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Exercise 61 Thie exericse defines the connecting homomorphism δ_n in Theorem 17.2 (The Long Exact Sequence in Cohomology). Let $0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0$ be a SES of cochain complexes. Let $a \in H^n(Z)$ and $a = z + \operatorname{im} d_n$ where $z \in \ker d_{n+1} : Z^n \to Z^{n+1}$.

1. Show that there exists $y \in Y^n$ such that $\beta_n(y) = z$ and a unique $x \in \ker d_{n+2} \subseteq X^{n+1}$ such that $\alpha(x) = d(y)$.

2. Let $z + \operatorname{im} d_n = z' + \operatorname{im} d_n$, and y, y', x, x' such that $\beta(y) = z$, $\beta(y') = z'$, $\alpha(x) = d(y)$ and $\alpha(x') = d(y')$. Show that $x + \operatorname{im} d_{n+1} = x' + \operatorname{im} d_{n+1}$.

3. Conclude that we have a map $\delta_n : H^n(Z) \to H^{n+1}(X)$ defined by $\delta_n(z + \operatorname{im} d_n) = x + \operatorname{im} d_{n+1}$.

4. Prove that the connecting homomorphism δ_n is a group homomorphism.

Proof.

1. Since we have SES of cochain complex, so $0 \to X^n \xrightarrow{\alpha_n} Y^n \xrightarrow{\beta_n} Z^n \to 0$ is a SES. By definition $\operatorname{im} \beta_n = \ker 0 = Z$, implying that β_n is surjective. Thus given $z \in \ker d_{n+1} \subseteq Z^n$, there exists $y \in Y^n$ such that $\beta_n(y) = z$.

Send $y \in Y^n$ along two paths we have

$$\beta_{n+1}(d^Y_{n+1}(y)) = d^Z_{n+1}(\beta_n(y)) = d^Z_{n+1}(z) = 0$$

since $z \in \ker d_{n+1}$. So $d^Y_{n+1}(y) \in \ker \beta_{n+1} = \operatorname{im} \alpha_{n+1}$. Note α is injective, so there exists a unique $x \in X^{n+1}$ such that $\alpha_{n+1}(x) = d^Y_{n+1}(y)$. We now check that $x \in \ker d^X_{n+2}$. Send x along two paths we have

$$\alpha_{n+2}(d^X_{n+2}(x)) = d^Y_{n+2}(\alpha_{n+1}(x) = d^Y_{n+2}(d^Y_{n+1})(y)) = 0$$

Note α is injective, so $\ker \alpha$ is trivial, we thus have $d^X_{n+2}(x) = 0$, so $x \in \ker d^X_{n+2}$.

2. Since $z + \operatorname{im} d_n = z' + \operatorname{im} d_n$, so $z - z' \in \operatorname{im} d_n$. Let $w \in Z^{n-1}$ such that $d^Z_n(w) = z - z'$. Since β is surjective, there exists $y_w \in Y^{n-1}$ such that $\beta_{n-1}(y_w) = w$. Send y_w along two paths we have

$$\beta_n(d^Y_n(y_z)) = d^Z_n(\beta_{n-1}(y_w)) = d^Z_n(w) = z - z' = \beta_n(y) - \beta_n(y') = \beta_n(y - y')$$

Rearranging we see $\beta_n(d^Y_n(y_z) - (y - y')) = 0$, so $d^Y_n(y_z) - (y - y') \in \ker \beta_n = \operatorname{im} \alpha_n$. Let $x_z \in X^n$ such that

$$\alpha_n(x_z) = d^Y_n(y_z) - (y - y')$$

Lastly, send x_z along two paths we have

$$\begin{aligned}
\alpha_{n+1}(d^X_{n+1}(x_z)) &= d^Y_{n+1}(\alpha_n(x_z)) \\
&= d^Y_{n+1}(d^Y_n(y_z) - (y - y')) \\
&= 0 - d^Y_{n+1}(y) + d^Y_{n+1}(y') \\
&= -\alpha_{n+1}(x) + \alpha_{n+1}(x') \\
&= \alpha_{n+1}(x' - x)
\end{aligned}$$

Since α is injective, we have that $x' - x = d^X_{n+1}(x_z)$, implying that $x' - x \in \operatorname{im} d^X_{n+1}$, so $x + \operatorname{im} d_{n+1} = x' + \operatorname{im} d_{n+1}$.

3. As suggested, consider the map $\delta_n : H^n(Z) \to H^{n+1}(X)$ defined by $\delta_n(z + \operatorname{im} d_n) = x + \operatorname{im} d_{n+1}$ where x is given as follow:

- In part 1, we have shown that by fixing z, there correspond some y (need not be unique). Also, there exists a unique x such that $\alpha(x) = d(y)$. Let this x be such that $\delta_n(z + \operatorname{im} d_n) = x + \operatorname{im} d_{n+1}$

- The elements of $H^n(Z)$ do take the form $z + \operatorname{im} d_n$, and the elements of $H^{n+1}(X)$ do take the form $x + \operatorname{im} d_{n+1}$, by the definition of cohomology.

- We have to check well-definedness, which is already settled in part 2.

- Moreover, we claim that $\delta_n(z + \operatorname{im} d_n)$ is independent of the choice of the middleman y where $\beta_n(y) = z$. Fix z, if we take two different pre-image of z, say y and y' such that $\beta_n(y) = z = \beta_n(y')$ and $\alpha(x) = d(y)$, $\alpha(x') = d(y')$, since we have $z + \operatorname{im} d_n = z + \operatorname{im} d_n$, applying part 2 tells us that $x + \operatorname{im} d_{n+1} = x' + \operatorname{im} d_{n+1}$, therefore no problem occurs.

4. Lastly, we show that the defined map δ_n is a group homomorphism. Let $z_1 + \operatorname{im} d_n, z_2 + \operatorname{im} d_n \in H^n(Z)$. Futher suppose that $\beta(y_1) = z_1, \beta(y_2) = z_2, \alpha(x_1) = d(y_1), \alpha(x_2) = d(y_2)$, where $x_1, x_2 \in \ker d_{n+2}$. Thus by definition we have

$$\delta_n(z_1 + \operatorname{im} d_n) = x_1 + \operatorname{im} d_{n+1} \quad \text{and} \quad \delta_n(z_2 + \operatorname{im} d_n) = x_2 + \operatorname{im} d_{n+1}$$

Note $\beta(y_1 + y_2) = z_1 + z_2$, and $\alpha(x_1 + x_2) = d(y_1 + y_2)$. Also $x_1 + x_2 \in \ker d_{n+2}$ since kernel is a subgroup. By definition, we see

$$\begin{aligned} \delta_n((z_1 + \operatorname{im} d_n) + (z_2 + \operatorname{im} d_n)) = \delta_n((z_1 + z_2) + \operatorname{im} d_n) &= (x_1 + x_2) + \operatorname{im} d_{n+1} \\ &= (x_1 + \operatorname{im} d_{n+1}) + (x_2 + \operatorname{im} d_{n+1}) \\ &= \delta_n(z_1 + \operatorname{im} d_n) + \delta_n(z_2 + \operatorname{im} d_n) \end{aligned}$$

This shows that δ_n is indeed a group homomorphism. □