NANYANG TECHNOLOGICAL UNIVERSITY

SPMS/DIVISION OF MATHEMATICAL SCIENCES

2025/26 Semester 2 MH4110 Partial Differential Equations Tutorial 3, 05 February

Problem 1 Solve the following first order PDE

$$x^2 v_x + xyv_y = v^2.$$

Problem 2 (Ex. 5 on Page 28) Consider the equation

$$u_x + yu_y = 0.$$

with the boundary condition $u(x, 0) = \phi(x)$.

(a) For $\phi(x) \equiv x$, show that no solution exists.

(b) For $\phi(x) \equiv 1$, show that there are many solutions.

Problem 3 Let's consider the PDE

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0, \tag{1}$$

which is a linear equation of order two in two variables with six real constant coefficients. We know that

- Equation (1) is of elliptic type, if $a_{11}a_{22} - a_{12}^2 > 0$. By a linear transform, it can be reduced to
$$u_{xx} + u_{yy} + \{\text{terms of lower order 1 or 0}\} = 0. \tag{2}$$
- Equation (1) is of hyperbolic type, if $a_{11}a_{22} - a_{12}^2 < 0$. By a linear transform, it can be reduced to
$$u_{xx} - u_{yy} + \{\text{terms of lower order 1 or 0}\} = 0. \tag{3}$$
- Equation (1) is of parabolic type, if $a_{11}a_{22} - a_{12}^2 = 0$. By a linear transform, it can be reduced to
$$u_{xx} + \{\text{terms of lower order 1 or 0}\} = 0, \tag{4}$$
(unless $a_{11} = a_{12} = a_{22} = 0$.)

Please explicitly show how to reduce (1) into the form of (2), (3), or (4). (*Hint: use the method of completing the square.*)

Problem 4 (Ex. 1 on Page 31) What is the type of each of the following equations?

(a) $u_{xx} - u_{xy} + 2u_y + u_{yy} - 3u_{yx} + 4u = 0$.

(b) $9u_{xx} + 6u_{xy} + u_{yy} + u_x = 0$.

Problem 5 (Ex. 2 on Page 31) Find the regions in the xy plane where the equation

$$(1+x)u_{xx} + 2xyu_{xy} - y^2u_{yy} = 0$$

is elliptic, hyperbolic, or parabolic. Sketch them.

Problem 6 (Ex. 6 on Page 32) Consider the equation $3u_y + u_{xy} = 0$.

(a) What is its type?
(b) Find the general solution. *Hint: Substitute* $v = u_y$.
(c) With the auxiliary conditions $u(x,0) = e^{-3x}$ and $u_y(x,0) = 0$, does a solution exist? Is it unique?

Problem 7 Reduce the elliptic equation

$$u_{xx} + 3u_{yy} - 2u_x + 24u_y + 5u = 0$$

to the form $v_{xx} + v_{yy} + cv = 0$.

Problem 8 (Ex. 2 on Page 38) Solve $u_{tt} = 3u_{xx}$, $u(x,0) = \ln(1+x^2)$, $u_t(x,0) = 4+x$.

Problem 9 (Ex. 7 on Page 38) If both ϕ and ψ are even functions of x, show that the solution $u(x,t)$ of the wave equation is also even in x for all t.

Problem 10 (Ex. 5 on Page 38: The hammer blow) Let $\phi(x) \equiv 0$ and $\psi(x) = 1$ for $|x| < a$ and $\psi(x) = 0$ for $|x| \geq a$. Sketch the string profile (u versus x) at each of the successive instants $t = a/2c, a/c, 3a/2c, 2a/c$, and $5a/c$.

Problem 11 Consider the equation

$$au_{tt} + bu_{xt} + cu_{xx} = 0, \quad -\infty < x < \infty, \quad t > 0, \tag{5}$$

$$u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \quad -\infty < x < \infty, \tag{6}$$

where a, b, and c are constants such that $ac < 0$. Show that the equation is hyperbolic, and derive the solution formula.