NANYANG TECHNOLOGICAL UNIVERSITY

SPMS/DIVISION OF MATHEMATICAL SCIENCES

2025/26 Semester 2 MH4110 Partial Differential Equations Tutorial 1, 22 January

Problem 1 Solve the equations

(a)
$$2\frac{dy}{dx} + (\tan x)y = \frac{(4x+5)^2}{\cos x}y^3$$

(b)
$$\frac{dy}{dx} + \frac{2}{x}y = (-x^2\cos x)y^2$$

[Solution:]

(a) We write the ODE in a standard form
$$\frac{dy}{dx} + \frac{\tan x}{2}y = \frac{(4x+5)^2}{2\cos x}y^3.$$
Dividing both sides by y^3 yields the equation
$$\frac{1}{y^3}\frac{dy}{dx} + \frac{\tan x}{2}\frac{1}{y^2} = \frac{(4x+5)^2}{2\cos x}.$$
We make the change of variable:
$$u = y^{-2} \Longrightarrow \frac{du}{dx} = -2y^{-3}\frac{dy}{dx} \Longrightarrow y^{-3}\frac{dy}{dx} = -\frac{1}{2}\frac{du}{dx},$$
Rewrite the new equation as the standard form:
$$\frac{du}{dx} - (\tan x)\cdot u = -\frac{(4x+5)^2}{\cos x}.$$
The integrating factor function can be chosen as
$$e^{\int -\tan x dx} = \cos x.$$
Thus,
$$\frac{d}{dx}(u\cos x) = -(4x+5)^2,$$
which has a general solution
$$u\cos x = -\frac{(4x+5)^3}{12} + C$$
or
$$y^{-2} = -\frac{(4x+5)^3}{12\cos x} + \frac{C}{\cos x}.$$

(b) For a Bernoulli's equation of order n in the standard form

$$\frac{dy}{dx} + p(x)y = q(x)y^n, \quad n \neq 1,$$

setting $u = y^{1-n}$, we transform it into a linear equation:

$$\frac{du}{dx} + \underbrace{(1-n)p(x)}_{P(x)}u(x) = \underbrace{(1-n)q(x)}_{Q(x)},$$

and the general solution is

$$y^{1-n} = e^{-(1-n)\int p(x)dx}\Big[(1-n)\int q(x)e^{(1-n)\int p(x)dx}dx + C\Big].$$

The ODE is in the standard form, where $n = 2$, and

$$p(x) = \frac{2}{x}, \qquad q(x) = -x^2 \cos x.$$

Using the solution formula, we obtain

$$\begin{aligned} y^{1-2} &= e^{-(1-2)\int \frac{2}{x}dx}\Big[(1-2)\int \left(-x^2\cos x\right) e^{(1-2)\int \frac{2}{x}dx}dx + C\Big] \\ &= e^{\int \frac{2}{x}dx}\Big[\int \left(x^2\cos x\right) e^{-\int \frac{2}{x}dx}dx + C\Big] \\ &= x^2\Big[\int \cos x dx + C\Big] \\ &= x^2 \sin x + Cx^2. \end{aligned}$$

Thus, the solution is

$$y = \frac{1}{x^2(\sin x + C)},$$

where C is an arbitrary constant.

Problem 2 (Ex. 2 on Page 5) Which of the following operators are linear?

(a) $\mathscr{L}u = u_x + xu_y$

(b) $\mathscr{L}u = u_x + uu_y$

(c) $\mathscr{L}u = u_x + u_y^2$

(d) $\mathscr{L}u = u_x + u_y + 1$

(e) $\mathscr{L}u = \sqrt{1+x^2}(\cos y)u_x + u_{yxy} - [\arctan(x/y)]u$

[Solution:] To check the linearity of an operator $\mathscr{L}$, we only need check if the following two equations are satisfied or not

$$\mathscr{L}(u+v) = \mathscr{L}u + \mathscr{L}v \qquad \mathscr{L}(cu) = c\mathscr{L}u.$$

Here u and v are the solutions to the discussing PDE. For a linear inhomogeneous PDE $\mathscr{L}u = g$, g is a function of independent variables.

(a) We have

$$\mathscr{L}(u+v) = (u+v)_x + x(u+v)_y = (u_x + xu_y) + (v_x + xv_y) = \mathscr{L}u + \mathscr{L}v,$$

and

$$\mathscr{L}(cu) = (cu)_x + x(cu)_y = c\,(u_x + xu_y) = c\mathscr{L}u.$$

So $\mathscr{L}$ here is a linear operator. The PDE (a) is linear homogeneous.

(b) By checking

$$\begin{aligned}\mathscr{L}(cu) &= (cu)_x + (cu)(cu)_y = cu_x + c^2 uu_y \\ &\neq c\,(u_x + uu_y) = c\mathscr{L}u,\end{aligned}$$

we know that $\mathscr{L}u = u_x + uu_y$ is nonlinear.

(c) We check

$$\begin{aligned}\mathscr{L}(u+v) &= (u+v)_x + (u+v)_y^2 = u_x + v_x + u_y^2 + v_y^2 + 2u_y v_y \\ &= \mathscr{L}u + \mathscr{L}v + 2u_y v_y \neq \mathscr{L}u + \mathscr{L}v\end{aligned}$$

and say that $\mathscr{L}u = u_x + u_y^2$ is nonlinear.

(d) We can check that

$$\begin{aligned}\mathscr{L}(u+v) &= (u+v)_x + (u+v)_y + 1 = u_x + v_x + u_y + v_y + 1 \\ &= \mathscr{L}u + \mathscr{L}v - 1 \neq \mathscr{L}u + \mathscr{L}v,\end{aligned}$$

indicating that $\mathscr{L}u = u_x + u_y + 1$ is nonlinear.

(e) We have

$$\mathscr{L}(u+v) = \sqrt{1+x^2}(\cos y)(u+v)_x + (u+v)_{yxy} - [\arctan(x/y)](u+v) = \mathscr{L}u + \mathscr{L}v,$$

and

$$\mathscr{L}(cu) = \sqrt{1+x^2}(\cos y)(cu)_x + (cu)_{yxy} - [\arctan(x/y)]cu = c\mathscr{L}u.$$

So $\mathscr{L}$ is a linear operator. The PDE (e) is linear homogeneous.

Problem 3 Prove that the first-order equation is linear.

$$a(x,y)u_x(x,y) + b(x,y)u_y(x,y) + c(x,y)u(x,y) = f(x,y)$$

[Solution:] To check the linearity of an operator $\mathscr{L}$, we only need check if the following two equations are satisfied or not

$$\mathscr{L}(u+v) = \mathscr{L}u + \mathscr{L}v \qquad \mathscr{L}(cu) = c\mathscr{L}u.$$

Here u and v are the solutions to the discussing PDE. For a linear inhomogeneous PDE $\mathscr{L}u = g$, g is a function of independent variables.

(a) We have

$$\begin{aligned}\mathscr{L}(u+v) =& a(x,y)[u_x(x,y)+v_x(x,y)]+b(x,y)[u_y(x,y)+u_y(x,y)]\\ &+c(x,y)[u(x,y)+v(x,y)]\\ =&a(x,y)u_x(x,y)+b(x,y)u_y(x,y)+c(x,y)u(x,y)\\ &+a(x,y)v_x(x,y)+b(x,y)v_y(x,y)+c(x,y)v(x,y)\\ =&\mathscr{L}u+\mathscr{L}v,\end{aligned}$$

and

(b)

$$\begin{aligned}\mathscr{L}(cu) &= a(x,y)[cu_x(x,y)]+b(x,y)[cu_y(x,y)]+c(x,y)[cu(x,y)]\\ &= c\left[a(x,y)u_x(x,y)+b(x,y)u_y(x,y)+c(x,y)u(x,y)\right]\\ &= c\mathscr{L}u.\end{aligned}$$

So $\mathscr{L}$ here is a linear operator. The PDE is linear non-homogeneous.

Problem 4 (Ex. 3 on Page 5) For each of the following equations, state the order and whether it is nonlinear, linear inhomogeneous, or linear homogeneous; provide reasons.

(a) $u_t - u_{xx} + 1 = 0$
(b) $u_t - u_{xx} + xu = 0$
(c) $u_t - u_{xxt} + uu_x = 0$
(d) $u_{tt} - u_{xx} + x^2 = 0$
(e) $iu_t - u_{xx} + u/x = 0$
(f) $u_x\left(1+u_x^2\right)^{-1/2} + u_y\left(1+u_y^2\right)^{-1/2} = 0$
(g) $u_x + e^y u_y = 0$
(h) $u_t + u_{xxxx} + \sqrt{1+u} = 0$

[Solution:]

- The order of a PDE is the order of the highest partial derivative that appears in the equation.
- To check the linearity of an operator $\mathscr{L}$, we only need check if the following two equations are satisfied or not

$$\mathscr{L}(u+v) = \mathscr{L}u + \mathscr{L}v \qquad \mathscr{L}(cu) = c\mathscr{L}u.$$

  Here u and v are the solutions to the discussing PDE. For a linear inhomogeneous PDE $\mathscr{L}u = g$, g is a function of independent variables.

(a) $u_t - u_{xx} + 1 = 0$ is a second order PDE because that its highest partial derivative u_{xx} is second order. If we assume that $g(t,x) = -1$ and define $\mathscr{L}$ as $\mathscr{L} = \partial/\partial t - \partial^2/\partial x^2$, then $u_t - u_{xx} + 1 = 0$ has a form of $\mathscr{L}u = g$. We further have

$$\mathscr{L}(u+v) = (u+v)_t - (u+v)_{xx} = (u_t - u_{xx}) + (v_t - v_{xx}) = \mathscr{L}u + \mathscr{L}v,$$

and

$$\mathscr{L}(cu) = (cu)_t - (cu)_{xx} = c\,(u_t - u_{xx}) = c\mathscr{L}u.$$

So $\mathscr{L}$ here is a linear operator. The PDE (a) is linear inhomogeneous given that $g(t,x) = -1$ is nonzero.

(b) Second order and linear homogeneous. The linear operator is $\mathscr{L} = \partial/\partial t - \partial^2/\partial x^2 + x$.

(c) Third order and nonlinear.

$$\begin{aligned}\mathscr{L}(cu) &= (cu)_t - (cu)_{xxt} + (cu)(cu)_x \\ &= c\,(u_t - u_{xxt} + cuu_x) \\ &\neq c\,(u_t - u_{xxt} + uu_x) = c\mathscr{L}u.\end{aligned}$$

(d) Second order and linear inhomogeneous. The linear operator is $\mathscr{L} = \partial^2/\partial t^2 - \partial^2/\partial x^2$ and the inhomogeneous term is $g(t,x) = -x^2$.

(e) Second order and linear homogeneous. The linear operator is $\mathscr{L} = i\partial/\partial t - \partial^2/\partial x^2 + 1/x$.

(f) First order and nonlinear.

(g) First order and linear homogeneous.

(h) Fourth order and nonlinear.

$$\begin{aligned}\mathscr{L}(u+v) &= (u+v)_t + (u+v)_{xxxx} + \sqrt{1+u+v} \\ &= \left(u_t - u_{xxxx} + \sqrt{1+u}\right) + \left(v_t - v_{xxxx} + \sqrt{1+v}\right) \\ &\quad + \sqrt{1+u+v} - \sqrt{1+u} - \sqrt{1+v} \\ &= \mathscr{L}(u) + \mathscr{L}(v) + \sqrt{1+u+v} - \sqrt{1+u} - \sqrt{1+v} \\ &\neq \mathscr{L}(u) + \mathscr{L}(v).\end{aligned}$$

Problem 5 (Ex. 4 on Page 6) Show that the difference of two solutions of an inhomogeneous linear equation $\mathscr{L}u = g$ with the same g is a solution of the homogeneous equation $\mathscr{L}u = 0$.

[Solution:] Assume that u_1 and u_2 are two different solutions of the linear equation $\mathscr{L}u = g$, then $\mathscr{L}u_1 = g$ and $\mathscr{L}u_2 = g$. Subtracting the latter equation from the former one gives

$$\mathscr{L}u_1 - \mathscr{L}u_2 = 0.$$

$\mathscr{L}$ is a linear operator, implying that $c\mathscr{L}u = \mathscr{L}(cu)$ for any constant c. So $-\mathscr{L}u_2 = -1 \cdot \mathscr{L}u_2 = \mathscr{L}(-1 \cdot u_2) = \mathscr{L}(-u_2)$. Again, that $\mathscr{L}$ is a linear operator implies that

$$\mathscr{L}u_1 + \mathscr{L}(-u_2) = \mathscr{L}(u_1 + (-u_2)) = \mathscr{L}(u_1 - u_2).$$

Therefore,

$$0 = \mathscr{L}u_1 - \mathscr{L}u_2 = \mathscr{L}u_1 + \mathscr{L}(-u_2) = \mathscr{L}(u_1 - u_2)$$

and $u_1 - u_2$ is a solution of the homogeneous equation $\mathscr{L}u = 0$.

Problem 6 (Ex. 12 on Page 6) Verify by direct substitution that

$$u_n(x, y) = \sin(nx)\sinh(ny)$$

is a solution of $u_{xx} + u_{yy} = 0$ for every $n > 0$.

[Solution:] It is clear that

$$\partial_{xx}u_n = -n^2 u_n, \quad \partial_{yy}u_n = n^2 u,$$

where we recall that

$$\text{sinh}s = \frac{e^s - e^{-s}}{2}, \ \ \text{cosh}s = \frac{e^s + e^{-s}}{2}, \ \ \frac{d}{ds}\text{sinh}s = \text{cosh}s, \ \ \frac{d}{ds}\text{cosh}s = \text{sinh}s.$$

Thus, the given u_n is a solution of the Laplace equation.