§1.3 Flows, Vibrations, and Diffusions

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- Heat equation
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Derive the following PDEs from physical principles

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- Wave equation
- Heat equation
- Laplace equation

Remark: Most often in physical problems, the independent variables are those of space x, y, z, and time t.

Example 1. Simple Transport

$$\int_0^b u(t,x)\,dx = \int_{ch}^{b+ch} u(t+h,x)\,dx$$

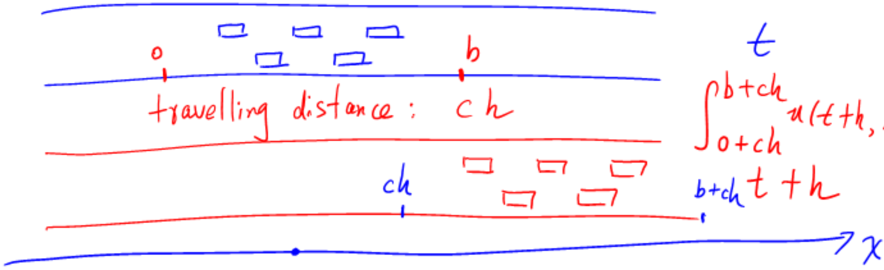
- On a very long and straight freeway, all the cars are running at the same speed c.

$$u(t,b) = u(t+h, b+ch)$$

- For simplicity, we assume that no cars enter or exit the freeway for quite a while.
- Let $u(x,t)$ be the car density at time t and position x. For instance $u(x,t)$ can be the number of cars at the time t in a distance of 1 km centered at x.

c m/s $\longrightarrow$

$$\int_0^b u(t,x)\,dx$$



$$\int_{0+ch}^{b+ch} u(t+h,x)\,dx$$

Example 1. Simple Transport (Cont'd)

- The number of cars in the interval $[0, b]$ at the time t is $M = \int_0^b u(x, t) dx$.

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- Differentiating with respect to b, we get

$$u(b,t) = u(b+ch, t+h). \tag{25}$$

$$LHS = \lim_{h\to 0} \frac{u(x+ch, t+h) - u(x+ch, t)}{h} + \lim_{h\to 0} \frac{u(x+ch, t) - u(x,t)}{h}$$

- The number of cars in the interval $[0, b]$ at the time t is $M = \int_0^b u(x,t)dx$.

$$= \frac{\partial u(x,t)}{\partial t} + \lim_{h\to 0} c\, \frac{u(x+ch,t) - u(x,t)}{ch}$$

- At the later time $t+h$, the same car has moved to the right by $c \cdot h$ km.

$$= \frac{\partial u(x,t)}{\partial t} + c\,\frac{\partial u(x,t)}{\partial x}$$

- Hence

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- Differentiating with respect to b, we get

A car at position b at time t will be at a new position $b+ch$ at a later time $t+h$

$$u(b,t) = u(b+ch, t+h). \tag{25}$$

- Change b to x, we have $u(x+ch, t+h) - u(x,t) = 0$.

$$\lim_{h\to 0} \frac{u(x+ch, t+h) - u(x+ch, t) + u(x+ch,t) - u(x,t)}{h} = 0$$

Example 1. Simple Transport (Cont'd)

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$$u(b,t) = u(b+ch, t+h). \tag{25}$$

- Change b to x, we have $u(x+ch, t+h) - u(x,t) = 0$.
- Dividing both sides by h and taking the limit as $h \to 0$, we get

$\#/sec \cdot m$

$$u_t(x,t) + cu_x(x,t) = 0. \tag{26}$$

$m/sec \cdot \#/m^2$

Example 1. Simple Transport (Cont'd)

Transport equation $u_t(x,t) - c\,u_x(x,t) = 0$

$$u_t(x,t) + cu_x(x,t) = 0. \tag{27}$$

The characteristic lines are

$$\frac{dx}{dt} = \frac{c}{1}$$

$$ct - x = W, \tag{28}$$

where W are arbitrary constants. The general solution is $f(ct - x)$. f is arbitrary. The traffic moves in the negative x direction

$$u(x,t) = u(x - ch, t + h)$$

$$\lim_{h \to 0} \frac{u(x,t) - u(x-ch,t) + u(x-ch,t) - u(x-ch,t+h)}{h}$$

$$= c\,u_x(x,t) - u_t(x,t)$$

Example 2. Vibrating string: Wave-motion

Consider a flexible, elastic homogeneous string or thread of length l, which undergoes relatively small transverse vibrations. For instance, it could be a guitar string or a plucked violin string.

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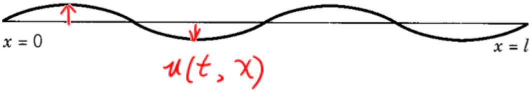
YouTube Example: Guitar Strings Oscillating in HD 60 fps https://www.youtube.com/watch?v=8YGQmV3NxMI.

Example 2. Vibrating string: Wave-motion (Cont'd)

ρdx

A plucked string. ρ is the linear density (units of mass per unit of length) and is constant along the entire length of the string. $u(t, x)$ is the displacement from equilibrium position at time t and position x.



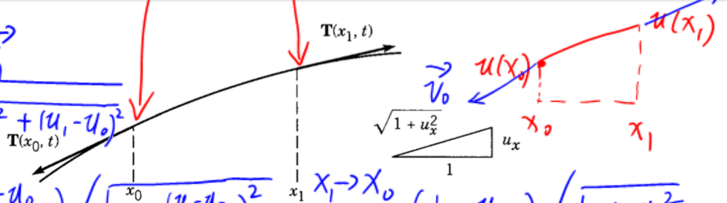
Example 2. Vibrating string: Wave-motion (Cont'd)

A plucked string. ρ is the linear density (units of mass per unit of length) and is constant along the entire length of the string. $u(t,x)$ is the displacement from equilibrium position at time t and position x.

$\vec{v}_1 = \left(x_1 - x_0,\ u(x_1) - u(x_0)\right)$

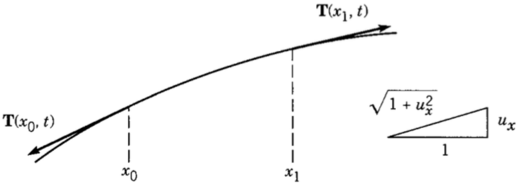
Ignore all the forces on the string except for its tension $\mathbf{T}(x,t)$. Consider the motion of a tiny portion of the string sitting atop the interval $[x_0, x_1]$.

$$\frac{\vec{v}_1}{|\vec{v}_1|} = \frac{\vec{v}_1}{\sqrt{(x_1-x_0)^2 + (u_1-u_0)^2}}$$



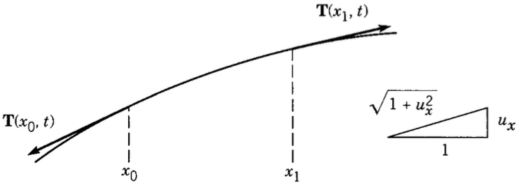
$$= \left(1,\ \frac{u_1-u_0}{x_1-x_0}\right) \Big/ \sqrt{1 + \left(\frac{u_1-u_0}{x_1-x_0}\right)^2} \ \xrightarrow{x_1 \to x_0}\ (1,\ u_x) \Big/ \sqrt{1+u_x^2}$$

Example 2. Vibrating string: Wave-motion (Cont'd)



Because the string is perfectly flexible, the tension $\mathbf{T}(x, t)$ is directed tangentially along the string.

Example 2. Vibrating string: Wave-motion (Cont'd)

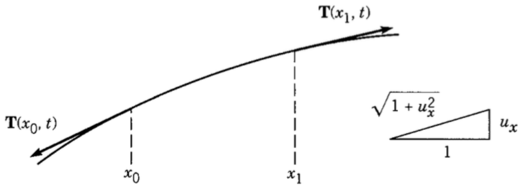


Because the string is perfectly flexible, the tension $\mathbf{T}(x,t)$ is directed tangentially along the string. Given the slope of the string is $u_x(x,t)$, the directions of the tension at the two ends x_0 and x_1 are

$$\mathbf{v}_0 = \left(-\frac{1}{\sqrt{1+u_x^2(x_0,t)}}, -\frac{u_x(x_0,t)}{\sqrt{1+u_x^2(x_0,t)}} \right), \tag{29}$$

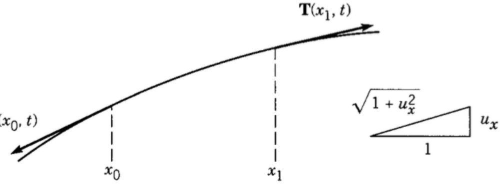
$$\mathbf{v}_1 = \left(\frac{1}{\sqrt{1+u_x^2(x_1,t)}}, \frac{u_x(x_1,t)}{\sqrt{1+u_x^2(x_1,t)}} \right). \tag{30}$$

Example 2. Vibrating string: Wave-motion (Cont'd)



Let $T(x, t)$ be the magnitude of the tension $\mathbf{T}(x, t)$.

Example 2. Vibrating string: Wave-motion (Cont'd)



F = ma

Let $T(x,t)$ be the magnitude of the tension $\mathbf{T}(x,t)$. In the longitudinal direction x, Newton's second law is

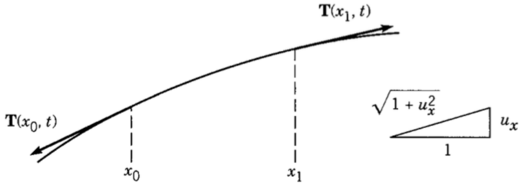
Net force in the x direction

$$a_x \int_{x_0}^{x_1} \underbrace{\rho dx}_{\text{mass}} = T(x_1,t)\frac{1}{\sqrt{1+u_x^2(x_1,t)}} - T(x_0,t)\frac{1}{\sqrt{1+u_x^2(x_0,t)}}. \tag{31}$$

Since we have assumed/observed that the motion is purely transverse, there is no longitudinal motion and hence the longitudinal acceleration $a_x = 0$. So

$$T(x_1,t)\frac{1}{\sqrt{1+u_x^2(x_1,t)}} = T(x_0,t)\frac{1}{\sqrt{1+u_x^2(x_0,t)}}. \tag{32}$$

Example 2. Vibrating string: Wave-motion (Cont'd)



In the transverse direction u, Newton's second law is

Net force

$$\int_{x_0}^{x_1} a_u(x,t)\underbrace{\rho dx}_{m} = T(x_1,t)\frac{u_x(x_1,t)}{\sqrt{1+u_x^2(x_1,t)}} - T(x_0,t)\frac{u_x(x_0,t)}{\sqrt{1+u_x^2(x_0,t)}}. \tag{33}$$

Here a_u is the transverse acceleration u_{tt}. Using the final relationship on the last slide, we have

$$\int_{x_0}^{x_1} u_{tt}(x,t)\rho dx = T(x_0,t)\frac{u_x(x_1,t)}{\sqrt{1+u_x^2(x_0,t)}} - T(x_0,t)\frac{u_x(x_0,t)}{\sqrt{1+u_x^2(x_0,t)}}. \tag{34}$$

Divide both sides by $x_1 - x_0$, we have

$$\frac{\int_{x_0}^{x_1} u_{tt}(x,t)\rho dx}{x_1 - x_0} = \frac{T(x_0,t)}{\sqrt{1+u_x^2(x_0,t)}} \frac{u_x(x_1,t) - u_x(x_0,t)}{x_1 - x_0}. \quad (35)$$

Passing to the limit $x_1 \to x_0$ gives

$$\rho u_{tt}(x_0,t) = \frac{T(x_0,t)}{\sqrt{1+u_x^2(x_0,t)}} u_{xx}(x_0,t). \quad (36)$$

$$\lim_{x_1 \to x_0} \frac{\int_{x_0}^{x_1} u_{tt}(x,t)\rho\, dx}{x_1 - x_0} = \lim_{x_1 \to x_0} \frac{\int_{x_0}^{x_1} u_{tt}(x_0,t)\rho\, dx}{x_1 - x_0}$$

$$= \rho u_{tt}(x_0,t)$$

Example 2. Vibrating string: Wave-motion (Cont'd)

Divide both sides by $x_1 - x_0$, we have

$$\frac{\int_{x_0}^{x_1} u_{tt}(x,t)\rho dx}{x_1 - x_0} = \frac{T(x_0,t)}{\sqrt{1+u_x^2(x_0,t)}} \frac{u_x(x_1,t) - u_x(x_0,t)}{x_1 - x_0}. \tag{35}$$

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Approximation: Assume that the motion is small, i.e., $|u_x| \approx 0$. Then,

$|u(x,t)| << 1$

$$|u_x| \approx 0 \quad \Rightarrow \quad \sqrt{1+u_x^2} = 1 + \frac{1}{2}u_x^2 + \cdots \approx 1.$$

Therefore, the equation in traverse direction becomes

$$\rho u_{tt}(x,t) = T(x,t)u_{xx}(x,t). \tag{37}$$

Example 2. Vibrating string: Wave-motion (Cont'd)

The first of the three fundamental PDEs of this course:

Wave equation

Assume that T is a constant, we obtain the wave equation:

$$u_{tt} = c^2 u_{xx} \quad \text{with} \quad c = \sqrt{\frac{T}{\rho}}, \tag{38}$$

where c is known as the **wave speed.**

Example 2. Vibrating string: Wave-motion (Cont'd)

Some variants

(i) If significant air resistance r is present, we have an extra term proportional to the speed u_t :

$$u_{tt} - c^2 u_{xx} + r u_t = 0, \quad r > 0. \tag{39}$$

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(iv) In the multiple dimensional case, u_{xx} is replaced by $\Delta u = u_{xx} + u_{yy}$ or $\Delta u = u_{xx} + u_{yy} + u_{zz}$. See Example 3 on Page 13 of the textbook for the derivation of the multidimensional models.

Example 5. Heat flow / Diffusion

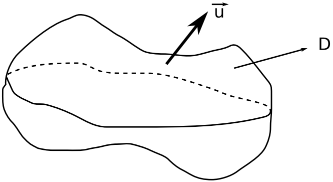
Let $D \in \mathbb{R}^3$ be a domain in space, and let $H(t)$ be the amount of heat (or energy) in D. Let $u(x, y, z, t)$ be the temperature at any point (x, y, z) at time t. Then

$$H(t) = \iiint_D c\rho u(x, y, z, t)\, dxdydz, \tag{42}$$

$\rho\, dxdydz$

mass

where ρ is the density of the material, and c is a "specific heat".



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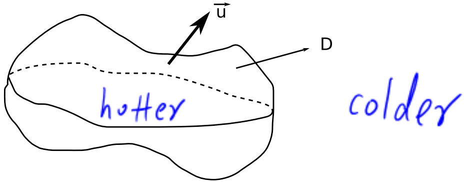

The change in the heat is

$$\frac{dH}{dt} = \iiint_D c\rho u_t(x, y, z, t)\, dxdydz. \tag{43}$$

Example 5. Heat flow / Diffusion (Cont'd)

Assume that outside the body the "space" is colder, so the energy "flux" (direction of heat flow) will go from inside to outside and will "diffuse" through the boundary ∂D.

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Heat flows from hot to cold regions proportionally to the temperature gradient.

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Therefore, we have

$$\frac{dH}{dt} = \iint_{\partial D} \kappa(\mathbf{n} \cdot \nabla u)\, dS, \tag{44}$$

where κ is a proportionality factor (the "heat conductivity").

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Divergence Theorem: Let D be a bounded spatial domain with a piecewise C^1 boundary surface S. Let $\mathbf{n}$ be the unit outward normal vector on S. Let $\mathbf{F}$ be any C^1 vector field on $\bar{D} = D \cup S$. Then

$$\iiint_D \nabla \cdot \mathbf{F} dxdydz = \iint_S \mathbf{F} \cdot \mathbf{n}\, dS. \tag{45}$$

Example 5. Heat flow / Diffusion (Cont'd)

Then by the divergence theorem (see (45)), we have

$$\frac{dH}{dt} = \iint_{\partial D} \kappa(\mathbf{n} \cdot \nabla u)\, dS = \iiint_D \nabla \cdot (\kappa \nabla u) dxdydz. \tag{46}$$

Fourier's law

Repeat equation (43) here

$$\frac{dH}{dt} = \iiint_D c\rho u_t(x, y, z, t)\, dxdydz.$$

Since D is an arbitrary domain, from the above two equations we get **the second of the three fundamental PDEs of this course**:

Heat equation

$$c\rho \frac{\partial u}{\partial t} = \nabla \cdot (\kappa \nabla u). \tag{47}$$

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If c, ρ and κ are constants, we obtain the equation:

$$\frac{\partial u}{\partial t} = k\Delta u, \quad k = \frac{\kappa}{c\rho}. \tag{48}$$

Example 6. Laplace equation

- A fireplace (with sufficient fuel supply) is burning to keep the room warm.

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- A fireplace (with sufficient fuel supply) is burning to keep the room warm.
- The heat is not expected to be evenly distributed throughout the room. The closer to the fireplace, the warmer you feel. $\partial u / \partial t = 0$
- The temperature of the room eventually reaches a steady state. At any position $\mathbf{x} = (x, y, z)$, the temperature remains the same. This steady state can be modeled by **Laplace equation**.

Example 6. Laplace equation (Cont'd)

Laplace equation is the third one of the three fundamental PDEs of this course. It can be derived from the heat equation (48) by assuming that the heat flow does not change with time, i.e., $u_t = 0$, so u does not depend on time any more in this situation.

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Laplace equation

$$\Delta u = u_{xx} + u_{yy} + u_{zz} = 0. \tag{49}$$

Its solutions are called harmonic solutions.

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The one-dimensional Laplace equation is $u_{xx} = 0$. Its general solution is a linear function: $u(x) = c_1 x + c_2$, for any constants c_1, c_2. The multi-dimensional Laplace equation is much more interesting and far more difficult to solve.

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Poisson equation

The inhomogeneous version of (49):

$$-\nabla \cdot \left(\delta(\vec{x}) \nabla u \right) = q\delta(\vec{x} - \vec{x}_s)$$

$$\Delta u = f, \tag{50}$$

where $f \neq 0$, is called the Poisson equation.

MH4110 PDE

Tutorial 01

Question 1

Solve the equations

(a)

$$2\frac{dy}{dx} + (\tan x)y = \frac{(4x+5)^2}{\cos x}y^3$$

(b)

$$\frac{dy}{dx} + \frac{2}{x}y = (-x^2\cos x)y^2$$

Question 1

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[Solution:]

(a) We write the ODE in a standard form

$$\frac{dy}{dx} + \frac{\tan x}{2}y = \frac{(4x+5)^2}{2\cos x}y^3.$$

Dividing both sides by y^3 yields the equation

$$\frac{1}{y^3}\frac{dy}{dx} + \frac{\tan x}{2}\frac{1}{y^2} = \frac{(4x+5)^2}{2\cos x}.$$

We make the change of variable:

$$u = y^{-2} \Longrightarrow \frac{du}{dx} = -2y^{-3}\frac{dy}{dx} \Longrightarrow y^{-3}\frac{dy}{dx} = -\frac{1}{2}\frac{du}{dx},$$

Rewrite the new equation as the standard form:

$$\frac{du}{dx} - (\tan x) \cdot u = -\frac{(4x+5)^2}{\cos x}.$$

The integrating factor function can be chosen as

$$e^{\int -\tan x dx} = \cos x.$$

Thus,

$$\frac{d}{dx}(u \cos x) = -(4x+5)^2,$$

which has a general solution

$$u \cos x = -\frac{(4x+5)^3}{12} + C$$

or

$$y^{-2} = -\frac{(4x+5)^3}{12 \cos x} + \frac{C}{\cos x}.$$

(b) For a Bernoulli's equation of order n in the standard form

$$\frac{dy}{dx} + p(x)y = q(x)y^n, \quad n \neq 1,$$

setting $u = y^{1-n}$, we transform it into a linear equation:

$$\frac{du}{dx} + \underbrace{(1-n)p(x)}_{P(x)}u(x) = \underbrace{(1-n)q(x)}_{Q(x)},$$

and the general solution is

$$y^{1-n} = e^{-(1-n)\int p(x)dx}\left[(1-n)\int q(x)e^{(1-n)\int p(x)dx}dx + C\right].$$

The ODE is in the standard form, where $n = 2$, and

$$p(x) = \frac{2}{x}, \qquad q(x) = -x^2 \cos x.$$

Using the solution formula, we obtain

$$\begin{aligned} y^{1-2} &= e^{-(1-2)\int \frac{2}{x}dx}\left[(1-2)\int \left(-x^2\cos x\right) e^{(1-2)\int \frac{2}{x}dx}dx + C\right] \\ &= e^{\int \frac{2}{x}dx}\left[\int \left(x^2\cos x\right) e^{-\int \frac{2}{x}dx}dx + C\right] \\ &= x^2\left[\int \cos x dx + C\right] = x^2 \sin x + Cx^2. \end{aligned}$$

Thus, the solution is $y = \frac{1}{x^2(\sin x + C)}$, where C is an arbitrary constant.

Question 2

Which of the following operators are linear?

(a) $\mathscr{L}u = u_x + xu_y$ ✓

(b) $\mathscr{L}u = u_x + uu_y$ ✗

(c) $\mathscr{L}u = u_x + u_y^2$ ✗

(d) $\mathscr{L}u = u_x + u_y + 1$ ✗

(e) $\mathscr{L}u = \sqrt{1+x^2}(\cos y)u_x + u_{yxy} - [\arctan(x/y)]u$ ✓

$\mathcal{L}(0) = 0$

$$\mathcal{L}(\alpha u + v) = \alpha \mathcal{L}u + \mathcal{L}v$$

$$(\alpha u + v)_x + (\alpha u + v)(\alpha u + v)_y$$

$$= \alpha u_x + v_x + \alpha^2 u u_y + \alpha u v_y + \alpha u_y v + v v_y$$

Question 3

Prove that the first-order equation is linear.

$$a(x,y)u_x(x,y) + b(x,y)u_y(x,y) + c(x,y)u(x,y) = f(x,y)$$

$$\mathcal{L} = a(x,y)\frac{\partial}{\partial x} + b(x,y)\frac{\partial}{\partial y} + c(x,y)$$

$$\begin{aligned}\mathcal{L}(\alpha u + v) &= a\frac{\partial}{\partial x}(\alpha u + v) + b\frac{\partial}{\partial y}(\alpha u + v) + c(\alpha u + v)\\ &= \underline{\alpha a u_x} + a v_x + \underline{\alpha b u_y} + b v_y + \underline{\alpha c u} + cv\\ &= \underline{\alpha \mathcal{L} u} + \mathcal{L} v\end{aligned}$$

Question 4

For each of the following equations, state the order and whether it is nonlinear, linear inhomogeneous, or linear homogeneous; provide reasons.

(a) $u_t - u_{xx} + 1 = 0$ 2 linear inhomogeneous

(b) $u_t - u_{xx} + xu = 0$ 2 linear homogeneous

(c) $u_t - u_{xxt} + uu_x = 0$ 3 nonlinear

(d) $u_{tt} - u_{xx} + x^2 = 0$ 2 linear inhomogeneous

(e) $iu_t - u_{xx} + u/x = 0$ 2 linear homogeneous

(f) $u_x \left(1 + u_x^2\right)^{-1/2} + u_y \left(1 + u_y^2\right)^{-1/2} = 0$ 1 nonlinear

(g) $u_x + e^y u_y = 0$ 1 linear homogeneous

(h) $u_t + u_{xxxx} + \sqrt{1 + u} = 0$ 4 nonlinear

Question 5

Show that the difference of two solutions of an inhomogeneous linear equation $\mathscr{L}u = g$ with the same g is a solution of the homogeneous equation $\mathscr{L}u = 0$.

$$\left.\begin{array}{l} \mathscr{L}u_1 = g \\ \mathscr{L}u_2 = g \end{array}\right\} \Rightarrow \mathscr{L}(u_1 - u_2)$$

$$= \mathscr{L}u_1 + (-1)\mathscr{L}u_2$$

$$= g - g$$

$$= 0$$

$$u_1 \neq u_2$$

Question 6

Verify by direct substitution that

$$u_n(x, y) = \sin(nx)\sinh(ny)$$

is a solution of $u_{xx} + u_{yy} = 0$ for every $n > 0$.

Question 6

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[Solution:] It is clear that

$$\partial_{xx} u_n = -n^2 u_n, \quad \partial_{yy} u_n = n^2 u,$$

where we recall that

$$\mathrm{sinh}s = \frac{e^s - e^{-s}}{2}, \;\; \mathrm{cosh}s = \frac{e^s + e^{-s}}{2}, \;\; \frac{d}{ds}\mathrm{sinh}s = \mathrm{cosh}s, \;\; \frac{d}{ds}\mathrm{cosh}s = \mathrm{sinh}s$$

Thus, the given u_n is a solution of the Laplace equation.