Initial value problem–Cauchy Problem

The initial-value problem (IVP) is to solve the wave equation

$$u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty, \quad t > 0 \tag{17}$$

with the initial conditions

$f(x) + g(x) = \phi(x)$

$$u(x, 0) = \phi(x), \quad u_t(x, 0) = \psi(x), \quad -\infty < x < \infty, \tag{18}$$

$t = 0$

where ϕ, ψ are arbitrary functions of single variable x, and together are called the initial data of the IVP.

$u(x,t) = f(x+ct) + g(x-ct)$

Solution of the initial-value problem

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The solution to the IVP is easily found from the general solution,

$$u(x, t) = f(x + ct) + g(x - ct). \tag{19}$$

All we need to do is to find f and g from the initial conditions of the IVP.

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The solution to the IVP is easily found from the general solution,

$$u_t(x,t) = cf'(x+ct) - cg'(x-ct)$$

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$$u_t(x,0) = \psi(x) = cf'(x) - cg'(x)$$

All we need to do is to find f and g from the initial conditions of the IVP.

- To check the first initial condition, set $t = 0$,

$$u(x,0) = f(x) + g(x) = \phi(x). \tag{20}$$

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- To check the first initial condition, set $t = 0$,

$$u(x,0) = f(x) + g(x) = \phi(x). \tag{20}$$

- To check the second initial condition, we differentiate (19), and set $t = 0$,

$$u_t(x,0) = cf'(x) - cg'(x) = \psi(x). \tag{21}$$

$f'(x) + g'(x) = \phi'(x)$

Solution of the initial-value problem (Cont'd)

- Equations (20) and (21) form a system of two equations with two unknown functions f and g.

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  1. We first differentiate (20),
  $$f'(x) + g'(x) = \phi'(x) \tag{22}$$
  $$f'(x) - g'(x) = \frac{\psi(x)}{c}$$
  2. Together with (21), we have
  $$f'(x) = \frac{\phi'(x)}{2} + \frac{\psi(x)}{2c}, \quad g'(x) = \frac{\phi'(x)}{2} - \frac{\psi(x)}{2c}. \tag{23}$$

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  1. We first differentiate (20),
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  2. Together with (21), we have $f(x) = \int_0^x \frac{\phi'(s)}{2} ds + \int_0^x \frac{\psi(s)}{2c} ds$
  $$f'(x) = \frac{\phi'(x)}{2} + \frac{\psi(x)}{2c}, \quad g'(x) = \frac{\phi'(x)}{2} - \frac{\psi(x)}{2c}. \tag{23}$$
  $= \frac{\phi(x)}{2} - \frac{\phi(0)}{2} + \frac{1}{2c}\int_0^x \psi(s) ds$
  3. Integrating, we get
  $$f(x) = \frac{\phi(x)}{2} + \frac{1}{2c}\int_0^x \psi(s)ds + A, \quad g(x) = \frac{\phi(x)}{2} - \frac{1}{2c}\int_0^x \psi(s)ds + B, \tag{24}$$
  where A, B are two constants.

$g(x) = \int_0^x \frac{\phi'(s)}{2} ds - \int_0^x \frac{\psi(s)}{2c} ds = \frac{\phi(x)}{2} - \frac{\phi(0)}{2} - \frac{1}{2c}\int_0^x \psi(s) ds$

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3. Integrating, we get

$$f(x) = \frac{\phi(x)}{2} + \frac{1}{2c}\int_0^x \psi(s)ds + A, \quad g(x) = \frac{\phi(x)}{2} - \frac{1}{2c}\int_0^x \psi(s)ds + B, \tag{24}$$

where A, B are two constants.

$f(x)+g(x)=\phi(x)+A+B$

4. Because of (20), we have $A+B=0$.

Solution of the initial-value problem (Cont'd)

- Substituting $x + ct$ into the formula for f and $x - ct$ into that of g in (24), we get

$$u(x,t) = \underbrace{\frac{\phi(x+ct)}{2} + \frac{1}{2c}\int_0^{x+ct}\psi(s)ds + A}_{f(x+ct)} + \underbrace{\frac{\phi(x-ct)}{2} - \frac{1}{2c}\int_0^{x-ct}\psi(s)ds + B}_{g(x-ct)}. \tag{25}$$

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- The above solution simplifies to

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)ds. \tag{26}$$

IVP

Solution of the initial-value problem (Cont'd)

- Substituting $x + ct$ into the formula for f and $x - ct$ into that of g in (24), we get

$$u(x,t) = \frac{\phi(x+ct)}{2} + \frac{1}{2c}\int_0^{x+ct} \psi(s)ds + A + \frac{\phi(x-ct)}{2} - \frac{1}{2c}\int_0^{x-ct} \psi(s)ds + B. \tag{25}$$

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$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds. \tag{26}$$

- Equation (26) is the solution formula for the IVP, due to d'Alambert in 1746. So it is also called d'Alambert formula.

Theorem: Consider the initial value problem:

$$\begin{aligned} &u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty, \\ &u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \quad -\infty < x < \infty, \end{aligned} \tag{27}$$

where ϕ, ψ are given functions. Its solution is

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds. \tag{28}$$

The solution formula (28) is known as the **d'Alembert formula**. Moreover, we call this initial value problem on $\mathbb{R} = (-\infty, \infty)$ a **Cauchy problem**.

Example 1

Solve the initial value problem (27) with the initial data

$$\phi(x) \equiv 0, \quad \psi(x) = \cos x. \tag{29}$$

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Using the trigonometric identities for the sine of a sum and difference of two angles, we can simplify the above to get

$$u(x,t) = (1/c)\cos x \sin ct.$$

Example A (not from the textbook)

Solve $u_{tt} = c^2 u_{xx}$ with $u(x,0) = e^x$ and $u_t(x,0) = \cos x$.

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Solve $u_{tt} = c^2 u_{xx}$ with $u(x,0) = e^x$ and $u_t(x,0) = \cos x$.

By the d'Alembert formula,

$$\begin{aligned} u(x,t) &= \frac{e^{x-ct} + e^{x+ct}}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \cos s \, ds \\ &= e^x \frac{e^{-ct} + e^{ct}}{2} + \frac{1}{2c}\big(\sin(x+ct) - \sin(x-ct)\big) \\ &= e^x \cosh(ct) + \frac{1}{c}\cos(x)\sin(ct). \end{aligned}$$

This gives the unique solution to this initial value problem.

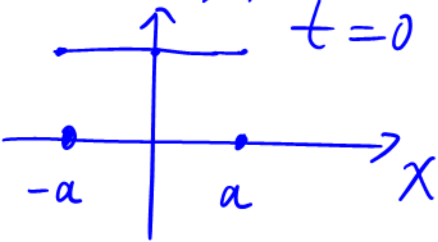
Example B: Initially at rest (not from the textbook)

Suppose that an infinite vibrating string is initially stretched into the shape of a single rectangular pulse is released at rest. The initial conditions are

$$u(x,0) = \phi(x) = \begin{cases} h, & |x| \le a, \\ 0, & |x| > a, \end{cases} \quad u_t(x,0) = \psi(x) = 0. \tag{30}$$

$a > 0$

Find the solution of the wave equation $u_{tt} - c^2 u_{xx} = 0$ under these conditions. $u(t,x)$



Example B: Initially at rest (not from the textbook) (Cont'd)

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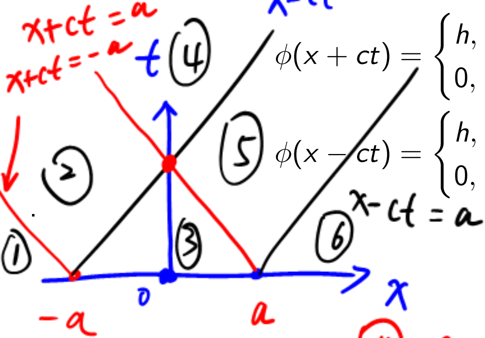
By the d'Alembert formula (32),

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right]. \qquad c > 0$$

Notice that

$x-ct=-a \qquad x-ct=a$



$$\phi(x+ct) = \begin{cases} h, & |x+ct| \le a, \\ 0, & |x+ct| > a, \end{cases} \iff -a \le x+ct \le a$$

$$\phi(x-ct) = \begin{cases} h, & |x-ct| \le a, \\ 0, & |x-ct| > a. \end{cases} \iff -a \le x-ct \le a$$

1 $u(x,t)=0$

2 $h/2$ 3 h

4 0 5 $h/2$ 6 0

Example B: Initially at rest (not from the textbook) (Cont'd)

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Hence, the solution

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right]$$

is piecewise defined in 4 different regions in the xt half-place (we consider only positive time $t \geq 0$).

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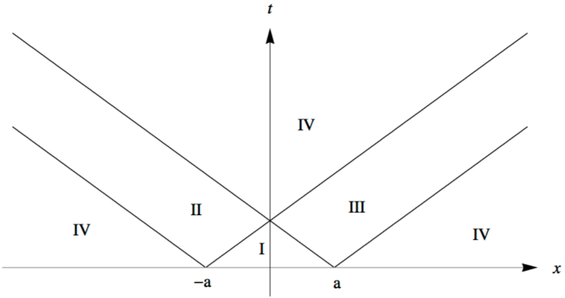
There four regions are

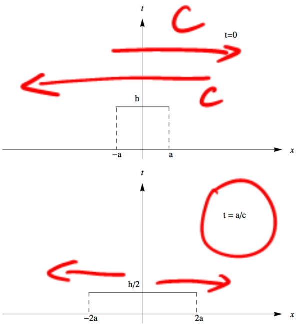
$$\begin{aligned}
\mathrm{I} &: \{|x+ct| \leq a, |x-ct| \leq a\}, \quad u(x,t) = h, \\
\mathrm{II} &: \{|x+ct| \leq a, |x-ct| > a\}, \quad u(x,t) = \tfrac{h}{2}, \\
\mathrm{III} &: \{|x+ct| > a, |x-ct| \leq a\}, \quad u(x,t) = \tfrac{h}{2}, \\
\mathrm{IV} &: \{|x+ct| > a, |x-ct| > a\}, \quad u(x,t) = 0.
\end{aligned}$$

Example B: Initially at rest (not from the textbook) (Cont'd)

$$
\begin{array}{rlll}
\mathrm{I} & : & \{|x+ct| \le a, |x-ct| \le a\}, & u(x,t) = h, \\
\mathrm{II} & : & \{|x+ct| \le a, |x-ct| > a\}, & u(x,t) = \frac{h}{2}, \\
\mathrm{III} & : & \{|x+ct| > a, |x-ct| \le a\}, & u(x,t) = \frac{h}{2}, \\
\mathrm{IV} & : & \{|x+ct| > a, |x-ct| > a\}, & u(x,t) = 0.
\end{array}
$$



Example B: Initially at rest (not from the textbook) (Cont'd)



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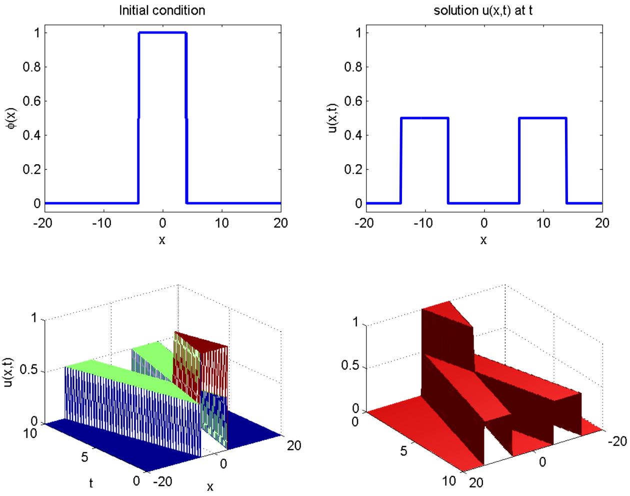

Example C

Solve

$$x^2 - 3xt - 4t^2 = (x-4t)(x+t)$$

$$u_{xx} - 3u_{xt} - 4u_{tt} = 0, \qquad \left(\frac{\partial}{\partial x} - 4\frac{\partial}{\partial t}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial t}\right) \tag{31}$$

$$u(x,0) = x^2, \quad u_t(x,0) = e^x.$$

$$A = \begin{pmatrix} 1 & -3/2 \\ -3/2 & -4 \end{pmatrix} \qquad \mathrm{Det}(A) = -4 - \frac{9}{4} < 0$$

Example C

Solve

$$\begin{aligned} &u_{xx} - 3u_{xt} - 4u_{tt} = 0, \\ &u(x,0) = x^2, \quad u_t(x,0) = e^x. \end{aligned} \tag{31}$$

We follow the idea of factoring the operator as for the wave equation.

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Factor the partial differential equation as

$$u_{xx} - 3u_{xt} - 4u_{tt} = (\partial_x + \partial_t)(\partial_x - 4\partial_t)u = 0.$$

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We follow the idea of factoring the operator as for the wave equation.

Factor the partial differential equation as

$$u_{xx} - 3u_{xt} - 4u_{tt} = (\partial_x + \partial_t)\overbrace{(\partial_x - 4\partial_t)u}^{v(x,t)} = 0.$$

Set $v = u_x - 4u_t$, then $v_x + v_t = 0$ and its general solution is $v(x,t) = h(x-t)$.

$$\frac{dt}{dx} = \frac{1}{1} \Rightarrow x - t = \text{const}.$$

$$\frac{dt}{dx} = \frac{-4}{1} \Rightarrow t + 4x = \text{const}$$

§2.1 The wave equation

Example C

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$$
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\end{aligned}
\tag{31}
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We follow the idea of factoring the operator as for the wave equation.

Factor the partial differential equation as

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try

Solve $u_x - 4u_t = h(x-t)$. One particular solution is $f(x-t)$ with $f'(s) = h(s)/5$. The general solution of the homogeneous equation $u_x - 4u_t = 0$ is $g(4x+t)$, so we have the general solution:

left going wave

right going wave

$$u(x,t) = f(x-t) + g(4x+t).$$

with speed $c = 4$

with speed $c = 1$

Example C (Cont'd)

Solve

$$
\begin{aligned}
&u_{xx} - 3u_{xt} - 4u_{tt} = 0,\\
&u(x,0) = x^2, \quad u_t(x,0) = e^x.
\end{aligned}
$$

Example C (Cont'd)

Solve

$$u_{xx} - 3u_{xt} - 4u_{tt} = 0,$$
$$u(x,0) = x^2, \quad u_t(x,0) = e^x.$$

Next, we impose the initial condition:

$$\begin{aligned}
& f(x) + g(4x) = x^2, \quad -f'(x) + g'(4x) = e^x; \\
\Rightarrow\ & f'(x) + 4g'(4x) = 2x, \quad -f'(x) + g'(4x) = e^x; \\
\Rightarrow\ & f'(x) = \frac{2}{5}x - \frac{4}{5}e^x, \quad g'(4x) = \frac{2}{5}x + \frac{1}{5}e^x; \\
\Rightarrow\ & f(x) = \frac{1}{5}x^2 - \frac{4}{5}e^x, \quad g'(s) = \frac{1}{10}s + \frac{1}{5}e^{s/4}; \\
\Rightarrow\ & f(x) = \frac{1}{5}x^2 - \frac{4}{5}e^x, \quad g(s) = \frac{1}{20}s^2 + \frac{4}{5}e^{s/4}.
\end{aligned}$$

Example C (Cont'd)

Solve

$$u_{xx} - 3u_{xt} - 4u_{tt} = 0,$$
$$u(x,0) = x^2, \quad u_t(x,0) = e^x.$$

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$$u_{xx} - 3u_{xt} - 4u_{tt} = 0,$$
$$u(x,0) = x^2, \quad u_t(x,0) = e^x.$$

Finally, we obtain the solution of this initial value problem is

$$\begin{aligned} u(x,t) &= \frac{1}{5}(x-t)^2 - \frac{4}{5}e^{x-t} + \frac{1}{20}(4x+t)^2 + \frac{4}{5}e^{x+t/4} \\ &= \frac{4}{5}\left(e^{x+t/4} - e^{x-t}\right) + x^2 + \frac{t^2}{4}. \end{aligned}$$