§1.2 First-order linear equations

$u = u(x, y)$

In general, we are interested in solving the first-order linear equation:

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y), \tag{13}$$

where a, b, f are continuous functions in some domain Ω. If $f \equiv 0$, then the equation is homogeneous, otherwise it is inhomogeneous. We will see that any linear first-order PDE can be reduced to an ODE, which will then allow us to tackle it with already familiar methods from ODEs.

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I. Constant coefficient case:

We start with one simplest case of (13) with a, b being constants and $c = f \equiv 0$. More precisely, we consider

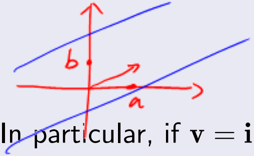
$\boxed{a^2+b^2=0 \Rightarrow a=b=0}$

$$au_x + bu_y = 0, \quad a^2 + b^2 \neq 0. \tag{14}$$

$(a,b)\cdot(u_x, u_y) = 0 \qquad \nabla u = (u_x, u_y)$

Directional derivative

Let $\mathbf{v} = (a, b) \neq \mathbf{0}$ be a given vector in $\mathbb{R}^2$. The directional derivative of u along $\mathbf{v}$ at (x, y) is defined by unit vector in the direction $\vec{v}$



$$\nabla_{\mathbf{v}} u = \nabla u \cdot \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{a u_x + b u_y}{\sqrt{a^2 + b^2}}. \qquad (15)$$

In particular, if $\mathbf{v} = \mathbf{i} = (1, 0)$, then it reduces to $\partial_x u$, while if $\mathbf{v} = \mathbf{j} = (0, 1)$, it becomes $\partial_y u$.

$a u_x + b u_y = 0$ $\Rightarrow \nabla_{\vec{v}} u = 0$

$$\frac{\vec{v}}{|\vec{v}|} = \frac{(a, b)}{\sqrt{a^2 + b^2}}$$

$$\frac{dy}{dx} = \frac{b}{a} \Rightarrow y = \frac{b}{a} x + c$$

$$u = u(c) = f\left(y - \frac{b}{a} x\right)$$

A. Geometric Method

Directional derivative

Let $\mathbf{v} = (a, b) \neq \mathbf{0}$ be a given vector in $\mathbb{R}^2$. The directional derivative of u along $\mathbf{v}$ at (x, y) is defined by

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In particular, if $\mathbf{v} = \mathbf{i} = (1, 0)$, then it reduces to $\partial_x u$, while if $\mathbf{v} = \mathbf{j} = (0, 1)$, it becomes $\partial_y u$.

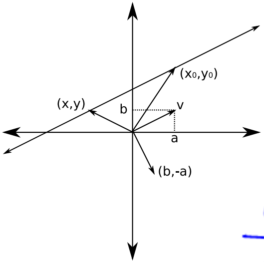
We can rewrite eqn. (14) as

$$au_x + bu_y = 0 \quad \Leftrightarrow \quad (a, b) \cdot \nabla u = 0.$$

Setting $\mathbf{v} = (a, b)$, we have

$$\nabla_{\mathbf{v}} u = 0. \tag{16}$$

A. Geometric Method



$\vec{v} = (a,b)$

$\frac{dy}{dx} = \frac{b}{a}$

$\nabla_{\mathbf{v}} u = 0$ means that $u(x,y)$ does not change along the direction (a,b), in other words, $u(x,y)$ must be a constant along the lines with this direction. The lines parallel to (i.e., tangent to) (a,b) have the equations:

$$bx - ay = c, \tag{17}$$

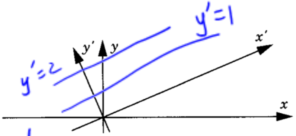
where c is an arbitrary constant. These lines are called the characteristic lines. If $u(x,y)$ does not change along these lines, the solution of (14) is

$$u(x,y)|_{bx-ay=c} = f(c) \quad \Rightarrow \quad u(x,y) = f(bx-ay), \tag{18}$$

where f is any function of one variable.

B. Coordinate Method

$a u_x + b u_y = 0$



$\frac{dy}{dx} = \frac{b}{a} \Rightarrow y = \frac{b}{a}x + C$

Change variables to $\boxed{x' = x}$

along $(1, -\frac{a}{b})$ $\Leftarrow$
$$x' = ax + by \qquad \underline{y' = bx - ay} \qquad (19)$$
$\rightarrow$ along the characteristic lines

Replace all x and y derivatives by x' and y' derivatives. By the chain rule,

$$\boxed{u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial x'}\underbrace{\frac{\partial x'}{\partial x}}_{a} + \frac{\partial u}{\partial y'}\underbrace{\frac{\partial y'}{\partial x}}_{b}} = a u_{x'} + b u_{y'}$$

and

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial x'}\underbrace{\frac{\partial x'}{\partial y}}_{b} + \frac{\partial u}{\partial y'}\underbrace{\frac{\partial y'}{\partial y}}_{-a} = b u_{x'} - a u_{y'}.$$

$\Rightarrow u_{x'} = 0$

Hence $a u_x + b u_y = a(a u_{x'} + b u_{y'}) + b(b u_{x'} - a u_{y'}) = (a^2 + b^2) u_{x'}$. Since $a^2 + b^2 \neq 0$, the equation takes the form $\underline{u_{x'} = 0}$ in the new (primed) variables. Thus the solution is $u = f(y') = f(bx - ay)$, with f an arbitrary function.

Example 1

Solve the PDE $4u_x - 3u_y = 0$, together with an auxiliary condition: $u(0, y) = y^3$.

$x' = x \qquad y' = bx - ay$

$au_x + bu_y = 0$ when $a = 0$

Since $a^2 + b^2 \neq 0$, $b \neq 0$. The PDE becomes $u_y = 0$

$u = f(x)$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x} = u_{x'} + b u_{y'}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y} = -a u_{y'}$$

$$0 = au_x + bu_y = a(u_{x'} + bu_{y'}) + b(-au_{y'}) = au_{x'}$$

$a \neq 0$, we have $u_{x'} = 0 \Rightarrow u = f(y') = f(bx - ay)$

§1.2 First-order linear equations: Constant coefficients

Example 1

Solve the PDE $4u_x - 3u_y = 0$, together with an auxiliary condition: $u(0, y) = y^3$.

We can rewrite the equation as

$$\nabla_{(4,-3)} u = 0. \qquad \frac{dy}{dx} = \frac{-3}{4}$$

This means that u is a constant along the direction $(4, -3)$. The characteristic lines are $-3x - 4y = c$. Thus the solution is

$c \in R$

$$u(x, y) = f(-3x - 4y). = g(3x+4y)$$

Since $u(0, y) = f(-4y) = y^3$. Let $w = -4y$. We find that $f(w) = -w^3/64$.

$\Leftrightarrow \; y = -\frac{w}{4}$

Therefore, the solution is

$$u(x, y) = -(-3x - 4y)^3/64 = (3x + 4y)^3/64$$

.

Example A (not from the textbook)

Solve $au_x + bu_y = c$, where a, b, c are constants and $a \neq 0$.

$au_x + bu_y = 0$ general solution $f(bx - ay)$

characteristic lines $\frac{dy}{dx} = \frac{b}{a} \Rightarrow y = \frac{b}{a}x + c$

New variables: $x' = x \quad y' = y - \frac{b}{a}x$

$u = \frac{c}{a}x + f(y - \frac{b}{a}x)$

$\Uparrow$

$u = \frac{c}{a}x' + f(y')$

$\Uparrow$

$u_x = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x} = \frac{\partial u}{\partial x'} - \frac{b}{a}\frac{\partial u}{\partial y'}$

$u_y = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y} = 0 + \frac{\partial u}{\partial y'}$

$au_x + bu_y = c \Rightarrow a(u_{x'} - \frac{b}{a}u_{y'}) + bu_{y'} = c \Rightarrow u_{x'} = \frac{c}{a}$

§1.2 First-order linear equations: Constant coefficients

$$\frac{c}{b}y - \frac{c}{a}x = -\frac{c}{ab}(bx - ay)$$

Example A (not from the textbook)

Solve $au_x + bu_y = c$, where a, b, c are constants and $a \neq 0$.

A is to be determined

We first look for a particular solution: $u_0(x, y) = Ax$. Substituting it into the equation leads to $aA + 0 = c$, so $A = c/a$. Thus,

$$u_1(x, y) = By$$

$$0 + bB = c \Rightarrow B = \frac{c}{b}$$

$$u_0(x, y) = \frac{c}{a}x$$

is a particular solution. On the other hand, the homogeneous equation: $au_x + bu_y = 0$ has the solution: $f(bx - ay)$. Therefore, the solution of the given equation is

$$u(x, y) = f(bx - ay) + \frac{c}{b}y$$

$$u(x, y) = f(bx - ay) + \frac{c}{a}x.$$

The idea used here is similar to the method of undetermined coefficients in ODE.

Example B (not from the textbook)

Solve $au_x + bu_y = cu$, where a, b, c are constants and $a \neq 0$.

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Solve $au_x + bu_y = cu$, where a, b, c are constants and $a \neq 0$.

$\frac{\partial \ln u}{\partial x} = \frac{1}{u} u_x$ $\qquad \frac{\partial \ln|u|}{\partial x} = \frac{1}{|u|} \frac{\partial |u|}{\partial x} = \begin{cases} \frac{1}{u}\frac{\partial u}{\partial x} & u>0 \\ \frac{1}{u}\frac{\partial u}{\partial x} & u<0 \end{cases}$

One solution is $u \equiv 0$. Now, we assume $u \neq 0$. We rewrite the equation as

$$a\frac{u_x}{u} + b\frac{u_y}{u} = c. \qquad u(x,y) = e^{\frac{c}{a}x}\, g(bx-ay)$$

$$\text{Let } v = \ln|u| \quad \Rightarrow \quad v_x = \frac{u_x}{u}, \quad v_y = \frac{u_y}{u}.$$

$$\Rightarrow \quad av_x + bv_y = c.$$

$$u(x,y) = \pm e^{f(bx-ay)} e^{\frac{c}{a}x}$$

From the previous example, we find

$$v(x,y) = f(bx-ay) + \frac{c}{a}x \quad \Rightarrow \quad |u(x,y)| = \exp\left(f(bx-ay) + \frac{c}{a}x\right),$$

which, together with $u \equiv 0$, is the solution of the given problem.

II. Variable coefficient case:

The equation $a(x,y)=1 \quad b(x,y)=y$

$$u_x + yu_y = 0 \tag{20}$$

is linear and homogeneous. We use the geometric method to find its general solution.

$$u_x + yu_y = 0 \iff (1, y)\cdot(u_x, u_y) = 0$$

$$\Updownarrow$$

$$\frac{(1,y)}{\sqrt{1+y^2}}\cdot \nabla u = 0$$

$$\frac{dy}{dx} = \frac{y}{1} \Rightarrow \frac{dy}{y} = dx$$

$$\ln|y| = x + c_1$$

$$|y| = e^{c_1}e^x$$

$$y = \pm e^{c_1}e^x \Rightarrow y = ce^x$$

§1.2 First-order linear equations

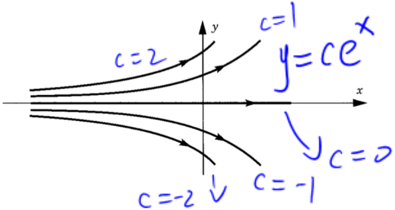
II. Variable coefficient case:

The equation

$$u_x + yu_y = 0 \tag{20}$$

is linear and homogeneous. We use the geometric method to find its general solution.

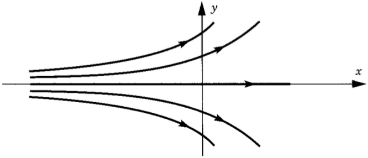
Equation (20) means that the directional derivative in the direction of $(1, y)$ is zero. The curves in the xy plane with $(1, y)$ as tangent vectors have slopes y.



$$u(x,y) = f(c) = f(ye^{-x})$$

Variable coefficient case (Cont'd):

The curves in the xy plane with $(1, y)$ as tangent vectors have slopes y.



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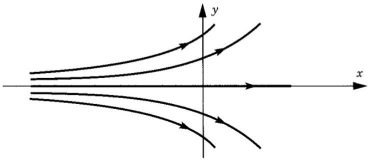

Their equations are

$$\frac{dy}{dx} = \frac{y}{1}.$$

§1.2 First-order linear equations

Variable coefficient case (Cont'd):

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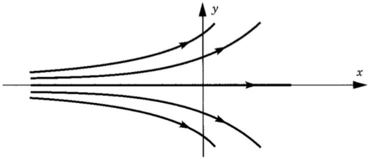

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This ODE has the solutions

$$y = Ce^x.$$

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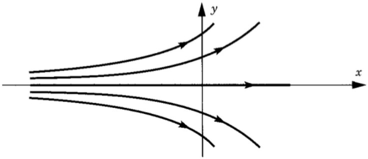


Their equations are

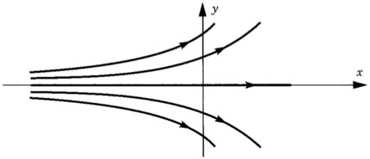
$$\frac{dy}{dx} = \frac{y}{1}.$$

This ODE has the solutions

$$y = Ce^x. \qquad C = ye^{-x}$$

There curves are called the characteristic curves of the PDE (20).

Variable coefficient case (Cont'd):



$x' = x \qquad y' = y e^{-x} \qquad u_x + y u_y = 0 \qquad u = f(y e^{-x})$

$$\Uparrow$$

$$u_x = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x} = u_{x'} - y e^{-x} u_{y'}$$

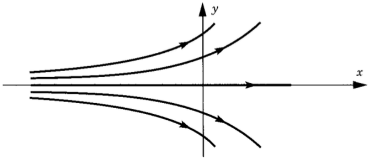
$$u = f(y')$$

$$u_y = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y} = 0 + e^{-x} u_{y'}$$

$$\Uparrow$$

$$u_x + y u_y = 0 \iff u_{x'} - y e^{-x} u_{y'} + y e^{-x} u_{y'} = 0 \Rightarrow u_{x'} = 0$$

Variable coefficient case (Cont'd):



$u(x, y)$ is constant on each characteristic curve $y = Ce^x$. So, $u(x, y)$ is only dependent on C, meaning that $u(x, y) = f(C)$ and f is an arbitrary function of a single variable. $y = Ce^x$ indicates that $C = e^{-x}y$. Hence, the general solution is

$$u(x, y) = f\left(e^{-x}y\right).$$

Example 2

Find the solution of $u_x + yu_y = 0$ that satisfies the auxiliary condition $u(0, y) = y^3$.

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We have found that the general solution to $u_x + yu_y = 0$ is

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$$u(x, y) = \underline{f(e^{-x}y)}.$$

Plugging $x = 0$ into it, we get

$$y^3 = u(0, y) = f(e^0 y) = f(y),$$

so that $f(y) = y^3$.

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Plugging $x = 0$ into it, we get

$$y^3 = u(0, y) = f(e^0 y) = f(y),$$

so that $f(y) = y^3$. Therefore,

$$u(x, y) = (e^{-x}y)^3 = e^{-3x}y^3.$$

Example 3

Solve the PDE

$$u_x + 2xy^2 u_y = 0. \tag{21}$$

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$u(x, y) = 0$

The characteristic curves satisfy the ODE

$$\frac{dy}{dx} = \frac{2xy^2}{1} = 2xy^2.$$

$y = 0$ is a solution

$y \neq 0 \quad \frac{dy}{y^2} = 2x\,dx \Rightarrow -\frac{1}{y} = x^2 - c \Rightarrow x^2 + \frac{1}{y} = c$

$f(x^2 + \frac{1}{y})$

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To solve the ODE, we separate variables: $dy/y^2 = 2xdx$; hence $-1/y = x^2 - C$, so that

$$y = \left(C - x^2\right)^{-1}.$$

There are the characteristic curves.

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There are the characteristic curves. $u(x,y)$ is constant on each characteristic curve. So $u(x,y) = f(C)$, where f is an arbitrary function.

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There are the characteristic curves. $u(x, y)$ is constant on each characteristic curve. So $u(x, y) = f(C)$, where f is an arbitrary function. Therefore, the general solution is

$$u(x, y) = f\left(x^2 + \frac{1}{y}\right).$$

In general, the equation

$$a(x, y)u_x + b(x, y)u_y = 0, \tag{22}$$

can be solved as long as the ODE

$$\frac{dy}{dx} = \frac{b(x, y)}{a(x, y)} \tag{23}$$

can be solved.

§1.2 First-order linear equations: Variable coefficients

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Moral

Solutions of PDEs generally depend on arbitrary functions (instead of arbitrary constants). You need an auxiliary condition if you want to determine a unique solution. Such conditions are usually called initial or boundary conditions.