IVP $\begin{cases} u_{tt} - c^2 u_{xx} = 0 \\ u(x,t_0) = \phi(x) \\ u_t(x,t_0) = \psi(x) \end{cases}$

Causality (Initial values travel with speeds bounded by c)

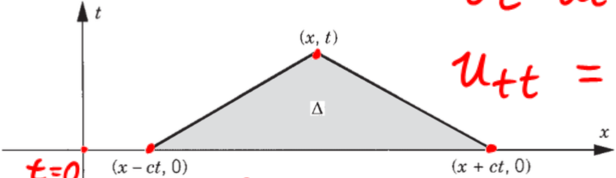
The value of the solution to the IVP (17) at a point (x,t) can be found from the d'Alembert formula,

$$u(x,t) = \frac{\phi(x+ct) + \phi(x-ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds. \qquad (32)$$

Let $\tau = t - t_0$

$u_t = \frac{\partial u}{\partial \tau} \cdot \frac{d\tau}{dt} = \frac{\partial u}{\partial \tau}$

$u_{tt} = \frac{\partial^2 u}{\partial \tau^2}$



$t=0$

IVP $\Rightarrow$ $u_{\tau\tau} - c^2 u_{xx} = 0$

$u(x,t_0) = u(x,\tau=0)$

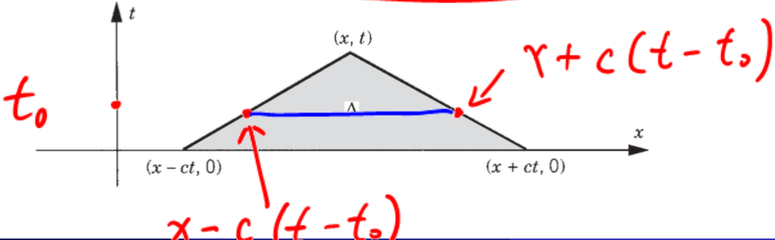
$\frac{\partial u}{\partial \tau}(x,\tau=0) = \psi$

$$u(x,t)=\frac{1}{2}\left[\phi(x+c(t-t_0))+\phi(x-c(t-t_0))\right]$$

Causality (Initial values travel with speeds bounded by c)

- $u(x,t)$ depends on the values of ϕ at only two points on the x axis, $(x-ct,0)$ and $(x+ct,0)$, and the values of ψ on the interval $[x-ct, x+ct]$. Thus, the interval $[x-ct,x+ct]$ is called **interval of dependence** for the point (x,t).

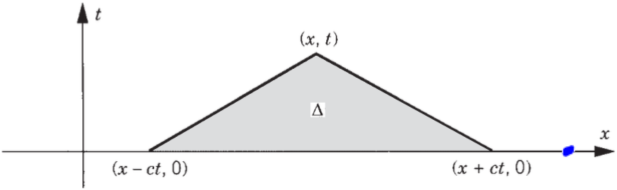
$$+\frac{1}{2c}\int_{x-c(t-t_0)}^{x+c(t-t_0)}\psi(s)\,ds$$



§2.2 Causality and energy

Causality (Initial values travel with speeds bounded by c)

- $u(x,t)$ depends on the values of ϕ at only two points on the x axis, $(x-ct,0)$ and $(x+ct,0)$, and the values of ψ on the interval $[x-ct,x+ct]$. Thus, the interval $[x-ct,x+ct]$ is called **interval of dependence** for the point (x,t).
- The triangular region with vertices at $x-ct$ and $x+ct$ on the x axis and the vertex (x,t) is called the **domain of dependence, or the past history of the point** (x,t).

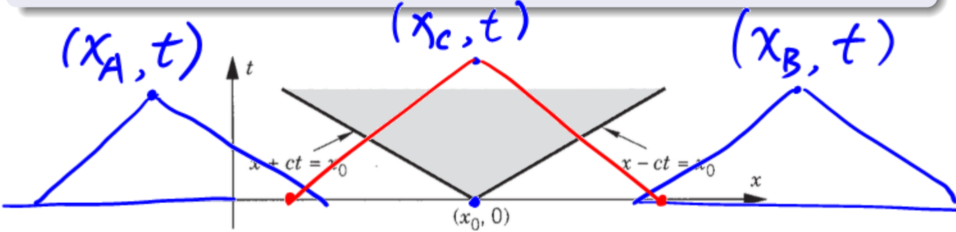


Causality (Initial values travel with speeds bounded by c)

- What are the points on the half-plane $t > 0$ that are influenced by the initial data at a fixed point $(x_0, 0)$? The set of all such points is called the **region of influence** of the point $(x_0, 0)$. It follows that the point $(x_0, 0)$ influences the value of the solution u at a point (x, t) if and only if

$$x - ct \leq x_0 \leq x + ct. \tag{33}$$

These are the points inside the forward characteristic cone that is defined by the point $(x_0, 0)$ and the edges $x - ct = x_0, x + ct = x_0$.



Example E (not from the textbook)

Analyze the solution to the IVP (17) with the following initial data

$$u(x,0) = \phi(x) \equiv 0, \quad u_t(x,0) = \psi(x) = \begin{cases} h, & |x| \le a, \\ 0, & |x| > a, \end{cases}. \tag{34}$$

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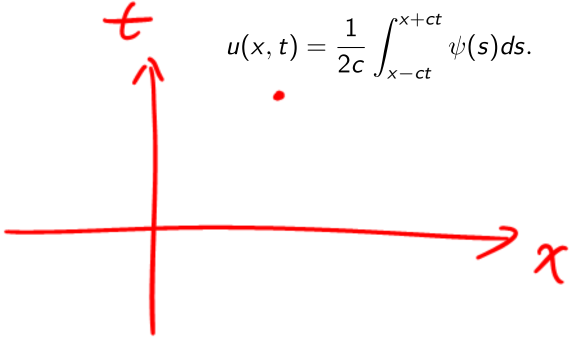
$|x| \le a$

$\Longleftrightarrow -a \le x \le a$

By the d'Alembert formula (32), we obtain the following solution

$$u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds.$$



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By the d'Alembert formula (32), we obtain the following solution

$$u(x,t) = \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s)ds.$$

Since the initial velocity $\psi(x)$ is nonzero only in the interval $[-a,a]$, the integral must be computed differently according to how the intervals $[-a,a]$ and $[x-ct, x+ct]$ intersect.

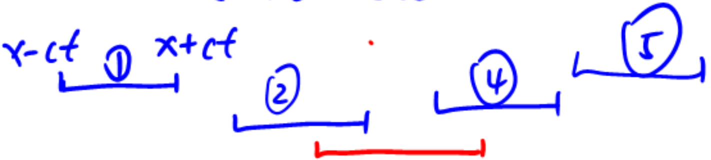
Example E (not from the textbook) (Cont'd)

① $x+ct<-a$ $\qquad \psi(s)=0$

② $x-ct\le -a\le x+ct\le a$

④ $-a<x-ct<a\le x+ct$

⑤ $a<x-ct$



$-a \qquad a$

③ $\begin{cases} \text{I}: & -a\le x-ct<x+ct\le a \qquad t\le \frac{a}{c} \\ \text{II}: & x-ct<-a<a<x+ct \qquad t>\frac{a}{c} \end{cases}$

Example E (not from the textbook) (Cont'd)

u has different values in the following 6 different cases:

$$
\begin{aligned}
\mathrm{I}:&\quad \{x-ct<x+ct<-a<a\}, && u(x,t)=0\\
\mathrm{II}:&\quad \{x-ct<-a<x+ct<a\}, && u(t,x)=\frac{1}{2c}\int_{-a}^{x+ct} h\,ds = h\frac{x+ct+a}{2c}\\
\mathrm{III}:&\quad \{x-ct<-a<a<x+ct\}, && u(t,x)=\frac{1}{2c}\int_{-a}^{a} h\,ds = h\frac{a}{c}\\
\mathrm{IV}:&\quad \{-a<x-ct<x+ct<a\}, && u(t,x)=\frac{1}{2c}\int_{x-ct}^{x+ct} h\,ds = ht\\
\mathrm{V}:&\quad \{-a<x-ct<a<x+ct\}, && u(t,x)=\frac{1}{2c}\int_{x-ct}^{a} h\,ds = h\frac{a-(x-ct)}{2c}\\
\mathrm{VI}:&\quad \{-a<a<x-ct<x+ct\}, && u(x,t)=0
\end{aligned}
$$

Example E (not from the textbook) (Cont'd)

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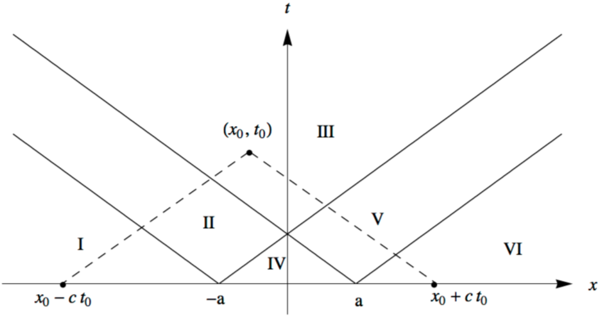

Energy and conservation of energy

Consider an infinite string with constant linear density ρ and tension magnitude T. Then the wave equation describing the vibrations of the string is:

$$\rho u_{tt} = T u_{xx}, \quad -\infty < x < \infty. \tag{35}$$

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- The kinetic energy is $mv^2/2$

$$\text{Kinetic Energy } \; E_K(t) = \frac{1}{2}\rho \int_{-\infty}^{\infty} u_t^2(x,t)\,dx, \tag{36}$$

§2.2 Causality and energy

Energy and conservation of energy

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- The potential energy is

$$\text{Potential Energy} \;\; E_P(t) = \frac{1}{2}T \int_{-\infty}^{\infty} u_x^2(x,t)dx. \tag{37}$$

Energy and conservation of energy

- The total energy of the string undergoing vibrations is

$$E(t) = E_K(t) + E_P(t) = \frac{1}{2}\int_{-\infty}^{\infty} (\rho u_t^2 + T u_x^2)\, dx, \quad t \geq 0. \tag{38}$$

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Differentiating $E(t)$ gives $\rho u_{tt} = T u_{xx}$

$$E'(t) = \int_{-\infty}^{\infty} \left(\rho u_t u_{tt} + T u_x u_{xt}\right) dx = \int_{-\infty}^{\infty} \left(u_t T u_{xx} + T u_x u_{xt}\right) dx. \tag{39}$$

$$= T \int_{-\infty}^{\infty} (u_t u_{xx} + u_x u_{xt})\, dx$$

$$= T \int_{-\infty}^{\infty} \frac{\partial}{\partial x}(u_t u_x)\, dx = T u_t u_x \Big|_{-\infty}^{\infty} = 0$$

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The integration by parts under the usual assumption that $u, u_x \to 0$ as $|x| \to \infty$, gives

$$\int_{-\infty}^{\infty} u_t T u_{xx} dx = T u_t u_x\big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} T u_x u_{xt} dx = -\int_{-\infty}^{\infty} T u_x u_{xt} dx. \tag{40}$$

Therefore, we have $E'(t) = 0$, indicating that the quantity $E(t)$ is conserved.

Energy and conservation of energy

Theorem: The significant consequence of (39) is that the **conservation of energy**:

$$\begin{aligned} E'(t) = 0 \quad &\Rightarrow \quad E(t) = \text{constant}, \quad \forall t \geq 0, \\ \Rightarrow \quad E(t) = E(0) &= \frac{1}{2}\int_{-\infty}^{\infty} \Big(\rho\psi^2(x) + T(\phi'(x))^2\Big)\,dx, \end{aligned} \tag{41}$$

where we used the initial data: $u(x,0) = \phi(x)$ and $u_t(x,0) = \psi(x)$. The conservation of energy implies conversion of the kinetic energy into the potential energy and back without a loss.

Energy and conservation of energy

The conservation of energy can be derived in another more mathematical way. Multiplying both sides of (35) and integrating the resulted equation by parts, we obtain $\rho u_{tt} = T u_{xx}$

$$
\begin{aligned}
0 &= \int_{-\infty}^{\infty} (\rho u_{tt} - T u_{xx}) u_t dx = \int_{-\infty}^{\infty} \left(\rho u_{tt} u_t + T u_x u_{xt}\right) dx \\
&= \int_{-\infty}^{\infty} \left(\frac{\rho}{2}\partial_t(u_t^2) + \frac{T}{2}\partial_t(u_x^2)\right) dx = \frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty} \left(\rho u_t^2 + T u_x^2\right) dx \qquad (42) \\
&= E'(t),
\end{aligned}
$$

where $E(t)$ is as defined in (38). This verifies (41).

Example F (not from the textbook)

Show that the initial value problem

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), & -\infty < x < \infty, \\ u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), & \end{cases} \tag{43}$$

has a unique solution.

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has a unique solution.

Arguing from the inverse, assume that the IVP (43) has two distinct solutions, u and v. $u_{tt} - c^2 u_{xx} = f(x,t)$ $v_{tt} - c^2 v_{xx} = f$

Then their difference $w = u - v$ will solve the following IVP

$$\begin{cases} w_{tt} - c^2 w_{xx} = 0, & -\infty < x < \infty, \\ w(x,0) = u(x,0) - v(x,0) = \phi(x) - \phi(x) \equiv 0, \\ w_t(x,0) = u_t(x,0) - v_t(x,0) = \psi(x) - \psi(x) \equiv 0, \end{cases}$$

$\Rightarrow w_x(x,0) = 0$

Example F (not from the textbook) (Cont'd)

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The energy associated with the solution w at time $t = 0$ is

$c^2 = \frac{T}{\rho}$

$$E[w](0) = \frac{1}{2} \int_{-\infty}^{\infty} \left[(w_t(x,0))^2 + c^2 (w_x(x,0))^2 \right] dx = 0.$$

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The energy associated with the solution w at time $t = 0$ is

$$E[w](0) = \frac{1}{2}\int_{-\infty}^{\infty}\left[(w_t(x,0))^2 + c^2(w_x(x,0))^2\right]dx = 0.$$

This differs from the energy defined above by a constant factor $1/\rho$ $(T/\rho = c^2)$, and is thus still a conserved quantity. It will be subsequently be zero at any later time as well. Thus,

$$E[w](t) = \frac{1}{2}\int_{-\infty}^{\infty}\left[(w_t(x,t))^2 + c^2(w_x(x,t))^2\right]dx = 0, \quad \forall t.$$

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But since the integrand in the expression of the energy is nonnegative, the only way the integral can be zero, is if the integrand is uniformly zero. That is,

$$\nabla w(t,x) = (w_t(x,t), w_x(t,x)) = 0, \quad \forall x, t.$$

Example F (not from the textbook) (Cont'd)

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$\nabla w(t,x) = (w_t(x,t), w_x(t,x)) = 0(\forall x, t)$ implies that w is constant for all values of x and t.

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Since $w(x,0) \equiv 0$, the constant value must be zero. Thus,

$$u(x,t) - v(x,t) = w(x,t) \equiv 0,$$

which is in contradiction with our initial assumption that u and v are different. This implies that the solution to the IVP (43) is unique.

MH4110 PDE

Tutorial 02

Question 1

Solve the first-order equation $2u_t + 3u_x = 0$ with the auxiliary condition $u = \sin x$ when $t = 0$.

$$\frac{dx}{dt} = \frac{3}{2} \Rightarrow x = \frac{3}{2}t + c$$

$$u(x,t) = f\left(x - \frac{3}{2}t\right)$$

Question 1

Solve the first-order equation $2u_t + 3u_x = 0$ with the auxiliary condition $u = \sin x$ when $t = 0$.

[Solution:] The characteristic lines are $3t - 2x = c$. The solution is constant along the characteristic lines. So, the general solution to $2u_t + 3u_x = 0$ is $f(3t - 2x)$. The solution also has to satisfy the additional condition (called initial condition), which we verify by plugging in $t = 0$ into the general solution.

$$\sin x = u(0, x) = f(-2x).$$

Let $w = -2x$, then $f(w) = \sin(-w/2)$ and hence

$$u(t, x) = \sin\left(\frac{-3t + 2x}{2}\right).$$

Question 2

Solve $3u_x - 5u_y = 0$ satisfying the condition $u(0, y) = \cos y$.

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[Solution:] The characteristic lines are $-5x - 3y = c$. The solution is constant along the characteristic lines. So, the general solution to $3u_x - 5u_y = 0$ is $f(-5x - 3y)$. The solution also has to satisfy the additional condition (called initial condition), which we verify by plugging in $x = 0$ into the general solution.

$$\cos y = u(0, y) = f(-3y).$$

So $f(w) = \cos(-w/3)$ and hence

$$u(x, y) = \cos\left(\frac{5x + 3y}{3}\right).$$

$$\frac{dy}{dx} = \frac{-5}{3} \qquad y = -\frac{5}{3}x + C$$

Question 3

Solve the equation $3u_y + u_{xy} = 0$.

Let $v = u_y$

$3v + v_x = 0$

Question 3

Solve the equation $3u_y + u_{xy} = 0$.

[Solution:] Let $v = u_y$. We can write the equation as a system:

$$u_y = v, \quad v_x + 3v = 0.$$

We first solve the second equation: $v_x + 3v = 0$. Treating y as a parameter, we obtain $v(x, y) = C(y)e^{-3x}$. Integrating v, we obtain the solution: $u(x, y) = D(y)e^{-3x} + E(x)$, where we denoted $D(y) = \int C(y)dy$. Here, D and E are arbitrary functions.

Question 4

Use both the geometric method and the coordinate method to solve the equation

$$u_x + u_y = 1.$$

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Use both the geometric method and the coordinate method to solve the equation

$$u_x + u_y = 1.$$

[Solution:]

(a) The geometric method

$$(1,1)\cdot(u_x, u_y) = 0$$

This is an inhomogeneous PDE. Assuming that a particular solution to this PDE takes a form as $u(x,y) = Ax$, we can plug it into the PDE and get $A = 1$. For the homogeneous PDE $u_x + u_y = 0$, its characteristic lines are given by $x - y = C$, where C is an arbitrary constant. So the solution of the homogeneous PDE is $u(x,y) = f(x-y)$. Thus, the solution of $u_x + u_y = 1$ is

$$u(x,y) = x + f(x-y),$$

where f is an arbitrary function.

Question 4 (continued)

Use both the geometric method and the coordinate method to solve the equation $u_x + u_y = 1$.

(b) The coordinate method

Change variables to $x' = x + y$ and $y' = x - y$. We have

$$\frac{\partial}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial x}\frac{\partial}{\partial y'} = \frac{\partial}{\partial x'} + \frac{\partial}{\partial y'},$$

and

$$\frac{\partial}{\partial y} = \frac{\partial x'}{\partial y}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial y}\frac{\partial}{\partial y'} = \frac{\partial}{\partial x'} - \frac{\partial}{\partial y'}.$$

So the PDE is transformed into

$$u_{x'} = \frac{1}{2}.$$

Consider y' as a parameter, this equation can be viewed as a first-order inhomogeneous ODE with the independent variable x'. Integrating with respect to x', we have

$$u = \frac{x'}{2} + f(y').$$

f is an arbitrary function of a single variable. Change back to the original coordinates, we have the solution

$$u(x, y) = \frac{x + y}{2} + f(x - y).$$

Question 5

Solve $au_x + bu_y = c$ using the coordinate method, where a, b, c are constants and $a \neq 0$.

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Solve $au_x + bu_y = c$ using the coordinate method, where a, b, c are constants and $a \neq 0$.

$\frac{dy}{dx} = \frac{b}{a} \Rightarrow bx - ay = \text{const.}$

[Solution:] Change variables to

$$x' = ax + by \qquad \boxed{y' = bx - ay}$$

Replace all x and y derivatives by x' and y' derivatives. By the chain rule,

$$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial x} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial x} = au_{x'} + bu_{y'}$$

and

$$u_y = \frac{\partial u}{\partial y} = \frac{\partial u}{\partial x'}\frac{\partial x'}{\partial y} + \frac{\partial u}{\partial y'}\frac{\partial y'}{\partial y} = bu_{x'} - au_{y'}.$$

Hence $c = au_x + bu_y = a(au_{x'} + bu_{y'}) + b(bu_{x'} - au_{y'}) = (a^2 + b^2)u_{x'}$. Since $a^2 + b^2 \neq 0$, the equation takes the form $\underline{u_{x'} = c/(a^2 + b^2)}$ in the new (primed) variables. Thus the solution is $u = cx'/(a^2 + b^2) + f(y')$, with f an arbitrary function. The solution in the original coordinates is $u = c(ax + by)/(a^2 + b^2) + f(bx - ay)$.

Question 6

(a) Solve the equation $yu_x + xu_y = 0$ with the condition $u(0, y) = y$.

(b) In which region of the xy plane is the solution uniquely determined?

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(b) In which region of the xy plane is the solution uniquely determined?

[Solution:] (a) The characteristic curves are given by

$$\frac{dy}{dx} = \frac{x}{y}.$$

$y = f(y^2)$ $y^2 \geq 0$

This is a separable ODE. Its solution is $y^2 = x^2 + C$, where C is any given constant. Then, the general solution of the original PDE is $u(x, y) = f(y^2 - x^2)$. The auxiliary condition says $u(0, y) = y$. Substituting $x = 0$ into the general solution, we have $y = f(y^2)$. We can only determine the explicit form of $f(y^2)$ as $f(y^2) = \sqrt{y^2} = y$ when $y \geq 0$ or $f(y^2) = -\sqrt{y^2} = y$ when $y < 0$. Thus,

$$u(x, y) = \begin{cases} \sqrt{y^2 - x^2}, & \text{if } y^2 \geq x^2 \text{ and } y \geq 0; \\ -\sqrt{y^2 - x^2}, & \text{if } y^2 \geq x^2 \text{ and } y < 0; \\ f(y^2 - x^2), & \text{if } y^2 < x^2. \end{cases}$$

(b) From the above discussion, we know that the solution is uniquely determined when $y^2 \geq x^2$.

Question 7

Find the general solution of

$$u_x + e^y\, u_y = 1.$$

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[Solution:] We look for the characteristic curve $y = y(x)$ satisfying

$$\frac{dy}{dx} = \frac{e^y}{1} = e^y \quad \Rightarrow \quad x + e^{-y} = C.$$

Hence, the general solution of the problem is

$$u(x, y) = f(x + e^{-y}).$$

Obviously, $u(x, y) = x$ is a particular solution. Thus, the solution of the original problem is

$$u(x, y) = f(x + e^{-y}) + x.$$

Question 8

Solve the equation

$$\sqrt{1-x^2}\,u_x + u_y = 0, \quad u(0,y) = y.$$

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$$\sqrt{1-x^2}\,u_x + u_y = 0, \quad u(0,y) = y.$$

[Solution:] We look for the characteristic curve $y = y(x)$ satisfying

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} \quad \Rightarrow \quad y(x) = \arcsin x + C.$$

Hence, the general solution of the problem is

$$u(x,y) = f(y - \arcsin x).$$

As $u(0,y) = f(y) = y$, we have that the solution is $u(x,y) = y - \arcsin x$.

Question 9

Find the general solution of the equation

$$\text{(a)} \qquad -2u_x + u_y + 3u = e^{x+y};$$

$$\text{(b)} \qquad xu_x - yu_y + y^2u = y^2, \qquad\qquad x, y \neq 0.$$

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[Solution:]

(a) Change variables to $x' = -2x + y$ and $y' = x + 2y$. We have

$$\frac{\partial}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial x}\frac{\partial}{\partial y'} = -2\frac{\partial}{\partial x'} + \frac{\partial}{\partial y'},$$

and

$$\frac{\partial}{\partial y} = \frac{\partial x'}{\partial y}\frac{\partial}{\partial x'} + \frac{\partial y'}{\partial y}\frac{\partial}{\partial y'} = \frac{\partial}{\partial x'} + 2\frac{\partial}{\partial y'}.$$

So the PDE is transformed into

$$5u_{x'} + 3u = e^{-\frac{x'}{5} + \frac{3y'}{5}}.$$

Consider y' as a parameter, this equation can be viewed as a first-order inhomogeneous ODE with the independent variable x'. Let $u = Ae^{-\frac{x'}{5}}$ be a particular solution, then we have

$$5u_{x'} + 3u = 5\left(-\frac{A}{5}e^{-\frac{x'}{5}}\right) + 3Ae^{-\frac{x'}{5}} = 2Ae^{-\frac{x'}{5}} = e^{-\frac{x'}{5} + \frac{3y'}{5}}.$$

(a) The coefficient A is determiend as

$$A = \frac{1}{2}e^{\frac{3y'}{5}}.$$

Thus, the particular solution is

$$u(x', y') = \frac{1}{2}e^{-\frac{x'}{5}+\frac{3y'}{5}}.$$

The general solution to the homogeneous equation $5u_{x'} + 3u = 0$ is $u(x', y') = f(y')e^{-\frac{3x'}{5}}$. The solution to the inhomogeneous equation is

$$u(x', y') = f(y')e^{-\frac{3x'}{5}} + \frac{1}{2}e^{-\frac{x'}{5}+\frac{3y'}{5}}.$$

Change back to the original coordinates, we have the solution

$$u(x, y) = f(x + 2y)e^{-\frac{3(-2x+y)}{5}} + \frac{1}{2}e^{-\frac{-2x+y}{5}+\frac{3(x+2y)}{5}},$$

which can be rewritten as

$$u(x, y) = f(x + 2y)e^{\frac{3}{5}(2x-y)} + \frac{1}{2}e^{x+y}.$$

(b) The characteristic curves are given by

$$\frac{dy}{dx} = -\frac{y}{x}.$$

Question 9 (continued)

(b) This is a separable ODE, which can be solved to obtain the general solution $xy = C$. Thus, our change of coordinates (no need to be orthogonal but should be non-degenerate) will be $x' = x$ and $y' = xy$. In these coordinates the equation takes the form

$$x' u_{x'} + \frac{y'^2}{x'^2} u = \frac{y'^2}{x'^2}, \qquad \text{or} \quad u_{x'} + \frac{y'^2}{x'^3} u = \frac{y'^2}{x'^3}.$$

Using the integrating factor

$$e^{\int \frac{y'^2}{x'^3} dx'} = e^{-\frac{y'^2}{2x'^2}},$$

the above equation can be written as

$$\left(e^{-\frac{y'^2}{2x'^2}} u\right)_{x'} = e^{-\frac{y'^2}{2x'^2}} \frac{y'^2}{x'^3}.$$

Integrating both sides in x', we arrive at

$$e^{-\frac{y'^2}{2x'^2}} u = \int e^{-\frac{y'^2}{2x'^2}} \frac{y'^2}{x'^3} dx' = e^{-\frac{y'^2}{2x'^2}} + f(y').$$

Thus, the general solution will be give by

$$u = e^{\frac{y'^2}{2x'^2}} \left[e^{-\frac{y'^2}{2x'^2}} + f(y')\right] = 1 + e^{\frac{y'^2}{2x'^2}} f(y').$$

Finally, returning back to (x, y) we obtain $u(x, y) = 1 + e^{\frac{y^2}{2}} f(xy)$.

Question 10

Use the coordinate method to solve the equation

$$u_x + 2u_y + (2x - y)u = 2x^2 + 3xy - 2y^2.$$

[Solution:] Change variables to $x' = x + 2y$ and $y' = 2x - y$. We have

$$x = (x' + 2y')/5, \quad y = (2x' - y')/5,$$

and

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial x'} + 2\frac{\partial}{\partial y'}, \quad \frac{\partial}{\partial y} = 2\frac{\partial}{\partial x'} - \frac{\partial}{\partial y'}.$$

So the PDE is transformed into

$$u_{x'} + \frac{y'}{5}u = \frac{x'y'}{5}.$$

Consider y' as a parameter, this equation can be viewed as a first-order inhomogeneous ODE with the independent variable x'. Let $P(x') = \int_0^{x'} \frac{y'}{5} ds = \frac{x'y'}{5}$. Multiplying both sides of the above equation by $e^{P(x')}$ we get

$$\frac{d}{dx'}\left[e^{\frac{x'y'}{5}}u\right] = \frac{x'y'}{5}e^{\frac{x'y'}{5}}.$$

Question 10 (continued)

Use the coordinate method to solve the equation

$$u_x + 2u_y + (2x - y)u = 2x^2 + 3xy - 2y^2.$$

[Solution:] Thus, after integrating with respect to x', we have

$$e^{\frac{x'y'}{5}} u = \left(x' - \frac{5}{y'}\right) e^{\frac{x'y'}{5}} + f(y'),$$

with f as an arbitrary function of a single variable. So

$$u(x', y') = \left(x' - \frac{5}{y'}\right) + e^{-\frac{x'y'}{5}} f(y').$$

Change back to the original coordinates, we have the solution

$$u(x, y) = \left(x + 2y - \frac{5}{2x - y}\right) + e^{-\frac{(x+2y)(2x-y)}{5}} f(2x - y).$$

Question 11

Find the solution to general constant coefficient linear first-order equations

$$au_x + bu_y + cu = g(x, y).$$

Question 11

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$$au_x + bu_y + cu = g(x, y).$$

[Solution:] Before we proceed, let's have a review on the following theorem from ODE

If the functions $p(x)$, $q(x)$ are continuous, then

$$y' = p(x)y + q(x), \tag{1}$$

has infinitely many solutions and every solution, $y(x)$, can be labeled by $r \in R$ as follows

$$y(x) = re^{P(x)} + e^{P(x)} \int e^{-P(x)} q(x)dx, \tag{2}$$

where we introduced the function $P(x) = \int p(x)dx$, any primitive of the function $p(x)$. □

We can use the characteristic coordinates $x' = ax + by$ and $y' = bx - ay$ to reduce the discussing equation into an inhomogeneous ODE ($G(x', y') = g(x(x', y'), y(x', y'))$ is assumed.)

$$(a^2 + b^2)u_{x'} + cu = G(x', y'), \quad \text{or} \quad u_{x'} = -\frac{c}{a^2 + b^2}u + \frac{G(x', y')}{a^2 + b^2}.$$

Question 11 (continued)

[Solution:] Directly using the above theorem, we have

$$P(x') = \int \left(-\frac{c}{a^2+b^2}\right) dx' = -\frac{c}{a^2+b^2}x' + r_1.$$

Here r_1 is constant in x' (depends only on y'). So

$$u(x', y') = r(y')e^{-\frac{c}{a^2+b^2}x' + r_1(y')} + e^{-\frac{c}{a^2+b^2}x' + r_1(y')} \int e^{\frac{c}{a^2+b^2}x' - r_1(y')} \frac{G(x', y')}{a^2+b^2} dx'.$$

It can be simplified into

$$u(x', y') = f(y')e^{-\frac{c}{a^2+b^2}x'} + e^{-\frac{c}{a^2+b^2}x'} \int e^{\frac{c}{a^2+b^2}x'} \frac{G(x', y')}{a^2+b^2} dx'.$$

Assume that

$$\int e^{\frac{c}{a^2+b^2}x'} \frac{G(x', y')}{a^2+b^2} dx' = M(y'),$$

where $M(y')$ is any primitive of the function $e^{\frac{c}{a^2+b^2}x'} \frac{G(x',y')}{a^2+b^2}$ with x' as the only independent variable and y' as a parameter. The solution is

$$u(x', y') = e^{-\frac{c}{a^2+b^2}x'} \left(f(y') + M(y')\right).$$

Replace x' and y' by their expressions in terms of x and y, we obtain that

$$u(x, y) = e^{-\frac{c}{a^2+b^2}(ax+by)} \left[f(bx - ay) + M(bx - ay)\right].$$

Question 12

Carefully derive the equation of a string in a medium in which the resistance is proportional to the velocity.

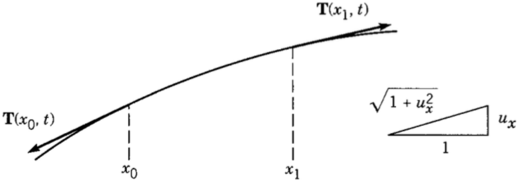
Question 12. Vibrating string with a resistance (Cont'd)

A plucked string. ρ is the linear density (units of mass per unit of length) and is constant along the entire length of the string. $u(t, x)$ is the displacement from equilibrium position at time t and position x.

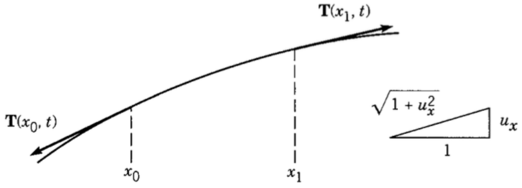
Question 12. Vibrating string with a resistance (Cont'd)

A plucked string. ρ is the linear density (units of mass per unit of length) and is constant along the entire length of the string. $u(t, x)$ is the displacement from equilibrium position at time t and position x.

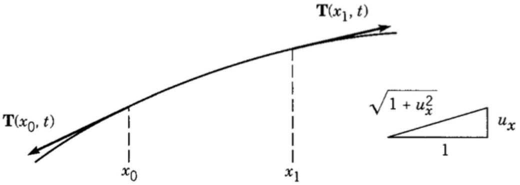
Ignore all the forces on the string except for its tension $\mathbf{T}(x, t)$. Consider the motion of a tiny portion of the string sitting atop the interval $[x_0, x_1]$.



Question 12. Vibrating string with a resistance (Cont'd)



Because the string is perfectly flexible, the tension $\mathbf{T}(x, t)$ is directed tangentially along the string.

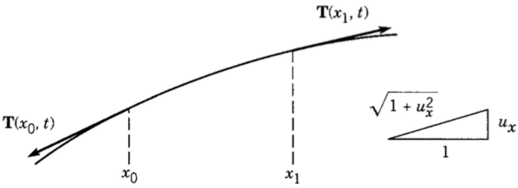


Because the string is perfectly flexible, the tension $\mathbf{T}(x,t)$ is directed tangentially along the string. Given the slope of the string is $u_x(x,t)$, the directions of the tension at the two ends x_0 and x_1 are

$$\mathbf{v}_0 = \left(-\frac{1}{\sqrt{1+u_x^2(x_0,t)}}, -\frac{u_x(x_0,t)}{\sqrt{1+u_x^2(x_0,t)}}\right), \tag{3}$$

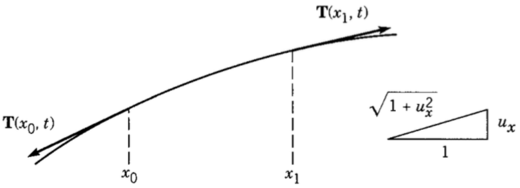
$$\mathbf{v}_1 = \left(\frac{1}{\sqrt{1+u_x^2(x_1,t)}}, \frac{u_x(x_1,t)}{\sqrt{1+u_x^2(x_1,t)}}\right). \tag{4}$$

Question 12. Vibrating string with a resistance (Cont'd)



Let $T(x, t)$ be the magnitude of the tension $\mathbf{T}(x, t)$.

Question 12. Vibrating string with a resistance (Cont'd)



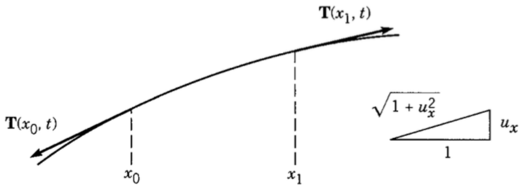
Let $T(x,t)$ be the magnitude of the tension $\mathbf{T}(x,t)$. In the longitudinal direction x, Newton's second law is

$$a_x \int_{x_0}^{x_1} \rho dx = T(x_1,t)\frac{1}{\sqrt{1+u_x^2(x_1,t)}} - T(x_0,t)\frac{1}{\sqrt{1+u_x^2(x_0,t)}}. \tag{5}$$

Since we have assumed/observed that the motion is purely transverse, there is no longitudinal motion and hence the longitudinal acceleration $a_x = 0$. So

$$T(x_1,t)\frac{1}{\sqrt{1+u_x^2(x_1,t)}} = T(x_0,t)\frac{1}{\sqrt{1+u_x^2(x_0,t)}}. \tag{6}$$

Question 12. Vibrating string with a resistance (Cont'd)



In the transverse direction u, the forces include the resistance and the tension; Newton's second law gives that

$$\int_{x_0}^{x_1} a_u(x,t)\rho dx = T(x_1,t)\frac{u_x(x_1,t)}{\sqrt{1+u_x^2(x_1,t)}} - T(x_0,t)\frac{u_x(x_0,t)}{\sqrt{1+u_x^2(x_0,t)}} - \int_{x_0}^{x_1} ru_t(x,t)dx. \quad (7)$$

Here a_u is the transverse acceleration u_{tt}. Using the final relationship on the last slide, we have

$$\int_{x_0}^{x_1} u_{tt}(x,t)\rho dx = T(x_0,t)\frac{u_x(x_1,t)}{\sqrt{1+u_x^2(x_0,t)}} - T(x_0,t)\frac{u_x(x_0,t)}{\sqrt{1+u_x^2(x_0,t)}} - \int_{x_0}^{x_1} ru_t(x,t)dx. \quad (8)$$

Divide both sides by $x_1 - x_0$, we have

$$\frac{\int_{x_0}^{x_1} u_{tt}(x,t)\rho dx}{x_1 - x_0} = \frac{T(x_0,t)}{\sqrt{1+u_x^2(x_0,t)}} \frac{u_x(x_1,t) - u_x(x_0,t)}{x_1 - x_0} - \frac{\int_{x_0}^{x_1} r u_t(x,t) dx}{x_1 - x_0}. \tag{9}$$

Passing to the limit $x_1 \to x_0$ gives

$$\rho u_{tt}(x_0,t) = \frac{T(x_0,t)}{\sqrt{1+u_x^2(x_0,t)}} u_{xx}(x_0,t) - r u_t(x_0,t). \tag{10}$$

Question 12. Vibrating string with a resistance (Cont'd)

Divide both sides by $x_1 - x_0$, we have

$$\frac{\int_{x_0}^{x_1} u_{tt}(x,t)\rho dx}{x_1 - x_0} = \frac{T(x_0,t)}{\sqrt{1+u_x^2(x_0,t)}} \frac{u_x(x_1,t) - u_x(x_0,t)}{x_1 - x_0} - \frac{\int_{x_0}^{x_1} r u_t(x,t) dx}{x_1 - x_0}. \tag{9}$$

Passing to the limit $x_1 \to x_0$ gives

$$\rho u_{tt}(x_0,t) = \frac{T(x_0,t)}{\sqrt{1+u_x^2(x_0,t)}} u_{xx}(x_0,t) - r u_t(x_0,t). \tag{10}$$

Approximation: Assume that the motion is small, i.e., $|u_x| \approx 0$. Then,

$$|u_x| \approx 0 \quad \Rightarrow \quad \sqrt{1+u_x^2} = 1 + \frac{1}{2}u_x^2 + \cdots \approx 1.$$

Therefore, the equation in traverse direction becomes

$$\rho u_{tt}(x,t) = T(x,t) u_{xx}(x,t) - r u_t(x,t). \tag{11}$$

Question 13

Consider heat flow in a long circular cylinder where the temperature depends only on t and on the distance r to the axis of the cylinder. Here $r = \sqrt{x^2 + y^2}$ is the cylindrical coordinate. From the three-dimensional heat equation derive the equation $u_t = k(u_{rr} + u_r/r)$.

Question 13

Consider heat flow in a long circular cylinder where the temperature depends only on t and on the distance r to the axis of the cylinder. Here $r = \sqrt{x^2 + y^2}$ is the cylindrical coordinate. From the three-dimensional heat equation derive the equation $u_t = k(u_{rr} + u_r/r)$.

[Solution:] The three-dimensional heat equation is $u_t = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$. The temperature depends only on t and on the distance r to the axis of the cylinder. That means the temperature is invariant in the z direction. The above heat equation can be simplified as

$$u_t = k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right).$$

Using the chain rule, we have

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} = \frac{\partial u}{\partial r}\frac{x}{r}$$

and

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\frac{x}{r}\right) = \frac{x}{r}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\right) + \frac{\partial u}{\partial r}\frac{\partial}{\partial x}\left(\frac{x}{r}\right) \\ &= \frac{x}{r}\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\right)\frac{\partial r}{\partial x} + \frac{\partial u}{\partial r}\frac{r - x\frac{\partial r}{\partial x}}{r^2} \\ &= \frac{x}{r}\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}\right)\frac{x}{r} + \frac{\partial u}{\partial r}\frac{r - x\frac{x}{r}}{r^2} = \frac{x^2}{r^2}\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r}\frac{r^2 - x^2}{r^3}.\end{aligned}$$

Question 13 (continued)

Consider heat flow in a long circular cylinder where the temperature depends only on t and on the distance r to the axis of the cylinder. Here $r = \sqrt{x^2 + y^2}$ is the cylindrical coordinate. From the three-dimensional heat equation derive the equation $u_t = k(u_{rr} + u_r/r)$.

[Solution:] Similarly,

$$\frac{\partial^2 u}{\partial y^2} = \frac{y^2}{r^2}\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r}\frac{r^2 - y^2}{r^3}.$$

Thus,

$$\begin{aligned} u_t &= k\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = k\left(\frac{x^2}{r^2}\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r}\frac{r^2 - x^2}{r^3} + \frac{y^2}{r^2}\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r}\frac{r^2 - y^2}{r^3}\right) \\ &= k\left(\frac{x^2 + y^2}{r^2}\frac{\partial^2 u}{\partial r^2} + \frac{\partial u}{\partial r}\frac{2r^2 - x^2 - y^2}{r^3}\right) \\ &= k\left(\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right). \end{aligned}$$

Question 14

Solve the following first order PDE and find where the solution is defined in the x - y plane.

$$u_x + xyu_y = 0, \qquad u(x,-1) = \frac{1}{2}x^2.$$

Question 14

Solve the following first order PDE and find where the solution is defined in the x - y plane.

$\frac{dy}{y} = x\,dx$

$$u_x + xyu_y = 0, \qquad u(x,-1) = \frac{1}{2}x^2.$$

[Solution:] The characteristic curves are given by $\frac{dy}{dx} = \frac{xy}{1}$. This is a separable ODE. Its solution is $\ln|y| = \frac{x^2}{2} + C$, where C is any given constant. Further we have $|y| = e^{x^2/2}e^C$ or $y = \pm e^{x^2/2}e^C$. If $D = \pm e^C$ is any given constant, then $y = De^{x^2/2}$. The characteristic curves are given by $ye^{-x^2/2} = D$. The general solution of the original PDE is $u(x,y) = \underline{f(ye^{-x^2/2})}$. The auxiliary condition says $u(x,-1) = \frac{1}{2}x^2$. Substituting $y = -1$ into the general solution, we have $\frac{1}{2}x^2 = f(\underline{-e^{-x^2/2}})$. We can only determine the explicit form of $f(z)$ when its independent variable z is on the half-closed interval $\underline{[-1,} \ 0)$ as $f(z) = -\ln(-z)$. Thus,

$$u(x,y) = \begin{cases} -\ln\left(-ye^{-x^2/2}\right) = \ln\left(-y^{-1}e^{x^2/2}\right) = \ln\left(-y^{-1}\right) + x^2/2, & \text{if } -1 \le ye^{-x^2/2} < 0; \\ f(ye^{-x^2/2}), & \text{otherwise.} \end{cases}$$

Or,

$$u(x,y) = \begin{cases} \ln\left(-y^{-1}\right) + x^2/2, & \text{if } -e^{x^2/2} \le y < 0; \\ f(ye^{-x^2/2}), & \text{otherwise} \end{cases}$$

From the above discussion, we know that the solution is uniquely determined when

Question 15

Solve the following first order PDE

$$x^2 u_x + xyu_y = u.$$

Question 15

Solve the following first order PDE

$$x^2 u_x + xyu_y = u.$$

[Solution:] The characteristic curves are given by

$$\frac{dy}{dx} = \frac{xy}{x^2} = \frac{y}{x}.$$

This is a separable ODE, which can be solved to obtain the general solution $y/x = C$. Thus, our change of coordinates (no need to be orthogonal but should be non-degenerate) will be

$$\begin{cases} x' = x, \\ y' = \frac{y}{x}. \end{cases}$$

Use the chain rule, we have

$$\begin{cases} \frac{\partial u}{\partial x} = \frac{\partial x'}{\partial x}\frac{\partial u}{\partial x'} + \frac{\partial y'}{\partial x}\frac{\partial u}{\partial y'} = 1 \cdot \frac{\partial u}{\partial x'} - \frac{y}{x^2}\frac{\partial u}{\partial y'} = \frac{\partial u}{\partial x'} - \frac{y'}{x'}\frac{\partial u}{\partial y'}, \\ \frac{\partial u}{\partial y} = \frac{\partial x'}{\partial y}\frac{\partial u}{\partial x'} + \frac{\partial y'}{\partial y}\frac{\partial u}{\partial y'} = 0 \cdot \frac{\partial u}{\partial x'} + \frac{1}{x}\frac{\partial u}{\partial y'} = \frac{1}{x'}\frac{\partial u}{\partial y'}. \end{cases}$$

In these coordinates the equation takes the form

$$x'^2\left(\frac{\partial u}{\partial x'} - \frac{y'}{x'}\frac{\partial u}{\partial y'}\right) + x'^2 y' \cdot \frac{1}{x'}\frac{\partial u}{\partial y'} = u,$$

Question 15 (continued)

Solve the following first order PDE

$$x^2 u_x + xy u_y = u.$$

[Solution:] or $u_{x'} - \frac{1}{x'^2} u = 0$. Using the integrating factor

$$e^{\int -\frac{1}{x'^2} dx'} = e^{\frac{1}{x'}},$$

the above equation can be written as

$$\left(e^{\frac{1}{x'}} u\right)_{x'} = 0.$$

Integrating both sides in x', we arrive at

$$e^{\frac{1}{x'}} u = f(y').$$

Thus, the general solution will be give by

$$u = e^{-\frac{1}{x'}} f(y').$$

Finally, substituting the expressions of x' and y' in terms of (x, y) into the solution, we obtain

$$u(x, y) = e^{-\frac{1}{x}} f(\frac{y}{x}).$$

One should again check by substitution that this is indeed a solution to the PDE.