MH4110: Partial Differential Equations

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Taken during AY 25/26 at

School of Physical and Mathematical Sciences
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Contents

0 Preface and Disclaimer

This note is written by me, a student who taken the course MH4110: Partial Differential Equations under Prof. Tong Ping during academic year 25/26 at the School of Physical and Mathematical Sciences of Nanyang Technological University in Singapore.

I hereby disclaim that the contents of this note are intended solely for my personal use and are not originally produced by me, except for the presentation of the definitions and the written proofs. I do not take responsibility for any grammatical or mathematical errors in this note. The note may be incomplete, and once the course is finished, I will no longer update it for any reason.

1 Where PDEs Come From

1.1 Notations, Definitions, and Basic Concepts

Throughout the notes we will adapt the following notations and symbol:

- $u_x = \dfrac{\partial u}{\partial x} = \partial_x u$
- $u_{xy} = \dfrac{\partial^2 u}{\partial x \partial y} = \partial_{xy} u$
- $\nabla u = \operatorname{grad} u = u_x \mathbf{i} + u_y \mathbf{j} = (u_x, u_y)$
- $\Delta u = u_{xx} + u_{yy}$
- $\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = u_x + v_y + w_z$ where $\mathbf{F} = (u, v, w)$
- $\nabla \times \mathbf{F} = \operatorname{curl} F = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ u & v & w \end{vmatrix}$
- Given $\vec{v} = (a, b) \neq \mathbf{0}$. The directional derivative of u along $\vec{v}$ at (x, y) is defined by

$$\nabla_{\vec{v}}\, u = \nabla u \cdot \frac{\vec{v}}{|\vec{v}|}$$

We assume that $u_{xy} = u_{yx}$. Some tools that will be useful are listed here:

- Multivariable chain rule
- Derivatives of integrals
- Green's Theorem
- Divergence Theorem

Definition 1.1.1 (Partial Differential Equation)**.** A partial differential equation (PDE) is an identity that relates

- the independent variables (i.e. the inputs): $x, y, \ldots$
- the dependent variable u
- the partial derrivatives of u

A PDE can be expressed in the form of $F(x, y, \ldots, u, u_x, u_y, \ldots, u_{xx}, u_{xy}, u_{yy}, \ldots) = 0$.

Definition 1.1.2 (Order)**.** The order of a PDE is the order of the highest partial derivative that appears in the equation.

Definition 1.1.3 (Solution)**.** A solution of a PDE is a function $u(x, y, \ldots)$ that satisfies the equation identically, possibly in some region of $x, y, \ldots$ variables.

Definition 1.1.4 (Linear operator)**.** Let $\mathcal{L}$ be an operator. We say $\mathcal{L}$ is a linear operator if the following is satisfied

$$\mathcal{L}(u + v) = \mathcal{L}u + \mathcal{L}v \quad \text{and} \quad \mathcal{L}(cu) = c\mathcal{L}u$$

where u, v are functions and c is constant.

Definition 1.1.5 (Linearity and Homogeneity)**.** Let $\mathcal{L}$ be a linear operator. Then the equation $\mathcal{L}u = g$ is said to be linear, where g is a given function of independent variables. Additionally, we say the equation is homogeneous linear equation if $g = 0$, and vice versa.

Theorem 1.1.6 (Superposition principle)**.** *Consider the equation $\mathcal{L}u = 0$. Suppose that $u_1, u_2, \ldots, u_n$ are solutions of the equations. Then the linear combination of the solutions is also a solution:*

$$\sum_{j=1}^{n} c_j u_j(x)$$

where c_j are constants. Additionally, the sum of a homogeneous solution and an inhomogeneous solution is an inhomogeneous solution.

The upshot of superposition principle is that the solution structure of a PDE is always in the form of **general solution** + **particular solution**, where the general solution is obtained simply solving the corresponding homogenous PDE. On the other hand, the particular solution might requires some hardwork to figure out. If the PDE is homogeneous, then the particular solution degenerates into 0.

1.2 First Order Linear Equation

Here, we will investigate the first-order linear PDE, which takes the form

$$a(x, y)u_x + b(x, y)u_y + c(x, y)u = f(x, y)$$

where a, b, f are continuous functions in some domain D. In particular, the two methods to solve such PDE are geometric methods and coordinate methods.

We will be using directional derivative. To interpret it, suppose given the directional derivative of u along $\vec{v}$ is $\nabla_{\vec{v}} u = g(x, y)$. This means that in the direction of $\vec{v}$, the gradient of u is $g(x, y)$.

<u>Type 1.1: $au_x + bu_y = 0$ with $a^2 + b^2 \neq 0$</u>

We first introduce the geometric method. Firstly, the equation can be rewritten as follow:

$$\begin{aligned} au_x + bu_y &= 0 \\ (a, b) \cdot \nabla u &= 0 \\ \nabla_{(a,b)} u &= 0 \end{aligned}$$

This suggests that $u(x, y)$ is constant along the direction (a, b). Take note that there are three variables here: x, y, and u, so one can picture that in a 3-dimensional cartesian coordinate system.

The line that parallel to (a, b) has the equation $bx - ay = c$, where c is constant. The family of such lines are called the *characteristic lines*. Since u is constant on these lines, the solution is then

$$u(x, y)|_{bx-ay=c} = f(c)$$

This means that if we only examine (x, y) that lies on the line $bx - ay = c$, the output u is an arbitrary function of one variable f. Of course, we can substitute c so that the solution becomes

$$u(x, y) = f(bx - ay)$$

The next method is coordinate method. This method heavily relies on the characteristic lines (or curves), which in this case we have $bx - ay = c$. The strategy is summarised here:

1. Changing variables, where one of the new variable has to be the form of characteristic lines (or curve).
2. Replace all x and y derivatives into the newly defined variables.
3. Rewrite the PDE and solve it.

We use the same example to demonstrate: consider $x' = x$ and $y' = bx - ay$. By chain rule we see

$$\begin{aligned} \frac{\partial u}{\partial x} &= u_{x'} \frac{\partial x'}{\partial x} + u_{y'} \frac{\partial y'}{\partial x} = u_{x'} + bu_{y'} \\ \frac{\partial u}{\partial y} &= u_{x'} \frac{\partial x'}{\partial y} + u_{y'} \frac{\partial y'}{\partial y} = -au_{y'} \end{aligned}$$

Then, rewrite the PDE $au_x + bu_y = 0$ into

$$a(u_{x'} + bu_{y'}) + b(-au_{y'}) = 0$$

which reduces to $au_{x'} = 0$. If $a \neq 0$, then $u_{x'} = 0$, and thus

$$u = f(y') = f(bx - ay)$$

If $a = 0$, then we must have $b \neq 0$. The PDE then becomes $bu_y = 0$, which reduces to $u_y = 0$. This gives $u = f(x)$, which is a specific form of $f(bx - ay)$. To conclude, we have the solution

$$u = f(bx - ay)$$

where f is an arbitrary single variable function.

Type 1.2: $au_x + bu_y = c$ with a, b, c as constants and $a \neq 0$

Recall that from Superposition Principle the solution structure of any PDE is always

$$\text{general solution} + \text{particular solution}$$

Here general solution is already obtained previously. It remains to make an educated guess on form of particular solution. With experience from ODE, it is clear that we can consider $u_0(x, y) = Ax$ where A is some constant. Substituting into the PDE we get

$$Aa + 0 = c \implies A = \frac{c}{a}$$

Thus a particular solution is $u_0 = \frac{c}{a}x$, and the full solution is

$$u = f(bx - ay) + \frac{c}{a}x$$

Type 1.3: $au_x + bu_y = cu$ with a, b, c as constants and $a \neq 0$

Using the characteristic equation method we listed down

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{cu}$$

Solving $\frac{dx}{a} = \frac{dy}{b}$ gives us

$$bx - ay = C$$

where C is a constant.

Next, relate dx (or dy) with du. Thus by solving $\frac{dx}{a} = \frac{du}{cu}$ we obtain solution

$$\ln|u| = \frac{c}{a}x + f(C')$$

which is equivalent to $u = e^{\frac{c}{a}x}g(C'')$ by removing the logarithm. Lastly, since C'' is an arbitrary constant, and previously we obtained that $bx - ay = C$ where C is constant. We can substitute C'' so that we now get

$$u = e^{\frac{c}{a}x}g(bx - ay)$$

Remark 1.2.1. Alternatively, one can choose the following method:

1. When $u = 0$, check that it is a solution.

2. When $u \neq 0$, we can divide u to obtain

$$a\frac{u_x}{u} + b\frac{u_y}{u} = c$$

3. By setting $v = \ln|u|$, we see that $v_x = \frac{u_x}{u} + \frac{u_y}{u}$

4. The PDE can be rewritten as $av_x + bv_y = c$, which the solution is fully known.

Type 2: $a(x,y)u_x + b(x,y)u_y = 0$ with $a^2 + b^2 \neq 0$

In the case that the coefficient of u_x and u_y are variables, similar to before, we can solve it in geometric method or coefficient method. The characteristic equation of this type is

$$\frac{dx}{a(x,y)} = \frac{dy}{b(x,y)} \iff \frac{dy}{dx} = \frac{b(x,y)}{a(x,y)}$$

In general the PDE can be solved as long as the ODE above can be solved.

Example 1.2.2. We solve $u_x + yu_y = 0$ in two methods. Regardless of which method we use, we need to solve the characteristic curves:

$$\frac{dy}{dx} = \frac{y}{1} \implies y = Ce^x \implies C = ye^{-x}$$

For the geometric method, we know that $u(x,y)$ is constant on $y = Ce^x$ where C is a constant. Thus $u(x,y) = f(C)$ where f is an arbitrary function of single variable. Rearranging and substituting we get general solution

$$u(x,y) = f(e^{-x}y)$$

For the coefficient method, we let $x' = x$ and $y' = ye^{-x}$. Using chain rule we obtain

$$u_x = u_{x'} - ye^{-x}u_{y'} \quad \text{and} \quad u_y = e^{-x}u_{y'}$$

Next, rewritting $u_x + yu_y = 0$ we get $u'_x = 0$, giving us the solution

$$u = f(y') = f(ye^{-x})$$

Example 1.2.3. We solve $u_x + 2xy^2u_y = 0$. First, the characteristic curve:

$$\frac{dy}{dx} = 2xy^2 \implies x^2 + \frac{1}{y} = C$$

where C is a constant. Note that the solving process of the characteristic curve involves dividing y, thus we need to check separately that whether $y = 0$ is a solution, which it indeed is.

Thus, using the geometric method, we see that the solution of the PDE is

$$u(x,y) = f(C) = f\left(x^2 + \frac{1}{y}\right)$$

Notice that this expression does not degenerate to another solution $y = 0$, thus the complete solution is

$$u(x,y) = f(C) = f\left(x^2 + \frac{1}{y}\right) \quad \text{or} \quad y = 0$$

In summary, first order linear PDE can be fully solved using the method of characteristic, and combined with geometric method or coefficient method. Say given $P(x,y)u_x + Q(x,y)u_y = R(x,y,u)$, the characteristic equations are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{du}{R}$$

1.3 Flows, Vibrations, and Diffusions

In this part, we introduce some PDEs that describe physical phenomenon. We omit the derivation of most PDEs here. One who is interested can check any standard PDE materials on his own.

Specifically, the three fundamental PDEs of this course are:

- Wave equation, which describes vibrations.
- Heat equation, which describes flows.
- Laplace equation, which describes diffusions.

Transport Equation: $u_t(x,t) + cu_x(x,t) = 0$

Transport equation describe the density of cars on a road way under ideal conditions. We will derive this equation here.

We assume that the road is long and straight, with all the cars running at the same speed c. Also we assume that no cars entering or exiting the road. Define $u(x,t)$ be the car density at time t and position x.

The number of cars in the interval $[0,b]$ at time t can be calculated as

$$M = \int_0^b u(x,t)dx$$

Since all the cars are in constant speed c, at the later time $t+h$ they move to the right by ch km. Thus we have

$$M = \int_0^b u(x,t)dx = \int_{ch}^{b+ch} u(x,t+h)dx$$

where differentiating both sides wrt b we get

$$u(b,t) = u(b+ch, t+h)$$

By changing the specific location b to an arbitrary position x and rearranging the equation we get

$$u(x+ch, t+h) - u(x,t) = 0$$

We are interested at the car density at some sudden time instead of a time interval. So we divide h and take limit $h \to 0$:

$$\begin{aligned}
\lim_{h\to 0} \frac{u(x+ch,t+h) - u(x,t)}{h} &= 0 \\
\lim_{h\to 0} \frac{u(x+ch,t+h) - u(x+ch,t) + u(x+ch,t) - u(x,t)}{h} &= 0 \\
\lim_{h\to 0} \frac{u(x+ch,t+h) - u(x+ch,t)}{h} + \lim_{h\to 0} \frac{u(x+ch,t) - u(x,t)}{h} &= 0 \\
\frac{\partial u(x,t)}{\partial t} + c\frac{\partial u(x,t)}{\partial x} &= 0
\end{aligned}$$

This gives us

$$u_t(x,t) + cu_x(x,t) = 0$$

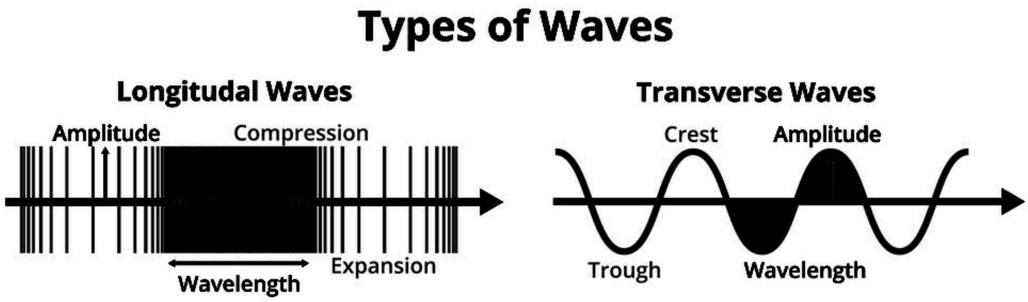
This is a first order linear PDE. The characteristic line is $ct - x = W$ where W are arbitrary constants, and the general solution is $f(ct-x)$ where f is arbitrary function.

Remark 1.3.1. Of course we can check if the derived equation is consistent or not, in the sense that if the units of two terms agree. Here u_t has the unit of numbers of car per second. Recall c is speed, so it has unit meter per second. Also u_x has unit of numbers of car per meter squared (due to the extra derivative $\partial/\partial x$). Together cu_x has unit of numbers of car per second.

Remark 1.3.2. In the case that the car moves to left, then c is taken to be negative.

Wave Equation: $u_{tt} = c^2 u_{xx}$

This is first of the three fundamental PDEs of this course. Wave equation gives the PDE that describe, clearly, waves. In general, there are two types of waves in 2D: transverse wave and longitudinal wave.



The wave equation introduced here describes transverse wave and assume that no involmen of longitudinal wave. First, we introduce several variables:

- Linear density ρ: unit of mass per unit of length.
- Then ρdx is then the unit of mass of length segment dx
- Let $u(t, x)$ be the displacement of strings from equilibrium position at time t and position x.
- We assume the linear density is constant throughout the string.
- We ignore all other forces except for the tension $\mathbf{T}(x, t)$.

Definition 1.3.3 (Wave Equation)**.** Assuming that T is a constant, the wave equation takes the form

$$u_{tt} = c^2 u_{xx}$$

where $c = \sqrt{\frac{T}{\rho}}$ is the wave speed.

Remark 1.3.4. We listed some variants of wave equations here.

- If considering air resistance r, the wave equation is then

$$u_{tt} - c^2 u_{xx} + r u_t = 0$$

- If there is a traverse elastic force, the wave equation becomes

$$u_{tt} - c^2 u_{xx} + ku = 0$$

- If there is an external force f, the wave equation becomes

$$u_{tt} - c^2 u_{xx} = f(x, t)$$

- There is wave equation for multidimensional case, but will not be mentioned here.

Heat Equation: $u_t = k(u_{xx} + u_{yy} + u_{zz})$

This is the second of the three fundamental PDEs of this course. Despite suggested by its name, heat equation can describe general flows. Again, we first introduce the variables needed:

- We work in a Minskowski space, i.e. spatial dimension (x, y, z) that describes the position and dimension t that describe the time. This is equivalent to $\mathbb{R}^4$.
- Let $u(x, y, z, t)$ be the temperature at point (x, y, z) and time t.
- Let D be a domain in $\mathbb{R}^3$ that represents the body, and $H(t)$ be the amount of heat (or energy) in D.
- Let ρ be the density of the material.
- Let c be the specific heat of the material (In physics, specific heat is the energy needed to raise one unit of mass of a substance by one unit of temperature. Simply speaking, it is just a constant.)

- Let κ be the heat conductivity of the material. (Another constant. Simply speaking, it descirbes the material's ability to conduct heat. Larger values represents better conductivity.)

For the sake of simplicity, we assume that the space ($\mathbb{R}^3$) outside the body (D) is colder. With that said, the direction of the heat flow, or more accurately, the energy flux, go from inside to outside.

Definition 1.3.5 (Heat Equation)**.** With all defined notations, the heat equation takes the form

$$c\rho u_t = \nabla \cdot (\kappa \nabla u)$$

In the case that c, ρ and κ are constant, we can combine the constant into $k = \frac{\kappa}{c\rho}$ and the equation becomes

$$u_t = k\triangle u$$

where recall $\triangle u = u_{xx} + u_{yy} + u_{zz}$.

Ultimately, the heat equation says that the heat flow u_t is propotional to $\triangle u$.

Remark 1.3.6 (Interepretation of heat equation.)**.** In 1D, we learned that the second derivative represents the curvature of the curve: concave or convex. In particular, if $\triangle u = u_{xx}$

- $= 0$, it means that it is a straight line. (The point is the same as its neighbors).
- > 0, it means that concave upwards ($\cup$ shape, the point is lower than its neighbors).
- < 0, it means concave downwords ($\cap$ shape, the point is higher than its neighbors)

In the 3D case, the expression $\triangle u = u_{xx} + u_{yy} + u_{zz}$ carries the same interpretation. The only difference is that we are now talking about 'temperature', so the higher the values of $|\triangle u|$, the larger the temperature difference between the point and its surroundings neighbors, and thus it is expected to have higher heat flow, i.e. higher u_t.

Laplace equation: $u_{xx} + u_{yy} + u_{zz} = 0$

Laplace equation describes diffusions. Think of a room with a heat source, it is clear that the heat is not homogeneous in the room: the closer the heat sources, the higher the heat. Assuming that the heat source exists in the room for long enough time, the temperature of the room is eventually homogeneous. In this case, the heat does not change with time, thus the heat flow u_t is 0. Therefore from heat equation we obtain:

Definition 1.3.7 (Laplace equations)**.** With same notation defined previously, the Laplace equation takes the form

$$\triangle u = u_{xx} + u_{yy} + u_{zz} = 0$$

and its solutions are called harmonic solutions.

In the case that the above PDEs is inhomogeneous, i.e. $\triangle u = f$ for some non-zero f, we called it the *Poisson equation.*

1.4 Initial and Boundary Conditions

As in ODE, the solution we get from a PDE actually comprises a whole family of functions that satisfies the given equation. In reality, some conditions are required to single out a solution. These conditions fall into two types: initial conditions and boundary conditions.

Intial condition specify the physical state at a particular time t_0. Simply speaking, it gives what happens exactly at same positions and time.

On the other hand, *boundary condition* gives a domain D for which the PDE is defined.

Example 1.4.1 (Examples of boundary conditions.)**.** Let D be a domain where a given PDE is defined. Let $\mathbf{n}$ be the unit normal vector pointing outwards on ∂D. Let $\Gamma \subseteq \partial D$.

- Dirichlet boundary condition: $u(\mathbf{x}, t) \mid_\Gamma = g(\mathbf{x}, t)$
- Neumann boundary condition: $u_\mathbf{n} \mid_\Gamma = g(\mathbf{x}, t)$

- Robin boundary condition: $(u_{\mathbf{n}} + au)|_{\Gamma} = g(\mathbf{x}, t)$

In any of the above example, we say the boundary conditions are homogeneous if $g(\mathbf{x}, t) = 0$, and inhomogeneous otherwise.

1.5 Well-posedness of a PDE

Suppose given a PDE in a domain D with a set of initial and/or boundary conditions. We say it is well-posed if all the following are met:

1. (Existence) There exists at least one solution.
2. (Uniqueness) There exists a unique solution.
3. (Stability) Small changes in the initial and/or boundary conditions lead to small changes in the output.

Well-posedness is an important property to have if one is using PDE to study the actual world. In particular, existence and uniqueness ensures that the model(PDE) is reliable, and stability gives basic control of the system due to continuous dependence.

1.6 Types of Second-Order Equations

We have investigated first-order linear PDE in the previous section, where we demonstrate how it can be reduced to an ODE problem via the method of characteristic curves. We now study second-order linear PDE. We will only go through the types of PDEs in this part, and the solution of each PDE will be presented in the next chapter.

Definition 1.6.1 (Second-order linear PDE)**.** A second-order linear PDE with n variables $x_1, x_2, \dots, x_n$ is a PDE that takes the form

$$\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} a_i u_{x_i} + a_0 u = 0 \tag{1}$$ {eqn: 2nd-or

where all constants a_{ij}, a_i, and a_0 are reals. Recall that we assumed that mixed derivatives are equal, thus we have $a_{ij} = a_{ji}$.

Definition 1.6.2 (Coefficient Matrix and Eigenvalues of PDEs)**.** Suppose given a second-order linear PDE of form (1). Its coefficient matrix is defined to be the $n \times n$ matrix where the (i, j)-th entry is the coefficient a_{ij}. The eigenvalues of a PDE is the eigenvalues of its coefficient matrix.

Definition 1.6.3 (Elliptic, Hyperbolic, Ultrahyperbolic, and Parabolic PDEs)**.** Suppose given a second-order linear PDE of form (1). We have the following classification of a given PDE:

- Elliptic: if all its eigenvalues are positive or negative.
- Hyperbolic: if all its eigenvalues are non-zero, and one of them has opposite sign from the other $n - 1$ eigenvalues.
- Ultrahyperbolic: if all its eigenvalues are non-zero, and at least two of them are positive as well as at least two of them are negative.
- Parabolic: if exactly one of them is zero, and all others have the same sign.

Note that wave equation, heat equation, and Laplace equation are all second-order linear PDEs. We can now have a classification of them

PDEs	Form	Eigenvalues	Types
Wave equation	$u_{tt} - c^2(u_{xx} + u_{yy} + u_{zz}) = 0$	$\left\{1^{(1)}, -c^{2(3)}\right\}$	Hyperbolic
Heat equation	$u_t - k(u_{xx} + u_{yy} + u_{zz}) = 0$	$\left\{0^{(1)}, -k^{(3)}\right\}$	Parabolic
Laplace equation	$u_{xx} + u_{yy} + u_{zz} = 0$	$\left\{1^{(4)}\right\}$	Elliptic

From linear algebra we know that the determinant $\det(A)$ of a matrix A equals to the product of all its eigenvalues. The following theorem is obtained from this relation:

Theorem 1.6.4. *Given a second-order linear PDE with two variables:*

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0$$

Denote its coefficient matrix as A. We have:

1. *If $\det(A) > 0$, then the PDE is of elliptic type.*
2. *If $\det(A) < 0$, then the PDE is of hyperbolic type.*
3. *If $\det(A) = 0$, then the PDE is of parabolic type.*

Improtant: This determinant test does not apply for second-order linear PDE with more than two variables!

2 Waves and Diffusions

2.1 The Wave Equation

We have defined wave equation is the previous chapter. Note that if we take the wave equation on $\mathbb{R}$, it takes the form of

$$u_{tt} = c^2 u_{xx}$$

where $x \in \mathbb{R}$ is a variable of reals, and the constant $c > 0$ is the wave speed. We will study its solution in this part.

Method 1: Factorization of the differential operator

Note that we can rewrite the equation into

$$u_{tt} - c^2 u_{xx} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u = 0$$

where what we done is to factorize the differential operators.

Next, define

$$\begin{aligned} v &= \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right) u \\ &= u_t + c u_x \end{aligned}$$

The wave equation can then be rewritten as

$$\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) v = 0 \implies v_t - c v_x = 0$$

This is a first-order linear PDE, which its solution is fully known.

By the method of characteristic, we see that

$$v(x,t) = h(x + ct)$$

where h is an arbitrary function. Recovering v we see

$$u_t + c u_x = h(x + ct)$$

This is again a first-order linear PDE, except that it is inhomogeneous. To solve this, we have to solve its general solution and particular solution.

Its general solution is $u = g(x - ct)$ where g is an arbitrary function. For the particular solution, we can verify that $u = f(x + ct)$ is a particular solution:

$$h(x + ct) = u_t + c u_x = c f'(x + ct) + c f'(x + ct) = 2c f'(x + ct)$$

where we see we can take function f so that $f'(x + ct) = \frac{1}{2c} h(x + ct)$.

Therefore the solution is

$$u(x,t) = f(x + ct) + g(x - ct)$$

where f and g are arbitrary functions.

Example 2.1.1. Suppose given $u_{xx} - 3u_{xt} - 4u_{tt} = 0$. This can be viewed as the same to factorize

$$x^2 - 3xt - 4t^2 = 0$$

which is $(x - 4t)(x + t) = 0$. Translating back, the given PDE can be factorized as

$$(\partial_x - 4\partial_t)(\partial_x + \partial_t) u = 0$$

To continue solving the PDE, simply let $v = (\partial_x + \partial_t) u$ so that the PDE can be rewritten as

$$v_x - 4v_t = 0$$

The remaining process to continue the solution is the same as described in the above paragraph, the details are omitted here.

Method 2: Characteristic Coordinates

We can introduce the characteristic coordinates

$$\xi = x + ct \quad \text{and} \quad \eta = x - ct$$

By chain rule, we obtain

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \quad \text{and} \quad \frac{\partial}{\partial t} = c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta}$$

as well as

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial \xi^2} + 2\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2} \quad \text{and} \quad \frac{\partial^2}{\partial t^2} = c^2\left(\frac{\partial^2}{\partial \xi^2} - 2\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2}\right)$$

Substituting these into wave equation we get

$$u_{tt} - c^2 u_{xx} = -4c^2 u_{\xi\eta} = 0 \implies u_{\xi\eta} = 0$$

Solving out, we get solution

$$u = f(\xi) + g(\eta) = f(x + ct) + g(x - ct)$$

where f and g are two arbitrary functions.

Remark 2.1.2. The above methods actually suggest that the wave equation has two families of characteristic lines: $x \pm ct = \text{constant}$. As the time t increases:

- $g(x - ct)$ is a wave that travels to the right at speed c.
- $f(x + ct)$ is a wave that travels to the left at speed c.

As wave equation is motivated from physical phenomenon, it is natural to discuss how to solve its IVP. Suppose given the initial condition

$$u(x, 0) = \phi(x) \quad \text{and} \quad u_t(x, 0) = \psi(x) \quad \text{and} \quad -\infty < x < \infty$$

where ϕ and ψ are arbitrary functions of single variable x, we want to give the exact solution to the wave equation. Since the solution $u = f(x + ct) + g(x - ct)$ is known, by substituting specific x and/or t, and taking derivative when necessarily, we obtain:

$$u(x, 0) = f(x) + g(x) = \phi(x) \implies f'(x) + g'(x) = \phi'(x)$$

and

$$u_t(x, 0) = cf'(x) - cg'(x) = \psi(x)$$

These two equations give

$$\begin{cases} f'(x) & = \frac{1}{2}\phi'(x) + \frac{1}{2c}\psi(x) \\ g'(x) & = \frac{1}{2}\phi'(x) - \frac{1}{2c}\psi(x) \end{cases}$$

Integrating, we get

$$\begin{cases} f(x) & = \frac{1}{2}\phi(x) + \frac{1}{2c}\int_0^x \psi(s)ds + A \\ g(x) & = \frac{1}{2}\phi(x) - \frac{1}{2c}\int_0^x \psi(s)ds + B \end{cases}$$

Taking sum of two equations, we get $f(x) + g(x) = \phi(x) + (A + B)$. However, by comparing with the initial condition $u(x, 0) = f(x) + g(x) = \phi(x)$, we see that $A + B = 0$.

Lastly, substitute the obtained f and g into the solution $u(x, t) = f(x + ct) + g(x - ct)$ and simply into

$$u(x, t) = \frac{1}{2}[\phi(x + ct) + \phi(x - ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds$$

This is the solution formula for the IVP of wave equation. In particular, the setup of the IVP is called *Cauchy Problem*, and the obtained solution formula is called the *d'Alambert Formula*.

Remark 2.1.3 (General d'Alambert Formula)**.** If the given IVP is not starting at $t = 0$, say $t = t_0 > 0$, one can perform a change of variable by setting $t' = t - t_0$, so that the d'Alambert Formula can be applied to solve the IVP. In general, the formula for solving such IVP is

$$u(x, t) = \frac{1}{2}[\phi(x + c(t - t_0)) + \phi(x - c(t - t_0))] + \frac{1}{2c}\int_{x-c(t-t_0)}^{x+c(t-t_0)} \psi(s)ds$$

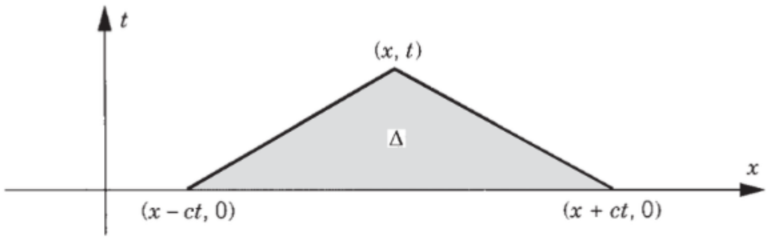
2.2 Causality and Energy

We have discussed d'Alambert formula previously. Note that $u(x,t)$ depends on the values of ϕ and ψ. In particular, the value $u(x,t)$ is affected by the following:

- value of ϕ at point $(x-ct,0)$
- value of ϕ at point $(x+ct,0)$
- value of ψ on the interval $[x-ct,x+ct]$.

The interval $[x-ct,x+ct]$ seems important as all informations are contained here.

Definition 2.2.1 (Interval of Dependence, Domain of Dependence)**.** The interval of dependece for point (x,t) is defined to be the interval $[x-ct,x+ct]$. The domain of dependence (or the past history) of the point (x,t) refers to the triangular region with vertices $(x-ct,0),(x-ct,0)$, and (x,t)



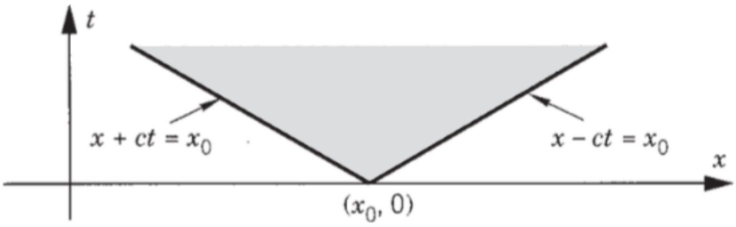
Remark 2.2.2. The triangular region is called the past history of point (x,t) for a reason. Consider that at the moment when the time equals to t, and we are in position x. This makes that everything within the triangular region representing 'the past' from when the time is t, since the second coordinate of all point within the triangular region is smaller than t.

Another physical interpretation of the triangular region is that, only the points inside the triangular region affects the value of $u(x,t)$, and hence the name domain of dependence. If something outside the region is altered, it won't affect the value of $u(x,t)$.

We can change our perspective by asking

> Suppose given a starting point $(x_0,0)$, for which point (x,t) where the value of $u(x,t)$ will be affected by the given point?

Clearly, many points will be affected, and these points are exactly the points within the shaded region in the following diagram:



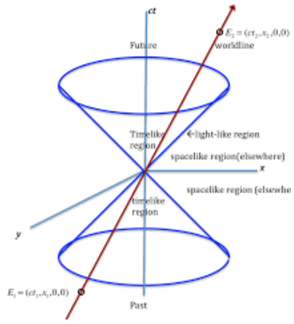
Definition 2.2.3 (Region of Influence)**.** The region of influence of the point $(x_0, 0)$ is defined to be the shaded region above.

The value $u(x,t)$ is affected by $(x_0, 0)$ if and only if

$$x - ct \leq x_0 \leq x + ct$$

given that the speed of the wave is bounded by c.

Remark 2.2.4 (Minkowski's light cone)**.** Since light has the property of wave-particle duality, photons obey the wave equation. The Minkowski's light cone, which illustrates causality from past and future, is related by the above concept.



Definition 2.2.5 (Kinetic Energy, Potential Energy, and Total Energy)**.** Consider an infinite string with constant linear density ρ and tension magnitude T. Inheritting the above notation:

- The kinetic energy at time t is defined as

$$E_K(t) = \frac{1}{2}\rho \int_{-\infty}^{\infty} u_t^2(x,t)\ dx$$

- The potential energy at time t is defined as

$$E_P(t) = \frac{1}{2}T \int_{-\infty}^{\infty} u_x^2(x,t)\ dx$$

The total energy at time t is just the sum of the kinetic energy and potential energy, both at time t:

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} (\rho u_t^2 + T u_x^2)\ dx$$

Remark 2.2.6 (Conservation of Energy)**.** It is well-known that energy is conserved in a closed system. Mathematically, it means that the rate of change of total energy is 0. Thus, showing the conservation of energy is equivalent to showing that $E'(t) = 0$.

So, we first obtain

$$E'(t) = \int_{-\infty}^{\infty} (\rho u_t u_{tt} + T u_x u_{xt})\ dx$$

Note that the wave equation gives $u_{tt} = c^2 u_{xx}$, where expanding out $c = \sqrt{\frac{T}{\rho}}$ we see that $\rho u_{tt} = T u_{xx}$. Substituting we get

$$E'(t) = \int_{-\infty}^{\infty} (T u_t u_{xx} + T u_x u_{xt})\ dx = T\int_{-\infty}^{\infty} (u_t u_{xx} + u_x u_{xt})\ dx$$

Note that $u_t u_{xx} + u_x u_{xt} = \dfrac{\partial}{\partial x}(u_t u_x)$, again substituting we get

$$E'(t) = T\int_{-\infty}^{\infty} \frac{\partial}{\partial x}(u_t u_x)\ dx$$

By Fundamental Theorem of Calculus, it can be evaluated as

$$E'(t) = T u_t u_x \Big|_{-\infty}^{\infty}$$

Of course, we assume that the energy is finite, so the wave flatten and rest eventually if we moved far enough away from the source $x = 0$. Recall that u_t represents velocity and u_x represents the slope of the wave. At $x = \pm\infty$, the string is now flat and still, which implies $u_x = 0$ and $u_t = 0$ respectively. Thus

$$E'(t) = T u_t u_x \Big|_{-\infty}^{\infty} = 0 - 0 = 0$$

The above can be summarized as follow:

Theorem 2.2.7 (Conservation of Energy and IVP)**.** *Adapting the same notation, we see that*

$$E(t) = constant \quad \forall t \geq 0$$

In addition, suppose given IVP $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$, then

$$E(t) = E(0) = \frac{1}{2}\int_{-\infty}^{\infty} \left(\rho\psi^2(x) + T(\phi'(x))^2\right)\, dx$$

2.3 The Diffusion Equation

The goal here is to solve the diffusion equation:

$$u_t = ku_{xx}$$

and obtain a solution formula, depending on the given initial data $u(x,0) = \phi(x)$. This is more difficult to solve comparing to the wave equation. The strategy is as follow:

1. Show the uniqueness of the solution.
2. Construct a solution to the problem.

We first give a theorem without proof:

Theorem 2.3.1 (Weak Maximum Principle)**.** *If $u(x,t)$ satisfies the diffusion equation $u_t = ku_{xx}$ in a rectangle $D = \{(x,t) : 0 \leq x \leq \ell,\ 0 \leq t \leq T\}$ in space-time, then the maximum value of $u(x,t)$ is achieved either initially at $t = 0$, or on the lateral sides, i.e. $x = 0$ or $x = \ell$.*

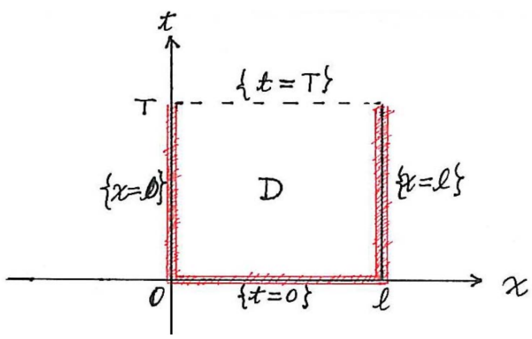
In other words, denote the union of the left, bottom, and right boundaries as:

$$\Gamma := \{(x,0) : 0 \leq x \leq \ell\} \cup \{(0,t) : 0 \leq t \leq T\} \cup \{(\ell,t) : 0 \leq t \leq T\}$$

Then

$$\max_{(x,t)\in\Gamma} u(x,t) = \max_{(x,t)\in D} u(x,t)$$

The notation of Γ is used in the later proof as well. In $x-t$ coordinate system, the union Γ of boundaries looks like this:



Remark 2.3.2.

1. The Strong Maximum Principle says that, the maximum *can and only can* achieved at Γ.
2. Note that Maximum Principle also implies Minimum Principle by applying it to the function $-u(x,t)$ by noting that $(-u)_t = k(-u)_{xx}$ and utilize the identity

$$\min\{u(x,t)\} = -\max\{-u(x,t)\}$$

3. From physical perspective, if applying to heat equation, the principle says that:

   The highest temperature in the interior body cannot exceed the highest initial temperature, or the highest temperature on the boundary.

We now show the uniqueness:

Theorem 2.3.3 (Uniqueness of Dirichlet Problem for Diffusion Equation)**.** *Consider the diffusion equation with Dirichlet boundary conditions:*

$$\begin{cases} u_t - ku_{xx} &= f(x,t) \ ,0 < x < \ell, t > 0 \\ u(x,0) &= \phi(x) \quad ,0 < x < \ell \\ u(0,t) &= g(t) \quad ,u(\ell,t) = h(t) \end{cases}$$

where f, ϕ, g, h are given functions. Then the above equation has unique solution.

Proof using Maximum Principle. Suppose that there are two solutions $u_1(x,t)$ and $u_2(x,t)$. Let $w := u_1 - u_2$, then w satisfies the homogenous equation with zero initial-boundary conditions:

$$\begin{cases} w_t - kw_{xx} &= 0 \quad ,0 < x < \ell, t > 0 \\ w(x,0) &= 0 \quad ,0 < x < \ell \\ w(0,t) &= 0 \quad ,w(\ell,t) = 0 \end{cases} \tag{2}$$

{eqn: proof

For any $T > 0$, define the rectangle $D = \{(x,t) : 0 \leq x \leq \ell, 0 \leq t \leq T\}$. Since w satisfeis the diffusion equation in the defined rectangle D, by Maximum Principle we see that

$$w(x,t) \leq \max_{(x',t')\in\Gamma} w(x',t') \quad \forall (x,t) \in D$$

Note that for any $(x',t') \in \Gamma$ it takes the form of $(x',0)$, $(0,t')$, or (ℓ, T), which by Equation (3) we see that $w(x',t') = 0$ regardless of which form. Together, we see that

$$w(x,t) \leq 0 \quad \forall (x,t) \in D$$

Similarly, we can show using Minimum Principle that

$$w(x,t) \geq \min_{(x',t')\in\Gamma} w(x',t') = 0 \quad \forall (x,t) \in D$$

This shows that $u_1(x,t) - u_2(x,t) = w(x,t) = 0$, implying that $u_1(x,t) = u_2(x,t)$ for all $0 < x < \ell$ and all $t > 0$. Thus the solution is unique. ☐

Proof using Energy Method. Suppose that there are two solutions $u_1(x,t)$ and $u_2(x,t)$. Let $w := u_1 - u_2$, then w satisfies the homogenous equation with zero initial-boundary conditions:

$$\begin{cases} w_t - kw_{xx} &= 0 \quad ,0 < x < \ell, t > 0 \\ w(x,0) &= 0 \quad ,0 < x < \ell \\ w(0,t) &= 0 \quad ,w(\ell,t) = 0 \end{cases} \tag{3}$$

{eqn: proof

The energy of w at time t is defined as

$$E[w](t) = \frac{1}{2} \int_0^\ell [w(x,t)]^2 \ dx$$

which is non-negative. Note that the initial condition $w(x,0) = 0$ implies $E[w](0) = 0$. Differentiating the energy w.r.t. time t, we get

$$\frac{\mathrm{d}E}{\mathrm{d}t} = \int_0^\ell ww_t \ dx$$

Note diffusion equation says $w_t = kw_{xx}$, substituting we get

$$\frac{\mathrm{d}E}{\mathrm{d}t} = k \int_0^\ell ww_{xx} \ dx$$

We can rewrite the integrand by noticing $ww_{xx} = \dfrac{\partial}{\partial x}(ww_x) - w_x^2$, and deriving it to get:

$$\begin{aligned}
\frac{\mathrm{d}E}{\mathrm{d}t} &= k\int_0^\ell ww_{xx}\ dx \\
&= \int_0^\ell k\left(\frac{\partial}{\partial x}(ww_x) - w_x^2\right)\ dx \\
&= k[w(x,t)w_x(x,t)]_{x=0}^{x=\ell} - \int_0^\ell w_x^2 dx \\
&\le k[w(x,t)w_x(x,t)]_{x=0}^{x=\ell}
\end{aligned}$$

where the last inequality is obtained by noting that $\int_0^\ell w_x^2$ is non-negative. Lastly, since $w(0,t) = 0$ ad $w(\ell,t) = 0$, thus $k[w(x,t)w_x(x,t)]_{x=0}^{x=\ell} = 0$. Together we have

$$\frac{\mathrm{d}E}{\mathrm{d}t} \le 0$$

This shows that $E[w](t)$ is a decreasing quantity. By the definition of decreasing, for all t we have

$$0 \le E[w](t) \le E[w](0) = 0$$

which forces that $E[w](t) = 0$. By definition of E, it implies that $w = 0$ and thus $u_1 = u_2$. Therefore the solution is unique. □

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Example 2.3.4. Consider the diffusion equation $u_t = u_x x$ with $(x,t) \in D := \{(x,t) : 0 < x < \ell, t > 0\}$ with boundary conditions:

$$u(0,t) = u(\ell,t) = 0 \quad \text{and} \quad u(x,0) = 4x(\ell - x)$$

We want to show that $0 < u(x,t) < \ell^2$ for all $(x,t) \in D$. By Maximum Principle, the maximum of u appears at the boundary Γ. The given condition $u(0,t) = u(\ell,t) = 0$ implies that the maximum does not occur on the lateral sides, since $u(x,0) = 4x(\ell - x)$ is greater than 0 at some x. It is easy to justify that the maximum of $4x(\ell - x)$ occurs at $x = \ell/2$, where $u(\ell/2, 0) = \ell^2$. Therefore by Maximum Principle we have $u(x,t) < \ell^2$ for all $(x,t) \in D$.

On the other hand, by Minimum Principle, the minimum of $u(x,t)$ must be achieved in Γ. Since $0 \le x \le \ell$ in Γ, so $u(x,0) = 4x(\ell - x) > 0$. But $u(0,t) = u(\ell,t) = 0$, thus 0 is the minimum value by Minimum Principle. Together $u(x,t) > 0$ for all $(x,t) \in D$.

Next, we use the energy method to show that $\int_0^\ell u^2\ dx$ is a strictly decreasing function of t. Define

$$w(t) := \int_0^\ell u^2(x,t)\ dx$$

It is equivalent to show that $\dfrac{\mathrm{d}}{\mathrm{d}t}w(t) < 0$. Taking derivative w.r.t. t we obtain:

$$\frac{\mathrm{d}w}{\mathrm{d}t} = \int_0^\ell 2uu_t\ dx = 2\int_0^\ell uu_{xx}\ dx$$

where the last equality is obatined by performing substitution by the given diffusion equation $u_t = u_x x$. Integration by parts give us:

$$\frac{\mathrm{d}w}{\mathrm{d}t} = 2[uu_x]_0^\ell - 2\int_0^\ell u_x^2\ dx = -2\int_0^\ell u_x^2\ dx$$

where the last equality is due to the given boundary conditions $u(0,t) = u(\ell,t) = 0$. Clealry $-2\int_0^\ell u_x^2\ dx$ is non-positive. We claim that it is non-zero. If not, then $u_x = 0$ implies that u is a constant. Since $u(0,t) = 0$, so u is a zero function. But we have shown that $u(x,t) > 0$, thus contradiction, implying that the integral is non-zero. Therefore

$$\frac{\mathrm{d}w}{\mathrm{d}t} = -2\int_0^\ell u_x^2\ dx < 0$$

We now turn to study stability. The general idea of stability is that, if in a system, "close" initial data implies "close" solutions, where the closeness can be measured by the L^2-distance. We first introduce the notion of distance between functions:

Definition 2.3.5 (L^2-distance)**.** Suppose given two functions f, g. The L^2-distance between f and g is defined to be

$$\operatorname{dist}(f,g) = \left(\int_0^\ell [f(x) - g(x)]^2 \; dx\right)^{\frac{1}{2}}$$

Again the setup is similar: suppose given diffusion equation $u_t - ku_{xx} = f(x,t)$ where $0 < x < \ell$ and $t > 0$, accompanied with initial boundary conditions:

$$u(x,0) = \phi(x), \; 0 < x < \ell \quad \text{and} \quad u(0,t) = g(t), \; u(\ell,t) = h(t)$$

Let $u_1(x,t)$ and $u_2(x,t)$ be two solutions generated by the equation with initial values $u_1(x,0) = \phi_1(x)$ and $u_2(x,0) = \phi_2(x)$ respectively.

Notice that $w := u_1 - u_2$ satisfies the diffusion equation $w_t - kw_{xx} = 0$ with initial boundary conditions:

$$w(x,0) = \phi_1(x) - \phi_2(x), \; 0 < x < \ell \quad \text{and} \quad w(0,t) = 0, \; w(\ell,t) = 0$$

Back in Example 2.3.4, we have shown that $\int_0^\ell w^2 \; dx$ is a strictly decreasing function of t, thus

$$\int_0^\ell (u_1(x,t) - u_2(x,t))^2 \; dx = \int_0^\ell w^2(x,t) \; dx \leq \int_0^\ell w^2(x,0) \; dx = \int_0^\ell (\phi_1(x) - \phi_2(x))^2 \; dx$$

By the definition of L^2-distance, we see that the above inequality implies

$$\operatorname{dist}(u_1, u_2) \leq \operatorname{dist}(\phi_1, \phi_2)$$

Thus the system is stable: if ϕ_1 and ϕ_2 are very close, then their L^2-distance will be small, and thus the L^2-distance of u_1 and u_2 is small, suggesting that the two solutions are close.

Remark 2.3.6 (Alternative formulation of Stability)**.** We can also apply Maximum Principle and Minimum Principle to show stability. Recall that the boundary conditions for $w = u_1 - u_2$ are

$$w(x,0) = \phi_1(x) - \phi_2(x), \; 0 < x < \ell \quad \text{and} \quad w(0,t) = w(\ell,t) = 0$$

So, by Maximum Principle we see

$$w(x,t) \leq \max(\phi_1 - \phi_2, 0)$$

Similarly Minimum Principle implies

$$w(x,t) \geq \min(\phi_1 - \phi_2, 0)$$

Together we have $\min(\phi_1 - \phi_2, 0) \leq w(x,t) \leq \max(\phi_1 - \phi_2, 0)$. Again, we see stability here: as ϕ_1 approaches ϕ_2, the difference $w = u_1 - u_2$ approaches to 0.

Recall that our goal is to solve the Cauchy problem on $\mathbb{R}$:

$$\begin{cases} u_t & = ku_x x, \; x \in \mathbb{R}, \; t > 0 \\ u(x,0) & = \phi(x) \end{cases} \tag{4}$$

{eqn: Diffus

We sketch the strategy to obtain the solution: we first solve the equation $u(x,0) = \phi(x)$ for a specific $\phi(x)$ of the form

$$\phi(x) = \begin{cases} 1, \; x > 0 \\ 0, \; x < 0 \end{cases}$$

Then we build the solution for general $\phi(x)$ from this particular one. Before starting, we list some properties that will be helpful later:

Proposition 2.3.7.

1. *[Spatial translations] Given a fixed y, the translate $u(x-y,t)$ of any solution $u(x,t)$ is another solution.*

2. *[Dilation (scaling)] Given constant $a>0$, the dilation $u(\sqrt{a}x, at)$ of any solution $u(x,t)$ is another solution.*

3. *[Differentiation] Any partial derivative of a solution is again a solution.*

4. *[Linear combinations] If $u_1, \dots, u_n$ are solutions of (4), then so is $u = c_1u_1 + \cdots + c_nu_n$ for any constants $c_1, \dots, c_n$.*

5. *[Convolution invariance] If $S(x,t)$ is a solution of (4), then so is*

$$v(x,t) = \int_{-\infty}^{\infty} S(x-y,t)g(y)dy =: S(\cdot,t) * g$$

for any function g.

Theorem 2.3.8 (Solution formula for the diffusion equation)**.** *The problem*

$$\begin{cases} u_t &= ku_{xx}, \ x \in \mathbb{R}, \ t>0 \\ u(x,0) &= \phi(x) \end{cases}$$

has solution

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp\left(-\frac{(x-y)^2}{4kt}\right) \phi(y) \ dy$$

Proof. Take the Heaviside step function

$$H(x) = \begin{cases} 1, \ x>0 \\ 0, \ x<0 \end{cases}$$

and consider the IVP with Heaviside step function as initial value

$$\begin{cases} Q_t &= kQ_{xx}, \ x \in \mathbb{R}, \ t>0 \\ Q(x,0) &= H(x) \end{cases}$$

If $Q(x,t)$ is a solution, then by Dilation property $Q(\sqrt{a}x, at)$ also solves the diffusion equation $u_t - ku_{xx} = 0$, but we do not know if it solves the specific IVP with Heaviside step function, thus checking is required. Note by assumption $Q(x,0) = H(x)$, and thus $Q(\sqrt{a}x, 0) = H(\sqrt{a}x)$, and it is easy to show that $H(\sqrt{a}x) = H(x)$. Thus we can conclude that $Q(\sqrt{a}x, at)$ also solves the IVP with Heaviside step function. We have showned previously that the solution of diffusion equation is unique, thus $Q(\sqrt{a}x, at) = Q(x,t)$ for all $x \in \mathbb{R}$ and $t>0$. This says that Q is invariant under dilation.

Fixed (x,t) and let $a = 1/t$, then

$$Q(x,t) = Q(\sqrt{a}x, at) = Q\left(\sqrt{\frac{1}{t}}x, \frac{1}{t}t\right) = Q\left(\sqrt{\frac{1}{t}}x, 1\right)$$

This shows that the value of Q only depends on $\frac{x}{\sqrt{t}}$. So, we shall look for $Q(x,t)$ of the special form

$$Q(x,t) = g(p) \quad \text{where} \quad p = \frac{x}{\sqrt{4kt}}$$

and g is some function of one variable. The $\sqrt{4k}$ is included for later convenience. With this, we can rewrite the IVP into an ODE of g using chain rule:

$$\begin{aligned} Q_t &= \frac{\mathrm{d}g}{\mathrm{d}p}\frac{\partial p}{\partial t} = -\frac{1}{2t}\frac{x}{\sqrt{4kt}}g'(p) = -\frac{1}{2t}pg'(p) \\ Q_x &= \frac{\mathrm{d}g}{\mathrm{d}p}\frac{\partial p}{\partial x} = \frac{1}{\sqrt{4kt}}g'(p) \\ Q_{xx} &= \frac{\mathrm{d}Q_x}{\mathrm{d}p}\frac{\partial p}{\partial x} = \frac{1}{4kt}g''(p) \end{aligned}$$

Since Q is a solution, we must have

$$0 = Q_t - kQ_{xx} = \frac{1}{t}\left(-\frac{1}{2}pg'(p) - \frac{1}{4}g''(p)\right)$$

Since $t > 0$ so we get $g''(p) + 2pg'(p) = 0$. This ODE can be solved as follows:

$$\begin{aligned}
\frac{g''}{g'} = -2p &\implies \ln|g'| = -p^2 + c_1 \\
&\implies g' = \pm e^{c_1} e^{-p^2} = C_1 e^{-p^2} \\
&\implies g(p) = C_1 \int e^{-p^2}\, dp + C_2
\end{aligned}$$

Thus we have the solution for $Q_t - kQ_{xx} = 0$ with $p = \frac{x}{\sqrt{4kt}}$

$$Q(x,t) = g(p) = C_1 \int e^{-p^2}\, dp + C_2 = C_1 \int_0^p e^{-s^2}\, dx + C_2$$

We now determine C_1 and C_2. By initial condition $Q(x, 0) = H(x)$ of the IVP:

$$\text{if } x > 0, \quad 1 = \lim_{t\to 0^+} Q = C_1 \int_0^\infty e^{-s^2}\, dx + C_2 = C_1 \frac{\sqrt{\pi}}{2} + C_2$$

$$\text{if } x < 0, \quad 0 = \lim_{t\to 0^+} Q = C_1 \int_0^\infty e^{-s^2}\, dx + C_2 = -C_1 \frac{\sqrt{\pi}}{2} + C_2$$

where we have applied the well-known integral formula $\int_0^\infty e^{-s^2}\, ds = \frac{\sqrt{\pi}}{2}$. Solving out we get

$$C_1 = \frac{1}{\sqrt{\pi}} \quad \text{and} \quad C_2 = \frac{1}{2}$$

Together we get

$$Q(x,t) = \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-s^2}\, dx + \frac{1}{2}$$

We can finally solve the general IVP. Define

$$S(x,t) = \frac{\partial Q}{\partial x} = \frac{1}{\sqrt{4k\pi t}} \exp\left(-\frac{x^2}{4kt}\right)$$

By the Differentiation property mentioned above $S(x,t)$ is a solution to $u_t - ku_{xx} = 0$. By the Convolution Invariance property, the following is also a solution:

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)\, dy, \ t > 0$$

It remains to check that $u(x,t)$ satisfies the initial condition $u(x,0) = \phi(x)$. We first rewrite u as follows:

$$\begin{aligned}
u(x,t) &= \int_{-\infty}^{\infty} \frac{\partial Q}{\partial x}(x-y,t)\phi(y)\, dy \\
&= -\int_{-\infty}^{\infty} \frac{\partial}{\partial y}[Q(x-y,t)]\phi(y)\, dy \\
&= \int_{-\infty}^{\infty} Q(x-y,t)\phi'(y)\, dy - [Q(x-y,t)\phi(y)]_{y=-\infty}^{y=\infty}
\end{aligned}$$

where the second last equality is due to $\dfrac{\partial Q}{\partial x} + \dfrac{\partial Q}{\partial y} = 0$, and the last equality is due to integration by parts. We assume that the limit $[Q(x-y,t)\phi(y)]_{y=-\infty}^{y=\infty}$ vanishes by assuming that $\phi(y) = 0$ for large $|y|$.

Recall we want to show $u(x,0)=\phi(x)$. By using the fact that $Q(x,0)=H(x)$, we see that

$$\begin{aligned}
u(x,0) &= \int_{-\infty}^{\infty} Q(x-y,0)\phi'(y)\ dy \\
&= \int_{-\infty}^{\infty} H(x-y)\phi'(y)\ dy \\
&= \int_{-\infty}^{x} \phi'(y)\ dy \\
&= \phi(x)-\phi(-\infty) \\
&= \phi(x)
\end{aligned}$$

where in the last line we use our assumption that $\phi(-\infty)=0$. This shows that the solution $u(x,t)$ satisfies the initial condition, thus solves the IVP. $\square$

Notably, the function

$$S(x,t)=\frac{1}{\sqrt{4k\pi t}}\exp\left(-\frac{x^2}{4kt}\right)$$

is known as *Gaussian kernel, fundamental solution, source function, Green's function,* or *propagator* of the heat equation. Essentially, the solution $u(x,t)$ is essentially a convolution of $S(x,t)$ with the initial value $\phi(x)$, which can be seen as a *weighting function* that 'distribute' $\phi(y)$:

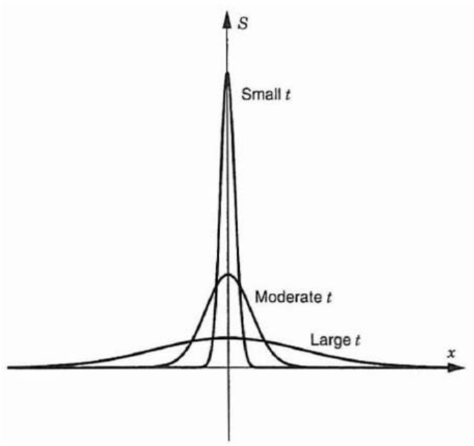
$$u(x,t)=\int_{-\infty}^{\infty} S(x-y,t)\phi(y)\ dy \simeq \sum_j S(x-y_j,t)\phi(y_j)\triangle y_j$$

where $\{y_j\}$ are some sampling points. This can be interpretated as follow:

> The source function $S(x-y,t)$ weights the contribution of $\phi(y)$ according to the distance of y from x and the elapsed time t. In other words, if point y is closer to x, then it has a bigger weight $S(x-y,t)$, and vice versa.

Remark 2.3.9.

- The name 'propagator' can be understood as propagating the initial data to the later time, giving the solution at any time $t>0$.
- Gaussian kernel $S(x,t)$ is an even function of x, and it is always positive.
- When t is large, $S(x,t)$ is very spread out; it is a thin tall spike of height $\frac{1}{\sqrt{4k\pi t}}$ when t is small.



- The area under the curve of $S(x,t)$ is 1 for any $t \geq 0$:

$$\begin{aligned}
\int_{-\infty}^{\infty} S(x,t)\ dx &= \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} \exp\left(-\frac{x^2}{4kt}\right)\ dx \\
&= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-q^2}\ dq \quad \text{where we substitute } q = \frac{x}{\sqrt{4kt}} \\
&= \frac{1}{\sqrt{\pi}} \sqrt{\pi} = 1
\end{aligned}$$

- The name 'fundamental solution' is given since $S(x,t)$ is the solution of the IVP:

$$\begin{cases} u_t = ku_{xx},\ x \in \mathbb{R},\ t > 0 \\ u(x,0) = \delta(x) \end{cases}$$

  where δ is the Dirac delta function. This follows from the fact that Q solves the IVP where $u(x,0) = H(x)$, thus $S(x,0) = Q_x(x,0) = H'(x) = \delta(x)$.

Together, we know when t is small, the source function $S(x,t)$ is a spike that the formula exaggerates the value of ϕ near x. Thus as t increases, the function $S(x,t)$ will spread out the initial value $\phi(x)$. This makes sense in the setting of trasfer of heat: as time increases, the heat is transfered and slowly became homogenous throughout the whole body.