MH4110 PDE

Tutorial 05

Question 1

Consider the diffusion equation on $(0, l)$ with the Robin boundary condition:

IBVP

$$
\begin{aligned}
&u_t = k u_{xx}, \quad 0 < x < l, \quad t > 0, \\
&u_x(0,t) - a_0 u(0,t) = 0, \quad u_x(l,t) + a_l u(l,t) = 0, \quad t > 0, \\
&u(x,0) = \phi(x), \quad 0 < x < l.
\end{aligned}
$$

If $a_0 > 0$ and $a_l > 0$, use the energy method to show that $\int_0^l u^2(x,t)\,dx$ decreases with respect to t.

$t > 0$

$a_l u(\ell, t) = -u_x(\ell, t)$

$x = 0$ $\quad \phi(x) \quad$ $x = \ell$

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If $a_0 > 0$ and $a_l > 0$, use the energy method to show that $\int_0^l u^2(x,t)\,dx$ decreases with respect to t.

$$E'(t) = 2\int_0^l u\,u_t\,dx$$

[Solution:] We have

$$u_x(0,t) = a_0 u(0,t), \quad a_0 > 0; \quad u_x(l,t) = -a_l u(l,t), \quad a_l > 0.$$

Multiplying the PDE by u and integrating from 0 to l gives

$$\frac{1}{2}E'(t) = \int_0^l u u_t\,dx = k\int_0^l u u_{xx}\,dx.$$

$$u u_{xx} = \frac{\partial}{\partial x}(u u_x) - u_x u_x$$

But $u u_t = \frac{1}{2}\frac{\partial}{\partial t}(u^2)$, so integrating by parts on the right hand side gives

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If $a_0 > 0$ and $a_l > 0$, use the energy method to show that $\int_0^l u^2(x,t)dx$ decreases with respect to t.

[Solution (continued):] But $uu_t = \frac{1}{2}\frac{\partial}{\partial t}(u^2)$, so integrating by parts on the right hand side gives

$$\frac{d}{dt}\left[\frac{1}{2}\int_0^l u^2 dx\right] = kuu_x\Big|_0^l - k\int_0^l u_x^2 dx = -ka_l u^2(l,t) - ka_0 u^2(0,t) - k\int_0^l u_x^2 dx,$$

where we used the boundary conditions. Note that all terms on the right hand side are ≤ 0, in particular so are the boundary terms. As $\frac{d}{dt}\int_0^l u^2(x,t)dx \leq 0$, $\int_0^l u^2(x,t)dx$ decreases with respect to t.

Question 2

Compute $\int_0^\infty e^{-x^2} dx$. (*Hint:* This is a function that cannot be integrated by formula. So use the following trick. Transform the double integral $\int_0^\infty e^{-x^2} dx \cdot \int_0^\infty e^{-y^2} dy$ into polar coordinates and you'll end up with a function that can be integrated easily.)

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[Solution:] We first try to calculate

$$\gamma = \left[\int_0^\infty e^{-x^2} dx\right]^2 = \int_0^\infty e^{-x^2} dx \cdot \int_0^\infty e^{-y^2} dy = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dxdy.$$

by introducing the polar coordinate system

$$r = \sqrt{x^2 + y^2} \qquad \theta = \arctan \frac{y}{x}.$$

The differential $dxdy$ represents an element of area in cartesian coordinates, with the domain of integration extending over the region $0 \le x < \infty$, $0 \le y < \infty$ in the xy-plane. An alternative representation of the last integral can be expressed in polar coordinates r, θ.

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[Solution (continued):] The figure on the next page shows the area corresponding to an increase in θ of $d\theta$ and an increase in r of dr. The small figure with sides of dr and $rd\theta$ is very nearly a rectangle, and has area $rdrd\theta$. Thus, we have

$$\gamma = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dxdy = \int_0^{\frac{\pi}{2}} \int_0^\infty e^{-r^2} rdrd\theta = \frac{1}{2} \int_0^{\pi/2} \int_0^\infty e^{-r^2} dr^2 d\theta.$$

Making the change of variables $r^2 = p$, we further have

$$\gamma = \frac{1}{2} \int_0^{\pi/2} \int_0^\infty e^{-p} dpd\theta = \frac{1}{2} \int_0^{\pi/2} \left(-e^{-p}\right) \Big|_0^\infty d\theta = \frac{1}{2} \int_0^{\pi/2} d\theta = \frac{\pi}{4}.$$

This gives

$$\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}.$$

Since e^{-x^2} is even

$$\int_{-\infty}^0 e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

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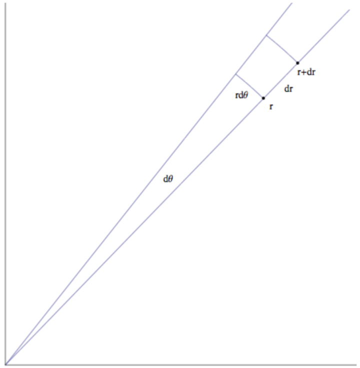

Question 3

Use the result of Problem 2 to show that $\int_{-\infty}^{\infty} e^{-p^2} dp = \sqrt{\pi}$. Then substitute $p = x/\sqrt{4kt}$ to show that

$$\int_{-\infty}^{\infty} S(x,t)dx = 1,$$

where $S(x,t)$ is the Gaussian kernel

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}} \exp\left(-\frac{x^2}{4kt}\right).$$

Let $-\frac{x^2}{4kt} = -p^2$ $\quad p = \frac{x}{\sqrt{4kt}}$

$\Rightarrow dx = \sqrt{4kt}\, dp$

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[Solution:] In Problem 2, we got that $\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$. Given that e^{-p^2} is an even function, we have

$$\int_{-\infty}^{\infty} e^{-p^2} dp = \int_{-\infty}^{0} e^{-p^2} dp + \int_0^{\infty} e^{-p^2} dp = 2\int_0^{\infty} e^{-p^2} dp = \sqrt{\pi}.$$

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[Solution (continued):] For

$$\int_{-\infty}^{\infty} S(x,t)dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} \exp\left(-\frac{x^2}{4kt}\right) dx,$$

making the change of variables $p = x/\sqrt{4kt}$ gives

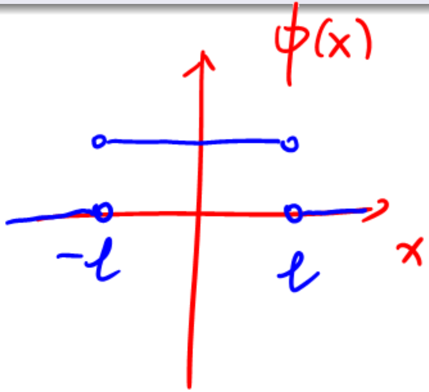
$$\int_{-\infty}^{\infty} S(x,t)dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(-p^2\right) dp = 1.$$

Question 4

Solve the diffusion equation: $u_t = ku_{xx}$ with the initial condition:

$$\phi(x) = 1, \quad |x| < l, \quad \phi(x) = 0, \quad |x| > l.$$

Write your answer in terms of the error function $\mathrm{Erf}(x)$.



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Write your answer in terms of the error function $\mathrm{Erf}(x)$.

[Solution:] We see that

$$\phi(x) = \begin{cases} 1, & \text{if } |x| < l, \\ 0, & \text{if } |x| > l, \end{cases}$$

$$\int_{-\infty}^{-\ell} + \int_{-\ell}^{\ell} + \int_{\ell}^{+\infty}$$

so

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-y)^2}{4kt}} \phi(y)dy = \int_{-l}^{l} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-y)^2}{4kt}} \phi(y)dy.$$

Let $p = \frac{x-y}{\sqrt{4kt}}$, so

$y = \ell$

$$dp = -\frac{1}{\sqrt{4kt}}\,dy$$

$$u(x,t) = -\frac{1}{\sqrt{\pi}} \int_{\frac{x+l}{\sqrt{4kt}}}^{\frac{x-l}{\sqrt{4kt}}} e^{-p^2} dp = \frac{1}{\sqrt{\pi}} \int_{\frac{x-l}{\sqrt{4kt}}}^{\frac{x+l}{\sqrt{4kt}}} e^{-p^2} dp.$$

$$\int_{\frac{x-\ell}{\sqrt{4kt}}}^{0} + \int_{0}^{\frac{x+\ell}{\sqrt{4kt}}}$$

$$-\sqrt{4kt}\,dp = dy$$

$y = -\ell$

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Write your answer in terms of the error function $\mathrm{Erf}(x)$.

[Solution (continued):] But this gives

$$u(x,t) = \frac{1}{\sqrt{\pi}}\left[\int_0^{\frac{x+l}{\sqrt{4kt}}} e^{-p^2}\,dp - \int_0^{\frac{x-l}{\sqrt{4kt}}} e^{-p^2}\,dp\right]$$

$$= \frac{1}{2}\left[\mathrm{Erf}\left(\frac{x+l}{\sqrt{4kt}}\right) - \mathrm{Erf}\left(\frac{x-l}{\sqrt{4kt}}\right)\right].$$

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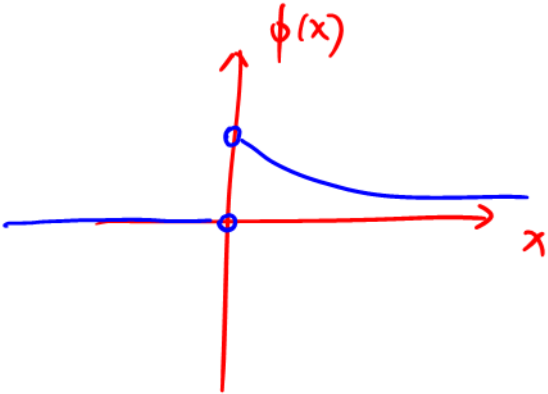
Solve the diffusion equation: $u_t = ku_{xx}$ with the initial condition:

$$\phi(x) = e^{-x}, \quad x > 0; \quad \phi(x) = 0, \quad x < 0.$$



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[Solution:] We see that

$$\phi(x) = \begin{cases} e^{-x}, & \text{if } x > 0, \\ 0, & \text{if } x < 0, \end{cases}$$

$\int_{-\infty}^{0} + \int_{0}^{\infty}$

so

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-y)^2}{4kt}} \phi(y)dy = \int_0^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-y)^2}{4kt}} e^{-y} dy.$$

The exponent is

$$-\frac{x^2 - 2xy + y^2 + 4kty}{4kt}.$$

Completing the square in the y variable, it is

$$-\frac{(y + 2kt - x)^2}{4kt} + kt - x.$$

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Solve the diffusion equation: $u_t = ku_{xx}$ with the initial condition:

$$\phi(x) = e^{-x}, \quad x > 0; \quad \phi(x) = 0, \quad x < 0.$$

[Solution (continued):] We let $p = \frac{y+2kt-x}{\sqrt{4kt}}$, so

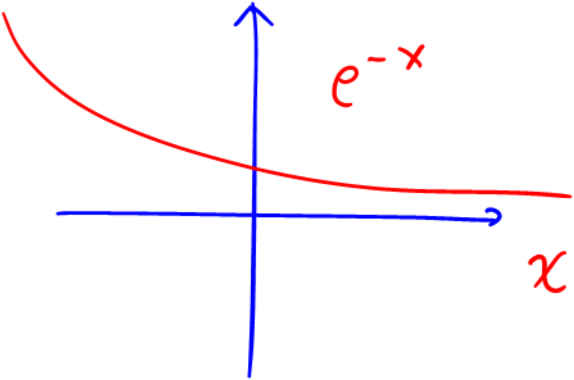
$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{kt-x} \int_0^\infty e^{-\frac{(y+2kt-x)^2}{4kt}}\, dy = \frac{1}{\sqrt{\pi}} e^{kt-x} \int_{\frac{2kt-x}{\sqrt{4kt}}}^\infty e^{-p^2}\, dp.$$

This gives

$$\begin{aligned} u(x,t) &= \frac{1}{\sqrt{\pi}} e^{kt-x} \int_{\frac{2kt-x}{\sqrt{4kt}}}^0 e^{-p^2}\, dp + \frac{1}{\sqrt{\pi}} e^{kt-x} \underbrace{\int_0^\infty e^{-p^2}\, dp}_{\frac{\sqrt{\pi}}{2}} \\ &= \frac{1}{\sqrt{\pi}} e^{kt-x} \int_0^{\frac{x-2kt}{\sqrt{4kt}}} e^{-p^2}\, dp + \frac{1}{2} e^{kt-x} \\ &= \frac{1}{2} e^{kt-x} \left[Erf\left(\frac{x-2kt}{\sqrt{4kt}}\right) + 1 \right]. \end{aligned}$$

Question 6

Solve the diffusion equation: $u_t = ku_{xx}$ with the initial condition $u(x, 0) = \phi(x) = e^{-x}$.



Question 6

Solve the diffusion equation: $u_t = ku_{xx}$ with the initial condition $u(x,0) = \phi(x) = e^{-x}$.

[Solution:] We see that $\phi(x) = e^{-x}$. So

$$u(x,t) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-y)^2}{4kt}} \phi(y)dy = \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{(x-y)^2}{4kt}} e^{-y} dy.$$

This is one of the few fortunate examples that can be integrated. The exponent is

$$-\frac{x^2 - 2xy + y^2 + 4kty}{4kt}.$$

Completing the square in the y variable, it is

$$-\frac{(y + 2kt - x)^2}{4kt} + kt - x.$$

We let $p = \frac{y+2kt-x}{\sqrt{4kt}}$, so

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{kt-x} \int_{-\infty}^{\infty} e^{-\frac{(y+2kt-x)^2}{4kt}} dy = \frac{1}{\sqrt{\pi}} e^{kt-x} \int_{-\infty}^{\infty} e^{-p^2} dp.$$

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[Solution (continued):] Completing the square in the y variable, it is

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We let $p = \frac{y+2kt-x}{\sqrt{4kt}}$, so

$$u(x, t) = \frac{1}{\sqrt{4\pi kt}} e^{kt-x} \int_{-\infty}^{\infty} e^{-\frac{(y+2kt-x)^2}{4kt}} dy = \frac{1}{\sqrt{\pi}} e^{kt-x} \int_{-\infty}^{\infty} e^{-p^2} dp.$$

This gives

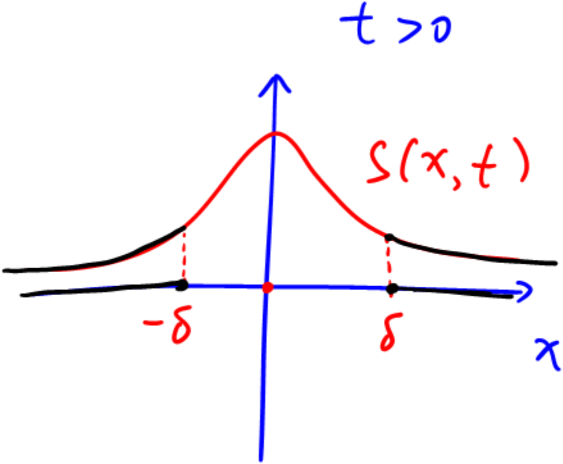
$$u(x, t) = e^{kt-x}.$$

By the maximum principle, a solution in a bounded interval cannot grow in time. However, this particular solution grows, rather than decays, in time. The reason is that the left side of the rod is initially very hot [$u(x, 0) \to +\infty$ as $x \to -\infty$] and the heat gradually diffuses throughout the rod.

Question 7

Show that for any fixed $\delta > 0$ (no matter how small),

$$\max_{\delta \le |x| < \infty} S(x,t) \to 0 \quad \text{as} \quad t \to 0.$$



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[Solution:] Given that

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}},$$

for any fixed $\delta > 0$, we know that when $\delta \le |x|$

attained at

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}} e^{-\frac{x^2}{4kt}} \le \frac{1}{\sqrt{4\pi kt}} e^{-\frac{\delta^2}{4kt}}.$$

$x = -\delta$ and

$x = \delta$

Taking the limit as $t \to 0^+$ and using the L'Hospital's rule, we have

$$\lim_{t\to 0^+} \frac{1}{\sqrt{4\pi kt}} e^{-\frac{\delta^2}{4kt}} = \lim_{t\to 0^+} \frac{\frac{1}{\sqrt{4\pi kt}}}{e^{\frac{\delta^2}{4kt}}} = \lim_{t\to 0^+} \frac{\frac{1}{\sqrt{4\pi k}}\left(-\frac{1}{2}\right)t^{-3/2}}{e^{\frac{\delta^2}{4kt}} \frac{\delta^2}{4k}(-1)t^{-2}} = \frac{\sqrt{k}}{\delta^2\sqrt{\pi}} \lim_{t\to 0^+} \frac{t^{1/2}}{e^{\frac{\delta^2}{4kt}}} = 0.$$

$S(x,t)$ is nonnegative. Based on the squeeze theorem for the limit, we have proved that for any fixed $\delta > 0$ (no matter how small)

$$\max_{\delta \le |x| < \infty} S(x,t) \to 0 \quad \text{as} \quad t \to 0.$$

Question 8

Solve the problem ($k > 0$):

$$u_t - ku_{xx} = \cos t, \quad x \in (-\infty, \infty), \quad t > 0,$$
$$u(x, 0) = e^{-x^2}, \quad x \in (-\infty, \infty).$$

Question 8

Solve the problem ($k > 0$): $V_t - kV_{xx} = \cos t$

$$u_t - ku_{xx} = \cos t, \quad x \in (-\infty, \infty), \quad t > 0, \quad \sin t$$

$$u(x, 0) = e^{-x^2}, \quad x \in (-\infty, \infty).$$

[Solution:] Because of the special form of the inhomogeneous equation, we look for a particular solution: $v = v(t)$. It can be easily verified that $v = \sin t$ is a particular solution to the inhomogeneous PDE $u_t - ku_{xx} = \cos t$. Let $w = u - v$, then w satisfies the following initial value problem

$$w_t - kw_{xx} = 0, \quad x \in (-\infty, \infty), \quad t > 0,$$
$$w(x, 0) = e^{-x^2}, \quad x \in (-\infty, \infty).$$

Using the solution formula for the homogeneous equation, we have

$$\begin{aligned} w(x, t) &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-y)^2}{4kt}\right] \exp(-y^2) dy \\ &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-y)^2 + 4kty^2}{4kt}\right] dy \\ &= \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp\left[-\frac{x^2 - 2xy + (1+4kt)y^2}{4kt}\right] dy \end{aligned}$$

Completing the squre of y,

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Solve the problem ($k > 0$):

$$u_t - ku_{xx} = \cos t, \quad x \in (-\infty, \infty), \quad t > 0,$$
$$u(x, 0) = e^{-x^2}, \quad x \in (-\infty, \infty).$$

[Solution (continued):] Completing the squre of y,

$$w(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp\left[-\frac{\left(y - \frac{x}{1+4kt}\right)^2 + \frac{4kt}{(1+4kt)^2}x^2}{\frac{4kt}{1+4kt}}\right] dy$$

$$= \exp\left[-\frac{\frac{4kt}{(1+4kt)^2}x^2}{\frac{4kt}{1+4kt}}\right] \cdot \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp\left[-\frac{\left(y - \frac{x}{1+4kt}\right)^2}{\frac{4kt}{1+4kt}}\right] dy$$

Making the change of variables

$$p = \frac{y - \frac{x}{1+4kt}}{\sqrt{\frac{4kt}{1+4kt}}},$$

we have

$$w(x,t) = \exp\left[-\frac{x^2}{1+4kt}\right] \cdot \frac{1}{\sqrt{4\pi kt}} \sqrt{\frac{4kt}{1+4kt}} \int_{-\infty}^{\infty} \exp\left[-p^2\right] dp.$$

Question 8

Solve the problem ($k > 0$):

$$u_t - ku_{xx} = \cos t, \quad x \in (-\infty, \infty), \quad t > 0,$$
$$u(x, 0) = e^{-x^2}, \quad x \in (-\infty, \infty).$$

[Solution (continued):] Using the fact that $\int_{-\infty}^{\infty} e^{-p^2} dp = \sqrt{\pi}$, we get the solution as

$$w(x, t) = \frac{1}{\sqrt{1 + 4kt}} \exp\left(-\frac{x^2}{1 + 4kt}\right).$$

Therefore, the solution of the original problem is given by

$$u(x, t) = w(x, t) + v(t) = \frac{1}{\sqrt{1 + 4kt}} \exp\left(-\frac{x^2}{1 + 4kt}\right) + \sin t.$$

The wave equation on the half line $D = (0, \infty)$

The initial/boundary value problem (IBVP) containing a Dirichlet boundary condition at the endpoint $x = 0$

IBVP

$$\begin{aligned}
&v_{tt} - c^2 v_{xx} = 0, \quad 0 < x < \infty, \ t > 0, \\
&v(x, 0) = \phi(x), \quad v_t(x, 0) = \psi(x), \quad \text{(initial condition at } t = 0\text{)}, \\
&v(0, t) = 0 \quad \text{(boundary condition at } x = 0\text{)},
\end{aligned} \tag{15}$$

$x = 0$ → a fixed boundary condition

where we assume that $\phi(0) = 0$ (consistent condition).

- If the solution to the above mixed initial/boundary value problem (15) exists, then it must be unique from an application of the energy method.
- For the vibrating string, the boundary condition of (15) means that the end of the string at $x = 0$ is held fixed.

The reflection method

- To solve the Dirichlet problem (15), the idea is again to extend the initial data, in this case ϕ, ψ, to the whole line.
- Since the boundary condition is in the Dirichlet form, one should take the odd extensions:

$\tilde{\phi}(-x) = \begin{cases} \phi(-x) \\ -\phi(x) \\ 0 \end{cases}$ $-x>0 \iff x<0$, $-x<0 \iff x>0$, $x=0$

$$\tilde{\phi}(x) = \begin{cases} \phi(x), & x > 0, \\ -\phi(-x), & x < 0, \\ 0, & x = 0, \end{cases} \quad \tilde{\psi}(x) = \begin{cases} \psi(x), & x > 0, \\ -\psi(-x), & x < 0, \\ 0, & x = 0. \end{cases} \tag{16}$$

- Then we solve the extended wave equation on the whole real axis:

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, & -\infty < x < \infty, \quad t > 0, \\ u(x,0) = \tilde{\phi}(x), \quad u_t(x,0) = \tilde{\psi}(x), & -\infty < x < \infty, \end{cases} \tag{17}$$

$u(0,t)=0$

Since $\tilde{\phi}(x)$ and $\tilde{\psi}(x)$ are odd, $u(x,t)$ is odd in x

The reflection method (Cont'd)

- Since the initial data of the above IVP are odd, we know that the solution of the IVP, $u(x,t)$, will also be odd in the x variable, and hence $u(0,t) = 0$ for all $t > 0$.
- Then defining the restriction of $u(x,t)$ to the positive half-line $x \geq 0$,

$$v(x,t) = u(x,t)\big|_{x \geq 0}, \tag{18}$$

  we automatically have that $v(0,t) = u(0,t) = 0$. So the boundary condition of the Dirichlet problem (15) is satisfied for v.
- The initial conditions are satisfied for v as well, since the restrictions of $\tilde{\phi}(x)$ and $\tilde{\psi}(x)$ to the positive half-line are $\phi(x)$ and $\psi(x)$ respectively.
- $v(x,t)$ solves the wave equation for $x > 0$, since $u(x,t)$ satisfies the wave equation for all $x \in R$, and in particular for $x > 0$.
- Therefore, $v(x,t)$ as defined in (18) is the unique solution to the IBVP (15).

The reflection method (Cont'd)

- Using the d'Alembert formula for the solution of (17), and taking the restriction (18), we have that for $x \geq 0$,

$$v(x,t) = \frac{1}{2}\left(\tilde{\phi}(x+ct) + \tilde{\phi}(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{\psi}(s)ds. \tag{19}$$

- For the IBVP (15), $x \geq 0$ and $t > 0$, so $x + ct \geq 0$. We only need to consider the sign of $x - ct$.

$\hookrightarrow \tilde{\phi}(x+ct) = \phi(x+ct)$

- If $x - ct > 0$, we have

$\Updownarrow$

$t < \frac{x}{c}$

$$v(x,t) = \frac{1}{2}\left(\phi(x+ct) + \phi(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds, \tag{20}$$

which is exactly d'Alembert formula.

Any information at x takes a time of $\frac{x}{c}$ to arrive at $x=0$

$t > \frac{x}{c}$ Any initial information at x has already arrived at $x=0$ after $\frac{x}{c}$ sec.

The reflection method (Cont'd)

- If $x - ct < 0$, and using (16) we can rewrite the solution (19) as

$$\begin{aligned}
v(x,t) &= \frac{1}{2}\left[\tilde{\phi}(\underbrace{x+ct}_{>0}) + \tilde{\phi}(x-ct)\right] + \frac{1}{2c}\left[\int_{x-ct}^{0} \tilde{\psi}(s)ds + \int_0^{x+ct} \tilde{\psi}(s)ds\right] \\
&= \frac{1}{2}\left[\phi(x+ct) - \phi(ct-x)\right] + \frac{1}{2c}\left[\int_{x-ct}^{0} -\psi(-s)ds + \int_0^{x+ct} \psi(s)ds\right] \\
&= \frac{1}{2}\left[\phi(x+ct) - \phi(ct-x)\right] + \frac{1}{2c}\left[\int_{ct-x}^{0} \psi(s)ds + \int_0^{x+ct} \psi(s)ds\right] \\
&= \frac{1}{2}\left[\phi(x+ct) - \phi(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi(s)ds
\end{aligned} \tag{21}$$

use $-s$ to replace s

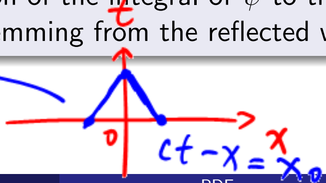
Solution formula

In summary, we have the solution formula for the IBVP (15):

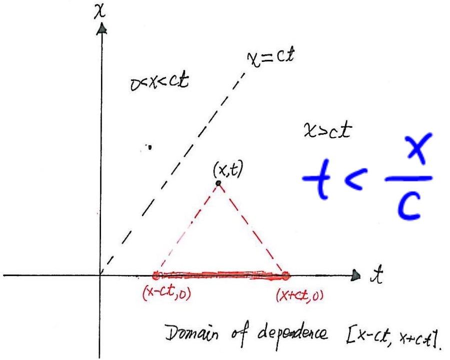
$$v(x,t) = \begin{cases} \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds, & x > ct, \\ \frac{1}{2}\left[\phi(x+ct) - \phi(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{ct+x} \psi(s)ds, & 0 < x < ct. \end{cases} \tag{22}$$

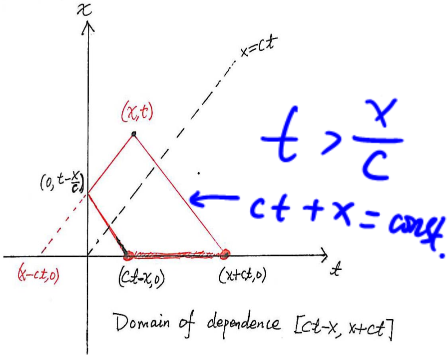
The minus sign in front of $\phi(ct-x)$ in the second expression above, as well as the reduction of the integral of ψ to the smaller interval are due to the cancellation stemming from the reflected wave.

$x - ct = x_0$



We find that the reflection happens when the left-going wave hits the boundary $x = 0$, and this results in the change of domain dependence (see the Figure below), as seen from the second formula of (22).





Example B (not in the textbook)

Consider the wave equation under the situation of initially at rest:

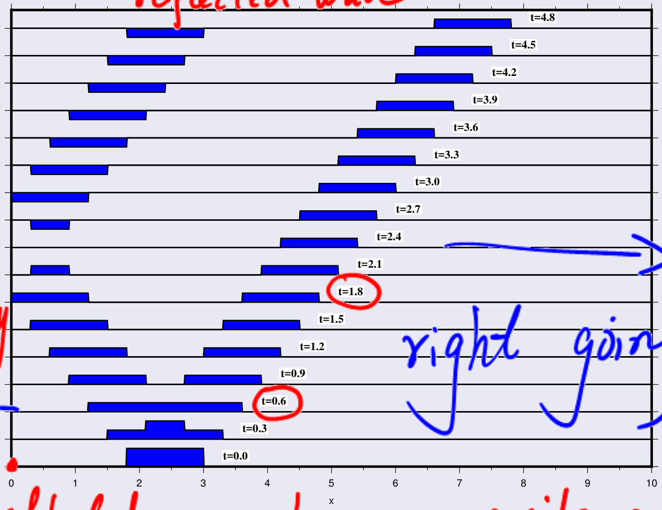
$$
\begin{aligned}
&u_{tt} - u_{xx} = 0, \quad 0 < x < \infty, \quad t > 0, \qquad c = 1 \\
&\underline{u(0,t) = 0,} \quad t > 0, \\
&u_t(x,0) = 0, \quad u(x,0) = \phi(x) = \begin{cases} 1, & 1.8 < x < 3, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
\tag{23}
$$

$$
u(x,t) = \begin{cases} [\phi(x+t) + \phi(x-t)]/2, & x > t \\ [\phi(x+t) - \phi(t-x)]/2 & t > x \end{cases}
$$

§3.2 Reflections of waves

Example B (Cont'd)

reflected wave



Fixed Boundary

right going

wave incoming the

The reflected wave has an opposite polarity to

The Neumann problem on the half-line

IBVP

$$
\begin{aligned}
&w_{tt} - c^2 w_{xx} = 0, \quad 0 < x < \infty, \quad t > 0, \\
&w_x(0,t) = 0, \quad t > 0, \\
&w(x,0) = \phi(x), \quad w_t(x,0) = \psi(x), \quad x > 0.
\end{aligned} \tag{25}
$$

We use the reflection method with even extensions to reduce the problem to an IVP on the whole line.

Solving the Neumann problem on the half-line

- Define the even extensions of the initial data

$$\tilde{\phi}(x) = \begin{cases} \phi(x), & x > 0, \\ \phi(-x), & x < 0, \end{cases} \quad \tilde{\psi}(x) = \begin{cases} \psi(x), & x > 0, \\ \psi(-x), & x < 0. \end{cases} \tag{26}$$

- Then we solve the extended wave equation on the whole real axis:

$$\text{IVP} \begin{cases} u_{tt} - c^2 u_{xx} = 0, \quad -\infty < x < \infty, \quad t > 0, \\ u(x,0) = \tilde{\phi}(x), \quad u_t(x,0) = \tilde{\psi}(x), \quad -\infty < x < \infty. \end{cases} \tag{27}$$

- Clearly, the solution $u(x,t)$ to the IVP (27) will be even in x, and since the derivative of an even function is odd, $u_x(x,t)$ will be odd in x, and hence $u_x(0,t) = 0$ for all $t > 0$.

Solving the Neumann problem on the half-line (Cont'd)

- Just like the case of the Dirichlet problem, the restriction

$$w(x,t) = u(x,t)\big|_{x\geq 0} \tag{28}$$

  will be the unique solution of the Neumann problem (25).
- Using d'Alembert formula for the solution of (27), and taking the restriction (28), we have that for $x \geq 0$,

$$w(x,t) = \frac{1}{2}\left[\tilde{\phi}(\underbrace{x+ct}_{\geq 0}) + \tilde{\phi}(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{\overbrace{x+ct}^{\geq 0}} \tilde{\psi}(s)ds. \tag{29}$$

- Once again, we need to consider the two cases $x > ct$ and $0 < x < ct$ separately.

§3.2 Reflections of waves

Solving the Neumann problem on the half-line (Cont'd)

- If $x - ct > 0$, we have

$t < \frac{x}{c}$

$$w(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds. \tag{30}$$

- If $x - ct < 0$, and using (26) we can rewrite the solution (29) as

$s<0$

$t > \frac{x}{c}$

$$\begin{aligned} w(x,t) &= \frac{1}{2}\left[\tilde{\phi}(x+ct) + \tilde{\phi}(x-ct)\right] + \frac{1}{2c}\left[\int_{x-ct}^{0} \tilde{\psi}(s)ds + \int_{0}^{x+ct} \tilde{\psi}(s)ds\right] \\ &= \frac{1}{2}\left[\phi(x+ct) + \phi(ct-x)\right] + \frac{1}{2c}\left[\int_{x-ct}^{0} \psi(-s)ds + \int_{0}^{x+ct} \psi(s)ds\right] \\ &= \frac{1}{2}\left[\phi(x+ct) + \phi(ct-x)\right] + \frac{1}{2c}\left[-\int_{ct-x}^{0} \psi(s)ds + \int_{0}^{x+ct} \psi(s)ds\right] \\ &= \frac{1}{2}\left[\phi(x+ct) + \phi(ct-x)\right] + \frac{1}{2c}\left[\int_{0}^{ct-x} \psi(s)ds + \int_{0}^{x+ct} \psi(s)ds\right] \end{aligned} \tag{31}$$

Solution formula

We have the solution formula for the Neumann problem on the half-line (25):

$$w(x,t)=\begin{cases}\dfrac{\phi(x+ct)+\phi(x-ct)}{2}+\dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct}\psi(s)ds, & x>ct,\\[2ex] \dfrac{\phi(x+ct)+\phi(ct-x)}{2}+\dfrac{1}{2c}\left(\displaystyle\int_0^{ct-x}+\int_0^{x+ct}\right)\psi(s)ds, & 0<x<ct.\end{cases} \tag{32}$$

The Neumann boundary condition corresponds to a vibrating string with a free end at $x=0$, since the string tension, which is proportional to the derivative $w_x(x,t)$, vanishes at $x=0$. In this case the reflected wave adds to the original wave, rather than canceling it.

Example C (not in the textbook)

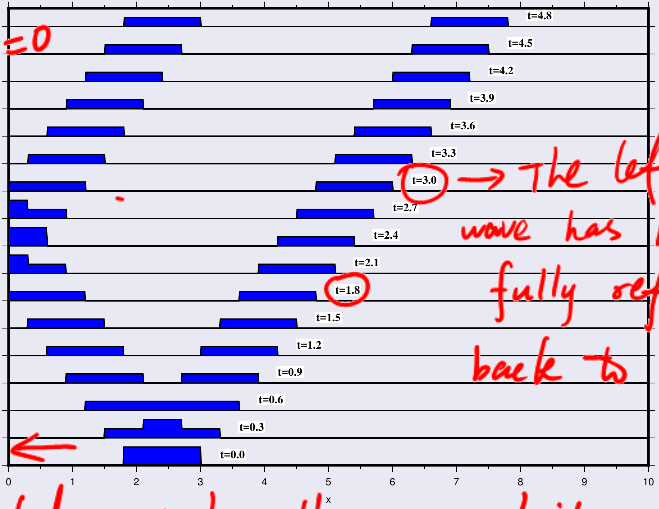
Consider the wave equation under the Neumann boundary condition:

$$
\begin{aligned}
&u_{tt} - u_{xx} = 0, \quad 0 < x < \infty, \quad t > 0, \\
&u_x(0,t) = 0, \quad t > 0, \\
&u_t(x,0) = 0, \quad u(x,0) = \phi(x) = \begin{cases} 1, & 1.8 < x < 3, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
\tag{33}
$$

Example C (Cont'd)

$u_x(0,t)=0$

free boundary at $x=0$



The left-going wave has been fully reflected back to $x>0$.

The reflected wave has the same polarity as the incoming wave.

Conclusion

1. We derived the solution to the wave equation on the half-line in much the same way as was done for the diffusion equation. That is, we reduced the initial/boundary value problem to the initial value problem over the whole line through appropriate extension of the initial data.
2. The characteristics nicely illustrate the reflection phenomenon. We saw that the characteristics that hit the initial data after reflection from the boundary wall $x = 0$ carry the values of the initial data with a minus sign in the case of the Dirichlet boundary conditions, and with a plus sign in the case of the Neumann boundary conditions. This corresponds to our intuition of reflected waves from a fixed end, and free end respectively.