MH4110 Partial Differential Equations

Chapter 1 - Where PDEs come from

Synopsis

1. Basic concepts.
2. First-Order Linear Equations.
3. Flows, Vibrations, and Diffusions.
4. Initial and Boundary Conditions.
5. Well-Posed Problems.
6. Types of Second-Order Equations.

Some notations

$$u_x = \frac{\partial u}{\partial x} = \partial_x u, \quad u_{xy} = \frac{\partial^2 u}{\partial x \partial y} = \partial_{xy} u, \tag{1}$$

$$\nabla u = \mathrm{grad}\, u = u_x \mathbf{i} + u_y \mathbf{j} = (u_x, u_y); \quad \Delta u = u_{xx} + u_{yy}, \tag{2}$$

$$\mathrm{div}\, \mathbf{F} = \nabla \cdot \mathbf{F} = u_x + v_y + w_z, \tag{3}$$

where $\mathbf{F} = (u, v, w)$, and

$$\nabla \times \mathbf{F} = \mathrm{curl}\, \mathbf{F} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} \tag{4}$$

$$= (w_y - v_z)\mathbf{i} + (u_z - w_x)\mathbf{j} + (v_x - u_y)\mathbf{k}.$$

Notice that

$$\mathrm{div}(\nabla u) = \Delta u. \tag{5}$$

§1.1 Basic concepts

Definition

A **partial differential equation**, PDE in short, is an identity that relates the independent variables x, y, $\cdots$, the dependent variable u, and the partial derivatives of u. A PDE can be expressed in the general form

$$F(x, y, \cdots, u, u_x, u_y, \cdots, u_{xx}, u_{xy}, u_{yy}, \cdots) = 0.$$

Example A (not in the textbook)

For a function $u(x, t)$ of one spatial variable x and the time variable t, the heat equation is

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = 0.$$

Examples of PDEs:

1. $u_x + u_y = 0$ (transport)
2. $u_x + yu_y = 0$ (transport)
3. $u_x + uu_y = 0$ (shock wave)
4. $u_{xx} + u_{yy} = \Delta u = 0$ (Laplace's equation)
5. $u_{tt} - u_{xx} + u^3 = 0$ (wave with interaction)
6. $u_t + uu_x + u_{xxx} = 0$ (dispersive wave)
7. $u_{tt} + u_{xxxx} = 0$ (vibrating bar)
8. $u_t - \mathrm{i}u_{xx} = 0 \quad (\mathrm{i} = \sqrt{-1})$ (quantum mechanics)
9. $|\nabla T(x, y, z)| = \frac{1}{c(x,y,z)}$ (Eikonal equation)

§1.1 Basic concepts

Definition

The **order** of a PDE is the order of the highest partial derivative that appears in the equation.

Definition

A **solution** of a PDE is a function $u(x, y, \cdots)$ that satisfies the equation identically, at least in some region of the $x, y, \cdots$ variables.

The second order heat equation

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = 0$$

has a solution

$$u(t, x) = 2\alpha t + x^2.$$

Definition

$\mathscr{L}$ is an operator. If

$$\mathscr{L}(u+v) = \mathscr{L}u + \mathscr{L}v \qquad \mathscr{L}(cu) = c\mathscr{L}u \tag{6}$$

holds for any functions u, v and any constant c, then $\mathscr{L}$ is called linear operator.

Definition

The equation

$$\mathscr{L}u = g \tag{7}$$

is called linear if $\mathscr{L}$ is a linear operator. g is a given function of the independent variables. If $g = 0$, equation (7) is called a homogeneous linear equation, otherwise it is called an inhomogeneous linear equation.

Superposition Principle

- For the equation $\mathscr{L} u = 0$, if u and v are both solutions, so is $(u + v)$. If $u_1, u_2, \cdots, u_n$ are all solutions, so is any linear combination

$$c_1 u_1(x) + c_2 u_2(x) + \cdots + c_n u_n(x) = \sum_{j=1}^{n} c_j u_j(x) \qquad (c_j = \text{constants}).$$

- The sum of a homogeneous solution and an inhomogeneous solution is an inhomogeneous solution.

Review on second order linear ODE with constant coefficients

Find the general solution of

$$y'' + ay' + by = 0 \tag{8}$$

where a and b are constants.

Look for a solution of the form $y = e^{\lambda x}$. Plugging into eqn. (8), we find that $e^{\lambda x}$ is a solution if and only if

$$\lambda^2 + a\lambda + b = 0.$$

1. If $a^2 - 4b > 0$, the characteristic function has two distinct real roots λ_1 and λ_2. The general solution is $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$.
2. If $a^2 - 4b = 0$, the characteristic function has one real root λ. The general solution is $y = c_1 e^{\lambda x} + c_2 x e^{\lambda x}$.
3. If $a^2 - 4b < 0$, the characteristic function has a pair of complex conjugate roots $\lambda = \alpha \pm i\beta$. The general solution is $y = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x)$.

Example 1

Find all $u(x, y)$ satisfying the equation $u_{xx} = 0$.

. . . .

Example 2

Solve the PDE $u_{xx} + u = 0$. Here $u = u(x, y)$.

. . . .

Example 3

Solve the PDE $u_{xy} = 0$. Here $u = u(x, y)$.

. . . .

§1.1 Basic concepts

Moral

- For an ODE of order m, we get m arbitrary constants in the solution.
- A PDE has arbitrary functions in its solution.

A few things to keep in mind

- Mixed derivatives are equal throughout this course: $u_{xy} = u_{yx}$.
- The chain rule is used frequently in PDEs; for instance,

$$\frac{\partial}{\partial x}[f(g(x,t))] = f'(g(x,t)) \cdot \frac{\partial g}{\partial x}(x,t).$$

- Derivatives of integrals like $I(t) = \int_{a(t)}^{b(t)} f(x,t)dx$

$$\frac{dI}{dt} = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}(x,t)dx + f(b(t),t)b'(t) - f(a(t),t)a'(t).$$

Green's Theorem

Let D be a bounded plane domain with a piecewise C^1 boundary curve C $=$ bdy D. Consider C to be parametrized so that it is traversed once with D on the left. Let $p(x,y)$ and $q(x,y)$ be any C^1 functions defined on $\bar{D} = D \cup C$. Then

$$\iint_D (q_x - p_y)dxdy = \int_C pdx + qdy. \tag{9}$$

Divergence Theorem

Let D be a bounded spatial domain with a piecewise C^1 boundary surface S. Let $\mathbf{n}$ be the unit outward normal vector on S. Let $\mathbf{f}(\mathbf{x})$ be any C^1 vector field on $\bar{D} = D \cup S$.Then

$$\iiint_D \nabla \cdot \mathbf{f} d\mathbf{x} = \iint_S \mathbf{f} \cdot \mathbf{n} dS. \tag{10}$$

Summary

A **differential equation**, DE in short, is an equation involving an unknown function and its derivatives. For example,

$$\frac{du}{dx} = 5x + 3 \quad (\text{or} \quad u' = 5x + 3), \tag{11}$$

and

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = \cos(xy) \quad (\text{or} \quad \Delta u = \cos(xy)). \tag{12}$$

- If the unknown function is a function of a single variable (e.g., $u(x), u(t), \cdots$) and the involved derivatives are ordinary derivatives, then this DE is an **ordinary differential equation (ODE)**, e.g., the equation (11).
- A **partial differential equation (PDE)** is one involving a function of two or more variables, in which the derivatives are partial derivatives, for example, the equation (12).

In general, we are interested in solving the first-order linear equation:

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y), \tag{13}$$

where a, b, f are continuous functions in some domain Ω. If $f \equiv 0$, then the equation is homogeneous, otherwise it is inhomogeneous. We will see that any linear first-order PDE can be reduced to an ODE, which will then allow us to tackle it with already familiar methods from ODEs.

I. Constant coefficient case:

We start with one simplest case of (13) with a, b being constants and $c = f \equiv 0$. More precisely, we consider

$$au_x + bu_y = 0, \quad a^2 + b^2 \neq 0. \tag{14}$$

A. Geometric Method

Directional derivative

Let $\mathbf{v} = (a, b) \neq \mathbf{0}$ be a given vector in $\mathbb{R}^2$. The directional derivative of u along $\mathbf{v}$ at (x, y) is defined by

$$\nabla_{\mathbf{v}} u = \nabla u \cdot \frac{\mathbf{v}}{|\mathbf{v}|} = \frac{au_x + bu_y}{\sqrt{a^2 + b^2}}. \tag{15}$$

In particular, if $\mathbf{v} = \mathbf{i} = (1, 0)$, then it reduces to $\partial_x u$, while if $\mathbf{v} = \mathbf{j} = (0, 1)$, it becomes $\partial_y u$.

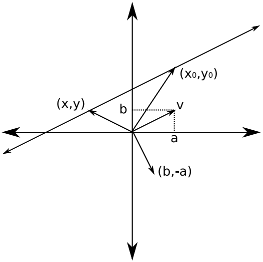
We can rewrite eqn. (14) as

$$au_x + bu_y = 0 \quad \Leftrightarrow \quad (a, b) \cdot \nabla u = 0.$$

Setting $\mathbf{v} = (a, b)$, we have

$$\nabla_{\mathbf{v}} u = 0. \tag{16}$$

A. Geometric Method



$\nabla_{\mathbf{v}} u = 0$ means that $u(x, y)$ does not change along the direction (a, b), in other words, $u(x, y)$ must be a constant along the lines with this direction. The lines parallel to (i.e., tangent to) (a, b) have the equations:

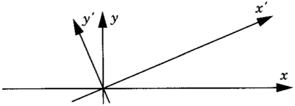
$$bx - ay = c, \tag{17}$$

where c is an arbitrary constant. These lines are called the characteristic lines. If $u(x, y)$ does not change along these lines, the solution of (14) is

$$u(x, y)|_{bx-ay=c} = f(c) \quad \Rightarrow \quad u(x, y) = f(bx - ay), \tag{18}$$

where f is any function of one variable.

B. Coordinate Method



Change variables to

$$x' = ax + by \qquad y' = bx - ay \tag{19}$$

Replace all x and y derivatives by x' and y' derivatives. By the chain rule,

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x} = au_{x'} + bu_{y'}$$

and

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y} = bu_{x'} - au_{y'}.$$

Hence $au_x + bu_y = a(au_{x'} + bu_{y'}) + b(bu_{x'} - au_{y'}) = (a^2 + b^2)u_{x'}$. Since $a^2 + b^2 \neq 0$, the equation takes the form $u_{x'} = 0$ in the new (primed) variables. Thus the solution is $u = f(y') = f(bx - ay)$, with f an arbitrary function.

§1.2 First-order linear equations: Constant coefficients

Example 1

Solve the PDE $4u_x - 3u_y = 0$, together with an auxiliary condition: $u(0, y) = y^3$.

. . . .

§1.2 First-order linear equations: Constant coefficients

Example A (not from the textbook)

Solve $au_x + bu_y = c$, where a, b, c are constants and $a \neq 0$.

. . . .

§1.2 First-order linear equations: Constant coefficients

Example B (not from the textbook)

Solve $au_x + bu_y = cu$, where a, b, c are constants and $a \neq 0$.

. . . .

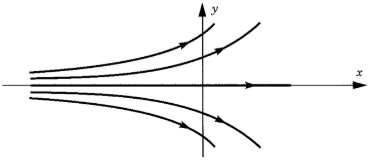
II. Variable coefficient case:

The equation

$$u_x + yu_y = 0 \tag{20}$$

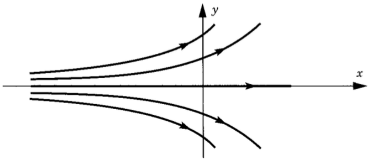
is linear and homogeneous. We use the geometric method to find its general solution.

Equation (20) means that the directional derivative in the direction of $(1, y)$ is zero. The curves in the xy plane with $(1, y)$ as tangent vectors have slopes y.



Variable coefficient case (Cont'd):

The curves in the xy plane with $(1, y)$ as tangent vectors have slopes y.



Their equations are

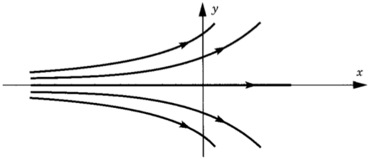
$$\frac{dy}{dx} = \frac{y}{1}.$$

This ODE has the solutions

$$y = Ce^{x}.$$

There curves are called the characteristic curves of the PDE (20).

Variable coefficient case (Cont'd):



$u(x,y)$ is constant on each characteristic curve $y = Ce^x$. So, $u(x,y)$ is only dependent on C, meaning that $u(x,y) = f(C)$ and f is an arbitrary function of a single variable. $y = Ce^x$ indicates that $C = e^{-x}y$. Hence, the general solution is

$$u(x,y) = f\left(e^{-x}y\right).$$

Example 2

Find the solution of $u_x + yu_y = 0$ that satisfies the auxiliary condition $u(0, y) = y^3$.

. . . .

Example 3

Solve the PDE

$$u_x + 2xy^2 u_y = 0. \tag{21}$$

. . . .

In general, the equation

$$a(x,y)u_x + b(x,y)u_y = 0, \tag{22}$$

can be solved as long as the ODE

$$\frac{dy}{dx} = \frac{b(x,y)}{a(x,y)} \tag{23}$$

can be solved.

Moral

Solutions of PDEs generally depend on arbitrary functions (instead of arbitrary constants). You need an auxiliary condition if you want to determine a unique solution. Such conditions are usually called initial or boundary conditions.

§1.3 Flows, Vibrations, and Diffusions

History: The subject of PDEs was practically a branch of physics until the 20th century. Many PDEs are from physical problems.

Derive the following PDEs from physical principles

- Transport equation
- Wave equation
- Heat equation
- Laplace equation

Remark: Most often in physical problems, the independent variables are those of space x, y, z, and time t.

Example 1. Simple Transport

- On a very long and straight freeway, all the cars are running at the same speed c.
- For simplicity, we assume that no cars enter or exit the freeway for quite a while.
- Let $u(x, t)$ be the car density at time t and position x. For instance $u(x, t)$ can be the number of cars at the time t in a distance of 1 km centered at x.

Example 1. Simple Transport (Cont'd)

- The number of cars in the interval $[0, b]$ at the time t is $M = \int_0^b u(x, t)dx$.
- At the later time $t + h$, the same car has moved to the right by $c \cdot h$ km.
- Hence

$$M = \int_0^b u(x, t)dx = \int_{ch}^{b+ch} u(x, t + h)dx. \tag{24}$$

- Differentiating with respect to b, we get

$$u(b, t) = u(b + ch, t + h). \tag{25}$$

- Change b to x, we have $u(x + ch, t + h) - u(x, t) = 0$.
- Dividing both sides by h and taking the limit as $h \to 0$, we get

$$u_t(x, t) + cu_x(x, t) = 0. \tag{26}$$

Example 1. Simple Transport (Cont'd)

Transport equation

$$u_t(x,t) + cu_x(x,t) = 0. \tag{27}$$

The characteristic lines are

$$ct - x = W, \tag{28}$$

where W are arbitrary constants. The general solution is $f(ct - x)$. f is arbitrary.

Example 2. Vibrating string: Wave-motion

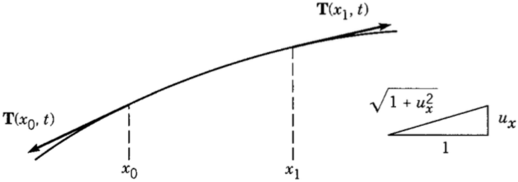
Consider a flexible, elastic homogeneous string or thread of length l, which undergoes relatively small transverse vibrations. For instance, it could be a guitar string or a plucked violin string.

YouTube Example: Guitar Strings Oscillating in HD 60 fps https://www.youtube.com/watch?v=8YGQmV3NxMI.

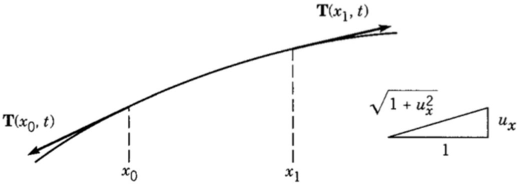
Example 2. Vibrating string: Wave-motion (Cont'd)

A plucked string. ρ is the linear density (units of mass per unit of length) and is constant along the entire length of the string. $u(t, x)$ is the displacement from equilibrium position at time t and position x.

Ignore all the forces on the string except for its tension $\mathbf{T}(x, t)$. Consider the motion of a tiny portion of the string sitting atop the interval $[x_0, x_1]$.



Example 2. Vibrating string: Wave-motion (Cont'd)

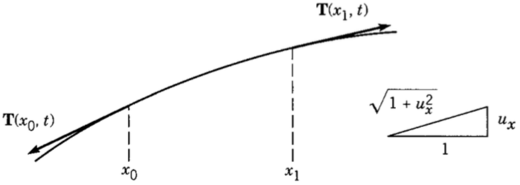


Because the string is perfectly flexible, the tension $\mathbf{T}(x,t)$ is directed tangentially along the string. Given the slope of the string is $u_x(x,t)$, the directions of the tension at the two ends x_0 and x_1 are

$$\mathbf{v}_0 = \left(-\frac{1}{\sqrt{1+u_x^2(x_0,t)}}, -\frac{u_x(x_0,t)}{\sqrt{1+u_x^2(x_0,t)}} \right), \tag{29}$$

$$\mathbf{v}_1 = \left(\frac{1}{\sqrt{1+u_x^2(x_1,t)}}, \frac{u_x(x_1,t)}{\sqrt{1+u_x^2(x_1,t)}} \right). \tag{30}$$

Example 2. Vibrating string: Wave-motion (Cont'd)



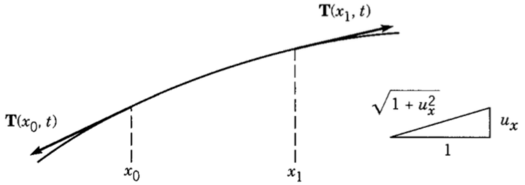
Let $T(x,t)$ be the magnitude of the tension $\mathbf{T}(x,t)$. In the longitudinal direction x, Newton's second law is

$$a_x \int_{x_0}^{x_1} \rho dx = T(x_1,t)\frac{1}{\sqrt{1+u_x^2(x_1,t)}} - T(x_0,t)\frac{1}{\sqrt{1+u_x^2(x_0,t)}}. \qquad (31)$$

Since we have assumed/observed that the motion is purely transverse, there is no longitudinal motion and hence the longitudinal acceleration $a_x = 0$. So

$$T(x_1,t)\frac{1}{\sqrt{1+u_x^2(x_1,t)}} = T(x_0,t)\frac{1}{\sqrt{1+u_x^2(x_0,t)}}. \qquad (32)$$

Example 2. Vibrating string: Wave-motion (Cont'd)



In the transverse direction u, Newton's second law is

$$\int_{x_0}^{x_1} a_u(x,t)\rho dx = T(x_1,t)\frac{u_x(x_1,t)}{\sqrt{1+u_x^2(x_1,t)}} - T(x_0,t)\frac{u_x(x_0,t)}{\sqrt{1+u_x^2(x_0,t)}}. \tag{33}$$

Here a_u is the transverse acceleration u_{tt}. Using the final relationship on the last slide, we have

$$\int_{x_0}^{x_1} u_{tt}(x,t)\rho dx = T(x_0,t)\frac{u_x(x_1,t)}{\sqrt{1+u_x^2(x_0,t)}} - T(x_0,t)\frac{u_x(x_0,t)}{\sqrt{1+u_x^2(x_0,t)}}. \tag{34}$$

Example 2. Vibrating string: Wave-motion (Cont'd)

Divide both sides by $x_1 - x_0$, we have

$$\frac{\int_{x_0}^{x_1} u_{tt}(x,t)\rho dx}{x_1 - x_0} = \frac{T(x_0,t)}{\sqrt{1+u_x^2(x_0,t)}} \frac{u_x(x_1,t) - u_x(x_0,t)}{x_1 - x_0}. \tag{35}$$

Passing to the limit $x_1 \to x_0$ gives

$$\rho u_{tt}(x_0,t) = \frac{T(x_0,t)}{\sqrt{1+u_x^2(x_0,t)}} u_{xx}(x_0,t). \tag{36}$$

Approximation: Assume that the motion is small, i.e., $|u_x| \approx 0$. Then,

$$|u_x| \approx 0 \quad \Rightarrow \quad \sqrt{1+u_x^2} = 1 + \frac{1}{2}u_x^2 + \cdots \approx 1.$$

Therefore, the equation in traverse direction becomes

$$\rho u_{tt}(x,t) = T(x,t) u_{xx}(x,t). \tag{37}$$

Example 2. Vibrating string: Wave-motion (Cont'd)

The first of the three fundamental PDEs of this course:

Wave equation

Assume that T is a constant, we obtain the wave equation:

$$u_{tt} = c^2 u_{xx} \quad \text{with} \quad c = \sqrt{\frac{T}{\rho}}, \tag{38}$$

where c is known as the **wave speed.**

Example 2. Vibrating string: Wave-motion (Cont'd)

Some variants

(i) If significant air resistance r is present, we have an extra term proportional to the speed u_t :

$$u_{tt} - c^2 u_{xx} + r u_t = 0, \quad r > 0. \tag{39}$$

(ii) If there is a traverse elastic force, we have an extra term proportional to the displacement u as in a coiled spring, thus

$$u_{tt} - c^2 u_{xx} + k u = 0, \quad k > 0. \tag{40}$$

(iii) If there is an external force, it appears as an extra term:

$$u_{tt} - c^2 u_{xx} = f(x, t). \tag{41}$$

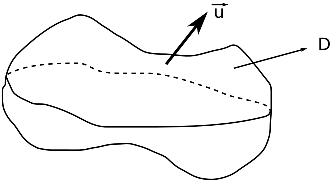
(iv) In the multiple dimensional case, u_{xx} is replaced by $\Delta u = u_{xx} + u_{yy}$ or $\Delta u = u_{xx} + u_{yy} + u_{zz}$. See Example 3 on Page 13 of the textbook for the derivation of the multidimensional models.

Example 5. Heat flow / Diffusion

Let $D \in \mathbb{R}^3$ be a domain in space, and let $H(t)$ be the amount of heat (or energy) in D. Let $u(x, y, z, t)$ be the temperature at any point (x, y, z) at time t. Then

$$H(t) = \iiint_D c\rho u(x, y, z, t)\, dxdydz, \tag{42}$$

where ρ is the density of the material, and c is a "specific heat".



The change in the heat is

$$\frac{dH}{dt} = \iiint_D c\rho u_t(x, y, z, t)\, dxdydz. \tag{43}$$

Example 5. Heat flow / Diffusion (Cont'd)

Assume that outside the body the "space" is colder, so the energy "flux" (direction of heat flow) will go from inside to outside and will "diffuse" through the boundary ∂D.

Fourier's law

Heat flows from hot to cold regions proportionally to the temperature gradient.

Therefore, we have

$$\frac{dH}{dt} = \iint_{\partial D} \kappa(\mathbf{n} \cdot \nabla u)\, dS, \tag{44}$$

where κ is a proportionality factor (the "heat conductivity").

Divergence Theorem: Let D be a bounded spatial domain with a piecewise C^1 boundary surface S. Let $\mathbf{n}$ be the unit outward normal vector on S. Let $\mathbf{F}$ be any C^1 vector field on $\bar{D} = D \cup S$. Then

$$\iiint_D \nabla \cdot \mathbf{F} dxdydz = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS. \tag{45}$$

Example 5. Heat flow / Diffusion (Cont'd)

Then by the divergence theorem (see (45)), we have

$$\frac{dH}{dt} = \iint_{\partial D} \kappa(\mathbf{n} \cdot \nabla u)\, dS = \iiint_D \nabla \cdot (\kappa \nabla u) dxdydz. \tag{46}$$

Repeat equation (43) here

$$\frac{dH}{dt} = \iiint_D c\rho u_t(x, y, z, t)\, dxdydz.$$

Since D is an arbitrary domain, from the above two equations we get **the second of the three fundamental PDEs of this course**:

Heat equation

$$c\rho \frac{\partial u}{\partial t} = \nabla \cdot (\kappa \nabla u). \tag{47}$$

If c, ρ and κ are constants, we obtain the equation:

$$\frac{\partial u}{\partial t} = k\Delta u, \quad k = \frac{\kappa}{c\rho}. \tag{48}$$

Example 6. Laplace equation

- A fireplace (with sufficient fuel supply) is burning to keep the room warm.
- The heat is not expected to be evenly distributed throughout the room. The closer to the fireplace, the warmer you feel.
- The temperature of the room eventually reaches a steady state. At any position $\mathbf{x} = (x, y, z)$, the temperature remains the same. This steady state can be modeled by **Laplace equation**.

Example 6. Laplace equation (Cont'd)

Laplace equation is the third one of the three fundamental PDEs of this course. It can be derived from the heat equation (48) by assuming that the heat flow does not change with time, i.e., $u_t = 0$, so u does not depend on time any more in this situation.

Laplace equation

$$\Delta u = u_{xx} + u_{yy} + u_{zz} = 0. \tag{49}$$

Its solutions are called harmonic solutions.

The one-dimensional Laplace equation is $u_{xx} = 0$. Its general solution is a linear function: $u(x) = c_1 x + c_2$, for any constants c_1, c_2. The multi-dimensional Laplace equation is much more interesting and far more difficult to solve.

Poisson equation

The inhomogeneous version of (49):

$$\Delta u = f, \tag{50}$$

where $f \neq 0$, is called the Poisson equation.

- PDEs typically have many solutions, so we need to impose some conditions in order to get a particular solution or to single out a solution.
- The conditions are from physical applications and they come in two varieties, initial conditions and boundary conditions.

Initial condition: Specify the physical state at a particular time t_0

- Heat/Diffusion equation: The initial condition (e.g., temperature, concentration, ...)

$$u_t - ku_{xx} = 0; \quad u(x,0) = \phi(x). \tag{51}$$

- Wave equation: A pair of initial conditions

$$u_{tt} - c^2 u_{xx} = 0; \quad u(x,0) = \phi(x), \;\; u_t(x,0) = \psi(x). \tag{52}$$

Boundary condition

Let D be a domain on which a PDE is defined. Let $\mathbf{n}$ be the unit outer normal vector on ∂D. Let $\Gamma \subseteq \partial D$. The three most important kinds of boundary conditions are

1. Dirichlet boundary condition: $u(\mathbf{x}, t)|_\Gamma = g(\mathbf{x}, t)$;
2. Neumann boundary condition: $\frac{\partial u}{\partial \mathbf{n}}\Big|_\Gamma = g(\mathbf{x}, t)$;
3. Robin boundary condition: $\left(\frac{\partial u}{\partial \mathbf{n}} + au\right)\Big|_\Gamma = g(\mathbf{x}, t)$.

On the whole boundary ∂D, one of these boundary conditions or a mixing of them could be imposed.

If the boundary data $g(\mathbf{x}, t) = 0$ are set to be constantly zero, the boundary conditions are said to be homogeneous, otherwise, they are inhomogeneous.

§1.5 Well-posedness of a PDE

A PDE in a domain D together with a set of initial and/or boundary conditions (or other auxiliary conditions) is said to be well-posed, if it meets the following three fundamental properties:

(i) **Existence** — There exists at least one solution to the differential equation.

(ii) **Uniqueness** — Physical processes are causal: given the state at some time we should be able to produce only one state at all later times. There exists a unique solution.

(iii) **Stability** — Small changes in the initial and/or boundary conditions (or other auxiliary conditions) should lead to small changes in the output. This means that if the data are changed a little, the corresponding solution changes only a little.

Example: A well-posed problem

A vibrating string with an external force, whose ends are moved in a specified way, satisfies the problem

$$Tu_{tt} - \rho u_{xx} = f(x,t) \tag{53}$$

with the initial and boundary conditions for $0 < x < L$:

$$\begin{aligned} u(x,0) &= \phi(x), \quad u_t(x,0) = \psi(x), \\ u(0,t) &= g(t), \quad u(L,t) = h(t). \end{aligned}$$

§1.6 Types of Second-Order Equations

Second-order linear PDE

Let's consider the PDE

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0. \tag{54}$$

This is a linear equation of order two in two variables with six real constant coefficients.

The factor 2 is introduced for convenience, largely due to the fact that

$$a_{12}u_{xy} + a_{12}u_{yx} = 2a_{12}u_{xy}. \tag{55}$$

§1.6 Types of Second-Order Equations

Theorem: Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{bmatrix}$. Then

- Equation (54) is of elliptic type, if $\det(A) = a_{11}a_{22} - a_{12}^2 > 0$. By a linear transform, it can be reduced to

$$u_{xx} + u_{yy} + \{\text{terms of lower order 1 or 0}\} = 0. \tag{56}$$

- Equation (54) is of hyperbolic type, if $\det(A) = a_{11}a_{22} - a_{12}^2 < 0$. By a linear transform, it can be reduced to

$$u_{xx} - u_{yy} + \{\text{terms of lower order 1 or 0}\} = 0. \tag{57}$$

- Equation (54) is of parabolic type, if $\det(A) = a_{11}a_{22} - a_{12}^2 = 0$. By a linear transform, it can be reduced to

$$u_{xx} + \{\text{terms of lower order 1 or 0}\} = 0, \tag{58}$$

(unless $a_{11} = a_{12} = a_{22} = 0$.)

§1.6 Types of Second-Order Equations

Representatives of three types

- Laplace equation:
$$u_{xx} + u_{yy} = 0 \tag{59}$$
We have $a_{11} = a_{22} = 1, a_{12} = 0$ so $\det(A) = 1$. It is elliptic type.

- Wave equation:
$$u_{tt} - c^2 u_{xx} = 0 \tag{60}$$
We have $a_{11} = 1, a_{22} = -c^2, a_{12} = 0$ so $\det(A) = -c^2$. It is hyperbolic type.

- Heat equation:
$$u_t - k u_{xx} = 0 \tag{61}$$
We have $a_{11} = 0, a_{22} = -k, a_{12} = 0$ so $\det(A) = 0$. It is parabolic type.

Example 1.

Classify each of the equations

$$\begin{aligned} &\text{(a)} \quad u_{xx} - 5u_{xy} = 0. \\ &\text{(b)} \quad 4u_{xx} - 12u_{xy} + 9u_{yy} + u_y = 0. \\ &\text{(c)} \quad 4u_{xx} + 6u_{xy} + 9u_{yy} = 0. \end{aligned}$$

Second-order linear PDE: The general case

Suppose that there are n variables, denoted $x_1, x_2, \cdots, x_n$, and the PDE is

$$\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} a_i u_{x_i} + a_0 u = 0. \tag{62}$$

with real constants a_{ij}, a_i, and a_0. Since the mixed derivatives are equal, we may as well assume that $a_{ij} = a_{ji}$. Let $A = (a_{ij})$ be the coefficient matrix. Further assume that the real numbers $d_1, \cdots, d_n$ are the eigenvalues of A.

§1.6 Types of Second-Order Equations

Definition

The PDE (62) is

- elliptic: if all the eigenvalues $d_1, \cdots, d_n$ are positive or negative.
- hyperbolic: if none of the eigenvalues $d_1, \cdots, d_n$ vanish and one of them has the opposite sign from the $(n-1)$ others.
- ultrahyperbolic: if none of the eigenvalues $d_1, \cdots, d_n$ vanish but at least two of them are positive and at least two are negative.
- parabolic: if exactly one of the eigenvalues $d_1, \cdots, d_n$ is zero and all the others have the same sign.

Example A

- Laplace equation: $u_{xx} + u_{yy} + u_{zz} = 0$ is elliptic because all the eigenvalues are 1.
- Wave equation: $u_{tt} - c^2(u_{xx} + u_{yy} + u_{zz}) = 0$ is hyperbolic because one eigenvalue is 1 and all the others are $-c^2$.
- Heat equation: $u_t - k(u_{xx} + u_{yy} + u_{zz}) = 0$ is parabolic because one eigenvalue is 0 and all the others are $-k$.

Summary

First-order linear equation

In general, we are interested in solving the PDEs:

$$a(x,y)u_x + b(x,y)u_y + c(x,y)u = f(x,y), \tag{63}$$

where a, b, c, f are constants or continuous functions in some domain Ω. The method of characteristics is usually used to solve (63).

Summary

Second-order linear equation

In general, we are interested in solving the PDEs:

$$\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} a_i u_{x_i} + a_0 u = 0. \tag{64}$$

where a_{ij}, a_i, and a_0 are constants or continuous functions in some domain Ω. We will mainly discuss the following three typical PDEs in our later lectures.

- Wave equation: $u_{tt} - c^2 \Delta u = 0$ (hyperbolic) Chapters 2-5.
- Heat equation: $u_t - k\Delta u = 0$ (parabolic) Chapters 2-5.
- Laplace equation: $\Delta u = 0$ (elliptic) Chapter 6.