§2.3 The diffusion equation

Stability

Stability of solutions is the third ingredient of well-posedness after existence and uniqueness. In general, we say that a system is stable if "close" initial data generate "close" solutions. To measure "closeness", we need a measure for distance of functions.

$|\delta\phi(x)| \ll 1$

Stability for the Dirichlet problem for the diffusion equation

Let us consider the diffusion equation:

$\phi(x) \to \phi(x) + \delta\phi(x)$

$x=0$ $\quad$ $x=l$ $\quad$ $t=0$

$$
\begin{aligned}
& u_t - k u_{xx} = f(x,t), \quad 0 < x < l, \quad t > 0, \\
& u(x,0) = \phi(x), \quad 0 < x < l, \\
& u(0,t) = g(t), \quad u(l,t) = h(t),
\end{aligned} \tag{53}
$$

$u(x,t) \to u(x,t) + \delta u(x,t)$

Let $u_1(x,t)$ and $u_2(x,t)$ be two solutions generated by the equation with the initial values $u_1(x,0) = \phi_1(x)$ and $u_2(x,0) = \phi_2(x)$, respectively.

$w = u_1 - u_2$

$w(x,t) \geq \min(\phi_1 - \phi_2, 0)$

Stability for the Dirichlet problem for the diffusion equation (Cont'd)

Suppose that we define the distance between two functions f, g as

$$\begin{cases} w_t - k w_{xx} = 0 \\ w(x,0) = \phi_1(x) - \phi_2(x) \end{cases}$$

$w(0,t) = w(l,t) = 0$

$$\mathrm{dist}(f,g) = \left(\int_0^l [f(x) - g(x)]^2 \, dx \right)^{1/2},$$

$w(x,t) \leq \max(\phi_1 - \phi_2, 0)$

which is called the L^2-distance.

$$\begin{cases} \frac{\partial}{\partial t} u_1 - k \frac{\partial^2 u_1}{\partial x^2} = f(x,t) \\ u_1(x,0) = \phi_1(x) \\ u_1(0,t) = g(t) \\ u_1(l,t) = h(t) \end{cases}$$

$$\begin{cases} \frac{\partial u_2}{\partial t} - k \frac{\partial u_2}{\partial x^2} = f \\ u_2(x,0) = \phi_2(x) \\ u_2(0,t) = g(t) \\ u_2(l,t) = h(t) \end{cases}$$

Stability for the Dirichlet problem for the diffusion equation (Cont'd)

Suppose that we define the distance between two functions f, g as

$$\mathrm{dist}(f,g) = \Big(\int_0^l [f(x) - g(x)]^2 \, dx \Big)^{1/2},$$

which is called the L^2-distance.

Notice that $w = u_1 - u_2$ solves the same equation (51) as in Example B but with a different initial condition $w(x,0) = \phi_1(x) - \phi_2(x)$. We have already shown that $\int_0^l w^2 dx$ is a strictly decreasing function of t in Example B. Thus, we have

$$\begin{aligned} & \int_0^l (u_1(x,t) - u_2(x,t))^2 dx \le \int_0^l (\phi_1(x) - \phi_2(x))^2 dx, \\ \Rightarrow \;\; & \mathrm{dist}(u_1, u_2) \le \mathrm{dist}(\phi_1, \phi_2), \quad \forall t \ge 0. \end{aligned}$$

Stability for the Dirichlet problem for the diffusion equation (Cont'd)

Suppose that we define the distance between two functions f, g as

$$\mathrm{dist}(f,g) = \Big(\int_0^l [f(x) - g(x)]^2 \, dx \Big)^{1/2},$$

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$$\int_0^l (u_1(x,t) - u_2(x,t))^2 dx \leq \int_0^l (\phi_1(x) - \phi_2(x))^2 dx,$$
$$\Rightarrow \quad \mathrm{dist}(u_1, u_2) \leq \mathrm{dist}(\phi_1, \phi_2), \quad \forall t \geq 0.$$

This means that the nearness of the initial conditions implies the nearness of the solutions.

Diffusion equation on the whole line

Our purpose is to solve the Cauchy problem on the real line:

$$u_t = ku_{xx}, \quad \underline{-\infty < x < \infty}, \quad \underline{t > 0}; \tag{54}$$

$$u(x, 0) = \underline{\phi}(x). \tag{55}$$

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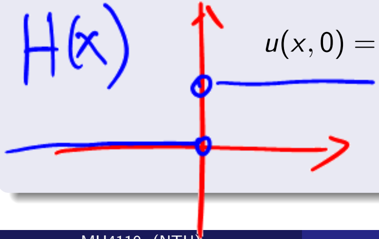
The method below will be very different from the one that we used for the wave equation. The main idea is to first solve the equation for a particular data $\phi(x)$ of the form

H(x)



$$u(x,0) = \phi(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases} \tag{56}$$

for any $a > 0$

$H(ax) = H(x)$

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Then we build the solution of (54)-(55) with general $\phi(x)$ from this particular one.

§2.3 The diffusion equation

The properties below can help derive a solution formula for the diffusion equation.

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The properties below can help derive a solution formula for the diffusion equation.

Invariance properties of the diffusion equation (54)

(i) **[Spatial translations]** The translate $u(x-y,t)$ of any solution $u(x,t)$ is another solution, for any fixed y.

If $\frac{\partial}{\partial t} u(x,t) = k \frac{\partial^2 u(x,t)}{\partial x^2}$

then $\frac{\partial}{\partial t} u(x-y,t) = k \frac{\partial^2 u(x-y,t)}{\partial x^2}$

§2.3 The diffusion equation

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Invariance properties of the diffusion equation (54)

(i) **[Spatial translations]** The translate $u(x-y,t)$ of any solution $u(x,t)$ is another solution, for any fixed y.

(ii) **[Dilation (scaling)]** The dilation $u(\sqrt{a}x, at)$ of any solution $u(x,t)$ is another solution, for any constant $a>0$.

$$\frac{\partial u(\sqrt{a}\,x, at)}{\partial t} = a\,u_t(\sqrt{a}\,x, at)$$

$$\frac{\partial u(\sqrt{a}\,x, at)}{\partial x} = \sqrt{a}\,u_x(\sqrt{a}\,x, at)$$

$$u_{xx}(\sqrt{a}\,x, at) = a\,u_{xx}(\sqrt{a}\,x, at)$$

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(iii) **[Differentiation]** Any partial derivative (e.g., $u_x, u_t, u_{xx}, u_{xt}, \cdots$) of a solution $u(x,t)$ is again a solution.

If $\quad u_t = k u_{xx}$

then $\quad \frac{\partial}{\partial x}(u_t) = \frac{\partial}{\partial x}(k u_{xx})$

$$\frac{\partial}{\partial t}(u_x) = k\left(\frac{\partial u}{\partial x}\right)_{xx}$$

§2.3 The diffusion equation

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(iv) **[Linear combinations]** If $u_1, u_2, \cdots, u_n$ are solutions of (54), then so is $u = c_1 u_1 + c_2 u_2 + \cdots + c_n u_n$ for any constants $c_1, c_2, \cdots, c_n$.

§2.3 The diffusion equation

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(iv) **[Linear combinations]** If $u_1, u_2, \cdots, u_n$ are solutions of (54), then so is $u = c_1u_1 + c_2u_2 + \cdots + c_nu_n$ for any constants $c_1, c_2, \cdots, c_n$.

(v) **[Convolution invariance]** If $S(x,t)$ is a solution of (54), then so is

$$v(x,t) = \int_{-\infty}^{\infty} S(x-y,t)g(y)dy = S(\cdot,t) * g, \tag{57}$$

for any function g.

Solution formula for the diffusion equation

Theorem The problem

$$u_t = k u_{xx}, \quad -\infty < x < \infty, \quad t > 0; \quad u(x,0) = \phi(x), \tag{58}$$

has the solution

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp\Big(-\frac{(x-y)^2}{4kt} \Big) \phi(y) dy. \tag{59}$$

Step 0: Starting with a particular IVP.

As a special initial data we take the following function

$$H(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0, \end{cases} \tag{60}$$

which is called the Heaviside step function. We first consider the initial value problem

$$Q_t = kQ_{xx}, \quad -\infty < x < \infty, \ \ t > 0; \quad Q(x,0) = H(x) \tag{61}$$

which will be solved in successive steps.

$Q(x,t)$

Solution derivation for the diffusion equation

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which will be solved in successive steps.

IVP (61) is the same as (58) but with a special initial data $H(x)$.

Solution derivation for the diffusion equation (Cont'd)

Step 1: Reduction to an ODE.

- If $Q(x,t)$ is a solution, then $Q(\sqrt{a}x, at)$ also solves $u_t - ku_{xx} = 0$ from the dilation property (ii) of the diffusion equation. But we cannot say that $Q(\sqrt{a}x, at)$ also solves the IVP (61) at this time.

$$Q(x,0) = H(x)$$

$$Q(\sqrt{a}\,x, 0) = ?$$

Solution derivation for the diffusion equation (Cont'd)

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- If $Q(x,t)$ is a solution, then $Q(\sqrt{a}x, at)$ also solves $u_t - ku_{xx} = 0$ from the dilation property (ii) of the diffusion equation. But we cannot say that $Q(\sqrt{a}x, at)$ also solves the IVP (61) at this time.
- Check the initial condition of $Q(\sqrt{a}x, at)$. Since $Q(x,0) = H(x)$, we have $Q(\sqrt{a}x, 0) = H(\sqrt{a}x)$. It is easy to notice that $H(\sqrt{a}x) = H(x)$. So $Q(\sqrt{a}x, 0) = H(x)$. It means that $Q(\sqrt{a}x, at)$ also solves the IVP (61).

Solution derivation for the diffusion equation (Cont'd)

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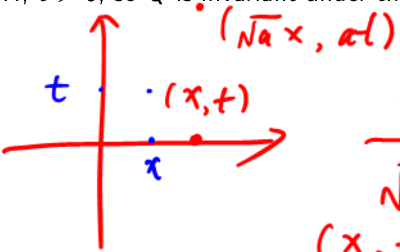
- The uniqueness of solutions then implies that $Q(\sqrt{a}x, at) = Q(x,t)$ for all $x \in R$, $t > 0$, so Q is invariant under the dilation $(x,t) \to (\sqrt{a}x, at)$ as well.



$$\frac{r}{\sqrt{t}}$$

(x,t)

$$\frac{\sqrt{a}x}{\sqrt{at}} = \frac{r}{\sqrt{t}}$$

$(\sqrt{a}x, at)$

Solution derivation for the diffusion equation (Cont'd)

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- For a fixed (x,t) and let $a = 1/t$, we have

$$Q(x,t) = Q(\sqrt{a}x, at) = Q(\sqrt{\frac{1}{t}}x, \frac{1}{t}t) = Q(\sqrt{\frac{1}{t}}x, 1). \tag{62}$$

So Q depends only on the ratio $x/\sqrt{t}$.

Solution derivation for the diffusion equation (Cont'd)

Step 1: Reduction to an ODE (Cont'd).

- We can thus look for $Q(x,t)$ of the special form

$$Q(x,t) = g(p) \qquad \text{where} \quad p = \frac{x}{\sqrt{4kt}}, \tag{63}$$

where g is a function of only one variable (to be determined). The $\sqrt{4k}$ is included only to simplify a later formula.

Solution derivation for the diffusion equation (Cont'd)

Step 1: Reduction to an ODE (Cont'd).

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- Using (63), we convert (61) into an ODE for g by use of the chain rule:

$$\begin{aligned} Q_t &= \frac{dg}{dp}\frac{\partial p}{\partial t} = -\frac{1}{2t}\frac{x}{\sqrt{4kt}}g'(p) = -\frac{1}{2t}pg'(p), \\ Q_x &= \frac{dg}{dp}\frac{\partial p}{\partial x} = \frac{1}{\sqrt{4kt}}g'(p), \\ Q_{xx} &= \frac{dQ_x}{dp}\frac{\partial p}{\partial x} = \frac{1}{4kt}g''(p). \end{aligned}$$

Solution derivation for the diffusion equation (Cont'd)

Step 1: Reduction to an ODE (Cont'd).

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- Thus, Q could be a solution only when

$t>0$

$$0 = Q_t - kQ_{xx} = \frac{1}{t}\left(-\frac{1}{2}pg'(p) - \frac{1}{4}g''(p)\right) \Rightarrow g''(p) + 2pg'(p) = 0.$$

Step 2: Solving the ODE.

$$g' = 0 \Rightarrow g(p) = C$$

The ODE $g''(p) + 2pg'(p) = 0$ can be solved as follows:

Integrate

$$\frac{g''}{g'} = -2p \quad \Rightarrow \quad \ln|g'| = -p^2 + c_1 \quad \Rightarrow \quad g' = \pm e^{c_1} e^{-p^2} = C_1 e^{-p^2},$$

$$\Rightarrow |g'| = e^{c_1} e^{-p^2}$$

$$\Rightarrow \quad g(p) = C_1 \int e^{-p^2} dp + C_2. = C_1 \int_0^p e^{-s^2} ds + C_2$$

C_1 is a real number

Therefore, by (63),

$$Q(x,t) = g(p) = C_1 \int e^{-p^2} dp + C_2,$$

which satisfies $Q_t - kQ_{xx} = 0$ with $p = \frac{x}{\sqrt{4kt}}$.

$$Q(x,t) = g(p) = C_1 \int_0^{\frac{x}{\sqrt{4kt}}} e^{-s^2} ds + C_2$$

Solution derivation for the diffusion equation (Cont'd)

Step 3: Checking the initial condition. We now impose the initial condition to find the unique solution for (61) by determining C_1 and C_2.

- For convenience, we write

$$Q(x,t) = g(p) = C_1 \int_0^p e^{-s^2} ds + C_2 \;\; \Rightarrow \;\; Q(x,t) = C_1 \int_0^{x/\sqrt{4kt}} e^{-s^2} ds + C_2, \;\; \forall t > 0. \tag{64}$$

Solution derivation for the diffusion equation (Cont'd)

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- By the initial conditions in (61), $x/\sqrt{4kt} \to +\infty$

$$\begin{aligned} &\text{if } \ x > 0, \quad 1 = \lim_{t \to 0^+} Q = C_1 \int_0^{\infty} e^{-s^2}\,ds + C_2 = C_1 \frac{\sqrt{\pi}}{2} + C_2, \\ &\text{if } \ x < 0, \quad 0 = \lim_{t \to 0^+} Q = C_1 \int_0^{-\infty} e^{-s^2}\,ds + C_2 = -C_1 \frac{\sqrt{\pi}}{2} + C_2, \end{aligned} \tag{65}$$

$x/\sqrt{4kt} \to -\infty$

where we used the known integral formula

$$H(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$

$$\int_0^{\infty} e^{-s^2}\,ds = \frac{\sqrt{\pi}}{2}. \tag{66}$$

Solution derivation for the diffusion equation (Cont'd)

Step 3: Checking the initial condition (Cont'd).

- Solving (65) leads to

$$C_1 = \frac{1}{\sqrt{\pi}}, \quad C_2 = \frac{1}{2}.$$

Step 3: Checking the initial condition (Cont'd).

$Q(x,0) = H(x)$

- Solving (65) leads to

$$C_1 = \frac{1}{\sqrt{\pi}}, \quad C_2 = \frac{1}{2}.$$

- Then the solution of (61) is $\rightarrow Q_t - kQ_{xx} = 0$

$$Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-s^2} ds, \quad t > 0, \ -\infty < x < \infty. \tag{67}$$

$$Q(\sqrt{a}\,x, at) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{\frac{\sqrt{a}\,x}{\sqrt{4kat}}} e^{-s^2} ds$$

$$= Q(x,t)$$

Solving the diffusion equation.

Step 4: Solving the general IVP.

- Define

$$S(x,t) = \frac{\partial Q}{\partial x} \Rightarrow \quad S(x,t) = \frac{1}{\sqrt{4\pi kt}} \exp\Big(-\frac{x^2}{4kt}\Big), \tag{68}$$

where $Q(x,t)$ is the solution of the particular IVP (61).

$$S(x,0) = \delta(x)$$

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}} \exp\left(-\frac{x^2}{4kt}\right)$$

$$= \frac{1/\sqrt{4\pi kt}}{e^{x^2/4kt}} \xrightarrow{t \to 0^+} \begin{cases} 0, & x \neq 0 \\ \lim_{t \to 0^+} \frac{1}{\sqrt{4\pi kt}} & x = 0 \end{cases}$$

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where $Q(x,t)$ is the solution of the particular IVP (61).

- By Property (iii), $S(x,t)$ is a solution of $u_t - ku_{xx} = 0$, and by Property (v), so is
$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy, \quad t > 0. \tag{69}$$

Step 4: Solving the general IVP.

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- We claim that this above u is the unique solution of the IVP (58). To verify this claim one only needs to check the initial condition of (58) as $u(x,0) = \phi(x)$.

Solution derivation for the diffusion equation (Cont'd)

Step 4: Solving the general IVP (Cont'd).

- Notice that $S(x,t) = \frac{\partial Q}{\partial x}$, we can rewrite u as follows

$$u(x,t) = \int_{-\infty}^{\infty} \frac{\partial Q}{\partial x}(x-y,t)\phi(y)dy = -\int_{-\infty}^{\infty} \frac{\partial}{\partial y}\left[Q(x-y,t)\right]\phi(y)dy$$
$$= +\int_{-\infty}^{\infty} Q(x-y,t)\phi'(y)dy - Q(x-y,t)\phi(y)\Big|_{y=-\infty}^{y=\infty}$$

upon integrating by parts.

$$-\frac{\partial Q(x-y,t)}{\partial y} = \frac{\partial Q(x-y,t)}{\partial x}$$

Solution derivation for the diffusion equation (Cont'd)

Step 4: Solving the general IVP (Cont'd).

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$$= + \int_{-\infty}^{\infty} Q(x-y,t)\phi'(y)dy - \underbrace{Q(x-y,t)\phi(y)\Big|_{y=-\infty}^{y=\infty}}_{0}$$

  upon integrating by parts.

- We assume these limits vanish. In particular, let's temporarily assume that $\phi(y)$ itself equals zero for $|y|$ large. Using that Q has the Heaviside function (60) as its initial data, we have

$$\int_{-\infty}^{x} + \int_{x}^{+\infty}$$

$$u(x,0) = \int_{-\infty}^{\infty} Q(x-y,0)\phi'(y)dy = \int_{-\infty}^{x} \phi'(y)dy = \phi(x),$$

  where we used the assumption $\phi(-\infty) = 0$.

$$Q(x-y,0) = H(x-y) = \begin{cases} 1 & x > y \\ 0 & x < y \end{cases}$$

Solution derivation for the diffusion equation (Cont'd)

Step 4: Solving the general IVP (Cont'd).

- Notice that $S(x,t) = \frac{\partial Q}{\partial x}$, we can rewrite u as follows

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- We assume these limits vanish. In particular, let's temporarily assume that $\phi(y)$ itself equals zero for $|y|$ large. Using that Q has the Heaviside function (60) as its initial data, we have

$$u(x,0) = \int_{-\infty}^{\infty} Q(x-y,0)\phi'(y)dy = \int_{-\infty}^{x} \phi'(y)dy = \phi(x),$$

  where we used the assumption $\phi(-\infty) = 0$.

- Therefore, we have proved (69) with $S(x,t)$ given by (68) is the solution of (58). This ends the derivation of the solution formula.

Case 1: $|y| > 100$ $|y-x| > 100$ Case 2: $|y| < 100$

Solution formula for the diffusion equation

Theorem (repeated) The problem Case 3: $|y| > 100$, $|y-x| < 100$

$$u_t = ku_{xx}, \quad -\infty < x < \infty, \quad t > 0; \quad u(x,0) = \phi(x),$$

has the solution

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp\Big(-\frac{(x-y)^2}{4kt}\Big)\phi(y)dy.$$

Let $k=1$ $\phi(x) = e^{-\sqrt{x^2+1}}$

$$\exp\left[-\frac{(x-y)^2}{4kt}\right] e^{-\sqrt{y^2+1}} = \begin{cases} \to 0 & |y| > 100 \\ & |y-x| > 100 \end{cases}$$

Solution formula for the diffusion equation

Theorem (repeated) The problem

$$u_t = ku_{xx}, \quad -\infty < x < \infty, \quad t > 0; \quad u(x,0) = \phi(x),$$

has the solution $u(x,t) = \int_{-\infty}^{+\infty} S(x-y,t)\phi(y)\,dy$

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp\Big(-\frac{(x-y)^2}{4kt}\Big)\phi(y)dy.$$

$S(x,t) = \frac{\partial}{\partial x} Q(x,t)$

The function

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}} \exp\Big(-\frac{x^2}{4kt}\Big) \tag{70}$$

is known as the Gaussian kernel, fundamental solution, source function, Green's function, or propagator of the heat equation. It gives a way of propagating the initial data ϕ to later times, giving the solution at any time $t > 0$.

Some properties of the kernel function $S(x, t)$

- The solution of the IVP (54)-(55) is a convolution of $S(x, t)$ with the initial value $\phi(x)$:

$$u(x,t) = S(\cdot, t) * \phi = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp\Big(- \frac{(x-y)^2}{4kt} \Big) \phi(y) dy. \qquad (71)$$

Hence, $S(x, t)$ is known as the **Gaussian kernel** of the diffusion equation.

§2.3 The diffusion equation

Some properties of the kernel function $S(x,t)$

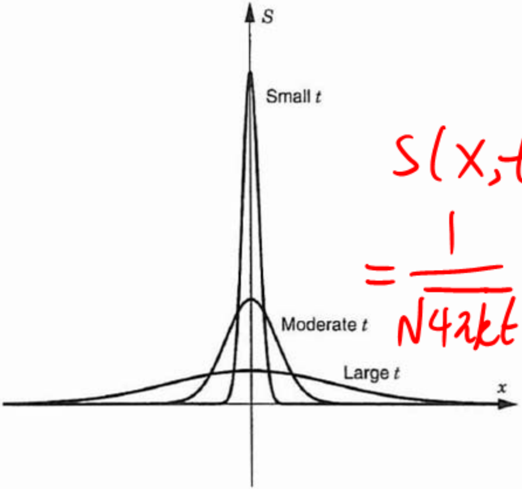
- The solution of the IVP (54)-(55) is a convolution of $S(x,t)$ with the initial value $\phi(x)$:

$$u(x,t) = S(\cdot,t) * \phi = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp\Big(- \frac{(x-y)^2}{4kt} \Big) \phi(y) dy. \tag{71}$$

  Hence, $S(x,t)$ is known as the **Gaussian kernel** of the diffusion equation.

- The Gaussian kernel $S(x,t)$ is an even function of x and it is always positive. For large t, $S(x,t)$ is very spread out. For small t, it is a very thin tall spike of height $\frac{1}{\sqrt{4\pi kt}}$. The area under the curve is 1 :

$$\begin{aligned} \int_{-\infty}^{\infty} S(x,t) dx &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp\Big(- \frac{x^2}{4kt} \Big) dx \\ \Big(\text{Let } q = \frac{x}{\sqrt{4kt}} \Big) \quad &= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-q^2} dq \overset{(66)}{=} 1, \quad \forall t \geq 0. \end{aligned}$$



$$S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}}$$

Figure 1

Some properties of the kernel function $S(x,t)$ (Cont'd)

$$s(x,t) = \frac{\partial Q(x,t)}{\partial x}$$

- $$S(x,t) = \frac{1}{\sqrt{4\pi kt}} \exp\Big(-\frac{x^2}{4kt}\Big)$$

  is also termed as the **fundamental solution** of the diffusion equation, as it is the solution of the initial value problem:

$$u_t = ku_{xx}, \quad -\infty < x < \infty, \quad t > 0; \quad u(x,0) = \delta(x), \tag{72}$$

  where δ is the Dirac delta function. This follows from the definition $S(x,t) = Q_x(x,t)$ (cf. (68)) and the fact that Q is the solution of the particular IVP (61), so $S(x,t)$ satisfies the initial condition

$$S(x,0) = Q_x(t,0) = \underline{H'(x) = \delta(x)}, \quad \text{where} \quad H(x) = \begin{cases} 1, & \text{if } x > 0, \\ 0, & \text{if } x < 0, \end{cases}$$

  where $H(x)$ is the Heaviside function.

§2.3 The diffusion equation

Some properties of the kernel function $S(x,t)$ (Cont'd)

- We see from the solution formula (71) that the solution u at a point (x,t) is influenced by the initial value $\phi(y)$ at all $y \in (-\infty, \infty)$.

  Indeed, we can view $S(x,t)$ as a weighting function:

  $y_{j+1} - y_j = \Delta y_j$

  $$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy \simeq \sum_j S(x-y_j,t)\phi(y_j)\Delta y_j, \qquad (73)$$

  $\{y_j\} \subset R$

  where $\{y_j\}$ are some sampling points. The source function $S(x-y,t)$ weights the contribution of $\phi(y)$ according to the distance of y from x and the elapsed time t. The contribution from a point y_1 closer to x has a bigger weight $S(x-y_1,t)$, than the contribution from a point y_2 farther away, which gets weighted by $S(x-y_2,t)$.

  For very small t, the source function is a spike so that the formula exaggerates the values of ϕ near x. For any $t>0$ the solution is a spread-out version of the initial values at $t=0$.