

**NANYANG TECHNOLOGICAL UNIVERSITY**  
**SPMS/DIVISION OF MATHEMATICAL SCIENCES**

2025/26 Semester 2      MH4110 Partial Differential Equations      Tutorial 6, 26 February

**Problem 1** Consider the heat equation

$$\frac{\partial u}{\partial t} - 4 \frac{\partial^2 u}{\partial x^2} = 0$$

- (a) Show that it is linear.
- (b) Show that it is parabolic.
- (c) If  $x^2 + at + bx$  is a solution, find the values of  $a$  and  $b$ .
- (d) If  $u(x, 0) = x^2 + 2x$ , find  $u(x, t)$ .

**Problem 2** Consider the PDE

$$u_x + 2u_y = 2x \quad (1)$$

- (a) Find the general solution to this PDE,
- (b) Find the solution to the PDE that additionally satisfies  $u(x, 0) = 5x^2$ .

**Problem 3** Find the general solution to this PDE

$$u_x + xu_y = 0. \quad (2)$$

**Problem 4** Consider the PDE

$$u_x + 2xu_y = xy^3. \quad (3)$$

- (a) Find the general solution to this PDE,
- (b) Find the solution to the PDE that additionally satisfies  $u(x, 0) = x^4$ .

**Problem 5** Let  $u(t, x)$  be the solution of the following IVP for the heat equation

$$\begin{cases} u_t - ku_{xx} = 0, & -\infty < x < \infty, \quad t \in R \\ u(x, 0) = \phi(x), & -\infty < x < \infty. \end{cases} \quad (4)$$

Show that if  $\phi$  is an odd function of  $x$ , then  $u(-x, t) = -u(x, t)$  for all  $x, t \in R$ .

**Problem 6** Find the solution  $u(t, x)$  to the following IVP for the heat equation

$$\begin{cases} u_t - ku_{xx} = 0, & -\infty < x < \infty, \quad t > 0 \\ u(x, 0) = \phi(x), & -\infty < x < \infty. \end{cases} \quad (5)$$

where  $\phi(x) = 3$  for  $0 < x < 2$  and  $\phi(x) = 0$  otherwise. Determine the maximum and minimum values of  $u(t, x)$  on the domain

$$R = \{(t, x) \in R^2 : -\infty < x < \infty, 0 \leq t < \infty\}.$$

**Problem 7** Find the solution  $u(t, x)$  to the following IBVP for the heat equation

$$\begin{cases} u_t - ku_{xx} = 0, & 0 < x < \infty, \quad t > 0 \\ u(x, 0) = \phi(x), & 0 < x < \infty, \\ u(0, t) = 0, & t \in R. \end{cases} \quad (6)$$

where  $\phi(x) = 3$  for  $0 < x < 2$  and  $\phi(x) = 0$  otherwise. Determine the maximum and minimum values of  $u(t, x)$  on the domain

$$R = \{(t, x) \in R^2 : 0 \leq x < \infty, 0 \leq t < \infty\}.$$

**Problem 8** Find the solution  $u(t, x)$  to the following IBVP for the heat equation

$$\begin{cases} u_t - ku_{xx} = 0, & 0 < x < \infty, \quad t > 0 \\ u(x, 0) = \phi(x), & 0 < x < \infty, \\ u_x(0, t) = 0, & t \in R. \end{cases} \quad (7)$$

where  $\phi(x) = 3$  for  $0 < x < 2$  and  $\phi(x) = 0$  otherwise. Determine the maximum and minimum values of  $u(t, x)$  on the domain

$$R = \{(t, x) \in R^2 : 0 \leq x < \infty, 0 \leq t < \infty\}.$$

**Problem 9** Find the general solution  $u(t, x)$  to the PDE

$$u_{xx} - 4u_{yy} = x^2, \quad -\infty < x < \infty, \quad -\infty < y < \infty. \quad (8)$$

**Problem 10** Find the solution  $u(t, x)$  to the following IVP for the wave equation

$$\begin{cases} u_{tt} - 4u_{xx} = 0, & -\infty < x < \infty, \quad t > 0 \\ u(x, 0) = x^2, \quad u_t(x, 0) = 1, & -\infty < x < \infty. \end{cases} \quad (9)$$

**Problem 11** Let  $u(t, x)$  be the solution of the following initial value problem for the wave equation

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & -\infty < x < \infty, \quad t > t_0 \\ u(x, t_0) = \phi(x), \quad u_t(x, t_0) = \psi(x), & -\infty < x < \infty. \end{cases}$$

If  $\phi(x)$  and  $\psi(x)$  are even functions of  $x$ , show that the solution  $u(x, t)$  of the wave equation is also even in  $x$ .

**Problem 12** Find the solution  $u(t, x)$  to the following IVP for the wave equation

$$\begin{cases} u_{tt} - 4u_{xx} = 0, & 0 < x < \infty, \quad t > 0 \\ u(x, 0) = x^2, \quad u_t(x, 0) = 1, & 0 < x < \infty \\ u(0, t) = 0, & t > 0. \end{cases} \quad (10)$$

**Problem 13** Find the solution  $u(t, x)$  to the following IVP for the wave equation

$$\begin{cases} u_{tt} - 4u_{xx} = 0, & 0 < x < \infty, \quad t > 0 \\ u(x, 0) = x^2, \quad u_t(x, 0) = 1, & 0 < x < \infty \\ u_x(0, t) = 0, & t > 0. \end{cases} \quad (11)$$

**Problem 14** Use the method of factoring the operator to find the general solution  $u(t, x)$  to the following PDE

$$u_{xx} + 6u_{xy} - 16u_{yy} = 16x + 2y. \quad (12)$$

**Problem 15** Classify each of the following PDE as hyperbolic, elliptic, or parabolic

- (a)  $u_{xx} + 2u_{xy} + u_{yy} + u_x + u_y = 0$ ,
- (b)  $u_{xx} + 2u_{xy} + 2u_{yy} + u_x + u_y = \sin(xy)$ ,
- (c)  $2u_{xx} - 4u_{xy} - 6u_{yy} + u_x = 0$ .