MH4110 Partial Differential Equations

Chapter 1 - Where PDEs come from

Synopsis

1. Basic concepts.
2. First-Order Linear Equations.
3. Flows, Vibrations, and Diffusions.
4. Initial and Boundary Conditions.
5. Well-Posed Problems.
6. Types of Second-Order Equations.

§1.1 Basic concepts

Some notations

$$u_x = \frac{\partial u}{\partial x} = \partial_x u, \quad u_{xy} = \frac{\partial^2 u}{\partial x \partial y} = \partial_{xy} u, \tag{1}$$

$$\nabla u = \operatorname{grad} u = u_x \mathbf{i} + u_y \mathbf{j} = (u_x, u_y); \quad \Delta u = u_{xx} + u_{yy}, \tag{2}$$

$$\operatorname{div} \mathbf{F} = \nabla \cdot \mathbf{F} = u_x + v_y + w_z, \tag{3}$$

where $\mathbf{F} = (u, v, w)$, and

$$\nabla \times \mathbf{F} = \operatorname{curl} \mathbf{F} = \begin{vmatrix} \mathrm{i} & \mathrm{j} & \mathrm{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} \tag{4}$$

$$= (w_y - v_z)\mathbf{i} + (u_z - w_x)\mathbf{j} + (v_x - u_y)\mathbf{k}.$$

Notice that

$$\operatorname{div}(\nabla u) = \Delta u. = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \tag{5}$$

Definition

A **partial differential equation**, PDE in short, is an identity that relates the independent variables x, y, $\cdots$, the dependent variable u, and the partial derivatives of u. A PDE can be expressed in the general form

$$F(\underbrace{x, y, \cdots}_{1}, \underbrace{u}_{2}, \underbrace{u_x, u_y, \cdots, u_{xx}, u_{xy}, u_{yy}, \cdots}_{3}) = 0.$$

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Example A (not in the textbook)

For a function $u(x, t)$ of one spatial variable x and the time variable t, the heat equation is

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = 0.$$

Examples of PDEs:

$$\left(\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}\right)(cu+v) = \left(c\frac{\partial}{\partial x}u + cy\frac{\partial}{\partial y}u\right) + \left(\frac{\partial}{\partial x}v + y\frac{\partial}{\partial y}v\right)$$

Linear operator

1. $u_x + u_y = 0$ (transport) 1st order PDE
2. $u_x + yu_y = 0$ (transport) $\mathcal{L} = \frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ $= c\mathcal{L}u + \mathcal{L}v$
3. $u_x + \underline{uu_y} = 0$ (shock wave) $\mathcal{L}(cu+v) = (cu+v)_x + (cu+v)(cu+v)_y \neq c\mathcal{L}u + \mathcal{L}v$
4. $u_{xx} + u_{yy} = \Delta u = 0$ (Laplace's equation)
5. $u_{tt} - u_{xx} + u^3 = 0$ (wave with interaction)
6. $u_t + uu_x + u_{xxx} = 0$ (dispersive wave)
7. $u_{tt} + u_{xxxx} = 0$ (vibrating bar) 4th order PDE
8. $u_t - \mathrm{i}u_{xx} = 0$ $(\mathrm{i} = \sqrt{-1})$ (quantum mechanics) 2nd order PDE
9. $|\nabla T(x, y, z)| = \frac{1}{c(x, y, z)}$ (Eikonal equation)

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$B((x,y,z), d)$

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The second order heat equation

$u(t,x) = 2\alpha t + x^2 + C_1$

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$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = 0$$

$LHS = \frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2}$

$= 2\alpha - \alpha \cdot 2$

$= 0$

$RHS = 0$

has a solution

$$u(t, x) = 2\alpha t + x^2.$$

$\frac{\partial u}{\partial x} = 2x$

$\frac{\partial^2 u}{\partial x^2} = 2$

x, y vectors

$A(x + \alpha y) = Ax + \alpha Ay$ A matrix

Definition

$\mathscr{L}$ is an operator. If

$$\mathscr{L}(u+v) = \mathscr{L}u + \mathscr{L}v \qquad \mathscr{L}(cu) = c\mathscr{L}u \tag{6}$$

holds for any functions u, v and any constant c, then $\mathscr{L}$ is called linear operator.

$$\Updownarrow$$

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$$Ax = b \begin{cases} b = 0 & \text{homogeneous} \\ b \neq 0 & \text{inhomogeneous} \end{cases}$$

Definition

The equation

$$\mathscr{L}u = g \tag{7}$$

is called linear if $\mathscr{L}$ is a linear operator. g is a given function of the independent variables. If $g = 0$, equation (7) is called a homogeneous linear equation, otherwise it is called an inhomogeneous linear equation.

Superposition Principle

- For the equation $\mathscr{L}u = 0$, if u and v are both solutions, so is $(u+v)$. If $u_1, u_2, \cdots, u_n$ are all solutions, so is any linear combination

$$c_1 u_1(x) + c_2 u_2(x) + \cdots + c_n u_n(x) = \sum_{j=1}^{n} c_j u_j(x) \qquad (c_j = \mathrm{constants}).$$

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- The sum of a homogeneous solution and an inhomogeneous solution is an inhomogeneous solution. $\mathcal{L}u_1 = 0$ $\quad \mathcal{L}u_2 = g$

$A x_1 = 0 \qquad A x_2 = b$

$A(x_1 + x_2) = b$

$\mathcal{L}(u_1 + u_2) = g$ ✓

Review on second order linear ODE with constant coefficients

Find the general solution of

$$y'' + ay' + by = 0 \tag{8}$$

where a and b are constants.

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Look for a solution of the form $y = e^{\lambda x}$. Plugging into eqn. (8), we find that $e^{\lambda x}$ is a solution if and only if

$$\lambda^2 + a\lambda + b = 0.$$

1. If $a^2 - 4b > 0$, the characteristic function has two distinct real roots λ_1 and λ_2. The general solution is $y = c_1 e^{\lambda_1 x} + c_2 e^{\lambda_2 x}$.
2. If $a^2 - 4b = 0$, the characteristic function has one real root λ. The general solution is $y = c_1 e^{\lambda x} + c_2 x e^{\lambda x}$.
3. If $a^2 - 4b < 0$, the characteristic function has a pair of complex conjugate roots $\lambda = \alpha \pm i\beta$. The general solution is $y = c_1 e^{\alpha x}\cos(\beta x) + c_2 e^{\alpha x}\sin(\beta x)$.

ODE - like equation

Example 1

Find all $u(x, y)$ satisfying the equation $u_{xx} = 0$.

If $u = u(x)$

then $u_{xx} = 0 \Rightarrow u_x = C_1 \Rightarrow u(x) = C_1 x + C_2$

$u = u(x, y)$

a constant in terms of x

$$u_{xx} = 0 \Longleftrightarrow \frac{\partial}{\partial x}(u_x) = 0 \Longrightarrow u_x = g(y)$$

$$u(x,y) = x\,g(y) + f(y)$$

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- $u_{xx} = 0$ indicates that u_x is independent of x. But u_x is dependent on y. So $u_x = f(y)$. $f(y)$ is arbitrary.
- Integrate again to get $u(x, y) = xf(y) + g(y)$, where $f(y)$ and $g(y)$ are arbitrary.

Example 2

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$\lambda^2 + 1 = 0 \quad \lambda_{1,2} = \pm i$

- If $u = u(x)$, we know $u(x) = c_1 \cos x + c_2 \sin x$, where c_1 and c_2 are arbitrary constants.

in terms of x

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- u is a function of x and y. The arbitrary constants c_1 and c_2 above should be replaced by the arbitrary functions $f(y)$ and $g(y)$.
- The solution is $u(x, y) = f(y) \cos x + g(y) \sin x$.

Example 3

Solve the PDE $u_{xy} = 0$. Here $u = u(x, y)$.

$$u_{xy} = 0 \iff \frac{\partial}{\partial y}(u_x) = 0$$

$$\Downarrow$$

$$u_x = f(x)$$

$$\hookrightarrow u(x,y) = \int_a^x f(s)\,ds + g(y)$$

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- Integrate in y, regarding x as fixed, we get

$$u(x, y) = F(y) + G(x)$$

where $F' = f$. Both $f(y)$ and $G(x)$ are arbitrary functions.

Moral

- For an ODE of order m, we get m arbitrary constants in the solution.
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A few things to keep in mind

- Mixed derivatives are equal throughout this course: $u_{xy} = u_{yx}$.
- The chain rule is used frequently in PDEs; for instance,

$$\frac{\partial}{\partial x}\left[f(g(x,t))\right] = f'(g(x,t)) \cdot \frac{\partial g}{\partial x}(x,t).$$

- Derivatives of integrals like $I(t) = \int_{a(t)}^{b(t)} f(x,t)dx$

$$\frac{dI}{dt} = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}(x,t)dx + f(b(t),t)b'(t) - f(a(t),t)a'(t).$$

Green's Theorem

Let D be a bounded plane domain with a piecewise C^1 boundary curve C $=$ bdy D. Consider C to be parametrized so that it is traversed once with D on the left. Let $p(x, y)$ and $q(x, y)$ be any C^1 functions defined on $\bar{D} = D \cup C$. Then

$$\iint_D (q_x - p_y)dxdy = \int_C pdx + qdy. \tag{9}$$

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Divergence Theorem

Let D be a bounded spatial domain with a piecewise C^1 boundary surface S. Let $\mathbf{n}$ be the unit outward normal vector on S. Let $\mathbf{f}(\mathbf{x})$ be any C^1 vector field on $\bar{D} = D \cup S$.Then

$$\iiint_D \nabla \cdot \mathbf{f} d\mathbf{x} = \iint_S \mathbf{f} \cdot \mathbf{n} dS. \tag{10}$$

Summary

A **differential equation**, DE in short, is an equation involving an unknown function and its derivatives. For example,

$$\frac{du}{dx} = 5x + 3 \quad (\text{or} \quad u' = 5x + 3), \tag{11}$$

and

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = \cos(xy) \quad (\text{or} \quad \Delta u = \cos(xy)). \tag{12}$$

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- If the unknown function is a function of a single variable (e.g., $u(x), u(t), \cdots$) and the involved derivatives are ordinary derivatives, then this DE is an **ordinary differential equation (ODE)**, e.g., the equation (11).

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- If the unknown function is a function of a single variable (e.g., $u(x), u(t), \cdots$) and the involved derivatives are ordinary derivatives, then this DE is an **ordinary differential equation (ODE)**, e.g., the equation (11).
- A **partial differential equation (PDE)** is one involving a function of two or more variables, in which the derivatives are partial derivatives, for example, the equation (12).