internal "hydraulic jumps". If the atmospheric analogy is correct, these correspond to "squall lines" or "pressure-jump lines". A model of a steady-state jump of this kind in a three-fluid system has been developed in the Hydrodynamics Laboratory of this university. Figure 1 is a photograph of this phenomenon. Further experimentation along these lines is now under way and the results will be published in a later issue of this journal.

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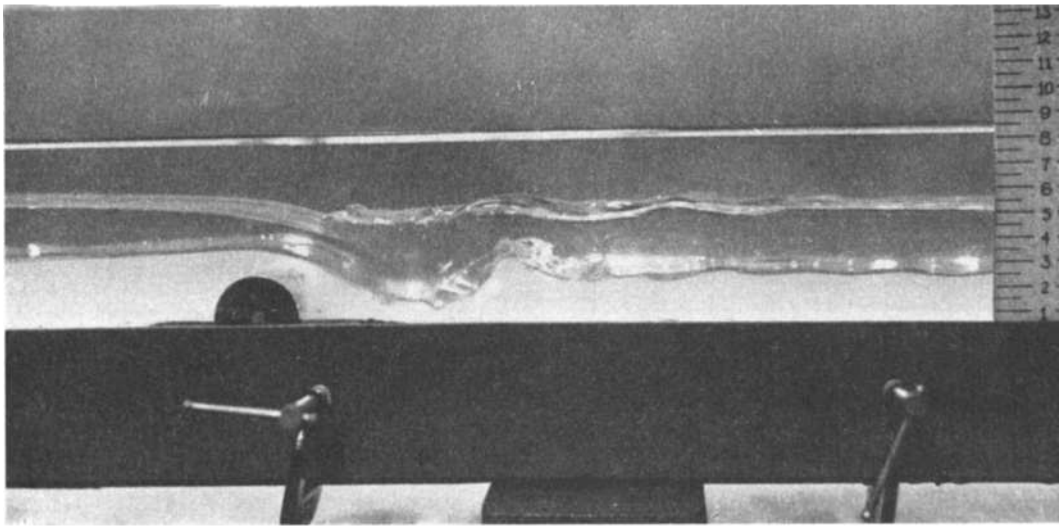

Fig. 1. Internal hydraulic jumps. One jump is located on the second interface just downstream of the obstacle. Another, larger one is on the first interface somewhat further downstream. The three liquids differ in specific gravity by 0.025, and the internal Froude number, using the overall density difference, is 0.19.

If a hydraulic jump be regarded as a manifestation of fluid instability, it is apparent that investigations that aim only to discover the existence and the nature of infinitesimal waves in a fluid will not in every case succeed in predicting stability and instability. For example, in the case of the flow of water in a channel, perturbation theory of the above type leads to the conclusion that no unstable waves exist. In reality, however, violent jumps may occur. The cause of the discrepancy is well known and is due to the tendency for *finite amplitude*, long waves to steepen indefinitely as they move along, and to develop eventually into waves of the shock type. In the case of a stratified fluid, a similar oversight results from a classical instability investigation. In the absence of shear, all small perturbations are neutral and the regime is considered on this basis to be stable. For this reason, while drawing freely on perturbation theory, an attempt will be made in this paper to give as much consideration as is possible to finite wave motions.

LYRA (1943) and QUENEY (1947) have studied stratified flows from the viewpoint of the stationary waves that mountain barriers might set up in a uniform current of stable fluid passing over them. Both of these authors considered the fluid as infinite in extent vertically. This is equivalent to the assumption that the wave lengths of the perturbations are small compared with the depth of the atmosphere. Since the waves set up by mountain barriers will have lengths of the order of the horizontal dimensions of the barrier, while the atmosphere may be considered to be about 10 km thick, this is not a very good assumption. On the other hand, Queney was able to incorporate the effect of the rotation of the earth in his study of a stratified fluid. This presents a considerable advantage for the application of the results to meteorological processes.

A paper by GÖRTLER (1943) considered this problem from a somewhat different viewpoint. In his work the linearized equations of motion were examined and the flows classified into elliptic, parabolic, and hyperbolic regimes, depending on the form of the resulting system of differential equations. Some interesting illustrations are given in his paper of an experiment using a stable salt solution in which the real characteristic lines of the hyperbolic case are generated by an oscillating source of disturbances.

More recently, ROSSBY (1951) and CRAYA (1951) have discussed the effect of stratification on the vertical concentration of momentum in a fluid current as a result of momentum losses at the underlying boundary. Rossby derived a criterion for a critical, internal Froude number above which the main effect is an increase in the depth of the current, but below which there is a strong tendency for the momentum to become concentrated in the upper levels.

The purpose of this paper is to present some results, primarily of a theoretical nature, pertaining to this general problem. In section 2 the exact, steady-state equations of motion of a current with an arbitrary initial vertical density and velocity gradients are integrated once. The resulting second-order vorticity equation is, of course, non-linear in general, but could probably be integrated numerically for any case of sufficient interest. The motion represented by the equation is examined quite generally in several ways. When thrown into a somewhat different form, a condition may be derived that the motion be uniquely determined by the form of the boundary. In two special cases uniqueness corresponds to instability, and a further investigation indicates that this may well be true in general. Sections 3 and 4 are devoted to a detailed study of a model with a density gradient and uniform basic velocity. Section 5 deals with a certain type of shearing flow.

2. General considerations

Figure 2 is a drawing of the theoretical model considered in this paper. The motion of the fluid is considered to be the same in all planes parallel to that of the printed page. The y-axis is directed upward and the x-axis is along the general direction of the current. The fluid, while not homogeneous, is incompressible. At great distances upstream the velocity is parallel to the plane bottom and is an arbitrary function $U(y)$. The density far upstream is given by $\varrho\ (y)$. Frictional forces are considered to be negligible except insofar as they may be required to make the problem determinate (see section 4).

It is to be noted that in the model the Coriolis forces vanish identically. This is a consequence of the assumption that the motion is strictly two-dimensional. If this assumption is dropped, one of the equations of motion contains a Coriolis term which bears a ratio to the inertial terms, of the order of w'/R_0u'.

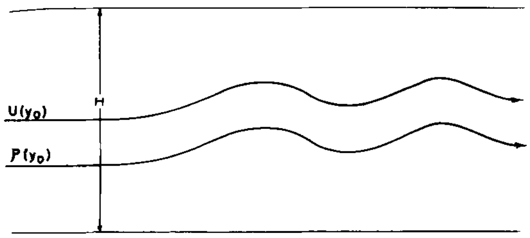

Fig. 2. Stratified fluid model. The velocity U and the density, ϱ, are arbitrary functions of the vertical coordinate alone, at a sufficiently great distance upstream. The upper and lower boundaries are rigid surfaces, although not necessarily plane surfaces.

R_0 is the Rossby number, U/fL (L is a measure of the scale of the motion), u' is a perturbed velocity component in the direction of the basic motion, w' is a perturbed velocity normal to the plane of fig. 2. If w'/R_0u' is small compared to 1, the neglect of the rotation terms is justifiable. As an example, assuming that w' is one-tenth of u', U is 10^3 cm. sec.$^{-1}$ and f is 10^{-4} sec.$^{-1}$, the scale of the motion, L, should be of the order of, or less than, 100 km. In general, however, the validity of applying the results of this paper to the atmosphere must be determined by an individual investigation along the above lines.

Assuming a steady state, the equations of motion are

$$\varrho \frac{\partial}{\partial x}\left(\frac{q^2}{2}\right) - \zeta \varrho v = -\frac{\partial p}{\partial x}, \tag{1}$$

$$\varrho \frac{\partial}{\partial y}\left(\frac{q^2}{2}\right) + \zeta \varrho u = -\frac{\partial p}{\partial y} - \varrho g, \tag{2}$$

where $q^2 = u^2 + v^2$, and ζ is the vorticity of the two-dimensional motion. Eliminating the pressure, we obtain

$$\frac{d\zeta}{dt} + \frac{1}{\varrho}\frac{\partial \varrho}{\partial x}\frac{\partial}{\partial y}\left(\frac{q^2}{2}\right) - \frac{1}{\varrho}\frac{\partial \varrho}{\partial y}\frac{\partial}{\partial x}\left(\frac{q^2}{2}\right) + \frac{g}{\varrho}\frac{\partial \varrho}{\partial x} = 0, \tag{3}$$

$$\frac{d}{dt} = u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}. \tag{4}$$

Since the motion is incompressible, there is a stream function $\psi\ (x, y)$ such that

$$u = -\frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \psi}{\partial x}. \tag{5}$$

The density, moreover, is conserved so that $\varrho = \varrho(\psi)$ The equation (3) then can be written

$$\frac{d\zeta}{dt} + \frac{1}{\varrho}\frac{d\varrho}{d\psi}\left(u\frac{\partial q^2/2}{\partial x} + v\frac{\partial q^2/2}{\partial y}\right) + \\ + \frac{g}{\varrho}\frac{d\varrho}{d\psi}\frac{dy}{dt} = 0. \tag{6}$$

Again, since ϱ and the derivative of ϱ with respect to ψ are not functions of time, we may write this as

$$\frac{d}{dt}\left[\zeta + \frac{1}{\varrho}\frac{d\varrho}{d\psi}\left(\frac{q^2}{2} + gy\right)\right] = 0. \tag{7}$$

As a consequence of the steady-state assumption, this may be integrated to give

$$\nabla^2 \psi + \frac{1}{\varrho}\frac{d\varrho}{d\psi}\left[\frac{(\nabla \psi)^2}{2} + gy\right] = H(\psi), \tag{8}$$

where $H(\psi)$ is a function of ψ to be determined by the conditions far upstream. Thus, if $y_0(\psi)$ is the height of the streamline ψ = const. and $\zeta(\psi)$ is the vorticity upstream, the final result is

$$\boxed{\nabla^2 \psi + \frac{1}{\varrho}\frac{d\varrho}{d\psi}\frac{(\nabla \psi)^2}{2} = \zeta(\psi) + \\ + \frac{1}{\varrho}\frac{d\varrho}{d\psi}\left[\frac{U^2}{2} + g(y_0 - y)\right].} \tag{9}$$

It should be noted that ζ, ϱ, U, and y_0 are all known functions of y_0 and hence of ψ. If proper consideration be given to the boundary conditions and other conditions which may be required to insure a unique solution, it seems quite feasible that (9) could be integrated numerically for any given basic distribution of density and velocity. It is to be noted that the assumption that the perturbed motion vanish at a sufficiently great distance upstream may be violated by some solutions satisfying (9). Such solutions will nevertheless satisfy the primitive equations of motion and continuity and are, therefore, dynamically possible.

The above equation assumes a familiar form if the fluid is homogeneous. Since ϱ is then a constant, (9) becomes

$$\nabla^2 \psi = \zeta(\psi). \tag{10}$$

This expresses the conservation of vorticity in the absence of solenoidal forces. It has been shown by LONG (1952) that an analogous equation, with the right-hand side a linear function of ψ, adequately describes the basic features of the perturbed motion of a spherical shell of rotating liquid. The waves in the rotating shell are "Rossby waves" and owe their existence to the special character of the basic absolute vorticity field. Similar wave motions may exist that satisfy (10) but they will be of a more general character, depending on the form of $\zeta(\psi)$. If the basic vorticity is zero or a constant, however, the perturbed motion will be irrotational and no wave motion can occur.

Equation (9) may be thrown into a different and useful form by substituting y_0 for the dependent variable ψ. If U does not vanish anywhere, the relation,

$$\frac{d}{d\psi} = -\frac{1}{U}\frac{d}{dy_0}, \tag{11}$$

leads to

$$\nabla^2 y_0 + \frac{1}{2}\left[(\nabla y_0)^2 - 1\right]\frac{d}{dy_0}(\ln U^2\varrho) = \\ = \frac{g}{U^2\varrho}\frac{d\varrho}{dy_0}(y_0 - y). \tag{12}$$

With the further substitution, $\delta = y_0 - y$,

$$\nabla^2 \delta + \frac{1}{2}\left[(\nabla \delta)^2 + 2\frac{\partial \delta}{\partial y}\right]\frac{d}{dy_0}(\ln U^2\varrho) = \\ = \frac{g}{U^2\varrho}\frac{d\varrho}{dy_0}\delta. \tag{13}$$

In this form we see one obvious model for which the equation is linear. This is one in which $U^2\varrho$ = const. and the density is linear in y_0. Equation (13) then reduces to

$$\nabla^2 \delta + \sigma^2 \delta = 0, \quad \sigma^2 = \frac{g\left|\frac{1}{\varrho}\frac{d\varrho}{dy_0}\right|}{U^2} = \text{const.} \tag{14}$$

This is the only case I have been able to discover for which the differential equation governing the motion of a stratified fluid is exactly linear.

Equation (12) may be used to derive a sufficient condition that a given solution for the motion in a channel be a unique steady solution. To show this, we replace our present model with a circular channel of very great radius in order to remove the necessity for defining the stream function at infinite distances along the x-axis. In the altered model the function $y_0(x, y)$ has cyclic continuity. With this understanding we may, however, neglect any centrifugal forces which arise, since the radius of curvature is made arbitrarily large.

Multiplying (12) by $U\varrho^{\frac{1}{2}}$ we obtain

$$\nabla^2 f = U\varrho^{\frac{1}{2}} \left[\frac{1}{2}\frac{d}{dy_0}(\ln U^2\varrho) + \right.$$

$$\left. + \frac{g}{U^2\varrho}\frac{d\varrho}{dy_0}(y_0 - y)\right] = G(f, y), \quad (15)$$

$$f(y_0) = \int_0^{y_0} U\varrho^{\frac{1}{2}}\, dy_0.$$

Assume now that there are two solutions y_{01}, y_{02} of (12) both of which are continuous and have continuous first and second derivatives and which take on the same values at the (perhaps irregular) bottom and top of the channel. Corresponding to these functions are $f(y_{01})$ and $f(y_{02})$, defined by the second equation in (15). They are obviously equal on the boundaries so that $f_0 = f(y_{02}) - f(y_{01})$ vanishes at the top and bottom of the channel. Substituting in (15) and subtracting, we obtain

$$\nabla^2 f_0 = \frac{\partial G}{\partial f} f_0, \quad (16)$$

where the partial derivative of $G(f, y)$ is with y held constant. According to the mean value theorem (COURANT, 1947), this derivative is evaluated at some value of f between $f(y_{01})$ and $f(y_{02})$. Multiplying (16) by f_0 and integrating over the whole region,

$$\iint \left[\nabla \cdot f_0 \nabla f_0 - (\nabla f_0)^2 - \frac{\partial G}{\partial f} f_0^2\right] dx\, dy = 0. \quad (17)$$

In view of the cyclic continuity and the disappearance of f_0 on the boundaries, this reduces to

$$\iint (\nabla f_0)^2 + \frac{\partial G}{\partial f} f_0^2\, dx\, dy = 0. \quad (18)$$

A sufficient condition that $f_0 \equiv 0$ or $f(y_{02}) \equiv f(y_{01})$ is that

$$\frac{\partial G}{\partial f} \geqslant 0. \quad (19)$$

If $f_0 \equiv 0$, however, using (15),

$$\int_{y_{01}}^{y_{02}} U\varrho^{\frac{1}{2}}\, dy_0 \equiv 0. \quad (20)$$

Since we have assumed that U does not vanish anywhere, the integrand is of one sign everywhere. Hence, $y_{02} \equiv y_{01}$ and the solution is unique. The inequality, (19) is, therefore, the sufficient condition that the motion in the channel be uniquely determined by the form of the boundaries. Expanding (19), the condition may be written

$$\frac{1}{U\varrho^{\frac{1}{2}}}\frac{d^2 U\varrho^{\frac{1}{2}}}{dy_0^2} + \frac{g}{U^2\varrho}\frac{d\varrho}{dy_0} +$$

$$+ \frac{g(y_0 - y)}{U\varrho^{\frac{1}{2}}}\frac{d}{dy_0}\left(\frac{1}{U\varrho^{\frac{1}{2}}}\frac{d\varrho}{dy_0}\right) \geqslant 0. \quad (21)$$

This can be simplified and made somewhat more intelligible if we assume that the disturbed motion is small. The last term in (21) may then be neglected and the condition becomes

$$\frac{1}{U\varrho}\frac{dU}{dy_0}\frac{d\varrho}{dy_0} + \frac{1}{U}\frac{d^2U}{dy_0^2} - \frac{1}{4\varrho^2}\left(\frac{d\varrho}{dy_0}\right)^2 +$$

$$+ \frac{1}{2\varrho}\frac{d^2\varrho}{dy_0^2} + \frac{g}{U^2\varrho}\frac{d\varrho}{dy_0} \geqslant 0. \quad (22)$$

The usefulness of this criterion is limited by the fact that it is not necessary and is, in fact, a weak condition in many cases. If, for example, we examine the model in which U is uniform and the density is $\varrho = \varrho_0 \exp(-\beta y_0)$ the inequality becomes $g\beta/U^2 \leqslant \beta^2/4$. In practical applications β^2 can be neglected and the uniqueness is assured only if β is zero or negative, i.e., if the density is constant or increases upward. It will be shown in the next section that a much stronger condition can be obtained in this case if the total depth of the channel is limited. Physically, the present condition is of interest, however. If the upper and lower boundaries of the channel are plane, one solution is the basic solution, $U = \text{const.}$ If the density increases upward, this is the only possible steady solution corresponding to the above inequality. Thus, any disturbance superimposed on the basic motion must progress, die out, or grow in amplitude. Since the stratification is definitely unstable, we know in advance that the last possibility is the one which will be realized.

A non-trivial condition is obtained if we assume the same density distribution as above but take as the velocity, $U = U_0 \exp a y_0$. A given solution is unique then if

$$\frac{g\beta}{U^2 a^2} \leqslant \left(1 - \frac{\beta}{2a}\right)^2. \tag{23}$$

Since β/a is small in cases of interest, and Ua is the velocity shear, this can be written

$$R_i \geqslant 1, \; R_i = \left(\frac{dU}{dy_0}\right)^2 \Big/ g\beta. \tag{24}$$

This expression is the Richardson number. According to RICHARDSON (1920) if this number exceeds one, the flow is unstable in the sense that turbulence will increase. We therefore have another case where uniqueness is associated with instability. In the turbulent flow examined by Richardson, the shear is that of the mean motion, of course.

It is possible to indicate, in general, that the criterion for uniqueness may also be a sufficient condition that a given steady motion in the channel is "secularly" unstable. This concept of stability was introduced by Thomson and Tait (see LAMB, 1932). The distinction between this and "ordinary" stability is that a given system may prove to be stable in the ordinary sense when considered frictionless but unstable in the secular or permanent sense when dissipative forces, however slight, are permitted. A simple, but beautiful, example of this is given by LAMB (1908). He investigated the stability of a small ball moving freely inside a hollow sphere rotating about a vertical axis in a uniform gravity field. If there is no friction whatever between the ball and the bowl, it is obvious that the only stable position is at the lower pole of the sphere, since the rotation has no significance in the problem. If there is any friction between the two, however, this position becomes unstable when a certain angular velocity is exceeded, and the ball will move to another latitude on the sphere. This second position is, then, secularly stable. The "practical stability" of the system in this example is assured by the condition that the "kinetic potential", $V - T_0$, be a minimum, where V is the potential energy and T_0 is the kinetic energy when the ball is at relative rest. Since

$$V - T_0 = -MgR\cos\Theta - \frac{1}{2} M\omega^2 R^2 \sin^2\Theta, \tag{25}$$

we have an extremum when $\sin\Theta = 0$ and when $\cos\Theta = g/\omega^2 R$. The condition that the first of these be a minimum is that $\omega < (g/R)^{\frac{1}{2}}$. If the angular velocity exceeds this value, LAMB showed that a ball starting initially at $\Theta = 0$ will, under the action of friction with the bowl, spiral outward from the pole until it reaches its secularly stable position of equilibrium. This action of the ball meets the ordinary specifications of instability in that the system departs permanently from its equilibrium when disturbed slightly.

Helmholtz (LAMB, 1932) applied this concept to a fluid system consisting of a homogeneous liquid in a channel with a free surface. He found that the condition for steady motion was that $V - T$ be stationary, where V is the integrated potential energy and T the kinetic energy. Furthermore, if the flow is sub-critical, i.e., if $U < (gH)^{\frac{1}{2}}$ this quantity is a minimum. If the flow is supercritical, $V - T$ is a maximum. A prior analysis indicates that if a small amount of friction is introduced, and it is assumed that it acts only to reduce the elevations of the free surface, then the maximum of $V - T$ means secular instability. This is somewhat surprising at first glance. Since the friction does not affect the flow directly, it is obvious that the choice of coordinate systems from which U is measured is immaterial. If we choose one that moves with the fluid, then the above result seems to show that a fluid basically at rest is unstable! The answer to this paradox seems to lie in the nature of the perturbation introduced in testing the maximum or minimum nature of $V-T$. If we introduced a free perturbation moving faster than $(gH)^{\frac{1}{2}}$ on a resting fluid, it would, of necessity, have a finite amplitude, since the maximum speed of an infinitesimal wave is $(gH)^{\frac{1}{2}}$. But it is well known that such a finite wave will develop into a surge (moving hydraulic jump) so that the resting fluid will be unstable in the sense considered in this paper. The perturbations considered in varying $V-T$, however, are those which are steady with respect to the coordinate system from which U is measured (although they need not be dynamically possible steady motions). If the value of $V-T$ is greater than in any disturbed state of this type, then the conclusion of Helmholtz is that the flow is unstable. In channel flow this instability is manifested in the form of hydraulic jumps.

Such a test of stability is a logical one from the viewpoint of the flows considered in this paper. We are interested primarily in the behavior of a stratified fluid passing over obstacles at the bottom. Since the upstream conditions are steady, the resulting theoretical perturbations will be steady. Whether or not they are stable will be tested by an approach similar to that used by Helmholtz.

The application of this principle to the model under consideration will be made in the following indirect manner: we first show that the dynamic differential equation (9) that must be satisfied by steady-state flow in the model can be derived by a variational method. To accomplish this, we seek the condition that the following integral be stationary:

$$I(\psi) = \iint \left[\varrho g(y-y_0) - \varrho \frac{U^2}{2} + \right.$$
$$\left. + \int \varrho g\, dy_0 - \varrho \frac{(\nabla \psi)^2}{2} \right] dx\,dy. \qquad (26)$$

In the variation the altered streamfunction $\psi(x, y) + \varepsilon(x, y)$ is assumed to be steady, and ε is to vanish at the top and bottom of the channel. Thus, both ψ and $\psi + \varepsilon$ satisfy the kinematic conditions at the boundaries. We again replace the straight channel with a circular one of arbitrarily large radius so that the assumed motions have cyclic continuity. There are a few points of delicacy in this variation problem that require a development from first principles. Thus, we write out the expression for the integral, $I(\psi + \varepsilon)$. This is accomplished by expanding ψ and functions of ψ in the integrand in a Taylor's series in powers of ε, neglecting terms of the order ε^3. For example,

$$U^2(\psi+\varepsilon) = U^2(\psi) + \frac{dU^2}{d\psi}\,\varepsilon + \frac{d^2U^2}{d\psi^2}\,\frac{\varepsilon^2}{2},$$
$$y_0(\psi+\varepsilon) = y_0(\psi) + \frac{dy_0}{d\psi}\,\varepsilon + \frac{d^2y_0}{d\psi^2}\,\frac{\varepsilon^2}{2},$$
$$= y_0(\psi) - \frac{\varepsilon}{U} + \frac{1}{U^2}\,\frac{dU}{d\psi}\,\frac{\varepsilon^2}{2}. \qquad (27)$$

To the order ε^2 we then obtain

$$I(\psi+\varepsilon) = I(\psi) + \iint \varepsilon \left\{ \varrho \nabla^2 \psi + \frac{d\varrho}{d\psi} \cdot \right.$$
$$\left. \cdot \left[\frac{(\nabla\psi)^2}{2} - \frac{U^2}{2} + g(y-y_0) \right] - \varrho U \frac{dU}{d\psi} \right\} dx\,dy$$
$$- \iint \varrho \frac{(\nabla\varepsilon)^2}{2} + \varepsilon^2 \left\{ \frac{g}{2}(y_0 - y)\frac{d^2\varrho}{d\psi^2} + \right.$$
$$+ \frac{1}{4}\frac{d^2\varrho}{d\psi^2}\left[U^2 - (\nabla\psi)^2\right] - \frac{1}{2}\frac{d\varrho}{d\psi}\nabla^2\psi +$$
$$+ \frac{\varrho}{2}\left(\frac{dU}{d\psi}\right)^2 + \frac{\varrho}{2}U\frac{d^2U}{d\psi^2} + U\frac{dU}{d\psi}\frac{d\varrho}{d\psi} -$$
$$\left. - \frac{g}{2U}\frac{d\varrho}{d\psi} \right\} dx\,dy. \qquad (28)$$

The condition that the integral be stationary is that the coefficient of ε vanish. This yields equation (9). *Hence the condition that the integral $I(\psi)$ be stationary is that the function ψ represent the stream function of a dynamically possible steady motion.* This is analogous to Helmholtz' result for channel flow. In his case the varied integral was the integrated difference between potential and kinetic energies, $V-T$. The integral $I(\psi)$ in (26) contains $-T$ in the last term in the integrand.

The remaining terms may be regarded as a generalized form of potential energy. This may be inferred from the Bernoulli equation for the stratified model, which is easily verified to be,

$$p + \varrho \frac{(\nabla \psi)^2}{2} + \varrho g (y - y_0) - \varrho \frac{U^2}{2} + \int \varrho g d y_0 = \text{const.} \tag{29}$$

For comparison, in steady, potential flow the equation is

$$p + \varrho \frac{(\nabla \psi)^2}{2} + V' = \text{const.}, \tag{30}$$

V' being the potential energy per unit volume. It is not apparent what physical interpretation is attached to the "potential energy" terms in the Bernoulli equation (29). Some insight is obtained from the above example of the ball moving in a rotating sphere. In that case the expression $V - T_0$ was varied, T_0 being the kinetic energy of the ball by virtue of the basic rotation. From the viewpoint of an observer in the sphere, however, $-T_0$ is nothing but the potential energy of the centrifugal "force" set up by the rotation. $V - T_0$, in this case, can very appropriately be regarded as a generalized potential energy.

The remaining question to be considered in the variation problem is the condition that the integral I be a maximum or minimum. Evidently it is a maximum if, in (28), the coefficient of ε vanishes and the coefficient of ε^2 is positive. A stronger condition is obtained, however, as follows: Since its integral over the region vanishes, we may add to the integrand of the second integral in (28) a term,

$$\nabla \cdot \frac{\varepsilon^2}{4} \frac{d\varrho}{d\psi} \nabla \psi = \frac{d\varrho}{d\psi} \frac{\varepsilon}{2} \nabla \varepsilon \cdot \nabla \psi + \frac{\varepsilon^2}{4} \frac{d^2\varrho}{d\psi^2} (\nabla \psi)^2 + \frac{\varepsilon^2}{4} \frac{d\varrho}{d\psi} \nabla^2 \psi. \tag{31}$$

Combining terms, (28) becomes:

$$I(\psi + \varepsilon) = I(\psi) - \frac{1}{2} \iint \varrho \left(\nabla \varepsilon + \frac{\varepsilon}{2 \varrho} \frac{d\varrho}{d\psi} \nabla \psi \right)^2 dx dy - \frac{1}{2} \iint \varepsilon^2 \left\{ -\frac{1}{2} \left[\nabla^2 \psi + \frac{1}{2 \varrho} \frac{d\varrho}{d\psi} (\nabla \psi)^2 \right] \cdot \frac{d\varrho}{d\psi} + \frac{d^2\varrho}{d\psi^2} \left[\frac{U^2}{2} + g (y_0 - y) \right] + \varrho \left(\frac{dU}{d\psi} \right)^2 + \varrho U \frac{d^2 U}{d\psi^2} + 2 U \frac{dU}{d\psi} \frac{d\varrho}{d\psi} - \frac{g}{U} \frac{d\varrho}{d\psi} \right\} dx dy. \tag{32}$$

The integral I will again be a maximum if the coefficient of ε^2 is positive. When the terms in the first square brackets in this coefficient are replaced by using equation (9) (or the condition that the coefficient of ε vanish), we obtain precisely the uniqueness condition (21).

I have not been able to demonstrate that the uniqueness condition implies instability in the secular sense, although it would seem from the above discussion to be very likely. It is to be hoped that further investigation will establish rigorously this suggested relationship.

3. General observations on the flow of liquid with uniform velocity distribution

One of the simplest cases included in (9) is one with a uniform velocity, U and a density distribution given by $\varrho = \varrho_0 \exp(-\beta y_0)$. We obtain

$$\nabla^2 \psi + \frac{\beta}{2 U} (\nabla \psi)^2 = \frac{\beta}{U} \left[\frac{U^2}{2} + g (y_0 - y) \right]. \tag{33}$$

If we introduce the perturbed stream function $\psi' = U y + \psi$, this becomes

$$\nabla^2 \psi' + \frac{\beta}{2 U} (\nabla \psi')^2 - \beta \frac{\partial \psi'}{\partial y} + \frac{\beta g}{U^2} \psi' = 0. \tag{34}$$

The middle terms in this expression are ordinarily exceedingly small. Thus, if the length scale of the phenomena is of the order of H, the ratio of the third to the last term is F^2, where F is the Froude number $U/(gH)^{\frac{1}{2}}$. In atmospheric motions, if H is taken as the height of the tropopause, F^2 is of the order of 10^{-2} to 10^{-3}. Furthermore, we may carry this term in the following development, since it is linear, but its effect on the motion is again found to be minor. The ratio of the second term to the last term is of the order of $F^2\delta/H$, where δ is the deviation of the stream line from its equilibrium height y_0. With a high degree of approximation, therefore, we may substitute for (34) the wave equation

$$\nabla^2\psi' + \alpha^2\psi' = 0,\ \alpha^2 = \frac{\beta g}{U^2}. \qquad (35)$$

Some important information on the type of motion represented by this equation can be obtained by assuming that the bottom of the channel is corrugated in the form of sinusoidal "bumps" of wave length $2\pi/k$ and amplitude γ. Writing

$$\psi' = \sum_{s=0}^{\infty} A_s \cos skx \sin\left[(\alpha^2 - s^2k^2)^{\frac{1}{2}}(y - H)\right] \qquad (36)$$

we have a general solution which vanishes at the top of the channel. The bottom boundary condition becomes

$$U\gamma \cos kx = \sum_{s=0}^{\infty} A_s \cos skx$$
$$\sin\left[(\alpha^2 - s^2k^2)^{\frac{1}{2}}(\gamma \cos kx - H)\right] \qquad (37)$$

If γ is small, we may neglect higher powers. Assuming this, we obtain

$$\psi' = -\frac{U\gamma}{\sin(\alpha^2 - k^2)^{\frac{1}{2}} H}\cos kx$$
$$\sin\left[(\alpha^2 - k^2)^{\frac{1}{2}}(y - H)\right], \qquad (38)$$

for a boundary form $\gamma \cos kx$. In this solution the oscillations have the same wave length as the bottom configuration, but the amplitude varies with height, and the waves will be inverted relative to the bottom waves after some vertical distance, provided $(\alpha^2 - k^2)H^2 > \pi^2$. Defining an internal Froude number, $F_i = U/(g\beta H^2)^{\frac{1}{2}}$, this inequality becomes $F_i^2 < (\pi^2 + k^2H^2)^{-1}$. If F_i exceeds π^{-1}, however, an inversion cannot occur, regardless of the wave length of the corrugations. This value, π^{-1}, of the internal Froude number would seem to be very significant. The analogous problem of the flow of a *homogeneous* fluid with a free surface over a corrugated bottom reveals that the criterion for the inversion of the free surface deformations over the bottom deformations is the value of the Froude number, $U/(gH)^{\frac{1}{2}} = 1$, which separates subcritical from supercritical flow (ROUSE, 1938).

In the case of open channel flow of a homogeneous fluid, a definition of critical flow more meaningful physically is the speed U, which coincides with the velocity of propagation of long gravity waves. If the stream velocity is less than this, long disturbances created by irregularities at the bottom will travel upstream until destroyed by frictional losses. If the stream velocity is greater than the critical speed, disturbances will be swept downstream, if of sufficiently small amplitude. Since the crests of finite long waves move faster than infinitesimal waves, the former will be swept downstream more slowly, absorbing the lower waves that come by. If, through this method of growth, a finite wave becomes high enough, it may reach a state in which it can maintain its position against the current. Since an elevation in such a wave moves more rapidly than a depression, there will eventually be a "breaking" and a "hydraulic jump" will then exist.

The importance of long waves in this discussion is that they are the waves which steepen as they progress and, therefore, eventually break, with the accompanying turbulence which characterizes jumps and surges. Short waves (deep-water waves), on the other hand, can move without change of shape when they have a finite amplitude.

When applied to stratified flows, these concepts become more complicated. The approximate speeds of infinitesimal waves in the model considered in this section are given by $c^2 = g\beta H^2/(n^2\pi^2 + k^2H^2)$. If we then define as a critical speed, the speed of long waves, we find an infinity of such speeds given by $g\beta H^2/n^2\pi^2$, where n is an integer greater than zero. The derivation of the wave formula shows that n is related to the number of nodal surfaces within the channel for the particular wave. The number of these surfaces is, in fact, $n - 1$. When $n = 1$ the waves extend from bottom to top, and the corresponding critical speed is given by $F_i = \pi^{-1}$. A flow with an internal Froude number slightly less than π^{-1} will be supercritical with respect to waves having one or more nodal surfaces between the bottom and top of the channel, and the analysis of water flow in a channel seems to indicate that internal jumps could result from the growth and breaking of these waves. The results of a very preliminary experiment using an obstacle in a three-fluid system support this tentative conclusion. A hydraulic jump at the second interface occurred just above the barrier for a Froude number of 0.14, which is well below the Froude number, $(1/6)^{\frac{1}{2}}$, corresponding to the fastest long waves which can exist in such a medium (see the discussion of a multi-fluid system at the end of this section). The experimental jump had a rather small amplitude and it seems likely that this would be true of all jumps that develop from the higher modes of oscillation ($n > 1$). The above suggestions have not been supported by rigorous theory or by careful experimentation, however, and should be regarded with reserve. It is to be hoped that the future experiments will shed light on these important questions.

It is of some interest to derive the upper critical Froude number by the use of another analogy to water flow in a channel. Thus, if an obstacle is inserted at the bottom of the channel, the resulting wave form at the free surface is indeterminate for subcritical flow, but entirely determined if the flow is supercritical. (The indeterminacy can be removed by appeal to viscosity). In the case of stratified flow, we may inquire into the conditions under which a solution is determined uniquely by specifying the bottom topography, $y = y_s$ (i.e. the values of ψ' on the boundary), and the surface $y = H$ as a streamline. As in the previous section, we assume that the fluid circulates in a circular channel of very great radius so that there is cyclic continuity. If we use the same arguments as those in deriving (21), then, assuming two solutions ψ_2' and ψ_1' of (35) whose difference is ψ_0, we find that

$$\iint \psi_0 \nabla^2 \psi_0 + \alpha^2 \psi_0^2 \, dx dy = 0. \tag{39}$$

The integral is taken over the entire region of flow. Integrating by parts,

$$-\iint (\nabla \psi_0)^2 - \alpha^2 \psi_0^2 \, dx dy = 0. \tag{40}$$

Now a quantity, $\nabla \cdot \psi_0^2 \nabla \eta \, (x, y)$, when integrated over the region, will vanish if η is sufficiently well-behaved in the region. Adding the integral of this expression to (40) we obtain[1]

$$\iint (\nabla \psi_0 - \psi_0 \nabla \eta)^2 - \psi_0^2 [(\nabla \eta)^2 + \\ + \nabla^2 \eta + \alpha^2] \, dx dy = 0. \tag{41}$$

The integrand of this expression will be everywhere non-negative (i.e. ψ_0 must vanish identically) if

$$(\nabla \eta)^2 + \nabla^2 \eta + \alpha_1^2 = 0, \tag{42}$$

where α_1^2 is slightly greater than or equal to α^2. Taking η as a function of y alone, the solution is

$$\eta = \ln [A \sin (\alpha_1 y + C)], \tag{43}$$

A and C being integration constants. Taking C as $-\pi$, then, provided that $\alpha_1 H < \pi$, the function η will be regular in the region and we must have a unique solution. This is equivalent to $F_i > \pi^{-1}$, the condition for supercritical flow derived above.

The work of Tepper and Freeman, mentioned in the first section of this paper, is concerned with the behavior of a system composed of a discrete number of layers of fluid of different density, rather than with continuous density distributions. It would seem that such a system is often a good approximation to the atmosphere and, in any case, is mathematically more convenient than any continuous distribution that could be fitted to the actual conditions. Furthermore, it appears that the most practical experimental model that could be set up to examine effects of stratification would be one with a given number of immiscible fluids of varying density layered in a channel. It is of interest, therefore, to investigate such discontinuous systems theoretically to obtain criteria for critical regimes of motion.

The following investigation was suggested by an approach due to R. R. Webb and reported by GREENHILL (1887). The critical speed will be taken as the speed of the fastest long waves which can exist in the stratified medium. In Greenhill's paper it is shown that

[1] This method of establishing the uniqueness of a solution is given in RIEMANN-WEBER (1925).

if a channel of $n+1$ equal layers of fluid is bounded above and below by rigid planes, the speeds c of long waves are given by the equation $|A| = 0$, where $|A|$ is the determinant:

$$\begin{vmatrix} A_1 & B_2 & 0 & 0 & \cdot & \cdot & \cdot \\ B_2 & A_2 & B_3 & 0 & \cdot & \cdot & \cdot \\ 0 & B_3 & A_3 & B_4 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & B_{n-1} & A_{n-1} & B_n \\ \cdot & \cdot & \cdot & \cdot & 0 & B_n & A_n \end{vmatrix} \tag{44}$$

and

$$A_s = \frac{c^2}{h}\,[2\,\varrho_1 + 2(s-1)\Delta] - g\Delta, \tag{45}$$

$$B_s = -\frac{c^2}{h}\,[\varrho_1 + (s-1)\Delta].$$

In the last equations h is the (constant) depth of each layer, ϱ_1 is the density of the top layer, and Δ is the (constant) difference of density from one layer to the next. The problem of solving the determinant is greatly simplified by assuming that Δ is very small (which it will be in any physical case of interest). If this assumption is made, it can be seen that the wave speeds will be given by an equation,

$$\frac{c^2}{h} = \frac{bg\Delta}{\varrho_1}, \tag{46}$$

where b is constant. Substituting back in the determinant the equation for b is

$$\begin{vmatrix} (2b-1) & -b & 0 & 0 & \cdot & \cdot \\ -b & (2b-1) & -b & 0 & \cdot & \cdot \\ 0 & -b & (2b-1) & -b & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = 0. \tag{47}$$

This determinant has n rows and columns. An equation of this type was solved by RAYLEIGH (1894). His method shows that

$$b = \frac{1}{4}\sec^2\frac{s\pi}{2(n+1)}, \quad s = 1, 2, \ldots n. \tag{48}$$

Of all these waves, the fastest are given by $s = n$. The critical speed becomes

$$c^2 = \frac{gh\Delta}{4\,\varrho_1}\sec^2\frac{n\pi}{2(n+1)}. \tag{49}$$

In order to compare this to the results of the calculation for a continuous density distribution we note that, if the density gradient in (22) is small, βH, in the continuous case, is close to $\Delta\varrho/\varrho_b$, where $\Delta\varrho$ is the overall density difference from top to bottom and ϱ_b is the density at the bottom. The critical speed is then given by

$$\frac{U^2}{gH\Delta\varrho/\varrho_b} = \frac{1}{\pi^2}. \tag{50}$$

To the same order, since $h = H/n+1$, and $\varrho_1 = \varrho_b - n\Delta$, (46) becomes

$$\frac{c^2}{gH\Delta\varrho/\varrho_b} = \frac{1}{4n(n+1)}\cdot$$

$$\cdot\sec^2\frac{n\pi}{2(n+1)} \xrightarrow[n\to\infty]{} \frac{1}{\pi^2}. \tag{51}$$

We thus obtain the same critical Froude number for this case as for the continuous density distribution. The critical numbers in the cases of two, three, and four layers are 0.50, 0.41, and 0.38 respectively.

4. Flow over an isolated barrier

In this section we will examine the flow of a stratified fluid with uniform velocity distribution over a small isolated barrier in the bottom of the cannel. Using equation (35), if γ is infinitesimal and the bottom deformations are given by $y_s = \gamma\exp(ikx)$, the resulting stream function is

$$\psi' = -\frac{U\gamma}{\sin(\alpha^2-k^2)^{\frac{1}{2}}H}\,e^{ikx}\sin\cdot$$

$$\cdot\,[(\alpha^2-k^2)^{\frac{1}{2}}(y-H)]. \tag{52}$$

The resulting linearity permits a build-up of functions satisfying the kinematic boundary conditions of an arbitrary bottom profile. Thus, if the bottom topography is

$$y_s = \int_{-\infty}^{\infty}\gamma(k)\,e^{ikx}\,dk, \tag{53}$$

the solution is

$$\psi' = -U \int_{-\infty}^{\infty} \gamma(k) e^{ikx} \frac{\sin[(\alpha^2 - k^2)^{\frac{1}{2}}(y-H)]}{\sin[(\alpha^2 - k^2)^{\frac{1}{2}} H]} dk. \tag{54}$$

By Fourier's Theorem,

$$\gamma(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} y_s(x') e^{-ikx'} dx', \tag{55}$$

so that

$$\psi' = -\frac{U}{2\pi} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} y_s(x') \cdot$$

$$\cdot \frac{\sin[(\alpha^2 - k^2)^{\frac{1}{2}}(y-H)]}{\sin[(\alpha^2 - k^2)^{\frac{1}{2}} H]} e^{ik(x-x')} dx'. \tag{56}$$

A simple, but interesting, boundary form may be taken as a low, rectangular obstacle with a height, a and width $2b$. Equation (56) then becomes

$$\psi' = -\frac{Ua}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{ik(x+b)}}{k} \cdot$$

$$\cdot \frac{\sin[(\alpha^2 - k^2)^{\frac{1}{2}}(y-H)]}{\sin[(\alpha^2 - k^2)^{\frac{1}{2}} H]} dk +$$

$$+ \frac{Ua}{2\pi i} \int_{-\infty}^{\infty} \frac{e^{ik(x-b)}}{k} \cdot$$

$$\cdot \frac{\sin[(\alpha^2 - k^2)^{\frac{1}{2}}(y-H)]}{\sin[(\alpha^2 - k^2)^{\frac{1}{2}} H]} dk = \psi_1' + \psi_2'. \tag{57}$$

The evaluation of these integrals is possible by contour integration in the complex plane, $\xi = k + iv$. For example,

$$I = \oint \frac{e^{i\xi(x+b)}}{\xi} \frac{\sin[(\alpha^2 - \xi^2)^{\frac{1}{2}}(y-H)]}{\sin[(\alpha^2 - \xi^2)^{\frac{1}{2}} H]} d\xi, \tag{58}$$

for any closed contour which includes the whole of the real axis, $v = 0$, includes the first integral in (57). The choice of the rest of the circuit depends on the sign of $x + b$. If this is positive, it proves advisable to adopt a contour consisting of a semi-circle, $|\xi| = R$, in

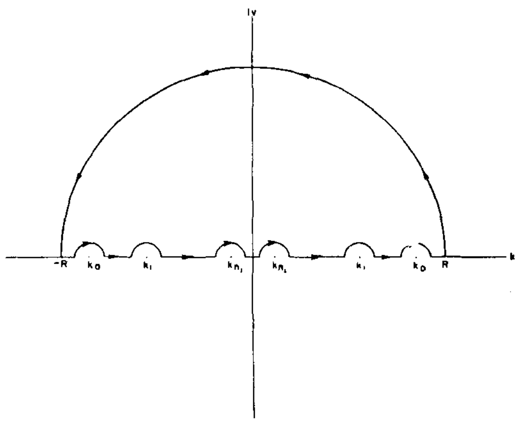

Fig. 3. Circuit in complex plane for evaluation of equation (59).

the upper half of the complex plane, plus the region of the real axis from $-R$ to R (see fig. 3). Then, letting $R \to \infty$, we would expect the integral along the real axis to converge to the first integral in (57). The success of the method depends on the behavior of (58) on the semi-circle. If $x + b > 0$, it may be shown that this part of the cyclic integral tends to zero as $R \to \infty$. For $x + b < 0$ we use a contour along the real axis and around a semi-circle in the lower half of the complex plane, ξ.

The above procedure is straightforward, but laborious, and we will indicate below the evaluation of ψ_1' in (57), for $x + b > 0$, and merely write down the results of the other integrations. Setting $\xi = Re^{i\varphi}$ on the semi-circle, (58) may then be written

$$I = i \int_0^{\pi} \exp[iR(x+b)e^{i\varphi}] \cdot$$

$$\cdot \frac{\sin[(\alpha^2 - R^2 e^{2i\varphi})^{\frac{1}{2}}(y-H)]}{\sin[(\alpha^2 - R^2 e^{2i\varphi})^{\frac{1}{2}} H]} d\varphi$$

$$+ \int_{-R}^{R} \frac{e^{ik(x+b)}}{k} \cdot$$

$$\cdot \frac{\sin[(\alpha^2 - k^2)^{\frac{1}{2}}(y-H)]}{\sin[(\alpha^2 - k^2)^{\frac{1}{2}} H]} dk = I_1 + I_2. \tag{59}$$

The integrand of I_2 is singular at a finite number of points,

$$k = 0,\ k_n = \pm\left(\alpha^2 - \frac{n^2\pi^2}{H^2}\right)^{\frac{1}{2}},\ n = 0, 1, 2, \ldots n_1, \tag{60}$$

where n_1 is the largest positive integer for which k_n is real. We enclose these points in small semi-circles of radius r and use the resulting circuit, shown in fig. 3, to evaluate I.

The value of the last integral in (59) may then be taken as its principal part, or the limit, as $r \to 0$, of the integrals along the parts of the real axis *excluded* by the semi-circles. Therefore,

$$-\frac{2\pi i}{Ua}\psi_1'(x + b > 0) =$$

$$= \lim_{R\to\infty} PI_2 = \lim_{R\to\infty} I - \lim_{R\to\infty} I_1$$

$$+ \sum \int_0^\pi \frac{e^{i\xi_n(x+b)}}{\xi_n} \frac{\sin\left[(\alpha^2 - \xi_n^2)^{\frac{1}{2}}(y - H)\right]}{\sin\left[(\alpha^2 - \xi_n^2)^{\frac{1}{2}} H\right]} d\xi_n. \tag{61}$$

In (61) the symbol, P, means "the principal part of", the summation is over all semi-circles on the real axis shown in fig. 3, and $\xi_n = k_n + re^{i\varphi}$.

The cyclic integral I over the modified circuit has a value equal to the sum of the residues at its poles. These poles are infinite in number and are given by

$$\xi = i\left(\frac{n^2\pi^2}{H^2} - \alpha^2\right)^{\frac{1}{2}},\ n = n_1 + 1, n_1 + 2, \ldots \tag{62}$$

The result is

$$I = i\sum_{n=n_1+1}^{\infty}(-1)^n \frac{2n\pi^2}{n^2\pi^2 - \alpha^2 H^2}\cdot$$

$$\cdot\exp\left[-\left(\frac{n^2\pi^2}{H^2} - \alpha^2\right)^{\frac{1}{2}}(x + b)\right]\sin\frac{n\pi}{H}(y - H). \tag{63}$$

As remarked above, for $x + b > 0$, $\lim I_1 = 0$ as R increases indefinitely. The integrals over the small circles in (61) yield the following:

$$i\pi\frac{\sin\alpha(y - H)}{\sin\alpha H} +$$

$$+ i\sum_{n=1}^{n_1}(-1)^{n+1}\frac{2n\pi^2}{\alpha^2 H^2 - n^2\pi^2}\cdot$$

$$\cdot\cos\left[\left(\alpha^2 - \frac{n^2\pi^2}{H^2}\right)^{\frac{1}{2}}(x + b)\right]\sin\frac{n\pi}{H}(y - H). \tag{64}$$

Using (61), (63), and (64), we obtain

$$\psi_1'(x + b > 0) = -\frac{Ua}{2}\frac{\sin\alpha(y - H)}{\sin\alpha H} -$$

$$-\frac{1}{2}\sum_{n=1}^{n_1}(-1)^{n+1}\frac{2Uan\pi}{\alpha^2 H^2 - n^2\pi^2}\cdot$$

$$\cdot\cos\left[\left(\alpha^2 - \frac{n^2\pi^2}{H^2}\right)^{\frac{1}{2}}(x + b)\right]\sin\frac{n\pi}{H}(y - H) +$$

$$+\frac{1}{2}\sum_{n=n_1+1}^{\infty}(-1)^{n+1}\frac{2Uan\pi}{n^2\pi^2 - \alpha^2 H^2}\cdot$$

$$\cdot\exp\left[-\left(\frac{n^2\pi^2}{H^2} - \alpha^2\right)^{\frac{1}{2}}(x + b)\right]\sin\frac{n\pi}{H}(y - H). \tag{65}$$

Applying the above method to the second integral in (57):

$$\psi_2'(x - b > 0) = \frac{Ua}{2}\frac{\sin\alpha(y - H)}{\sin\alpha H} +$$

$$+\frac{1}{2}\sum_{n=1}^{n_1}(-1)^{n+1}\frac{2Uan\pi}{\alpha^2 H^2 - n^2\pi^2}\cdot$$

$$\cdot\cos\left[\left(\alpha^2 - \frac{n^2\pi^2}{H^2}\right)^{\frac{1}{2}}(x + b)\right]\sin\frac{n\pi}{H}(y - H) -$$

$$-\frac{1}{2}\sum_{n=n_1+1}^{\infty}(-1)^{n+1}\frac{2Uan\pi}{n^2\pi^2 - \alpha^2 H^2}\cdot$$

$$\cdot\exp\left[-\left(\frac{n^2\pi^2}{H^2} - \alpha^2\right)^{\frac{1}{2}}(x - b)\right]\sin\frac{n\pi}{H}(y - H). \tag{66}$$

$$\psi_2'(x - b < 0) = -\frac{Ua}{2}\frac{\sin\alpha(y - H)}{\sin\alpha H} -$$

$$-\frac{1}{2}\sum_{n=1}^{n_1}(-1)^{n+1}\frac{2Uan\pi}{\alpha^2 H^2 - n^2\pi^2}\cdot$$

$$\cdot\cos\left[\left(\alpha^2 - \frac{n^2\pi^2}{H^2}\right)^{\frac{1}{2}}(x - b)\right]\sin\frac{n\pi}{H}(y - H) +$$

$$+\frac{1}{2}\sum_{n=n_1+1}^{\infty}(-1)^{n+1}\frac{2\,Uan\pi}{n^2\pi^2-\alpha^2H^2}\cdot$$
$$\cdot\exp\left[\left(\frac{n^2\pi^2}{H^2}-\alpha^2\right)^{\frac{1}{2}}(x-b)\right]\sin\frac{n\pi}{H}(y-H). \tag{67}$$

We may combine (65), (66), and (67) to obtain the solution for the two regions: $x > b$, and $-b < x < b$. The solution for $x < -b$ follows at once from the observation that (57) requires that ψ' be symmetric about the line, $x = 0$.

The assumption that the obstacle has a very small height permits the addition to the particular solution above, of any free oscillations that satisfy the conditions that ψ' vanish at $y = 0$ and $y = H$. It is obvious that these do not exist if $\alpha H < \pi$. Therefore, if $F_i > \pi^{-1}$ *the solution is unique.*

The indeterminacy that arises when $F_i < \pi^{-1}$ can be removed by considering the analogous problem of subcritical flow over barriers in a water channel. An investigation by RAYLEIGH (1883) revealed that the indeterminacy is eliminated by introducing viscosity. Then, as the frictional coefficient tends to zero, the resulting asymptotic solution is fully determined. This "practical solution" differs from the perfect fluid motion only in that the *upstream waves are absent.* Adopting this to our case, we annul the upstream oscillations by adding to the whole channel a series of free oscillations:

$$-\sum_{n=1}^{n_1}(-1)^n\frac{2\,Uan\pi}{\alpha^2H^2-n^2\pi^2}\sin\left(\alpha^2-\frac{n^2\pi^2}{H^2}\right)^{\frac{1}{2}}b\cdot$$
$$\cdot\sin\left(\alpha^2-\frac{n^2\pi^2}{H^2}\right)^{\frac{1}{2}}x\sin\frac{n\pi}{H}(y-H). \tag{68}$$

The final solution is

$$\psi'(x>b) = -\sum_{n=1}^{n_1}(-1)^n\frac{4\,Uan\pi}{\alpha^2H^2-n^2\pi^2}\cdot$$
$$\cdot\sin\left[\left(\alpha^2-\frac{n^2\pi^2}{H^2}\right)^{\frac{1}{2}}b\right]\cdot$$
$$\cdot\sin\left[\left(\alpha^2-\frac{n^2\pi^2}{H^2}\right)^{\frac{1}{2}}x\right]\sin\frac{n\pi}{H}(y-H)+$$
$$+\sum_{n=n_1+1}^{\infty}(-1)^n\frac{2\,Uan\pi}{n^2\pi^2-\alpha^2H^2}\cdot$$
$$\cdot\sinh\left(\frac{n^2\pi^2}{H^2}-\alpha^2\right)^{\frac{1}{2}}b\cdot$$
$$\cdot\exp\left[-\left(\frac{n^2\pi^2}{H^2}-\alpha^2\right)^{\frac{1}{2}}x\right]\sin\frac{n\pi}{H}(y-H). \tag{69}$$

$$\psi'(-b<x<b) = -Ua\frac{\sin\alpha(y-H)}{\sin\alpha H}+$$
$$+\sum_{n=1}^{n_1}(-1)^n\frac{2\,Uan\pi}{\alpha^2H^2-n^2\pi^2}\cdot$$
$$\cdot\cos\left[\left(\alpha^2-\frac{n^2\pi^2}{H^2}\right)^{\frac{1}{2}}(x+b)\right]\sin\frac{n\pi}{H}(y-H)-$$
$$-\sum_{n=n_1+1}^{\infty}(-1)^n\frac{2\,Uan\pi}{n^2\pi^2-\alpha^2H^2}\cdot$$
$$\cdot\exp\left[-\left(\frac{n^2\pi^2}{H^2}-\alpha^2\right)^{\frac{1}{2}}b\right]\cdot$$
$$\cdot\cosh\left(\frac{n^2\pi^2}{H^2}-\alpha^2\right)^{\frac{1}{2}}x\sin\frac{n\pi}{H}(y-H). \tag{70}$$

$$\psi'(x<-b) = \sum_{n=n_1+1}^{\infty}(-1)^n\frac{2\,Uan\pi}{n^2\pi^2-\alpha^2H^2}\cdot$$
$$\cdot\exp\left[\left(\frac{n^2\pi^2}{H^2}-\alpha^2\right)^{\frac{1}{2}}x\right]\cdot$$
$$\cdot\sinh\left(\frac{n^2\pi^2}{H^2}-\alpha^2\right)^{\frac{1}{2}}b\sin\frac{n\pi}{H}(y-H). \tag{71}$$

When $F_i < \pi^{-1}$ the obstacle produces downstream waves. The amplitudes are proportional to $n/(F_i^{-2}-n^2\pi^2)$, which is a maximum for $n = n_1$. When $n_1 = 1$, the total motion is

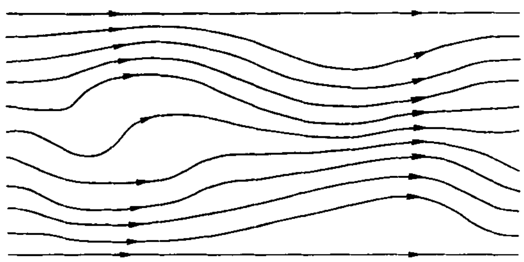

Fig. 4. Streamlines at some distance downstream of an obstacle whose area in the plane of the motion, $Q = H^2/10\,\pi$. The internal Froude number, $F^2 = 1/5\,\pi^2$.

rather simply described. One part is a local disturbance, dying out rapidly on both sides of the obstacle; the other is a single wave of length, $\lambda = 2\pi H \ (F_i^{-2} - \pi^2)^{-\frac{1}{2}}$ downstream, When $n_1 > 1$, two or more waves will exist with respective strengths indicated in (69). Figure 4 is an example of the flow when $F_i^2 = 1/5\pi^2$, ($n_1 = 2$). It represents a section of the channel sufficiently far downstream that only the oscillatory terms in (69) are sensible. A short calculation shows that the wave terms are already of the order of ten times the remaining terms in (69) at a distance $H/4$ downstream.

5. The effect of shear on the motion of a stable fluid

Several serious objections may be raised to any application of the results of the previous section to the atmosphere. Normally the flow from the surface to the stratosphere has a very considerable shear of velocity with height. In the westerlies, for example, the flow is roughly in the same direction at all levels but several times faster aloft than near the surface.

Many studies have been made of the stability of shearing flow with density gradients, notably by RICHARDSON (1920), TAYLOR (1931), and GOLDSTEIN (1931). The stability of the motion is assured, according to Richardson, if the number

$$R_i = \left(\frac{dU}{dy}\right)^2 \Big/ g\beta < 1. \qquad (72)$$

This result is dependent on the assumption of the conservation of momentum of the perturbed motion.

Tylor's analysis was greatly complicated by mathematical difficulties. He could only state in fact that, for a fluid extending above a rigid plane to infinity, progressive waves can exist if $R_i < 4$, and that no waves of any kind can exist if $R_i > 4$.

In this section an analysis similar to the one in section 3 will be applied to a current moving between two rigid walls with a density distribution, $\varrho = \varrho_0 \exp(-\beta y_0)$, and with $U = U_0 \exp(a y_0)$. This velocity distribution is somewhat different than Taylor's (linear shear), but the advantage is to avoid some of the mathematical difficulties of the latter distribution.

If we assume that the perturbed motion is small, we obtain from (13)

$$\nabla^2\delta + (2a - \beta)\frac{\partial\delta}{\partial y} + \frac{g\beta}{U_0^2 e^{2ay}}\delta = 0. \qquad (73)$$

Assuming $\delta = \chi(y) e^{ikx}$, the equation for χ is

$$\chi'' + \chi'(2a - \beta) + \chi\left(\frac{g\beta}{U_0^2 e^{2ay}} - k^2\right) = 0. \qquad (74)$$

The solution may be verified to be

$$\chi = \exp\left[-\left(a - \frac{\beta}{2}\right)y\right] \cdot$$
$$\cdot Z_m\left[\left(\frac{g\beta}{U_0^2 a^2}\right)^{\frac{1}{2}} e^{-ay}\right],\ m^2 = \frac{k^2}{a^2} + \left(1 - \frac{\beta}{2a}\right)^2, \qquad (75)$$

where Z_m is a cylinder function,

$$Z_m = A_m J_m + B_m N_m. \qquad (76)$$

If we consider that the fluid is bounded by a wall at $y = 0$ and $y = H$, then the possibility of free, steady perturbations depends on the existence of two zeros of the following function:

$$A_m(R_i, aH) = N_m(e^{-aH} R_i^{-\frac{1}{2}})\, J_m(R_i^{-\frac{1}{2}}) -$$
$$- N_m(R_i^{-\frac{1}{2}})\, J_m(e^{-aH} R_i^{-\frac{1}{2}}). \qquad (77)$$

Its zeros are given by JAHNKE and EMDE (1945). The tables indicate that the first zeros increase in size with m. The index m, however, has a minimum of 1 if we assume that $\beta < < a$, a reasonable assumption in most cases of interest. The largest values of R_i permitting this type of motion are therefore those corresponding to the zeros of (74) with $m = 1$. It appears that the results of this investigation are much clearer if, instead of seeking critical Richardson numbers, we consider the values of the function

$$K = \frac{(e^{aH} - 1)}{e^{aH} R_i^{\frac{1}{2}}} = \frac{1}{\overline{F}_i}\frac{\overline{U}^2}{U_1 U_0} \qquad (78)$$

when the function A_1 is zero. $\overline{F}_i$ is the mean Froude number, $\overline{U}/(g\beta H^2)^{\frac{1}{2}}$, $\overline{U}$ is the mean velocity of the flow, U_1 is the velocity at the top of the channel, and U_0 is the velocity at the bottom. The values of K corresponding to the zeros of Λ_1 vary very little when aH changes from zero to ∞. For the first zeros K increases from π at $aH = 0$, to only 3.8317 at $aH = \infty$. $\overline{F}_i$ changes even more slowly so that, if the velocity at the top is less than about $4\,U_0$, the critical value of $\overline{F}_i$ is given by

$$\overline{F}_i = \frac{1}{3}\ (\pm\ .02). \tag{79}$$

Such a range will probably include all cases of physical interest. When the shear is zero the critical value of $\overline{F}_i$ is π^{-1}, as we would expect from the results of the previous section. We may formulate the above results as follows: *The criterion for supercritical flow in a channel is* $\overline{F}_i = \overline{U}/(g\beta H^2)^{\frac{1}{2}} > 1/3$, *approximately, where* $\overline{U}$ *is the mean velocity of the flow.*

This conclusion is based, of course, on the assumption of a particular velocity distribution in the channel, namely, $U = U_0\, e^{ay_0}$. It would have been preferable, no doubt, to have carried through the analysis with a linear shear, but this presents considerable mathematical difficulties. It does not seem likely, however, that the above result would be greatly altered by this change in the model. Certainly, if the shear is small, such a generalization must be valid, since the shear will then be practically linear.

The above results seem to indicate that critical states of shearing flow, and quite possibly the instability of a given flow, depend not alone on the Richardson number but also on the depth of the flow if the depth is finite. In the case of the shear considered in this section, a channel of infinite depth has a critical Richardson number of approximately one (see section 2). The critical state in a finite channel, however, is best described by a mean Froude number.

6. Meteorological implications

The models investigated in this paper are far simpler than anything likely to be encountered in the atmosphere. The imposition of a rigid upper surface is a very important feature of the problem, yet in the atmosphere the closest approach to such a surface is the tropopause, which is more like a fluid interface. In addition, such effects as those due to compressibility and the earth's rotation are neglected. The results of the investigation, therefore, are not likely to have much quantitative application to the atmosphere. Regarded simply as an investigation of internal gravity waves, it is probable that certain of the phenomena examined, in particular internal hydraulic jumps, occur in the atmosphere and are qualitatively explainable by the approach used in this paper. If so, more reasonable models can be examined based on equation (9) which is valid quite generally. In addition, other extensions of this kind are obvious. For example, the procedure used in deriving critical flows in a multiple-layer system could be used to find such criteria in a three-fluid system approximating the atmosphere. The layers need not be of equal depth and the first two layers would represent the troposphere with a subsidence inversion. The upper layer could be used for the stratosphere.

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