Name: Loo Wee Lun

Exercise 36 Let D be a right $\mathbb{Z}$-module and $m \in \mathbb{Z}$. We aim to prove that

$$D \otimes_{\mathbb{Z}} (\mathbb{Z}/m\mathbb{Z}) \cong D/mD$$

where $mD = \{md : d \in D\}$.

1. Define a map $\beta : D \times (\mathbb{Z}/m\mathbb{Z}) \to D/mD$ by $\beta(d, \overline{a}) = ad + mD$. Prove that β is well-defined and $\mathbb{Z}$-balanced.

2. By the Universal Property of Tensor Product, we have a group homomorphism $\Phi : D \otimes_{\mathbb{Z}} (\mathbb{Z}/m\mathbb{Z}) \to D/mD$ such that $\Phi(d \otimes \overline{a}) = ad + mD$. Prove that Φ is an isomorphism.

Proof.

Define $\beta : D \times (\mathbb{Z}/m\mathbb{Z}) \to D/mD$ by $\beta(d, \overline{a}) = ad + mD$. It is clear that mD is a right $\mathbb{Z}$-submodule of D. We first show that β is well-defined. Let $a_1, a_2 \in \mathbb{Z}$ such that $\overline{a}_1 = \overline{a}_2$ in $\mathbb{Z}/m\mathbb{Z}$. We claim that $\beta(d, \overline{a}_1) = \beta(d, \overline{a}_2)$. Note

$$\begin{aligned}
\overline{a}_1 = \overline{a}_2 &\implies a_1 - a_2 \in m\mathbb{Z} \\
&\implies a_1 d - a_2 d = (a_1 - a_2)\, d \in mD \ \forall d \in D \\
&\implies a_1 d + mD = a_2 d + mD \\
&\implies \beta(d, \overline{a}_1) = \beta(d, \overline{a}_2)
\end{aligned}$$

This shows that β is well-defined.

Next we show that β is $\mathbb{Z}$-balanced. First, additivity in the first argument:

$$\begin{aligned}
\beta(d_1 + d_2, \overline{a}) &= a(d_1 + d_2) + mD = (ad_1 + ad_2) + mD \\
&= (ad_1 + mD) + (ad_2 + mD) \\
&= \beta(d_1, \overline{a}) + \beta(d_2, \overline{a})
\end{aligned}$$

Next, note that $\overline{a}_1 + \overline{a}_2 = \overline{a_1 + a_2}$. We then show additivity in the second argument:

$$\begin{aligned}
\beta(d, \overline{a}_1 + \overline{a}_2) &= \beta(d, \overline{a_1 + a_2}) \\
&= (a_1 + a_2)d + mD \\
&= (a_1 d + a_2 d) + mD \\
&= (ad_1 + mD) + (ad_2 + mD) \\
&= \beta(d_1, \overline{a}) + \beta(d_2, \overline{a})
\end{aligned}$$

Thirdly, to show compatibility with scalar multiplication, by applying additivity in the first and second argument, which we have proved to be true, we see that

$$\begin{aligned}
\beta(d \cdot n, \overline{a}) &= \beta(nd, \overline{a}) \\
&= \beta(\underbrace{d + \cdots + d}_{n \text{ times}}, \overline{a}) \\
&= \underbrace{\beta(d, \overline{a}) + \cdots + \beta(d, \overline{a})}_{n \text{ times}} \\
&= \beta(d, \underbrace{\overline{a}, \ldots, \overline{a}}_{n \text{ times}}) \\
&= \beta(d, n\overline{a}) \\
&= \beta(d, n \cdot \overline{a})
\end{aligned}$$

Altogether, this shows that β is $\mathbb{Z}$-balanced, which proves the first statement.

For the second statement, as mentioned in the statement, by the Universal Property of Tensor Product we have a group homomorphism $\Phi : D \otimes_{\mathbb{Z}} (\mathbb{Z}/m\mathbb{Z}) \to D/mD$ such that $\Phi(d \otimes \overline{a}) = ad + mD$. We will show that it is an isomorphism. Firstly, note that Φ is already well-defined by the statement of Universal Property of Tensor Product.

Showing injectivity of Φ is equivalent to showing that it has a trivial kernel, i.e. $\ker \Phi = \{0\}$. First note that since $\otimes_{\mathbb{Z}}$ is $\mathbb{Z}$-balanced, we have that

$$\sum_i (d_i \otimes \overline{a}_i) = \sum_i (d_i \otimes (a_i \cdot \overline{1})) = \sum_i ((d_i \cdot a_i) \otimes \overline{1}) = \sum_i (a_i d_i \otimes \overline{1}) \tag{1}$$

Suppose $\sum_i (d_i \otimes \overline{a}_i) \in \ker \Phi$ where i is finite, then

$$\Phi\left(\sum_i (d_i \otimes \overline{a}_i)\right) = \Phi\left(\sum_i (a_i d_i \otimes \overline{1})\right) = \sum_i a_i d_i + mD = 0 + mD$$

This implies that $\sum_i a_i d_i \in mD$, so write $\sum_i a_i d_i = md$ for some $d \in D$. Substituting back to Equation 1, we get that

$$\sum_i (d_i \otimes \overline{a}_i) = \sum_i (a_i d_i \otimes \overline{1}) \overset{(*)}{=} \left(\sum_i a_i d_i\right) \otimes \overline{1} = md \otimes \overline{1} = d \otimes \overline{m}$$

where at $(*)$ we apply the additivity in the first argument, one of the property of tensor product being $\mathbb{Z}$-balanced. Note $\overline{m} = \overline{0}$ in $\mathbb{Z}/m\mathbb{Z}$. To conclude, we have that

$$d \otimes \overline{a} = d \otimes \overline{0} = 0$$

This shows that $\ker \Phi$ is trivial, and thus Φ is injective.

Next, for surjectivity, suppose given $ad + mD \in D/mD$. Then simply consider $d \otimes \overline{a} \in D \otimes_{\mathbb{Z}} (\mathbb{Z}/m\mathbb{Z})$ and see that

$$\Phi(d \otimes \overline{a}) = ad + mD$$

This proves surjectivity of Φ.

Altogether, we have shown that Φ is an isomorphism, and thus

$$D \otimes_{\mathbb{Z}} (\mathbb{Z}/m\mathbb{Z}) \cong D/mD$$

as required. $\square$