MH4110 Partial Differential Equations

Course Overview

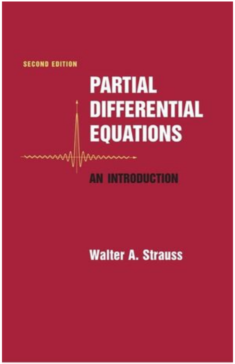
data $\longrightarrow$ Info

$\vec{d} \longrightarrow \vec{m}$

Welcome to MH4110!

Course Information

- Textbook: Partial Differential Equations: An Introduction, *Walter Strauss*, 2nd edition.



- Synopsis: Chapters 1-8 (mainly starred sections).

Course Information

- Instructors: Tong Ping

- Time:
  - Lecture: Mondays 2:30 pm–4:20 pm, SPMS-LT5,
    Thursdays 1:30 pm–2:20 pm, SPMS-LT5.
  - Tutorial: Thursdays 2:30 pm–3:20 pm, SPMS-LT5, Weeks 2-13.

- Office hours
  - Office hours are held on Thursdays from 3:30–5:30 pm at SPMS-MAS 04-17 (Tel: 6513 7457). Zoom or in-person meetings can be arranged by email at tongping@ntu.edu.sg.

Course Evaluation

- 10% Group Presentation: Students should form groups of 3–4 members and prepare a presentation on one of the following topics:
  - Option A: A rigorous and step-by-step derivation of Maxwell's equations.
  - Option B: Paper review – independently identify a scientific paper that applies machine learning to solve a partial differential equation (PDE). The group should explain the problem setting, the methodology, and how machine learning is used to solve the PDE.

  Each group will present during the tutorial session on an assigned Thursday. The presentation duration must not exceed 30 minutes.
- 20% Midterm Test: 9 March 2026 (Monday, Week 8), 90 minutes. Coverage includes lectures up to Week 7 (approximately Chapters 1–4). One double-sided A4 cheatsheet (2 pages) is permitted.
- 10% Quiz: 6 April 2026 (Monday, Week 12), 30 minutes. Open-book.
- 60% Final exam: Chapters 1-7, strictly open (1 double-sided A4 cheatsheat, 2 pages).

About MH4110: Partial Differential Equations

Goals

- Formulate PDEs
- Solve PDEs analytically or numerically
- Analyse the solutions

Prerequisites:

- Calculus, Linear Algebra, Ordinary Differential Equations (ODEs).

Observations and useful tips from ODEs

- DEs arise from mathematical modeling of real problems, and most of the models are PDEs.
- Only special types of DEs can be solved analytically, so the solutions of most DEs have to resort to numerical means.
- For solvable DEs, it is essential to identify the type and use the right technique to resolve the underlying problem.
  Find the right KEY to open the DOOR!
- Solving PDEs is much more complicated than solving ODEs.
- ODEs can be viewed as a special case of PDEs.

Why MH4110: Partial Differential Equations?

PDEs is a subject at the forefront of research in modern science.

Wide applications

PDEs can be used to describe a wide variety of phenomena such as sound, heat, electrodynamics, fluid dynamics, elasticity, quantum mechanics, or earthquakes.

- Heat equation: $u_t - \alpha\left(u_{xx} + u_{yy} + u_{zz}\right) = f(t, x, y, z)$

  Natural Convection in a room with a heater

  https://www.youtube.com/watch?v=Zr8WbV8F_5g

$$u(t, x, y, z)$$

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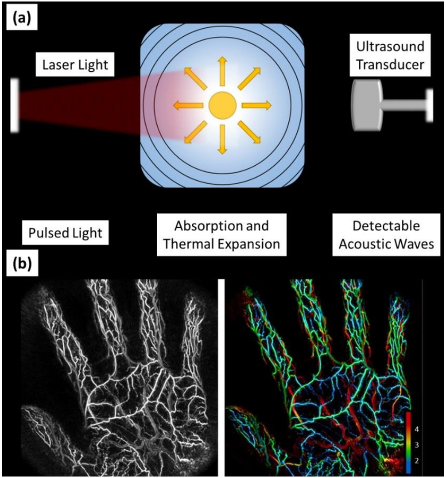
- Wave equation: $u_{tt} - c^2\left(u_{xx} + u_{yy} + u_{zz}\right) = f(t, x, y, z)$

  Seismic Waves from 2011 Tohoku Japan earthquake
  https://www.youtube.com/watch?v=ZCAOMMjjoN4

Why MH4110: Partial Differential Equations?

- Biomedical Imaging (Photoacoustic Imaging)



A schematic of photoacoustic imaging with two example images. (a) Shown is how pulsed light is absorbed by a biological tissue, which then causes thermal expansion and the emission of acoustic waves that can be detected by an ultrasound transducer. (b) A maximum intensity projected photoacoustic image of the palm of an individual's hand alongside the same image with vessel depth represented through color coding. Numbering on the bottom right of the figure represents vessel depth in mm. Permission for reuse granted under the Creative Commons Attribution License.

Deegan and Wang, Physics in Medicine and Biology, 2019.

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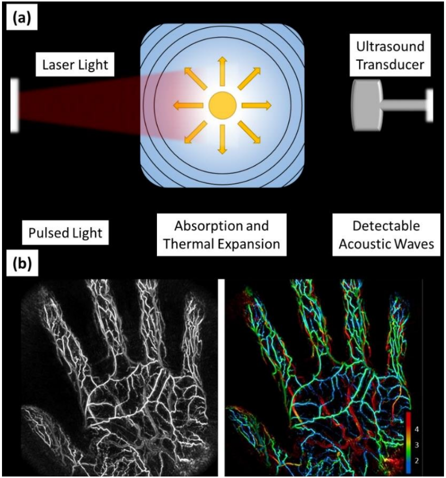

- The general photoacoustic equation

$$\left(\nabla^2 - \frac{1}{v_s^2}\frac{\partial^2}{\partial t^2}\right) p(\mathbf{r},\ \mathbf{t}) = -\frac{\beta}{\kappa \mathbf{v_s^2}}\frac{\partial^2 \mathbf{T}(\mathbf{r},\ \mathbf{t})}{\partial \mathbf{t}^2},$$

where $p(\mathbf{r},\ \mathbf{t})$ denotes the acoustic pressure at location $\mathbf{r}$ and time t and T denotes the temperature rise. The left-hand side of this equation describes the wave propagation, whereas the right-hand side represents the source term.

- The thermal equation

$$\rho C_v \frac{\partial T(\mathbf{r},\ \mathbf{t})}{\partial t} = H(\mathbf{r},\ \mathbf{t}).$$

Here $H(\mathbf{r},\ \mathbf{t})$ is the heating function defined as the thermal energy converted per unit volume and per unit time.

Wang, L. V. and Wu, H.-i.; Biomedical Optics: Principles and Imaging. (Wiley-Interscience, 2007)

Why MH4110: Partial Differential Equations?

Finance: Pricing of Financial Derivatives

16 Jan 26

A call option is a contract giving the owner the right but not the obligation to buy a specified amount of an underlying security (e.g., stock) at a specified price (strike) with a specified time.

1.30 USD/share 190 USD

The price $c(t, S(t))$ of a call option on a stock $S(t)$ with strike K at expiration time T satisfies the **Black-Scholes-Merton partial differential equation**

$$c_t(t,x) + rxc_x(t,x) + \frac{1}{2}\sigma^2 x^2 c_{xx}(t,x) = rc(t,x),\ t \in [0,T),\ x \geq 0,$$

which satisfies the **terminal condition**

NVDA

$$c(T,x) = (x-K)^+.$$

184.86 USD

and the following **boundary conditions**

$$c(t,0) = 0, \quad \lim_{x\to\infty}\left(c(t,x) - x + Ke^{-r(T-t)}\right) = 0, \qquad t \in [0,T).$$

12 Jan 2026

$S(t)$ is the price of the underlying stock. σ denotes the volatility of the stock price. r is the interest rate of the money market.

MH4110 Partial Differential Equations

ODE Review

ODE Review

Definition

An **ordinary differential equation**, ODE for short, is a relation containing one real variable x, the real dependent variable y, and some of its derivatives $y', y'', \cdots, y^{(n)}, \cdots$, with respect to x.

1

2

3

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Definition

The **order** of an ODE is defined to be the order of the highest derivative that appears in the equation. Thus, an n-th order ODE has the general form

$$F(\underbrace{x}_{1}, \underbrace{y}_{2}, \underbrace{y', \cdots, y^{(n)}}_{3}) = 0.$$

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(a, b)

Definition

A functional relation $y = y(x)$ between the dependent variable y and the independent variable x that satisfies the given ODE in some interval J is called a **solution** of the given ODE on J.

local $=$ $(-\infty, a)$

Remarks

A general solution of an n-th order ODE depends on n arbitrary constants, i.e. the solution y depends on x and n real constants $c_1, \cdots, c_n$.

$$\frac{dy}{dx} = x$$

$$y = \frac{1}{2}x^2 + C$$

Ordinary Differential Equations

Remarks

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Definition

An n-th order ODE is **linear** if it can be written in the form

$$a_0(x)y^{(n)} + a_1(x)y^{(n-1)} + \cdots + a_n(x)y = r(x). \tag{1}$$

The functions $a_j(x)$, $0 \leq j \leq n$ are called coefficients of the equation.

independent of y and its derivatives

Ordinary Differential Equations

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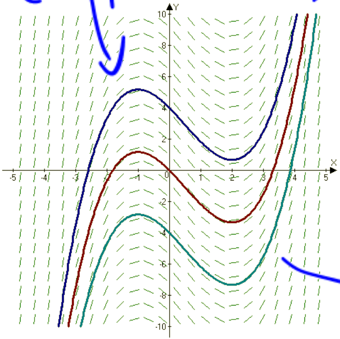
We shall always assume that $a_0(x) \neq 0$ in any interval in which the equation is defined. If $r(x) = 0$, (1) is called a **homogeneous** equation. If $r(x) \neq 0$, (1) is said to be a **non-homogeneous** equation, and $r(x)$ is called the non-homogeneous term. $Ax = b$

Integral curves

The solution of a first order ODE $y' = f(x, y)$ represents a one-parameter family of curves in the xy-plane. These are called *integral curves.*

$$y = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 2x + C$$

$C = 4$



$C = 0$

$C = -4$

Figure: Three integral curves for the slope field corresponding to the differential equation $y' = x^2 - x - 2$.

Ordinary Differential Equations

Separable equations

Typical separable equation can be written as

$$y' = \frac{f(x)}{g(y)}, \quad \text{or} \quad g(y)dy = f(x)dx.$$

$$dy/dx = f(x)/g(y)$$

The solution is given by

$$\int g(y)dy = \int f(x)dx + c.$$

Ordinary Differential Equations

Separable equations

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General solution: $y = ce^{-\frac{1}{2}x^2}$ $c \in \mathbb{R}$

$$\int g(y)dy = \int f(x)dx + c.$$

$y=0$ is a solution to $y' = -xy$

Exercise 1

Solve $y' = -xy$, $y(0) = 1$. $\longrightarrow c = 1$

$\frac{dy}{dx} = -xy \overset{y \neq 0}{\Longrightarrow} \frac{dy}{y} = -x\,dx$

$\ln|y| = -\frac{1}{2}x^2 + c_1$

$|y| = e^{-\frac{1}{2}x^2 + c_1}$

$y = \pm e^{c_1} e^{-\frac{1}{2}x^2} = ce^{-\frac{x^2}{2}}$ $(c \neq 0)$

The equation $y' = f\left(\frac{y}{x}\right)$ can be reduced to a separable equation by letting $u = \frac{y}{x}$, i.e. $y = xu$. So $f(u) = y' = u + xu'$,

$$\int \frac{du}{f(u) - u} = \int \frac{dx}{x} + c.$$

$$\frac{dy}{dx} = \frac{d}{dx}(xu) = u + x\frac{du}{dx}$$

$$\Big\downarrow \; y' = f\left(\frac{y}{x}\right)$$

$$u + x\frac{du}{dx} = f(u) \longrightarrow \frac{du}{f(u) - u} = \frac{dx}{x}$$

$\ln(1+u^2)\cdot|x| = C_1 \rightarrow (1+y^2/x^2)|x| = e^{C_1}$

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$(1+\frac{y^2}{x^2})x = \pm e^{C_1}$

$x^2+y^2 = cx \ (c \neq 0)$

If $x^2+y^2=0$ is a solution ?

$\rightarrow 2x+2yy'=0$

$2x^2+2xyy'=0$

Exercise 2

Solve $2xyy' + x^2 - y^2 = 0$.

$x^2+y^2 = cx, \ c \in \mathbb{R}$

$2\frac{xy}{x^2}y' + 1 - \frac{y^2}{x^2} = 0$

$2\frac{y}{x}y' + 1 - \frac{y^2}{x^2} = 0$

$2u(u+xu') + 1 - u^2 = 0$

$2uxu' + 1 + u^2 = 0$

$\frac{2u}{1+u^2}du = -\frac{dx}{x}$

$\ln(1+u^2) = -\ln|x| + C_1$

$$\frac{dy}{dx} = f(x,y)$$

Exact equations

We can write a first order ODE in the following form

$$M(x,y)dx + N(x,y)dy = 0. \tag{2}$$

Equation (2) is called *exact* if there exists a function $u(x,y)$ such that

$$M(x,y)dx + N(x,y)dy = du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy.$$

Once equation (2) is exact, the general solution is given by $u(x,y) = c$, where c is an arbitrary constant.

$$\frac{\partial u}{\partial x} = M(x,y) \qquad \frac{\partial u}{\partial y} = N(x,y)$$

Ordinary Differential Equations

Theorem

Assume M and N together with their first partial derivatives are continuous in the rectangle S: $|x - x_0| < a$, $|y - y_0| < b$. A necessary and sufficient condition for $M(x,y)dx + N(x,y)dy = 0$ to be exact is

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \qquad \text{for all } (x,y) \text{ in } S. \tag{3}$$

When equation (3) is satisfied, a general solution of $M(x,y)dx + N(x,y)dy = 0$ is given by $u(x,y) = c$, where

$$u(x,y) = \int_{x_0}^{x} M(s,y)ds + \int_{y_0}^{y} N(x_0,t)dt \tag{4}$$

and c is an arbitrary constant.

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Exercise 3

$M(x,y) = x^3 + 3xy^2$ $\quad \frac{\partial M}{\partial y} = 6xy$

Solve $(x^3 + 3xy^2)dx + (3x^2y + y^3)dy = 0$.

$N(x,y) = 3x^2y + y^3$ $\quad \frac{\partial N}{\partial x} = 6xy$

Ordinary Differential Equations

Integrating factors

A non-zero function $\mu(x, y)$ is an *integrating factor* of equation (2) if the equivalent differential equation

$$\mu(x, y)M(x, y)dx + \mu(x, y)N(x, y)dy = 0. \tag{5}$$

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If μ is an integrating factor of equation (2) then $(\mu M)_y = (\mu N)_x$, i.e.

$$N\mu_x - M\mu_y = \mu(M_y - N_x). \tag{6}$$

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$v = x, y, \frac{y}{x}, xy$

One may look for an integrating factor of the form $\mu = \mu(v)$, where v is a known function of x and y. Plugging into equation (6) we find

$$\frac{1}{\mu} = \frac{M_y - N_x}{N v_x - M v_y}.$$

Ordinary Differential Equations

If $\frac{M_y - N_x}{N v_x - M v_y}$ is a function of v alone, say, $\phi(v)$, then

$$\mu = e^{\int^v \phi(v) dv}$$

is an integrating factor of equation (2).

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- Let $v = x$. If $\frac{M_y - N_x}{N}$ is a function of x alone, say, $v = \phi_1(x)$, then $\mu = e^{\int^x \phi_1(x) dx}$ is an integrating factor of equation (2).

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- Let $v = y$. If $-\frac{M_y - N_x}{M}$ is a function of y alone, say, $v = \phi_2(y)$, then $\mu = e^{\int^y \phi_2(y)dy}$ is an integrating factor of equation (2).

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- Let $v = xy$. If $\frac{M_y - N_x}{yN - xM}$ is a function of xy alone, say, $\phi_3(xy)$, then $\mu = e^{\int^{xy} \phi_3(v) dv}$ is an integrating factor of equation (2).

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Exercise 4

Solve $(x^2y + y + 1) + x(1 + x^2)y' = 0$

First Order Linear Equations

Homogeneous equations

A first order homogeneous linear equation is of the form

$$y' + p(x)y = 0, \tag{7}$$

where $p(x)$ is a continuous function on an interval J.

First Order Linear Equations

Homogeneous equations

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$$y' + p(x)y = 0, \tag{7}$$

where $p(x)$ is a continuous function on an interval J.

Let $P(x) = \int_a^x p(s)ds$. Multiplying (12) by $e^{P(x)}$, we get

$$\frac{d}{dx}\left[e^{P(x)}y\right] = 0,$$

$e^{P(x)}p(x)\,y + e^{P(x)}y' = 0$

so $e^{P(x)}y = c$. The general solution of (12) is given by

$$y(x) = ce^{-\int_a^x p(s)ds}.$$

First Order Linear Equations

Non-homogeneous equations

A first order non-homogeneous linear equation is of the form

$$y' + p(x)y = q(x), \tag{8}$$

where $p(x)$ and $q(x)$ are continuous functions on an interval J.

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Let $P(x) = \int_a^x p(s)ds$. Multiplying (8) by $e^{P(x)}$, we get

$$\frac{d}{dx}\left[e^{P(x)}y\right] = e^{P(x)}q(x).$$

Thus,

$$e^{P(x)}y = \int_a^x e^{P(t)}q(t)dt + c.$$

The general solution of (8) is given by

$$y(x) = e^{-P(x)}\left[\int_a^x e^{P(t)}q(t)dt + c\right], \text{ where } P(x) = \int_a^x p(s)ds.$$

First Order Linear Equations

Exercise 5

Solve

$$y' + y = x. \tag{9}$$

First Order Linear Equations

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Solve

$$y' + y = x. \tag{9}$$

In this problem $p(x) = 1$ and $q(x) = x$. Let $P(x) = \int_a^x 1ds = x - a$. Multiplying (9) by $e^{P(x)} = e^{x-a}$, we get

$$\frac{d}{dx}\left[e^{x-a}y\right] = e^{x-a}x.$$

Thus,

$$e^{x-a}y = \int_a^x e^{t-a}tdt + c_0 = te^{t-a}\Big|_{t=a}^{t=x} - \int_a^x e^{t-a}dt + c_0 = (x-1)e^{x-a} + c_1.$$

The general solution of (9) is given by

$$y(x) = x - 1 + c_1 e^{-(x-a)} = x - 1 + ce^{-x}.$$

Bernoulli's Equation

Definition

The Bernoulli's equation of order n in the standard form reads

$$\frac{dy}{dx} + p(x)y = q(x)y^n, \tag{10}$$

where $n \neq 0, 1$ is a real number and which is nonlinear.

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Solution technique:

- Make a change of variable $u = y^{1-n}$
- Convert the equation to a linear equation in the new variable u
- Obtain a solution formula similar to the first-order linear equation

Dividing (10) by y^n, yields the equation

$$y^{-n}\frac{dy}{dx} + p(x)y^{1-n} = q(x). \tag{11}$$

We make the change of variable:

$$u = y^{1-n} \Longrightarrow \frac{du}{dx} = (1-n)y^{-n}\frac{dy}{dx} \Longrightarrow y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{du}{dx},$$

and eliminate y and y' from (11):

$$y^{-n}\frac{dy}{dx} + p(x)y^{1-n} = q(x) \Longrightarrow \frac{1}{1-n}\frac{du}{dx} + p(x)u(x) = q(x).$$

Rewrite the new equation as the standard form:

$$\frac{du}{dx} + \underbrace{(1-n)p(x)}_{P(x)}u(x) = \underbrace{(1-n)q(x)}_{Q(x)}.$$

It is a linear equation with the unknown function $u(x)$, and

$$P(x) = (1-n)p(x), \quad Q(x) = (1-n)q(x).$$

Using the formula for linear DE, we find

$$\begin{aligned} u(x) &= e^{-\int P(x)dx}\left[\int Q(x)e^{\int P(x)dx}dx + C\right] \\ &= e^{-(1-n)\int p(x)dx}\left[(1-n)\int q(x)e^{(1-n)\int p(x)dx}dx + C\right]. \end{aligned}$$

Finally we change the variable back:

$$y^{1-n} = e^{-(1-n)\int p(x)dx}\left[(1-n)\int q(x)e^{(1-n)\int p(x)dx}dx + C\right].$$

Summary of Solution Formula

Theorem

For a Bernoulli's equation of order n in the standard form

$$\frac{dy}{dx} + p(x)y = q(x)y^n, \quad n \neq 1,$$

setting $u = y^{1-n}$, we transform it into a linear equation:

$$\frac{du}{dx} + \underbrace{(1-n)p(x)}_{P(x)}u(x) = \underbrace{(1-n)q(x)}_{Q(x)},$$

and the general solution is

$$y^{1-n} = e^{-(1-n)\int p(x)dx}\left[(1-n)\int q(x)e^{(1-n)\int p(x)dx}dx + C\right].$$

Exercises

Solve the equations

$$\text{(i)} \quad 2\frac{dy}{dx} + (\tan x)y = \frac{(4x+5)^2}{\cos x}y^3.$$

and

$$\text{(ii)} \quad \frac{dy}{dx} + \frac{2}{x}y = (-x^2\cos x)y^2.$$

Homogeneous DE

Definition.

A function $f(x, y)$ is said to be a homogeneous function (of degree zero), if it satisfies

$$f(tx, ty) = f(x, y), \quad \forall t > 0.$$

Example: Check that

$$f(x, y) = \frac{x^2 - y^2}{2xy + y^2} = \frac{1 - \left(\frac{y}{x}\right)^2}{2\frac{y}{x} + \left(\frac{y}{x}\right)^2}$$

is homogeneous of degree zero.

$$f(tx, ty) = \frac{(tx)^2 - (ty)^2}{2t^2xy + (ty)^2} = f(x, y)$$

Observation: If we factor an x^2 term from the numerator and denominator, then the above function can be written in the form

$$f(x,y) = \frac{x^2\left[1-(y/x)^2\right]}{x^2\left[2y/x+(y/x)^2\right]} = \frac{1-(y/x)^2}{2y/x+(y/x)^2}.$$

Thus f can be considered as a function depending only on the single variable $V = y/x$, that is,

$$f(x,y) = F(V) = \frac{1-V^2}{2V+V^2}.$$

A function $f(x,y)$ is a homogeneous function (of degree zero), if and only if it depends on $V = y/x$, i.e.,

$$f(x,y) = F(y/x) = F(V).$$

Homogeneous DE: Definition

Definition:

If $f(x, y)$ is a homogeneous function (of degree zero), then the DE

$$\frac{dy}{dx} = f(x, y) = F(y/x). \tag{12}$$

is called a homogeneous first-order DE.

For example,

$$y' = \frac{x^2 - y^2}{2xy + y^2}$$

is a homogeneous DE.

Solution Technique

Let

$$V(x) = \frac{y}{x} \implies y = xV \implies \frac{dy}{dx} = V + x\frac{dV}{dx}. \tag{13}$$

Substituting it into the equation (12), we therefore obtain

$$V + x\frac{dV}{dx} = F(V),$$

or equivalently,

$$x\frac{dV}{dx} = F(V) - V.$$

The functions can now be separated to yield

$$\frac{1}{F(V) - V}dV = \frac{1}{x}dx \implies \int \frac{1}{F(V) - V}dV = \ln|x| + C.$$

Substituting the valuable $V = y/x$ back, we obtain the solution.

Solve

$$\frac{dy}{dx} = \frac{4x + y}{x - 4y}.$$

Solve

$$\frac{dy}{dx} = \frac{4x + y}{x - 4y}.$$

Solution. The function on the right-hand side is homogeneous of degree zero, so it is a homogeneous DE. Substitute $y = xV$

$$\frac{d}{dx}(xV) = \frac{4x + xV}{x - 4xV} = \frac{4 + V}{1 - 4V}.$$

That is

$$x\frac{dV}{dx} + V = \frac{4 + V}{1 - 4V},$$

or equivalently,

$$x\frac{dV}{dx} = \frac{4(1 + V^2)}{1 - 4V}.$$

Separating the variables gives

$$\frac{1 - 4V}{4(1 + V^2)}dV = \frac{1}{x}dx.$$

Variable Substitution

Use the variable substitution $u = xy$ to solve the initial value problem

$$\frac{dy}{dx} + \frac{y}{x} = \frac{xy}{1 + x^2y^2}, \quad y(1) = 1.$$

Variable Substitution

Solution: By the substitution $u = xy$, we have

$$\frac{du}{dx} = y + x\frac{dy}{dx} \quad \Rightarrow \quad \frac{dy}{dx} + \frac{y}{x} = \frac{1}{x}\frac{du}{dx}.$$

We therefore obtain the new equation:

$$\frac{1}{x}\frac{du}{dx} = \frac{u}{1+u^2} \quad \Rightarrow \quad \frac{1+u^2}{u}du = xdx.$$

It is separable, so the general solution is

$$\int \frac{1+u^2}{u}du = \int xdx + C \quad \Rightarrow \quad \ln|u| + \frac{u^2}{2} = \frac{x^2}{2} + C.$$

Variable Substitution

That is

$$2\ln|u| + u^2 = x^2 + 2C \quad \Rightarrow \quad 2\ln|xy| + x^2y^2 = x^2 + 2C.$$

By the initial condition, we have $C = 0$. So the solution is

$$2\ln|xy| + x^2y^2 = x^2$$

or equivalently, we formulate the solution as

$$x^2y^2 = e^{x^2 - x^2y^2}.$$

This completes the solution.

Second order linear ODE with constant coefficients

Find the general solution of

$$y'' + ay' + by = 0 \tag{14}$$

where a and b are constants.

Second order linear ODE with constant coefficients

Find the general solution of

$$y'' + ay' + by = 0 \tag{14}$$

where a and b are constants.

Look for a solution of the form $y = e^{\lambda x}$. Plugging into eqn. (14), we find that $e^{\lambda x}$ is a solution if and only if

$$\lambda^2 + a\lambda + b = 0.$$

1. If $a^2 - 4b > 0$, the characteristic function has two distinct real roots λ_1 and λ_2. The general solution is $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$.
2. If $a^2 - 4b = 0$, the characteristic function has one real root λ. The general solution is $y = c_1 e^{\lambda x} + c_2 x e^{\lambda x}$.
3. If $a^2 - 4b < 0$, the characteristic function has a pair of complex conjugate roots $\lambda = \alpha \pm i\beta$. The general solution is $y = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x)$.

Exercise 6

Solve

$$y'' + y' - 2y = 0, \quad y(0) = 4, \quad y'(0) = -5. \tag{15}$$

Exercise 7

Solve

$$y'' - 4y' + 4y = 0, \quad y(0) = 3, \quad y'(0) = 1. \tag{16}$$

Exercise 8

Solve

$$y'' - 2y' + 10y = 0. \tag{17}$$

Reducible Second-order DE

- Consider the second-order DE:

$$\frac{d^2y}{dx^2} = F\left(x, y, \frac{dy}{dx}\right), \tag{18}$$

where F is a known function.

- Introduce

$$v = dy/dx \Longrightarrow d^2y/dx^2 = dv/dx.$$

We can rewrite it as an equivalent system of 1st-order DEs:

$$\begin{cases} \frac{dy}{dx} = v \\ \frac{dv}{dx} = F(x, y, v) \end{cases} \tag{19}$$

- Exercise: Write the second-order DE as a system:

$$y'' = x \sin y' + e^x y + 1.$$

- In general, the second DE can not be solved directly, since the system involves three variables, namely, x, y and v.
- We shall explore the possibility to solve the DE, if it only involves two variables.

Nonhomogeneous DE

Theorem

Consider the nonhomogeneuous DE:

$$y'' + \underline{p(x)}y' + \underline{q(x)}y = \underline{f(x)}, \tag{20}$$

where p, q and f are given continuous functions on I. If $y = y_1(x)$ is a solution to the associated homogeneous DE:

$$y'' + p(x)y' + q(x)y = 0. \tag{21}$$

Then $y_2(x) = u(x)y_1(x)$ is the (general) solution of (20), where

$$u'(x) = v(x) \quad \text{and} \quad v' + \left(2\frac{y_1'}{y_1} + p\right)v = \frac{f}{y_1}, \tag{22}$$

with the solution

$$v(x) = \frac{1}{y_1^2(x)I(x)}\left(\int y_1(x)f(x)I(x)dx + C\right), \quad I(x) = e^{\int p(x)dx}.$$