MH4110 Partial Differential Equations

ODE Review

ODE Review

Definition

An **ordinary differential equation**, ODE for short, is a relation containing one real variable x, the real dependent variable y, and some of its derivatives y', y'', $\cdots$, $y^{(n)}$, $\cdots$, with respect to x.

Definition

The **order** of an ODE is defined to be the order of the highest derivative that appears in the equation. Thus, an n-th order ODE has the general form

$$F(x, y, y', \cdots, y^{(n)}) = 0.$$

Definition

A functional relation $y = y(x)$ between the dependent variable y and the independent variable x that satisfies the given ODE in some interval J is called a **solution** of the given ODE on J.

Ordinary Differential Equations

Remarks

A general solution of an n-th order ODE depends on n arbitrary constants, i.e. the solution y depends on x and n real constants $c_1, \cdots, c_n$.

Definition

An n-th order ODE is **linear** if it can be written in the form

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = r(x). \tag{1}$$

The functions $a_j(x)$, $0 \le j \le n$ are called coefficients of the equation.

We shall always assume that $a_0(x) \neq 0$ in any interval in which the equation is defined. If $r(x) = 0$, (1) is called a **homogeneous** equation. If $r(x) \neq 0$, (1) is said to be a **non-homogeneous** equation, and $r(x)$ is called the non-homogeneous term.

Integral curves

The solution of a first order ODE $y' = f(x, y)$ represents a one-parameter family of curves in the xy-plane. These are called *integral curves.*

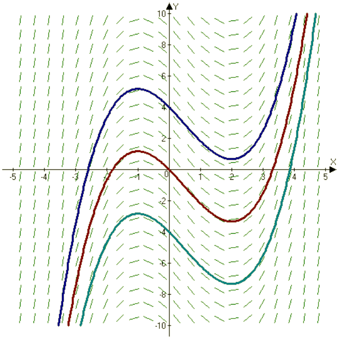

Figure: Three integral curves for the slope field corresponding to the differential equation $y' = x^2 - x - 2$.

Ordinary Differential Equations

Separable equations

Typical separable equation can be written as

$$y' = \frac{f(x)}{g(y)}, \quad \text{or} \quad g(y)dy = f(x)dx.$$

The solution is given by

$$\int g(y)dy = \int f(x)dx + c.$$

Exercise 1

Solve $y' = -xy$, $y(0) = 1$.

Ordinary Differential Equations

The equation $y' = f\left(\frac{y}{x}\right)$ can be reduced to a separable equation by letting $u = \frac{y}{x}$, i.e. $y = xu$. So $f(u) = y' = u + xu'$,

$$\int \frac{du}{f(u) - u} = \int \frac{dx}{x} + c.$$

Exercise 2

Solve $2xyy' + x^2 - y^2 = 0$.

Exact equations

We can write a first order ODE in the following form

$$M(x,y)dx + N(x,y)dy = 0. \tag{2}$$

Equation (2) is called *exact* if there exists a function $u(x,y)$ such that

$$M(x,y)dx + N(x,y)dy = du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$$

Once equation (2) is exact, the general solution is given by $u(x,y) = c$, where c is an arbitrary constant.

Ordinary Differential Equations

Theorem

Assume M and N together with their first partial derivatives are continuous in the rectangle S: $|x - x_0| < a$, $|y - y_0| < b$. A necessary and sufficient condition for $M(x,y)dx + N(x,y)dy = 0$ to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \text{for all } (x,y) \text{ in } S. \tag{3}$$

When equation (3) is satisfied, a general solution of $M(x,y)dx + N(x,y)dy = 0$ is given by $u(x,y) = c$, where

$$u(x,y) = \int_{x_0}^{x} M(s,y)ds + \int_{y_0}^{y} N(x_0,t)dt \tag{4}$$

and c is an arbitrary constant.

Exercise 3

Solve $(x^3 + 3xy^2)dx + (3x^2y + y^3)dy = 0$.

Ordinary Differential Equations

Integrating factors

A non-zero function $\mu(x, y)$ is an *integrating factor* of equation (2) if the equivalent differential equation

$$\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0. \tag{5}$$

is exact.

If μ is an integrating factor of equation (2) then $(\mu M)_y = (\mu N)_x$, i.e.

$$N\mu_x - M\mu_y = \mu(M_y - N_x). \tag{6}$$

One may look for an integrating factor of the form $\mu = \mu(v)$, where v is a known function of x and y. Plugging into equation (6) we find

$$\frac{1}{\mu} = \frac{M_y - N_x}{Nv_x - Mv_y}.$$

Ordinary Differential Equations

If $\frac{M_y - N_x}{Nv_x - Mv_y}$ is a function of v alone, say, $\phi(v)$, then

$$\mu = e^{\int^v \phi(v)dv}$$

is an integrating factor of equation (2).

- Let $v = x$. If $\frac{M_y - N_x}{N}$ is a function of x alone, say, $v = \phi_1(x)$, then $\mu = e^{\int^x \phi_1(x)dx}$ is an integrating factor of equation (2).
- Let $v = y$. If $-\frac{M_y - N_x}{M}$ is a function of y alone, say, $v = \phi_2(y)$, then $\mu = e^{\int^y \phi_2(y)dy}$ is an integrating factor of equation (2).
- Let $v = xy$. If $\frac{M_y - N_x}{yN - xM}$ is a function of xy alone, say, $\phi_3(xy)$, then $\mu = e^{\int^{xy} \phi_3(v)dv}$ is an integrating factor of equation (2).

Exercise 4

Solve $(x^2y + y + 1) + x(1 + x^2)y' = 0$

First Order Linear Equations

Homogeneous equations

A first order homogeneous linear equation is of the form

$$y' + p(x)y = 0, \tag{7}$$

where $p(x)$ is a continuous function on an interval J.

Let $P(x) = \int_a^x p(s)ds$. Multiplying (12) by $e^{P(x)}$, we get

$$\frac{d}{dx}\left[e^{P(x)}y\right] = 0,$$

so $e^{P(x)}y = c$. The general solution of (12) is given by

$$y(x) = ce^{-\int_a^x p(s)ds}.$$

First Order Linear Equations

Non-homogeneous equations

A first order non-homogeneous linear equation is of the form

$$y' + p(x)y = q(x), \tag{8}$$

where $p(x)$ and $q(x)$ are continuous functions on an interval J.

Let $P(x) = \int_a^x p(s)ds$. Multiplying (8) by $e^{P(x)}$, we get

$$\frac{d}{dx}\left[e^{P(x)}y\right] = e^{P(x)}q(x).$$

Thus,

$$e^{P(x)}y = \int_a^x e^{P(t)}q(t)dt + c.$$

The general solution of (8) is given by

$$y(x) = e^{-P(x)}\left[\int_a^x e^{P(t)}q(t)dt + c\right], \text{ where } P(x) = \int_a^x p(s)ds.$$

First Order Linear Equations

Exercise 5

Solve

$$y' + y = x. \tag{9}$$

Bernoulli's Equation

Definition

The Bernoulli's equation of order n in the standard form reads

$$\frac{dy}{dx} + p(x)y = q(x)y^n, \tag{10}$$

where $n \neq 0, 1$ is a real number and which is nonlinear.

Solution technique:

- Make a change of variable $u = y^{1-n}$
- Convert the equation to a linear equation in the new variable u
- Obtain a solution formula similar to the first-order linear equation

Dividing (10) by y^n, yields the equation

$$y^{-n}\frac{dy}{dx} + p(x)y^{1-n} = q(x). \tag{11}$$

We make the change of variable:

$$u = y^{1-n} \Longrightarrow \frac{du}{dx} = (1-n)y^{-n}\frac{dy}{dx} \Longrightarrow y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{du}{dx},$$

and eliminate y and y' from (11):

$$y^{-n}\frac{dy}{dx} + p(x)y^{1-n} = q(x) \Longrightarrow \frac{1}{1-n}\frac{du}{dx} + p(x)u(x) = q(x).$$

Rewrite the new equation as the standard form:

$$\frac{du}{dx} + \underbrace{(1-n)p(x)}_{P(x)}u(x) = \underbrace{(1-n)q(x)}_{Q(x)}.$$

It is a linear equation with the unknown function $u(x)$, and

$$P(x) = (1-n)p(x), \quad Q(x) = (1-n)q(x).$$

Using the formula for linear DE, we find

$$\begin{aligned} u(x) &= e^{-\int P(x)dx}\Big[\int Q(x)e^{\int P(x)dx}dx + C\Big] \\ &= e^{-(1-n)\int p(x)dx}\Big[(1-n)\int q(x)e^{(1-n)\int p(x)dx}dx + C\Big]. \end{aligned}$$

Finally we change the variable back:

$$y^{1-n} = e^{-(1-n)\int p(x)dx}\Big[(1-n)\int q(x)e^{(1-n)\int p(x)dx}dx + C\Big].$$

Summary of Solution Formula

Theorem

For a Bernoulli's equation of order n in the standard form

$$\frac{dy}{dx} + p(x)y = q(x)y^n, \quad n \neq 1,$$

setting $u = y^{1-n}$, we transform it into a linear equation:

$$\frac{du}{dx} + \underbrace{(1-n)p(x)}_{P(x)}u(x) = \underbrace{(1-n)q(x)}_{Q(x)},$$

and the general solution is

$$y^{1-n} = e^{-(1-n)\int p(x)dx}\left[(1-n)\int q(x)e^{(1-n)\int p(x)dx}dx + C\right].$$

Exercises

Solve the equations

$$\text{(i)} \quad 2\frac{dy}{dx} + (\tan x)y = \frac{(4x+5)^2}{\cos x} y^3.$$

and

$$\text{(ii)} \quad \frac{dy}{dx} + \frac{2}{x} y = (-x^2 \cos x) y^2.$$

Homogeneous DE

Definition.

A function $f(x, y)$ is said to be a homogeneous function (of degree zero), if it satisfies

$$f(tx, ty) = f(x, y), \quad \forall t > 0.$$

Example: Check that

$$f(x, y) = \frac{x^2 - y^2}{2xy + y^2}$$

is homogeneous of degree zero.

Observation: If we factor an x^2 term from the numerator and denominator, then the above function can be written in the form

$$f(x,y) = \frac{x^2\left[1-(y/x)^2\right]}{x^2\left[2y/x+(y/x)^2\right]} = \frac{1-(y/x)^2}{2y/x+(y/x)^2}.$$

Thus f can be considered as a function depending only on the single variable $V = y/x$, that is,

$$f(x,y) = F(V) = \frac{1-V^2}{2V+V^2}.$$

A function $f(x,y)$ is a homogeneous function (of degree zero), if and only if it depends on $V = y/x$, i.e.,

$$f(x,y) = F(y/x) = F(V).$$

Homogeneous DE: Definition

Definition:

If $f(x,y)$ is a homogeneous function (of degree zero), then the DE

$$\frac{dy}{dx} = f(x,y) = F(y/x). \tag{12}$$

is called a homogeneous first-order DE.

For example,

$$y' = \frac{x^2 - y^2}{2xy + y^2}$$

is a homogeneous DE.

Solution Technique

Let

$$V(x) = \frac{y}{x} \implies y = xV \implies \frac{dy}{dx} = V + x\frac{dV}{dx}. \tag{13}$$

Substituting it into the equation (12), we therefore obtain

$$V + x\frac{dV}{dx} = F(V),$$

or equivalently,

$$x\frac{dV}{dx} = F(V) - V.$$

The functions can now be separated to yield

$$\frac{1}{F(V) - V}dV = \frac{1}{x}dx \implies \int \frac{1}{F(V) - V}dV = \ln|x| + C.$$

Substituting the valuable $V = y/x$ back, we obtain the solution.

Solve

$$\frac{dy}{dx} = \frac{4x + y}{x - 4y}.$$

Solution. The function on the right-hand side is homogeneous of degree zero, so it is a homogeneous DE. Substitute $y = xV$

$$\frac{d}{dx}(xV) = \frac{4x + xV}{x - 4xV} = \frac{4 + V}{1 - 4V}.$$

That is

$$x\frac{dV}{dx} + V = \frac{4 + V}{1 - 4V},$$

or equivalently,

$$x\frac{dV}{dx} = \frac{4(1 + V^2)}{1 - 4V}.$$

Separating the variables gives

$$\frac{1 - 4V}{4(1 + V^2)}dV = \frac{1}{x}dx.$$

Variable Substitution

Use the variable substitution $u = xy$ to solve the initial value problem

$$\frac{dy}{dx} + \frac{y}{x} = \frac{xy}{1 + x^2y^2}, \quad y(1) = 1.$$

Variable Substitution

Solution: By the substitution $u = xy$, we have

$$\frac{du}{dx} = y + x\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} + \frac{y}{x} = \frac{1}{x}\frac{du}{dx}.$$

We therefore obtain the new equation:

$$\frac{1}{x}\frac{du}{dx} = \frac{u}{1+u^2} \quad \Rightarrow \quad \frac{1+u^2}{u}du = xdx.$$

It is separable, so the general solution is

$$\int \frac{1+u^2}{u}du = \int xdx + C \quad \Rightarrow \quad \ln|u| + \frac{u^2}{2} = \frac{x^2}{2} + C.$$

Variable Substitution

That is

$$2\ln|u| + u^2 = x^2 + 2C \quad \Rightarrow \quad 2\ln|xy| + x^2y^2 = x^2 + 2C.$$

By the initial condition, we have $C = 0$. So the solution is

$$2\ln|xy| + x^2y^2 = x^2$$

or equivalently, we formulate the solution as

$$x^2y^2 = e^{x^2 - x^2y^2}.$$

This completes the solution.

Second order linear ODE with constant coefficients

Find the general solution of

$$y'' + ay' + by = 0 \tag{14}$$

where a and b are constants.

Look for a solution of the form $y = e^{\lambda x}$. Plugging into eqn. (14), we find that $e^{\lambda x}$ is a solution if and only if

$$\lambda^2 + a\lambda + b = 0.$$

1. If $a^2 - 4b > 0$, the characteristic function has two distinct real roots λ_1 and λ_2. The general solution is $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$.
2. If $a^2 - 4b = 0$, the characteristic function has one real root λ. The general solution is $y = c_1 e^{\lambda x} + c_2 x e^{\lambda x}$.
3. If $a^2 - 4b < 0$, the characteristic function has a pair of complex conjugate roots $\lambda = \alpha \pm i\beta$. The general solution is $y = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x)$.

Second Order Linear Equations with constant coefficients

Exercise 6

Solve

$$y'' + y' - 2y = 0, \quad y(0) = 4, \quad y'(0) = -5. \tag{15}$$

Second Order Linear Equations with constant coefficients

Exercise 7

Solve

$$y^{''} - 4y' + 4y = 0, \quad y(0) = 3, \quad y'(0) = 1. \tag{16}$$

Second Order Linear Equations with constant coefficients

Exercise 8

Solve
$$y'' - 2y' + 10y = 0. \tag{17}$$

Reducible Second-order DE

- Consider the second-order DE:

$$\frac{d^2y}{dx^2} = F\left(x, y, \frac{dy}{dx}\right), \tag{18}$$

where F is a known function.

- Introduce

$$v = dy/dx \Longrightarrow d^2y/dx^2 = dv/dx.$$

We can rewrite it as an equivalent system of 1st-order DEs:

$$\begin{cases} \frac{dy}{dx} = v \\ \frac{dv}{dx} = F(x, y, v) \end{cases} \tag{19}$$

- Exercise: Write the second-order DE as a system:

$$y'' = x \sin y' + e^x y + 1.$$

- In general, the second DE can not be solved directly, since the system involves three variables, namely, x, y and v.
- We shall explore the possibility to solve the DE, if it only involves two variables.

Nonhomogeneous DE

Theorem

Consider the nonhomogeneuous DE:

$$y'' + p(x)y' + q(x)y = f(x), \tag{20}$$

where p, q and f are given continuous functions on I. If $y = y_1(x)$ is a solution to the associated homogeneous DE:

$$y'' + p(x)y' + q(x)y = 0. \tag{21}$$

Then $y_2(x) = u(x)y_1(x)$ is the (general) solution of (20), where

$$u'(x) = v(x) \quad \text{and} \quad v' + \left(2\frac{y_1'}{y_1} + p\right)v = \frac{f}{y_1}, \tag{22}$$

with the solution

$$v(x) = \frac{1}{y_1^2(x)I(x)}\left(\int y_1(x)f(x)I(x)dx + C\right), \quad I(x) = e^{\int p(x)dx}.$$