NANYANG TECHNOLOGICAL UNIVERSITY

SPMS/DIVISION OF MATHEMATICAL SCIENCES

2025/26 Semester 2 MH4110 Partial Differential Equations Tutorial 4, 12 February

Problem 1 Consider the wave equation:

$$u_{tt} - 25u_{xx} = 0$$

on the whole plane. Find the domain of dependence of $u(x,t)$ at $(x,t) = (1,5)$, and find the domain of influence of the interval $[1,5]$.

Problem 2 Let $\phi(x) = e^{-x^2}$ and $\psi(x) = 0$. The wave speed is $c = 1$. Sketch the string profile (u versus x) at each of the successive instants $t = 0, 2, 4, 6, 8$, and 10.

Problem 3 (Ex. 10 on Page 38) Solve $u_{xx} + u_{xt} - 20u_{tt} = 0$, $u(x,0) = \phi(x)$, $u_t(x,0) = \psi(x)$.

Problem 4 (Ex. 11 on Page 38) Find the general solution of $3u_{tt} + 10u_{xt} + 3u_{xx} = \sin(x+t)$.

Problem 5 (i) Show that

$$v(x,t) = \frac{1}{2c}\int_0^t \Big(\int_{x-c(t-\tau)}^{x+c(t-\tau)} f(\xi,\tau)\,d\xi\Big)d\tau. \tag{1}$$

is a solution of the Cauchy problem:

$$\begin{aligned} &v_{tt} - c^2 v_{xx} = f(x,t), \quad x \in (-\infty,\infty),\ t > 0, \\ &v(x,0) = 0, \quad v_t(x,0) = 0, \quad x \in (-\infty,\infty). \end{aligned} \tag{2}$$

Hint: use the derivative formula:

$$\frac{d}{dt}\int_{a(t)}^{b(t)} G(\xi,t)d\xi = G(b(t),t)b'(t) - G(a(t),t)a'(t) + \int_{a(t)}^{b(t)} \frac{\partial}{\partial t}G(\xi,t)\,d\xi. \tag{3}$$

(ii) Consider the general nonhomogeneous equation:

$$\begin{aligned} &u_{tt} - c^2 u_{xx} = f(x,t), \quad x \in (-\infty,\infty),\ t > 0, \\ &u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \quad x \in (-\infty,\infty). \end{aligned} \tag{4}$$

Show that the solution u of (4) can be decomposed as $u = v + w$, where v is the solution of (2), i.e., given by (1), and w is the solution of

$$\begin{aligned} &w_{tt} - c^2 w_{xx} = 0, \quad x \in (-\infty,\infty),\ t > 0, \\ &w(x,0) = \phi(x), \quad w_t(x,0) = \psi(x), \quad x \in (-\infty,\infty). \end{aligned} \tag{5}$$

(iii) Derive the solution formula for (4).

Problem 6 Solve

$$\begin{aligned} &u_{tt} - 4u_{xx} = \sin(x+t), \quad x \in (-\infty, \infty),\ t > 0, \\ &u(x,0) = x^2, \quad u_t(x,0) = e^x, \quad x \in (-\infty, \infty). \end{aligned} \tag{6}$$

Problem 7 Solve the problem:

$$\begin{aligned} &u_{tt} - u_{xx} = e^{-t}, \quad x \in (-\infty, \infty), \ \ t > 0, \\ &u(x,0) = e^{-x^2} + \cos x, \quad x \in (-\infty, \infty), \\ &u_t(x,0) = 0, \quad x \in (-\infty, \infty). \end{aligned}$$

Problem 8 Use the characteristic coordinate method to solve the equation

$$u_x + y u_y + 2xu = 1.$$

Problem 9 (Ex. 6 on Page 41) For the damped string:

$$u_{tt} - c^2 u_{xx} + r u_t = 0,$$

where $r > 0$, show that the energy decreases.

Problem 10 Show that the maximum principle is not true for the equation: $u_t = x u_{xx}$, which has a variable coefficient. Verify that $u = -2xt - x^2$ is a solution. Find the location of its maximum in the closed rectangle:

$$D = \big\{(x,t) :\ -2 \le x \le 2,\ 0 \le t \le 1\big\}.$$

Problem 11 $S(x,t)$ is a solution of the diffusion equation

$$u_t - k u_{xx} = 0. \tag{7}$$

Show that

$$v(x,t) = \int_{-\infty}^{\infty} S(x-y,t) g(y) dy = S(\cdot, t) * g \tag{8}$$

is also a solution of (7) for any function g.