Diffusion equation

We would like to solve the diffusion equation (i.e., heat equation)

$$u_t = ku_{xx} \tag{44}$$

and obtain a solution formula depending on the given initial data $u(x,0) = \phi(x)$. Here k is a positive constant.

$-\infty < x < \infty$

§2.3 The diffusion equation

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Compared with the wave equation, it is more challenging to solve and has quite different mathematical properties.

$$\begin{cases} u_t = k u_{xx} & -\infty < x < \infty \\ u(x,0) = \phi(x) \end{cases}$$

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Approach to be taken:

- **Step One** Show the uniqueness of the solution by using the Maximum Principle or Energy Method.
- **Step Two** Construct a special solution of this problem, which is thereby the unique one we want.

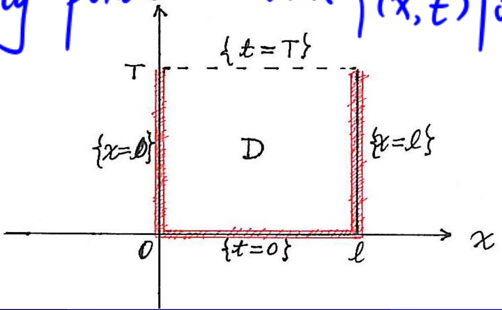
(without proof.)

Theorem: Maximum Principle. If $u(x,t)$ satisfies the diffusion equation $u_t = ku_{xx}$ in a rectangle, say,

$$D = \{(x,t) \;:\; 0 \le x \le l,\; 0 \le t \le T\},$$

in space-time, then the maximum value of $u(x,t)$ is assumed either initially (i.e., at $t=0$) or on the lateral sides (i.e., $x=0$ or $x=l$).

not at any points within $\{(x,t) \mid 0<x<l, 0<t\le T\}$



Comments on Maximum Principle

- This result is actually a weaker version of the maximum principle, as it does not specify that if the maximum can be attained in the interior of D. A stronger version asserts that **the maximum can not be assumed anywhere inside the rectangle but only on the bottom or the lateral sides (unless u is a constant).** However, the strong form is much more difficult to prove.

§2.3 The diffusion equation $u_t = k u_{xx}$

Comments on Maximum Principle $\Rightarrow (-u)_t = k(-u)_{xx}$

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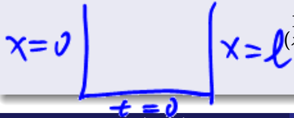
- The maximum of $u(x,t)$ over the three sides must be equal to the maximum of the $u(x,t)$ over the entire rectangle. If we denote the set of points comprising the three sides by

$$\Gamma = \{(x,t) \in D | t = 0\} \cup \{(x,t) \in D | x = 0\} \cup \{(x,t) \in D | x = l\}, \tag{45}$$

then the maximum principle can be written as



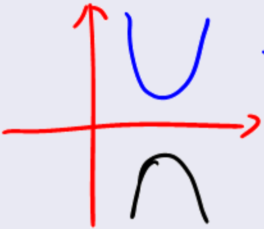
$$\max_{(x,t)\in\Gamma} \{u(x,t)\} = \max_{(x,t)\in D} \{u(x,t)\}. \tag{46}$$

Comments on Maximum Principle (Cont'd)

- The **maximum principle** also implies a **minimum principle**, since one can apply it to the function $-u(x, t)$, which also solves the diffusion equation, and make use of the following identity,

$$\min\{u(x, t)\} = -\max\{-u(x, t)\}.$$

  Thus, the minima points of the function $u(x, t)$ will exactly coincide with the maxima points of $-u(x, t)$, of which, by the maximum principle, there must necessarily be in Γ.



$y = (x-2)^2 + 1$

$y = -(x-2)^2 - 1$

§2.3 The diffusion equation

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- For the heat equation, the physical significance of the maximum principle is clear: *the highest temperature in the interior of the body can not exceed the highest initial temperature or the highest temperature on the boundary.* If you think of the heat conduction phenomena in a thin rod, since the initial temperature, as well as the temperature at the endpoints will dissipate through conduction of heat, and at no point the temperature can rise above the highest initial or end-point temperature.

$\min_D u(x,t) = -2kT$ at $(1,1)$

Example A (Exercise 2.3.1 on Page 45)

Verify that $u(x,t) = 1 - x^2 - 2kt$ is a solution to the diffusion equation: $u_t = ku_{xx}$. Find the locations of its maximum and its minimum in the closed rectangle: $D = \{(x,t) : 0 \le x \le 1, 0 \le t \le T\}$.

$u_t = -2k \qquad u_x = -2x \qquad u_{xx} = -2$

So $u_t - ku_{xx} = (-2k) - k(-2) = 0$

$1 - x^2 - 2kt$ is a solution to $u_t = ku_{xx}$

$\max_D u(x,t) = 1$ at $(0,0)$

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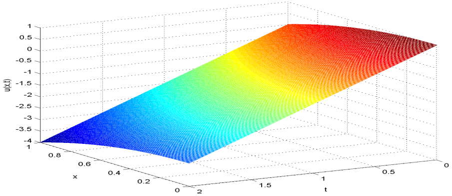
Solution: It is easy to verify that $u(x,t) = 1 - x^2 - 2kt$ is a solution. Then by the maximum principle, its maximum and minimum should be assumed at the bottom or lateral sides. We find

$$u|_{t=0} = 1 - x^2 \in [0,1], \quad u|_{x=0} = 1 - 2kt \in [1-2kT, 1], \quad u|_{x=1} = -2kt. \in [-2kT, 0]$$

Thus, the maximum is $u(0,0) = 1$ and the minimum is $u(1,T) = -2kT$.



Uniqueness

Theorem. The diffusion equation with Dirichlet boundary conditions:

IBVP

$$\begin{aligned} &u_t - ku_{xx} = f(x,t), \quad 0 < x < l, \quad t > 0, \\ &u(x,0) = \phi(x), \quad 0 < x < l, \\ &u(0,t) = g(t), \quad u(l,t) = h(t), \end{aligned} \tag{47}$$

where f, ϕ, g, h are given functions, has a unique solution.

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The maximum principle or the energy method can be used to give a proof of **uniqueness for the Dirichlet problem for the diffusion equation**.

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The maximum principle or the energy method can be used to give a proof of **uniqueness for the Dirichlet problem for the diffusion equation**.

Uniqueness means that any solution is completely determined by its initial and boundary conditions.

Uniqueness by Maximum Principle–Proof

Proof 1 Suppose that (47) has two solutions $u_1(x,t)$ and $u_2(x,t)$. Let $w = u_1 - u_2$, then w satisfies the homogeneous equation with zero initial-boundary conditions:

$$
\begin{aligned}
&w_t - kw_{xx} = 0, \quad 0 < x < l, \quad t > 0, \\
&w(x,0) = 0, \quad 0 < x < l, \\
&w(0,t) = 0, \quad w(l,t) = 0.
\end{aligned}
\tag{48}
$$

$$
\left.\begin{aligned}
(u_1)_t - k(u_1)_{xx} &= f(x,t) \\
(u_2)_t - k(u_2)_{xx} &= f(x,t)
\end{aligned}\right\} \Rightarrow w_t - kw_{xx} = 0
$$

$$
\left.\begin{aligned}
u_1(x,0) &= \phi(x) \\
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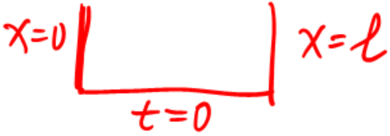
Taking any $T > 0$, we apply the maximum principle to the rectangle

$$D = \{(x,t) : 0 \le x \le l,\ 0 \le t \le T\},$$

and find that

$$w(x,t) \le \max_{(x,t)\in\Gamma} w(x,t) = 0, \quad \forall (x,t) \in D,$$

where Γ is the boundaries as defined in (45).



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$$w(x,t) \ge \min_{(x,t)\in\Gamma} w(x,t) = 0, \quad \forall (x,t) \in D,$$

Therefore, $w(x,t) \equiv 0$. Since T is arbitrary, $w(x,t) \equiv 0$ holds for all $0 < x < l$ and all $t > 0$. This means $u_1(x,t) \equiv u_2(x,t)$, so (47) has a unique solution.

Uniqueness by Energy Method – Proof

Proof 2 Suppose that (47) has two solutions $u_1(x,t)$ and $u_2(x,t)$. Let $w = u_1 - u_2$, then w satisfies the homogeneous equation with zero initial-boundary conditions (48).

Uniqueness by Energy Method – Proof

Proof 2 Suppose that (47) has two solutions $u_1(x, t)$ and $u_2(x, t)$. Let $w = u_1 - u_2$, then w satisfies the homogeneous equation with zero initial-boundary conditions (48). Define the following **energy**

$$E[w](t) = \frac{1}{2}\int_0^l [w(x, t)]^2 dx, \tag{49}$$

which is always nonnegative and $E[w](0) = 0$ because of the initial condition $w(x, 0) = 0$.

$$E[w](0) = \frac{1}{2}\int_0^l [w(x,0)]^2 dx = 0$$

$$E[w](t) \geq 0$$

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which is always nonnegative and $E[w](0) = 0$ because of the initial condition $w(x,0) = 0$. Differentiating the energy with respect to time, using the diffusion equation, then integrating by parts, we get

$w_t = k w_{xx}$

$w(0,t) = 0$

$$\frac{d}{dt}E = \int_0^l w \underline{w_t} dx = k\int_0^l w w_{xx} dx = k\int_0^l \left[\frac{\partial}{\partial x}(w w_x) - w_x^2\right] dx = k w w_x \Big|_0^l - \int_0^l w_x^2 dx \le 0, \tag{50}$$

$w(l,t) = 0$

$k w w_x \Big|_0^l$ is zero, since the boundary conditions $w(0,t) = 0$ and $w(l,t) = 0$. (49) is therefore a nonnegative and decreasing quantity, i.e., $0 \le E[w](t) \le E[w](0) = 0$.

$(w w_x)_x = w w_{xx} + w_x w_x$

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$kww_x \Big|_0^l$ is zero, since the boundary conditions $w(0,t) = 0$ and $w(l,t) = 0$. (49) is therefore a nonnegative and decreasing quantity, i.e., $0 \leq E[w](t) \leq E[w](0) = 0$. Consequently, $\int_0^l w^2(x,t)dx = 0$ for all $t \geq 0$. This yields $w \equiv 0$, i.e., $u_1 \equiv u_2$, so (47) has a unique solution.

Example B (not in the textbook)

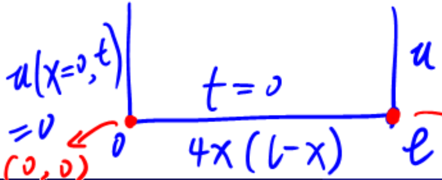
Consider the diffusion equation:

$$\begin{aligned} &u_t = u_{xx}, \quad (x,t) \in D := \{(x,t) : 0 < x < l,\ t > 0\}, \\ &u(0,t) = u(l,t) = 0, \quad u(x,0) = \underline{\underline{4x(l-x)}}. = -4x^2 + 4\ell x \end{aligned} \tag{51}$$

(i) Show that $0 < u(x,t) < l^2$ for all $(x,t) \in D$. $= -(2x-\ell)^2 + \ell^2$

(ii) Use the energy method to show that $\int_0^l u^2 dx$ is a strictly decreasing function of t.

$u(x,0) = \ell^2 - (2x-\ell)^2$ has a max. ℓ^2 at $x = \ell/2$



$u(x=\ell,t) = 0$

$(\ell, 0)$

a min 0 at $x = 0, \ell$

Example B (not in the textbook)

Consider the diffusion equation:

$k=1$

$$\begin{aligned} &u_t = u_{xx}, \quad (x,t) \in D := \{(x,t) : 0 < x < l,\ t > 0\}, \\ &u(0,t) = u(l,t) = 0, \quad u(x,0) = 4x(l-x). \end{aligned} \tag{51}$$

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Proof (i) Since u is zero on the lateral sides and the maximum value at $t = 0$ is $u(l/2, 0) = l^2$, the strong maximum principle implies that $u(x,t) < l^2$ for all $(x,t) \in D$. Since the minimum value at $t = 0$ is $u(0,0) = 0$, the strong minimum principle implies that $u(x,t) > 0$, for all $(x,t) \in D$. $0 < u(x,t) < l^2$ for all $(x,t) \in D$.

Example B (Cont'd)

(ii) It suffices to show that

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In fact, we find that $u_t = u_{xx}$

$$\frac{d}{dt}\int_0^l u^2(x,t)dx = 2\int_0^l u(x,t)u_t(x,t)dx = 2\int_0^l u(x,t)u_{xx}(x,t)dx$$

$$= \cancel{2u(x,t)u_x(x,t)\Big|_{x=0}^{x=l}} - 2\int_0^l u_x^2(x,t)dx = -2\int_0^l u_x^2(x,t)dx. \leq 0$$

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$$= 2u(x,t)u_x(x,t)\Big|_{x=0}^{x=l} - 2\int_0^l u_x^2(x,t)dx = -2\int_0^l u_x^2(x,t)dx.$$

This integral can not be zero since this would imply $u_x = 0$, i.e., u is a constant. Since $u(0,t) = 0$, the constant would be zero. But by (i), u is positive. Therefore, (52) holds, that is, $\int_0^l u^2 dx$ is a strictly decreasing function of t.