$$\begin{cases} u_t - k u_{xx} = 0 & x \in (-\infty, \infty) \\ & t > 0 \\ u(x, 0) = \phi(x) \end{cases}$$

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y, t)\,\phi(y)\,dy$$

$$S(x,t) = \frac{1}{\sqrt{4\pi k t}}\, e^{-\frac{x^2}{4kt}}$$

(even in x)

If $\phi(x) = \phi(-x)$ even

$$u(-x,t) = \int_{-\infty}^{\infty} S(-x-y, t)\,\phi(y)\,dy \qquad (*)$$

use $-y$ to replace y in $(*)$

$$u(-x,t)=\int_{-\infty}^{+\infty} S(-x+y,t)\phi(-y)dy$$

Since $\phi(-y)=\phi(y)$

we have

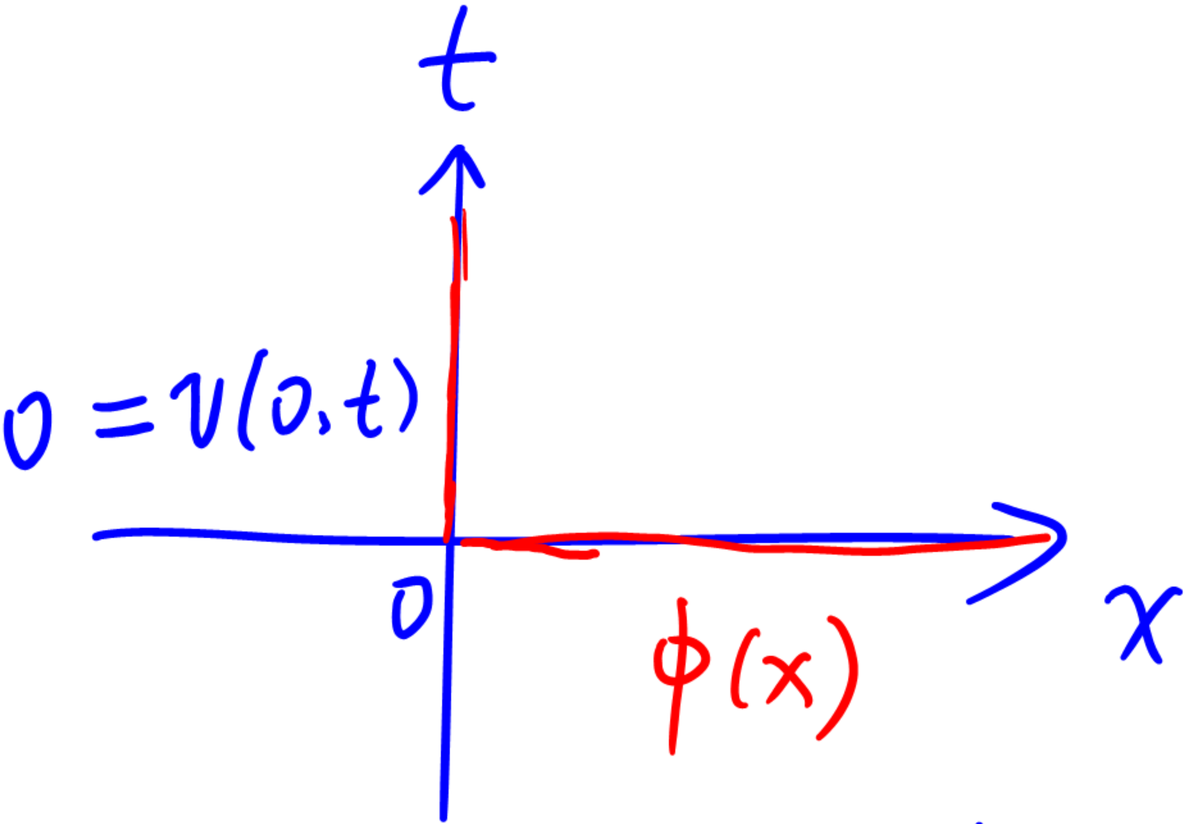
$$u(-x,t)=\int_{-\infty}^{+\infty} S(-x+y,t)\phi(y)dy$$

$$=\int_{-\infty}^{+\infty} S(x-y,t)\phi(y)dy$$

$$=u(x,t)$$

$$\begin{cases} v_t - k v_{xx} = 0 & 0 < x < \infty \\ & t > 0 \\ v(x, 0) = \phi(x) & v(0,t) = 0 \end{cases}$$

IBVP



Extension

$\psi(x) = \phi(x)$

when $x > 0$

$$\begin{cases} u_t - k u_{xx} = 0 \\ \qquad x \in R \quad t > 0 \\ u(x, 0) = \psi(x) \\ u(0, t) = 0 \end{cases}$$

Odd functions:

$$f(-x) = -f(x)$$

Substitute $x=0$

we get $f(0) = -f(0)$

$$\Rightarrow f(0) = 0$$

Odd extension

$$\psi(x) = \begin{cases} \phi(x) & 0 < x < \infty \\ 0 & x = 0 \\ -\phi(-x) & x < 0 \end{cases}$$

$$u(x,t)=\int_{-\infty}^{\infty}S(x-y,t)\psi(y)dy$$

$$=\int_{-\infty}^{0}S(x-y,t)\psi(y)dy$$

$$+\int_{0}^{\infty}S(x-y,t)\psi(y)dy$$

$$=-\int_{-\infty}^{0}S(x-y,t)\phi(-y)dy$$

$$+\int_{0}^{\infty}S(x-y,t)\phi(y)dy$$

$$=-\int_{0}^{+\infty}S(x+y,t)\phi(y)dy$$

$$+\int_{0}^{\infty}S(x-y,t)\phi(y)dy$$

$$u(x,t)$$

$$= \int_0^{+\infty} \left[S(x-y,t) - S(x+y,t) \right] \psi(y)\, dy$$

IBVP

$$\begin{cases} w_t - k w_{xx} = 0 & 0 < x < \infty \\ & t > 0 \\ w(x,0) = \phi(x) & w_x(0,t) = 0 \end{cases}$$

Extension

$\psi(x) = \phi(x)$

for $x > 0$

$$\begin{cases} u_t - k u_{xx} = 0 & x \in \mathbb{R} \\ & 0 < t \\ u(x,0) = \psi(x) & \\ u_x(0,t) = 0 & \end{cases}$$

Even function $f(x)$

$$f(-x) = f(x)$$

Differentiate both sides

$$-f'(-x) = f'(x)$$

Substitute $x=0$

$$-f'(0) = f'(0)$$

$$\Rightarrow f'(0) = 0$$

Even extension of $\phi(x)$

$$\psi(x) = \begin{cases} \phi(x) & 0 < x < \infty \\ \phi(0) & x = 0 \\ \phi(-x) & x < 0 \end{cases}$$

$$u(x,t) = \int_{-\infty}^{+\infty} S(x-y,t)\,\psi(y)\,dy$$

$$= \int_{-\infty}^{0} S(x-y,t)\,\phi(-y)\,dy$$

$$+ \int_{0}^{+\infty} S(x-y,t)\,\phi(y)\,dy$$

$$= \int_{0}^{+\infty} S(x+y,t)\,\phi(y)\,dy$$

$$+ \int_{0}^{+\infty} S(x-y,t)\,\phi(y)\,dy$$

$$u(x,t)$$

$$= \int_0^{+\infty} \left[S(x-y,t) + S(x+y,t) \right] \phi(y)\, dy$$

MH4110 PDE

Tutorial 04

Question 1

Consider the wave equation:

$$u_{tt} - 25u_{xx} = 0$$

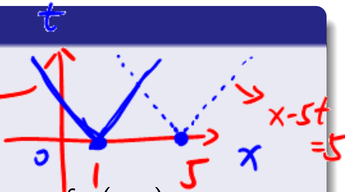
on the whole plane. Find the domain of dependence of $u(x, t)$ at $(x, t) = (1, 5)$, and find the domain of influence of the interval $[1, 5]$.

Question 1

Consider the wave equation:

$x+5t=1$

$$u_{tt} - 25u_{xx} = 0$$



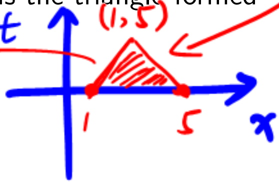
on the whole plane. Find the domain of dependence of $u(x,t)$ at $(x,t) = (1,5)$, and find the domain of influence of the interval $[1,5]$.

The speed of the wave is $c = 5$, and hence the characteristic lines of this equation are $x \pm 5t = C$. The two characteristic lines passing through $(x,t) = (1,5)$ can be found as $x - 5t = -24$ and $x + 5t = 26$, respectively. The intersection points of the lines with x-axis are $(-24, 0)$ and $(26, 0)$. Therefore, the domain of independence is the triangle formed by vertices $[-24, 0]$, $[26, 0]$, and $[1, 5]$.

$x-5t=-24$

The domain of influence is formed by x-axis, and

$$x + 5t \geq 1, \quad x - 5t \leq 5,$$



i.e.,

$$R = \{(x,t) : 1 - 5t \leq x \leq 5 + 5t, \ t \geq 0\}.$$

Question 2

Let $\phi(x) = e^{-x^2}$ and $\psi(x) = 0$. The wave speed is $c = 1$. Sketch the string profile (u versus x) at each of the successive instants $t = 0$, 2, 4, 6, 8, and 10.

Question 2

Let $\phi(x) = e^{-x^2}$ and $\psi(x) = 0$. The wave speed is $c = 1$. Sketch the string profile (u versus x) at each of the successive instants $t = 0$, 2, 4, 6, 8, and 10.

By the d'Alembert formula,

$$u(x,t) = \frac{\phi(x-ct)+\phi(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds.$$

Given $\phi(x) = e^{-x^2}$, $\psi(x) = 0$ and $c = 1$, we have

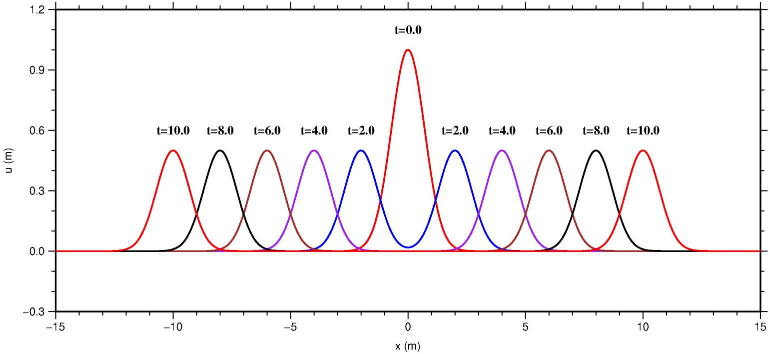
$$u(x,t) = \frac{e^{-(x-t)^2} + e^{-(x+t)^2}}{2}.$$

Question 2

Let $\phi(x) = e^{-x^2}$ and $\psi(x) = 0$. The wave speed is $c = 1$. Sketch the string profile (u versus x) at each of the successive instants $t = 0, 2, 4, 6, 8$, and 10.

The solution is

$$u(x,t) = \frac{e^{-(x-t)^2} + e^{-(x+t)^2}}{2}.$$



Question 3

Solve $u_{xx} + u_{xt} - 20u_{tt} = 0$, $u(x,0) = \phi(x)$, $u_t(x,0) = \psi(x)$.

$$\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & -20 \end{pmatrix}$$

Question 3

Solve $u_{xx} + u_{xt} - 20u_{tt} = 0$, $u(x,0) = \phi(x)$, $u_t(x,0) = \psi(x)$.

We have derived a d'Alembert-type formula

$$u(x,t) = \frac{\alpha\phi(x-\beta t) - \beta\phi(x-\alpha t)}{\alpha-\beta} + \frac{1}{\alpha-\beta}\int_{x-\alpha t}^{x-\beta t}\psi(s)ds,$$

where $\alpha = (b+\sqrt{b^2-4ac})/(2a)$ and $\beta = (b-\sqrt{b^2-4ac})/(2a)$. In this problem, we have $a=-20$, $b=c=1$, and $b^2-4ac = 81 > 0$. So $\alpha = -1/4$ and $\beta = 1/5$. The solution is therefore

$$\begin{aligned} u(x,t) &= \frac{-\frac{1}{4}\phi(x-\frac{1}{5}t) - \frac{1}{5}\phi(x+\frac{1}{4}t)}{-\frac{1}{4}-\frac{1}{5}} + \frac{1}{-\frac{1}{4}-\frac{1}{5}}\int_{x+\frac{1}{4}t}^{x-\frac{1}{5}t}\psi(s)ds \\ &= \frac{5\phi(x-\frac{1}{5}t) + 4\phi(x+\frac{1}{4}t)}{9} + \frac{20}{9}\int_{x-\frac{t}{5}}^{x+\frac{t}{4}}\psi(s)ds. \end{aligned}$$

Question 4

Find the general solution of $3u_{tt} + 10u_{xt} + 3u_{xx} = \sin(x + t)$.

Question 4

Find the general solution of $3u_{tt} + 10u_{xt} + 3u_{xx} = \sin(x+t)$.

Assume that the inhomogeneous PDE has a particular solution taking the form as $u(x,t) = A\sin(x+t)$ with the undetermined coefficient A. Substituting $u(x,t) = A\sin(x+t)$ into the PDE gives $\sin(x+t) = 3u_{tt} + 10u_{xt} + 3u_{xx} = -16A\sin(x+t)$. Thus, the coefficient is $A = -1/16$ and the particular solution is $u(x,t) = -\frac{1}{16}\sin(x+t)$.
We now turn to the homogeneous equation $3u_{tt} + 10u_{xt} + 3u_{xx} = 0$. Problem 11 of Tutorial 3 gives the general solution to the homogeneous equation $au_{tt} + bu_{xt} + cu_{xx} = 0$ $(b^2 - 4ac > 0)$ as

$$u(x,t) = f(x - \alpha t) + g(x - \beta t),$$

where $\alpha = (b + \sqrt{b^2 - 4ac})/(2a)$ and $\beta = (b - \sqrt{b^2 - 4ac})/(2a)$. In this problem, we have $a = c = 3$, $b = 10$, and $b^2 - 4ac = 64 > 0$. So $\alpha = 3$ and $\beta = 1/3$. The solution of the homogeneous problem is therefore

$$u(x,t) = f(x - 3t) + g(x - \frac{1}{3}t).$$

The solution to the original problem is the sum of the particular solution and the general solution of the homogeneous PDE as

$$u(x,t) = f(x - 3t) + g(x - \frac{1}{3}t) - \frac{1}{16}\sin(x+t),$$

where f and g are two arbitrary functions.

Question 5

$g(x,t,t)=0$

(i) Show that

$$v(x,t) = \frac{1}{2c}\int_0^t \Big(\int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\xi,\tau)\, d\xi \Big) d\tau. \tag{1}$$

$g(x,t,\tau)$

is a solution of the Cauchy problem:

$$\begin{aligned} &v_{tt} - c^2 v_{xx} = f(x,t), \quad x \in (-\infty,\infty), \ t>0, \\ &v(x,0) = 0, \quad v_t(x,0) = 0, \quad x \in (-\infty,\infty). \end{aligned} \tag{2}$$

Hint: use the derivative formula:

$v(x,t) = \frac{1}{2c}\int_0^t g(x,t,\tau)\,d\tau$

$$\frac{d}{dt}\int_{a(t)}^{b(t)} G(\xi,t)d\xi = G(b(t),t)b'(t) - G(a(t),t)a'(t) + \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} G(\xi,t)\, d\xi. \tag{3}$$

(ii) Consider the general nonhomogeneous equation:

$\frac{\partial v}{\partial t} = \frac{1}{2c} g(x,t,t) + \frac{1}{2c}\int_0^t g_t\, dt$

$$\begin{aligned} &u_{tt} - c^2 u_{xx} = f(x,t), \quad x \in (-\infty,\infty), \ t>0, \\ &u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \quad x \in (-\infty,\infty). \end{aligned} \tag{4}$$

Show that the solution u of (4) can be decomposed as $u = v + w$, where v is the solution of (2), i.e., given by (1), and w is the solution of

$= \frac{1}{2c}\int_0^t g_t\, dt$

$$\begin{aligned} &w_{tt} - c^2 w_{xx} = 0, \quad x \in (-\infty,\infty), \ t>0, \\ &w(x,0) = \phi(x), \quad w_t(x,0) = \psi(x), \quad x \in (-\infty,\infty). \end{aligned} \tag{5}$$

$g_t = f(x+c(t-\tau),\tau)c + f(x-c(t-\tau),\tau)c$

(iii) Derive the solution formula for (4).

(i) Clearly, $v(x,0)=0$. We find from (3) that

$$
\begin{aligned}
v_t(x,t) &= \frac{1}{2c}\int_x^x f(\xi,t)d\xi + \frac{1}{2}\int_0^t \Big(f(x+c(t-\tau),\tau)+f(x-c(t-\tau),\tau)\Big)d\tau \\
&= \frac{1}{2}\int_0^t \Big(f(x+c(t-\tau),\tau)+f(x-c(t-\tau),\tau)\Big)d\tau.
\end{aligned}
$$

Therefore, $v_t(x,0)=0$.

By taking the second derivative with respect to t, we have

$$
v_{tt}(x,t) = f(x,t) + \frac{c}{2}\int_0^t \Big(f_x(x+c(t-\tau),\tau)-f_x(x-c(t-\tau),\tau)\Big)d\tau.
$$

Similarly,

$$
\begin{aligned}
v_x(x,t) &= \frac{1}{2c}\int_0^t \Big(f(x+c(t-\tau),\tau)-f(x-c(t-\tau),\tau)\Big)d\tau, \\
v_{xx}(x,t) &= \frac{1}{2c}\int_0^t \Big(f_x(x+c(t-\tau),\tau)-f_x(x-c(t-\tau),\tau)\Big)d\tau.
\end{aligned}
$$

Therefore, $v(x,t)$ is a solution of the nonhomogeneous equation (2).

(ii) Adding the equations (2) and (5), we find $v + w$ satisfies (4).

(iii) By the d'Alambert's formula, we have

$$w(x,t) = \frac{\phi(x+ct) + \phi(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds.$$

Hence, the solution of (4) is

$$u(x,t) = w(x,t) + v(x,t) = \frac{\phi(x+ct) + \phi(x-ct)}{2} + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds$$
$$+ \frac{1}{2c} \int_0^t \Big(\int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\xi,\tau)\, d\xi \Big) d\tau.$$

Question 6

Solve

$$\begin{aligned} &u_{tt} - 4u_{xx} = \sin(x+t), \quad x \in (-\infty, \infty),\ t > 0, \\ &u(x,0) = x^2, \quad u_t(x,0) = e^x, \quad x \in (-\infty, \infty). \end{aligned} \tag{6}$$

Directly substitute $f(x,t) = \sin(x+t)$, $\phi(x) = x^2$, and $\psi(x) = e^x$ into the solution formula

$$u(x,t) = \frac{\phi(x+ct) + \phi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds + \frac{1}{2c}\int_0^t \Big(\int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\xi,\tau)\, d\xi \Big) d\tau,$$

we have

$$\begin{aligned} u(x,t) &= \frac{(x+ct)^2 + (x-ct)^2}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} e^s ds + \frac{1}{2c}\int_0^t \Big(\int_{x-c(t-\tau)}^{x+c(t-\tau)} \sin(\xi+\tau)\, d\xi \Big) d\tau, \\ &= x^2 + c^2t^2 + \frac{1}{2c}\left(e^{x+ct} - e^{x-ct}\right) \\ &\quad + \frac{1}{2c}\int_0^t \Big(\cos(x - c(t-\tau) + \tau) - \cos(x + c(t-\tau) + \tau) \Big) d\tau, \\ &= x^2 + c^2t^2 + \frac{1}{2c}\left(e^{x+ct} - e^{x-ct}\right) \\ &\quad + \frac{1}{2c(1+c)} \sin(x - c(t-\tau) + \tau)\Big|_{\tau=0}^{\tau=t} - \frac{1}{2c(1-c)} \sin(x + c(t-\tau) + \tau)\Big|_{\tau=0}^{\tau=t} \end{aligned}$$

Question 6

Solve

$$
\begin{aligned}
&u_{tt} - 4u_{xx} = \sin(x+t), \quad x \in (-\infty, \infty),\ t > 0, \\
&u(x,0) = x^2, \quad u_t(x,0) = e^x, \quad x \in (-\infty, \infty).
\end{aligned} \tag{7}
$$

Directly substitute $f(x,t) = \sin(x+t)$, $\phi(x) = x^2$, and $\psi(x) = e^x$ into the solution formula

$$u(x,t) = \frac{\phi(x+ct) + \phi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds + \frac{1}{2c}\int_0^t \Big(\int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\xi,\tau)\, d\xi \Big) d\tau,$$

we have

$$
\begin{aligned}
u(x,t) = {} & x^2 + c^2t^2 + \frac{1}{2c}\left(e^{x+ct} - e^{x-ct}\right) \\
& + \frac{1}{2c(1+c)}[\sin(x+t) - \sin(x-ct)] - \frac{1}{2c(1-c)}[\sin(x+t) - \sin(x+ct)]
\end{aligned}
$$

We know that the speed of the wave is $c = 2$, then

$$u(x,t) = x^2 + 4t^2 + \frac{1}{4}\left(e^{x+2t} - e^{x-2t}\right) + \frac{1}{12}[\sin(x+t) - \sin(x-2t)] + \frac{1}{4}[\sin(x+t) - \sin(x+2t$$

Question 7

Solve the problem:

$$
\begin{aligned}
&u_{tt} - u_{xx} = e^{-t}, \quad x \in (-\infty, \infty), \ \ t > 0, \\
&u(x,0) = e^{-x^2} + \cos x, \quad x \in (-\infty, \infty), \\
&u_t(x,0) = 0, \quad x \in (-\infty, \infty).
\end{aligned}
$$

Question 7

Solve the problem:

$$\begin{aligned} &u_{tt} - u_{xx} = e^{-t}, \quad x \in (-\infty, \infty), \quad t > 0, \\ &u(x,0) = e^{-x^2} + \cos x, \quad x \in (-\infty, \infty), \\ &u_t(x,0) = 0, \quad x \in (-\infty, \infty). \end{aligned}$$

We can use the solution formula from Problem 5. The solution formula is

$$u(x,t) = \frac{\phi(x+ct) + \phi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds + \frac{1}{2c}\int_0^t \Big(\int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\xi,\tau)\,d\xi \Big) d\tau.$$

In this problem, we have $c = 1$, $\phi(x) = e^{-x^2} + \cos x$, $\psi(x) = 0$, and $f(x,t) = e^{-t}$. Therefore, the solution is

$$\begin{aligned} u(x,t) &= \frac{e^{-(x+t)^2} + \cos(x+t) + e^{-(x-t)^2} + \cos(x-t)}{2} + \frac{1}{2}\int_0^t \Big(\int_{x-(t-\tau)}^{x+(t-\tau)} e^{-\tau}\,d\xi \Big) d\tau \\ &= \frac{e^{-(x+t)^2} + \cos(x+t) + e^{-(x-t)^2} + \cos(x-t)}{2} + \int_0^t (t-\tau)e^{-\tau}\,d\tau \\ &= \frac{e^{-(x+t)^2} + e^{-(x-t)^2}}{2} + \cos x \cos t + \int_0^t (t-\tau)e^{-\tau}\,d\tau. \end{aligned}$$

Question 7

Solve the problem:

$$\begin{aligned}
&u_{tt} - u_{xx} = e^{-t}, \quad x \in (-\infty, \infty), \ \ t > 0, \\
&u(x,0) = e^{-x^2} + \cos x, \quad x \in (-\infty, \infty), \\
&u_t(x,0) = 0, \quad x \in (-\infty, \infty).
\end{aligned}$$

Consider that

$$\frac{d}{d\tau}(t-\tau)e^{-\tau} = -(t-\tau)e^{-\tau} - e^{-\tau},$$

we have

$$\int_0^t (t-\tau)e^{-\tau}\,d\tau = -(t-\tau)e^{-\tau}\Big|_{\tau=0}^{\tau=t} - \int_0^t e^{-\tau}\,d\tau = t + e^{-t} - 1.$$

So, the solution is

$$u(x,t) = \frac{e^{-(x+t)^2} + e^{-(x-t)^2}}{2} + \cos x \cos t + t + e^{-t} - 1.$$

Question 8

Use the characteristic coordinate method to solve the equation

$$u_x + yu_y + 2xu = 1.$$

Question 8

Use the characteristic coordinate method to solve the equation

$$u_x + yu_y + 2xu = 1.$$

The characteristic curves are given by

$$\frac{dy}{dx} = \frac{y}{1}.$$

The ODE has the solutions

$$y = Ce^x.$$

Let

$$\begin{cases} \xi = x, \\ \eta = e^{-x}y. \end{cases}$$

Using the chain rule, we have

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial \xi}{\partial x}\frac{\partial u}{\partial \xi} + \frac{\partial \eta}{\partial x}\frac{\partial u}{\partial \eta} = 1 \cdot \frac{\partial u}{\partial \xi} - e^{-x}y \cdot \frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial \xi} - e^{-x}y\frac{\partial u}{\partial \eta}, \\ \\ \frac{\partial u}{\partial y} = \frac{\partial \xi}{\partial y}\frac{\partial u}{\partial \xi} + \frac{\partial \eta}{\partial y}\frac{\partial u}{\partial \eta} = 0 \cdot \frac{\partial u}{\partial \xi} + e^{-x} \cdot \frac{\partial u}{\partial \eta} = e^{-x}\frac{\partial u}{\partial \eta}. \end{cases}$$

In the new coordinates the equation takes the form

$$u_\xi + 2\xi u\left(x(\xi,\eta), y(\xi,\eta)\right) = 1.$$

Question 8 (Cont'd)

Use the characteristic coordinate method to solve the equation

$$u_x + yu_y + 2xu = 1.$$

Using the integrating factor

$$e^{\int 2\xi d\xi} = e^{\xi^2},$$

the above equation can be written as

$$\left(e^{\xi^2} u\right)_\xi = e^{\xi^2}.$$

Integrating both sides in ξ, we arrive at

$$e^{\xi^2} u = \int_0^\xi e^{s^2} ds + f(\eta).$$

Thus, the general solution will be give by

$$u = e^{-\xi^2} \int_0^\xi e^{s^2} ds + e^{-\xi^2} f(\eta).$$

Finally, substituting the expressions of ξ and η in terms of (x, y) into the solution, we obtain

$$u(x, y) = e^{-x^2} \int_0^x e^{s^2} ds + e^{-x^2} f(e^{-x} y).$$

One should again check by substitution that this is indeed a solution to the PDE.

Question 9

For the damped string:

$$u_{tt} - c^2 u_{xx} + r u_t = 0,$$

where $r > 0$, show that the energy decreases.

Question 9

For the damped string:

$$u_{tt} - c^2 u_{xx} + r u_t = 0,$$

where $r > 0$, show that the energy decreases.

The kinetic energy, potential energy, and the total energy are defined as follows:

$$\text{Kinetic Energy} \quad E_K(t) = \frac{1}{2}\rho \int_{-\infty}^{\infty} u_t^2(x,t)\,dx,$$

$$c^2 = \frac{T}{\rho}$$

$$\text{Potential Energy} \quad E_P(t) = \frac{1}{2} T \int_{-\infty}^{\infty} u_x^2(x,t)\,dx.$$

The total energy of the string undergoing vibrations is

$$E(t) = E_K(t) + E_P(t) = \frac{1}{2}\int_{-\infty}^{\infty} (\rho u_t^2 + T u_x^2)\,dx, \quad t \geq 0.$$

Question 9

For the damped string:

$$u_{tt} - c^2 u_{xx} + r u_t = 0, \Rightarrow u_{tt} = c^2 u_{xx} - r u_t$$

where $r > 0$, show that the energy decreases.

$$= \frac{T}{\rho} u_{xx} - r u_t$$

Differentiating $E(t)$ gives

$$E'(t) = \int_{-\infty}^{\infty} (\rho u_t u_{tt} + T u_x u_{xt})\, dx = \int_{-\infty}^{\infty} (u_t [T u_{xx} - \rho r u_t] + T u_x u_{xt})\, dx.$$

The integration by parts under the usual assumption that $u, u_x \to 0$ as $|x| \to \infty$, gives

$$\int_{-\infty}^{\infty} u_t T u_{xx} dx = T u_t u_x|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} T u_x u_{xt} dx = -\int_{-\infty}^{\infty} T u_x u_{xt} dx.$$

Therefore, we have

$$E'(t) = -\int_{-\infty}^{\infty} \rho r u_t^2\, dx \leq 0,$$

indicating that the energy decreases.

Question 10

Show that the maximum principle is not true for the equation: $u_t = xu_{xx}$, which has a variable coefficient. Verify that $u = -2xt - x^2$ is a solution. Find the location of its maximum in the closed rectangle:

$$D = \{(x,t) : -2 \le x \le 2,\ 0 \le t \le 1\}.$$

Question 10

Show that the maximum principle is not true for the equation: $u_t = xu_{xx}$, which has a variable coefficient. Verify that $u = -2xt - x^2$ is a solution. Find the location of its maximum in the closed rectangle:

$$D = \{(x,t) : -2 \le x \le 2,\ 0 \le t \le 1\}.$$

It is straightforward to verify that $u = -2xt - x^2$ is a solution. We can use calculus to find its maximum and minimum in the rectangle D. To do this, we check the interior using $u_t = u_x = 0$ to get the point $(0,0)$ where $u(0,0) = 0$. Then we check the four sides:

$$u|_{x=-2} = 4t - 4, \quad u|_{x=2} = -4t - 4, \quad u|_{t=0} = -x^2, \quad u|_{t=1} = -2x - x^2.$$

We find that the maximum value is 1 at $(x,t) = (-1,1)$, which is on the top of D, and the minimum value is -8, at $(2,1)$, which is at the corner point. The maximum occurs on the top of D, which violates the maximum principle.

Question 11

$S(x, t)$ is a solution of the diffusion equation $u_t - ku_{xx} = 0$. Show that

$$v(x, t) = \int_{-\infty}^{\infty} S(x - y, t)g(y)dy = S(\cdot, t) * g \tag{8}$$

is also a solution of the diffusion equation for any function g.

Question 11

$S(x,t)$ is a solution of the diffusion equation $u_t - ku_{xx} = 0$. Show that

$$v(x,t) = \int_{-\infty}^{\infty} S(x-y,t)g(y)dy = S(\cdot,t) * g \tag{8}$$

is also a solution of the diffusion equation for any function g.

Since $S(x,t)$ is a a solution of the diffusion equation, we have

$$S_t - kS_{xx} = 0.$$

We calculate v_t

$$v_t(x,t) = \int_{-\infty}^{\infty} \frac{\partial S(x-y,t)}{\partial t} g(y)dy = k\int_{-\infty}^{\infty} \frac{\partial^2 S(x-y,t)}{\partial x^2} g(y)dy$$

and v_{xx}

$$v_{xx}(x,t) = \frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty} S(x-y,t)g(y)dy = \int_{-\infty}^{\infty} \frac{\partial^2 S(x-y,t)}{\partial x^2} g(y)dy.$$

It is obvious that $v_t - kv_{xx} = 0$. Therefore,

$$v(x,t) = \int_{-\infty}^{\infty} S(x-y,t)g(y)dy = S(\cdot,t) * g$$

is also a solution of the diffusion equation for any function g.