# Tutorial #6

41) ${}_{\mathbb{Z}}N$, $\mathbb{Z}^{\times} = \mathbb{Z}/\{0\}$, $\sim$ relation on $\mathbb{Z}^{\times} \times N$ by $(m,x) \sim (n,y) \iff t(my - nx) = 0$ for some $0 \neq t \in \mathbb{Z}$

(i) Prove $\sim$ is an equivalence relation, let $\mathbb{Z}^{-1}N$ be set of equiv classes

① $(m,x) \sim (m,x)$ is true since any $t$ will work $t(mx - mx) = t \cdot 0 = 0$

② $(n,y) \sim (m,x)$

③ $(m,x) \sim (n,y)$, $(n,y) \sim (p,z)$ $\exists t_1$ s.t. $t_1(my - nx) = 0$ and $t_2(nz - py) = 0$

since $n \neq 0$ we have $t_1 t_2 n(mz - px) = t_1 t_2 nmz - t_1 t_2 npx$

$= t_1 m(t_2 nz) - t_2 p(t_1 nx) = t_1 m(t_2 py) - t_2 p(t_1 my) = 0$

(ii) Prove $\mathbb{Z}^{-1}N$ is a $\mathbb{Z}$-mod with $[(m,x)] + [(n,y)] = [(mn, my + nx)]$

① Well defined: Suppose $(m,x) \sim (m',x')$ and $(n,y) \sim (n',y')$ then

$(mn, my + nx) \sim (m'n', m'y' + n'x')$ $\quad t(mx' - m'x) = 0 = s(ny' - n'y)$

We have $ts(m'n')(my + nx) = tm'm(sn'y) + sn'n(tm'x)$

$= tm'm(sny') + sn'n(tmx')$

$= ts(mn)(m'y' + n'x')$ as desired

② Identity $[(1,0)]$ is identity since $[(m,x)] + [(1,0)] = [(m,x)]$

③ Inverse of $[(m,x)]$ is $[(m,-x)]$ since $[(m,x)] + [(m,-x)] = [(m^2, 0)]$ and $(m^2, 0) \sim (1,0)$

④ Assoc and comm are tedious but clear

⑤ Define $r \cdot [(m,x)] = [(m, rx)]$, check the axioms

(iii) Show $\beta : \mathbb{Q} \times N \to \mathbb{Z}^{-1}N$ where $\beta(\frac{a}{b}, x) = [(b, ax)]$ is $\mathbb{Z}$-balanced and induces $\mathbb{Z}$-mod homo $f : \mathbb{Q} \otimes_R N \to \mathbb{Z}^{-1}N$ s.t. $f(\frac{a}{b} \otimes x) = [(b, ax)]$

① Well defined: Let $\frac{a}{b} = \frac{a'}{b'} \iff ab' = a'b$, then $b'(ax) = (ab')x = (a'b)x = b(a'x)$

hence $[(b, ax)] = [(b', a'x)]$ since $(b, ax) \sim (b', a'x)$ $\quad t(ba'x - b'ax) = 0$

② $\mathbb{Z}$-balanced: $\beta(r \cdot \frac{a}{b}, x) = [(b, rax)]$

$\beta(\frac{a}{b}, rx) = [(b, arx)]$

③ By $\boxed{T10}$ $\exists! f : \mathbb{Q} \otimes_R N \to \mathbb{Z}^{-1}N$ s.t. $f(\frac{a}{b} \otimes x) = \beta(\frac{a}{b}, x) = [(b, ax)]$

iv) $g : \mathbb{Z}^{-1}N \to \mathbb{Q} \otimes_R N$ by $g([(m,x)]) = \frac{1}{m} \otimes x$

① Well defined: $(m,x) \sim (n,y) \Rightarrow t(my - nx) = 0$ then $\frac{1}{m} \otimes x = \frac{tn}{tnm} \otimes x$

$= \frac{1}{tnm} \otimes tnx = \frac{1}{tnm} \otimes tmy = \frac{tm}{tnm} \otimes y = \frac{1}{n} \otimes y$

② $g$ is group homo

③ $gf(\frac{a}{b} \otimes x) = g([(b, ax)]) = \frac{1}{b} \otimes ax = \frac{a}{b} \otimes x$

$fg([(m,x)]) = f(\frac{1}{m} \otimes x) = [(m, x)]$

v) Conclude $\frac{1}{m} \otimes x = 0$ in $\mathbb{Q} \otimes_R N \iff rx = 0$ for $0 \neq r \in \mathbb{Z}$

Suppose $\frac{1}{m} \otimes x = 0$, hence $0 = f(0) = f(\frac{1}{m} \otimes x) = [(m,x)]$, 0 element in $\mathbb{Z}^{-1}N$

additive 0 $\to$ is $[(1,0)]$ hence $[(m,x)] = [(1,0)] \iff (m,x) \sim (1,0) \iff t(m \cdot 0 + 1 \cdot x) = -tx$

hence let $r = -t$, suppose $rx = 0$ then $\frac{1}{m} \otimes x = \frac{r}{rm} \otimes x = \frac{1}{rm} \otimes rx$

$= \frac{1}{rm} \otimes 0 = 0$

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42) $\boxed{\text{Cor } 10.16}$: $R$ is comm, $M_1, \dots, M_n$ are $R$-mods

$i : \prod M_i \to \otimes M_i$, $i(m_1, \dots, m_n) = m_1 \otimes \dots \otimes m_n$ then

① $\forall$ R-mod homo $\Phi : \otimes M_i \to L$, $\varphi = \Phi i$ is $n$-multilinear from $\prod M_i \to L$

② If $\varphi : \prod M_i \to L$ is an $n$-multilinear map, $\exists!$ R-mod homo $\Phi : \otimes M_i \to L$ s.t. $\varphi = \Phi i$

Proof : $\otimes M_i = \left(\overset{n-1}{\otimes} M_i\right) \otimes M_n$ which we can think of as 2 R-mods

① By $\boxed{\text{Cor } 12}$ $\Phi : \left(\overset{n-1}{\otimes} M_i\right) \otimes M_n \to L$ R-mod homo corresponds to R-bilinear map $\varphi$ s.t. $\varphi = \Phi i$. By $\boxed{\text{Cor } 15}$ we can rearrange tensor product as we wish, hence the $n$-multilinear makes sence

② Consider $\varphi : \left(\prod_{i=1}^{n-1} M_i\right) \times M_n \to L$ is bilinear, by $\boxed{\text{Cor } 12}$ $\Rightarrow \exists! \Phi : \otimes M_i \to L$ s.t. $\varphi = \Phi i$

$\boxed{\text{Prop } 10.21}$: Read pg 374

43)
$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "\phi"] \arrow[d, "\alpha"] & Y \arrow[r, "\psi"] \arrow[d, "\beta"] & Z \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & X' \arrow[r, "\phi'"] \arrow[d, "\alpha'"] & Y' \arrow[r, "\psi'"] \arrow[d, "\beta'"] & Z' \arrow[r] \arrow[d, "\gamma'"] & 0 \\
0 \arrow[r] & X'' \arrow[r] & Y'' \arrow[r] & Z'' \arrow[r] & 0
\end{tikzcd}$$

$(\alpha, \beta, \gamma)$ and $(\alpha', \beta', \gamma')$ are homo of SES

(i) Prove $(\alpha'\alpha, \beta'\beta, \gamma'\gamma)$ is homo of SES

Both halfs commute, hence the direct path will commute

(ii) Suppose $(\alpha, \beta, \gamma)$ is an iso, Prove $(\alpha^{-1}, \beta^{-1}, \gamma^{-1})$ is iso

We just need to insure that the reverse diagram commutes, since if $\alpha$ is iso so is $\alpha^{-1}$. That is we have $\alpha\phi' = \phi\beta$ and $\psi'\beta = \gamma\psi$.

Now $\phi'\beta^{-1} = \alpha^{-1}\phi$ comes from $\alpha^{-1}(\alpha\phi')\beta^{-1} = \alpha^{-1}(\phi\beta)\beta^{-1}$, similarly for other relation

44) If $f_1$ is surj, $f_2, f_4$ are inj $\Rightarrow f_3$ is surj

$$\begin{tikzcd}
m_1 \arrow[r] & m_2 \arrow[r] & m_3 = g_2(m_2) \arrow[r] & g_3(m_3) = 0 \\
M_1 \arrow[r, "g_1"] \arrow[d, "f_1"] & M_2 \arrow[r, "g_2"] \arrow[d, "f_2"] & M_3 \arrow[r, "g_3"] \arrow[d, "f_3"] & M_4 \arrow[d, "f_4"] \\
N_1 \arrow[r, "h_1"] & N_2 \arrow[r, "h_2"] & N_3 \arrow[r, "h_3"] & N_4 \\
n_1 \arrow[r] & f_2(m_2) \arrow[r] & f_3(m_3) = 0 \arrow[r] & h_3(f_3(m_3)) = 0
\end{tikzcd}$$

Focus on this diagram

$\boxed{\text{WTS: } \ker(f_3) = \{0\}}$

Let $m_3 \in M_3$ s.t. $f_3(m_3) = 0$

Then $h_3(f_3(m_3)) = 0$

By commutativity $f_4(g_3(m_3)) = h_3(f_3(m_3)) = 0$, hence by inj of $f_4$ $g_3(m_3) = 0$

By exactness $\ker(g_3) = \operatorname{Im}(g_2)$, since $g_3(m_3) = 0 \Rightarrow m_3 \in \ker(g_3)$ hence $\exists m_2$ s.t. $g_2(m_2) = m_3$. By comm $h_2(f_2(m_2)) = f_3(g_2(m_2)) = f_3(m_3) = 0$. By exact $f_2(m_2) \in \ker(h_2) \Rightarrow \exists n_1 \in N_1$ s.t. $h_1(n_1) = f_2(m_2)$. Since $f_1$ is surj, $\exists m_1 \in M_1$ s.t. $f_1(m_1) = n_1$ by comm $f_2(g_1(m_1)) = h_1(f_1(m_1)) = h_1(n_1) = f_2(m_2)$, since $f_2$ is inj $\Rightarrow g_1(m_1) = m_2$, hence $m_3 = g_2(g_1(m_1))$. By exactness $g_2 g_1 = 0$ hence $m_3 = 0(m_1) = 0$ ∎

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45) $1 \to A_n \xrightarrow{\iota} S_n \xrightarrow{sgn} \{\pm 1\} \to 1$

(i) show $\exists$ group homo $\gamma : \{\pm 1\} \to S_n$ s.t. $sgn(\gamma) = Id_{\{\pm 1\}}$

Let $J = (1,2) \in S_n$ then $\gamma(1) \in 1_{S_n}$, $\gamma(-1) = J$. This is a group homo since $h(-1 \times -1) = 1_{S_n} = h(-1)h(-1) = (1\,2)(1\,2)$. Then $sgn(\gamma(1)) = sgn(1_{S_n}) = 1$ and $sgn(\gamma(-1)) = sgn((1,2)) = -1$

(ii) Show that when $n \geq 3$, there is no group homo $\delta : S_n \to A_n$ s.t. $\delta i = Id_{A_n}$

Suppose such a group homo does exist, since $\delta i = Id_{A_n} \Rightarrow \delta$ is surjective hence by 1st iso $S_n / Ker(\delta) \cong A_n \Rightarrow |S_n| / |A_n| = 2 = |Ker(\delta)|$, hence $Ker(\delta) = \{1_{S_n}, k\}$ where $k^2 = 1$. $\boxed{\text{Remember, kernel of group homo is normal}}$

$Ker(\delta)$ is normal hence $\forall g \in S_n$, $gkg^{-1} = 1_{S_n}$ or $k$, first case is impossible since $order(gkg^{-1}) = order(k)$, hence $gkg^{-1} = k \Rightarrow gk = kg \Rightarrow k \in Z(S_n)$

But for $n \geq 3$ $Z(S_n) = \{1\}$ $\to\leftarrow$

46) Let $V = \mathbb{Z}/2\mathbb{Z}$ consider exact $\mathbb{Z} \xrightarrow{\pi} V \to 0$. Show $Hom_{\mathbb{Z}}(V, \mathbb{Z})$ is trivial and hence $\pi^* : Hom_{\mathbb{Z}}(V, \mathbb{Z}) \to Hom_{\mathbb{Z}}(V, V)$ is not surj

Let $\phi \in Hom_{\mathbb{Z}}(V, \mathbb{Z})$, $\phi(\bar{1}) = n$ $\quad 0 = f(\bar{0}) = f(\bar{1} + \bar{1}) = f(\bar{1}) + f(\bar{1}) = 2n \Rightarrow n = 0$

hence $\phi$ is trivial, By $\boxed{\text{Ex18}}$ $Hom_{\mathbb{Z}}(V, V) \cong C^2$

47) $\{F_i : i \in I\}$ is collection of free mods

(i) Prove $\oplus F_i$ is a free mod

Let $F_i = F(S_i)$ where $S_i$ is generating set, then we $\boxed{\text{claim}: \oplus F_i = F(\cup S_i)}$

Now if $0 \neq x \in \oplus F_i = \{x_i, x_i \in F_i \Rightarrow x_i = \sum r_{i,l} s_{i,l}$ in $F_i$, hence $\oplus F_i$ is free on $\cup S_i$

(ii) $\forall i \in I$, Let $P_i$ be direct summand of $F_i$. Assume $F_i = P_i \oplus Q_i$ for some submodule $Q_i$ of $F_i$. Prove that $\oplus F_i = (\oplus P_i) \oplus (\oplus Q_i)$

Let $P = \oplus P_i$, $Q = \oplus Q_i$ clearly $P \oplus Q \subseteq F$. Now if $x = (x_i) \in \oplus F_i$, let $x_i = p_i + q_i \in P_i + Q_i$, then $(x_i) = (p_i) + (q_i) \in P + Q \Rightarrow F \subseteq P \oplus Q$. Let $x \in P \cap Q$

$\Rightarrow x_i \in P_i, Q_i \Rightarrow x_i \in P_i \cap Q_i \subseteq F_i = P_i \oplus Q_i$ hence $x_i = 0 \Rightarrow (x = (x_i) = 0$

(iii) (By $\boxed{\text{Prop 30}}$ F is proj if it is the direct summand of free mods)

Now each $F_i$ is proj by assumption, then by (ii) $\oplus F_i = (\oplus P_i) \oplus (\oplus Q_i)$ each of which are free mods by (i), hence $\oplus F_i$ is proj by $\boxed{\text{Prop30}}$

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48) Prove R-mod is a category. Conclude $\mathbb{Z}$-mod cat is Ab

$Obj(R\text{-mod}) = \{\text{all left } R\text{-mods}\}$

$Mor_{R\text{-mod}}(X,Y) = Hom_R(X,Y)$

(1) Show composition is well defined: $f: X \to Y$, $g: Y \to Z$ then $g \circ f: X \to Z$ $\boxed{\text{Prop 2.2}}$

(2) Associative: $h: Z \to W$, then $(h \circ g) \circ f = h \circ (g \circ f)$

(3) $Hom_R(X,X)$ contains $1_X$: $1_X(x) = x$; $1_X(rx + y) = 1_X(rx) + 1_X(y) = rx + y$ $= r 1_X(x) + 1_Y(y)$ hence $1_X \in Hom_R(X,X)$. For any $f: X \to Y$ $1_Y f = f$, $f 1_X = f$

We know $\mathbb{Z}$-mod iff Abelian group, hence $\mathbb{Z}$-mod $\cong$ Ab as categories