MH4110 Partial Differential Equations

Chapter 3 - Reflections and Sources

Outline

- Part 1
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  - Diffusion on the half-line with a Neumann boundary condtion
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  - Waves on the half-line with a Neumann boundary condtion
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  - Diffusion with a source on the half-line
  - Waves with a source on the whole line
  - Waves with a source on the half-line

The diffusion equation on the half line $D = (0, \infty)$

The initial/boundary value problem (IBVP) containing a Dirichlet boundary condition at the endpoint $x = 0$

$$\begin{aligned} &v_t - kv_{xx} = 0, \quad 0 < x < \infty, \quad t > 0, \\ &v(x,0) = \phi(x) \quad \text{(initial condition at } t = 0\text{)}, \\ &v(0,t) = 0 \quad \text{(boundary condition at } x = 0\text{)}, \end{aligned} \tag{1}$$

where we assume that $\phi(0) = 0$ (consistent condition).

- If the solution to the above mixed initial/boundary value problem (1) exists, then it must be unique from an application of the maximum principle.
- In terms of the heat conduction, one can think of v in (1) as the temperature in an infinite rod, one end of which is kept at a constant zero temperature. The initial temperature of the rod is then given by $\phi(x)$.
- The homogeneous boundary condition is necessary for the method of odd extensions. The inhomogeneous $v(0,t) = g(t) \neq 0$ will be discussed later.

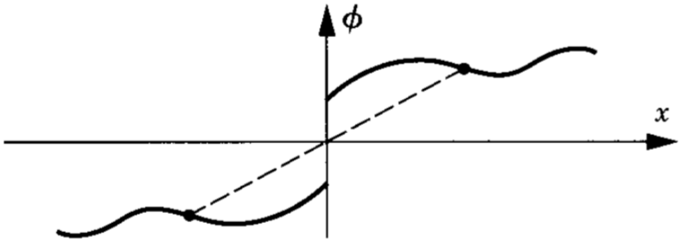
§3.1 Diffusion on the half-line

- To solve the IBVP (1) and derive a solution formula, instead of constructing the solution from scratch, it makes sense to try to reduce this problem to the IVP on the whole line, for which we already have a solution formula.
- The above idea is achieved by extending the initial data $\phi(x)$ to the whole line.
- The extension should make sure at least that the boundary condition of (1) is automatically satisfied by the solution $u(x,t)$ to the IVP on the whole line. That means $u(0,t) = 0$.
- By definition, a function $\phi(x)$ is odd, if $\phi(-x) = -\phi(x)$. Plugging in $x = 0$ into this definition, one gets $\phi(0) = 0$ for any odd function.
- We can prove that the solution $u(x,t)$ to the diffusion IVP with odd initial data is itself odd in the x variable.
- Therefore, if one extends $\phi(x)$ to an odd function on the whole line, then the solution with the extended initial data automatically satisfies the boundary condition of (1).

Method of odd extensions

Extend the initial data ϕ to an *odd function* ψ defined by

$$\psi(x) = \begin{cases} \phi(x), & \text{for } x > 0, \\ -\phi(-x), & \text{for } x < 0, \\ 0, & \text{for } x = 0, \end{cases} \tag{2}$$



Method of odd extensions (Cont'd)

After extension, we consider the Cauchy problem:

$$\begin{aligned} &u_t - ku_{xx} = 0, \quad -\infty < x < \infty, \ \ t > 0, \\ &u(x,0) = \psi(x), \quad -\infty < x < \infty, \end{aligned} \tag{3}$$

where ψ is defined in (2).

By the solution formula, we have

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\psi(y)dy, \tag{4}$$

where

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}} \exp\Big(-\frac{x^2}{4kt}\Big). \tag{5}$$

§3.1 Diffusion on the half-line

Method of odd extensions (Cont'd)

Restricting the x variable to only the non-negative half-line produces the function

$$v(x,t) = u(x,t)\big|_{x\geq 0}. \tag{6}$$

- $v(x,t)$ solves the diffusion equation on the positive half-line, since so does $u(x,t)$.
- Furthermore,

  $$v(x,0) = u(x,0)\big|_{x>0} = \psi(x)\big|_{x>0} = \phi(x), \tag{7}$$

  indicating that the initial condition is satisfied.
- $v(0,t) = u(0,t) = 0$, since $u(x,t)$ is an odd function of x.

So $v(x,t)$ defined in (6) satisfies the initial and boundary conditions of (1) and it is also the unique solution.

Method of odd extensions (Cont'd)

Breaking the integral (4) into two parts, $y > 0$ and $y < 0$, we obtain

$$\begin{aligned} u(x,t) &= \int_0^\infty S(x-y,t)\psi(y)dy + \int_{-\infty}^0 S(x-y,t)\psi(y)dy \\ &= \int_0^\infty S(x-y,t)\phi(y)dy - \int_{-\infty}^0 S(x-y,t)\phi(-y)dy \\ &= \int_0^\infty S(x-y,t)\phi(y)dy - \int_0^\infty S(x+y,t)\phi(y)dy \\ &= \int_0^\infty \left[S(x-y,t) - S(x+y,t)\right]\phi(y)dy. \end{aligned}$$

Solution formula

Now, we restrict the solution $u(x,t)$ to the half-line. For $0 < x < \infty$ and $0 < t < \infty$, we have the solution formula for the IBVP (1) as follows

$$v(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left[\exp\left(-\frac{(x-y)^2}{4kt} \right) - \exp\left(-\frac{(x+y)^2}{4kt} \right) \right] \phi(y) dy. \quad (8)$$

Remark

The zero boundary condition at $x = 0$, i.e., $v(0,t) = 0$ is important for implementing the method of odd extensions or the reflection method.

Example A (not in the textbook)

Solve the equation:

$$\begin{aligned}
&v_t - kv_{xx} = 0, \quad 0 < x < \infty, \quad t > 0, \\
&v(x,0) = 0, \quad 0 < x < \infty, \\
&v(0,t) = -1, \quad t > 0.
\end{aligned}$$

Express the solution in terms of the error function $\mathrm{Erf}(x)$.

Example A (Cont'd)

Example A (Cont'd)

The diffusion equation on the half line $D = (0, \infty)$

The initial/boundary value problem (IBVP) containing a Neumann boundary condition at the endpoint $x = 0$

$$\begin{aligned} &w_t - k w_{xx} = 0, \quad 0 < x < \infty, \quad t > 0, \\ &w(x,0) = \phi(x) \quad \text{(initial condition at } t = 0\text{)}, \\ &w_x(0,t) = 0 \quad \text{(boundary condition at } x = 0\text{)}. \end{aligned} \tag{9}$$

- If the solution to the above mixed initial/boundary value problem (9) exists, then it must be unique from an application of the maximum principle.
- To find the solution of (9), we employ a similar idea used in the case of the Dirichlet problem. That is, we seek to reduce the IBVP to an IVP on the whole line by extending the initial data $\phi(x)$ to the negative half-axis in such a way that the boundary condition is automatically satisfied.

How to do the extension

- Notice that if $\phi(x)$ is an even function, i.e. $\phi(-x) = \phi(x)$, then its derivative function will be odd.
- Indeed, differentiating in the definition of the even function, we get $-\phi'(-x) = \phi'(x)$, which is the same as $\phi'(-x) = -\phi'(x)$.
- Hence, for an arbitrary even function $\phi(x)$, $\phi^{'}(0) = 0$.
- It is now clear that extending the initial data so that the resulting function is even will produce solutions to the IVP on the whole line that automatically satisfy the Neumann condition of (9).

Therefore, we make an *even extension* of the initial data:

$$\varphi(x) = \begin{cases} \phi(x), & \text{for} \quad x \geq 0, \\ \phi(-x), & \text{for} \quad x \leq 0. \end{cases} \tag{10}$$

It is clear that we have $\varphi'(0) = 0$.

Method of even extensions

After extension, we consider the Cauchy problem:

$$\begin{aligned} &u_t - ku_{xx} = 0, \quad -\infty < x < \infty, \quad t > 0, \\ &u(x,0) = \varphi(x), \quad -\infty < x < \infty, \end{aligned} \tag{11}$$

where φ is defined in (10).

By the solution formula, we have

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\varphi(y)dy, \tag{12}$$

where

$$S(x,t) = \frac{1}{\sqrt{4\pi kt}}\exp\Big(-\frac{x^2}{4kt}\Big). \tag{13}$$

Method of even extensions (Cont'd)

Breaking the integral (12) into two parts, $y > 0$ and $y < 0$, we obtain

$$\begin{aligned} u(x,t) &= \int_0^{\infty} S(x-y,t)\varphi(y)dy + \int_{-\infty}^{0} S(x-y,t)\varphi(y)dy \\ &= \int_0^{\infty} S(x-y,t)\phi(y)dy + \int_{-\infty}^{0} S(x-y,t)\phi(-y)dy \\ &= \int_0^{\infty} S(x-y,t)\phi(y)dy + \int_0^{\infty} S(x+y,t)\phi(y)dy \\ &= \int_0^{\infty} \left[S(x-y,t) + S(x+y,t)\right]\phi(y)dy. \end{aligned}$$

§3.1 Diffusion on the half-line

Solution formula

Now, we restrict the solution $u(x,t)$ to the half-line. For $0 < x < \infty$ and $0 < t < \infty$, we have the solution formula for the IBVP (9) as follows

$$w(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_0^\infty \left[\exp\Big(-\frac{(x-y)^2}{4kt} \Big) + \exp\Big(-\frac{(x+y)^2}{4kt} \Big) \right] \phi(y)\,dy. \tag{14}$$

Remark

The zero boundary condition at $x = 0$, i.e., $w_x(0,t) = 0$ is important for implementing the method of even extensions or the reflection method.

Example 2

Solve the equation:

$$\begin{aligned}
&w_t - kw_{xx} = 0, \quad 0 < x < \infty, \quad t > 0, \\
&w(x,0) = 1, \quad 0 < x < \infty, \\
&w_x(0,t) = 0, \quad t > 0.
\end{aligned}$$

In terms of heat conduction, this describes the temperature dynamics with identically 1 initial temperature, and no heat loss at the endpoint. Note that the heat flux is proportional to the spatial derivative of the temperature.

Example 2 (Cont'd)

§3.1 Diffusion on the half-line

Conclusion

1. We derived the solution to the diffusion equation on the half-line by reducing the initial/boundary value problem (IBVP) to the initial value problem (IVP) over the whole line through appropriate extension of the initial data.
2. In the case of zero Dirichlet boundary condition the odd extension of the initial data automatically guarantees that the solution will satisfy the boundary condition.
3. While for the case of zero Neumann boundary condition the appropriate choice is the even extension.
4. This reflection method relies on the fact that the solution to the diffusion equation on the whole line with odd initial data is odd, while the solution with even initial data is even.

§3.2 Reflections of waves

The wave equation on the half line $D = (0, \infty)$

The initial/boundary value problem (IBVP) containing a Dirichlet boundary condition at the endpoint $x = 0$

$$\begin{aligned} &v_{tt} - c^2 v_{xx} = 0, \quad 0 < x < \infty, \ \ t > 0, \\ &v(x,0) = \phi(x), \quad v_t(x,0) = \psi(x), \quad \text{(initial condition at } t = 0\text{)}, \\ &v(0,t) = 0 \quad \text{(boundary condition at } x = 0\text{)}, \end{aligned} \tag{15}$$

where we assume that $\phi(0) = 0$ (consistent condition).

- If the solution to the above mixed initial/boundary value problem (15) exists, then it must be unique from an application of the energy method.
- For the vibrating string, the boundary condition of (15) means that the end of the string at $x = 0$ is held fixed.

§3.2 Reflections of waves

The reflection method

- To solve the Dirichlet problem (15), the idea is again to extend the initial data, in this case ϕ, ψ, to the whole line.
- Since the boundary condition is in the Dirichlet form, one should take the odd extensions:

$$\tilde{\phi}(x) = \begin{cases} \phi(x), & x > 0, \\ -\phi(-x), & x < 0, \\ 0, & x = 0, \end{cases} \quad \tilde{\psi}(x) = \begin{cases} \psi(x), & x > 0, \\ -\psi(-x), & x < 0, \\ 0, & x = 0. \end{cases} \tag{16}$$

- Then we solve the extended wave equation on the whole real axis:

$$\begin{aligned} &u_{tt} - c^2 u_{xx} = 0, \quad -\infty < x < \infty, \quad t > 0, \\ &u(x,0) = \tilde{\phi}(x), \quad u_t(x,0) = \tilde{\psi}(x), \quad -\infty < x < \infty, \end{aligned} \tag{17}$$

The reflection method (Cont'd)

- Since the initial data of the above IVP are odd, we know that the solution of the IVP, $u(x, t)$, will also be odd in the x variable, and hence $u(0, t) = 0$ for all $t > 0$.
- Then defining the restriction of $u(x, t)$ to the positive half-line $x \geq 0$,

$$v(x, t) = u(x, t)\big|_{x \geq 0}, \tag{18}$$

  we automatically have that $v(0, t) = u(0, t) = 0$. So the boundary condition of the Dirichlet problem (15) is satisfied for v.
- The initial conditions are satisfied for v as well, since the restrictions of $\tilde{\phi}(x)$ and $\tilde{\psi}(x)$ to the positive half-line are $\phi(x)$ and $\psi(x)$ respectively.
- $v(x, t)$ solves the wave equation for $x > 0$, since $u(x, t)$ satisfies the wave equation for all $x \in R$, and in particular for $x > 0$.
- Therefore, $v(x, t)$ as defined in (18) is the unique solution to the IBVP (15).

The reflection method (Cont'd)

- Using the d'Alembert formula for the solution of (17), and taking the restriction (18), we have that for $x \geq 0$,

$$v(x,t) = \frac{1}{2}\big(\tilde{\phi}(x+ct) + \tilde{\phi}(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{\psi}(s)ds. \tag{19}$$

- For the IBVP (15), $x \geq 0$ and $t > 0$, so $x + ct \geq 0$. We only need to consider the sign of $x - ct$.
- If $x - ct > 0$, we have

$$v(x,t) = \frac{1}{2}\big(\phi(x+ct) + \phi(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds, \tag{20}$$

which is exactly d'Alembert formula.

The reflection method (Cont'd)

- If $x - ct < 0$, and using (16) we can rewrite the solution (19) as

$$\begin{aligned}
v(x,t) &= \frac{1}{2}\left[\tilde{\phi}(x+ct) + \tilde{\phi}(x-ct)\right] + \frac{1}{2c}\left[\int_{x-ct}^{0} \tilde{\psi}(s)ds + \int_{0}^{x+ct} \tilde{\psi}(s)ds\right] \\
&= \frac{1}{2}\left[\phi(x+ct) - \phi(ct-x)\right] + \frac{1}{2c}\left[\int_{x-ct}^{0} -\psi(-s)ds + \int_{0}^{x+ct} \psi(s)ds\right] \\
&= \frac{1}{2}\left[\phi(x+ct) - \phi(ct-x)\right] + \frac{1}{2c}\left[\int_{ct-x}^{0} \psi(s)ds + \int_{0}^{x+ct} \psi(s)ds\right] \\
&= \frac{1}{2}\left[\phi(x+ct) - \phi(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi(s)ds
\end{aligned} \tag{21}$$

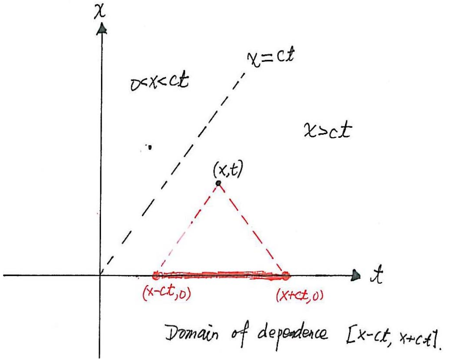
Solution formula

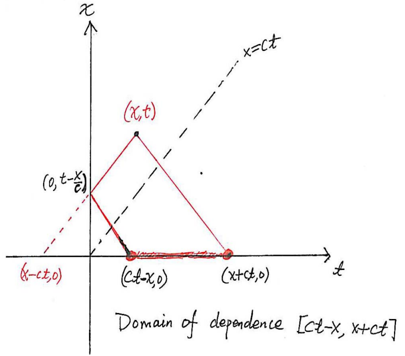
In summary, we have the solution formula for the IBVP (15):

$$v(x,t) = \begin{cases} \frac{1}{2}\left[\phi(x+ct)+\phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)ds, & x > ct, \\ \frac{1}{2}\left[\phi(x+ct)-\phi(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{ct+x}\psi(s)ds, & 0 < x < ct. \end{cases} \tag{22}$$

The minus sign in front of $\phi(ct-x)$ in the second expression above, as well as the reduction of the integral of ψ to the smaller interval are due to the cancellation stemming from the reflected wave.

We find that the reflection happens when the left-going wave hits the boundary $x = 0$, and this results in the change of domain dependence (see the Figure below), as seen from the second formula of (22).





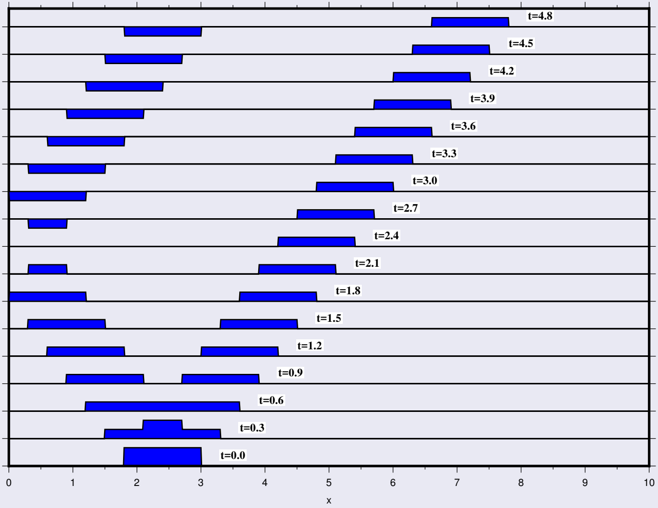
Example B (not in the textbook)

Consider the wave equation under the situation of initially at rest:

$$\begin{aligned} & u_{tt} - u_{xx} = 0, \quad 0 < x < \infty, \quad t > 0, \\ & u(0,t) = 0, \quad t > 0, \\ & u_t(x,0) = 0, \quad u(x,0) = \phi(x) = \begin{cases} 1, & 1.8 < x < 3, \\ 0, & \text{otherwise.} \end{cases} \end{aligned} \tag{23}$$

§3.2 Reflections of waves

Example B (Cont'd)



The Neumann problem on the half-line

$$\begin{aligned} &w_{tt} - c^2 w_{xx} = 0, \quad 0 < x < \infty, \ \ t > 0, \\ &w_x(0,t) = 0, \quad t > 0, \\ &w(x,0) = \phi(x), \ \ w_t(x,0) = \psi(x), \ \ x > 0. \end{aligned} \tag{25}$$

We use the reflection method with even extensions to reduce the problem to an IVP on the whole line.

Solving the Neumann problem on the half-line

- Define the even extensions of the initial data

$$\tilde{\phi}(x) = \begin{cases} \phi(x), & x > 0, \\ \phi(-x), & x < 0, \end{cases} \quad \tilde{\psi}(x) = \begin{cases} \psi(x), & x > 0, \\ \psi(-x), & x < 0. \end{cases} \tag{26}$$

- Then we solve the extended wave equation on the whole real axis:

$$\begin{aligned} &u_{tt} - c^2 u_{xx} = 0, \quad -\infty < x < \infty, \ \ t > 0, \\ &u(x,0) = \tilde{\phi}(x), \quad u_t(x,0) = \tilde{\psi}(x), \quad -\infty < x < \infty. \end{aligned} \tag{27}$$

- Clearly, the solution $u(x,t)$ to the IVP (27) will be even in x, and since the derivative of an even function is odd, $u_x(x,t)$ will be odd in x, and hence $u_x(0,t) = 0$ for all $t > 0$.

Solving the Neumann problem on the half-line (Cont'd)

- Just like the case of the Dirichlet problem, the restriction

$$w(x,t) = u(x,t)\big|_{x\geq 0} \tag{28}$$

  will be the unique solution of the Neumann problem (25).

- Using d'Alembert formula for the solution of (27), and taking the restriction (28), we have that for $x \geq 0$,

$$w(x,t) = \frac{1}{2}\left[\tilde{\phi}(x+ct) + \tilde{\phi}(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{\psi}(s)ds. \tag{29}$$

- Once again, we need to consider the two cases $x > ct$ and $0 < x < ct$ separately.

Solving the Neumann problem on the half-line (Cont'd)

- If $x - ct > 0$, we have

$$w(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)ds. \tag{30}$$

- If $x - ct < 0$, and using (26) we can rewrite the solution (29) as

$$\begin{aligned} w(x,t) &= \frac{1}{2}\left[\tilde{\phi}(x+ct) + \tilde{\phi}(x-ct)\right] + \frac{1}{2c}\left[\int_{x-ct}^{0} \tilde{\psi}(s)ds + \int_{0}^{x+ct} \tilde{\psi}(s)ds\right] \\ &= \frac{1}{2}\left[\phi(x+ct) + \phi(ct-x)\right] + \frac{1}{2c}\left[\int_{x-ct}^{0} \psi(-s)ds + \int_{0}^{x+ct} \psi(s)ds\right] \\ &= \frac{1}{2}\left[\phi(x+ct) + \phi(ct-x)\right] + \frac{1}{2c}\left[-\int_{ct-x}^{0} \psi(s)ds + \int_{0}^{x+ct} \psi(s)ds\right] \\ &= \frac{1}{2}\left[\phi(x+ct) + \phi(ct-x)\right] + \frac{1}{2c}\left[\int_{0}^{ct-x} \psi(s)ds + \int_{0}^{x+ct} \psi(s)ds\right] \end{aligned} \tag{31}$$

Solution formula

We have the solution formula for the Neumann problem on the half-line (25):

$$w(x,t) = \begin{cases} \dfrac{\phi(x+ct)+\phi(x-ct)}{2} + \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} \psi(s)ds, & x > ct, \\ \dfrac{\phi(x+ct)+\phi(ct-x)}{2} + \dfrac{1}{2c}\left(\displaystyle\int_0^{ct-x} + \int_0^{x+ct}\right)\psi(s)ds, & 0 < x < ct. \end{cases} \tag{32}$$

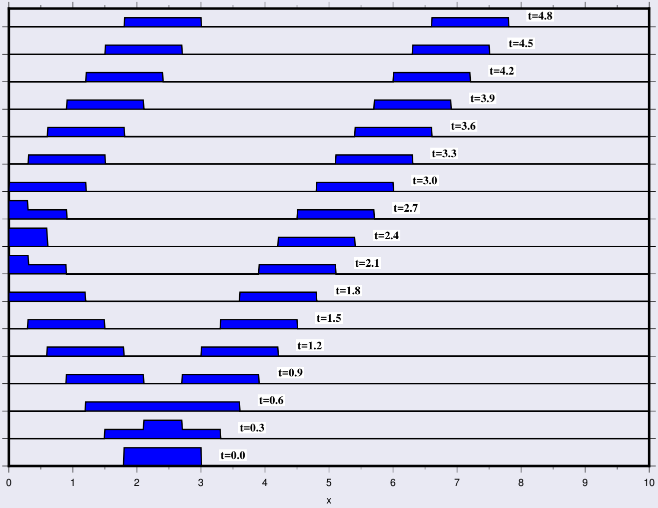
The Neumann boundary condition corresponds to a vibrating string with a free end at $x = 0$, since the string tension, which is proportional to the derivative $w_x(x,t)$, vanishes at $x = 0$. In this case the reflected wave adds to the original wave, rather than canceling it.

Example C (not in the textbook)

Consider the wave equation under the Neumann boundary condition:

$$
\begin{aligned}
&u_{tt} - u_{xx} = 0, \quad 0 < x < \infty, \;\; t > 0, \\
&u_x(0,t) = 0, \quad t > 0, \\
&u_t(x,0) = 0, \quad u(x,0) = \phi(x) = \begin{cases} 1, & 1.8 < x < 3, \\ 0, & \text{otherwise.} \end{cases}
\end{aligned}
\tag{33}
$$

Example C (Cont'd)



§3.2 Reflections of waves

Conclusion

1. We derived the solution to the wave equation on the half-line in much the same way as was done for the diffusion equation. That is, we reduced the initial/boundary value problem to the initial value problem over the whole line through appropriate extension of the initial data.
2. The characteristics nicely illustrate the reflection phenomenon. We saw that the characteristics that hit the initial data after reflection from the boundary wall $x = 0$ carry the values of the initial data with a minus sign in the case of the Dirichlet boundary conditions, and with a plus sign in the case of the Neumann boundary conditions. This corresponds to our intuition of reflected waves from a fixed end, and free end respectively.

Diffusion with a source on the whole line

Consider the inhomogeneous diffusion equation on the whole line,

$$\begin{aligned} & u_t - k u_{xx} = f(x,t), \quad -\infty < x < \infty, \quad t > 0, \\ & u(x,0) = \phi(x) \quad \text{(initial condition at } t = 0) \end{aligned} \tag{35}$$

with $f(x,t)$ and $\phi(x)$ arbitrary given functions.

- The right hand side of the equation, $f(x,t)$, is called the source term.
- In terms of heat conduction, the source term measures the physical effect of an external heat source. It has units of heat flux (left hand side of the equation has the units of u_t, i.e. change in temperature per unit time), thus it gives the instantaneous temperature change due to an external heat source.

§3.3 Diffusion with a source

Theorem 1. The solution of (35) is,

$$u(x,t)=\int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy+\int_{0}^{t}\int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)dyds, \tag{36}$$

where $S(x,t)$ is the Gaussian kernel

$$S(x,t)=\frac{1}{\sqrt{4\pi kt}}e^{-\frac{x^2}{4kt}}. \tag{37}$$

- In terms of heat conduction, the physical meaning of the expression $\int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy$ in (36) is that the heat kernel averages out the initial temperature distribution along the entire rod.
- $f(x,t)$ is the external heat source. It must be averaged out, too; not only over the entire rod, but over time as well, since the heat contribution at an earlier time will affect the temperatures at all later times.
- The time integration is only over the interval $[0,t]$, since the heat contribution at later times can not affect the temperature at time t.

Proof of Theorem 1. We show that (36) indeed solves problem (35) by a direct substitution.

- Differentiate (36) with respect to t, we have

$$
\begin{aligned}
\frac{\partial}{\partial t}u(x,t) = & \frac{\partial}{\partial t}\int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy \\
& + \frac{\partial}{\partial t}\int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)dyds \\
= & \int_{-\infty}^{\infty} \frac{\partial}{\partial t}S(x-y,t)\phi(y)dy + \int_{-\infty}^{\infty} S(x-y,0)f(y,t)dy \\
& + \int_0^t \int_{-\infty}^{\infty} \frac{\partial}{\partial t}S(x-y,t-s)f(y,s)dyds
\end{aligned}
\tag{38}
$$

- Recall that Gaussian kernel $S(x,t)$ solves the diffusion equation and has the Dirac delta function as its initial data, i.e. $S_t = kS_{xx}$, and $S(x-y,0) = \delta(x-y)$.

Proof of Theorem 1 (Cont'd).

- Hence,

$$\begin{aligned} \frac{\partial}{\partial t}u(x,t) &= \int_{-\infty}^{\infty} k\frac{\partial^2}{\partial x^2}S(x-y,t)\phi(y)dy + \int_{-\infty}^{\infty}\delta(x-y)f(y,t)dy \\ &\quad + \int_0^t\int_{-\infty}^{\infty} k\frac{\partial^2}{\partial x^2}S(x-y,t-s)f(y,s)dyds \\ &= k\frac{\partial^2}{\partial x^2}\int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy + f(x,t) \\ &\quad + k\frac{\partial^2}{\partial x^2}\int_0^t\int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)dyds \\ &= k\frac{\partial^2}{\partial x^2}u(x,t) + f(x,t). \end{aligned} \tag{39}$$

- Expression (39) indicates that $u(x,t)$ defined in (36) solves the inhomogeneous diffusion equation $u_t - ku_{xx} = f(x,t)$.

Proof of Theorem 1 (Cont'd).

- We verify the initial condition. The solution is

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy + \int_0^t \int_{-\infty}^{\infty} S(x-y,t-s)f(y,s)dyds.$$

  Letting $t \to 0$, the first term becomes

$$\int_{-\infty}^{\infty} S(x-y,0)\phi(y)dy = \int_{-\infty}^{\infty} \delta(x-y)\phi(y)dy = \phi(x).$$

  The second term is an integral from 0 to 0. Therefore, $u(x,0) = \phi(x)$ satisfies the initial condition. This proves that (36) is the unique solution of the IVP (35).

- The solution has an explicit form as

$$u(x,t) = \frac{1}{\sqrt{4\pi kt}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4kt}} \phi(y)dy + \int_0^t \int_{-\infty}^{\infty} \frac{e^{-\frac{(x-y)^2}{4k(t-s)}}}{\sqrt{4\pi k(t-s)}} f(y,s)dyds. \tag{40}$$

Example A (not in the textbook)

Find the solution of the inhomogeneous problem,

$$\begin{aligned} &u_t - ku_{xx} = \delta(x-1)\delta(t-2), \quad -\infty < x < \infty, \;\; t > 0, \\ &u(x,0) = 0, \qquad t > 0. \end{aligned} \tag{41}$$

§3.3 Diffusion with a source

§3.3 Diffusion with a source

Diffusion with a source on a half-line

Consider the more complicated Dirichlet problem of a boundary source $h(t)$ on the half-line; that is,

$$\begin{aligned} &v_t - kv_{xx} = f(x,t), \quad 0 < x < \infty, \quad t > 0, \\ &v(0,t) = h(t) \quad \text{(boundary condition at } x = 0\text{)}, \\ &v(x,0) = \phi(x) \quad \text{(initial condition at } t = 0\text{)}. \end{aligned} \tag{42}$$

- For inhomogeneous diffusion on the half-line we can use the method of reflection as well.
- We may use the subtraction device to reduce (42) to a simpler problem with a homogeneous boundary condition. Let

$$V(x,t) = v(x,t) - h(t). \tag{43}$$

Diffusion with a source on a half-line (Cont'd)

- We can check that $V(x,t) = v(x,t) - h(t)$ satisfies

$$\begin{aligned} &V_t - kV_{xx} = f(x,t) - h'(t), \quad 0 < x < \infty, \quad t > 0, \\ &V(0,t) = 0 \quad \text{(boundary condition at } x = 0), \\ &V(x,0) = \phi(x) - h(0) \quad \text{(initial condition at } t = 0). \end{aligned} \tag{44}$$

  This new problem (44) has a homogeneous boundary condition to which we can apply the method of reflection or odd extensions.

- Once we find $V(x,t)$, we can recover $v(x,t)$ by $v(x,t) = V(x,t) + h(t)$.
- The domain of independent variables (x,t) in this case is a quarter-plane with specified conditions on both of its half-lines. If they do not agree at the corner [i.e., if $\phi(0) \neq h(0)$], then the solution is discontinuous there (but continuous everywhere else). This is physically sensible. Think for instance, of suddenly at $t = 0$ sticking a hot iron bar into a cold bath.

Diffusion with a source on a half-line (Cont'd)

For the inhomogeneous Neumann problem on the half line

$$\begin{aligned} &w_t - kw_{xx} = f(x,t), \quad 0 < x < \infty, \quad t > 0, \\ &w_x(0,t) = h(t) \quad \text{(boundary condition at } x = 0), \\ &w(x,0) = \phi(x) \quad \text{(initial condition at } t = 0). \end{aligned} \tag{45}$$

- Let $W(x,t) = w(x,t) - xh(t)$, we can check that $W(x,t)$ satisfies

$$\begin{aligned} &W_t - kW_{xx} = f(x,t) - xh'(t), \quad 0 < x < \infty, \quad t > 0, \\ &W_x(0,t) = 0 \quad \text{(boundary condition at } x = 0), \\ &W(x,0) = \phi(x) - xh(0) \quad \text{(initial condition at } t = 0). \end{aligned} \tag{46}$$

  This new problem (46) has a homogeneous boundary condition to which we can apply the method of reflection or even extensions.
- $w(x,t)$ can be recovered by $w(x,t) = W(x,t) + xh(t)$ once $W(x,t)$ is found.

§3.4 Waves with a source

Waves with a source on the whole line

Consider the inhomogeneous wave equation

$$\begin{aligned} &u_{tt} - c^2 u_{xx} = f(x,t), \quad -\infty < x < \infty, \quad t > 0, \\ &u(x,0) = \phi(x), \quad u_t(x,0) = \psi(x), \end{aligned} \tag{47}$$

where $f(x)$, $\phi(x)$, and $\psi(x)$ are arbitrary functions.

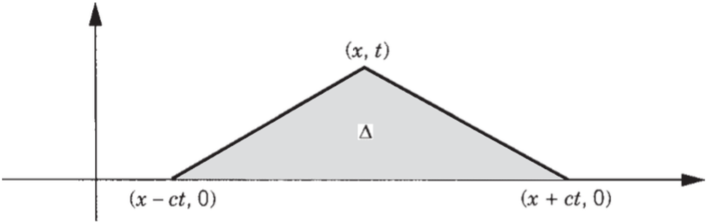
- Similar to the inhomogeneous heat equation, the right hand side of the equation, $f(x,t)$, is called the source term.
- In the case of the string vibrations this source term measures the external force (per unit mass) applied on the string, and the equation again arises from Newton's second law, in which one now has a nonzero external force.

§3.4 Waves with a source

Theorem 1. The unique solution of (47) is,

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy \\ + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds. \tag{48}$$

The iterated integral in (48) is over the characteristic triangle (see below).



Proof of Theorem 1. We integrate f over the past history triangle (characteristic triangle) Δ. Thus

$$\iint_{\Delta} f dxdt = \iint_{\Delta} (u_{tt} - c^2 u_{xx}) dxdt. \tag{49}$$

But Green's theorem says that

$$\iint_{\Delta} (P_x - Q_t) dxdt = \int_{\text{boundary}} Pdt + Qdx \tag{50}$$

for any functions P and Q, where the line integral on the boundary is taken counterclockwise.

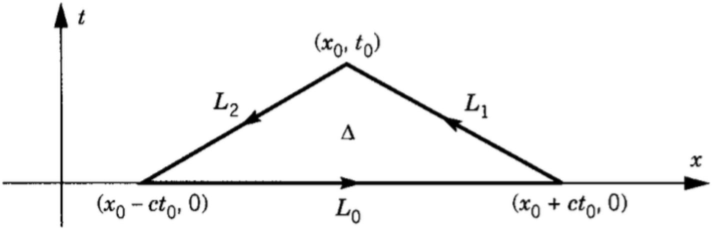
We assume that $P = -c^2 u_x$ and $Q = -u_t$, then

$$P_x - Q_t = -c^2 u_{xx} + u_{tt}. \tag{51}$$

Proof of Theorem 1 (Cont'd). Thus (49) can be rewritten as

$$\iint_{\Delta} f dx dt = \int_{L_0+L_1+L_2} (-c^2 u_x dt - u_t dx). \tag{52}$$

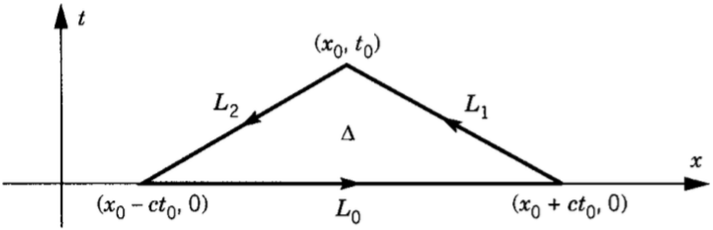
This is the sum of three line integrals over straight line segments (see Figure below). We evaluate each piece separately.



Proof of Theorem 1 (Cont'd). On L_0,

$t = 0$, $dt = 0$, and $u_t(x,t) = u_t(x,0) = \psi(x)$, so that

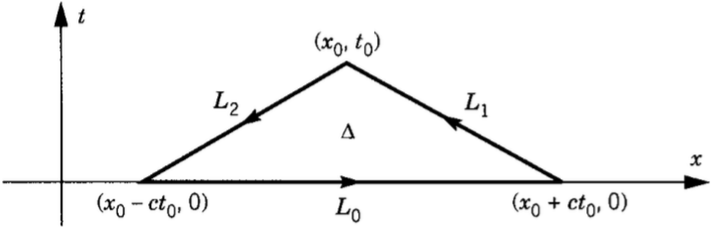
$$\int_{L_0} (-c^2 u_x dt - u_t dx) = \int_{L_0} (-u_t dx) = -\int_{x_0-ct_0}^{x_0+ct_0} \psi(x) dx. \tag{53}$$



Proof of Theorem 1 (Cont'd). On L_1,

$x + ct = x_0 + ct_0 \longrightarrow dx + cdt = 0$ or $dt = -dx/c$ or $dx = -cdt$. Thus

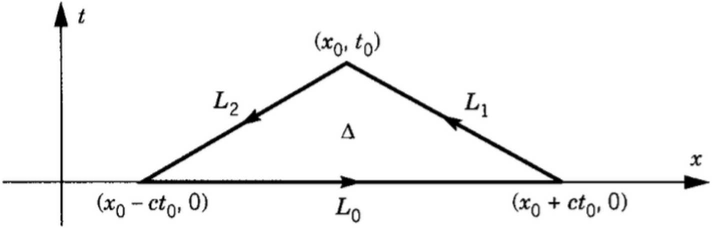
$$\begin{aligned}\int_{L_1} (-c^2 u_x dt - u_t dx) &= \int_{L_1} (cu_x dx + cu_t dt) = c \int_{L_1} du \\ &= cu(x_0, t_0) - cu(x_0 + ct_0, 0) \\ &= cu(x_0, t_0) - c\phi(x_0 + ct_0).\end{aligned} \tag{54}$$



Proof of Theorem 1 (Cont'd). On L_2,

$x - ct = x_0 - ct_0 \longrightarrow dx - cdt = 0$ or $dt = dx/c$ or $dx = cdt$. Thus

$$\begin{aligned} \int_{L_2} (-c^2 u_x dt - u_t dx) &= \int_{L_2} (-cu_x dx - cu_t dt) = -c \int_{L_2} du \\ &= -cu(x_0 - ct_0, 0) + cu(x_0, t_0) \\ &= cu(x_0, t_0) - c\phi(x_0 - ct_0). \end{aligned} \tag{55}$$



Proof of Theorem 1 (Cont'd).

(52) can be calculated by adding up (53), (54), and (55) as

$$\begin{aligned}\iint_\Delta f dx dt = &- \int_{x_0-ct_0}^{x_0+ct_0} \psi(x)dx + cu(x_0,t_0) - c\phi(x_0+ct_0) \\ &+ cu(x_0,t_0) - c\phi(x_0-ct_0).\end{aligned} \tag{56}$$

Thus

$$\begin{aligned}u(x_0,t_0) = &\frac{1}{2}\left[\phi(x_0-ct_0)+\phi(x_0+ct_0)\right] + \frac{1}{2c}\int_{x_0-ct_0}^{x_0+ct_0} \psi(x)dx \\ &+ \frac{1}{2c}\iint_\Delta f dx dt.\end{aligned} \tag{57}$$

(x_0, t_0) can be any point in the half plane $t \geq 0$. (57) completes the proof.

Example B (not in the textbook)

Solve the inhomogeneous wave initial value problem,

$$\begin{aligned} &u_{tt} - c^2 u_{xx} = \delta(t-2)\cos x, \quad -\infty < x < \infty, \quad t > 0, \\ &u(x,0) = u_t(x,0) = 0, \qquad t > 0. \end{aligned} \tag{58}$$

§3.4 Waves with a source

Waves with a source on a half-line

Consider the Dirichlet problem of a boundary source $h(t)$ on the half-line; that is,

$$\begin{aligned} &v_{tt} - c^2 v_{xx} = f(x,t), \quad 0 < x < \infty, \ \ t > 0, \\ &v(0,t) = h(t) \quad \text{(boundary condition at } x = 0\text{)}, \\ &v(x,0) = \phi(x), \quad v_t(x,0) = \psi(x), \quad \text{(initial condition at } t = 0\text{)}. \end{aligned} \tag{59}$$

- One can employ the subtraction method that we used for the heat equation to reduce the problem to one with zero Dirichlet data, and then use the reflection method to derive a solution formula for the reduced problem.
- Let
$$V(x,t) = v(x,t) - h(t). \tag{60}$$

Waves with a source on a half-line (Cont'd)

- We can check that $V(x,t) = v(x,t) - h(t)$ satisfies

$$
\begin{aligned}
&V_{tt} - c^2 V_{xx} = f(x,t) - h''(t), \quad 0 < x < \infty, \quad t > 0,\\
&V(0,t) = 0,\\
&V(x,0) = \phi(x) - h(0), \quad V_t(x,0) = \psi(x) - h'(0).
\end{aligned} \tag{61}
$$

  This new problem (61) has a homogeneous boundary condition to which we can apply the method of reflection or odd extensions.

- For simplicity, let $F(x,t) = f(x,t) - h''(t)$, $\Phi(x) = \phi(x) - h(0)$, and $\Psi(x) = \psi(x) - h'(0)$.
- Then (61) becomes

$$
\begin{aligned}
&V_{tt} - c^2 V_{xx} = F(x,t), \quad 0 < x < \infty, \quad t > 0,\\
&V(0,t) = 0,\\
&V(x,0) = \Phi(x), \quad V_t(x,0) = \Psi(x).
\end{aligned} \tag{62}
$$

Waves with a source on a half-line (Cont'd)

- Take the odd extensions:

$$\tilde{F}(x,t)=\begin{cases}F(x,t), & x>0,\\ -F(-x,t), & x<0,\\ 0, & x=0,\end{cases}\quad \tilde{\Phi}(x)=\begin{cases}\Phi(x), & x>0,\\ -\Phi(-x), & x<0,\\ 0, & x=0,\end{cases}$$
$$\tilde{\Psi}(x)=\begin{cases}\Psi(x), & x>0,\\ -\Psi(-x), & x<0,\\ 0, & x=0.\end{cases} \tag{63}$$

- Then we solve the extended wave equation on the whole real axis:

$$\begin{aligned}&U_{tt}-c^2U_{xx}=\tilde{F}(x,t), \quad -\infty<x<\infty, \ \ t>0,\\ &U(x,0)=\tilde{\Phi}(x), \quad U_t(x,0)=\tilde{\Psi}(x), \quad -\infty<x<\infty,\end{aligned} \tag{64}$$

Waves with a source on a half-line (Cont'd)

Using the solution formula (48), we have the solution to the extended wave equation (64) on the whole real axis; that is

$$\begin{aligned} U(x,t) = \frac{1}{2}[\tilde{\Phi}(x+ct) + \tilde{\Phi}(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \tilde{\Psi}(y)dy \\ + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} \tilde{F}(y,s)dyds. \end{aligned} \tag{65}$$

We only consider the solution $U(x,t) = V(x,t) = v(x,t) - h(t)$ on the positive axis $x > 0$. Two different cases should be discussed.

(I) If $x > ct$ $(x > 0)$, then

$$\begin{aligned} V(x,t) = \frac{1}{2}[\Phi(x+ct) + \Phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \Psi(y)dy \\ + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} F(y,s)dyds. \end{aligned} \tag{66}$$

Waves with a source on a half-line (Cont'd)

Since $V(x,t) = v(x,t) - h(t)$, (66) can be rewritten as

$$\begin{aligned} v(x,t) - h(t) &= \frac{1}{2}[\phi(x+ct) - h(0) + \phi(x-ct) - h(0)] \\ &\quad + \frac{1}{2c}\int_{x-ct}^{x+ct} [\psi(y) - h'(0)]\, dy + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} [f(y,s) - h''(s)]\, dyds \\ &= \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds \\ &\quad - h(0) - h'(0)t - \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} h''(s)dyds \\ &= \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds \\ &\quad - h(t). \end{aligned} \tag{67}$$

Waves with a source on a half-line (Cont'd)

(II) If $x < ct$ $(x > 0)$, then

$$
\begin{aligned}
V(x,t) &= \frac{1}{2}[\Phi(x+ct) + \tilde{\Phi}(x-ct)] + \frac{1}{2c}\int_0^{x+ct}\Psi(y)dy + \frac{1}{2c}\int_{x-ct}^{0}\tilde{\Psi}(y)dy \\
&\quad + \frac{1}{2c}\int_{t-\frac{x}{c}}^{t}\int_{x-c(t-s)}^{x+c(t-s)} F(y,s)dyds + \frac{1}{2c}\int_0^{t-\frac{x}{c}}\int_0^{x+c(t-s)} F(y,s)dyds \\
&\quad + \frac{1}{2c}\int_0^{t-\frac{x}{c}}\int_{x-c(t-s)}^{0} \tilde{F}(y,s)dyds \\
&= \frac{1}{2}[\Phi(x+ct) - \Phi(ct-x)] + \frac{1}{2c}\int_0^{x+ct}\Psi(y)dy - \frac{1}{2c}\int_{x-ct}^{0}\Psi(-y)dy \\
&\quad + \frac{1}{2c}\int_{t-\frac{x}{c}}^{t}\int_{x-c(t-s)}^{x+c(t-s)} F(y,s)dyds + \frac{1}{2c}\int_0^{t-\frac{x}{c}}\int_0^{x+c(t-s)} F(y,s)dyds \\
&\quad - \frac{1}{2c}\int_0^{t-\frac{x}{c}}\int_{x-c(t-s)}^{0} F(-y,s)dyds.
\end{aligned}
$$

Waves with a source on a half-line (Cont'd)

Make the change of variables in the blue integrals as y to $-y$, we get

$$\begin{aligned}
V(x,t) &= \frac{1}{2}[\Phi(x+ct)-\Phi(ct-x)] + \frac{1}{2c}\int_0^{x+ct}\Psi(y)dy + \frac{1}{2c}\int_{-x+ct}^{0}\Psi(y)dy \\
&\quad + \frac{1}{2c}\int_{t-\frac{x}{c}}^{t}\int_{x-c(t-s)}^{x+c(t-s)}F(y,s)dyds + \frac{1}{2c}\int_0^{t-\frac{x}{c}}\int_0^{x+c(t-s)}F(y,s)dyds \\
&\quad + \frac{1}{2c}\int_0^{t-\frac{x}{c}}\int_{-x+c(t-s)}^{0}F(y,s)dyds \\
&= \frac{1}{2}[\Phi(x+ct)-\Phi(ct-x)] + \frac{1}{2c}\int_{ct-x}^{x+ct}\Psi(y)dy \\
&\quad + \frac{1}{2c}\left\{\int_{t-\frac{x}{c}}^{t}\int_{x-c(t-s)}^{x+c(t-s)} + \int_0^{t-\frac{x}{c}}\int_{c(t-s)-x}^{x+c(t-s)}\right\}F(y,s)dyds
\end{aligned} \tag{69}$$

Waves with a source on a half-line (Cont'd)

Substituting $F(x,t) = f(x,t) - h''(t)$, $\Phi(x) = \phi(x) - h(0)$, and $\Psi(x) = \psi(x) - h'(0)$ into (69) gives

$$\begin{aligned}
V(x,t) &= \frac{1}{2}[\phi(x+ct) - h(0) - \phi(ct-x) + h(0)] + \frac{1}{2c}\int_{ct-x}^{x+ct}[\psi(y) - h'(0)]\,dy \\
&\quad + \frac{1}{2c}\left\{\int_{t-\frac{x}{c}}^{t}\int_{x-c(t-s)}^{x+c(t-s)} + \int_0^{t-\frac{x}{c}}\int_{c(t-s)-x}^{x+c(t-s)}\right\}[f(y,s) - h''(s)]\,dyds \\
&= \frac{1}{2}[\phi(x+ct) - \phi(ct-x)] + \frac{1}{2c}\int_{ct-x}^{x+ct}\psi(y)dy - \frac{1}{2c}\int_{ct-x}^{x+ct}h'(0)dy \\
&\quad + \frac{1}{2c}\left\{\int_{t-\frac{x}{c}}^{t}\int_{x-c(t-s)}^{x+c(t-s)} + \int_0^{t-\frac{x}{c}}\int_{c(t-s)-x}^{x+c(t-s)}\right\}f(y,s)dyds \\
&\quad - \frac{1}{2c}\left\{\int_{t-\frac{x}{c}}^{t}\int_{x-c(t-s)}^{x+c(t-s)} + \int_0^{t-\frac{x}{c}}\int_{c(t-s)-x}^{x+c(t-s)}\right\}h''(s)dyds
\end{aligned} \tag{70}$$

Waves with a source on a half-line (Cont'd)

(70) can be further simplified as

$$\begin{aligned} V(x,t) = {} & \frac{1}{2}[\phi(x+ct) - \phi(ct-x)] + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi(y)dy \\ & + \frac{1}{2c}\left\{\int_{t-\frac{x}{c}}^{t}\int_{x-c(t-s)}^{x+c(t-s)} + \int_{0}^{t-\frac{x}{c}}\int_{c(t-s)-x}^{x+c(t-s)}\right\} f(y,s)dyds \\ & -h(t) + h(t-\frac{x}{c}). \end{aligned} \tag{71}$$

Since $V(x,t) = v(x,t) - h(t)$, when $x < ct$ $(x > 0)$ we have

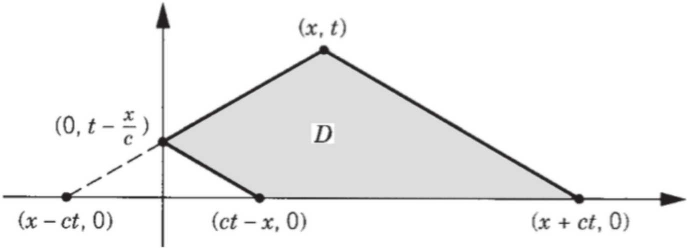
$$\begin{aligned} v(x,t) = {} & h(t-\frac{x}{c}) + \frac{1}{2}[\phi(x+ct) - \phi(ct-x)] + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi(y)dy \\ & + \frac{1}{2c}\left\{\int_{t-\frac{x}{c}}^{t}\int_{x-c(t-s)}^{x+c(t-s)} + \int_{0}^{t-\frac{x}{c}}\int_{c(t-s)-x}^{x+c(t-s)}\right\} f(y,s)dyds. \end{aligned} \tag{72}$$

Waves with a source on a half-line (Cont'd)

The sum of the two double integrals

$$\frac{1}{2c}\left\{\int_{t-\frac{x}{c}}^{t}\int_{x-c(t-s)}^{x+c(t-s)}+\int_{0}^{t-\frac{x}{c}}\int_{c(t-s)-x}^{x+c(t-s)}\right\}f(y,s)dyds. \tag{73}$$

is over the dependence domain of (x, t).



Waves with a source on a half-line (Cont'd)

Therefore, the solution to the Dirichlet problem (59) of a boundary source $h(t)$ on the half-line ($x > 0$) is

when $x > ct$

$$v(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dyds. \tag{74}$$

when $x < ct$

$$v(x,t) = h(t - \frac{x}{c}) + \frac{1}{2}[\phi(x+ct) - \phi(ct-x)] + \frac{1}{2c}\int_{ct-x}^{x+ct} \psi(y)dy + \frac{1}{2c}\left\{\int_{t-\frac{x}{c}}^{t}\int_{x-c(t-s)}^{x+c(t-s)} + \int_0^{t-\frac{x}{c}}\int_{c(t-s)-x}^{x+c(t-s)}\right\} f(y,s)dyds. \tag{75}$$

§3.4 Waves with a source

Waves with a source on a half-line

Consider the Neumann problem of a boundary source $g(t)$ on the half-line; that is,

$$\begin{aligned} &w_{tt} - c^2 w_{xx} = f(x,t), \quad 0 < x < \infty, \quad t > 0, \\ &w_x(0,t) = g(t) \quad \text{(boundary condition at } x = 0), \\ &w(x,0) = \phi(x), \quad w_t(x,0) = \psi(x), \quad \text{(initial condition at } t = 0). \end{aligned} \tag{76}$$

- We can also use the subtraction method to reduce the problem to one with zero Neumann data.
- Let

$$W(x,t) = w(x,t) - xg(t). \tag{77}$$

- Then use the reflection method to derive a solution formula for the reduced problem.