NANYANG TECHNOLOGICAL UNIVERSITY

SPMS/DIVISION OF MATHEMATICAL SCIENCES

2025/26 Semester 2 MH4110 Partial Differential Equations Tutorial 4, 12 February

Problem 1 Consider the wave equation:

$$u_{tt} - 25u_{xx} = 0$$

on the whole plane. Find the domain of dependence of $u(x,t)$ at $(x,t) = (1,5)$, and find the domain of influence of the interval $[1,5]$.

[Solution:] The speed of the wave is $c = 5$, and hence the characteristic lines of this equation are $x \pm 5t = C$. The two characteristic lines passing through $(x,t) = (1,5)$ can be found as $x - 5t = -24$ and $x + 5t = 26$, respectively. The intersection points of the lines with x-axis are $(-24,0)$ and $(26,0)$. Therefore, the domain of independence is the triangle formed by vertices $[-24,0]$, $[26,0]$, and $[1,5]$.

The domain of influence is formed by x-axis, and

$$x + 5t \geq 1, \quad x - 5t \leq 5,$$

i.e.,

$$R = \Big\{(x,t) : 1 - 5t \leq x \leq 5 + 5t,\ t \geq 0\Big\}.$$

Problem 2 Let $\phi(x) = e^{-x^2}$ and $\psi(x) = 0$. The wave speed is $c = 1$. Sketch the string profile (u versus x) at each of the successive instants $t = 0, 2, 4, 6, 8$, and 10.

[Solution:] By the d'Alembert formula,

$$u(x,t) = \frac{\phi(x-ct) + \phi(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds.$$

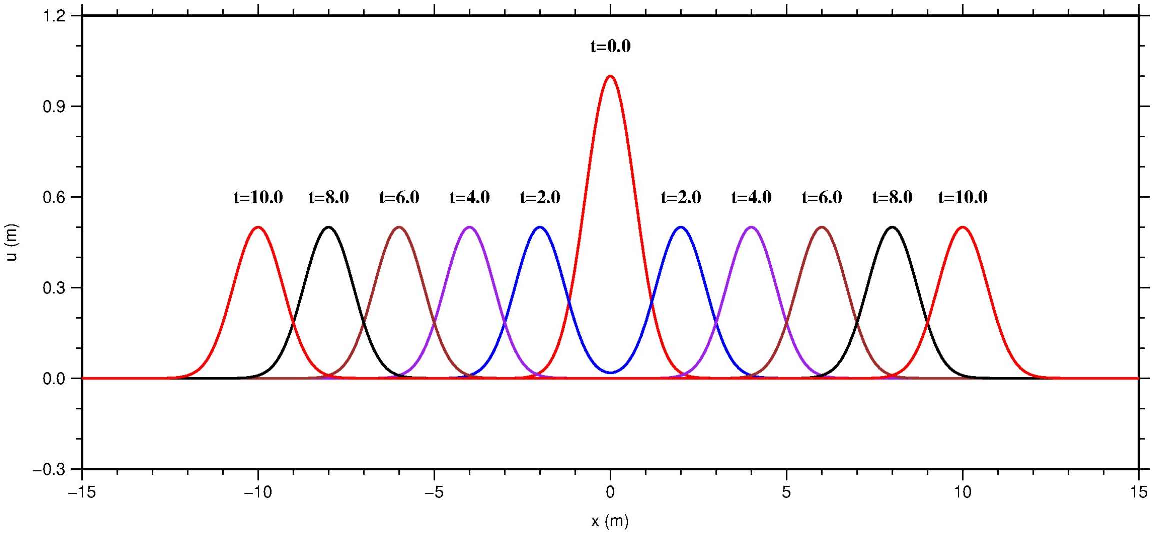
Given $\phi(x) = e^{-x^2}$, $\psi(x) = 0$ and $c = 1$, we have

$$u(x,t) = \frac{e^{-(x-t)^2} + e^{-(x+t)^2}}{2}.$$

Problem 3 (Ex. 10 on Page 38) Solve $u_{xx} + u_{xt} - 20u_{tt} = 0$, $u(x,0) = \phi(x)$, $u_t(x,0) = \psi(x)$.

[Solution:] We have derived a d'Alembert-type formula

$$u(x,t) = \frac{\alpha\phi(x-\beta t) - \beta\phi(x-\alpha t)}{\alpha - \beta} + \frac{1}{\alpha-\beta}\int_{x-\alpha t}^{x-\beta t} \psi(s)ds,$$



where $\alpha = (b + \sqrt{b^2 - 4ac})/(2a)$ and $\beta = (b - \sqrt{b^2 - 4ac})/(2a)$. In this problem, we have $a = -20$, $b = c = 1$, and $b^2 - 4ac = 81 > 0$. So $\alpha = -1/4$ and $\beta = 1/5$. The solution is therefore

$$\begin{aligned} u(x,t) &= \frac{-\frac{1}{4}\phi(x - \frac{1}{5}t) - \frac{1}{5}\phi(x + \frac{1}{4}t)}{-\frac{1}{4} - \frac{1}{5}} + \frac{1}{-\frac{1}{4} - \frac{1}{5}} \int_{x+\frac{1}{4}t}^{x-\frac{1}{5}t} \psi(s)ds \\ &= \frac{5\phi(x - \frac{1}{5}t) + 4\phi(x + \frac{1}{4}t)}{9} + \frac{20}{9} \int_{x-\frac{t}{5}}^{x+\frac{t}{4}} \psi(s)ds. \end{aligned}$$

Problem 4 (Ex. 11 on Page 38) Find the general solution of $3u_{tt} + 10u_{xt} + 3u_{xx} = \sin(x+t)$.

[Solution:] Assume that the inhomogeneous PDE has a particular solution taking the form as $u(x,t) = A\sin(x+t)$ with the undetermined coefficient A. Substituting $u(x,t) = A\sin(x+t)$ into the PDE gives $\sin(x+t) = 3u_{tt} + 10u_{xt} + 3u_{xx} = -16A\sin(x+t)$. Thus, the coefficient is $A = -1/16$ and the particular solution is $u(x,t) = -\frac{1}{16}\sin(x+t)$.

We now turn to the homogeneous equation $3u_{tt} + 10u_{xt} + 3u_{xx} = 0$. Problem 9 Tutorial 3 gives the general solution to the homogeneous equation $au_{tt} + bu_{xt} + cu_{xx} = 0$ ($b^2 - 4ac > 0$) as

$$u(x,t) = f(x - \alpha t) + g(x - \beta t),$$

where $\alpha = (b + \sqrt{b^2 - 4ac})/(2a)$ and $\beta = (b - \sqrt{b^2 - 4ac})/(2a)$. In this problem, we have $a = c = 3$, $b = 10$, and $b^2 - 4ac = 64 > 0$. So $\alpha = 3$ and $\beta = 1/3$. The solution of the homogeneous problem is therefore

$$u(x,t) = f(x - 3t) + g(x - \frac{1}{3}t).$$

The solution to the original problem is the sum of the particular solution and the general solution of the homogeneous PDE as

$$u(x,t) = f(x-3t) + g(x - \frac{1}{3}t) - \frac{1}{16}\sin(x+t),$$

where f and g are two arbitrary functions.

Problem 5 (i) Show that

$$v(x,t) = \frac{1}{2c}\int_0^t \Big(\int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\xi,\tau)\,d\xi\Big)d\tau. \tag{1}$$

is a solution of the Cauchy problem:

$$\begin{aligned} &v_{tt} - c^2 v_{xx} = f(x,t), \quad x \in (-\infty,\infty),\ t>0,\\ &v(x,0) = 0, \quad v_t(x,0) = 0, \quad x \in (-\infty,\infty). \end{aligned} \tag{2}$$

Hint: use the derivative formula:

$$\frac{d}{dt}\int_{a(t)}^{b(t)} G(\xi,t)d\xi = G(b(t),t)b'(t) - G(a(t),t)a'(t) + \int_{a(t)}^{b(t)} \frac{\partial}{\partial t}G(\xi,t)\,d\xi. \tag{3}$$

(ii) Consider the general nonhomogeneous equation:

$$\begin{aligned} &u_{tt} - c^2 u_{xx} = f(x,t), \quad x \in (-\infty,\infty),\ t>0,\\ &u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \quad x \in (-\infty,\infty). \end{aligned} \tag{4}$$

Show that the solution u of (4) can be decomposed as $u = v + w$, where v is the solution of (2), i.e., given by (1), and w is the solution of

$$\begin{aligned} &w_{tt} - c^2 w_{xx} = 0, \quad x \in (-\infty,\infty),\ t>0,\\ &w(x,0) = \phi(x), \quad w_t(x,0) = \psi(x), \quad x \in (-\infty,\infty). \end{aligned} \tag{5}$$

(iii) Derive the solution formula for (4).

[Solution:]

(i) Clearly, $v(x,0) = 0$. We find from (3) that

$$\begin{aligned} v_t(x,t) &= \frac{1}{2c}\int_x^x f(\xi,t)d\xi + \frac{1}{2}\int_0^t \Big(f(x+c(t-\tau),\tau) + f(x-c(t-\tau),\tau)\Big)d\tau\\ &= \frac{1}{2}\int_0^t \Big(f(x+c(t-\tau),\tau) + f(x-c(t-\tau),\tau)\Big)d\tau. \end{aligned}$$

Therefore, $v_t(x,0) = 0$. By taking the second derivative with respect to t, we have

$$v_{tt}(x,t) = f(x,t) + \frac{c}{2}\int_0^t \Big(f_x(x+c(t-\tau),\tau) - f_x(x-c(t-\tau),\tau)\Big)d\tau.$$

Similarly,

$$v_x(x,t) = \frac{1}{2c}\int_0^t \Big(f(x+c(t-\tau),\tau) - f(x-c(t-\tau),\tau)\Big)d\tau,$$
$$v_{xx}(x,t) = \frac{1}{2c}\int_0^t \Big(f_x(x+c(t-\tau),\tau) - f_x(x-c(t-\tau),\tau)\Big)d\tau.$$

Therefore, $v(x,t)$ is a solution of the nonhomogeneous equation (2).

(ii) Adding the equations (2) and (5), we find $v+w$ satisfies (4).

(iii) By the d'Alembert's formula, we have

$$w(x,t) = \frac{\phi(x+ct)+\phi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)ds.$$

Hence, the solution of (4) is

$$u(x,t) = w(x,t)+v(x,t) = \frac{\phi(x+ct)+\phi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)ds$$
$$+\frac{1}{2c}\int_0^t \Big(\int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\xi,\tau)\,d\xi\Big)d\tau.$$

Problem 6 Solve

$$\begin{aligned} &u_{tt} - 4u_{xx} = \sin(x+t), \quad x\in(-\infty,\infty),\ t>0,\\ &u(x,0) = x^2, \quad u_t(x,0) = e^x, \quad x\in(-\infty,\infty). \end{aligned} \tag{6}$$

[Solution:] Directly substitute $f(x,t) = \sin(x+t)$, $\phi(x) = x^2$, and $\psi(x) = e^x$ into the solution formula

$$u(x,t) = \frac{\phi(x+ct)+\phi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)ds + \frac{1}{2c}\int_0^t \Big(\int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\xi,\tau)\,d\xi\Big)d\tau,$$

we have

$$\begin{aligned} u(x,t) &= \frac{(x+ct)^2+(x-ct)^2}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} e^s ds + \frac{1}{2c}\int_0^t\Big(\int_{x-c(t-\tau)}^{x+c(t-\tau)}\sin(\xi+\tau)\,d\xi\Big)d\tau,\\ &= x^2+c^2t^2+\frac{1}{2c}\left(e^{x+ct}-e^{x-ct}\right)\\ &\quad+\frac{1}{2c}\int_0^t\Big(\cos(x-c(t-\tau)+\tau)-\cos(x+c(t-\tau)+\tau)\Big)d\tau,\\ &= x^2+c^2t^2+\frac{1}{2c}\left(e^{x+ct}-e^{x-ct}\right)\\ &\quad+\frac{1}{2c(1+c)}\sin(x-c(t-\tau)+\tau)\Big|_{\tau=0}^{\tau=t} - \frac{1}{2c(1-c)}\sin(x+c(t-\tau)+\tau)\Big|_{\tau=0}^{\tau=t}\\ &= x^2+c^2t^2+\frac{1}{2c}\left(e^{x+ct}-e^{x-ct}\right)\\ &\quad+\frac{1}{2c(1+c)}\left[\sin(x+t)-\sin(x-ct)\right] - \frac{1}{2c(1-c)}\left[\sin(x+t)-\sin(x+ct)\right] \end{aligned}$$

We know that the speed of the wave is $c = 2$, then

$$u(x,t) = x^2 + 4t^2 + \frac{1}{4}\left(e^{x+2t} - e^{x-2t}\right) + \frac{1}{12}\left[\sin(x+t) - \sin(x-2t)\right] + \frac{1}{4}\left[\sin(x+t) - \sin(x+2t)\right].$$

Problem 7 Solve the problem:

$$\begin{aligned} &u_{tt} - u_{xx} = e^{-t}, \quad x \in (-\infty, \infty), \ \ t > 0, \\ &u(x,0) = e^{-x^2} + \cos x, \quad x \in (-\infty, \infty), \\ &u_t(x,0) = 0, \quad x \in (-\infty, \infty). \end{aligned}$$

[Solution 1:]

We can use the solution formula from Problem 5 to directly write out the solution to this problem. The solution formula is

$$u(x,t) = \frac{\phi(x+ct) + \phi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds + \frac{1}{2c}\int_0^t \Big(\int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\xi,\tau)\, d\xi\Big) d\tau.$$

In this problem, we have $c = 1$, $\phi(x) = e^{-x^2} + \cos x$, $\psi(x) = 0$, and $f(x,t) = e^{-t}$. Therefore, the solution is

$$\begin{aligned} u(x,t) &= \frac{e^{-(x+t)^2} + \cos(x+t) + e^{-(x-t)^2} + \cos(x-t)}{2} + \frac{1}{2}\int_0^t \Big(\int_{x-(t-\tau)}^{x+(t-\tau)} e^{-\tau}\, d\xi\Big) d\tau \\ &= \frac{e^{-(x+t)^2} + \cos(x+t) + e^{-(x-t)^2} + \cos(x-t)}{2} + \int_0^t (t-\tau)e^{-\tau} d\tau \\ &= \frac{e^{-(x+t)^2} + e^{-(x-t)^2}}{2} + \cos x \cos t + \int_0^t (t-\tau)e^{-\tau} d\tau. \end{aligned}$$

Consider that

$$\frac{d}{d\tau}(t-\tau)e^{-\tau} = -(t-\tau)e^{-\tau} - e^{-\tau},$$

we have

$$\int_0^t (t-\tau)e^{-\tau} d\tau = -(t-\tau)e^{-\tau}\Big|_{\tau=0}^{\tau=t} - \int_0^t e^{-\tau} d\tau = t + e^{-t} - 1.$$

So, the solution is

$$u(x,t) = \frac{e^{-(x+t)^2} + e^{-(x-t)^2}}{2} + \cos x \cos t + t + e^{-t} - 1.$$

[Solution 2:]

We can also use a different way to solve this problem (but it only works for some special cases). Because of the special form of the inhomogeneous equation, we look for a particular solution: $v = v(t)$. It can be easily verified that $v = e^{-t}$ is a particular solution to the inhomogeneous PDE $u_{tt} - u_{xx} = e^{-t}$. Let $w = u - v$, then w satisfies the following initial value problem

$$\begin{aligned} &w_{tt} - w_{xx} = 0, \quad x \in (-\infty, \infty), \ \ t > 0, \\ &w(x,0) = e^{-x^2} + \cos x - 1, \quad x \in (-\infty, \infty), \\ &w_t(x,0) = 1, \quad x \in (-\infty, \infty). \end{aligned}$$

Using the d'Alembert's formula for the homogeneous equation, we have

$$w(x,t) = \frac{1}{2}\left[\left(e^{-(x+t)^2} + \cos(x+t) - 1\right) + \left(e^{-(x-t)^2} + \cos(x-t) - 1\right)\right] + \frac{1}{2}\int_{x-t}^{x+t} ds,$$

and the solution of the original problem is given by

$$u(x,t) = w(x,t) + v(t) = \frac{1}{2}\left[\left(e^{-(x+t)^2} + \cos(x+t) - 1\right) + \left(e^{-(x-t)^2} + \cos(x-t) - 1\right)\right] + t + e^{-t},$$

and can be further simplified as

$$u(x,t) = \frac{1}{2}\left[e^{-(x+t)^2} + e^{-(x-t)^2}\right] + \cos x \cos t + t + e^{-t} - 1.$$

Problem 8 Use the characteristic coordinate method to solve the equation

$$u_x + y u_y + 2xu = 1.$$

[Solution:] The characteristic curves are given by

$$\frac{dy}{dx} = \frac{y}{1}.$$

The ODE has the solutions

$$y = Ce^x.$$

Let

$$\begin{cases} \xi = x, \\ \eta = e^{-x} y. \end{cases}$$

Using the chain rule, we have

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial u}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial u}{\partial \eta} = 1 \cdot \frac{\partial u}{\partial \xi} - e^{-x} y \cdot \frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial \xi} - e^{-x} y \frac{\partial u}{\partial \eta}, \\ \\ \frac{\partial u}{\partial y} = \frac{\partial \xi}{\partial y}\frac{\partial u}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial u}{\partial \eta} = 0 \cdot \frac{\partial u}{\partial \xi} + e^{-x} \cdot \frac{\partial u}{\partial \eta} = e^{-x}\frac{\partial u}{\partial \eta}. \end{cases}$$

In the new coordinates the equation takes the form

$$u_\xi + 2\xi u\left(x(\xi,\eta), y(\xi,\eta)\right) = 1.$$

Using the integrating factor

$$e^{\int 2\xi d\xi} = e^{\xi^2},$$

the above equation can be written as

$$\left(e^{\xi^2} u\right)_\xi = e^{\xi^2}.$$

Integrating both sides in ξ, we arrive at

$$e^{\xi^2} u = \int_0^{\xi} e^{s^2} ds + f(\eta).$$

Thus, the general solution will be give by

$$u = e^{-\xi^2} \int_0^{\xi} e^{s^2} ds + e^{-\xi^2} f(\eta).$$

Finally, substituting the expressions of ξ and η in terms of (x, y) into the solution, we obtain

$$u(x, y) = e^{-x^2} \int_0^{x} e^{s^2} ds + e^{-x^2} f(e^{-x} y).$$

One should again check by substitution that this is indeed a solution to the PDE.

Problem 9 (Ex. 6 on Page 41) For the damped string:

$$u_{tt} - c^2 u_{xx} + r u_t = 0,$$

where $r > 0$, show that the energy decreases.

[Solution:] We can express the speed c in terms of density ρ and tension magnitude T as $c^2 = T/\rho$. The damped wave equation can be rewritten as

$$\rho u_{tt} - T u_{xx} + \rho r u_t = 0.$$

The kinetic energy, potential energy, and the total energy are defined as follows:

- The kinetic energy is $mv^2/2$:

$$\text{Kinetic Energy} \ \ E_K(t) = \frac{1}{2}\rho \int_{-\infty}^{\infty} u_t^2(x, t) dx,$$

- The potential energy is

$$\text{Potential Energy} \ \ E_P(t) = \frac{1}{2} T \int_{-\infty}^{\infty} u_x^2(x, t) dx.$$

- The total energy of the string undergoing vibrations is

$$E(t) = E_K(t) + E_P(t) = \frac{1}{2} \int_{-\infty}^{\infty} (\rho u_t^2 + T u_x^2)\, dx, \quad t \geq 0.$$

Differentiating $E(t)$ gives

$$E'(t) = \int_{-\infty}^{\infty} \big(\rho u_t u_{tt} + T u_x u_{xt}\big)\, dx = \int_{-\infty}^{\infty} \big(u_t [T u_{xx} - \rho r u_t] + T u_x u_{xt}\big)\, dx.$$

The integration by parts under the usual assumption that $u, u_x \to 0$ as $|x| \to \infty$, gives

$$\int_{-\infty}^{\infty} u_t T u_{xx} dx = T u_t u_x |_{-\infty}^{\infty} - \int_{-\infty}^{\infty} T u_x u_{xt} dx = - \int_{-\infty}^{\infty} T u_x u_{xt} dx.$$

Therefore, we have

$$E'(t) = - \int_{-\infty}^{\infty} \rho r u_t^2\, dx \leq 0,$$

indicating that the energy decreases.

Problem 10 Show that the maximum principle is not true for the equation: $u_t = xu_{xx}$, which has a variable coefficient. Verify that $u = -2xt - x^2$ is a solution. Find the location of its maximum in the closed rectangle:

$$D = \Big\{(x,t) : \ -2 \le x \le 2, \ 0 \le t \le 1\Big\}.$$

[Solution:] It is straightforward to verify that $u = -2xt - x^2$ is a solution. We can use calculus to find its maximum and minimum in the rectangle D. To do this, we check the interior using $u_t = u_x = 0$ to get the point $(0,0)$ where $u(0,0) = 0$. Then we check the four sides:

$$u|_{x=-2} = 4t - 4, \ \ u|_{x=2} = -4t - 4, \ \ u|_{t=0} = -x^2, \quad u|_{t=1} = -2x - x^2.$$

We find that the maximum value is 1 at $(x,t) = (-1,1)$, which is on the top of D, and the minimum value is -8, at $(2,1)$, which is at the corner point. The maximum occurs on the top of D, which violates the maximum principle.

Problem 11 $S(x,t)$ is a solution of the diffusion equation

$$u_t - ku_{xx} = 0. \tag{7}$$

Show that

$$v(x,t) = \int_{-\infty}^{\infty} S(x-y,t)g(y)dy = S(\cdot,t) * g \tag{8}$$

is also a solution of (7) for any function g.

[Solution:] Since $S(x,t)$ is a a solution of the diffusion equation, we have

$$S_t - kS_{xx} = 0.$$

We calculate v_t

$$v_t(x,t) = \int_{-\infty}^{\infty} \frac{\partial S(x-y,t)}{\partial t} g(y)dy = k\int_{-\infty}^{\infty} \frac{\partial^2 S(x-y,t)}{\partial x^2} g(y)dy$$

and v_{xx}

$$v_{xx}(x,t) = \frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty} S(x-y,t)g(y)dy = \int_{-\infty}^{\infty} \frac{\partial^2 S(x-y,t)}{\partial x^2} g(y)dy.$$

It is obvious that

$$v_t - kv_{xx} = 0.$$

Therefore,

$$v(x,t) = \int_{-\infty}^{\infty} S(x-y,t)g(y)dy = S(\cdot,t) * g$$

is also a solution of (7) for any function g.