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Exercise 12. Consider the group algebra RG. Define the RG-action on R by, for each $x \in R$,

$$\left(\sum_{g\in G} r_g g\right) * x = \sum_{g\in G} r_g x$$

Prove that R is an RG-module. The module is called the trivial RG-module (for the group algebra RG).

Proof.

It is clear that $(R, +, \cdot)$ is an abelian group with respect to the operation $+$.

Let $\sum_{g\in G} r_g g$ and $\sum_{g\in G} s_g g$ be two elements of RG. Let $x, y \in R$. We now verify the axioms of RG-module:

Firstly:

$$\begin{aligned}
\left(\sum_{g\in G} r_g g + \sum_{g\in G} s_g g\right) * x &= \left(\sum_{g\in G} (r_g + s_g) g\right) * x \\
&= \sum_{g\in G} (r_g + s_g) x \\
&= \sum_{g\in G} (r_g x + s_g x) \\
&= \left(\sum_{g\in G} r_g x\right) + \left(\sum_{g\in G} s_g x\right) \\
&= \left(\sum_{g\in G} r_g g\right) * x + \left(\sum_{g\in G} s_g g\right) * x
\end{aligned}$$

Secondly:

$$\begin{aligned}
\left(\sum_{g\in G} r_g g\right) * (x + y) &= \sum_{g\in G} r_g (x + y) \\
&= \sum_{g\in G} (r_g x + r_g y) \\
&= \left(\sum_{g\in G} r_g x\right) + \left(\sum_{g\in G} r_g y\right) \\
&= \left(\sum_{g\in G} r_g g\right) * x + \left(\sum_{g\in G} r_g g\right) * y
\end{aligned}$$

Before proceeding, we shall rewrite the multiplication of elements of group algebra RG to be:

$$\left(\sum_{g\in G} r_g g\right) \left(\sum_{h\in G} s_h h\right) = \sum_{k\in G} \left(\sum_{gh=k} r_g s_h\right) k$$

and thus the two definitions of multiplication are equal:

$$\sum_{g\in G} \sum_{h\in G} (r_{gh} s_{h^{-1}} g) = \sum_{k\in G} \sum_{gh=k} (r_g s_h k)$$

Thirdly:

$$\begin{aligned}
\left(\sum_{g\in G} r_g g\right) * \left(\left(\sum_{h\in G} s_h h\right) * (x)\right) &= \left(\sum_{g\in G} r_g g\right) * \left(\sum_{h\in G} s_h x\right) \\
&= \sum_{g\in G}\left(r_g \left(\sum_{h\in G} s_h x\right)\right) \\
&= \sum_{g\in G}\sum_{h\in G}(r_g s_h)x \\
&= \sum_{g,h\in G}(r_g s_h)x
\end{aligned}$$

On the other hand, we have that

$$\begin{aligned}
\left(\left(\sum_{g\in G} r_g g\right)\left(\sum_{h\in G} s_h h\right)\right) * x &= \sum_{k\in G}\left(\sum_{gh=k} r_g s_h\right) k * x \\
&= \sum_{k\in G}\sum_{h=g^{-1}k}(r_g s_h)x \\
&= \sum_{g,h\in G}(r_g s_h)x
\end{aligned}$$

and therefore we see that

$$\left(\sum_{g\in G} r_g g\right) * \left(\left(\sum_{h\in G} s_h h\right) * (x)\right) = \left(\left(\sum_{g\in G} r_g g\right)\left(\sum_{h\in G} s_h h\right)\right) * x$$

Lastly, note RG is unital. In Exercise 8, we have shown that the identity for RG is $1_R e_G$:

$$1_R e_G * x = 1_R \cdot x = x$$

Therefore, this shows that R is an RG-module. $\square$