Tutorial #4 $\quad\quad$ $\overset{\infty}{\underset{i=1}{\oplus}}\mathbb{Z}$

25) Let M be free Module $\mathbb{Z}\times\mathbb{Z}\times\cdots$, $R = End_{\mathbb{Z}}(M)$

i) $\phi(a_1,a_2,\ldots) = (0,a_1,a_2,\ldots)$, $\psi(a_1,a_2,\ldots) = (a_2,a_3,\ldots)$

$\psi(\phi(a_1,a_2,\ldots)) = \psi(0,a_1,a_2,\ldots) = (a_1,a_2,\ldots) \Rightarrow \psi\phi = 1_M$

$\phi(\psi(a_1,a_2,\ldots)) = \phi(a_2,a_3,\ldots) = (0,a_2,a_3,\ldots) \Rightarrow \phi\psi \neq 1_M$

Since $\psi,\phi \in End_{\mathbb{Z}}(M)$ and $\psi\phi \neq \phi\psi \Rightarrow R$ is not comm

ii) $\alpha_1(a_1,a_2,a_3,\ldots) = (a_1,a_3,\ldots)$ $\quad$ $\alpha_2(a_1,a_2,\ldots) = (a_2,a_4,\ldots)$

Prove $\{\alpha_1,\alpha_2\}$ is a free basis of ${}_RR$

$\boxed{\text{WTS: } \forall 0\neq x\in R,\ \exists!\, r_i\in R,\ \exists!\, a_i\in\{\alpha_1,\alpha_2\} \text{ s.t. } x=\sum r_i\alpha_i}$ $\quad$ These are easy to check

we follow the hint $\alpha_i\beta_i = 1_R$, $\alpha_1\beta_2 = \alpha_2\beta_1 = 0$, $\beta_1\alpha_1 + \beta_2\alpha_2 = 1_R$

where $\beta_1(a_1,a_2,\ldots) = (a_1,0,a_2,0,\ldots)$, $\beta_2(a_1,a_2,\ldots) = (0,a_1,0,a_2,\ldots)$

Now take $x\in R$, $x = x\cdot 1_R = x\beta_1\alpha_1 + x\beta_2\alpha_2$, since $x,\beta_i\in R \Rightarrow$

$x = (x\beta_1)\alpha_1 + (x\beta_2)\alpha_2$ is of the desired form, Now we are left to prove uniqueness

Suppose $x = x_1\alpha_1 + x_2\alpha_2$, then $x\beta_1 = (x_1\alpha_1)\beta_1 + (x_2\alpha_2)\beta_1 = x_1\cdot 1_R + 0 = x_1$

and $x\beta_2 = x_1\alpha_1\beta_2 + x_2\alpha_2\beta_2 = 0 + x_2\cdot 1_R = x_2$ ∎

iii) Prove $R \cong R^2$ as R-mods

We have shown that we can uniquely represent $x\in R$ by $x\beta_1\alpha_1 + x\beta_2\alpha_2$

Let $f: R\to R^2$ defined by $f(x) = (x\beta_1, x\beta_2)$

① Show f is R-mod homo, let $x,y\in R$, $r\in R$ then

$\phi(rx+y) = ((rx+y)\beta_1, (rx+y)\beta_2) = r(x\beta_1,x\beta_2) + (y\beta_1,y\beta_2) = r\cdot f(x) + f(y)$

② Show f is bij

Let $(x,y)\in R^2$ then $f(x\alpha_1 + y\alpha_2) = (x\alpha_1\beta_1 + y\alpha_2\beta_1,\ x\alpha_1\beta_2 + y\alpha_2\beta_2)$

$= (x+0,\ 0+y) = (x,y) \Rightarrow$ surj

Let $x\in\ker(f)$, $f(x) = 0 = (x\beta_1, x\beta_2)$ but $x = x\beta_1\alpha_1 + x\beta_2\alpha_2$

if $(x\beta_1,x\beta_2) = 0 \Rightarrow x\beta_1 = x\beta_2 = 0 \Rightarrow x = 0\alpha_1 + 0\alpha_2 = 0 \Rightarrow \ker(f) = \{0\}$

$\Rightarrow f$ is inj ∎

26) Prove $F(A) = \{f: A\to R$ with finite support$\}$ is an R-mod and $\Phi: R(A)\to M$ is R-mod homo

Very simple just check def

$F(A)$ is free on A, let its free basis be $\{e_a : a\in A\}$ where $e_a = 1$ at a, 0 elsewhere

hence $x\in F(A)$ has form $x = \sum_{a\in A} r_a e_a$.

F(A) is R-mod: $(F(A),+)$ is clearly abelian group since $0 = \sum 0e_a$, $-x = \sum -r_a e_a$

$(r+s)\cdot x = r\cdot x + s\cdot x$, $r(x+y) = r\cdot x + r\cdot y$ $\quad$ $rs\cdot x = r\cdot(s\cdot x)$ where $x,y\in F(A)$, $r,s\in R$

F(A) is R-mod homo: Let $\Phi(x) = \Phi(\sum r_a e_a) = \sum r_a f(a)$, by def of free x is

uniquely represented hence Φ is well defined, then $\Phi(rx+y) = \Phi(\sum(rr_a + s_a)e_a)$

$= \sum(rr_a + s_a)f(a) = r\sum r_a f(a) + \sum s_a f(a) = r\Phi(x) + \Phi(y)$ ∎ $\quad$ $y = \sum s_a e_a$

Date No.

★ 27) Let A, B be sets of same cardinality. Prove $F(A), F(B)$ are isomorphic

Let $\phi: A \to B$ be a bijection by Universal property we have

View diagonals as $j\phi, i\phi^{-1}$

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[d, leftrightarrow, "\phi"'] & F(A) \arrow[d, shift right, "\Psi"'] \\ B \arrow[r, hook, "j"'] & F(B) \arrow[u, shift right, "\Phi"'] \end{tikzcd}$$

Now $\Phi(\Psi i) = (\Phi j)\phi^{-1} = i\phi^{-1}\phi = i$, similarly $\Psi \Phi j = j$, by uniqueness it must be that $\Phi\Psi = 1_{F(A)}$, $\Psi\Phi = 1_{F(B)}$ $\Rightarrow$ $F(A) \cong F(B)$

28) $_RM$ is called $\boxed{\text{torsion module}}$, if for every $m \in M$, $\exists\, 0 \neq r \in R$ s.t. $r \cdot m = 0$

Prove every finite abelian group is a torsion $\mathbb{Z}$-module. Give counterex for converse

Let M be a finite abelian group, we know it may be viewed as a $\mathbb{Z}$-module where $n \cdot g = \overbrace{g + \dots + g}^{n \text{ times}}$, now for any $g \in G$ $\langle g \rangle \subseteq M$ hence $|\langle g \rangle| = k < \infty$ $\Rightarrow$ $\exists k$ s.t. $k \cdot g = 0$

Counterexample: $M = \bigoplus_{p \text{ prime}} \mathbb{Z}/p\mathbb{Z}$, for any $x \in M$, $p \cdot x = 0$ but M is infinite

29) $_RM$ is called $\boxed{\text{irreducible}}$ if $M \neq 0$ and the only submodules are 0 and M

Suppose M irred, $0 \neq m \in M$, Prove $M \cong Rm$

Let $\varphi: R \to M$ be R-mod homo, $\varphi(r) = r \cdot m$, now $m \in M \Rightarrow M \neq \emptyset$ hence $\varphi(R)$ which is a submodule of M must be M itself, By 1st iso theorem $R/\ker(\varphi) \cong \varphi(R) = M$, but clearly $Rm = \{r \cdot m : r \in R\} = \varphi(R)$ $\Rightarrow M = \varphi(R) = Rm$ as an isomorphism

30) Find all irreducible $\mathbb{Z}$-modules

Let M be an irred $\mathbb{Z}$-mod, consider the submodule generated by an element m, $(m) = \{n \cdot m : n \in \mathbb{Z}\} = \mathbb{Z}m$, if $m \neq 0$ then by irred we have $\mathbb{Z}m = M$ $\Rightarrow$ M is a cyclic abelian group.

It is not $\mathbb{Z}$ since this has non trivial submodules $2\mathbb{Z}$ for example $\Rightarrow$ $\mathbb{Z}$ must be finite, If $M = C_{n_1} \times \dots \times C_{n_k}$ then C_{n_i} is non-trivial subgroup $\Rightarrow M = C_n$, but if n is not prime every $d | n$, C_d is non-trivial subgroup we are left with $M = C_p$, p is prime

Date No.

31) $e \in R$ is $\boxed{\text{idempotent}}$ if $e^2 = e$. It is called central idempotent if it is in the center

Let e be a central idempotent and ${}_RM$. Prove $M = eM \oplus (1-e)M$

Clearly $eM, (1-e)M$ are submodules of M, then $m = em + 1-em \in eM \oplus (1-e)M$

$\Rightarrow M = eM + (1-e)M$. Then if $x \in eM \cap (1-e)M \Rightarrow x = em = (1-e)m'$

$\Rightarrow e^2m = (e-e^2)m = (e-e)m = 0$ but $e^2m = em \Rightarrow x = 0 \Rightarrow eM \cap (1-e)M = \{0\}$

that is $M = eM \oplus (1-e)M$

32) $M_R, {}_RN$, L is abelian group. $\beta : M \times N \to L$ is R-balanced. Show that group homo $\Phi : F(M \times N) \to L$ obtained from universal property

$$\begin{tikzcd} M\times n \arrow[r, "i"] \arrow[dr, "\beta"'] & F(M\times N) \arrow[d, "\Phi"] \\ & L \end{tikzcd}$$

of the free $\mathbb{Z}$-mods maps subgroup H, L is in $\ker(\Phi)$, hence $\psi : F(M\times N)/H \to L$ is group homo

$i(m,n) = (m,n)$ from universal property we obtain Φ s.t. $\Phi i(m,n) = \Phi(m,n) = \beta(m,n)$

Now we want to show $\Phi(H) \subseteq \ker(\Phi)$ that is show every generator of H goes to 0

① $\Phi((m+m',n) - (m,n) - (m',n)) = \beta(m+m',n) - \beta(m,n) - \beta(m',n)$ since R-mod homo

$= 0$ by property of R-balanced β

② $\Phi((m,n+n') - (m,n) - (m,n')) = \beta(m,n+n') - \beta(m,n) - \beta(m,n') = 0$

③ $\Phi((mr,n) - (m,rn)) = \beta(mr,n) - \beta(m,rn) = 0$

$\Big\}$ same argument as ①

Since each generator goes to 0, $\forall h \in H$, $\Phi(h) = 0 \Rightarrow H \subseteq \ker(\Phi)$, hence we get the induced group homomorphism $\psi : F(M\times N)/H \cong M \otimes_R N \to L$, This property is formally known as universal property of Quotient group