NANYANG TECHNOLOGICAL UNIVERSITY

SPMS/DIVISION OF MATHEMATICAL SCIENCES

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Problem 1 (Ex. 1 on Page 9). Solve the first-order equation

$$2u_t + 3u_x = 0$$

subject to the initial condition $u(0, x) = \sin x$.
Solution. The characteristic curves satisfy

$$3t - 2x = c.$$

The solution is constant along each characteristic curve, so the general solution has the form

$$u(t, x) = f(3t - 2x).$$

Imposing the initial condition at $t = 0$, we obtain

$$\sin x = u(0, x) = f(-2x).$$

Letting $w = -2x$, we find $f(w) = \sin(-w/2)$. Hence,

$$u(t, x) = \sin\left(\frac{-3t + 2x}{2}\right).$$

Problem 2. Solve

$$3u_x - 5u_y = 0$$

subject to the condition $u(0, y) = \cos y$.
Solution. The characteristic curves are given by

$$-5x - 3y = c.$$

Thus, the general solution is

$$u(x, y) = f(-5x - 3y).$$

Applying the condition at $x = 0$,

$$\cos y = u(0, y) = f(-3y),$$

which implies $f(w) = \cos(-w/3)$. Therefore,

$$u(x, y) = \cos\left(\frac{5x + 3y}{3}\right).$$

Problem 3 (Ex. 2 on Page 9) Solve the equation $3u_y + u_{xy} = 0$.

[Solution:] Letting $v = u_y$, we can write the equation as a system:

$$u_y = v, \quad v_x + 3v = 0.$$

We first solve the second equation: $v_x + 3v = 0$. Treating y as a parameter, we obtain $v(x, y) = C(y)e^{-3x}$. Integrating v, we obtain the solution: $u(x, y) = D(y)e^{-3x} + E(x)$, where we denoted $D(y) = \int C(y)dy$. Here, D and E are arbitrary functions.

Problem 4 (Modified from Ex. 9 on Page 10) Use both the geometric method and the coordinate method to solve the equation

$$u_x + u_y = 1.$$

[Solution:]

(a) The geometric method. This is an inhomogeneous PDE. Assuming that a particular solution to this PDE takes a form as $u(x, y) = Ax$, we can plug it into the PDE and get $A = 1$. For the homogeneous PDE $u_x + u_y = 0$, its characteristic lines are given by $x - y = C$, where C is an arbitrary constant. So the solution of the homogeneous PDE is $u(x, y) = f(x - y)$. Thus, the solution of $u_x + u_y = 1$ is

$$u(x, y) = x + f(x - y),$$

where f is an arbitrary function.

(b) The coordinate method
Change variables to

$$x' = x + y \qquad \text{and} \qquad y' = x - y,$$

we have

$$\frac{\partial}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial x}\frac{\partial}{\partial y'} = \frac{\partial}{\partial x'} + \frac{\partial}{\partial y'},$$

and

$$\frac{\partial}{\partial y} = \frac{\partial x'}{\partial y}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial y}\frac{\partial}{\partial y'} = \frac{\partial}{\partial x'} - \frac{\partial}{\partial y'}.$$

So the PDE is transformed into

$$u_{x'} = \frac{1}{2}.$$

Consider y' as a parameter, this equation can be viewed as a first-order inhomogeneous ODE with the independent variable x'. Integrating with respect to x', we have

$$u = \frac{x'}{2} + f(y').$$

f is an arbitrary function of a single variable. Change back to the original coordinates, we have the solution

$$u(x, y) = \frac{x + y}{2} + f(x - y).$$

Problem 5 Solve $au_x + bu_y = c$ using the coordinate method, where a, b, c are constants and $a \neq 0$.

[Solution:] Change variables to

$$x' = ax + by \qquad y' = bx - ay$$

Replace all x and y derivatives by x' and y' derivatives. By the chain rule,

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x} = au_{x'} + bu_{y'}$$

and

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y} = bu_{x'} - au_{y'}.$$

Hence $c = au_x + bu_y = a(au_{x'} + bu_{y'}) + b(bu_{x'} - au_{y'}) = (a^2+b^2)u_{x'}$. Since $a^2 + b^2 \neq 0$, the equation takes the form $u_{x'} = c/(a^2+b^2)$ in the new (primed) variables. Thus the solution is $u = cx'/(a^2+b^2) + f(y')$, with f an arbitrary function. The solution in the original coordinates is $u = c(ax+by)/(a^2+b^2) + f(bx - ay)$.

Problem 6 Solve the equation $yu_x + xu_y = 0$ with the condition $u(0, y) = y$. In which region of the xy plane is the solution uniquely determined?
[Solution:] (a) The characteristic curves are given by $\frac{dy}{dx} = \frac{x}{y}$. This is a separable ODE. Its solution is $y^2 = x^2 + C$, where C is any given constant. Then, the general solution of the original PDE is $u(x, y) = f(y^2 - x^2)$. The auxiliary condition says $u(0, y) = y$. Substituting $x = 0$ into the general solution, we have $y = f(y^2)$. We can only determine the explicit form of $f(y^2)$ as $f(y^2) = \sqrt{y^2} = y$ when $y \geq 0$ or $f(y^2) = -\sqrt{y^2} = y$ when $y < 0$. Thus,

$$u(x,y) = \begin{cases} \sqrt{y^2 - x^2}, & \text{if } y^2 \geq x^2 \text{ and } y \geq 0; \\ -\sqrt{y^2 - x^2}, & \text{if } y^2 \geq x^2 \text{ and } y < 0; \\ f(y^2 - x^2), & \text{if } y^2 < x^2. \end{cases}$$

(b) From the above discussion, we know that the solution is uniquely determined when $y^2 \geq x^2$.

Problem 7 Find the general solution of

$$u_x + e^y\, u_y = 1.$$

[Solution:] We look for the characteristic curve $y = y(x)$ satisfying

$$\frac{dy}{dx} = \frac{e^y}{1} = e^y \quad \Rightarrow \quad x + e^{-y} = C.$$

Hence, the general solution of the problem is

$$u(x, y) = f(x + e^{-y}).$$

Obviously, $u(x, y) = x$ is a particular solution. Thus, the solution of the original problem is

$$u(x, y) = f(x + e^{-y}) + x.$$

Problem 8 (Ex. 6 on Page 10) Solve the equation

$$\sqrt{1-x^2}\, u_x + u_y = 0, \quad u(0, y) = y.$$

[Solution:] We look for the characteristic curve $y = y(x)$ satisfying

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \quad \Rightarrow \quad y(x) = \arcsin x + C.$$

Hence, the general solution of the problem is

$$u(x, y) = f(y - \arcsin x).$$

As $u(0, y) = f(y) = y$, we have that the solution is $u(x, y) = y - \arcsin x$.

Problem 9 Find the general solution of the equation

$$\text{(a)} \qquad -2u_x + u_y + 3u = e^{x+y};$$

$$\text{(b)} \qquad xu_x - yu_y + y^2u = y^2, \qquad\qquad x, y \neq 0.$$

[Solution:]

(a) Change variables to

$$x' = -2x + y \qquad \text{and} \qquad y' = x + 2y,$$

we have

$$\frac{\partial}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial x}\frac{\partial}{\partial y'} = -2\frac{\partial}{\partial x'} + \frac{\partial}{\partial y'},$$

and

$$\frac{\partial}{\partial y} = \frac{\partial x'}{\partial y}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial y}\frac{\partial}{\partial y'} = \frac{\partial}{\partial x'} + 2\frac{\partial}{\partial y'}.$$

So the PDE is transformed into

$$5u_{x'} + 3u = e^{-\frac{x'}{5}+\frac{3y'}{5}}.$$

Consider y' as a parameter, this equation can be viewed as a first-order inhomogeneous ODE with the independent variable x'. Let $u = Ae^{-\frac{x'}{5}}$ be a particular solution, then we have

$$5u_{x'} + 3u = 5\left(-\frac{A}{5}e^{-\frac{x'}{5}}\right) + 3Ae^{-\frac{x'}{5}} = 2Ae^{-\frac{x'}{5}} = e^{-\frac{x'}{5}+\frac{3y'}{5}}.$$

The coefficient A is determiend as

$$A = \frac{1}{2}e^{\frac{3y'}{5}}.$$

Thus, the particular solution is

$$u(x', y') = \frac{1}{2}e^{-\frac{x'}{5}+\frac{3y'}{5}}.$$

The general solution to the homogeneous equation $5u_{x'}+3u = 0$ is $u(x', y') = f(y')e^{-\frac{3x'}{5}}$. The solution to the inhomogeneous equation is

$$u(x', y') = f(y')e^{-\frac{3x'}{5}} + \frac{1}{2}e^{-\frac{x'}{5}+\frac{3y'}{5}}.$$

Change back to the original coordinates, we have the solution

$$u(x, y) = f(x + 2y)e^{-\frac{3(-2x+y)}{5}} + \frac{1}{2}e^{-\frac{-2x+y}{5}+\frac{3(x+2y)}{5}},$$

which can be rewritten as

$$u(x, y) = f(x + 2y)e^{\frac{3}{5}(2x-y)} + \frac{1}{2}e^{x+y}.$$

(b) The characteristic curves are given by

$$\frac{dy}{dx} = -\frac{y}{x}.$$

This is a separable ODE, which can be solved to obtain the general solution $xy = C$. Thus, our change of coordinates (no need to be orthogonal but should be non-degenerate) will be

$$\begin{cases} x' = x, \\ y' = xy. \end{cases}$$

In these coordinates the equation takes the form

$$x'u_{x'} + \frac{y'^2}{x'^2}u = \frac{y'^2}{x'^2}, \qquad \text{or} \quad u_{x'} + \frac{y'^2}{x'^3}u = \frac{y'^2}{x'^3}.$$

Using the integrating factor

$$e^{\int \frac{y'^2}{x'^3} dx'} = e^{-\frac{y'^2}{2x'^2}},$$

the above equation can be written as

$$\left(e^{-\frac{y'^2}{2x'^2}} u \right)_{x'} = e^{-\frac{y'^2}{2x'^2}} \frac{y'^2}{x'^3}.$$

Integrating both sides in x', we arrive at

$$e^{-\frac{y'^2}{2x'^2}} u = \int e^{-\frac{y'^2}{2x'^2}} \frac{y'^2}{x'^3} dx' = e^{-\frac{y'^2}{2x'^2}} + f(y').$$

Thus, the general solution will be give by

$$u = e^{\frac{y'^2}{2x'^2}} \left[e^{-\frac{y'^2}{2x'^2}} + f(y') \right] = 1 + e^{\frac{y'^2}{2x'^2}} f(y').$$

Finally, substituting the expressions of x' and y' in terms of (x, y) into the solution, we obtain

$$u(x, y) = 1 + e^{\frac{y^2}{2}} f(xy).$$

One should again check by substitution that this is indeed a solution to the PDE.

Problem 10 (Ex. 13 on Page 10) Use the coordinate method to solve the equation

$$u_x + 2u_y + (2x - y)u = 2x^2 + 3xy - 2y^2.$$

[Solution:] Change variables to

$$x' = x + 2y \qquad y' = 2x - y,$$

we have

$$x = (x' + 2y')/5, \quad y = (2x' - y')/5,$$

and

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'} + 2\frac{\partial}{\partial y'}, \quad \frac{\partial}{\partial y} = 2\frac{\partial}{\partial x'} - \frac{\partial}{\partial y'}.$$

So the PDE is transformed into

$$u_{x'} + \frac{y'}{5} u = \frac{x'y'}{5}.$$

Consider y' as a parameter, this equation can be viewed as a first-order inhomogeneous ODE with the independent variable x'. Let $P(x') = \int_0^{x'} \frac{y'}{5} ds = \frac{x'y'}{5}$. Multiplying both sides of the above equation by $e^{P(x')}$ we get

$$\frac{d}{dx'} \left[e^{\frac{x'y'}{5}} u \right] = \frac{x'y'}{5} e^{\frac{x'y'}{5}}.$$

Thus, after integrating with respect to x', we have

$$e^{\frac{x'y'}{5}} u = \left(x' - \frac{5}{y'} \right) e^{\frac{x'y'}{5}} + f(y'),$$

with f as an arbitrary function of a single variable. So

$$u(x', y') = \left(x' - \frac{5}{y'} \right) + e^{-\frac{x'y'}{5}} f(y').$$

Change back to the original coordinates, we have the solution

$$u(x,y) = \left(x + 2y - \frac{5}{2x-y}\right) + e^{-\frac{(x+2y)(2x-y)}{5}} f(2x-y).$$

Problem 11 Find the solution to general constant coefficient linear first-order equations

$$au_x + bu_y + cu = g(x,y).$$

[Solution:] Before we proceed, let's have a review on the following theorem from ODE

Theorem 1. *If the functions $p(x)$, $q(x)$ are continuous, then*

$$y' = p(x)y + q(x), \tag{1}$$

has infinitely many solutions and every solution, $y(x)$, can be labeled by $r \in R$ as follows

$$y(x) = re^{P(x)} + e^{P(x)} \int e^{-P(x)} q(x)dx, \tag{2}$$

where we introduced the function $P(x) = \int p(x)dx$, any primitive of the function $p(x)$. □

We can use the characteristic coordinates $x' = ax + by$ and $y' = bx - ay$ to reduce the discussing equation into an inhomogeneous ODE ($G(x',y') = g(x(x',y'), y(x',y'))$ is assumed.)

$$(a^2+b^2)u_{x'} + cu = G(x',y'), \quad \text{or} \quad u_{x'} = -\frac{c}{a^2+b^2}u + \frac{G(x',y')}{a^2+b^2}.$$

Directly using the above theorem, we have

$$P(x') = \int \left(-\frac{c}{a^2+b^2}\right) dx' = -\frac{c}{a^2+b^2}x' + r_1.$$

Here r_1 is constant in x' (depends only on y'). So

$$u(x',y') = r(y')e^{-\frac{c}{a^2+b^2}x' + r_1(y')} + e^{-\frac{c}{a^2+b^2}x' + r_1(y')} \int e^{\frac{c}{a^2+b^2}x' - r_1(y')} \frac{G(x',y')}{a^2+b^2} dx'.$$

It can be simplified into

$$u(x',y') = f(y')e^{-\frac{c}{a^2+b^2}x'} + e^{-\frac{c}{a^2+b^2}x'} \int e^{\frac{c}{a^2+b^2}x'} \frac{G(x',y')}{a^2+b^2} dx'.$$

Assume that

$$\int e^{\frac{c}{a^2+b^2}x'} \frac{G(x',y')}{a^2+b^2} dx' = M(y'),$$

where $M(y')$ is any primitive of the function $e^{\frac{c}{a^2+b^2}x'} \frac{G(x',y')}{a^2+b^2}$ with x' as the only independent variable and y' as a parameter. The solution is

$$u(x',y') = e^{-\frac{c}{a^2+b^2}x'} \left(f(y') + M(y')\right).$$

Replace x' and y' by their expressions in terms of x and y, we obtain that

$$u(x,y) = e^{-\frac{c}{a^2+b^2}(ax+by)} \left[f(bx-ay) + M(bx-ay)\right].$$

f is an arbitrary function of a single variable.

Problem 12 (Ex. 1 on Page 19) Carefully derive the equation of a string in a medium in which the resistance is proportional to the velocity. [Solution:] The derivation is almost the same as the one shown in Example 2 on Pages 5-11 of Handout 2. The only difference is that equation (10) should be modified into

$$\int_{x_0}^{x_1} a_u(x,t)\rho dx = T(x_1,t)\frac{u_x(x_1,t)}{\sqrt{1+u_x^2(x_1,t)}} - T(x_0,t)\frac{u_x(x_0,t)}{\sqrt{1+u_x^2(x_0,t)}} - \int_{x_0}^{x_1} ru_t(x,t)dx.$$

The red part of the above equation is the resistance. Following the same steps as in Example 2, we can finally get the wave equation with resistance as

$$u_{tt} - c^2 u_{xx} + ru_t = 0, \quad r > 0.$$

Problem 13 (Ex. 6 on Page 19) Consider heat flow in a long circular cylinder where the temperature depends only on t and on the distance r to the axis of the cylinder. Here $r = \sqrt{x^2+y^2}$ is the cylindrical coordinate. From the three-dimensional heat equation derive the equation $u_t = k(u_{rr} + u_r/r)$. [Solution:] The three-dimensional heat equation is

$$u_t = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right).$$

The temperature depends only on t and on the distance r to the axis of the cylinder. That means the temperature is invariant in the z direction. The above heat equation can be simplified as

$$u_t = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right).$$

Using the chain rule, we have

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} = \frac{\partial u}{\partial r}\frac{x}{r}$$

and

$$\begin{aligned}
\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\frac{x}{r}\right) &= \frac{x}{r}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\right) + \frac{\partial u}{\partial r}\frac{\partial}{\partial x}\left(\frac{x}{r}\right) \\
&= \frac{x}{r}\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\right)\frac{\partial r}{\partial x} + \frac{\partial u}{\partial r}\frac{r - x\frac{\partial r}{\partial x}}{r^2} \\
&= \frac{x}{r}\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\right)\frac{x}{r} + \frac{\partial u}{\partial r}\frac{r - x\frac{x}{r}}{r^2} \\
&= \frac{x^2}{r^2}\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r}\frac{r^2 - x^2}{r^3}.
\end{aligned}$$

Similarly,

$$\frac{\partial^2 u}{\partial y^2} = \frac{y^2}{r^2}\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r}\frac{r^2 - y^2}{r^3}.$$

Thus,

$$\begin{aligned}
u_t = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) &= k\left(\frac{x^2}{r^2}\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r}\frac{r^2 - x^2}{r^3} + \frac{y^2}{r^2}\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r}\frac{r^2 - y^2}{r^3}\right) \\
&= k\left(\frac{x^2+y^2}{r^2}\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r}\frac{2r^2 - x^2 - y^2}{r^3}\right) \\
&= k\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right).
\end{aligned}$$

Problem 14 Solve the following first order PDE and find where the solution is defined in the x - y plane.

$$u_x + xyu_y = 0, \qquad u(x, -1) = \frac{1}{2}x^2.$$

[Solution:] The characteristic curves are given by $\frac{dy}{dx} = \frac{xy}{1}$. This is a separable ODE. Its solution is $\ln|y| = \frac{x^2}{2} + C$, where C is any given constant. Further we have $|y| = e^{x^2/2}e^C$ or $y = \pm e^{x^2/2}e^C$. If $D = \pm e^C$ is any given constant, then $y = De^{x^2/2}$. The characteristic curves are given by $ye^{-x^2/2} = D$. The general solution of the original PDE is $u(x, y) = f(ye^{-x^2/2})$. The auxiliary condition says $u(x, -1) = \frac{1}{2}x^2$. Substituting $y = -1$ into the general solution, we have $\frac{1}{2}x^2 = f(-e^{-x^2/2})$. We can only determine the explicit form of $f(z)$ when its independent variable z is on the half-closed interval $[-1, 0)$ as $f(z) = -\ln(-z)$. Thus,

$$u(x, y) = \begin{cases} -\ln\left(-ye^{-x^2/2}\right) = \ln\left(-y^{-1}e^{x^2/2}\right) = \ln\left(-y^{-1}\right) + x^2/2, & \text{if } -1 \le ye^{-x^2/2} < 0; \\ f(ye^{-x^2/2}), & \text{otherwise.} \end{cases}$$

Or,

$$u(x, y) = \begin{cases} \ln\left(-y^{-1}\right) + x^2/2, & \text{if } -e^{x^2/2} \le y < 0; \\ f(ye^{-x^2/2}), & \text{otherwise} \end{cases}$$

From the above discussion, we know that the solution is uniquely determined when $-e^{x^2/2} \le y < 0$.

Problem 15 Solve the following first order PDE

$$x^2u_x + xyu_y = u.$$

[Solution:] The characteristic curves are given by

$$\frac{dy}{dx} = \frac{xy}{x^2} = \frac{y}{x}.$$

This is a separable ODE, which can be solved to obtain the general solution $y/x = C$. Thus, our change of coordinates (no need to be orthogonal but should be non-degenerate) will be

$$\begin{cases} x' = x, \\ y' = \frac{y}{x}. \end{cases}$$

Use the chain rule, we have

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial u}{\partial x'} + \frac{\partial y'}{\partial x}\frac{\partial u}{\partial y'} = 1 \cdot \frac{\partial u}{\partial x'} - \frac{y}{x^2}\frac{\partial u}{\partial y'} = \frac{\partial u}{\partial x'} - \frac{y'}{x'}\frac{\partial u}{\partial y'}, \\ \frac{\partial u}{\partial y} = \frac{\partial x'}{\partial y}\frac{\partial u}{\partial x'} + \frac{\partial y'}{\partial y}\frac{\partial u}{\partial y'} = 0 \cdot \frac{\partial u}{\partial x'} + \frac{1}{x}\frac{\partial u}{\partial y'} = \frac{1}{x'}\frac{\partial u}{\partial y'}. \end{cases}$$

In these coordinates the equation takes the form

$$x'^2\left(\frac{\partial u}{\partial x'} - \frac{y'}{x'}\frac{\partial u}{\partial y'}\right) + x'^2y' \cdot \frac{1}{x'}\frac{\partial u}{\partial y'} = u,$$

or

$$u_{x'} - \frac{1}{x'^2}u = 0.$$

Using the integrating factor

$$e^{\int -\frac{1}{x'^2}dx'} = e^{\frac{1}{x'}},$$

the above equation can be written as

$$\left(e^{\frac{1}{x'}}u\right)_{x'} = 0.$$

Integrating both sides in x', we arrive at

$$e^{\frac{1}{x'}}u = f(y').$$

Thus, the general solution will be give by

$$u = e^{-\frac{1}{x'}}f(y').$$

Finally, substituting the expressions of x' and y' in terms of (x, y) into the solution, we obtain

$$u(x, y) = e^{-\frac{1}{x}}f(\frac{y}{x}).$$

One should again check by substitution that this is indeed a solution to the PDE.