Date No.

Simple 9) i) Let A be R-algebra. Prove on ${}_RA$ we have $r\cdot(ab) = (r\cdot a)b = a(r\cdot b)$ $\forall r\in R,\ a,b\in A$

Let $f: R \to A$ be the ring homo defining A as R-algebra,

* (Recall $r\cdot a = f(r)a$ when we think of A as an R-module)

$$r\cdot(ab) = f(r)ab = \begin{cases} (f(r)a)b = (r\cdot a)b \\ af(r)b = a(r\cdot b) \end{cases} \blacksquare$$

ii) Conversely, A is ring and ${}_RA$ satisfies $r\cdot(ab) = (r\cdot a)b = a(r\cdot b)$.

Prove A is R-algebra with $f: R \to A$ defined by $f(r) = r\cdot 1_A$

A is unital ring by def (1_A exists), Now show it's a ring homo s.t. $f(R) \subseteq Z(A)$

$f(1_R) = 1_R \cdot 1_A = 1_A$,

$f(r+s) = (r+s)\cdot 1_A = r\cdot 1_A + s\cdot 1_A = f(r) + f(s)$

$f(rs) = (rs)\cdot 1_A = (r\cdot 1_A)(s\cdot 1_A) = f(r)f(s)$

$f(r)a = (r\cdot 1_A)a = r\cdot(1_A a) = r\cdot a = r\cdot(a 1_A) = a(r\cdot 1_A) = af(r)$ hence $f(R) \subseteq Z(A)$ $\blacksquare$

Simple 10) Let $\phi: V \to W$ be an R-mod homo. Prove $\ker(\phi)$, $\mathrm{Im}(\phi)$ are submodules of V, W resp

$\boxed{\text{Use Submodule Criterion}}$ Since $\phi(0_V) = 0_W$ it is clear that $0_V \in \ker(\phi)$, $0_W \in \mathrm{Im}(\phi)$

so they are both non-empty. Let $x, y \in \ker(\phi)$, $r \in R$, then

$\phi(x+ry) = \phi(x) + r\phi(y) = 0 + r\cdot 0 = 0 \in \ker(\phi)$

Let $x = \phi(u), y = \phi(w) \in \mathrm{Im}(\phi)$, $r \in R$, then

$x + ry = \phi(u) + r\phi(w) = \phi(u + rw) \in \mathrm{Im}(\phi)$ $\blacksquare$

Simple 11) Prove $\ker(\phi) = \{0_V\}$ iff ϕ is injective

($\Rightarrow$) Suppose $\phi(u) = \phi(v)$ then $\phi(u-v) = 0 \Rightarrow u-v \in \ker(\phi) \Rightarrow u-v = 0_V \Rightarrow u = v$

($\Leftarrow$) Let $v \in \ker(\phi)$ then $\phi(v) = 0_W = \phi(0_V)$, hence by injectivity $0_V = v$ $\blacksquare$

Only associativity is hard to check

12) Define RG-action on R by $\left(\sum r_g g\right) * x = \sum r_g x$. Prove R is RG-mod

$\boxed{\text{Check axioms}}$ ① $\left(\sum r_g g + \sum s_g g\right) * x = \sum (r_g + s_g) g * x = \sum (r_g + s_g) x = \sum r_g g * x + \sum s_g g * x$

② $\sum r_g g * (x + y) = \sum r_g x + \sum r_g y = \left(\sum r_g g\right) * x + \left(\sum r_g g\right) * y$

③ $\sum_{g\in G} r_g g * \left(\sum_{h\in G} s_h h * x\right) = \sum_{g\in G} r_g g * \sum_{h\in G} s_h x = \sum_{g,h\in G} r_g s_h x = \sum_{k\in G} \left(\sum_{gh=k} r_g s_h\right) k * x$

$= \left(\sum r_g g\right)\left(\sum s_h h\right) x$ since R is comm

Date No.

Simple 13) V, W are R-mod. Prove $Hom_R(V,W)$ is abelian group with $(\phi+\psi)(x) = \phi(x)+\psi(x)$

don't forget to check well-defined

$\boxed{Hom_R(V,W) \text{ is set of all R-mod homo from } V \to W}$ $\boxed{\text{Check abelian group axioms}}$

① Well defined: $(\phi+\psi)(rx+y) = \phi(rx+y) + \psi(rx+y) = r\phi(x)+\phi(y) + r\psi(x)+\psi(y)$

$r \in R,\ x,y \in V$ $= r(\phi+\psi)(x+y)$. $\phi+\psi \in Hom_R(V,W)$

② Associativity: Trivial

③ Identity: $0(v) = 0_W$, then $(\phi+0)x = \phi(x) + 0(x) = \phi(x)$

④ Inverse: $(-\phi)(x) = -\phi(x)$, $(\phi+(-\phi))(x) = \phi(x) + (-\phi(x)) = 0_W$

⑤ Comm: Trivial

Simple 14) X, Y, V are R-mod $\beta : X \to Y$ is R-mod homo

i) Prove $\beta_* : Hom_R(V,X) \to Hom_R(V,Y)$, $\beta_*(f) = \beta \circ f$ is group homo

Let $f,g \in Hom_R(V,X)$, $v \in V$ then $\boxed{\text{Check group homo } \beta_*(f+g) = \beta_* f + \beta_* g}$

$\beta_*(f+g)(v) = \beta((f+g)(v)) = \beta(f(v)+g(v)) = \beta(f(v)) + \beta(g(v))$

$(\beta\circ f)(v) + (\beta\circ f)(v) = (\beta\circ f + \beta\circ g)(v) = (\beta_*(f) + \beta_*(g))(v)$

ii) Prove $\beta^* : Hom_R(Y,V) \to Hom_R(X,V)$, $\beta^*(f) = f\circ\beta$ is group homo

Let $f,g \in Hom_R(Y,V)$, $x \in X$

$\beta^*(f+g)(x) = (f+g)\circ\beta(x) = (f+g)(\beta(x)) = f(\beta(x)) + g(\beta(x)) = (f\circ\beta)(x) + (g\circ\beta)(x)$

$= (\beta^*(f) + \beta^*(g))(x)$

Simple 15) F is field. Show F-mod homo are linear transformations over F

$\boxed{\text{Compare def}}$ We know F-mod are just vector spaces over F.

Def of linear transformation, let $\phi : V \to W$, V, W are F-mod s.t. $\forall u,v \in V, \lambda \in F$

$\phi(u+v) = \phi(u) + \phi(v)$, $\phi(\lambda u) = \lambda\phi(u)$ which is the definition of F-mod homo

Simple 16) Show $\mathbb{Z}$-mod homo are abelian group homo

By def of $\mathbb{Z}$-mod homo, if $f : A \to B$ where ${}_{\mathbb{Z}}A, {}_{\mathbb{Z}}B$ then $f(a+a') = f(a) + f(a')$

which implies f is a group homo and A, B are Abelian group by def of $\mathbb{Z}$-mod