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**Initial condition:** Specify the physical state at a particular time $t_0$

- Heat/Diffusion equation: The initial condition (e.g., temperature, concentration, ....)

$$u_t - ku_{xx} = 0; \quad u(x,0) = \phi(x). \tag{51}$$

$$u = u(t, \vec{x})$$

$$u(t = t_0, \vec{x}) = \phi(\vec{x})$$

$$u(\vec{x})\Big|_{\partial\Omega \backslash \Gamma} = 0$$

$$\frac{\partial u}{\partial \vec{n}}\Big|_{\Gamma} = 0$$

$$-\nabla\cdot\left(\delta(\vec{x})\, u(\vec{x})\right) = f(\vec{x})$$

$\partial\Omega / \Gamma$

air $\Gamma$

material $\Omega$

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- Wave equation: A pair of initial conditions

$$u_{tt} - c^2 u_{xx} = 0; \quad u(x,0) = \phi(x), \;\; u_t(x,0) = \psi(x). \tag{52}$$

## Boundary condition

Let $D$ be a domain on which a PDE is defined. Let $\mathbf{n}$ be the unit outer normal vector on $\partial D$. Let $\Gamma \subseteq \partial D$. The three most important kinds of boundary conditions are

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directional derivative

( flux across the boundary $\Gamma$)

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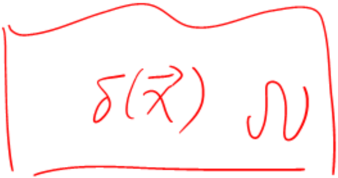

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On the whole boundary $\partial D$, one of these boundary conditions or a mixing of them could be imposed.

If the boundary data $g(\mathbf{x}, t) = 0$ are set to be constantly zero, the boundary conditions are said to be homogeneous, otherwise, they are inhomogeneous.

# §1.5 Well-posedness of a PDE

A PDE in a domain $D$ together with a set of initial and/or boundary conditions (or other auxiliary conditions) is said to be well-posed, if it meets the following three fundamental properties:

(i) **Existence** — There exists at least one solution to the differential equation.

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(iii) **Stability** — Small changes in the initial and/or boundary conditions (or other auxiliary conditions) should lead to small changes in the output. This means that if the data are changed a little, the corresponding solution changes only a little. $|\Delta\vec{b}| \ll 1$

$A\vec{x} = \vec{b}$ If $\vec{b}$ changes to $\vec{b} + \Delta\vec{b}$
how about $\vec{x}$

# §1.5 Well-posedness of a PDE

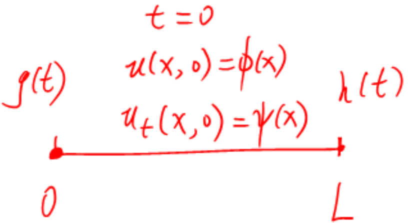
## Example: A well-posed problem

A vibrating string with an external force, whose ends are moved in a specified way, satisfies the problem

$$Tu_{tt} - \rho u_{xx} = f(x,t) \tag{53}$$

with the initial and boundary conditions for $0 < x < L$:

$$\begin{aligned} u(x,0) &= \phi(x), \quad & u_t(x,0) &= \psi(x), \\ u(0,t) &= g(t), \quad & u(L,t) &= h(t). \end{aligned}$$



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$|\Delta u|_{L_2} < \alpha |g_1 - g|_{L_2}$

- Existence and uniqueness would mean that there is exactly one solution $u(x,t)$ for arbitrary (differentiable) functions $f$, $\phi$, $\psi$, $g$, and $h$.
- Stability would mean that if any of these five functions are slightly perturbed, then $u$ is also changed only slightly.

$g(t) \longrightarrow g_1(t)$

$u(t;x) \rightarrow u(t,x) + \Delta u$

# §1.6 Types of Second-Order Equations

$u = u(x, y)$

## Second-order linear PDE

Let's consider the PDE

$$a_{11} u_{xx} + 2a_{12} u_{xy} + a_{22} u_{yy} + a_1 u_x + a_2 u_y + a_0 u = 0. \tag{54}$$

This is a linear equation of order two in two variables with six real constant coefficients.

The factor 2 is introduced for convenience, largely due to the fact that

$$a_{12} u_{xy} + a_{12} u_{yx} = 2a_{12} u_{xy}. \tag{55}$$

$u_{xy} = u_{yx}$ is assumed

**Theorem**: Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$. Then

- Equation (54) is of elliptic type, if $\det(A) = a_{11}a_{22} - a_{12}^2 > 0$. By a linear transform, it can be reduced to

$$u_{xx} + u_{yy} + \{\text{terms of lower order 1 or 0}\} = 0. \tag{56}$$

$\Delta u(x,y) = 0 \iff u_{xx} + u_{yy} = 0$

$A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad$ $\mathrm{Det}(A) = 1 > 0$

Eigenvalues : $\lambda_1 = \lambda_2 = 1 > 0$

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- Equation (54) is of hyperbolic type, if $\det(A) = a_{11}a_{22} - a_{12}^2 < 0$. By a linear transform, it can be reduced to

$$u_{xx} - u_{yy} + \{\text{terms of lower order 1 or 0}\} = 0. \tag{57}$$

$u_{tt} - c^2 u_{xx} = 0$

$A = \begin{pmatrix} 1 & 0 \\ 0 & -c^2 \end{pmatrix}$ $\quad Det(A) = -c^2 < 0$

$\lambda_1 = 1 > 0, \lambda_2 = -c^2 < 0$

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- Equation (54) is of parabolic type, if $\det(A) = a_{11}a_{22} - a_{12}^2 = 0$. By a linear transform, it can be reduced to

$u_t - k u_{xx} = 0$

$$u_{xx} + \{\text{terms of lower order 1 or 0}\} = 0, \tag{58}$$

(unless $a_{11} = a_{12} = a_{22} = 0$.)

$A = \begin{pmatrix} 0 & 0 \\ 0 & -k \end{pmatrix}$ $\quad \mathrm{Det}(A) = 0$

$\lambda_1 = 0 \quad \lambda_2 = -k$

## Representatives of three types

- Laplace equation:

$$u_{xx} + u_{yy} = 0 \tag{59}$$

We have $a_{11} = a_{22} = 1, a_{12} = 0$ so $\det(A) = 1$. It is elliptic type.

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- Laplace equation:
$$u_{xx} + u_{yy} = 0 \tag{59}$$
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- Wave equation:
$$u_{tt} - c^2 u_{xx} = 0 \tag{60}$$
We have $a_{11} = 1, a_{22} = -c^2, a_{12} = 0$ so $\det(A) = -c^2$. It is hyperbolic type.

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- Heat equation:

$$u_t - k u_{xx} = 0 \tag{61}$$

We have $a_{11} = 0, a_{22} = -k, a_{12} = 0$ so $\det(A) = 0$. It is parabolic type.

**Example 1.**

Classify each of the equations

(a) $u_{xx} - 5u_{xy} = 0.$

(b) $4u_{xx} - 12u_{xy} + 9u_{yy} + u_y = 0.$

(c) $4u_{xx} + 6u_{xy} + 9u_{yy} = 0.$

## §1.6 Types of Second-Order Equations

$(1-\lambda)(-\lambda) - 25/4 = 0$

**Example 1.**

Classify each of the equations

$\lambda^2 - \lambda - 25/4 = 0$

$B = \begin{pmatrix} 4 & -6 \\ -6 & 9 \end{pmatrix}$

(a) $u_{xx} - 5u_{xy} = 0.$

(b) $4u_{xx} - 12u_{xy} + 9u_{yy} + u_y = 0.$

(c) $4u_{xx} + 6u_{xy} + 9u_{yy} = 0.$

$(\lambda - \frac{1}{2})^2 = \frac{13}{2}$

$\lambda_1 = \frac{1}{2} + \sqrt{\frac{13}{2}} > 0$

$\lambda_2 = \frac{1}{2} - \sqrt{\frac{13}{2}} < 0$

**Solution**: We see that

(a) $a_{11} = 1, a_{22} = 0, a_{12} = -5/2 \Rightarrow a_{11}a_{22} - a_{12}^2 = -25/4 < 0 \Rightarrow$ hyperbolic equation.

$A = \begin{pmatrix} 1 & -5/2 \\ -5/2 & 0 \end{pmatrix}$

$\mathrm{Det}(A) = -25/4 < 0$

# §1.6 Types of Second-Order Equations

## Example 1.

Classify each of the equations

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(b) $a_{11} = 4, a_{22} = 9, a_{12} = -6 \Rightarrow a_{11}a_{22} - a_{12}^2 = 0 \Rightarrow$ parabolic equation.

$$\lambda_{1,2} = \frac{13 \pm \sqrt{61}}{2} > 0$$

$$C = \begin{pmatrix} 4 & 3 \\ 3 & 9 \end{pmatrix} \qquad \mathrm{Det}(C) = 27 > 0$$

$$(4-\lambda)(9-\alpha) - 9 = 0 \Rightarrow \lambda^2 - 13\lambda + 27 = 0$$

**Example 1.**

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$$\begin{aligned} &\text{(a)}\quad u_{xx} - 5u_{xy} = 0.\\ &\text{(b)}\quad 4u_{xx} - 12u_{xy} + 9u_{yy} + u_y = 0.\\ &\text{(c)}\quad 4u_{xx} + 6u_{xy} + 9u_{yy} = 0. \end{aligned}$$

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(a) $a_{11} = 1, a_{22} = 0, a_{12} = -5/2 \Rightarrow a_{11}a_{22} - a_{12}^2 = -25/4 < 0 \Rightarrow$ hyperbolic equation.

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(c) $a_{11} = 4, a_{22} = 9, a_{12} = 3 \Rightarrow a_{11}a_{22} - a_{12}^2 = 27 > 0 \Rightarrow$ elliptic equation.

# §1.6 Types of Second-Order Equations

## Second-order linear PDE: The general case

Suppose that there are $n$ variables, denoted $x_1, x_2, \cdots, x_n$, and the PDE is

$$\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} a_i u_{x_i} + a_0 u = 0. \tag{62}$$

with real constants $a_{ij}$, $a_i$, and $a_0$. Since the mixed derivatives are equal, we may as well assume that $a_{ij} = a_{ji}$. Let $A = (a_{ij})$ be the coefficient matrix. Further assume that the real numbers $d_1, \cdots, d_n$ are the eigenvalues of $A$.

**Definition**

The PDE (62) is

- elliptic: if all the eigenvalues $d_1, \cdots, d_n$ are positive or negative.

have the same sign

## Definition

The PDE (62) is

- elliptic: if all the eigenvalues $d_1, \cdots, d_n$ are positive or negative.
- hyperbolic: if none of the eigenvalues $d_1, \cdots, d_n$ vanish and one of them has the opposite sign from the $(n-1)$ others.

$$\prod_{i=1}^{n} d_i = \det(A) \neq 0$$

$$u_{tt} - c^2(u_{xx} + u_{yy} + u_{zz}) = 0$$

$$\lambda_1 = 1 > 0 \quad \lambda_2 = \lambda_3 = \lambda_4 = -c^2 < 0 \qquad \begin{pmatrix} 1 & & & \\ & -c^2 & & \\ & & -c^2 & \\ & & & -c^2 \end{pmatrix}$$

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- ultrahyperbolic: if none of the eigenvalues $d_1, \cdots, d_n$ vanish but at least two of them are positive and at least two are negative.

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- ultrahyperbolic: if none of the eigenvalues $d_1, \cdots, d_n$ vanish but at least two of them are positive and at least two are negative.
- parabolic: if exactly one of the eigenvalues $d_1, \cdots, d_n$ is zero and all the others have the same sign.

$$u_t - k(u_{xx} + u_{yy} + u_{zz}) = f(t, x)$$

$$A = \begin{pmatrix} 0 & & & \\ & -k & & \\ & & -k & \\ & & & -k \end{pmatrix}$$

$$\lambda_1 = 0, \quad \lambda_2 = \lambda_3 = \lambda_4 = -k$$

**Example A**

- Laplace equation: $u_{xx} + u_{yy} + u_{zz} = 0$ is elliptic because all the eigenvalues are 1.

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- Laplace equation: $u_{xx} + u_{yy} + u_{zz} = 0$ is elliptic because all the eigenvalues are 1.
- Wave equation: $u_{tt} - c^2(u_{xx} + u_{yy} + u_{zz}) = 0$ is hyperbolic because one eigenvalue is 1 and all the others are $-c^2$.
- Heat equation: $u_t - k(u_{xx} + u_{yy} + u_{zz}) = 0$ is parabolic because one eigenvalue is 0 and all the others are $-k$.

# Summary

## First-order linear equation

In general, we are interested in solving the PDEs:

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y), \tag{63}$$

where $a, b, c, f$ are constants or continuous functions in some domain $\Omega$.
The method of characteristics is usually used to solve (63).

# Summary

## Second-order linear equation

In general, we are interested in solving the PDEs:

$$\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} a_i u_{x_i} + a_0 u = 0. \tag{64}$$

where $a_{ij}$, $a_i$, and $a_0$ are constants or continuous functions in some domain $\Omega$. We will mainly discuss the following three typical PDEs in our later lectures.

- Wave equation: $u_{tt} - c^2 \Delta u = 0$ (hyperbolic) Chapters 2-5.
- Heat equation: $u_t - k\Delta u = 0$ (parabolic) Chapters 2-5.
- Laplace equation: $\Delta u = 0$ (elliptic) Chapter 6.

# MH4110 Partial Differential Equations

## Chapter 2 - Waves and diffusions

# Synopsis

1. Wave equation: General solution, d'Alambert's formula.
2. Wave equation: Causality, The energy method.
3. Heat equation: Maximum principle, Uniqueness, and Stability.
4. Heat equation: The solution in an integral form, Interpretation of the solution.
5. Comparison of wave and heat equations.

# §2.1 The wave equation

## Wave equation

The wave equation on the whole real line takes the form

$$u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty, \tag{1}$$

where the constant $c > 0$ is the wave speed. Physically, you can imagine a very long string in a transverse motion. It describes the dynamics of the amplitude $u(x, t)$ of the point at position $x$ on the string at time $t$.

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We use two methods to derive the general solution of (1):

1. Factorization of the differential operator

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We use two methods to derive the general solution of (1):

1. Factorization of the differential operator → reduce the PDE into 2 1st order PDEs
2. Characteristic coordinates

2nd order

Method 1: Factorization of the differential operator

## Method 1: Factorization of the differential operator

Observe that the second order linear operator of the wave equation factors into two first order operators

$u_t + c u_x$

$$u_{tt} - c^2 u_{xx} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\underline{\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u} = 0. \tag{2}$$

$$\mathcal{L} = \frac{\partial^2}{\partial t^2} - c^2\frac{\partial^2}{\partial x^2} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)$$

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(u_t + c u_x\right) = \frac{\partial}{\partial t}\left(u_t + c u_x\right) - c\frac{\partial}{\partial x}\left(u_t + c u_x\right)$$

# §2.1 The wave equation

## Method 1: Factorization of the differential operator

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$$u_{tt} - c^2 u_{xx} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\underbrace{\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u}_{v(t,x)} = 0. \tag{2}$$

Define

$$v = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u = u_t + cu_x. \tag{3}$$

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Define

$$v = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u = u_t + cu_x. \tag{3}$$

Then (2) can be rewritten as

$$u_{tt} - c^2 u_{xx} = 0 \xRightarrow{v = u_t + cu_x} \underbrace{v_t - cv_x = 0.}_{} \tag{4}$$

## Method 1: Factorization of the differential operator

Observe that the second order linear operator of the wave equation factors into two first order operators

$$u_{tt} - c^2 u_{xx} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u = 0. \tag{2}$$

Define

$$v = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u = u_t + cu_x. \tag{3}$$

Then (2) can be rewritten as

$$v_t - cv_x = 0. \tag{4}$$

characteristic lines

$ct + x = \text{const.}$

Therefore, we can solve (4) to find $v$, and then find $u$ by solving (3).

$v(t, x) = h(ct + x)$

Method 1: Factorization of the differential operator (Cont'd)

Applying **the method of characteristics** to (4) (see Chapter 1) leads to

$$v(x, t) = h(x + ct), \tag{5}$$

where $h$ is an arbitrary function.

## Method 1: Factorization of the differential operator (Cont'd)

Applying **the method of characteristics** to (4) (see Chapter 1) leads to

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Plugging (5) into (3) gives

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where $h$ is an arbitrary function.

By assumption: $f(x+ct)$ is a particular solution (with $f$ TBD)

Plugging (5) into (3) gives

$$u_t + cu_x = h(x+ct). \tag{6}$$

It is a linear inhomogeneous equation, whose general solution is a particular solution plus the general solution to the homogeneous equation: $u_t + cu_x = 0$.

characteristic lines: $ct - x = \text{const.}$

$\Rightarrow$ General solution: $g(ct-x)$

Method 1: Factorization of the differential operator (Cont'd)

The general solution of $u_t + cu_x = 0$ is $g(x - ct)$, where $g$ is an arbitrary function.

## Method 1: Factorization of the differential operator (Cont'd)

The general solution of $u_t + cu_x = 0$ is $g(x - ct)$, where $g$ is an arbitrary function.

We can check directly by differentiation that $u = f(x + ct)$ is a particular solution of (6):

$$LHS = u_t + cu_x = cf'(x + ct) + cf'(x + ct), = 2cf'(x+ct) \tag{7}$$

where $f'$ is the ordinary derivative of a function of one variable and $f'(s)$ can be taken as $f'(s) = h(s)/(2c)$.

$$RHS = h(x+ct)$$

$$LHS = RHS \Longrightarrow 2cf'(x+ct) = h(x+ct)$$

## Method 1: Factorization of the differential operator (Cont'd)

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Finally, we obtain the solution of (1) is $f(x) = \int_0^x \frac{h(s)}{2c}\, ds$

$$u(x, t) = \underbrace{f(x + ct)}_{\text{particular}} + \underbrace{g(x - ct)}_{\text{general solution}}, \tag{8}$$

where $f$, $g$ are arbitrary functions. particular general solution

**Theorem**: The general solution of

$$u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty, \tag{9}$$

is

$$u(x,t) = f(x+ct) + g(x-ct), \tag{10}$$

General Solu.

where $f, g$ are two arbitrary functions.

$$u_t - c\,u_x = 0, \quad -\infty < x < \infty \Rightarrow f(x+ct)$$

$$u_t + c\,u_x = 0, \quad -\infty < x < \infty \Rightarrow g(x-ct)$$

Method 2: Characteristic coordinates

## Method 2: Characteristic coordinates

$$u_{tt} - c^2 u_{xx} = 0$$

Introduce the characteristic coordinates

$$\xi = x + ct, \quad \eta = x - ct, \tag{11}$$

By the chain rule, we have

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial u}{\partial t} - c\frac{\partial u}{\partial x}\right) = 0$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}, \quad \frac{\partial}{\partial t} = c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta}, \tag{12}$$

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\underbrace{\frac{\partial \xi}{\partial x}}_{=1} + \frac{\partial}{\partial \eta}\underbrace{\frac{\partial \eta}{\partial x}}_{=1}$$

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and hence

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial \xi^2} + 2\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2}, \quad \frac{\partial^2}{\partial t^2} = c^2\left[\frac{\partial^2}{\partial \xi^2} - 2\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2}\right]. \tag{13}$$

## Method 2: Characteristic coordinates (Cont'd)

So wave equation (1) or (9) takes the form

$$u_{tt} - c^2 u_{xx} = -4c^2 u_{\xi\eta} = 0, \tag{14}$$

which means that $u_{\xi\eta} = 0$ since $c \neq 0$.

$$\Downarrow$$

$$\frac{\partial}{\partial \eta}(u_\xi) = 0 \Rightarrow u_\xi = h(\xi)$$

$$\Rightarrow u = f(\xi) + g(\eta)$$

## Method 2: Characteristic coordinates (Cont'd)

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which means that $u_{\xi\eta} = 0$ since $c \neq 0$.
The solution of the transformed equation (14) is

$$u = f(\xi) + g(\eta). \tag{15}$$

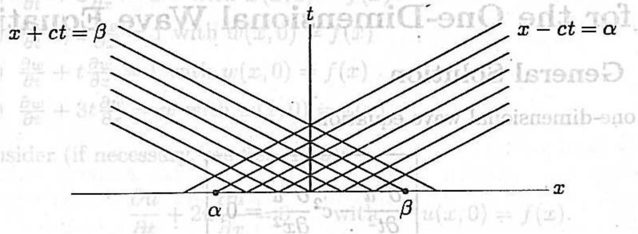
Switch back to the original variables $(x, t)$, and we obtain the general solution

$$u(x,t) = f(x+ct) + g(x-ct), \tag{16}$$

where $f, g$ are two arbitrary functions.

## Geometry of the wave equation

- The wave equation (1) has *two* families of characteristic lines: $x \pm ct$ =constant (see Figure below).



# §2.1 The wave equation

## Geometry of the wave equation

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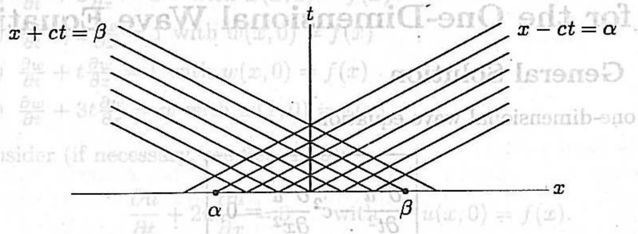
- Part of the solution is constant along the corresponding characteristic line.



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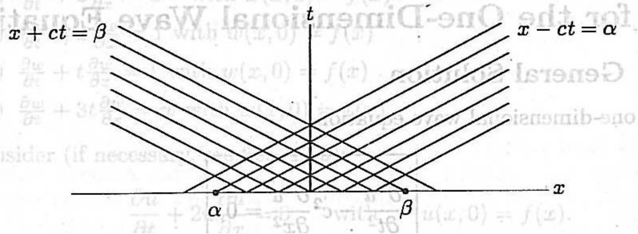

# §2.1 The wave equation

$u_{tt} - c^2 u_{xx} = 0$

## Geometry of the wave equation

- The wave equation (1) has *two* families of characteristic lines: $x \pm ct =$constant (see Figure below).
- Part of the solution is constant along the corresponding characteristic line.
- One, $g(x - ct)$ (is a constant along $x - ct = \alpha$), is a wave of arbitrary shape traveling to the **right** at speed $c$.
- The other, $f(x + ct)$ (is a constant along $x + ct = \beta$), is a wave of another arbitrary shape traveling to the **left** at speed $c$.