MH4110 Partial Differential Equations

Chapter 2 - Waves and diffusions

Synopsis

1. Wave equation: General solution, d'Alambert's formula.
2. Wave equation: Causality, The energy method.
3. Heat equation: Maximum principle, Uniqueness, and Stability.
4. Heat equation: The solution in an integral form, Interpretation of the solution.
5. Comparison of wave and heat equations.

§2.1 The wave equation

Wave equation

The wave equation on the whole real line takes the form

$$u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty, \tag{1}$$

where the constant $c > 0$ is the wave speed. Physically, you can imagine a very long string in a transverse motion. It describes the dynamics of the amplitude $u(x, t)$ of the point at position x on the string at time t.

We use two methods to derive the general solution of (1):

1. Factorization of the differential operator
2. Characteristic coordinates

Method 1: Factorization of the differential operator

Observe that the second order linear operator of the wave equation factors into two first order operators

$$u_{tt} - c^2 u_{xx} = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u = 0. \tag{2}$$

Define

$$v = \left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)u = u_t + cu_x. \tag{3}$$

Then (2) can be rewritten as

$$v_t - cv_x = 0. \tag{4}$$

Therefore, we can solve (4) to find v, and then find u by solving (3).

Method 1: Factorization of the differential operator (Cont'd)

Applying **the method of characteristics** to (4) (see Chapter 1) leads to

$$v(x,t) = h(x+ct), \tag{5}$$

where h is an arbitrary function.

Plugging (5) into (3) gives

$$u_t + cu_x = h(x+ct). \tag{6}$$

It is a linear inhomogeneous equation, whose general solution is a particular solution plus the general solution to the homogeneous equation: $u_t + cu_x = 0$.

Method 1: Factorization of the differential operator (Cont'd)

The general solution of $u_t + cu_x = 0$ is $g(x - ct)$, where g is an arbitrary function.

We can check directly by differentiation that $u = f(x + ct)$ is a particular solution of (6):

$$u_t + cu_x = cf'(x + ct) + cf'(x + ct), \tag{7}$$

where f' is the ordinary derivative of a function of one variable and $f'(s)$ can be taken as $f'(s) = h(s)/(2c)$.

Finally, we obtain the solution of (1) is

$$u(x, t) = f(x + ct) + g(x - ct), \tag{8}$$

where f, g are arbitrary functions.

Theorem: The general solution of

$$u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty, \tag{9}$$

is

$$u(x,t) = f(x+ct) + g(x-ct), \tag{10}$$

where f, g are two arbitrary functions.

Method 2: Characteristic coordinates

Introduce the characteristic coordinates

$$\xi = x + ct, \quad \eta = x - ct, \tag{11}$$

By the chain rule, we have

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}, \quad \frac{\partial}{\partial t} = c\frac{\partial}{\partial \xi} - c\frac{\partial}{\partial \eta}, \tag{12}$$

and hence

$$\frac{\partial^2}{\partial x^2} = \frac{\partial^2}{\partial \xi^2} + 2\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2}, \quad \frac{\partial^2}{\partial t^2} = c^2\left[\frac{\partial^2}{\partial \xi^2} - 2\frac{\partial^2}{\partial \xi \partial \eta} + \frac{\partial^2}{\partial \eta^2}\right]. \tag{13}$$

Method 2: Characteristic coordinates (Cont'd)

So wave equation (1) or (9) takes the form

$$u_{tt} - c^2 u_{xx} = -4c^2 u_{\xi\eta} = 0, \tag{14}$$

which means that $u_{\xi\eta} = 0$ since $c \neq 0$.
The solution of the transformed equation (14) is

$$u = f(\xi) + g(\eta). \tag{15}$$

Switch back to the original variables (x, t), and we obtain the general solution

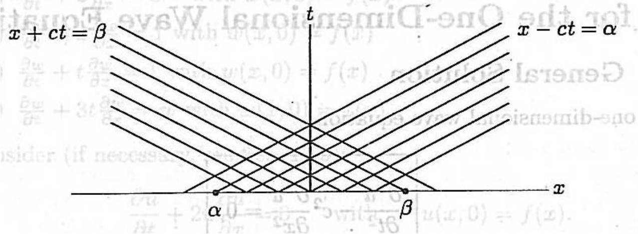
$$u(x,t) = f(x + ct) + g(x - ct), \tag{16}$$

where f, g are two arbitrary functions.

§2.1 The wave equation

Geometry of the wave equation

- The wave equation (1) has *two* families of characteristic lines: $x \pm ct =$constant (see Figure below).
- Part of the solution is constant along the corresponding characteristic line.
- One, $g(x - ct)$ (is a constant along $x - ct = \alpha$), is a wave of arbitrary shape traveling to the **right** at speed c.
- The other, $f(x + ct)$ (is a constant along $x + ct = \beta$), is a wave of another arbitrary shape traveling to the **left** at speed c.



Initial value problem–Cauchy Problem

The initial-value problem (IVP) is to solve the wave equation

$$u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty, \tag{17}$$

with the initial conditions

$$u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \quad -\infty < x < \infty, \tag{18}$$

where ϕ, ψ are arbitrary functions of single variable x, and together are called the initial data of the IVP.

Solution of the initial-value problem

The solution to the IVP is easily found from the general solution,

$$u(x,t) = f(x+ct) + g(x-ct). \tag{19}$$

All we need to do is to find f and g from the initial conditions of the IVP.

- To check the first initial condition, set $t = 0$,

$$u(x,0) = f(x) + g(x) = \phi(x). \tag{20}$$

- To check the second initial condition, we differentiate (19), and set $t = 0$,

$$u_t(x,0) = cf'(x) - cg'(x) = \psi(x). \tag{21}$$

Solution of the initial-value problem (Cont'd)

- Equations (20) and (21) form a system of two equations with two unknown functions f and g.

  1. We first differentiate (20),
  $$f'(x) + g'(x) = \phi'(x) \tag{22}$$
  2. Together with (21), we have
  $$f'(x) = \frac{\phi'(x)}{2} + \frac{\psi(x)}{2c}, \quad g'(x) = \frac{\phi'(x)}{2} - \frac{\psi(x)}{2c}. \tag{23}$$
  3. Integrating, we get
  $$f(x) = \frac{\phi(x)}{2} + \frac{1}{2c}\int_0^x \psi(s)ds + A, \quad g(x) = \frac{\phi(x)}{2} - \frac{1}{2c}\int_0^x \psi(s)ds + B, \tag{24}$$
  where A, B are two constants.
  4. Because of (20), we have $A + B = 0$.

Solution of the initial-value problem (Cont'd)

- Substituting $x + ct$ into the formula for f and $x - ct$ into that of g in (24), we get

$$u(x,t) = \frac{\phi(x+ct)}{2} + \frac{1}{2c}\int_0^{x+ct} \psi(s)ds + A + \frac{\phi(x-ct)}{2} - \frac{1}{2c}\int_0^{x-ct} \psi(s)ds + B. \tag{25}$$

- The above solution simplifies to

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds. \tag{26}$$

- Equation (26) is the solution formula for the IVP, due to d'Alambert in 1746. So it is also called d'Alambert formula.

Theorem: Consider the initial value problem:

$$\begin{aligned} &u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty, \\ &u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \quad -\infty < x < \infty, \end{aligned} \tag{27}$$

where ϕ, ψ are given functions. Its solution is

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds. \tag{28}$$

The solution formula (28) is known as the **d'Alembert formula**. Moreover, we call this initial value problem on $\mathbb{R} = (-\infty, \infty)$ a **Cauchy problem**.

Example 1

Solve the initial value problem (27) with the initial data

$$\phi(x) \equiv 0, \quad \psi(x) = \cos x. \tag{29}$$

Example A (not from the textbook)

Solve $u_{tt} = c^2 u_{xx}$ with $u(x,0) = e^x$ and $u_t(x,0) = \cos x$.

Example B: Initially at rest (not from the textbook)

Suppose that an infinite vibrating string is initially stretched into the shape of a single rectangular pulse is released at rest. The initial conditions are

$$u(x,0) = \phi(x) = \begin{cases} h, & |x| \le a, \\ 0, & |x| > a, \end{cases} \quad u_t(x,0) = \psi(x) = 0. \tag{30}$$

Find the solution of the wave equation $u_{tt} - c^2 u_{xx} = 0$ under these conditions.

Example B: Initially at rest (not from the textbook) (Cont'd)

By the d'Alembert formula (32),

$$u(x,t) = \frac{1}{2}\big[\phi(x+ct) + \phi(x-ct)\big].$$

Notice that

$$\phi(x+ct) = \begin{cases} h, & |x+ct| \le a, \\ 0, & |x+ct| > a, \end{cases}$$

$$\phi(x-ct) = \begin{cases} h, & |x-ct| \le a, \\ 0, & |x-ct| > a. \end{cases}$$

.

Example B: Initially at rest (not from the textbook) (Cont'd)

Hence, the solution

$$u(x,t) = \frac{1}{2}\big[\phi(x+ct) + \phi(x-ct)\big]$$

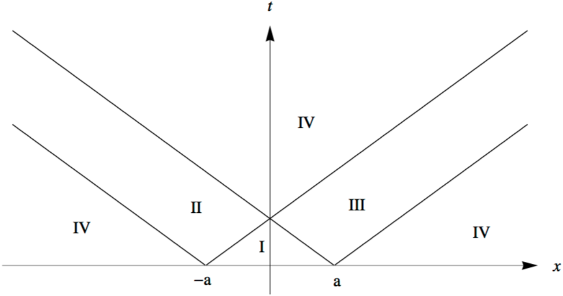
is piecewise defined in 4 different regions in the xt half-place (we consider only positive time $t \geq 0$).

There four regions are

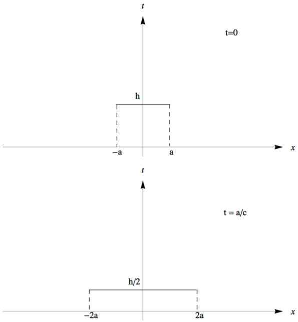
$$\begin{aligned} \mathrm{I} &: \{|x+ct| \leq a, |x-ct| \leq a\}, \quad && u(x,t) = h, \\ \mathrm{II} &: \{|x+ct| \leq a, |x-ct| > a\}, \quad && u(x,t) = \tfrac{h}{2}, \\ \mathrm{III} &: \{|x+ct| > a, |x-ct| \leq a\}, \quad && u(x,t) = \tfrac{h}{2}, \\ \mathrm{IV} &: \{|x+ct| > a, |x-ct| > a\}, \quad && u(x,t) = 0. \end{aligned}$$

Example B: Initially at rest (not from the textbook) (Cont'd)

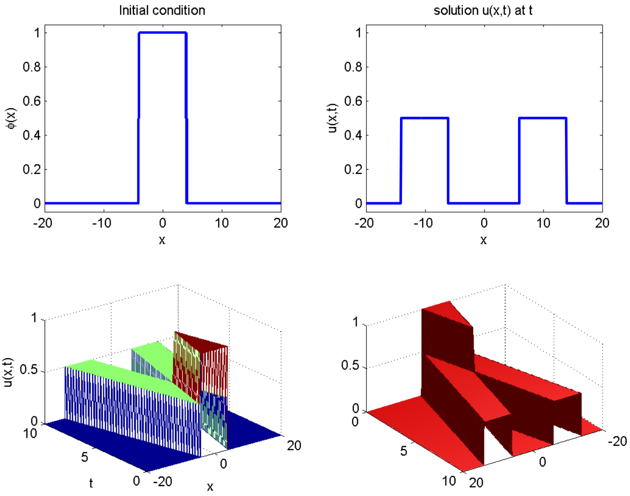
$$
\begin{aligned}
\text{I} &: \{|x+ct| \le a, |x-ct| \le a\}, & u(x,t) &= h, \\
\text{II} &: \{|x+ct| \le a, |x-ct| > a\}, & u(x,t) &= \tfrac{h}{2}, \\
\text{III} &: \{|x+ct| > a, |x-ct| \le a\}, & u(x,t) &= \tfrac{h}{2}, \\
\text{IV} &: \{|x+ct| > a, |x-ct| > a\}, & u(x,t) &= 0.
\end{aligned}
$$



Example B: Initially at rest (not from the textbook) (Cont'd)



Example B: Initially at rest (not from the textbook) (Cont'd)



Example C

Solve

$$\begin{aligned} &u_{xx} - 3u_{xt} - 4u_{tt} = 0, \\ &u(x,0) = x^2, \quad u_t(x,0) = e^x. \end{aligned} \tag{31}$$

Example C (Cont'd)

Solve

$$
\begin{aligned}
&u_{xx} - 3u_{xt} - 4u_{tt} = 0, \\
&u(x,0) = x^2, \quad u_t(x,0) = e^x.
\end{aligned}
$$

Example C (Cont'd)

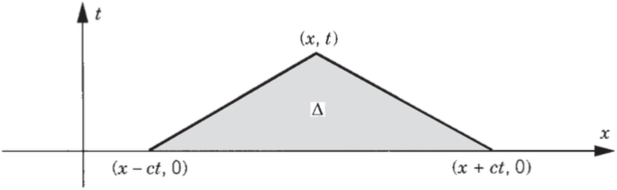
Solve

$$u_{xx} - 3u_{xt} - 4u_{tt} = 0,$$
$$u(x,0) = x^2, \quad u_t(x,0) = e^x.$$

Causality (Initial values travel with speeds bounded by c)

The value of the solution to the IVP (17) at a point (x, t) can be found from the d'Alembert formula,

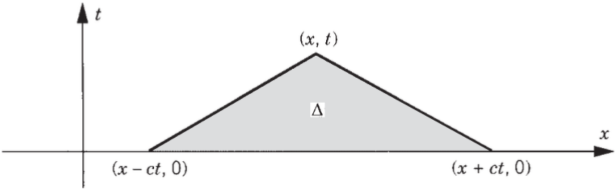
$$u(x,t) = \frac{\phi(x+ct)+\phi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds. \qquad (32)$$



§2.2 Causality and energy

Causality (Initial values travel with speeds bounded by c)

- $u(x,t)$ depends on the values of ϕ at only two points on the x axis, $(x-ct,0)$ and $(x+ct,0)$, and the values of ψ on the interval $[x-ct,x+ct]$. Thus, the interval $[x-ct,x+ct]$ is called **interval of dependence** for the point (x,t).
- The triangular region with vertices at $x-ct$ and $x+ct$ on the x axis and the vertex (x,t) is called the **domain of dependence, or the past history of the point** (x,t).

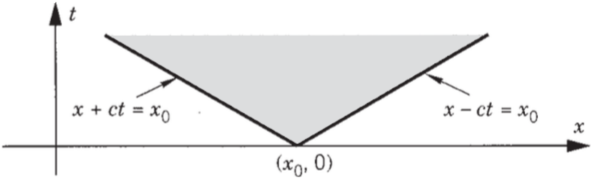


Causality (Initial values travel with speeds bounded by c)

- What are the points on the half-plane $t > 0$ that are influenced by the initial data at a fixed point $(x_0, 0)$? The set of all such points is called the **region of influence** of the point $(x_0, 0)$. It follows that the point $(x_0, 0)$ influences the value of the solution u at a point (x, t) if and only if

$$x - ct \leq x_0 \leq x + ct. \tag{33}$$

These are the points inside the forward characteristic cone that is defined by the point $(x_0, 0)$ and the edges $x - ct = x_0, x + ct = x_0$.



Example E (not from the textbook)

Analyze the solution to the IVP (17) with the following initial data

$$u(x,0) = \phi(x) \equiv 0, \quad u_t(x,0) = \psi(x) = \begin{cases} h, & |x| \le a, \\ 0, & |x| > a, \end{cases}. \tag{34}$$

Example E (not from the textbook) (Cont'd)

Example E (not from the textbook) (Cont'd)

Energy and conservation of energy

Consider an infinite string with constant linear density ρ and tension magnitude T. Then the wave equation describing the vibrations of the string is:

$$\rho u_{tt} = T u_{xx}, \quad -\infty < x < \infty. \tag{35}$$

We study the involved **kinetic energy** and **potential energy** of this system.

- The kinetic energy is $mv^2/2$:

$$\text{Kinetic Energy} \;\; E_K(t) = \frac{1}{2}\rho \int_{-\infty}^{\infty} u_t^2(x,t)dx, \tag{36}$$

- The potential energy is

$$\text{Potential Energy} \;\; E_P(t) = \frac{1}{2} T \int_{-\infty}^{\infty} u_x^2(x,t)dx. \tag{37}$$

Energy and conservation of energy

- The total energy of the string undergoing vibrations is

$$E(t) = E_K(t) + E_P(t) = \frac{1}{2}\int_{-\infty}^{\infty} (\rho u_t^2 + T u_x^2)\, dx, \quad t \geq 0. \tag{38}$$

Differentiating $E(t)$ gives

$$E'(t) = \int_{-\infty}^{\infty} \left(\rho u_t u_{tt} + T u_x u_{xt}\right) dx = \int_{-\infty}^{\infty} \left(u_t T u_{xx} + T u_x u_{xt}\right) dx. \tag{39}$$

The integration by parts under the usual assumption that $u, u_x \to 0$ as $|x| \to \infty$, gives

$$\int_{-\infty}^{\infty} u_t T u_{xx} dx = T u_t u_x\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} T u_x u_{xt} dx = -\int_{-\infty}^{\infty} T u_x u_{xt} dx. \tag{40}$$

Therefore, we have $E'(t) = 0$, indicating that the quantity $E(t)$ is conserved.

Energy and conservation of energy

Theorem: The significant consequence of (39) is that the **conservation of energy**:

$$\begin{aligned} E'(t) = 0 \quad &\Rightarrow \quad E(t) = \text{constant}, \quad \forall t \geq 0, \\ \Rightarrow \quad E(t) = E(0) &= \frac{1}{2} \int_{-\infty}^{\infty} \left(\rho \psi^2(x) + T(\phi'(x))^2 \right) dx, \end{aligned} \tag{41}$$

where we used the initial data: $u(x, 0) = \phi(x)$ and $u_t(x, 0) = \psi(x)$. The conservation of energy implies conversion of the kinetic energy into the potential energy and back without a loss.

Energy and conservation of energy

The conservation of energy can be derived in another more mathematical way. Multiplying both sides of (35) and integrating the resulted equation by parts, we obtain

$$\begin{aligned} 0 &= \int_{-\infty}^{\infty} (\rho u_{tt} - T u_{xx}) u_t dx = \int_{-\infty}^{\infty} \left(\rho u_{tt} u_t + T u_x u_{xt}\right) dx \\ &= \int_{-\infty}^{\infty} \left(\frac{\rho}{2}\partial_t(u_t^2) + \frac{T}{2}\partial_t(u_x^2)\right) dx = \frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} \left(\rho u_t^2 + T u_x^2\right) dx \\ &= E'(t), \end{aligned} \tag{42}$$

where $E(t)$ is as defined in (38). This verifies (41).

Example F (not from the textbook)

Show that the initial value problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), & -\infty < x < \infty, \\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), & \end{cases} \tag{43}$$

has a unique solution.

Example F (not from the textbook) (Cont'd)

Example F (not from the textbook) (Cont'd)

Diffusion equation

We would like to solve the diffusion equation (i.e., heat equation)

$$u_t = ku_{xx} \tag{44}$$

and obtain a solution formula depending on the given initial data $u(x, 0) = \phi(x)$. Here k is a positive constant.

Compared with the wave equation, it is more challenging to solve and has quite different mathematical properties.

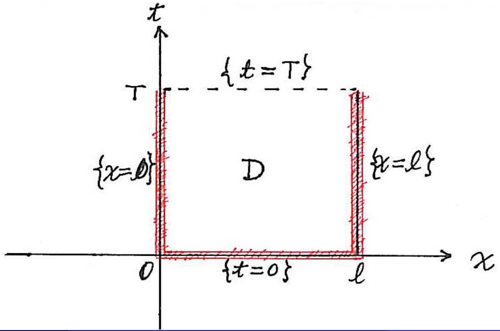
Approach to be taken:

- **Step One** Show the uniqueness of the solution by using the Maximum Principle or Energy Method.
- **Step Two** Construct a special solution of this problem, which is thereby the unique one we want.

Theorem: Maximum Principle. If $u(x,t)$ satisfies the diffusion equation $u_t = ku_{xx}$ in a rectangle, say,

$$D = \{(x,t) \ : \ 0 \le x \le l, \ 0 \le t \le T\},$$

in space-time, then the maximum value of $u(x,t)$ is assumed either initially (i.e., at $t=0$) or on the lateral sides (i.e., $x=0$ or $x=l$).



Comments on Maximum Principle

- This result is actually a weaker version of the maximum principle, as it does not specify that if the maximum can be attained in the interior of D. A stronger version asserts that **the maximum can not be assumed anywhere inside the rectangle but only on the bottom or the lateral sides (unless u is a constant).** However, the strong form is much more difficult to prove.
- The maximum of $u(x,t)$ over the three sides must be equal to the maximum of the $u(x,t)$ over the entire rectangle. If we denote the set of points comprising the three sides by

$$\Gamma = \{(x,t) \in D | t = 0\} \cup \{(x,t) \in D | x = 0\} \cup \{(x,t) \in D | x = l\}, \quad (45)$$

then the maximum principle can be written as

$$\max_{(x,t)\in\Gamma} \{u(x,t)\} = \max_{(x,t)\in D} \{u(x,t)\}. \quad (46)$$

Comments on Maximum Principle (Cont'd)

- The **maximum principle** also implies a **minimum principle**, since one can apply it to the function $-u(x,t)$, which also solves the diffusion equation, and make use of the following identity,

$$\min\{u(x,t)\} = -\max\{-u(x,t)\}.$$

  Thus, the minima points of the function $u(x,t)$ will exactly coincide with the maxima points of $-u(x,t)$, of which, by the maximum principle, there must necessarily be in Γ.

- For the heat equation, the physical significance of the maximum principle is clear: *the highest temperature in the interior of the body can not exceed the highest initial temperature or the highest temperature on the boundary.* If you think of the heat conduction phenomena in a thin rod, since the initial temperature, as well as the temperature at the endpoints will dissipate through conduction of heat, and at no point the temperature can rise above the highest initial or end-point temperature.

Example A (Exercise 2.3.1 on Page 45)

Verify that $u(x,t) = 1 - x^2 - 2kt$ is a solution to the diffusion equation: $u_t = ku_{xx}$. Find the locations of its maximum and its minimum in the closed rectangle: $D = \{(x,t) : 0 \leq x \leq 1, 0 \leq t \leq T\}$.

Uniqueness

Theorem. The diffusion equation with Dirichlet boundary conditions:

$$\begin{aligned} &u_t - ku_{xx} = f(x,t), \quad 0 < x < l, \ \ t > 0, \\ &u(x,0) = \phi(x), \quad 0 < x < l, \\ &u(0,t) = g(t), \quad u(l,t) = h(t), \end{aligned} \tag{47}$$

where f, ϕ, g, h are given functions, has a unique solution.

The maximum principle or the energy method can be used to give a proof of **uniqueness for the Dirichlet problem for the diffusion equation**.

Uniqueness means that any solution is completely determined by its initial and boundary conditions.

Uniqueness by Maximum Principle–Proof

Proof 1 Suppose that (47) has two solutions $u_1(x,t)$ and $u_2(x,t)$. Let $w = u_1 - u_2$, then w satisfies the homogeneous equation with zero initial-boundary conditions:

$$\begin{aligned} &w_t - k w_{xx} = 0, \quad 0 < x < l, \ \ t > 0, \\ &w(x,0) = 0, \quad 0 < x < l, \\ &w(0,t) = 0, \quad w(l,t) = 0. \end{aligned} \tag{48}$$

Taking any $T > 0$, we apply the maximum principle to the rectangle

$$D = \{(x,t) : 0 \le x \le l, \ 0 \le t \le T\},$$

and find that

$$w(x,t) \le \max_{(x,t)\in\Gamma} w(x,t) = 0, \quad \forall (x,t) \in D,$$

where Γ is the boundaries as defined in (45). The same type of argument for the minimum value shows that

$$w(x,t) \ge \min_{(x,t)\in\Gamma} w(x,t) = 0, \quad \forall (x,t) \in D,$$

Therefore, $w(x,t) \equiv 0$. Since T is arbitrary, $w(x,t) \equiv 0$ holds for all $0 < x < l$ and all $t > 0$. This means $u_1(x,t) \equiv u_2(x,t)$, so (47) has a unique solution.

Uniqueness by Energy Method – Proof

Proof 2 Suppose that (47) has two solutions $u_1(x,t)$ and $u_2(x,t)$. Let $w = u_1 - u_2$, then w satisfies the homogeneous equation with zero initial-boundary conditions (48). Define the following **energy**

$$E[w](t) = \frac{1}{2}\int_0^l [w(x,t)]^2 dx, \tag{49}$$

which is always nonnegative and $E[w](0) = 0$ because of the initial condition $w(x,0) = 0$. Differentiating the energy with respect to time, using the diffusion equation, then integrating by parts, we get

$$\frac{d}{dt}E = \int_0^l ww_t dx = k\int_0^l ww_{xx} dx = k\int_0^l \left[\frac{\partial}{\partial x}(ww_x) - w_x^2\right] dx = kww_x \Big|_0^l - \int_0^l w_x^2 dx \le 0, \tag{50}$$

$kww_x \Big|_0^l$ is zero, since the boundary conditions $w(0,t) = 0$ and $w(l,t) = 0$. (49) is therefore a nonnegative and decreasing quantity, i.e., $0 \le E[w](t) \le E[w](0) = 0$. Consequently, $\int_0^l w^2(x,t)dx = 0$ for all $t \ge 0$. This yields $w \equiv 0$, i.e., $u_1 \equiv u_2$, so (47) has a unique solution.

Example B (not in the textbook)

Consider the diffusion equation:

$$\begin{aligned} &u_t = u_{xx}, \quad (x,t) \in D := \{(x,t) : 0 < x < l,\ t > 0\}, \\ &u(0,t) = u(l,t) = 0, \quad u(x,0) = 4x(l-x). \end{aligned} \tag{51}$$

(i) Show that $0 < u(x,t) < l^2$ for all $(x,t) \in D$.

(ii) Use the energy method to show that $\int_0^l u^2 dx$ is a strictly decreasing function of t.

Proof (i) Since u is zero on the lateral sides and the maximum value at $t = 0$ is $u(l/2, 0) = l^2$, the strong maximum principle implies that $u(x,t) < l^2$ for all $(x,t) \in D$. Since the minimum value at $t = 0$ is $u(0,0) = 0$, the strong minimum principle implies that $u(x,t) > 0$, for all $(x,t) \in D$. $0 < u(x,t) < l^2$ for all $(x,t) \in D$.

Example B (Cont'd)

(ii) It suffices to show that

$$\frac{d}{dt}\int_0^l u^2(x,t)dx < 0. \tag{52}$$

In fact, we find that

$$\frac{d}{dt}\int_0^l u^2(x,t)dx = 2\int_0^l u(x,t)u_t(x,t)dx = 2\int_0^l u(x,t)u_{xx}(x,t)dx$$
$$= 2u(x,t)u_x(x,t)\Big|_{x=0}^{x=l} - 2\int_0^l u_x^2(x,t)dx = -2\int_0^l u_x^2(x,t)dx.$$

This integral can not be zero since this would imply $u_x = 0$, i.e., u is a constant. Since $u(0,t) = 0$, the constant would be zero. But by (i), u is positive. Therefore, (52) holds, that is, $\int_0^l u^2 dx$ is a strictly decreasing function of t.

Stability

Stability of solutions is the third ingredient of well-posedness after existence and uniqueness. In general, we say that a system is stable if "close" initial data generate "close" solutions. To measure "closeness", we need a measure for distance of functions.

Stability for the Dirichlet problem for the diffusion equation

Let us consider the diffusion equation:

$$\begin{aligned} &u_t - ku_{xx} = f(x,t), \quad 0 < x < l, \quad t > 0, \\ &u(x,0) = \phi(x), \quad 0 < x < l, \\ &u(0,t) = g(t), \quad u(l,t) = h(t), \end{aligned} \tag{53}$$

Let $u_1(x,t)$ and $u_2(x,t)$ be two solutions generated by the equation with the initial values $u_1(x,0) = \phi_1(x)$ and $u_2(x,0) = \phi_2(x)$, respectively.

Stability for the Dirichlet problem for the diffusion equation (Cont'd)

Suppose that we define the distance between two functions f, g as

$$\mathrm{dist}(f,g) = \Big(\int_0^l [f(x) - g(x)]^2 \, dx \Big)^{1/2},$$

which is called the L^2-distance.

Notice that $w = u_1 - u_2$ solves the same equation (51) as in Example B but with a different initial condition $w(x,0) = \phi_1(x) - \phi_2(x)$. We have already shown that $\int_0^l w^2 dx$ is a strictly decreasing function of t in Example B. Thus, we have

$$\int_0^l (u_1(x,t) - u_2(x,t))^2 dx \leq \int_0^l (\phi_1(x) - \phi_2(x))^2 dx,$$
$$\Rightarrow \quad \mathrm{dist}(u_1, u_2) \leq \mathrm{dist}(\phi_1, \phi_2), \quad \forall t \geq 0.$$

This means that the nearness of the initial conditions implies the nearness of the solutions.

Diffusion equation on the whole line

Our purpose is to solve the Cauchy problem on the real line:

$$u_t = ku_{xx}, \quad -\infty < x < \infty, \;\; t > 0; \tag{54}$$

$$u(x,0) = \phi(x). \tag{55}$$

The method below will be very different from the one that we used for the wave equation. The main idea is to first solve the equation for a particular data $\phi(x)$ of the form

$$u(x,0) = \phi(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases} \tag{56}$$

Then we build the solution of (54)-(55) with general $\phi(x)$ from this particular one.

§2.3 The diffusion equation

The properties below can help derive a solution formula for the diffusion equation.

Invariance properties of the diffusion equation (54)

(i) **[Spatial translations]** The translate $u(x-y,t)$ of any solution $u(x,t)$ is another solution, for any fixed y.

(ii) **[Dilation (scaling)]** The dilation $u(\sqrt{a}x, at)$ of any solution $u(x,t)$ is another solution, for any constant $a>0$.

(iii) **[Differentiation]** Any partial derivative (e.g., $u_x, u_t, u_{xx}, u_{xt}, \cdots$) of a solution $u(x,t)$ is again a solution.

(iv) **[Linear combinations]** If $u_1, u_2, \cdots, u_n$ are solutions of (54), then so is $u = c_1u_1 + c_2u_2 + \cdots + c_nu_n$ for any constants $c_1, c_2, \cdots, c_n$.

(v) **[Convolution invariance]** If $S(x,t)$ is a solution of (54), then so is

$$v(x,t) = \int_{-\infty}^{\infty} S(x-y,t)g(y)dy = S(\cdot,t) * g, \tag{57}$$

for any function g.

Solution formula for the diffusion equation

Theorem The problem

$$u_t = ku_{xx}, \quad -\infty < x < \infty, \quad t > 0; \quad u(x,0) = \phi(x), \tag{58}$$

has the solution

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp\Big(-\frac{(x-y)^2}{4kt} \Big) \phi(y) dy. \tag{59}$$

Solution derivation for the diffusion equation

Step 0: Starting with a particular IVP.

As a special initial data we take the following function

$$H(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0, \end{cases} \tag{60}$$

which is called the Heaviside step function. We first consider the initial value problem

$$Q_t = kQ_{xx}, \quad -\infty < x < \infty, \ \ t > 0; \quad Q(x,0) = H(x) \tag{61}$$

which will be solved in successive steps.

IVP (61) is the same as (58) but with a special initial data $H(x)$.

Solution derivation for the diffusion equation (Cont'd)

Step 1: Reduction to an ODE.

- If $Q(x,t)$ is a solution, then $Q(\sqrt{a}x, at)$ also solves $u_t - ku_{xx} = 0$ from the dilation property (ii) of the diffusion equation. But we cannot say that $Q(\sqrt{a}x, at)$ also solves the IVP (61) at this time.
- Check the initial condition of $Q(\sqrt{a}x, at)$. Since $Q(x,0) = H(x)$, we have $Q(\sqrt{a}x, 0) = H(\sqrt{a}x)$. It is easy to notice that $H(\sqrt{a}x) = H(x)$. So $Q(\sqrt{a}x, 0) = H(x)$. It means that $Q(\sqrt{a}x, at)$ also solves the IVP (61).
- The uniqueness of solutions then implies that $Q(\sqrt{a}x, at) = Q(x,t)$ for all $x \in R$, $t > 0$, so Q is invariant under the dilation $(x,t) \to (\sqrt{a}x, at)$ as well.
- For a fixed (x,t) and let $a = 1/t$, we have

$$Q(x,t) = Q(\sqrt{a}x, at) = Q(\sqrt{\frac{1}{t}}x, \frac{1}{t}t) = Q(\sqrt{\frac{1}{t}}x, 1). \tag{62}$$

So Q depends only on the ratio $x/\sqrt{t}$.

Solution derivation for the diffusion equation (Cont'd)

Step 1: Reduction to an ODE (Cont'd).

- We can thus look for $Q(x,t)$ of the special form

$$Q(x,t) = g(p) \qquad \text{where} \quad p = \frac{x}{\sqrt{4kt}}, \tag{63}$$

  where g is a function of only one variable (to be determined). The $\sqrt{4k}$ is included only to simplify a later formula.

- Using (63), we convert (61) into an ODE for g by use of the chain rule:

$$\begin{aligned} Q_t &= \frac{dg}{dp}\frac{\partial p}{\partial t} = -\frac{1}{2t}\frac{x}{\sqrt{4kt}}g'(p) = -\frac{1}{2t}pg'(p), \\ Q_x &= \frac{dg}{dp}\frac{\partial p}{\partial x} = \frac{1}{\sqrt{4kt}}g'(p), \\ Q_{xx} &= \frac{dQ_x}{dp}\frac{\partial p}{\partial x} = \frac{1}{4kt}g''(p). \end{aligned}$$

- Thus, Q could be a solution only when

$$0 = Q_t - kQ_{xx} = \frac{1}{t}\Big(-\frac{1}{2}pg'(p) - \frac{1}{4}g''(p)\Big) \;\Rightarrow\; g''(p) + 2pg'(p) = 0.$$

Solution derivation for the diffusion equation (Cont'd)

Step 2: Solving the ODE.

The ODE $g''(p) + 2pg'(p) = 0$ can be solved as follows:

$$\frac{g''}{g'} = -2p \;\Rightarrow\; \ln|g'| = -p^2 + c_1 \;\Rightarrow\; g' = \pm e^{c_1} e^{-p^2} = C_1 e^{-p^2},$$
$$\Rightarrow\; g(p) = C_1 \int e^{-p^2} dp + C_2.$$

Therefore, by (63),

$$Q(x,t) = g(p) = C_1 \int e^{-p^2} dp + C_2,$$

which satisfies $Q_t - kQ_{xx} = 0$ with $p = \frac{x}{\sqrt{4kt}}$.

Solution derivation for the diffusion equation (Cont'd)

Step 3: Checking the initial condition. We now impose the initial condition to find the unique solution for (61) by determining C_1 and C_2.

- For convenience, we write

$$Q(x,t) = g(p) = C_1 \int_0^p e^{-s^2} ds + C_2 \;\; \Rightarrow \;\; Q(x,t) = C_1 \int_0^{x/\sqrt{4kt}} e^{-s^2} ds + C_2, \;\; \forall t > 0. \tag{64}$$

- By the initial conditions in (61),

$$\begin{aligned} &\text{if} \;\; x > 0, \quad 1 = \lim_{t \to 0^+} Q = C_1 \int_0^{\infty} e^{-s^2} ds + C_2 = C_1 \frac{\sqrt{\pi}}{2} + C_2, \\ &\text{if} \;\; x < 0, \quad 0 = \lim_{t \to 0^+} Q = C_1 \int_0^{-\infty} e^{-s^2} ds + C_2 = -C_1 \frac{\sqrt{\pi}}{2} + C_2, \end{aligned} \tag{65}$$

where we used the known integral formula

$$\int_0^{\infty} e^{-s^2} ds = \frac{\sqrt{\pi}}{2}. \tag{66}$$

Solution derivation for the diffusion equation (Cont'd)

Step 3: Checking the initial condition (Cont'd).

- Solving (65) leads to

$$C_1 = \frac{1}{\sqrt{\pi}}, \quad C_2 = \frac{1}{2}.$$

- Then the solution of (61) is

$$Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-s^2} ds, \quad t > 0, \ -\infty < x < \infty. \tag{67}$$

Solution derivation for the diffusion equation (Cont'd)

Step 4: Solving the general IVP.

- Define
$$S(x,t) = \frac{\partial Q}{\partial x} \quad \Rightarrow \quad S(x,t) = \frac{1}{\sqrt{4\pi k t}} \exp\Big(-\frac{x^2}{4kt}\Big), \tag{68}$$
where $Q(x,t)$ is the solution of the particular IVP (61).
- By Property (iii), $S(x,t)$ is a solution of $u_t - ku_{xx} = 0$, and by Property (v), so is
$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy, \quad t > 0. \tag{69}$$
- We claim that this above u is the unique solution of the IVP (58). To verify this claim one only needs to check the initial condition of (58) as $u(x,0) = \phi(x)$.

Step 4: Solving the general IVP (Cont'd).

- Notice that $S(x,t) = \frac{\partial Q}{\partial x}$, we can rewrite u as follows

$$u(x,t) = \int_{-\infty}^{\infty} \frac{\partial Q}{\partial x}(x-y,t)\phi(y)dy = -\int_{-\infty}^{\infty} \frac{\partial}{\partial y}\left[Q(x-y,t)\right]\phi(y)dy$$
$$= +\int_{-\infty}^{\infty} Q(x-y,t)\phi'(y)dy - Q(x-y,t)\phi(y)\Big|_{y=-\infty}^{y=\infty}$$

  upon integrating by parts.

- We assume these limits vanish. In particular, let's temporarily assume that $\phi(y)$ itself equals zero for $|y|$ large. Using that Q has the Heaviside function (60) as its initial data, we have

$$u(x,0) = \int_{-\infty}^{\infty} Q(x-y,0)\phi'(y)dy = \int_{-\infty}^{x} \phi'(y)dy = \phi(x),$$

  where we used the assumption $\phi(-\infty) = 0$.

- Therefore, we have proved (69) with $S(x,t)$ given by (68) is the solution of (58). This ends the derivation of the solution formula.

Solution formula for the diffusion equation

Theorem (repeated) The problem

$$u_t = ku_{xx}, \quad -\infty < x < \infty, \quad t > 0; \quad u(x,0) = \phi(x),$$

has the solution

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp\Big(-\frac{(x-y)^2}{4kt}\Big)\phi(y)dy.$$

The function

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}} \exp\Big(-\frac{x^2}{4kt}\Big) \tag{70}$$

is known as the Gaussian kernel, fundamental solution, source function, Green's function, or propagator of the heat equation. It gives a way of propagating the initial data ϕ to later times, giving the solution at any time $t > 0$.

Some properties of the kernel function $S(x,t)$

- The solution of the IVP (54)-(55) is a convolution of $S(x,t)$ with the initial value $\phi(x)$:

$$u(x,t) = S(\cdot,t) * \phi = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp\Big(-\frac{(x-y)^2}{4kt}\Big)\phi(y)dy. \tag{71}$$

  Hence, $S(x,t)$ is known as the **Gaussian kernel** of the diffusion equation.

- The Gaussian kernel $S(x,t)$ is an even function of x and it is always positive. For large t, $S(x,t)$ is very spread out. For small t, it is a very thin tall spike of height $\frac{1}{\sqrt{4\pi kt}}$. The area under the curve is 1 :

$$\int_{-\infty}^{\infty} S(x,t)dx = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp\Big(-\frac{x^2}{4kt}\Big)dx$$

$$\Big(\text{Let } q = \frac{x}{\sqrt{4kt}}\Big) \quad = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-q^2}dq \overset{(66)}{=} 1, \quad \forall t \geq 0.$$

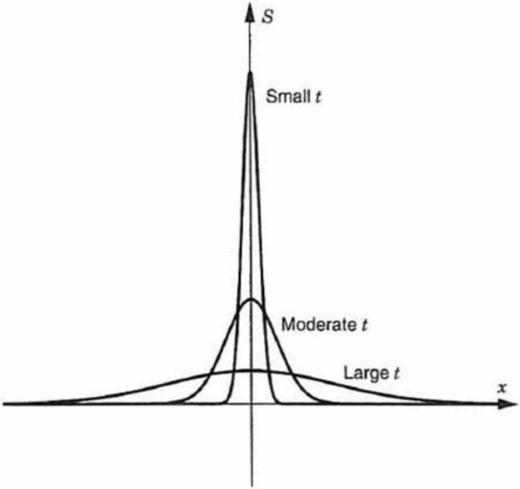


Figure 1

Some properties of the kernel function $S(x,t)$ (Cont'd)

- $$S(x,t) = \frac{1}{\sqrt{4\pi kt}} \exp\Big(-\frac{x^2}{4kt}\Big)$$

  is also termed as the **fundamental solution** of the diffusion equation, as it is the solution of the initial value problem:

  $$u_t = ku_{xx}, \quad -\infty < x < \infty, \quad t > 0; \quad u(x,0) = \delta(x), \tag{72}$$

  where δ is the Dirac delta function. This follows from the definition $S(x,t) = Q_x(x,t)$ (cf. (68)) and the fact that Q is the solution of the particular IVP (61), so $S(x,t)$ satisfies the initial condition

  $$S(x,0) = Q_x(t,0) = H'(x) = \delta(x), \quad \text{where} \quad H(x) = \begin{cases} 1, & \text{if} \quad x > 0, \\ 0, & \text{if} \quad x < 0, \end{cases}$$

  where $H(x)$ is the Heaviside function.

Some properties of the kernel function $S(x, t)$ (Cont'd)

- We see from the solution formula (71) that the solution u at a point (x, t) is influenced by the initial value $\phi(y)$ at all $y \in (-\infty, \infty)$.

  Indeed, we can view $S(x, t)$ as a weighting function:

  $$u(x, t) = \int_{-\infty}^{\infty} S(x - y, t)\phi(y)dy \simeq \sum_j S(x - y_j, t)\phi(y_j)\Delta y_j, \tag{73}$$

  where $\{y_j\}$ are some sampling points. The source function $S(x - y, t)$ weights the contribution of $\phi(y)$ according to the distance of y from x and the elapsed time t. The contribution from a point y_1 closer to x has a bigger weight $S(x - y_1, t)$, than the contribution from a point y_2 farther away, which gets weighted by $S(x - y_2, t)$.

  For very small t, the source function is a spike so that the formula exaggerates the values of ϕ near x. For any $t > 0$ the solution is a spread-out version of the initial values at $t = 0$.

Error function

It is usually impossible to evaluate integral

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} \exp\Big(-\frac{(x-y)^2}{4kt}\Big)\phi(y)dy.$$

completely in terms of elementary functions. Answers to particular problems, that is, to particular initial data $\phi(x)$, are sometimes expressible in terms of the error function of statistics.

The error function is defined as

$$\mathrm{Erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-p^2} dp. \tag{74}$$

When $x = 0$, $\mathrm{Erf}(0) = 0$; as $x \to \infty$, $\mathrm{Erf}(x) \to 1$.

Example 1

Let $Q(x,t)$ be the function defined in (67) as,

$$Q(x,t) = \frac{1}{2} + \frac{1}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-s^2}\, ds, \quad t > 0, \;\; -\infty < x < \infty,$$

then

$$Q(x,t) = \frac{1}{2} + \frac{1}{2} \mathrm{Erf}\left(\frac{x}{\sqrt{4kt}} \right). \tag{75}$$

Example C (not in the textbook)

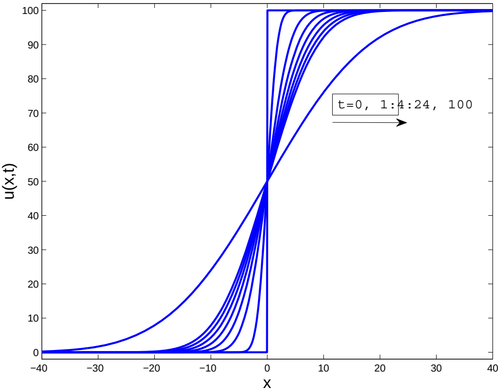
Use the error function to express the solution of

$$u_t = ku_{xx}, \quad -\infty < x < \infty, \quad u(x,0) = \phi(x) = \begin{cases} 100, & x > 0, \\ 0, & x < 0. \end{cases} \tag{76}$$

Solution By the solution formula, we have

$$\begin{aligned} u(x,t) &= \frac{100}{\sqrt{4\pi kt}} \int_0^\infty \exp\Big(-\frac{(x-y)^2}{4kt}\Big) dy \quad \Big(\text{Letting } q = \frac{y-x}{\sqrt{4kt}}\Big) \\ &= \frac{100}{\sqrt{\pi}} \int_{-x/\sqrt{4kt}}^\infty e^{-q^2} dq = \frac{100}{\sqrt{\pi}} \int_0^\infty e^{-q^2} dq + \frac{100}{\sqrt{\pi}} \int_{-x/\sqrt{4kt}}^0 e^{-q^2} dq \\ &= 50 + \frac{100}{\sqrt{\pi}} \int_0^{x/\sqrt{4kt}} e^{-q^2} dq \\ &= 50\Big(1 + \mathrm{Erf}\Big(\frac{x}{\sqrt{4kt}}\Big)\Big). \end{aligned}$$

Example C (not in the textbook) (Cont'd)



Observe that although the initial condition $\phi(x)$ is a step function, the solution is very smooth for $t > 0$. In fact, we find from the solution formula (59) that $u(x,t)$ is sufficiently smooth for $t > 0$ and any x, as all the partial derivatives of u exist. Hence, the Gaussian kernel is called a smoothing kernel.

§2.4 Comparison of waves and diffusions

- We have seen that the basic property of waves is that the information (e.g., initial data) gets transported in both directions at a finite speed. The basic property of diffusions is that the initial information gets spread out in a smooth fashion and gradually disappears.
- There is no maximum principle for the wave equation (why? see the example below), while the solution of the diffusion equation satisfies the maximum principle.
- For the wave equation, the energy is preserved as a constant, while for the diffusion equation, the energy decays to zero.
- For the diffusion equation is not well-posed for $t < 0$,[1] while the wave equation is well-posed for all t.

[1] We verify that $u_n(x,t) = \frac{1}{n}\sin(nx)e^{-n^2kt}$ is a solution to the diffusion equation. However, it goes to infinity for $t < 0$.

§2.4 Comparison of waves and diffusions

The fundamental properties of the wave and diffusion equations are summarized in the table below.

	Property	Waves	Diffusions
(i)	Speed of propagation?	Finite ($\leq c$)	Infinite
(ii)	Singularities for $t > 0$?	Transported along characteristics (speed $= c$)	Lost immediately
(iii)	Well-posed for $t > 0$?	Yes	Yes (at least for bounded solutions)
(iv)	Well-posed for $t < 0$?	Yes	No
(v)	Maximum principle	No	Yes
(vi)	Behavior as $t \to +\infty$?	Energy is constant so does not decay	Decays to zero (if ϕ integrable)
(vii)	Information	Transported	Lost gradually

Example D (not in the textbook)

Show that there is no maximum principle for the wave equation.

Solution We consider the wave equation $u_{tt} = u_{xx}$ with the initial condition:

$$u(x,0) = 0, \quad u_t(x,0) = \psi(x) = \begin{cases} 1, & |x| < 1, \\ 0, & |x| > 1. \end{cases}$$

Since there is no boundary, the maximum principle would state that the maximum is attained initially.

But the solution is zero initially, and takes on positive values for $t > 0$ based on the d'Alembert formula. Therefore, the maximum principle does not hold for the wave equation.

Summary

In the last several lectures we solved the initial value problems associated with the wave and diffusion equations on the whole line $x \in R$.

Waves

The solution to the wave initial-value problem on the whole line

$$\begin{aligned} & u_{tt} = c^2 u_{xx}, \quad -\infty < x < \infty, \\ & u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \quad -\infty < x < \infty, \end{aligned} \tag{77}$$

is given by d'Alembert formula

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds. \tag{78}$$

Summary (Cont'd)

Diffusions

The solution to the diffusion (heat) initial-value problem on the whole line

$$\begin{aligned} u_t &= k u_{xx}, \quad -\infty < x < \infty, \\ u(x,0) &= \phi(x), \quad -\infty < x < \infty, \end{aligned} \tag{79}$$

is given by the formula

$$u(x,t) = \frac{1}{\sqrt{4\pi k t}} \int_{-\infty}^{\infty} \exp\Big(-\frac{(x-y)^2}{4kt}\Big)\phi(y)dy. \tag{80}$$

The fundamental solution or Gaussian kernel

$$S(x,t) = \frac{1}{\sqrt{4\pi k t}} \exp\Big(-\frac{x^2}{4kt}\Big) \tag{81}$$

has the Dirac delta function $\delta(x)$ as its initial data.