NANYANG TECHNOLOGICAL UNIVERSITY

SPMS/DIVISION OF MATHEMATICAL SCIENCES

2025/26 Semester 2 MH4110 Partial Differential Equations Tutorial 5, 19 February

Problem 1 Consider the diffusion equation on $(0, l)$ with the Robin boundary condition:

$$\begin{aligned} &u_t = ku_{xx}, \quad 0 < x < l, \ \ t > 0, \\ &u_x(0,t) - a_0 u(0,t) = 0, \quad u_x(l,t) + a_l u(l,t) = 0, \quad t > 0, \\ &u(x,0) = \phi(x), \quad 0 < x < l. \end{aligned}$$

If $a_0 > 0$ and $a_l > 0$, use the energy method to show that $\int_0^l u^2(x,t)dx$ decreases with respect to t.

Problem 2 (Ex. 6 on Page 52) Compute $\int_0^\infty e^{-x^2}dx$. (*Hint:* This is a function that cannot be integrated by formula. So use the following trick. Transform the double integral $\int_0^\infty e^{-x^2}dx \cdot \int_0^\infty e^{-y^2}dy$ into polar coordinates and you'll end up with a function that can be integrated easily.)

Problem 3 (Ex. 7 on Page 52) Use the result of Problem 2 to show that $\int_{-\infty}^\infty e^{-p^2}dp = \sqrt{\pi}$. Then substitute $p = x/\sqrt{4kt}$ to show that

$$\int_{-\infty}^\infty S(x,t)dx = 1,$$

where $S(x,t)$ is the Gaussian kernel

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}} \exp\left(-\frac{x^2}{4kt}\right).$$

Problem 4 Solve the diffusion equation: $u_t = ku_{xx}$ with the initial condition:

$$\phi(x) = 1, \quad |x| < l, \quad \phi(x) = 0, \quad |x| > l.$$

Write your answer in terms of the error function $\mathrm{Erf}(x)$.

Problem 5 Solve the diffusion equation: $u_t = ku_{xx}$ with the initial condition:

$$\phi(x) = e^{-x}, \quad x > 0; \quad \phi(x) = 0, \quad x < 0.$$

Problem 6 Solve the diffusion equation: $u_t = ku_{xx}$ with the initial condition $u(x,0) = \phi(x) = e^{-x}$.

Problem 7 Show that for any fixed $\delta > 0$ (no matter how small),

$$\max_{\delta \leq |x| < \infty} S(x,t) \to 0 \quad \text{as} \quad t \to 0.$$

Problem 8 Solve the problem ($k > 0$):

$$\begin{aligned} &u_t - ku_{xx} = \cos t, \quad x \in (-\infty, \infty), \;\; t > 0, \\ &u(x,0) = e^{-x^2}, \quad x \in (-\infty, \infty). \end{aligned}$$