# Homological Algebra tutorial

Date No.

① $R, S$ are rings, $V, W$ are $R, S$ modules resp

i) Suppose $R$ is comm, show that $v \star r = r \cdot v$ defines a right $R$ module

$\boxed{\text{Simply check axioms}}$, let $r, s \in R$, $v, u \in V$, Remember $r \cdot v$ comes from $V$ as left $R$-module

① $v \star (r+s) = (r+s) \cdot v = r \cdot v + s \cdot v = v \star r + v \star s$

② $v \star (r \star s) = (r \star s) \cdot v = (s \cdot r) \cdot v = s \cdot (r \cdot v) = (r \cdot v) \star s = (v \star r) \star s$

③ $(v+u) \star r = r \cdot (v+u) = r \cdot v + r \cdot u = v \star r + u \star r$

ii) Verify that ${}_R R$ is an $R$-module

$\boxed{\text{Simply check def}}$ ring is Abelian under $+$ by definition, now check 1-3

① $(r+s) \cdot m = (r+s)m = rm + sm$ Remember $rm = r \cdot m$ where $\cdot$ is defined in the ring axioms!!

iii) Let $\alpha : R \to S$ be a ring homomorphism, prove $r \star w = \alpha(r) \cdot w$ defines $R$-mod

$\boxed{\text{Check axioms}}$ Let $r_1, r_2 \in R$, $w_1, w_2 \in W$

① $(r_1 + r_2) \star w_1 = \alpha(r_1 + r_2) \cdot w = \alpha(r_1) \cdot w + \alpha(r_2) \cdot w = r_1 \star w + r_2 \star w$ $\alpha$ is additive

② $(r_1 r_2) \star w_1 = \alpha(r_1 r_2) \cdot w_1 = (\alpha(r_1)\alpha(r_2)) \cdot w_1 \overset{*}{=} \alpha(r_1) \cdot (\alpha(r_2) \cdot w_1) = r_1 \star (r_2 \star w_1)$

③ $r_1 \star (w_1 + w_2) = \alpha(r_1) \cdot (w_1 + w_2) = \alpha(r_1) \cdot w_1 + \alpha(r_1) \cdot w_2 = r_1 \star w_1 + r_1 \star w_2$

* note we used $S$-module associativity

iv) $\pi : R \to S$ is surj ring homo, $Ann(V) = \{ r \in R ; r \cdot v = 0, \forall v \in V \}$

Prove $Ann(V)$ is ideal of $R$.

Let $r, s \in Ann(V)$, $v \in V$

① $(Ann(V), +)$ is subgroup of $(R, +)$ : $(r+s) \cdot v = r \cdot v + s \cdot v = 0 + 0 = 0$

② $\forall t \in R$, $\forall r \in Ann(V)$ $tr \in Ann(V)$ : $(tr) \cdot v = t \cdot (r \cdot v) = t \cdot 0 = 0$

iv.1) Suppose $\ker(\pi) \subseteq Ann(V)$. Prove $s \star v = r \cdot v$ defines $S$-module $\pi(r) = s$

① Check well defined : Let $\pi(r_1) = s = \pi(r_2)$, then $r_1 - r_2 \in \ker(\pi) \subseteq Ann(V)$

$\Rightarrow 0 = (r_1 - r_2) \cdot v = r_1 \cdot v - r_2 \cdot v \Rightarrow r_1 \cdot v = r_2 \cdot v$

② $\boxed{\text{Check module axioms}}$ Let $\pi(r_1) = s_1$, $\pi(r_2) = s_2$ $\pi(r_1 r_2) = s_1 s_2$ $\pi(r_1 + r_2) = s_1 + s_2$

①$(s_1 + s_2) \star v = (r_1 + r_2) \cdot v = r_1 \cdot v + r_2 \cdot v = s_1 \star v + s_2 \star v$

v) $G$ is abelian group, Prove $G$ is $\mathbb{Z}$-module $\cdot : \mathbb{Z} \times G \to G$

Date No.

2) Let $F$ be a field. Prove every vector space $V$ over $F$ is an $F$-module and subspaces of $V$ are $F$-submodules

$\boxed{\text{Compare definitions}}$ A vector space over $F$ is precisely $(V,+)$ abelian group with $F \times V \to V$, $(\lambda, v) \to \lambda v$ satisfying the $F$-module axioms

3) Let $V$ be an $R$-module. Show that $0 \cdot v = 0$ and $(-1) \cdot v = -v$ $\forall v \in V$

$0 = 0 + 0 \in R$ hence $0 \cdot v = (0+0) \cdot v = 0 \cdot v + 0 \cdot v \Rightarrow 0 \cdot v = 0$

$0 = r - r \in R$ hence $0 \cdot v = (r-r) \cdot v = (r \cdot v) + (-r) \cdot v$, hence $(-r) \cdot v = -(r \cdot v)$

If $R$ is unital we get $-(1 \cdot v) = -v = -1 \cdot v$

4) $V$ is $R$-module

i) Sum of 2 submodule, $U, W$ is a submodule

$\boxed{\text{Apply submodule Criterion}}$ ① $U + W = \{u + w : u \in U, w \in W\} \neq \emptyset$ since $U, W \neq \emptyset$

② Let $u_1, u_2 \in U$, $w_1, w_2 \in W$, $r \in R$, then $(u_1 + w_1) + r(u_2 + w_2)$
$= (u_1 + ru_2) + (w_1 + rw_2) \in U + W$

ii) Intersection of any non-empty collection of submodules of $V$ is submodule

Let $\{U_i\}_{i \in S}$ be collection ① $0 \in U_i \Rightarrow 0 \in \bigcap_{i \in S} U_i = U$

② If $x, y \in U \Rightarrow x, y \in U_i$ $\forall i \Rightarrow x + ry \in U_i$ $\forall i \Rightarrow x + ry \in U$

5) Take $G = C_6 \times C_{10}$ as $\mathbb{Z}$-module. Find $\mathrm{Ann}(G) = \{r \in \mathbb{Z} : r \cdot n = 0, n \in G\}$

$r \cdot (a,b) = (ra, rb)$ we want $ra = 0 = rb$ $\forall a \in C_6$, $b \in C_{10}$ $\Rightarrow$

$6 | r$, $10 | r \Rightarrow 30 | r$, hence $\mathrm{Ann}(G) \subseteq 30\mathbb{Z}$ and clearly

$30 \cdot (a,b) = (0,0)$ $\forall (a,b) \in G \Rightarrow 30\mathbb{Z} \subseteq \mathrm{Ann}(G)$

6) $\{V_i\}_{i \in I}$ is collection of $R$-modules

i) $\prod V_i = \{(v_i)_{i \in I} : v_i \in V_i\}$ $(v_i) + (w_i) = (v_i + w_i)$, $r \star (v_i) = (r \cdot v_i)$ show $\prod V_i$ is $R$-module

$\boxed{\text{Check axioms}}$ ① $(r+s) \star (v_i) = ((r+s) \cdot v_i) \overset{\vee}{=} (r \cdot v_i) + (s \cdot v_i) = r \star (v_i) + s \star (v_i)$

$(r \cdot v_i + s \cdot v_i)$

② $(rs) \star (v_i) = (rs \cdot v_i) = (r \cdot (s \cdot v_i)) = r \star (s \star v_i)$

③ $r \star ((v_i) + (w_i)) = r \star (v_i + w_i) = (r \cdot (v_i + w_i)) = (r \cdot v_i + r \cdot w_i)$
$= (r \cdot v_i) + (r \cdot w_i) = r \star v_i + r \star w_i$

$(xy)z = \left(\sum_{g\in G}\left(\sum_{h\in G} r_{gh}s_{h^{-1}}\right)g\right)\left(\sum t_g g\right) = \sum_{g\in G}\left(\sum_{i\in G} q_{g i} t_{h^{-1}}\right)g \qquad q_g = \sum_{h\in G} r_{gh}s_{h^{-1}}$

$= \sum_{g\in G}\left(\sum_{i\in G}\left(\sum_{h\in G} r_{gih}s_{h^{-1}}\right)t_{h^{-1}}\right)g$



Date No.

ii) $\prod V_i \supseteq \bigoplus V_i = \{(v_i) : \text{finitely many } v_i \neq 0\}$

$\boxed{\text{Apply Submodule Criterion}}$ ① $0 \in \bigoplus V_i$ clearly

② Let $(x_i),(y_i) \in \bigoplus V_i$, $r \in R$ now $x + ry \in \bigoplus V_i$ since finitely many $x_i \neq 0$, $y_i \neq 0$ hence finitely many $x_i + ry_i \neq 0$

If $|I| < \infty$ then by def if $(v_i) \in \prod V_i \Rightarrow (v_i) \in \bigoplus V_i$ hence $\prod V_i = \bigoplus V_i$

7) $I$ is left ideal of $R$, $V$ is $R$-module, Prove $IV = \{\sum a_i \cdot v_i : a_i \in I, v_i \in V\}$ is a submodule of $V$

$\boxed{\text{Apply Submodule Criterion}}$ ① $0 \in I, V$ hence $0 \in IV$ $\quad$ $rb_i \in I$ by def of ideal

② Let $\sum_{i=1}^{m} a_i \cdot v_i$, $\sum_{i=1}^{n} b_i \cdot w_i \in IV$, $r \in R$ $\quad m > n$

then $\sum_{i=1}^{m} a_i \cdot v_i + r\sum_{i=1}^{n} b_i \cdot w_i = \sum_{i=1}^{m+n} c_i \cdot u_i$ where $c_i = \begin{cases} a_i, & i \leq m \\ rb_i, & i > m \end{cases}$ $u_i = \begin{cases} v_i, & i \leq m \\ w_i, & i > m \end{cases}$

8) $G$ is a group, $R$ is a ring. $RG = \{\sum_{g\in G} r_g g : r_g \in R, \text{finite number of} \neq 0\text{'s}\}$

Show $RG$ is an $R$-algebra, $\phi : R \to RG$, $\phi(r) = re_G$.

① Show $RG$ is a unital ring

(1.1) $RG$ is an Abelian group under addition since $0 \in R$, hence $\sum_{g\in G} 0 \cdot g$ is zero in $RG$

Let $x = \sum r_g g$, $y = \sum s_g g$, $z = \sum t_g g$

(1.2) Distributivity: $x(y+z) = (\sum r_g g)(\sum (s_g + t_g)g) = \sum_{g\in G}\left(\sum_{h\in G} r_{gh} q_{h^{-1}}\right)g$ where $q_g = s_g + t_g$

$= \sum_{g\in G}\left(\sum_{h\in G} r_{gh}s_{h^{-1}} + r_{gh}t_{h^{-1}}\right)g = xy + xz$

(1.3) Associativity: $\boxed{\text{Unbelivably painful: try reformulate definition}}$

We redefine multiplication as $\left(\sum_{g\in G} r_g g\right)\left(\sum_{h\in G} s_h h\right) = \sum_{k\in G}\left(\sum_{gh=k} r_g s_h\right)k$

In our given rule take $b = h^{-1} \Leftrightarrow h = b^{-1}$ to get $\sum_{b\in G} r_{gb^{-1}}s_b = \sum_{\substack{a,b\in G \\ ab=g}} r_a s_b$ since $ab = g \Leftrightarrow a = gb^{-1}$

Now $(xy)z = \left(\sum_{k\in G}\left(\sum_{gh=k} r_g s_h\right)k\right)\left(\sum_{i\in G} t_i i\right) = \sum_{k\in G}\left(\sum_{ghi=k} r_g s_h t_i\right)k = \sum_{\substack{ghi=k \\ k\in G}} r_g s_h t_i (gh)i$

$= \sum_{ghi=k} r_g s_h t_i g(hi) = x(yz)$

(1.4) $1_{RG} = 1_R e_G$, $\boxed{e_G = \sum_{g\in G} c_g g, \ c_{e_G} = 1, \text{ else } 0}$ hence $1_{RG}x = \sum_{g\in G}\left(\sum_{h\in G} c_{gh} s_{h^{-1}}\right)g$

$= \sum_{g\in G} s_g g$ since $c_{gh} = 1 \Leftrightarrow h = g^{-1} \Leftrightarrow h^{-1} = g$

② Check $\phi$ is ring homo s.t. $\phi(R) \subseteq Z(RG)$

① $\phi(1_R) = 1_R e_G = 1_{RG}$ ② $\phi(r+s) = (r+s)e_G = \phi(r) + \phi(s)$ ③ $\phi(rs) = rse_G = (re_G)(se_G) = \phi(r)\phi(s)$

Because $R$ is comm $\phi(r)x = (re_G)(\sum r_g g) = \sum r r_g g = \sum r_g r g = x(re_G) = x\phi(r)$ hence $\phi(R) \subseteq Z(RG)$. ∎