Error function

It is usually impossible to evaluate integral

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp\Big(-\frac{(x-y)^2}{4kt}\Big)\phi(y)dy.$$

completely in terms of elementary functions. Answers to particular problems, that is, to particular initial data $\phi(x)$, are sometimes expressible in terms of the error function of statistics.

$$\phi(x) = e^{-\sqrt{x^2+1}}$$

$$\int_0^\infty e^{-p^2}\,dp = \frac{\sqrt{\pi}}{2}$$

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$\mathrm{Dom}(\mathrm{Erf}(x)) = \mathbb{R}$

The error function is defined as

increasing function

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2}\,dp. \qquad [\mathrm{Erf}(x)]' = \frac{2}{\sqrt{\pi}} e^{-x^2} > 0 \tag{74}$$

When $x = 0$, $\mathrm{Erf}(0) = 0$; as $x \to \infty$, $\mathrm{Erf}(x) \to 1$.

Range $(-1, 1)$

$$I = \int_0^\infty e^{-x^2}\,dx \qquad I^2 = \int_0^\infty e^{-x^2}\,dx \int_0^\infty e^{-y^2}\,dy$$

Example 1

Let $Q(x,t)$ be the function defined in (67) as,

$$= \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$$

$$Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-s^2}\,ds, \quad t > 0, \quad -\infty < x < \infty,$$

$$= \int_0^\infty \int_0^{\pi/2} e^{-r^2}\,r\,dr\,d\theta$$

then

$$Q(x,t) = \frac{1}{2} + \frac{1}{2}\mathrm{Erf}\left(\frac{x}{\sqrt{4kt}}\right). \tag{75}$$

$$= \frac{\pi}{2} \int_0^\infty \frac{1}{2} e^{-r^2}\,dr^2$$

$$= \frac{\pi}{4}$$

$u(x,t) = 100\, Q(x,t)$

$\phi(x) = 100\, H(x)$

Example C (not in the textbook)

Use the error function to express the solution of

$$u_t = ku_{xx}, \quad -\infty < x < \infty, \quad u(x,0) = \phi(x) = \begin{cases} 100, & x > 0, \\ 0, & x < 0. \end{cases} \tag{76}$$

Solution By the solution formula, we have

$$u(x,t) = \frac{100}{\sqrt{4\pi kt}} \int_0^\infty \exp\left(-\frac{(x-y)^2}{4kt}\right) dy \quad \left(\text{Letting } q = \frac{y-x}{\sqrt{4kt}}\right)$$

$$= \frac{100}{\sqrt{\pi}} \int_{-x/\sqrt{4kt}}^\infty e^{-q^2}\, dq = \underline{\frac{100}{\sqrt{\pi}} \int_0^\infty e^{-q^2}\, dq} + \frac{100}{\sqrt{\pi}} \int_{-x/\sqrt{4kt}}^0 e^{-q^2}\, dq$$

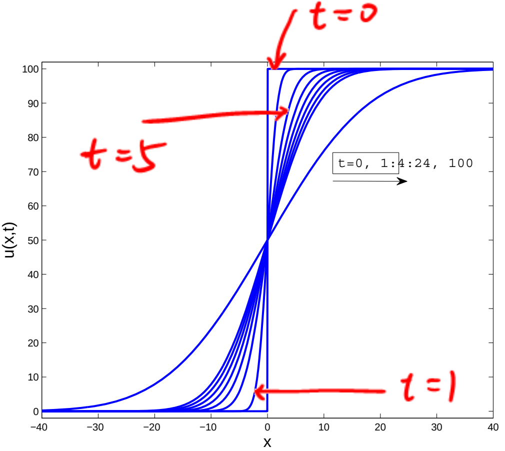
even function

$\frac{\sqrt{\pi}}{2}$

$$= 50 + \frac{100}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-q^2}\, dq$$

$$= 50\left(1 + \mathrm{Erf}\left(\frac{x}{\sqrt{4kt}}\right)\right). = 100\, Q(x,t)$$

Example C (not in the textbook) (Cont'd)



Observe that although the initial condition $\phi(x)$ is a step function, the solution is very smooth for $t > 0$. In fact, we find from the solution formula (59) that $u(x,t)$ is sufficiently smooth for $t > 0$ and any x, as all the partial derivatives of u exist. Hence, the Gaussian kernel is called a smoothing kernel.

§2.4 Comparison of waves and diffusions

- We have seen that the basic property of waves is that the information (e.g., initial data) gets transported in both directions at a finite speed. The basic property of diffusions is that the initial information gets spread out in a smooth fashion and gradually disappears.

[1] We verify that $u_n(x,t) = \frac{1}{n}\sin(nx)e^{-n^2kt}$ is a solution to the diffusion equation. However, it goes to infinity for $t < 0$.

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- There is no maximum principle for the wave equation (why? see the example below), while the solution of the diffusion equation satisfies the maximum principle.

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- We have seen that the basic property of waves is that the information (e.g., initial data) gets transported in both directions at a finite speed. The basic property of diffusions is that the initial information gets spread out in a smooth fashion and gradually disappears.
- There is no maximum principle for the wave equation (why? see the example below), while the solution of the diffusion equation satisfies the maximum principle.
- For the wave equation, the energy is preserved as a constant, while for the diffusion equation, the energy decays to zero.
- For the diffusion equation is not well-posed for $t < 0$,[1] while the wave equation is well-posed for all t.

can be reversed in time

[1] We verify that $u_n(x,t) = \frac{1}{n}\sin(nx)e^{-n^2kt}$ is a solution to the diffusion equation. However, it goes to infinity for $t < 0$.

§2.4 Comparison of waves and diffusions

The fundamental properties of the wave and diffusion equations are summarized in the table below.

Property		Waves	Diffusions
(i)	Speed of propagation?	Finite ($\leq c$)	Infinite
(ii)	Singularities for $t > 0$?	Transported along characteristics (speed $= c$)	Lost immediately
(iii)	Well-posed for $t > 0$?	Yes	Yes (at least for bounded solutions)
(iv)	Well-posed for $t < 0$?	Yes	No
(v)	Maximum principle	No	Yes
(vi)	Behavior as $t \to +\infty$?	Energy is constant so does not decay	Decays to zero (if ϕ integrable)
(vii)	Information	Transported	Lost gradually

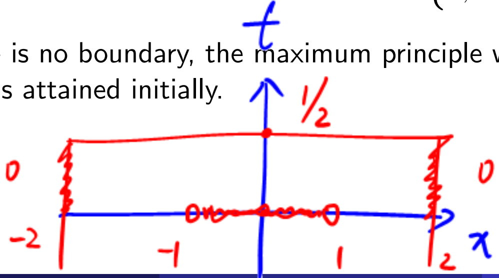
Example D (not in the textbook)

Show that there is no maximum principle for the wave equation.

Solution We consider the wave equation $u_{tt} = u_{xx}$ with the initial condition:

$$u(x,0) = 0, \quad u_t(x,0) = \psi(x) = \begin{cases} 1, & |x| < 1, \\ 0, & |x| > 1. \end{cases}$$

Since there is no boundary, the maximum principle would state that the maximum is attained initially.



$C = 1$

Example D (not in the textbook)

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Since there is no boundary, the maximum principle would state that the maximum is attained initially.

But the solution is zero initially, and takes on positive values for $t > 0$ based on the d'Alembert formula. Therefore, the maximum principle does not hold for the wave equation.

Summary

In the last several lectures we solved the initial value problems associated with the wave and diffusion equations on the whole line $x \in R$.

Waves

The solution to the wave initial-value problem on the whole line

$$\begin{aligned} &u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty, \\ &u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \quad -\infty < x < \infty, \end{aligned} \tag{77}$$

is given by d'Alembert formula

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds. \tag{78}$$

Summary (Cont'd)

Diffusions

The solution to the diffusion (heat) initial-value problem on the whole line

$$\begin{aligned} u_t &= ku_{xx}, \quad -\infty < x < \infty, \\ u(x,0) &= \phi(x), \quad -\infty < x < \infty, \end{aligned} \tag{79}$$

is given by the formula

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp\Big(- \frac{(x-y)^2}{4kt} \Big) \phi(y) dy. \tag{80}$$

The fundamental solution or Gaussian kernel

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}} \exp\Big(- \frac{x^2}{4kt} \Big) \tag{81}$$

has the Dirac delta function $\delta(x)$ as its initial data.

MH4110 PDE

Tutorial 03

Question 1

Solve the following first order PDE

$$x^2 v_x + xy v_y = v^2.$$

Question 1

Solve the following first order PDE

$v=0$ is a solution

$$x^2 v_x + xyv_y = v^2. \qquad \frac{v_x}{v^2} = \frac{\partial}{\partial x}\left(-\frac{1}{v}\right)$$

Solution: We define $u = -1/v$ to change the PDE into $x^2 u_x + xyu_y = 1$, and then solve it using the characteristic method. The characteristic curves are given by

$$\frac{dy}{dx} = \frac{xy}{x^2} = \frac{y}{x}. \qquad \frac{v_y}{v^2} = \frac{\partial}{\partial y}\left(-\frac{1}{v}\right)$$

This is a separable ODE, which can be solved to obtain the general solution $y/x = C$. Thus, our change of coordinates (no need to be orthogonal but should be non-degenerate) will be

$$\begin{cases} x' = x, \\ y' = \frac{y}{x}. \end{cases}$$

Use the chain rule, we have

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial u}{\partial x'} + \frac{\partial y'}{\partial x}\frac{\partial u}{\partial y'} = 1 \cdot \frac{\partial u}{\partial x'} - \frac{y}{x^2}\frac{\partial u}{\partial y'} = \frac{\partial u}{\partial x'} - \frac{y'}{x'}\frac{\partial u}{\partial y'}, \\ \frac{\partial u}{\partial y} = \frac{\partial x'}{\partial y}\frac{\partial u}{\partial x'} + \frac{\partial y'}{\partial y}\frac{\partial u}{\partial y'} = 0 \cdot \frac{\partial u}{\partial x'} + \frac{1}{x}\frac{\partial u}{\partial y'} = \frac{1}{x'}\frac{\partial u}{\partial y'}. \end{cases}$$

In these coordinates the equation takes the form

$$x'^2\left(\frac{\partial u}{\partial x'} - \frac{y'}{x'}\frac{\partial u}{\partial y'}\right) + x'^2 y' \cdot \frac{1}{x'}\frac{\partial u}{\partial y'} = u,$$

Question 1 (cont'd)

or

$$\boxed{u_{x'} - \frac{1}{x'^2} u = 0.}$$

Using the integrating factor

$$e^{\int -\frac{1}{x'^2} dx'} = e^{\frac{1}{x'}},$$

the above equation can be written as

$$\left(e^{\frac{1}{x'}} u\right)_{x'} = 0.$$

Integrating both sides in x', we arrive at

$$e^{\frac{1}{x'}} u = f(y').$$

Thus, the general solution will be give by

$$u = e^{-\frac{1}{x'}} f(y').$$

Finally, substituting the expressions of x' and y' in terms of (x, y) into the solution, we obtain

$$u(x, y) = e^{-\frac{1}{x}} f(\frac{y}{x})$$

or

$$v(x, y) = -e^{\frac{1}{x}} g(\frac{y}{x})$$

$v = -\frac{1}{u}$

One should again check by substitution that this is indeed a solution to the PDE.

Question 2

Consider the equation

$$u_x + yu_y = 0.$$

with the boundary condition $u(x,0) = \phi(x)$.

(a) For $\phi(x) \equiv x$, show that no solution exists.

(b) For $\phi(x) \equiv 1$, show that there are many solutions.

Question 2

Consider the equation

$$u_x + yu_y = 0.$$

$u(x,y) = f(e^{-x}y)$

with the boundary condition $u(x,0) = \phi(x)$.

(a) For $\phi(x) \equiv x$, show that no solution exists.

(b) For $\phi(x) \equiv 1$, show that there are many solutions.

Solution: We look for the characteristic curve $y = y(x)$ satisfying

$$\frac{dy}{dx} = y \quad \Rightarrow \quad y(x) = Ce^x.$$

Hence, the general solution of the problem is

$$u(x,y) = f(e^{-x}y).$$

Given the boundary condition $u(x,0) = \phi(x)$, we have $u(x,0) = f(0) = \phi(x)$.

(a) If $\phi(x) \equiv x$, this contradicts the fact that $u(x,0) = f(0)$ is a constant. So no solution exists.

(b) If $\phi(x) \equiv 1$, that means $f(0) = 1$. There are many arbitrary functions satisfying $f(0) = 1$. For example, $f(x) = 1 + \sum_{i=1}^{n} a_i x^{n-i}$ (a_i are constants and n is an integer). So there are many solutions.

Question 3

Let's consider the PDE

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0, \tag{1}$$

which is a linear equation of order two in two variables with six real constant coefficients. We know that

Equation (1) is of elliptic type, if $a_{11}a_{22} - a_{12}^2 > 0$. By a linear transform, it can be reduced to

$$u_{xx} + u_{yy} + \{\text{terms of lower order 1 or 0}\} = 0. \tag{2}$$

Equation (1) is of hyperbolic type, if $a_{11}a_{22} - a_{12}^2 < 0$. By a linear transform, it can be reduced to

$$u_{xx} - u_{yy} + \{\text{terms of lower order 1 or 0}\} = 0. \tag{3}$$

Equation (1) is of parabolic type, if $a_{11}a_{22} - a_{12}^2 = 0$. By a linear transform, it can be reduced to

$$u_{xx} + \{\text{terms of lower order 1 or 0}\} = 0, \tag{4}$$

(unless $a_{11} = a_{12} = a_{22} = 0$.)

Please explicitly show how to reduce (1) into the form of (2), (3), or (4). (*Hint: use the method of completing the square.*)

Question 3 (continued)

Solution: Since the linear transform does not change the order of partial derivatives, we assume that $a_1 = a_2 = a_0 = 0$. Without loss of generality, we assume that $a_{11} = 1$. Then the second-order part is

$$u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} = \left(\partial_x^2 + 2a_{12}\partial_{xy} + a_{22}\partial_y^2\right)u$$
$$= \left\{(\partial_x + a_{12}\partial_y)^2 + (-a_{12}^2 + a_{22})\partial_y^2\right\}u.$$

Now, we know the classification of three cases is based on the sign of $a_{22} - a_{12}^2$.
We look for a suitable linear transform of the form:

$$x = \alpha\xi + \beta\eta, \quad y = \gamma\xi + \delta\eta. \tag{5}$$

with α, β, γ, δ to be determined

Notice that

$$u_\xi = u_x\frac{\partial x}{\partial \xi} + u_y\frac{\partial y}{\partial \xi} = \alpha u_x + \gamma u_y,$$
$$u_\eta = u_x\frac{\partial x}{\partial \eta} + u_y\frac{\partial y}{\partial \eta} = \beta u_x + \delta u_y,$$
$$\Rightarrow \quad \partial_\xi = \alpha\partial_x + \gamma\partial_y, \quad \partial_\eta = \beta\partial_x + \delta\partial_y.$$

$\alpha = 1$ $\qquad \beta = 0$

$\gamma = a_{12}$ $\qquad \delta = \sqrt{|a_{22} - a_{12}^2|}$

Question 3 (continued)

We take

$$\alpha = 1, \quad \gamma = a_{12}, \quad \beta = 0, \quad \delta = \sqrt{|a_{22} - a_{12}^2|}, \tag{6}$$

in (5), i.e., the transform:

$$x = \xi, \quad y = a_{12}\xi + \sqrt{|a_{22} - a_{12}^2|}\eta. \tag{7}$$

If $a_{22} - a_{12}^2 > 0$ (elliptic case), we can transform the equation to $u_{\xi\xi} + u_{\eta\eta} = 0$.

If $a_{22} - a_{12}^2 < 0$ (hyperbolic case), we can use the same transform (6) to convert the equation to $u_{\xi\xi} - u_{\eta\eta} = 0$.

If $a_{22} - a_{12}^2 = 0$ (parabolic case), we cannot use the transform (6) to convert the equation to $u_{\xi\xi} = 0$ as it is degenerate. The characteristic lines corresponding to the linear operator $\partial_x + a_{12}\partial_y$ are $a_{12}x - y = C$. Thus, we can use the transform:

$$\xi = x, \quad \eta = a_{12}x - y \tag{8}$$

to reduce the equatio to $u_{\xi\xi}$=0.

Question 4

What is the type of each of the following equations?

(a) $u_{xx} - u_{xy} + 2u_y + u_{yy} - 3u_{yx} + 4u = 0.$

(b) $9u_{xx} + 6u_{xy} + u_{yy} + u_x = 0.$

Question 4

What is the type of each of the following equations?

(a) $u_{xx} - u_{xy} + 2u_y + u_{yy} - 3u_{yx} + 4u = 0.$

(b) $9u_{xx} + 6u_{xy} + u_{yy} + u_x = 0.$

Solution:

(a) Assume that the mixed partial derivatives u_{xy} and u_{yx} are equal to each other, the PDE $u_{xx} - u_{xy} + 2u_y + u_{yy} - 3u_{yx} + 4u = 0$ can be simplified into $u_{xx} - 4u_{xy} + u_{yy} + 2u_y + 4u = 0$. The coefficient matrix is therefore

$$A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}.$$

Its determinant is $\det(A) = 1 \times 1 - (-2) \times (-2) = -3 < 0$. The PDE is of hyperbolic type.

(b) The coefficient matrix is

$$A = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}.$$

Its determinant is $\det(A) = 9 \times 1 - 3 \times 3 = 0$. The PDE is of parabolic type.

Question 5

Find the regions in the xy plane where the equation

$$(1+x)u_{xx} + 2xyu_{xy} - y^2u_{yy} = 0$$

is elliptic, hyperbolic, or parabolic. Sketch them.

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is elliptic, hyperbolic, or parabolic. Sketch them.

Solution: The coefficient matrix is

$$A = \begin{bmatrix} 1+x & xy \\ xy & -y^2 \end{bmatrix}.$$

Its determinant is

$\det(A) = (1+x)(-y^2) - (xy)^2 = -(x^2+x+1)y^2 = -\left[(x+\frac{1}{2})^2 + \frac{3}{4}\right] y^2.$

We have

If $y = 0$, then $\det(A) = 0$. The PDE is of parabolic type.

If $y \neq 0$, then $\det(A) < 0$. The PDE is of hyperbolic type.

Question 6

Consider the equation $3u_y + u_{xy} = 0$.

(a) What is its type?

(b) Find the general solution. *Hint: Substitute* $v = u_y$.

(c) With the auxiliary conditions $u(x,0) = e^{-3x}$ and $u_y(x,0) = 0$, does a solution exist? Is it unique?

Question 6

Consider the equation $3u_y + u_{xy} = 0$.

$$A = \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix}$$

(a) What is its type?

(b) Find the general solution. *Hint: Substitute $v = u_y$.*

(c) With the auxiliary conditions $u(x,0) = e^{-3x}$ and $u_y(x,0) = 0$, does a solution exist? Is it unique?

Solution:

(a) $a_{11} = a_{22} = 0$ and $a_{12} = 1/2$. So $a_{11}a_{22} - a_{12}^2 = -1/4 < 0$. The equation is hyperbolic.

(b) Set $v = u_y$, the original equation can be rewritten as $3v + v_x = 0$. Its general solution is $v(x,y) = f(y)e^{-3x}$ with f as an arbitrary function. Integrating over y, we have the general solution to $u_y = v$ as $u(x,y) = F(y)e^{-3x} + G(x)$. Here F (a primitive function of f) and G are both arbitrary functions of a single variable.

(c) Given that $u(x,0) = e^{-3x}$, we have $u(x,0) = F(0)e^{-3x} + G(x) = e^{-3x}$, so $G(x) = [1 - F(0)]e^{-3x}$. Additionally, $u_y(x,0) = 0$ implies that $u_y(x,0) = F'(0)e^{-3x} = 0$ and hence $F'(0) = 0$. Therefore, the solution is $u(x,y) = [F(y) + 1 - F(0)]e^{-3x}$ with the function $F(y)$ satisfying $F'(0) = 0$. There are many possibilities for $F(y)$. For example, $F(y) = cy^2$ or $F(y) = cy^3$ (c is an arbitrary constant). The solution exists but is not unique.

$$u(x,y) = \int_0^y f(s)e^{-3x}\,ds + G(x)$$

Question 7

Reduce the elliptic equation

$$u_{xx} + 3u_{yy} - 2u_x + 24u_y + 5u = 0$$

to the form $v_{xx} + v_{yy} + cv = 0$.

$A = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$ $\quad \mathrm{Det}(A) = 3 > 0$

$B = \begin{pmatrix} 1 & \\ & 1 \end{pmatrix}$ $\quad \mathrm{Det}(B) > 0$

Question 7

Reduce the elliptic equation

$$u_{xx} + 3u_{yy} - 2u_x + 24u_y + 5u = 0$$

to the form $v_{xx} + v_{yy} + cv = 0$.

Solution: We look for a suitable linear transform of the form:

$$\xi = ax + by, \quad \eta = cx + dy.$$

The Jacobian determinant of the linear transform is

$$J = \left|\frac{\partial(\xi,\eta)}{\partial(x,y)}\right| = ad - bc.$$

To have a non-degenerate transform, $ad \neq bc$ is required. The first and second order derivatives in the $\xi - \eta$ coordinates can be expressed as

$$u_x = u_\xi \frac{\partial \xi}{\partial x} + u_\eta \frac{\partial \eta}{\partial x} = au_\xi + cu_\eta,$$
$$u_y = u_\xi \frac{\partial \xi}{\partial y} + u_\eta \frac{\partial \eta}{\partial y} = bu_\xi + du_\eta,$$

Question 7 (continued)

and further

$$
\begin{aligned}
u_{xx} &= \left(au_\xi + cu_\eta\right)_\xi \frac{\partial \xi}{\partial x} + \left(au_\xi + cu_\eta\right)_\eta \frac{\partial \eta}{\partial x} = \left(a^2 u_{\xi\xi} + acu_{\xi\eta}\right) + \left(acu_{\xi\eta} + c^2 u_{\eta\eta}\right) \\
&= a^2 u_{\xi\xi} + 2acu_{\xi\eta} + c^2 u_{\eta\eta}, \\
u_{yy} &= \left(bu_\xi + du_\eta\right)_\xi \frac{\partial \xi}{\partial y} + \left(bu_\xi + du_\eta\right)_\eta \frac{\partial \eta}{\partial y} = \left(b^2 u_{\xi\xi} + bdu_{\xi\eta}\right) + \left(bdu_{\xi\eta} + d^2 u_{\eta\eta}\right) \\
&= b^2 u_{\xi\xi} + 2bdu_{\xi\eta} + d^2 u_{\eta\eta}.
\end{aligned}
$$

So, the PDE can be rewritten as

$$
\begin{aligned}
0 &= u_{xx} + 3u_{yy} - 2u_x + 24u_y + 5u \\
&= \left(a^2 u_{\xi\xi} + 2acu_{\xi\eta} + c^2 u_{\eta\eta}\right) + 3\left(b^2 u_{\xi\xi} + 2bdu_{\xi\eta} + d^2 u_{\eta\eta}\right) \\
&\quad - 2\left(au_\xi + cu_\eta\right) + 24\left(bu_\xi + du_\eta\right) + 5u \\
&= (a^2 + 3b^2)u_{\xi\xi} + (2ac + 6bd)u_{\xi\eta} + (c^2 + 3d^2)u_{\eta\eta} + (-2a + 24b)u_\xi + (-2c + 24d)u_\eta + 5u
\end{aligned}
$$

To make the above equation have a form of $v_{xx} + v_{yy} + cv = 0$, $2ac + 6bd = 0$, $-2a + 24b = 0$, and $-2c + 24d = 0$ are required. These imply that $a = b = 0$ or $c = d = 0$. The linear transform would be a degenerate one. We cannot find a linear transform to change the PDE into the desired form.

Question 7 (continued)

Alternatively, let $u = v(x,y)e^{\alpha x+\beta y}$, then

$$u_x = v_x e^{\alpha x+\beta y} + \alpha v e^{\alpha x+\beta y}, \qquad u_y = v_y e^{\alpha x+\beta y} + \beta v e^{\alpha x+\beta y}$$

and

$$\begin{aligned} u_{xx} &= v_{xx} e^{\alpha x+\beta y} + \alpha v_x e^{\alpha x+\beta y} + \alpha v_x e^{\alpha x+\beta y} + \alpha^2 v e^{\alpha x+\beta y}, \\ u_{yy} &= v_{yy} e^{\alpha x+\beta y} + \beta v_y e^{\alpha x+\beta y} + \beta v_y e^{\alpha x+\beta y} + \beta^2 v e^{\alpha x+\beta y}. \end{aligned}$$

The original PDE can be rewritten as

$$\begin{aligned} 0 &= u_{xx} + 3u_{yy} - 2u_x + 24u_y + 5u \\ &= \left(v_{xx} e^{\alpha x+\beta y} + \alpha v_x e^{\alpha x+\beta y} + \alpha v_x e^{\alpha x+\beta y} + \alpha^2 v e^{\alpha x+\beta y}\right) \\ &+ 3\left(v_{yy} e^{\alpha x+\beta y} + \beta v_y e^{\alpha x+\beta y} + \beta v_y e^{\alpha x+\beta y} + \beta^2 v e^{\alpha x+\beta y}\right) \\ &- 2\left(v_x e^{\alpha x+\beta y} + \alpha v e^{\alpha x+\beta y}\right) + 24\left(v_y e^{\alpha x+\beta y} + \beta v e^{\alpha x+\beta y}\right) + 5v e^{\alpha x+\beta y}, \end{aligned}$$

which can be simplified as

$$\begin{aligned} 0 &= \left(v_{xx} + 2\alpha v_x + \alpha^2 v\right) + 3\left(v_{yy} + 2\beta v_y + \beta^2 v\right) - 2\left(v_x + \alpha v\right) + 24\left(v_y + \beta v\right) + 5v \\ &= v_{xx} + 3v_{yy} + (2\alpha - 2)v_x + (6\beta + 24)v_y + (\alpha^2 + 3\beta^2 - 2\alpha + 24\beta + 5)v \end{aligned}$$

Assuming that $\alpha = 1$ and $\beta = -4$, the change of variable $u = ve^{x-4y}$ leads to a new PDE as

$$v_{xx} + 3v_{yy} - 44v = 0.$$

Question 8

Solve $u_{tt} = 3u_{xx},\ u(x,0) = \ln(1+x^2),\ u_t(x,0) = 4 + x$.

Question 8

Solve $u_{tt} = 3u_{xx},\ u(x,0) = \ln(1+x^2),\ u_t(x,0) = 4+x.$

Solution: In this problem the speed of the wave is $c = \sqrt{3}$. By the d'Alembert formula,

$$u(x,t) = \frac{\phi(x-ct)+\phi(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds.$$

Since $\phi(x) = \ln(1+x^2)$ and $\psi(x) = 4+x$, we have

$$\begin{aligned}
u(x,t) &= \frac{\ln[1+(x-ct)^2] + \ln[1+(x+ct)^2]}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct}(4+s)ds \\
&= \frac{1}{2}\ln[1+(x-ct)^2+(x+ct)^2+(x^2-c^2t^2)^2] \\
&+ \frac{1}{2c}\left[4(x+ct)+\frac{1}{2}(x+ct)^2 - 4(x-ct) - \frac{1}{2}(x-ct)^2\right] \\
&= \frac{1}{2}\ln[1+2x^2+2c^2t^2+(x^2-c^2t^2)^2] + \frac{1}{2c}[8ct+2ctx] \\
&= \frac{1}{2}\ln[1+2x^2+2c^2t^2+(x^2-c^2t^2)^2] + (4+x)t.
\end{aligned}$$

Thus, the solution is

$$\begin{aligned}
u(x,t) &= \frac{1}{2}\ln[1+2x^2+2c^2t^2+(x^2-c^2t^2)^2] + (4+x)t \\
&= \frac{1}{2}\ln[1+2x^2+6t^2+(x^2-3t^2)^2] + (4+x)t.
\end{aligned}$$

Question 9

If both ϕ and ψ are even functions of x, show that the solution $u(x, t)$ of the wave equation is also even in x for all t.

Given $\phi(x) = \phi(-x)$ $\psi(x) = \psi(-x)$

Prove $u(x, t) = u(-x, t)$

Question 9

If both ϕ and ψ are even functions of x, show that the solution $u(x,t)$ of the wave equation is also even in x for all t.

Solution: By the d'Alembert formula,

$$u(x,t) = \frac{\phi(x-ct)+\phi(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds.$$

Since $\phi(-s) = \phi(s)$ and $\psi(-s) = \psi(s)$, we have $= \phi(x+ct)$

$$u(-x,t) = \frac{\phi(-x-ct)+\phi(-x+ct)}{2} + \frac{1}{2c}\int_{-x-ct}^{-x+ct} \psi(s)ds$$

Make a change of variables as $s = -y$ and note that $ds = -dy$, then

$$\begin{aligned} u(-x,t) &= \frac{\phi(x+ct)+\phi(x-ct)}{2} - \frac{1}{2c}\int_{x+ct}^{x-ct} \psi(-y)dy \\ &= \frac{\phi(x+ct)+\phi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy \\ &= u(x,t). \end{aligned}$$

This means $u(x,t)$ is even in x for all t.

$$\frac{1}{2c}\int_{-x-ct}^{-x+ct} \psi(s)ds \overset{s=-y}{=\!=} \frac{1}{2c}\int_{x+ct}^{x-ct} \psi(-y)\,d(-y)$$

Question 10 (The hammer blow)

Let $\phi(x) \equiv 0$ and $\psi(x) = 1$ for $|x| < a$ and $\psi(x) = 0$ for $|x| \geq a$. Sketch the string profile (u versus x) at each of the successive instants $t = a/2c, a/c, 3a/2c, 2a/c$, and $5a/c$.

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Solution: By the d'Alembert formula,

$$u(x,t) = \frac{\phi(x-ct) + \phi(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds.$$

Given $\phi(x) \equiv 0$ and $\psi(x) = 1$ for $|x| < a$ and $\psi(x) = 0$ for $|x| \geq a$, we have

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds.$$

Since the initial velocity $\psi(x)$ is nonzero only in the interval $[-a, a]$, the integral must be computed differently according to how the intervals $[-a, a]$ and $[x-ct, x+ct]$ intersect.

Question 10 (The hammer blow) (cont'd)

For a small value of t (only if $ct \leq a$), the two intervals $[-a, a]$ and $[x - ct, x + ct]$ intersect in the following 5 different ways

If $x - ct < x + ct \leq -a < a$, then

$$u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds = 0.$$

If $x - ct \leq -a < x + ct \leq a$, then

$$u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds = \frac{1}{2c} \int_{-a}^{x+ct} 1 ds = \frac{x + ct + a}{2c}.$$

If $-a \leq x - ct < x + ct \leq a$, then

$$u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds = \frac{1}{2c} \int_{x-ct}^{x+ct} 1 ds = t.$$

If $-a \leq x - ct \leq a < x + ct$, then

$$u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds = \frac{1}{2c} \int_{x-ct}^{a} 1 ds = \frac{-x + ct + a}{2c}.$$

If $-a < a \leq x - ct < x + ct$, then

$$u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds = 0.$$

Question 10 (The hammer blow) (cont'd)

Solution : For a large value of t (only if $ct > a$), the two intervals $[-a, a]$ and $[x-ct, x+ct]$ also intersect in 5 different ways

If $x - ct < x + ct \leq -a < a$, then

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds = 0.$$

If $x - ct \leq -a < x + ct \leq a$, then

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds = \frac{1}{2c}\int_{-a}^{x+ct} 1ds = \frac{x+ct+a}{2c}.$$

If $x - ct \leq -a < a \leq x + ct$, then

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds = \frac{1}{2c}\int_{-a}^{a} 1ds = \frac{a}{c}.$$

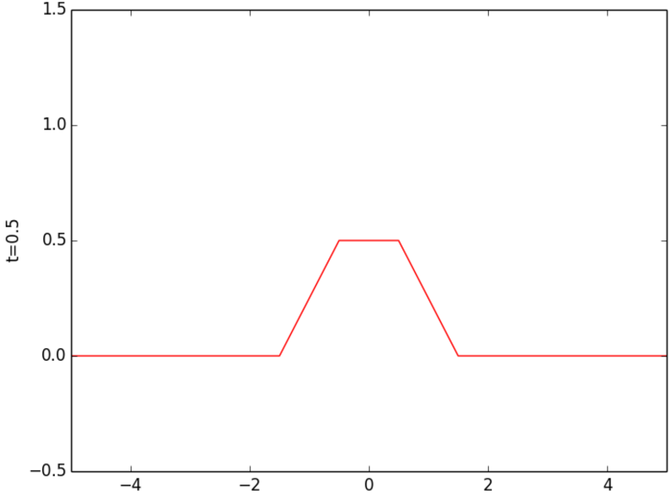
If $-a \leq x - ct \leq a < x + ct$, then

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds = \frac{1}{2c}\int_{x-ct}^{a} 1ds = \frac{-x+ct+a}{2c}.$$

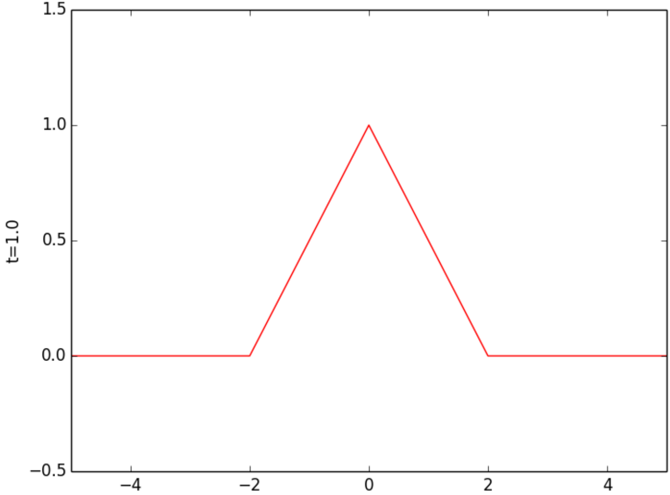
If $-a < a \leq x - ct < x + ct$, then

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds = 0.$$

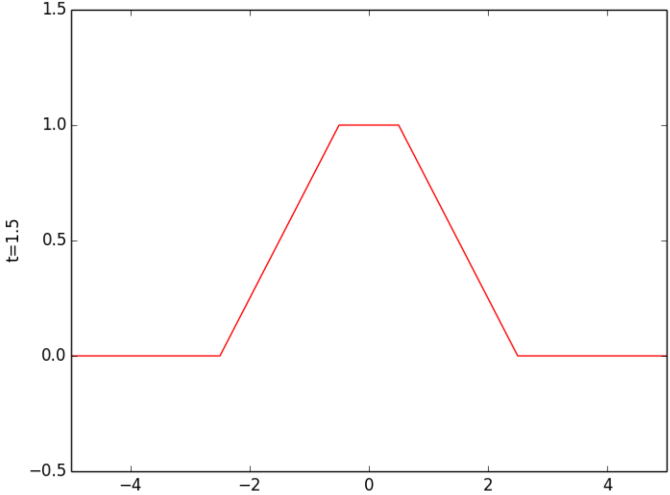
Question 10 (The hammer blow) (cont'd)



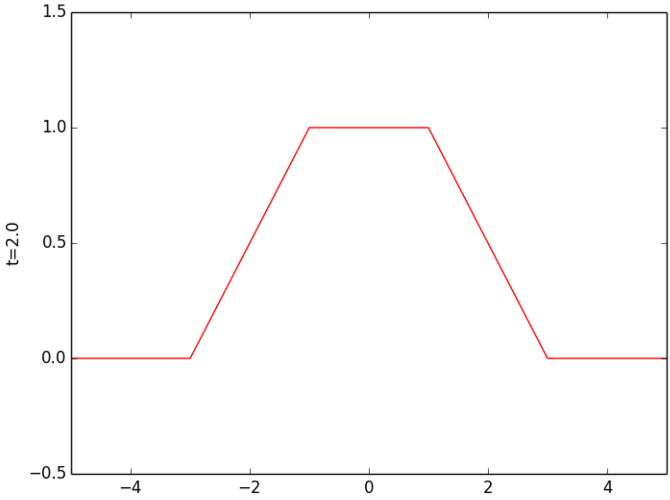
Question 10 (The hammer blow) (cont'd)



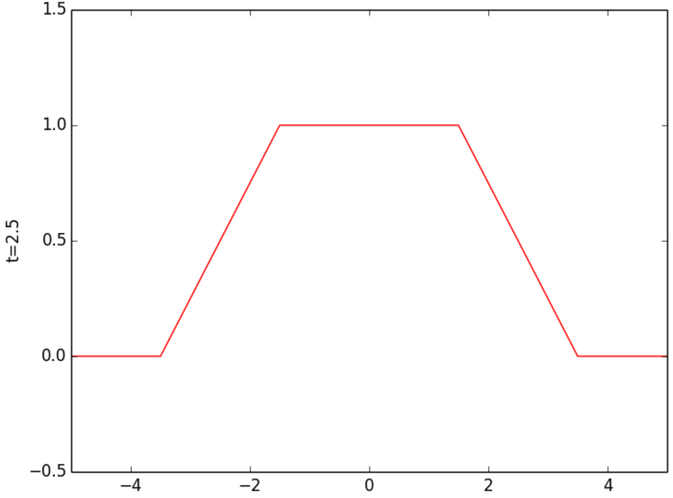
Question 10 (The hammer blow) (cont'd)



Question 10 (The hammer blow) (cont'd)



Question 10 (The hammer blow) (cont'd)



Question 11

Consider the equation

$$au_{tt} + bu_{xt} + cu_{xx} = 0, \quad -\infty < x < \infty, \quad t > 0, \tag{9}$$

$$u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \quad -\infty < x < \infty, \tag{10}$$

where a, b, and c are constants such that $ac < 0$. Show that the equation is hyperbolic, and derive the solution formula.

$$A = \begin{pmatrix} a & \frac{b}{2} \\ \frac{b}{2} & c \end{pmatrix}$$

$$\mathrm{Det}(A) = ac - \frac{b^2}{4}$$

Since $ac < 0$,

$$\mathrm{Det}(A) < 0$$

Question 11

Consider the equation $ay^2 + by + c = 0 \Rightarrow y_{1,2} = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$$au_{tt} + bu_{xt} + cu_{xx} = 0, \quad -\infty < x < \infty, \quad t > 0, \tag{9}$$

$$u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \quad -\infty < x < \infty, \tag{10}$$

where a, b, and c are constants such that $ac < 0$. Show that the equation is hyperbolic, and derive the solution formula. Then $a(y-y_1)(y-y_2)=0$

Solution: By the definition in Chapter 1, we see that the associated determinant is $ac - b^2/4 < 0$, so the equation is hyperbolic.
Letting $\alpha = (b + \sqrt{b^2 - 4ac})/(2a)$ and $\beta = (b - \sqrt{b^2 - 4ac})/(2a)$, we find that $\alpha + \beta = b/a$ and $\alpha\beta = c/a$. We factor the differential operator as

$$\left(a\partial_t^2 + b\partial_t\partial_x + c\partial_x^2\right)u = a(\partial_t + \alpha\partial_x)\underline{(\partial_t + \beta\partial_x)u} = 0.$$

Following the idea of solving the wave equation, we introduce $v = (\partial_t + \beta\partial_x)u$, and so we have

$$a(v_t + \alpha v_x) = 0 \quad \Rightarrow \quad v(x,t) = h(x - \alpha t).$$

We now solve

$$u_t + \beta u_x = h(x - \alpha t).$$

Question 11 (continued)

Notice that it has a particular solution of the form $u = f(x - \alpha t)$ with $f'(s) = h(s)/(\beta - \alpha)$, and the general solution of $u_t + \beta u_x = 0$ is $g(x - \beta t)$. Therefore, the general solution of (9) is

$$u(x,t) = f(x - \alpha t) + g(x - \beta t).$$

Now, we impose the initial conditions (10), and find that

$$f(x) + g(x) = \phi(x), \quad -\alpha f'(x) - \beta g'(x) = \psi(x).$$

Solving the system:

$$f'(x) + g'(x) = \phi'(x), \quad -\alpha f'(x) - \beta g'(x) = \psi(x),$$

leads to

$$f(x) = -\frac{\beta}{\alpha - \beta}\phi(x) - \frac{1}{\alpha - \beta}\int_0^x \psi(s)ds + A,$$
$$g(x) = \frac{\alpha}{\alpha - \beta}\phi(x) + \frac{1}{\alpha - \beta}\int_0^x \psi(s)ds + B,$$

where $A + B = 0$.
Finally, we obtain the d'Alembert-type formula

$$u(x,t) = \frac{\alpha\phi(x - \beta t) - \beta\phi(x - \alpha t)}{\alpha - \beta} + \frac{1}{\alpha - \beta}\int_{x-\alpha t}^{x-\beta t} \psi(s)ds,$$