NANYANG TECHNOLOGICAL UNIVERSITY

SPMS/DIVISION OF MATHEMATICAL SCIENCES

2025/26 Semester 2 MH4110 Partial Differential Equations Tutorial 3, 05 February

Problem 1 Solve the following first order PDE

$$x^2 v_x + xyv_y = v^2.$$

[Solution:] We define $u = -1/v$ to change the PDE into $x^2 u_x + xyu_y = 1$, and then solve it using the characteristic method. The characteristic curves are given by

$$\frac{dy}{dx} = \frac{xy}{x^2} = \frac{y}{x}.$$

This is a separable ODE, which can be solved to obtain the general solution $y/x = C$. Thus, our change of coordinates (no need to be orthogonal but should be non-degenerate) will be

$$\begin{cases} x' = x, \\ y' = \frac{y}{x}. \end{cases}$$

Use the chain rule, we have

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial u}{\partial x'} + \frac{\partial y'}{\partial x}\frac{\partial u}{\partial y'} = 1 \cdot \frac{\partial u}{\partial x'} - \frac{y}{x^2}\frac{\partial u}{\partial y'} = \frac{\partial u}{\partial x'} - \frac{y'}{x'}\frac{\partial u}{\partial y'}, \\ \frac{\partial u}{\partial y} = \frac{\partial x'}{\partial y}\frac{\partial u}{\partial x'} + \frac{\partial y'}{\partial y}\frac{\partial u}{\partial y'} = 0 \cdot \frac{\partial u}{\partial x'} + \frac{1}{x}\frac{\partial u}{\partial y'} = \frac{1}{x'}\frac{\partial u}{\partial y'}. \end{cases}$$

In these coordinates the equation takes the form

$$x'^2 \left(\frac{\partial u}{\partial x'} - \frac{y'}{x'}\frac{\partial u}{\partial y'} \right) + x'^2 y' \cdot \frac{1}{x'}\frac{\partial u}{\partial y'} = 1,$$

or

$$u_{x'} = \frac{1}{x'^2}.$$

Integrating both sides in x', we arrive at

$$u = -\frac{1}{x'} - f(y').$$

Finally, substituting the expressions of x' and y' in terms of (x, y) into the solution, we obtain

$$u(x, y) = -\frac{1}{x} - f\left(\frac{y}{x}\right).$$

or

$$v(x, y) = \frac{x}{1 + xf\left(\frac{y}{x}\right)}.$$

One should again check by substitution that this is indeed a solution to the PDE.

Problem 2 (Ex. 5 on Page 28) Consider the equation

$$u_x + yu_y = 0.$$

with the boundary condition $u(x,0) = \phi(x)$.

(a) For $\phi(x) \equiv x$, show that no solution exists.
(b) For $\phi(x) \equiv 1$, show that there are many solutions.

[Solution:] We look for the characteristic curve $y = y(x)$ satisfying

$$\frac{dy}{dx} = y \quad \Rightarrow \quad y(x) = Ce^x.$$

Hence, the general solution of the problem is

$$u(x,y) = f(e^{-x}y).$$

Given the boundary condition $u(x,0) = \phi(x)$, we have $u(x,0) = f(0) = \phi(x)$.

(a) If $\phi(x) \equiv x$, this contradicts the fact that $u(x,0) = f(0)$ is a constant. So no solution exists.
(b) If $\phi(x) \equiv 1$, that means $f(0) = 1$. There are many arbitrary functions satisfying $f(0) = 1$. For example, $f(x) = 1 + \sum_{i=1}^{n} a_i x^{n-i}$ (a_i are constants and n is an integer). So there are many solutions.

Problem 3 Let's consider the PDE

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + a_1u_x + a_2u_y + a_0u = 0, \tag{1}$$

which is a linear equation of order two in two variables with six real constant coefficients. We know that

- Equation (1) is of elliptic type, if $a_{11}a_{22} - a_{12}^2 > 0$. By a linear transform, it can be reduced to
$$u_{xx} + u_{yy} + \{\text{terms of lower order 1 or 0}\} = 0. \tag{2}$$
- Equation (1) is of hyperbolic type, if $a_{11}a_{22} - a_{12}^2 < 0$. By a linear transform, it can be reduced to
$$u_{xx} - u_{yy} + \{\text{terms of lower order 1 or 0}\} = 0. \tag{3}$$
- Equation (1) is of parabolic type, if $a_{11}a_{22} - a_{12}^2 = 0$. By a linear transform, it can be reduced to
$$u_{xx} + \{\text{terms of lower order 1 or 0}\} = 0, \tag{4}$$
(unless $a_{11} = a_{12} = a_{22} = 0$.)

Please explicitly show how to reduce (1) into the form of (2), (3), or (4). (*Hint: use the method of completing the square.*)

[Solution:] Since the linear transform does not change the order of partial derivatives, we assume that $a_1 = a_2 = a_0 = 0$. Without loss of generality, we assume that $a_{11} = 1$. Then the second-order part is

$$\begin{aligned} u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} &= \left(\partial_x^2 + 2a_{12}\partial_{xy} + a_{22}\partial_y^2\right)u \\ &= \left\{(\partial_x + a_{12}\partial_y)^2 + (-a_{12}^2 + a_{22})\partial_y^2\right\}u. \end{aligned}$$

Now, we know the classification of three cases is based on the sign of $a_{22} - a_{12}^2$.

We look for a suitable linear transform of the form:

$$x = \alpha\xi + \beta\eta, \quad y = \gamma\xi + \delta\eta. \tag{5}$$

Notice that

$$\begin{aligned} u_\xi &= u_x\frac{\partial x}{\partial \xi} + u_y\frac{\partial y}{\partial \xi} = \alpha u_x + \gamma u_y, \\ u_\eta &= u_x\frac{\partial x}{\partial \eta} + u_y\frac{\partial y}{\partial \eta} = \beta u_x + \delta u_y, \\ \Rightarrow \quad &\partial_\xi = \alpha\partial_x + \gamma\partial_y, \quad \partial_\eta = \beta\partial_x + \delta\partial_y. \end{aligned}$$

We take

$$\alpha = 1, \quad \gamma = a_{12}, \quad \beta = 0, \quad \delta = \sqrt{|a_{22} - a_{12}^2|}, \tag{6}$$

in (5), i.e., the transform:

$$x = \xi, \quad y = a_{12}\xi + \sqrt{|a_{22} - a_{12}^2|}\eta. \tag{7}$$

- If $a_{22} - a_{12}^2 > 0$ (elliptic case), we can transform the equation to $u_{\xi\xi} + u_{\eta\eta} = 0$.
- If $a_{22} - a_{12}^2 < 0$ (hyperbolic case), we can use the same transform (6) to convert the equation to $u_{\xi\xi} - u_{\eta\eta} = 0$.
- If $a_{22} - a_{12}^2 = 0$ (parabolic case), we cannot use the transform (6) to convert the equation to $u_{\xi\xi} = 0$ as it is degenerate. The characteristic lines corresponding to the linear operator $\partial_x + a_{12}\partial_y$ are $a_{12}x - y = C$. Thus, we can use the transform:

$$\xi = x, \quad \eta = a_{12}x - y \tag{8}$$

  to reduce the equatio to $u_{\xi\xi}$=0.

You also can refer to the proof of Theorem 1 on Page 29 of the reference book.

Problem 4 (Ex. 1 on Page 31) What is the type of each of the following equations?

(a) $u_{xx} - u_{xy} + 2u_y + u_{yy} - 3u_{yx} + 4u = 0$.

(b) $9u_{xx} + 6u_{xy} + u_{yy} + u_x = 0$.

[Solution:]

(a) Assume that the mixed partial derivatives u_{xy} and u_{yx} are equal to each other, the PDE $u_{xx} - u_{xy} + 2u_y + u_{yy} - 3u_{yx} + 4u = 0$ can be simplified into $u_{xx} - 4u_{xy} + u_{yy} + 2u_y + 4u = 0$. The coefficient matrix is therefore

$$A = \begin{bmatrix} 1 & -2 \\ -2 & 1 \end{bmatrix}.$$

Its determinant is $\det(A) = 1 \times 1 - (-2) \times (-2) = -3 < 0$. The PDE is of hyperbolic type.

(b) The coefficient matrix is

$$A = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}.$$

Its determinant is $\det(A) = 9 \times 1 - 3 \times 3 = 0$. The PDE is of parabolic type.

Problem 5 (Ex. 2 on Page 31) Find the regions in the xy plane where the equation

$$(1+x)u_{xx} + 2xyu_{xy} - y^2 u_{yy} = 0$$

is elliptic, hyperbolic, or parabolic. Sketch them.

[Solution:]

The coefficient matrix is

$$A = \begin{bmatrix} 1+x & xy \\ xy & -y^2 \end{bmatrix}.$$

Its determinant is $\det(A) = (1+x)(-y^2) - (xy)^2 = -(x^2+x+1)y^2 = -\left[(x+\frac{1}{2})^2 + \frac{3}{4}\right] y^2$. We have

- If $y = 0$, then $\det(A) = 0$. The PDE is of parabolic type.
- If $y \neq 0$, then $\det(A) < 0$. The PDE is of hyperbolic type.

Problem 6 (Ex. 6 on Page 32) Consider the equation $3u_y + u_{xy} = 0$.

(a) What is its type?

(b) Find the general solution. *Hint: Substitute* $v = u_y$.

(c) With the auxiliary conditions $u(x,0) = e^{-3x}$ and $u_y(x,0) = 0$, does a solution exist? Is it unique?

[Solution:]

(a) We see that $a_{11} = 0, a_{12} = 1/2, a_{22} = 0$, so we have $a_{11}a_{22} - a_{12}^2 = -1/4 < 0$. The equation is of hyperbolic type.

(b) Set $v = u_y$, the original equation $3u_y + u_{xy} = 0$ can be rewritten as $3v + v_x = 0$. Its general solution is $v(x,y) = f(y)e^{-3x}$ with f as an arbitrary function. Integrating over y, we have the general solution to $u_y = v$ as $u(x,y) = F(y)e^{-3x} + G(x)$. Here F (a primitive function of f) and G are both arbitrary functions of a single variable.

(c) Given that $u(x,0) = e^{-3x}$, we have $u(x,0) = F(0)e^{-3x} + G(x) = e^{-3x}$, so $G(x) = [1 - F(0)]e^{-3x}$. Additionally, $u_y(x,0) = 0$ implies that $u_y(x,0) = F'(0)e^{-3x} = 0$ and hence $F'(0) = 0$. Therefore, the solution is $u(x,y) = [F(y) + 1 - F(0)]e^{-3x}$ with the function $F(y)$ satisfying $F'(0) = 0$. There are many possibilities for $F(y)$. For example, $F(y) = cy^2$ or $F(y) = cy^3$ (c is an arbitrary constant). The solution exists but is not unique.

Problem 7 Reduce the elliptic equation

$$u_{xx} + 3u_{yy} - 2u_x + 24u_y + 5u = 0$$

to the form $v_{xx} + v_{yy} + cv = 0$.

[Solution:] We look for a suitable linear transform of the form:

$$\xi = ax + by, \quad \eta = cx + dy.$$

The Jacobian determinant of the linear transform is

$$J = |\frac{\partial(\xi, \eta)}{\partial(x, y)}| = ad - bc.$$

To have a non-degenerate transform, $ad \neq bc$ is required. The first and second order derivatives in the $\xi - \eta$ coordinates can be expressed as

$$\begin{aligned} u_x &= u_\xi \frac{\partial \xi}{\partial x} + u_\eta \frac{\partial \eta}{\partial x} = au_\xi + cu_\eta, \\ u_y &= u_\xi \frac{\partial \xi}{\partial y} + u_\eta \frac{\partial \eta}{\partial y} = bu_\xi + du_\eta, \end{aligned}$$

and further

$$\begin{aligned} u_{xx} &= (au_\xi + cu_\eta)_\xi \frac{\partial \xi}{\partial x} + (au_\xi + cu_\eta)_\eta \frac{\partial \eta}{\partial x} = \left(a^2 u_{\xi\xi} + acu_{\xi\eta}\right) + \left(acu_{\xi\eta} + c^2 u_{\eta\eta}\right) \\ &= a^2 u_{\xi\xi} + 2acu_{\xi\eta} + c^2 u_{\eta\eta}, \\ u_{yy} &= (bu_\xi + du_\eta)_\xi \frac{\partial \xi}{\partial y} + (bu_\xi + du_\eta)_\eta \frac{\partial \eta}{\partial y} = \left(b^2 u_{\xi\xi} + bdu_{\xi\eta}\right) + \left(bdu_{\xi\eta} + d^2 u_{\eta\eta}\right) \\ &= b^2 u_{\xi\xi} + 2bdu_{\xi\eta} + d^2 u_{\eta\eta}. \end{aligned}$$

So, the PDE can be rewritten as

$$\begin{aligned} 0 &= u_{xx} + 3u_{yy} - 2u_x + 24u_y + 5u \\ &= \left(a^2 u_{\xi\xi} + 2acu_{\xi\eta} + c^2 u_{\eta\eta}\right) + 3\left(b^2 u_{\xi\xi} + 2bdu_{\xi\eta} + d^2 u_{\eta\eta}\right) \\ &\quad - 2\left(au_\xi + cu_\eta\right) + 24\left(bu_\xi + du_\eta\right) + 5u \\ &= (a^2 + 3b^2)u_{\xi\xi} + (2ac + 6bd)u_{\xi\eta} + (c^2 + 3d^2)u_{\eta\eta} + (-2a + 24b)u_\xi + (-2c + 24d)u_\eta + 5u \end{aligned}$$

To make the above equation have a form of $v_{xx} + v_{yy} + cv = 0$, $2ac + 6bd = 0$, $-2a + 24b = 0$, and $-2c + 24d = 0$ are required. These imply that $a = b = 0$ or $c = d = 0$. The linear transform would be a degenerate one. We cannot find a linear transform to change the PDE into the desired form.

Alternatively, let $u = v(x, y)e^{\alpha x + \beta y}$, then

$$u_x = v_x e^{\alpha x + \beta y} + \alpha v e^{\alpha x + \beta y}, \qquad u_y = v_y e^{\alpha x + \beta y} + \beta v e^{\alpha x + \beta y}$$

and

$$u_{xx} = v_{xx}e^{\alpha x+\beta y} + \alpha v_x e^{\alpha x+\beta y} + \alpha v_x e^{\alpha x+\beta y} + \alpha^2 v e^{\alpha x+\beta y},$$
$$u_{yy} = v_{yy}e^{\alpha x+\beta y} + \beta v_y e^{\alpha x+\beta y} + \beta v_y e^{\alpha x+\beta y} + \beta^2 v e^{\alpha x+\beta y}.$$

The original PDE can be rewritten as

$$\begin{aligned}
0 &= u_{xx} + 3u_{yy} - 2u_x + 24u_y + 5u \\
&= \left(v_{xx}e^{\alpha x+\beta y} + \alpha v_x e^{\alpha x+\beta y} + \alpha v_x e^{\alpha x+\beta y} + \alpha^2 v e^{\alpha x+\beta y}\right) \\
&+ 3\left(v_{yy}e^{\alpha x+\beta y} + \beta v_y e^{\alpha x+\beta y} + \beta v_y e^{\alpha x+\beta y} + \beta^2 v e^{\alpha x+\beta y}\right) \\
&- 2\left(v_x e^{\alpha x+\beta y} + \alpha v e^{\alpha x+\beta y}\right) + 24\left(v_y e^{\alpha x+\beta y} + \beta v e^{\alpha x+\beta y}\right) + 5ve^{\alpha x+\beta y},
\end{aligned}$$

which can be simplified as

$$\begin{aligned}
0 &= \left(v_{xx} + 2\alpha v_x + \alpha^2 v\right) + 3\left(v_{yy} + 2\beta v_y + \beta^2 v\right) - 2\left(v_x + \alpha v\right) + 24\left(v_y + \beta v\right) + 5v \\
&= v_{xx} + 3v_{yy} + (2\alpha - 2)v_x + (6\beta + 24)v_y + (\alpha^2 + 3\beta^2 - 2\alpha + 24\beta + 5)v
\end{aligned}$$

Assuming that $\alpha = 1$ and $\beta = -4$, the change of variable $u = ve^{x-4y}$ leads to a new PDE as

$$v_{xx} + 3v_{yy} - 44v = 0.$$

Letting $y' = \sqrt{1/3}y$, we have $v_{yy} = v_{y'y'}(dy'/dy)^2 = (1/3)v_{y'y'}$. In the $x-0-y'$ coordinates, the above PDE has the form

$$v_{xx} + v_{y'y'} - 44v = 0.$$

Problem 8 (Ex. 2 on Page 38) Solve $u_{tt} = 3u_{xx}$, $u(x,0) = \ln(1+x^2)$, $u_t(x,0) = 4+x$.
[Solution:] In this problem the speed of the wave is $c = \sqrt{3}$. By the d'Alembert formula,

$$u(x,t) = \frac{\phi(x-ct) + \phi(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds.$$

Since $\phi(x) = \ln(1+x^2)$ and $\psi(x) = 4+x$, we have

$$\begin{aligned}
u(x,t) &= \frac{\ln[1+(x-ct)^2] + \ln[1+(x+ct)^2]}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct}(4+s)ds \\
&= \frac{1}{2}\ln[1 + (x-ct)^2 + (x+ct)^2 + (x^2 - c^2t^2)^2] \\
&+ \frac{1}{2c}\left[4(x+ct) + \frac{1}{2}(x+ct)^2 - 4(x-ct) - \frac{1}{2}(x-ct)^2\right] \\
&= \frac{1}{2}\ln[1 + 2x^2 + 2c^2t^2 + (x^2 - c^2t^2)^2] + \frac{1}{2c}[8ct + 2ctx] \\
&= \frac{1}{2}\ln[1 + 2x^2 + 2c^2t^2 + (x^2 - c^2t^2)^2] + (4+x)t.
\end{aligned}$$

Thus, the solution is

$$\begin{aligned} u(x,t) &= \frac{1}{2}\ln[1+2x^2+2c^2t^2+(x^2-c^2t^2)^2]+(4+x)t \\ &= \frac{1}{2}\ln[1+2x^2+6t^2+(x^2-3t^2)^2]+(4+x)t. \end{aligned}$$

Problem 9 (Ex. 7 on Page 38) If both ϕ and ψ are even functions of x, show that the solution $u(x,t)$ of the wave equation is also even in x for all t.

[Solution:] By the d'Alembert formula,

$$u(x,t) = \frac{\phi(x-ct)+\phi(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)ds.$$

Since $\phi(-s) = \phi(s)$ and $\psi(-s) = \psi(s)$, we have

$$u(-x,t) = \frac{\phi(-x-ct)+\phi(-x+ct)}{2} + \frac{1}{2c}\int_{-x-ct}^{-x+ct}\psi(s)ds$$

Make a change of variables as $s = -y$ and note that $ds = -dy$, then

$$\begin{aligned} u(-x,t) &= \frac{\phi(x+ct)+\phi(x-ct)}{2} - \frac{1}{2c}\int_{x+ct}^{x-ct}\psi(-y)dy \\ &= \frac{\phi(x+ct)+\phi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(y)dy \\ &= u(x,t). \end{aligned}$$

This means $u(x,t)$ is even in x for all t.

Problem 10 (Ex. 5 on Page 38: The hammer blow) Let $\phi(x) \equiv 0$ and $\psi(x) = 1$ for $|x| < a$ and $\psi(x) = 0$ for $|x| \geq a$. Sketch the string profile (u versus x) at each of the successive instants $t = a/2c, a/c, 3a/2c, 2a/c$, and $5a/c$.

[Solution:] By the d'Alembert formula,

$$u(x,t) = \frac{\phi(x-ct)+\phi(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)ds.$$

Given $\phi(x) \equiv 0$ and $\psi(x) = 1$ for $|x| < a$ and $\psi(x) = 0$ for $|x| \geq a$, we have

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)ds.$$

Since the initial velocity $\psi(x)$ is nonzero only in the interval $[-a,a]$, the integral must be computed differently according to how the intervals $[-a,a]$ and $[x-ct, x+ct]$ intersect.

- For a small value of t (only if $ct \leq a$), the two intervals $[-a,a]$ and $[x-ct, x+ct]$ intersect in the following 5 different ways

- If $x-ct < x+ct \le -a < a$, then

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds = 0.$$

- If $x-ct \le -a < x+ct \le a$, then

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds = \frac{1}{2c}\int_{-a}^{x+ct} 1ds = \frac{x+ct+a}{2c}.$$

- If $-a \le x-ct < x+ct \le a$, then

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds = \frac{1}{2c}\int_{x-ct}^{x+ct} 1ds = t.$$

- If $-a \le x-ct \le a < x+ct$, then

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds = \frac{1}{2c}\int_{x-ct}^{a} 1ds = \frac{-x+ct+a}{2c}.$$

- If $-a < a \le x-ct < x+ct$, then

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds = 0.$$

- For a big value of t (only if $ct > a$), the two intervals $[-a,a]$ and $[x-ct, x+ct]$ also intersect in 5 different ways
  - If $x-ct < x+ct \le -a < a$, then

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds = 0.$$

  - If $x-ct \le -a < x+ct \le a$, then

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds = \frac{1}{2c}\int_{-a}^{x+ct} 1ds = \frac{x+ct+a}{2c}.$$

  - If $x-ct \le -a < a \le x+ct$, then

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds = \frac{1}{2c}\int_{-a}^{a} 1ds = \frac{a}{c}.$$

  - If $-a \le x-ct \le a < x+ct$, then

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds = \frac{1}{2c}\int_{x-ct}^{a} 1ds = \frac{-x+ct+a}{2c}.$$

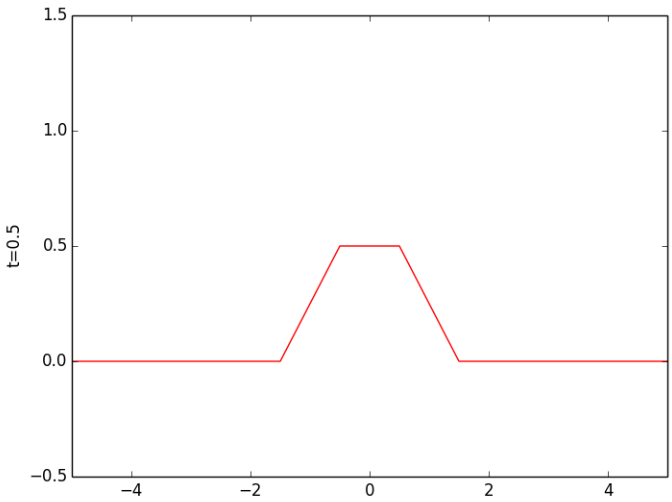
  - If $-a < a \le x-ct < x+ct$, then

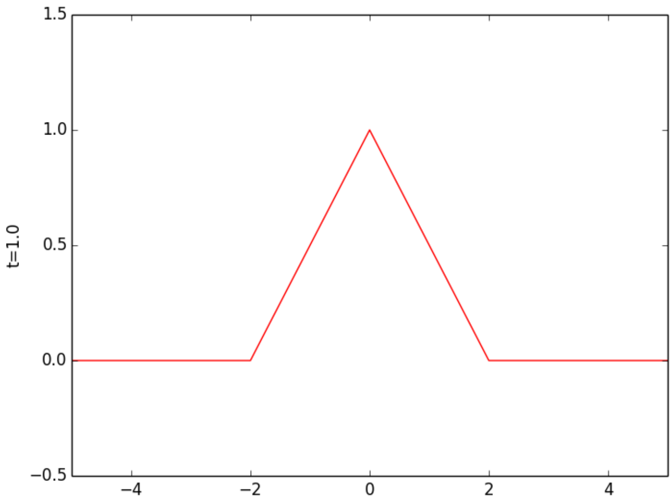
$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds = 0.$$

Problem 11 Consider the equation

$$au_{tt} + bu_{xt} + cu_{xx} = 0, \quad -\infty < x < \infty, \quad t > 0, \tag{9}$$

$$u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \quad -\infty < x < \infty, \tag{10}$$





where a, b, and c are constants such that $ac < 0$. Show that the equation is hyperbolic, and derive the solution formula.

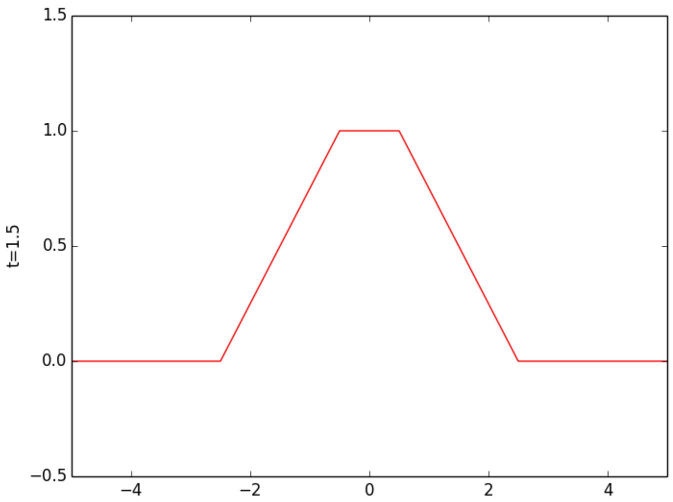
[Solution:] By the definition in Chapter 1, we see that the associated determinant is $ac - b^2/4 < 0$, so the equation is hyperbolic.

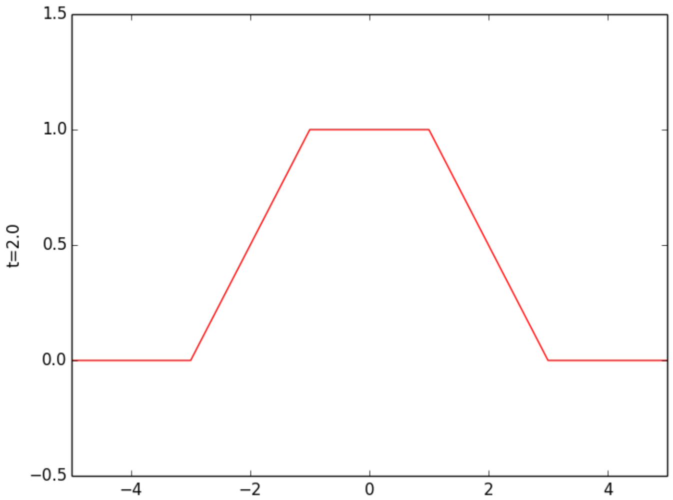
Letting $\alpha = (b + \sqrt{b^2 - 4ac})/(2a)$ and $\beta = (b - \sqrt{b^2 - 4ac})/(2a)$, we find that $\alpha + \beta = b/a$ and $\alpha\beta = c/a$. We factor the differential operator as

$$\left(a\partial_t^2 + b\partial_t\partial_x + c\partial_x^2\right)u = a(\partial_t + \alpha\partial_x)(\partial_t + \beta\partial_x)u = 0.$$

Following the idea of solving the wave equation, we introduce $v = (\partial_t + \beta\partial_x)u$, and so we have

$$a(v_t + \alpha v_x) = 0 \quad \Rightarrow \quad v(x,t) = h(x - \alpha t).$$





We now solve

$$u_t + \beta u_x = h(x - \alpha t).$$

Notice that it has a particular solution of the form $u = f(x - \alpha t)$ with $f'(s) = h(s)/(\beta - \alpha)$, and the general solution of $u_t + \beta u_x = 0$ is $g(x - \beta t)$. Therefore, the general solution of (9) is

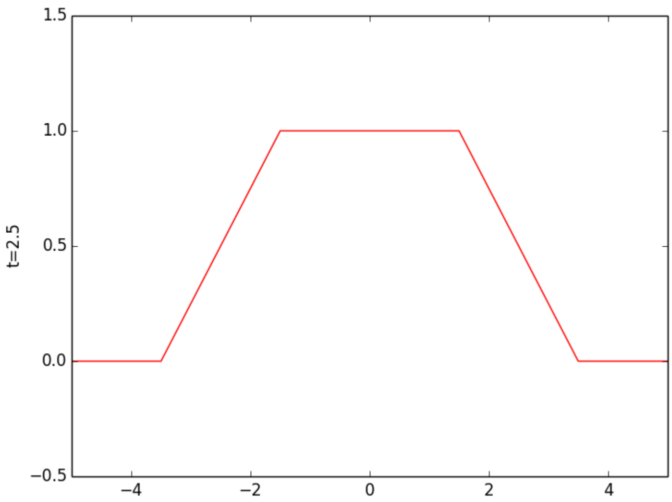
$$u(x,t) = f(x - \alpha t) + g(x - \beta t).$$

Now, we impose the initial conditions (10), and find that

$$f(x) + g(x) = \phi(x), \quad -\alpha f'(x) - \beta g'(x) = \psi(x).$$

Solving the system:

$$f'(x) + g'(x) = \phi'(x), \quad -\alpha f'(x) - \beta g'(x) = \psi(x),$$



leads to

$$f(x) = -\frac{\beta}{\alpha-\beta}\phi(x) - \frac{1}{\alpha-\beta}\int_0^x \psi(s)ds + A,$$
$$g(x) = \frac{\alpha}{\alpha-\beta}\phi(x) + \frac{1}{\alpha-\beta}\int_0^x \psi(s)ds + B,$$

where $A+B=0$.

Finally, we obtain the d'Alembert-type formula

$$u(x,t) = \frac{\alpha\phi(x-\beta t) - \beta\phi(x-\alpha t)}{\alpha-\beta} + \frac{1}{\alpha-\beta}\int_{x-\alpha t}^{x-\beta t} \psi(s)ds,$$

where $\alpha = (b+\sqrt{b^2-4ac})/(2a)$ and $\beta = (b-\sqrt{b^2-4ac})/(2a)$.