Solutions to the Exercises

Solution 1 We only check (iv). The rest are easy. Let $r, s \in \mathsf{Ann}(V)$ and $v \in V$. We have $(r+s)\cdot v = r\cdot v + s\cdot v = 0+0 = 0$ and $(rs)\cdot v = r\cdot(s\cdot v) = r\cdot 0 = 0$. So $\mathsf{Ann}(V)$ is a subring of R. Let $t \in R$. We have $(tr)\cdot v = t\cdot(r\cdot v) = t\cdot 0 = 0$ and $(rt)\cdot v = r\cdot(t\cdot v) = 0$. So $\mathsf{Ann}(V)$ is an ideal of R. We check the S-action is well-defined. Let $\pi(r) = s = \pi(r')$. Then $r - r' \in \ker \pi \subseteq \mathsf{Ann}(V)$. So

$$r\cdot v - r'\cdot v = (r-r')\cdot v = 0,$$

i.e., $r\cdot v = r'\cdot v$. This shows that the S-action is well-defined. Furthermore, let $\pi(r) = s$, $\pi(r') = s'$ and $v, w \in V$ (so that $\pi(rr') = ss'$ and $\pi(r+r') = s+s'$), we have

$$\begin{aligned} s*(v+w) &= r\cdot(v+w) = r\cdot v + r\cdot w = s*v + s*w, \\ (ss')*v &= (rr')\cdot v = r\cdot(r'\cdot v) = s*(s*v), \\ (s+s')*v &= (r+r')\cdot v = r\cdot v + r'\cdot v = s*v + s'*v. \end{aligned}$$

Solution 2 Comparing the definitions of module and vector space, we obtain that a vector space over F is an F-module. Therefore, subspaces of a vector space are submodules.

Solution 3 Left as exercise.

Solution 4 (i) The sum $U + W$ is nonempty because both U, W are nonempty. Let $u, u' \in U$, $w, w' \in W$ and $r \in R$. We have

$$(u+w) + r(u'+w') = (u+ru') + (w+rw') \in U + W.$$

(ii) Let $\{U_i\}_{i\in I}$ be a collection of submodules of V. Since $0 \in U_i$, we have $0 \in \bigcap U_i$. Let $r \in R$ and $x, y \in \bigcap U_i$. We have $x + ry \in U_i$ for each $i \in I$ and hence $x + ry \in \bigcap U_i$.

Solution 5 We claim that $\mathsf{Ann}(G) = 30\mathbb{Z}$ since $30 = \mathsf{lcm}(6, 10)$. Let $C_6 = \langle a\rangle$ and $C_{10} = \langle b\rangle$ and let $n \in \mathbb{Z}$ such that $n \in \mathsf{Ann}(G)$. We have $n\cdot(a, b) = (0, 0)$. So $na = 0$ and $nb = 0$. So $6 \mid n$ and $10 \mid n$. So $\mathsf{lcm}(6, 10) \mid n$, i.e., $30 \mid n$. So $\mathsf{Ann}(G) \subseteq 30\mathbb{Z}$. Conversely, $30m\cdot(ka, \ell b) = m(k30a + \ell 30b) = m(0, 0) = (0, 0)$. So $30\mathbb{Z} \subseteq \mathsf{Ann}(G)$.

Solution 6 Left as exercise.

Solution 7 We have IV is not empty because $0 \in IV$. For $\sum a_i v_i, \sum b_j w_j \in IV$ and $r \in R$, we have

$$\sum a_i \cdot v_i + r\sum b_j w_j = \sum a_i\cdot v_i + \sum (rb_j)w_j \in IV$$

because $rb_j \in I$.

Solution 8 Left as exercise.

Solution 9 (i) Let $f : R \to A$ be the ring homomorphism defining A as an R-algebra. So

$$\begin{aligned} r \cdot (ab) &= f(r)(ab) = (f(r)a)b = (r \cdot a)b \\ r \cdot (ab) &= f(r)(ab) = (f(r)a)b = (af(r))b = a(f(r)b) = a(r \cdot b). \end{aligned}$$

(ii) Let $r, s \in R$ and $a \in A$. We have

$$\begin{aligned} f(r+s) &= (r+s) \cdot 1_A = r \cdot 1_A + s \cdot 1_A = f(r) + f(s), \\ f(rs) &= (rs) \cdot 1_A = 1_A((rs) \cdot 1_A) = 1_A(r \cdot (s \cdot 1_A)) = (r \cdot 1_A)(s \cdot 1_A) = f(r)f(s). \\ f(1_R) &= 1_R \cdot 1_A = 1_A, \\ f(r)a &= (r \cdot 1_A)a = r \cdot (1_A a) = r \cdot (a 1_A) = a(r \cdot 1_A) = af(r) \end{aligned}$$

Solution 10 Left as exercise.

Solution 11 Left as exercise.

Solution 12 Left as exercise.

Solution 13 Left as exercise.

Solution 14 We only prove part (i). Let $f : V \to X$. The composition $\beta f : V \to Y$ is an R-module homomorphism. Furthermore, for any $v \in V$, we have

$$\begin{aligned} \beta_*(f+g)(v) &= \beta \circ (f+g)(v) = \beta((f+g)(v)) = \beta(f(v) + g(v)) = \beta(f(v)) + \beta(g(v)) \\ &= (\beta \circ f)(v) + (\beta \circ g)(v) = (\beta \circ f + \beta \circ g)(v) = (\beta_*(f) + \beta_*(g))(v). \end{aligned}$$

Solution 15 Left as exercise.

Solution 16 Left as exercise.

Solution 17 Let $v, w \in V$ and $s \in R$. We have

$$\begin{aligned} \lambda_r(v+w) &= r \cdot (v+w) = r \cdot v + r \cdot w = \lambda_r(v) + \lambda_r(w), \\ \lambda_r(s \cdot v) &= r \cdot (s \cdot v) = (rs) \cdot v = (sr) \cdot v = s \cdot (r \cdot v) = s \cdot \lambda_r(v). \end{aligned}$$

Without the assumption that $r \in Z(R)$, we would not have $rs = sr$. Consider the matrix ring $\mathrm{Mat}_2(\mathbb{R})$ and V is the natural module where V is the set of (2×1)-matrices over $\mathbb{R}$ where the ring acts by matrix multiplication. Let $r = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $s = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. If λ_r were an R-module homomorphism, we would have

$$\begin{aligned} \lambda_r(s \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}) &= s \cdot \lambda_r(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) \\ rsv &= srv \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned}$$

Solution 18 Since $\mathbb{Z}$-module homomorphisms are just abelian group homomorphisms, given $\phi : V \to V$, we have either $\phi(1) = 0$ or $\phi(1) = 1$. In the first case, ϕ is the trivial homomorphism. The other is the identity map.

Suppose now that $\phi \in \mathsf{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$. Since $\mathbb{Z}/m\mathbb{Z}$ is cyclic, ϕ is determined by its value on 1. Suppose that $\phi(1) = k \in \mathbb{Z}/n\mathbb{Z}$ where $0 \leq k \leq n-1$. Let $d = \gcd(m, n)$. We claim that $k = \frac{n}{d}j$ where $j = 0, \ldots, d-1$. We have

$$0 = \phi(0) = \phi(m) = m\phi(1) = mk.$$

Therefore, we have $n \mid mk$ and hence $\frac{n}{d} \mid \frac{m}{d}k$. Since $\gcd(\frac{n}{d}, \frac{m}{d}) = 1$, we have $\frac{n}{d} \mid k$. So $k = \frac{n}{d}j$ where $j = 0, \ldots, d-1$ because $k \in [0, n-1]$. Given such k, we only need to check that ϕ is well-defined. Suppose that $a \equiv b(\mod m)$. We have $n \mid \frac{nm}{d}$ and hence

$$\phi(a) = ka = \frac{n}{d}ja = \frac{n}{d}j(b + \ell m) = kb + \frac{nm}{d}j = kb = \phi(b).$$

We define

$$\Phi : \mathsf{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \to \mathbb{Z}/d\mathbb{Z}$$

by $\Phi(\phi) = k$ if $\phi(1) = k$. This is a bijection. We leave it to you to check that Φ is a group homomorphism.

Solution 19 (i) We have seen that $\mathsf{Hom}_R(R, V)$ is an abelian group. The action of R on $\mathsf{Hom}_R(R, V)$ is given by $(r * \phi)(s) = \phi(sr)$. We define $\Phi : \mathsf{Hom}_R(R, V) \to V$ by $\Phi(\alpha) := \alpha(1)$ and $\Psi : V \to \mathsf{Hom}_R(R, V)$ by $\Psi(v) := r \mapsto r \cdot v$. For $r, s \in R$, $v \in V$ and $\alpha \in \mathsf{Hom}_R(R, V)$, we have

$$\begin{aligned}
\Phi(r * \alpha) &= (r * \alpha)(1) = \alpha(r) = r \cdot \alpha(1) = r \cdot \Phi(\alpha), \\
\Psi(v)(rs) &= (rs) \cdot v = r \cdot (s \cdot v) = r\dot{\Psi}(v)(s), \\
\Psi(r \cdot v)(s) &= s \cdot (r \cdot v) = (sr) \cdot v = \Psi(v)(sr) = (r * \Psi(v))(s).
\end{aligned}$$

These show that Ψ is well-defined and both Φ and Ψ are R-module homomorphism. Furthermore, we have

$$\begin{aligned}
(\Psi \circ \Phi)(\alpha)(r) &= \Psi(\alpha(1))(r) = r \cdot \alpha(1) = \alpha(r), \\
(\Phi \circ \Psi)(v) &= (\Psi(v))(1) = 1 \cdot v = v.
\end{aligned}$$

So Ψ is the inverse of Φ.

(ii) Let $\Phi : \mathsf{End}_R(R) \to R^{\mathsf{op}}$ be defined by $\Phi(\alpha) = \alpha(1)$. We denote $*$ for the product in R^{op}. We first check that Φ is surjective. For $r \in R$, we claim that $\alpha(s) := sr$ is an R-module endomorphism of R. For $t \in R$,

$$\alpha(ts) = (ts)r = t(sr) = t\alpha(s).$$

So $\Phi(\alpha) = \alpha(1) = r$ and hence Φ is surjective. For $\alpha, \beta \in \mathsf{End}_R(R)$, we have

$$\begin{aligned}
\Phi(\alpha + \beta) &= (\alpha + \beta)(1) = \alpha(1) + \beta(1) = \Phi(\alpha) + \Phi(\beta), \\
\Phi(\alpha\beta) &= (\alpha\beta)(1) = \alpha(\beta(1)) = \beta(1)\alpha(1) = \alpha(1) * \beta(1) = \Phi(\alpha) * \Phi(\beta), \\
\Phi(\mathrm{Id}) &= \mathrm{Id}(1) = 1.
\end{aligned}$$

So Φ is a ring homomorphism. Let $\alpha \in \ker \Phi$. We have $\alpha(1) = 0$. So $\alpha(r) = r\alpha(1) = 0$. So $\alpha = 0$. This shows that Φ is injective.

Solution 20 Define $\phi : V_1 \times \cdots \times V_n \to (V_1/U_1) \times \cdots \times (V_n/U_n)$ by $\phi(v_1, \ldots, v_n) = (v_1 + U_1, \ldots, v_n + U_n)$. Clearly ϕ is surjective and

$$\ker \phi = \{(v_1, \ldots, v_n) : v_i \in U_i\} = U_1 \times \cdots \times U_n.$$

We only need to check that ϕ is an R-module homomorphism and the result then follows using the first isomorphism theorem. For $r \in R$, we have

$$\begin{aligned}\phi(r(v_1, \ldots, v_n)) &= \phi(rv_1, \ldots, rv_n)\\ &= (rv_1 + U_1, \ldots, rv_n + U_n)\\ &= r(v_1 + U_1, \ldots, v_n + U_n)\\ &= r\phi(v_1, \ldots, v_n).\end{aligned}$$

Solution 21 Let $U_i = IR$. We claim that $U := U_1 \times \cdots \times U_n = IR^n$ as submodule of R^n. Notice that

$$IR^n = \{a_1 \cdot v_1 + \cdots + a_m \cdot v_m : a_1, \ldots, a_m \in I,\ v_1, \ldots, v_m \in R^n,\ m \in \mathbb{N}\}.$$

By Exercise 7, IR is a submodule of R and hence U is a submodule of R^n. For $u \in U$, we have $u = (u_1, \ldots, u_n)$ where $u_i \in IR$. So $u_i = \sum a_{i,j} r_{i,j}$ (finite sum) with $a_{i,j} \in I$ and $r_{i,j}$. So

$$u = (\sum a_{1,j} r_{1,j}, \ldots, \sum a_{n,j} r_{n,j}) = \sum a_{i,j} e_{i,j} \in IR^n$$

where $e_{i,j}$ denotes the element $(0, \ldots, 0, r_{i,j}, 0, \ldots, 0)$ in R^n where the nonzero component occurs at the ith position. Therefore, $U \subseteq IR^n$. Conversely, for $a_1 \cdot v_1 + \cdots + a_m \cdot v_m \in IR^n$, we have

$$a_1 \cdot v_1 + \cdots + a_m \cdot v_m = \sum_{j=1}^{m} (a_j v_{1,j}, \ldots, a_j v_{n,j}) \in U$$

as $a_j v_{i,j} \in U_i$ where $v_j = (v_{1,j}, \ldots, v_{n,j})$. So $IR^n \subseteq U$. Now the desired isomorphism follows using Exercise 20.

Solution 22 Left as exercise. For part (ii), you need our assumption that R is unital.

Solution 23 Suppose that M/N and N are generated by the finite sets A and B respectively. Let $A = \{a_1 + N, \ldots, a_m + N\}$ and $B = \{b_1, \ldots, b_n\}$. We claim that M be generated by $\{a_1, \ldots, a_m, b_1, \ldots, b_n\}$. Let $m \in M$. Then $m + N = \sum r_i(a_i + N)$ for some $r_i \in R$. Since $m - \sum r_i a_i \in N$, we have $m - \sum r_i a_i = \sum s_j b_j$ for some $s_j \in R$. So $m = \sum r_i a_i + \sum s_j b_j$.

Solution 24 Let I be a maximal ideal of R. Since I annihilates the module R^n, we can view $R^n/IR^n \cong (R/I)^n$ as R/I-module. Since $F := R/I$ is a field, it is a vector space over F. The vector space $(R/I)^n$ has dimension n. Therefore, if $R^m \cong R^n$, then $(R/I)^m \cong (R/I)^n$ as vector spaces over F and hence $m = n$ by linear algebra. The converse is clear.

Solution 25 Part (i) is easy. For part (ii),

$$\begin{aligned}
\alpha_1\beta_1(a_1,a_2,a_3,\ldots) &= \alpha_1(a_1,0,a_2,0,\ldots) = (a_1,a_2,\ldots),\\
\alpha_2\beta_2(a_1,a_2,a_3,\ldots) &= \alpha_2(0,a_1,0,a_2,\ldots) = (a_1,a_2,\ldots),\\
\alpha_1\beta_2(a_1,a_2,a_3,\ldots) &= \alpha_1(0,a_1,0,a_2,\ldots) = (0,0,\ldots),\\
\alpha_2\beta_1(a_1,a_2,a_3,\ldots) &= \alpha_2(a_1,0,a_2,0,\ldots) = (0,0,\ldots),\\
(\beta_1\alpha_1+\beta_2\alpha_2)(a_1,a_2,a_3,\ldots) &= (\beta_1\alpha_1)(a_1,a_2,a_3,\ldots) + (\beta_2\alpha_2)(a_1,a_2,a_3,\ldots)\\
&= \beta_1(a_1,a_3,a_5,\ldots) + \beta_2(a_2,a_4,a_6,\ldots)\\
&= (a_1,0,a_3,0,a_5,\ldots) + (0,a_2,0,a_4,\ldots)\\
&= (a_1,a_2,a_3,a_4,\ldots).
\end{aligned}$$

For any $x \in R$, we have $x = x\cdot 1 = (x\beta_1)\alpha_1 + (x\beta_2)\alpha_2$. Suppose that $x = x_1\alpha_1 + x_2\alpha_2$. We have

$$\begin{aligned}
x\beta_2 &= (x_1\alpha_1 + x_2\alpha_2)\beta_2 = x_1(\alpha_1\beta_2) + x_2(\alpha_2\beta_2) = 0 + x_2 = x_2,\\
x\beta_1 &= (x_1\alpha_1 + x_2\alpha_2)\beta_1 = x_1(\alpha_1\beta_1) + x_2(\alpha_2\beta_1) = x_1 + 0 = x_1.
\end{aligned}$$

Therefore, $\{\alpha_1,\alpha_2\}$ is a free basis for ${}_RR$.

For part (iii), define $\phi : R \to R^2$ by $\phi(x) = (x\beta_1, x\beta_2)$. For $\xi \in R$, we have

$$\begin{aligned}
\phi(x+y) &= ((x+y)\beta_1,(x+y)\beta_2) = (x\beta_1 + y\beta_1, x\beta_2 + y\beta_2) = (x\beta_1,x\beta_2) + (y\beta_1,y\beta_2)\\
&= \phi(x) + \phi(y),\\
\phi(\xi x) &= (\xi x\beta_1, \xi x\beta_2) = \xi(x\beta_1,x\beta_2) = \xi\phi(x).
\end{aligned}$$

So ϕ is an R-module homomorphism. For $(y,z) \in R^2$, we have

$$\begin{aligned}
\phi(y\alpha_1 + z\alpha_2) &= ((y\alpha_1 + z\alpha_2)\beta_1, (y\alpha_1 + z\alpha_2)\beta_2)\\
&= (y(\alpha_1\beta_1) + z(\alpha_2\beta_1), y(\alpha_1\beta_2) + z(\alpha_2\beta_2))\\
&= (y+0, 0+z) = (y,z).
\end{aligned}$$

So ϕ is surjective. Suppose that $\phi(x) = 0$. We have $x\beta_1 = 0 = x\beta_2$. So

$$x = x\cdot 1 = (x\beta_1)\alpha_1 + (x\beta_2)\alpha_2 = 0 + 0 = 0.$$

So ϕ is injective.

Solution 26 Left as exercise.

Solution 27 Let $\phi : A \to B$ be a bijective function. By the universal property, we have

$$\begin{tikzcd}
A \arrow[r, hook, "\iota"] \arrow[d, "\phi"] & F(A) \arrow[d, shift right, "\Psi"'] \\
B \arrow[r, hook, "\jmath"] & F(B) \arrow[u, shift right, "\Phi"']
\end{tikzcd}$$

where $\Psi\iota = \jmath\phi$ and $\Phi\jmath = \iota\phi^{-1}$. We also have $\Phi\Psi\iota = \Phi\jmath\phi = \iota\phi^{-1}\phi = \iota$. Also, $\Psi\Phi\jmath = \jmath$. By the uniqueness, $\Phi\Psi = 1_{F(A)}$ and $\Psi\Phi = 1_{F(B)}$. So $F(A) \cong F(B)$.

Solution 28 Let M be a finite abelian group. By the classification of the finite abelian group, we may assume that $M = C_{n_1} \times \cdots \times C_{n_k}$ for some positive integers $n_1, \dots, n_k$. Let $\ell = \mathrm{lcm}(n_1, \dots, n_k)$. Then $\ell \cdot m = 0$. For the counterexample, let M be the infinite product of C_2, i.e., $M = C_2 \times C_2 \times \cdots$. For any $(m_i)_{i\in\mathbb{N}}$, we have $2 \cdot (m_i) = (2 \cdot m_i) = 0$.

Solution 29 Notice that Rm is a nonzero submodule of M since R contains an identity. So $M = Rm$ by definition.

Solution 30 They are the cyclic groups of prime orders. Let M be an irreducible $\mathbb{Z}$-module. The subgroups of M are the submodules of M. Suppose that M is infinite. Let $0 \neq m \in M$. Then $M = \mathbb{Z}m$. Since M is infinite, there is no nonzero integer k such that $km = 0$. But $\langle 2m \rangle$ is a submodule of M. So $m = k(2m)$ for some $k \in \mathbb{N}$, i.e., $(2k-1)m = 0$. This is a contradiction. So M must be finite. By the classification of finite abelian group, M is a direct product of cyclic groups. Since every copy of the direct product is a subgroup of M and M is irreducible, there is only one copy, i.e., $M \cong C_n$. The subgroups of C_n correspond to divisors of n. So n must be prime.

Solution 31 We first check that eM is a submodule of M. Clearly, $e0 = 0 \in eM$. So $eM \neq \emptyset$. For $em, em' \in eM$ and $r \in R$, we have

$$em + r(em') = em + (re)m' = em + (erm') = e(m + rm') \in eM.$$

Similarly, $(1-e)M$ is a submodule of M. For $m \in M$, we have $m = em + (1-e)m$. So $M = eM + (1-e)M$. Suppose that $x \in eM \cap (1-e)M$. Then $(1-e)m = x = em'$. So

$$\begin{aligned} e(1-e)m &= e^2m' \\ (e-e)m &= em' \\ 0 &= em' = x. \end{aligned}$$

So $eM \cap (1-e)M = \{0\}$. As such, $M = eM \oplus (1-e)M$.

Solution 32 Left as exercise.

Solution 33 Left as exercise.

Solution 34 We first verify that $\phi * r$ belongs in $\mathrm{Hom}_S(Y, Z)$. For any $s \in S$ and $y \in Y$, we have

$$(\phi * r)(ys) = \phi(r(ys)) = \phi((ry)s) = \phi(ry)s = (\phi * r)(y)s.$$

So $\phi * r \in \mathrm{Hom}_S(Y, Z)$. For $r, r' \in R$, we have

$$\begin{aligned} (\phi * (r + r'))(y) &= \phi((r + r')y) = \phi(ry + r'y) = \phi(ry) + \phi(r'y) = (\phi * r)(y) + (\phi * r')(y) \\ &= (\phi * r + \phi * r')(y). \end{aligned}$$

The other axiom is left as an exercise to check.

Solution 35 In our examples, we have proved that $R \otimes_R N \cong N$ for any R-module N. So $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q} \cong \mathbb{Q}$ and $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \cong \mathbb{C}$. Define $\beta : \mathbb{Q} \to \mathbb{Q}$ by $\beta(x) = x$. This is clearly an $\mathbb{Z}$-module homomorphism. By Theorem 10.8, there exists an $\mathbb{Q}$-module homomorphism $\Phi : \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$ such that $\Phi(1 \otimes x) = x$. In particular, Φ is surjective. For any $\frac{r}{s} \otimes x, \frac{a}{b} \otimes y \in \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$, we have

$$\frac{r}{s} \otimes x + \frac{a}{b} \otimes y = \frac{rb}{sb} \otimes x + \frac{sa}{sb} \otimes y = \frac{1}{sb} \otimes (rbx) + \frac{1}{sb} \otimes (say) = \frac{1}{sb} \otimes (rbx + say).$$

So every element in $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ can be expressed as a simple tensor. As such, if $x \otimes y \in \ker \Phi$, we have

$$0 = \Phi(x \otimes y) = \Phi(x(1 \otimes y)) = x\Phi(1 \otimes y) = xy.$$

So either $x = 0$ or $y = 0$. As such, $x \otimes y = 0$. This shows that Φ is injective. So $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$ as $\mathbb{Q}$-modules.

On the other hand, we show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ surjects onto $\mathbb{C} \oplus \mathbb{C}$ as $\mathbb{C}$-modules. Similar as before, let $L = \mathbb{C} \oplus \mathbb{C}$ and $\beta : \mathbb{C} \to \mathbb{C} \oplus \mathbb{C}$ defined by $\beta(a + ib) = (a, ib)$ where $a, b \in \mathbb{R}$. For $r \in \mathbb{R}$ and $a + ib \in \mathbb{C}$, we have

$$\beta(r(a + ib)) = (ra, rib) = r(a, ib) = r\beta(a + ib).$$

So β is an $\mathbb{R}$-module homomorphism. By Theorem 10.8, there exists an $\mathbb{C}$-module homomorphism $\Phi : \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C} \oplus \mathbb{C}$ such that $\Phi\iota = \beta$. In particular, $\Phi(1 \otimes (a + ib)) = \Phi\iota(a + ib) = \beta(a + ib) = (a, ib)$. So $(1, 0), (0, i) \in \operatorname{im} \Phi$. Since Φ is an $\mathbb{C}$-module homomorphism,

$$\Phi(i \otimes i) = \Phi(i(1 \otimes i)) = i(0, i) = (0, -1),$$
$$\Phi(i \otimes 1) = \Phi(i(1 \otimes 1)) = i(1, 0) = (i, 0).$$

So Φ is surjective. This shows that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ cannot be isomorphic to $\mathbb{C}$ as $\mathbb{C}$-modules.

Solution 36 (i): Suppose that $\overline{a} = \overline{a'}$, i.e., $a - a' = km$ for some $k \in \mathbb{Z}$. We have $ad + mD = (a' + km)d + mD = a'd + mD$. So β is well-defined. Also, for example,

$$\beta(dn, \overline{a}) = a(dn) + mD = (na)d + mD = \beta(d, \overline{na}).$$

So β is $\mathbb{Z}$-balanced.

(ii): By Theorem 10.10, we have a group homomorphism $\Phi : D \otimes_{\mathbb{Z}} (\mathbb{Z}/m\mathbb{Z}) \to D/mD$ such that $\Phi(d \otimes \overline{a}) = ad + mD$. Taking $a = 1$, we have that Φ is surjective. Furthermore,

$$d \otimes \overline{a} = d \otimes a\overline{1} = (da) \otimes \overline{1}.$$

Let $d \otimes 1 \in \ker \Phi$, we have $\Phi(d \otimes \overline{1}) = d + mD = mD$. Then $d \in mD$, i.e., $d = md'$. So $d \otimes 1 = (d'm) \otimes 1 = d' \otimes \overline{m} = d' \otimes 0 = 0$. So Φ is injective.

Solution 37 Left as exercise.

Solution 38 Let $\beta : M \times N \to N \otimes_R M$ be given by $\beta(m, n) = n \otimes m$. This is R-bilinear because

$$\begin{aligned}\beta(rm + r'm', n) &= n \otimes (rm + r'm') = n \otimes (rm) + n \otimes (r'm') = (nr) \otimes m + (nr') \otimes m' \\ &= (rn) \otimes m + (r'n) \otimes m' = r(n \otimes m) + r'(n \otimes m') = r\beta(m, n) + r'\beta(m', n).\end{aligned}$$

Similarly, we can prove that $\beta(m, rn+r'n') = r\beta(m, n)+r'\beta(m, n')$. By the universal property, we have an R-module homomorphism $\phi : M \otimes_R N \to N \otimes_R M$ such that $\phi(m \otimes n) = n \otimes m$. Similarly, we have an R-module homomorphism $\psi : N \otimes_R M \to M \otimes_R N$ such that $\psi(n \otimes m) = m \otimes n$. By the uniqueness of the universal property,

$$\begin{tikzcd} M \times N \arrow[r, "\iota"] \arrow[dr, "\iota"'] & M \otimes_R N \arrow[d, equal] \\ & M \otimes_R N \end{tikzcd}$$

Since $\psi\phi(m \otimes n) = m \otimes n$, we have $\psi\phi = 1$. Similarly, $\phi\psi = 1$. So ϕ is an isomorphism.

Solution 39 Let $\beta : M \times (\bigoplus_{i\in I} N_i) \to \bigoplus_{i\in I}(M \otimes_R N_i)$ be defined as

$$\beta(m, (n_i)) = (m \otimes n_i)_{i\in I}.$$

We have β is R-balanced. For example,

$$\beta(mr, (n_i)) = ((mr) \otimes n_i)_{i\in I} = (m \otimes (rn_i))_{i\in I} = \beta(m, (rn_i)_{i\in I}) = \beta(m, r(n_i)_{i\in I}).$$

As such, we have a group homomorphism $\Phi : M \otimes_R (\bigoplus_{i\in I} N_i) \to \bigoplus_{i\in I}(M \otimes_R N_i)$ such that $\Phi(m, (n_i)_{i\in I}) = (m \otimes n_i)_{i\in I}$. Similarly, for each $i \in I$, we define $\beta_i : M \times_R N_i \to M \otimes_R (\bigoplus_{i\in I} N_i)$ by $\beta_i(m_i, n_i) = m_i \otimes (n_j^{(i)})_{j\in I}$ where $n_j = 0$ if $j \neq i$. Therefore, there exist group homomorphisms $\Psi_i : M \otimes_R N_i \to M \otimes_R (\bigoplus_{i\in I} N_i)$ such that $\Psi_i(m \otimes n_i) = m_i \otimes (n_j^{(i)})$. Now let $\Psi = \oplus\Psi_i : \bigoplus_{i\in I}(M \otimes_R N_i) \to M \otimes_R (\bigoplus_{i\in I} N_i)$ so that $\Psi(m_i \otimes n_i)_{i\in I} = \sum m \otimes (n_j^{(i)})_{i\in I}$. (This is where the proof fails if we replace direct sum with direct product because the sum $\sum m \otimes (n_j^{(i)})_{i\in I}$ could end up with infinite sum and it does not make sense in $M \otimes_R (\prod_{i\in I} N_i)$). It is now routine to check that Ψ and Φ are inverses of each other.

Solution 40 *We have $\mathbb{Q} \otimes_{\mathbb{Z}} N_i = 0$ because

$$x \otimes y = \frac{x}{2^i} \cdot 2^i \otimes y = \frac{x}{2^i} \otimes 2^i \cdot y = \frac{x}{2^i} \otimes 0 = 0.$$

Therefore, $\prod_{i\in I}(\mathbb{Q} \otimes N_i) = 0$. On the other hand, we claim that $\mathbb{Q} \otimes_{\mathbb{Z}} \prod_{i\in I} N_i \neq 0$.

For this, we prove the following statement: Let N be a left nonzero torsion-free $\mathbb{Z}$-module. Then $\mathbb{Q} \otimes_{\mathbb{Z}} N$ is nonzero.

We first define an equivalence relation on $\mathbb{Q} \times N$. Let $(\frac{r}{s}, x) \sim (\frac{a}{b}, y)$ if and only if $(rb)x = (as)y$. Suppose that $\frac{r'}{s'} = \frac{r}{s}$ and $\frac{a'}{b'} = \frac{a}{b}$. We have

$$(sb)(r'b'x - s'a'y) = (rs'bb'x - ss'ab'y) = 0.$$

Since N is torsion-free and $\mathbb{Z} \ni sb \neq 0$, we have $r'b'x = s'a'y$. So the relation $\sim$ is well-defined. Clearly, $(\frac{r}{s}, x) \sim (\frac{r}{s}, x)$ and, $(\frac{r}{s}, x) \sim (\frac{a}{b}, y)$ if and only if $(\frac{a}{b}, y) \sim (\frac{r}{s}, x)$. Furthermore, suppose that $(\frac{r}{s}, x) \sim (\frac{a}{b}, y)$ and $(\frac{a}{b}, y) \sim (\frac{c}{d}, z)$. We have

$$b(rdx - csz) = dasy - sady = 0.$$

Again, since N is torsion-free, we have $rdx = csz$. So $\sim$ is an equivalence relation.

Let $\widetilde{N} = \mathbb{Q} \times N/\sim$ be the set of equivalence classes and we write $[q, x]$ for the equivalence class containing (q, x). It is an $\mathbb{Q}$-module (vector space) where

$$[\frac{a}{b}, x] + [\frac{r}{s}, y] = [\frac{1}{bs}, asx + bry],$$
$$\frac{a}{b}[\frac{r}{s}, y] = [\frac{ar}{bs}, y].$$

The zero element is $[0, 0]$. We claim that $[1, x] \neq [0, 0]$ for $x \neq 0$. If not, we have $x = 1 \cdot 0 = 0$, a contradiction. So $\widetilde{N} \neq 0$ because $N \neq 0$.

We define $\beta : \mathbb{Q} \times N \to \widetilde{N}$ by $\beta(q, x) = [q, x]$. The map β is well-defined and surjective. Next, we shall show that β is $\mathbb{Z}$-balanced. For $n \in \mathbb{Z}$, $\frac{a}{b}, \frac{a'}{b'} \in \mathbb{Q}$ and $x, x \in N$, we have

$$\begin{aligned}
\beta(\frac{a}{b}n, x) &= [\frac{a}{b}n, x] = [\frac{a}{b}, nx] = \beta(\frac{a}{b}, nx), \\
\beta(\frac{a}{b} + \frac{a'}{b'}, x) &= \beta(\frac{ab' + a'b}{bb'}, x) = [\frac{ab' + a'b}{bb'}, x] \\
&= [\frac{1}{bb'}, ab'x + a'bx] = [\frac{a}{b}, x] + [\frac{a'}{b'}, x] = \beta(\frac{a}{b}, x) + \beta(\frac{a'}{b'}, x) \\
\beta(\frac{a}{b}, x + x') &= [\frac{a}{b}, x + x'] = [\frac{ab}{bb}, x + x'] = [\frac{1}{bb}, abx' + abx] \\
&= [\frac{a}{b}, x] + [\frac{a}{b}, x']\beta(\frac{a}{b}, x) + \beta(\frac{a}{b}, x').
\end{aligned}$$

By Theorem 10.10, there exists a group homomorphism $\Phi : \mathbb{Q} \otimes_{\mathbb{Z}} N \to \widetilde{N}$ such that $\beta = \Phi \circ \iota$ where $\iota : \mathbb{Q} \times N \to \mathbb{Q} \otimes_{\mathbb{Z}} N$. Since β is surjective and $\widetilde{N} \neq 0$, we have Φ is surjective and hence $\mathbb{Q} \otimes_{\mathbb{Z}} N \neq 0$.

For arbitrary $\mathbb{Z}$-module, let

$$N' = \{x \in N : nx = 0 \text{ for some } 0 \neq n \in \mathbb{Z}\}.$$

This is a submodule (the torsion submodule) of N. The quotient N/N' is a torsion-free $\mathbb{Z}$-module because, if $n(x + N') = N'$ for some $n \neq 0$, then $nx \in N'$ and hence $x \in N'$. Define the map $\gamma : \mathbb{Q} \times N \to \mathbb{Q} \otimes_{\mathbb{Z}} N'$ by $\gamma(q, x) = q \otimes (x + N')$. We claim that γ is a $\mathbb{Z}$-balanced.

$$\begin{aligned}
\gamma(q, x + x') &= q \otimes ((x + x') + N') = q \otimes (x + N') + q \otimes (x' + N') = \gamma(q, x) + \gamma(q, x'), \\
\gamma(q + q', x) &= (q + q') \otimes (x + N') = q \otimes (x + N') + q' \otimes (x + N') = \gamma(q, x) + \gamma(q', x), \\
\gamma(qn, x) &= qn \otimes (x + N') = q \otimes (nx + N') = \gamma(q, nx).
\end{aligned}$$

By Theorem 10.10 again, there exists a group homomorphism $\Phi' : \mathbb{Q} \otimes_{\mathbb{Z}} N \to \mathbb{Q} \otimes_{\mathbb{Z}} (N/N')$. Since γ is also surjective, we have Φ' is surjective. Suppose also that $N/N' \neq 0$. By our statement, $\mathbb{Q} \otimes_{\mathbb{Z}} (N/N') \neq 0$ and hence $\mathbb{Q} \otimes_{\mathbb{Z}} N \neq 0$.

We now apply the statement to our case. Let $N = \prod_{i \in I} N_i$. The submodule N' consists of sequences $(x_i)_{i \in I}$ such that x_i's are almost all zero. Therefore, N/N' is nonzero. For example, it contains $(1)_{i \in I}$. So $\mathbb{Q} \otimes_{\mathbb{Z}} N \neq 0$. This proves our claim.

Solution 41 (i): It is clear that $(m, x) \sim (n, y)$ if and only if $(n, y) \sim (m, x)$. Also, $(m, x) \sim (m, x)$ because $1(mx - mx) = 0$. Suppose now that $(m, x) \sim (n, y) \sim (k, z)$, i.e., $t(my - nx) = 0$ and

$s(nz - ky) = 0$ for some nonzero $s, t \in \mathbb{Z}$. Since $n \neq 0$, we have

$$tsn(mz - kx) = (tm)(snz) - ks(tnx) = tm(sky) - ks(tmy) = 0.$$

So $(m, x) \sim (k, z)$. This shows that $\sim$ is an equivalence relation.

(ii): We only check well-definedness. The rest is left as exercise. Suppose that $(m, x) \sim (m', x')$ and $(n, y) \sim (n', y')$, i.e., $t(mx' - m'x) = 0 = s(ny' - n'y)$ for some nonzero integers s, t. We have

$$ts(m'n')(my + nx) = tm'm(sn'y) + sn'n(tm'x) = tm'm(sny') + sn'n(tmx') = ts(mn)(m'y' + n'x').$$

(iii): Again, we only check that the map β is well-defined. The rest is left as exercise. Suppose that $\frac{a}{b} = \frac{a'}{b'}$, i.e., $ab' = a'b$. We have

$$b'(ax) = (b'a)x = (a'b)x = b(a'x).$$

So $[(b, ax)] = [(b', a'x)]$. By Theorem 10.10, there exists an abelian group homomorphism $f : \mathbb{Q} \otimes_{\mathbb{Z}} N \to \mathbb{Z}^{-1}N$ such that $f(\frac{a}{b} \otimes x) = [(b, ax)]$.

(iv): Suppose that $(m, x) \sim (m', x')$, i.e., $t(mx' - m'x) = 0$. We have

$$\frac{1}{m} \otimes x = \frac{tm'}{tmm'} \otimes x = \frac{1}{tmm'} \otimes (tm'x) = \frac{1}{mm'} \otimes (tmx') = \frac{tm}{tmm'} \otimes x' = \frac{1}{m'} \otimes x'.$$

So g is well-defined. It is left as an exercise to check that g is a group homomorphism and it is the inverse of f.

(v): Suppose that $\frac{1}{m} \otimes x = 0$. By the isomorphism in part (iv), we have

$$0 = f(0) = f(\frac{1}{m} \otimes x) = [(m, x)].$$

The zero element in $\mathbb{Z}^{-1}N$ is $[(1, 0)]$. So $(m, x) \sim (1, 0)$, i.e., $t(x - 0) = 0$ for some $0 \neq t \in \mathbb{Z}$. Suppose now that $rx = 0$ for some $0 \neq r \in \mathbb{Z}$. We have

$$\frac{1}{m} \otimes x = \frac{r}{rm} \otimes x = \frac{1}{rm} \otimes (rx) = \frac{1}{rm} \otimes 0 = 0.$$

Solution 42 Self-study.

Solution 43 Left as exercise.

Solution 44 Suppose that $f_3(m) = 0$ for some $m \in M_3$. Therefore, $fg(m) = hf(m) = 0$. Since f_4 is injective, we have $g(m) = 0$. Since it is exact at M_2, there exists $m' \in M_2$ such that $g(m') = m$. We have $hf(m') = fg(m') = f(m) = 0$. So $f(m') \in \ker h_2 = \operatorname{im} h_1$ and hence there exists $n \in N_1$ such that $h(n) = f(m')$. But f_1 is surjective. There exists $m'' \in M_1$ such that $f(m'') = n$. We have $fg(m'') = hf(m'') = h(n) = f(m')$. Since f_2 is injective, we have $m' = g(m'')$. So $m = g(m') = g^2(m'') = 0$. So f_3 is injective.

Solution 45 (i) Define $\gamma(1) = 1$ and $\gamma(-1) = (1, 2)$. This is a group homomorphism. We have $\operatorname{sgn} \gamma(-1) = \operatorname{sgn}((1, 2)) = -1$. So $\operatorname{sgn} \gamma = \mathrm{id}_{\{\pm 1\}}$. (ii) Suppose that we have a group homomorphism such that $\delta\iota = \mathrm{id}_{A_n}$. In particular, δ is surjective and there is a normal subgroup N of S_n such that $|N| = |S_n|/|A_n| = 2$. So $N = \langle \tau \rangle$ where τ is a permutation of order 2, i.e., $\tau = (a_1, b_1)(a_2, b_2) \cdots (a_k, b_k)$ as a product of disjoint cycles where $k \geq 1$. Since $n \geq 3$, there exists $c \notin \{a_1, b_1\}$. Since N is closed under conjugation, we must have $\tau = (a_1, c)\tau(a_1, c) := \sigma$. But $\tau : b_1 \mapsto a_1$ while $\sigma : b_1 \mapsto c$. A contradiction. So there is no such δ.