Tutorial #3

Date No.

17) Let $_RV$, $r \in Z(R)$, $\lambda_r : V \to V$ defined by $\lambda_r(v) = r \cdot v$. Show λ_r is R-mod homo

Prove false without assumption $r \in Z(R)$

$\boxed{\text{Check axioms}}$ $\lambda_r(v+w) = r \cdot (v+w) = r \cdot v + r \cdot w = \lambda_r(v) + \lambda_r(w)$

$\lambda_r(r'v) = r \cdot (r'v) = rr'v \overset{*}{=} r'rv = r'\lambda_r(v)$

we used $rr' = r'r$ since $r \in Z(R)$

☆ 18) $V = \mathbb{Z}/2\mathbb{Z}$, compute $End_{\mathbb{Z}}(V)$. Prove $Hom_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/\gcd(m,n)\mathbb{Z}$

$End_{\mathbb{Z}}(V) = Hom_{\mathbb{Z}}(V,V)$. By (Ex 16) $\mathbb{Z}$-mod homo are abelian group homo, the only homo between $V \to V$ is $\phi(1) = 0$ trivial map, or $\phi(1) = 1$ the identity map

Hence $End_{\mathbb{Z}}(V) = \{0, id\} \cong \mathbb{Z}/2\mathbb{Z}$

In general let $X = \mathbb{Z}/m\mathbb{Z}$, $Y = \mathbb{Z}/n\mathbb{Z}$ if $\phi \in Hom_{\mathbb{Z}}(X,Y)$ we can fully determine ϕ by its action on 1. Suppose $\phi(1) = k$, $0 \le k \le n-1$, then we know

$0 = \phi(0) = \phi(m) = \phi(m \cdot 1) = m\phi(1) = mk$ since ϕ is R-mod homo, hence $n \mid mk$

Now let $d = \gcd(m,n)$, then $\gcd(\frac{n}{d}, \frac{m}{d}) = 1$ together with $\frac{n}{d} \mid \frac{m}{d}k \Rightarrow \frac{n}{d} \mid k$

hence $k = \frac{n}{d}j$ $0 \le j \le d-1$.

Now we check ϕ is well defined, let $a \equiv b \pmod m$

$\phi(a) = ak = k(b + \ell m) = kb + k\ell m = \phi(b) + k\ell m$ $\boxed{\text{WTS: } k\ell m = 0}$

$k\ell m = n(\frac{m}{d})j\ell \equiv 0 \pmod n$ since $\frac{m}{d} \in \mathbb{Z}$

Hence $\Phi : Hom_{\mathbb{Z}}(X,Y) \to \mathbb{Z}/d\mathbb{Z}$ is defined by $\Phi(\phi) = k$ is $\phi(1) = k$

Inj: let $\phi_1, \phi_2 \in Hom_{\mathbb{Z}}(X,Y)$ s.t. $\Phi(\phi_1) = \Phi(\phi_2) = k \Rightarrow \phi_1(1) = \phi_2(1) = k \Rightarrow \phi_1 = \phi_2$

Srj: Let $x \in \mathbb{Z}/d\mathbb{Z}$ take $\phi \in Hom_{\mathbb{Z}}(X,Y)$ s.t. $\Phi(\phi) = x$, $\phi(1) = x$

19) Let R be comm.

i) Let $_RV$, Prove $Hom_R(R,V) \cong V$ as R-mods where $\alpha \to \alpha(1)$

By (P2.2) $Hom_R(R,V)$ is abelian group where $(r\psi)(m) = r\psi(m)$

☆ $\begin{cases} \text{Now define } \Psi : Hom_R(R,V) \to V \text{ where } \Psi(\alpha) = \alpha(1), \text{ then by (P2.1)} \\ \Psi(\phi + r\psi) = (\phi + r\psi)(1) = \phi(1) + r\psi(1) = \Psi(\phi) + r\Psi(\psi) \Rightarrow \Psi \text{ is R-mod homo} \end{cases}$

Then define $\Psi^{-1} : V \to Hom_R(R,V)$ where $\Psi^{-1}(v) = \phi_v$ where $\phi_v(r) = r \cdot v$

$\Psi^{-1}(v_1 + rv_2)(s) = \phi_{v_1 + rv_2}(s) = s \cdot (v_1 + rv_2) = sv_1 + srv_2 = sv_1 + rsv_2$

$= \phi_{v_1}(s) + r\phi_{v_2}(s) = \Psi^{-1}(v_1)(s) + r\Psi^{-1}(v_2)(s) = (\Psi^{-1}(v_1) + r\Psi^{-1}(v_2))(s)$

so Ψ^{-1} is also R-mod homo

Now $\Psi(\Psi^{-1}(v)) = \Psi(\phi_v) = \phi_v(1) = 1 \cdot v = v$ that is Ψ^{-1} is

$\Psi^{-1}(\Psi(\phi))(v) = \Psi^{-1}(\phi(1))(v) = \phi_{\phi(1)}(v) = v \cdot \phi(1) = \phi(v)$ really inverse of Ψ

hence this is an isomorphism

(☆ Actully if we drop comm then $End_R(R) \cong R^{op}$ try to prove it!)

19) ii) Prove $End_R(R) \cong R$ as Rings

from i) we have $End_R(R) \cong R$ as R-modules (Think of R regular module)

That is we have $\Phi : End_R(R) \to R$ defined by $\Phi(f) = f(1)$ is bijective, it preserves addition, and scalar multiplication in R. The only thing remaining to upgrade this to a ring iso is that Φ respects multiplication, that is $\Phi(f \circ g) = \Phi(f)\Phi(g)$

$\Phi(f\circ g) = (f \circ g)(1) = f(g(1))$, since f is R-mod homo $r := g(1)$ we have :

$$= f(r) = rf(1) = g(1)f(1) = f(1)g(1) = \Phi(f)\Phi(g) \blacksquare$$

☆ 20) $n \in \mathbb{Z}^+$. $\forall i \le n$ U_i is submod of R-mod V_i Prove

$(V_1 \times \dots \times V_n)/(U_1 \times \dots \times U_n) \cong (V_1/U_1) \times \dots \times (V_n/U_n)$ $\boxed{\text{This is screaming 1st iso theo}}$

Define $\phi : \prod V_i \to \prod (V_i/U_i)$ by $\phi(v_1, \dots, v_n) = (v_1 + U_1, \dots, v_n + U_n)$

Clearly ϕ is surj, now $0 = \phi(v_1, \dots, v_n) = (v_1 + U_1, \dots, v_n + U_n) \iff v_i \in U_i$

that is $\ker(\phi) = \prod U_i$, so if ϕ is R-mod homo then result follows from 1st iso theo.

Let $x, y \in \prod V_i$, $r \in R$ $\phi(x + ry) = \phi(x_1 + ry_1, \dots, x_n + ry_n)$

$= (x_1 + ry_1 + U_1, \dots, x_n + ry_n + U_n) = (x_1 + U_1, \dots, x_n + U_n) + (ry_1 + U_1, \dots, ry_n + U_n)$

$= \phi(x) + r\phi(y)$ as desired $\blacksquare$

21) I is left ideal of R, consider free R-mod R^n of rank n. Prove

$R^n / IR^n \cong \prod^n R/IR$

$\boxed{\text{The form looks similar to Ex 20}}$ R is an R-mod IR is a submodule of R

by $\boxed{\text{Ex 7}}$ all we need to show is that $IR^n = (IR)^n$ and apply $\boxed{\text{Ex 20}}$

$IR^n = \{\sum_{i=1}^m a_i \cdot r_i : a_i \in I, r_i \in R^n, m \in \mathbb{N}\}$ $IR = \{\sum^k b_i \cdot r_i : b_i \in I, r_i \in R, k \in \mathbb{N}\}$

$(\subseteq)$: Let $x \in IR^n$, then $x = \sum_{i=1}^m a_i \cdot r_i$ now $r_i = (r_{i,1}, \dots, r_{i,n})$ hence

$x = \sum_{i=1}^m (a_i r_{1,i}, \dots, a_i r_{n,i}) \in (IR)^n$ since $a_i r_{1,i} \in IR$

$(\supseteq)$: Let $x \in (IR)^n$ then $x = (x_1, \dots, x_n)$ where $x_i = \sum_{j=1}^n b_{i,j} r_{i,j}$, $b_{i,j} \in I$, $r_{i,j} \in R$

Let $e_{i,j} = (0, \dots, 0, r_{i,j}, 0, \dots 0) \in R^n$ where $r_{i,j}$ appears in i-th position then

$x = \sum_{i,j} b_{i,j} e_{i,j} \in IR^n$ $\blacksquare$

22) $A \subseteq V$, $_RV$

(Almost same as Ex 7)

i) Prove RA is submodule of V, $RA = \{\sum_{i=1}^n r_i a_i : a_i \in A, r_i \in R, m \in \mathbb{N}\}$

$\boxed{\text{Submodule Criterion}}$ $0 \in RA$ since take $m = 0$, let $x = \sum^n r_i a_i$ $y = \sum^m s_i b_i$

Then $x + y = \sum_{i=1}^{m+n} \beta_i \alpha_i$ $\alpha_i = \begin{cases} a_i, & i \le n \\ b_i & i > n \end{cases}$ $\beta_i = \begin{cases} r_i, & i \le n \\ s_i, & i > n \end{cases}$

$rx = r\sum r_i a_i = \sum (rr_i) a_i$ $\blacksquare$

22) ii) Prove $A \subseteq RA$

We require R to be unital then $\forall a \in A$ $a = 1_R \cdot a \in RA$

iii) Prove RA is smallest submodule of V containing A

Let U be a submodule of V containing A, now if $x \in RA$ x is finite R-linear combination of elements in A, but since U is submodule containing A it is closed under $+$ and scalar mult, hence $x \in U$. From this the intersection statement is clear ∎

23) N is submodule of M. Prove M is finitely generated if both N, M/N are

Suppose M is finitely gen by $\{a_1, \ldots, a_n\}$, M/N is finitely gen by $\{b_1 + N, \ldots, b_m + N\}$, $\boxed{\text{Claim: } M \text{ is finitely gen by } \{a_1, \ldots, a_n, b_1, \ldots, b_m\}}$

Let $m \in M$, then $m + N = \sum_i r_i (b_i + N)$ hence $m - \sum_i r_i b_i \in N$

$\Rightarrow m - \sum r_i b_i = \sum_j s_j a_j$, $s_j \in R$ $\Rightarrow$ $m = \sum_i r_i b_i + \sum s_j a_j$ ∎

★ 24) R is comm. Show $R^m \cong R^n$ iff $n = m$

($\Leftarrow$) If $m = n$ then $R^m \cong R^n$ is clear, take identity map

($\Rightarrow$) $\boxed{\text{Follow Hint}}$ Let I be a maximal ideal of R, now I prove intermediate claim

Let M be an R-mod, then we know M/IM is an R-mod with action $r \cdot (m + IM) = rm + IM$, but it can also be viewed as an R/I-module with action $(r + I) \cdot (m + IM) = rm + IM$.

This means $R^n / IR^n \cong (R/I)^n$ as an R/I-module. Now by maximality of I, $R/I = F$ is a field hence $(R/I)^n$ can be viewed as an n-dimensional vector space. Finally $R^n \cong R^n \Leftrightarrow R^m / IR^m \cong R^n / IR^n \Leftrightarrow (R/I)^m \cong (R/I)^n$ as vector spaces, which implies $m = n$ ∎