Tutorial #8 Date No.

57) (i) $\mathrm{Hom}_R(\oplus V_i, W) \cong \prod \mathrm{Hom}(V_i, W)$

Define $\Phi: \mathrm{Hom}_R(\oplus V_i, W) \to \prod \mathrm{Hom}_R(V_i, W)$ by $\Phi(f) = (f i_i)$ where $i_i: V_i \to \oplus V_i$ is canonical inclusion.

① $\Phi(f) \in \prod \mathrm{Hom}(V_i, W)$ (Note it may not be in $\oplus \mathrm{Hom}(V_i, W)$ since $f i_i$ could be non-zero for infinite i)

② Group homo: $\Phi(f+g) = ((f+g)i_i) = (f i_i + g i_i) = (f i_i) + (g i_i) = \Phi(f) + \Phi(g)$

Define $\Psi: \prod \mathrm{Hom}_R(V_i, W) \to \mathrm{Hom}_R(\oplus V_i, W)$ by $\Psi((f_i))((v_i)) = \sum f_i(v_i)$

$(v_i) \in \oplus V_i$ hence finitely many non-zero hence $f_i(v_i)$ is finitely many non-zero and the sum works

① $\Psi\Phi(f)((v_i)) = \Psi((f i_i))((v_i)) = \sum f i_i(v_i) = f((v_i))$

② $\Phi\Psi((f_i))((v_i)) = \Phi(\sum f_i(v_i)) = \Phi(\sum f i_i(v_i))$ where $f = (f_i)$

$= \Phi(f(v_i)) = (f i_i(v_i)) = f((v_i)) = ((f_i))((v_i))$

Hence Φ, Ψ are inverses

(ii) $\mathrm{Hom}_R(V, \prod W_j) \cong \prod \mathrm{Hom}_R(V, W_j)$

$\Phi: \mathrm{Hom}_R(V, \prod W_j) \to \prod \mathrm{Hom}(V, W_j)$ where $\Phi(f) = (\pi_j f)$ where $\pi_j: \prod W_j \to W_j$

$\Psi: \prod \mathrm{Hom}_R(V, W_j) \to \mathrm{Hom}_R(V, \prod W_j)$ where $\Psi((f_j))(v) = (f_j(v))$

Same as above

58) G is finite group, $H \leq G$. k is a field, consider group algebras kG, kH

(i) Consider kG as right kH-mod. Let $\{x_1, \dots, x_m\}$ be complete set of left coset representatives of H in G. Show $kG \cong \oplus \mathrm{span}_k\{x_i h : h \in H\} \cong \oplus kH$

Recall kG is a vector space over k with basis $\{g : g \in G\}$. Now we have $G = \sqcup x_i H = \{g : g \in G\}$ hence kG is direct sum of subspaces $H_i = \mathrm{span}_k\{x_i h : h \in H\}$, that is $kG = \oplus H_i$.

$\boxed{\text{WTS: } H_i \cong kH}$ Firstly H_i is right kH submod of kG, let $v = \sum a_h x_i h \in H_i$, $b = \sum b_{h'} h' \in kH$, then $v \cdot b = x_i \sum_{h,h'} a_h b_{h'} (hh') \in H_i$. Now there is iso $\varphi_i: kH \to H_i$, $\varphi_i(\sum a_h h) = \sum a_h x_i h = x_i(\sum a_h h)$

① Group homo: $\varphi_i(a \cdot b) = x_i ab = \varphi_i(a) \cdot b$, othere axioms are easy

② $\varphi_i(a) = 0 \Rightarrow x_i a = 0 \Rightarrow a = 0$, if $a \in H_i \Rightarrow a = \sum a_h x_i h$ so φ_i is bij

Hence $H_i \cong kH$, and $kG = \oplus H_i \cong \oplus kH$

(ii) Conclude kG is free right kH-mod

kH is a free right kH-mod, hence by Ex47: $\oplus H_i$ is a free right kH-mod $\Rightarrow kG = \oplus H_i$ is free

$$\begin{array}{ccccccc}
\cdots \longrightarrow & X^n & \xrightarrow{d^X_{n+1}} & X^{n+1} & \xrightarrow{d^X_{n+2}} & \cdots \\
& \downarrow{\scriptstyle \alpha^n} & & \downarrow{\scriptstyle \alpha^{n+1}} & & \\
\longrightarrow & Y^n & \xrightarrow{d^Y_{n+1}} & Y^{n+1} & \xrightarrow{d^Y_{n+2}} & \cdots \\
& \downarrow{\scriptstyle \beta^n} & & \downarrow{\scriptstyle \beta^{n+1}} & & \\
\longrightarrow & Z^n & \xrightarrow{d^Z_{n+1}} & Z^{n+1} & \xrightarrow{d^Z_{n+2}} & \cdots
\end{array}$$

For any left kH-mod V, the kG-mod $\mathrm{Ind}_H^G(V) = kG \otimes_{kH} V$

(iii) For any exact seq $0 \to X \to Y \to Z \to 0$ of kH-mods. Prove
$0 \to \mathrm{Ind}_H^G(X) \to \mathrm{Ind}_H^G(Y) \to \mathrm{Ind}_H^G(Z) \to 0$
Since by (ii) kG is free right R-mod $\Rightarrow$ by $\boxed{\text{Cor 42}}$ kG is flat kH-mod, hence by $\boxed{\text{Prop 40.1}}$ we get the desired result

(iv) Conclude we have exact covariant functor $\mathrm{Ind}_H^G : kH\text{-mod} \to kG\text{-mod}$
$\boxed{\text{Cor 41}}$ gives us this since ${}_{kG}kG_{kH}$ and we get functor
$\mathrm{Ind}_H^G : kH\text{-mod} \to kG\text{-mod}$

59) Q is inj R-mod. Prove that is $\cdots \to V_{n+1} \xrightarrow{d_{n+1}} V_n \xrightarrow{d_n} V_{n-1} \to \cdots$ is exact chain complex then $\cdots \to \mathrm{Hom}(V_{n-1}, Q) \xrightarrow{d_n^*} \mathrm{Hom}(V_n, Q) \xrightarrow{d_{n+1}^*} \mathrm{Hom}(V_{n+1}, Q) \to \cdots$ is exact cochain complex

$\boxed{\text{WTS: } \mathrm{Im}(d_n^*) = \ker(d_{n+1}^*)}$ $\subseteq$: $d_{n+1}^* \circ d_n^* = (d_n d_{n+1})^* = 0$ since $\mathrm{Im}(d_{n+1}) = \ker(d_n)$
hence $\mathrm{Im}(d_n^*) \subseteq \ker(d_{n+1}^*)$

$\supseteq$: Let $\phi \in \ker(d_{n+1}^*)$, that is $\phi \in \mathrm{Hom}(V_n, Q)$ and $d_{n+1}^*(\phi) = \phi d_{n+1} = 0$
$\boxed{\text{WTS: } \phi \in \mathrm{Im}(d_n^*), \text{ that is } \exists \psi \in \mathrm{Hom}(V_{n-1}, Q) \text{ s.t. } d_n^*(\psi) = \psi d_n = \phi}$

$$\begin{array}{ccc}
\mathrm{Im}(d_n) & \xrightarrow{i} & V_{n-1} \\
\downarrow{\scriptstyle \theta} & \swarrow{\scriptstyle \psi} & \\
Q & &
\end{array}$$

Define $\theta : \mathrm{Im}(d_n) \to Q$ where $\theta(d_n(v)) = \phi(v)$, this is well defined since if $d_n(v) = d_n(v') \Rightarrow v - v' \in \ker(d_n) = \mathrm{Im}(d_{n+1})$ hence $\phi(v - v') = 0 = \phi(v) - \phi(v')$. By Inj of Q $\exists \psi : V_{n-1} \to Q$ s.t. $\psi i = \theta$. Now $d_n^*(\psi)(v) = \psi d_n(v) = \psi i d_n(v) = \theta d_n(v) = \phi(v)$ as desired $\blacksquare$

60) D is flat right R-mod. Prove that is $\to V_{n+1} \xrightarrow{d_{n+1}} V_n \xrightarrow{d_n} V_{n-1} \to \cdots$ is exact then $\to D \otimes_R V_{n+1} \xrightarrow{1 \otimes d_{n+1}} D \otimes_R V_n \xrightarrow{1 \otimes d_n} V_{n-1} \to \cdots$ is exact
Take SES $0 \to \mathrm{Im}(d_{n+1}) \xrightarrow{i} V_n \xrightarrow{d_n} \mathrm{Im}(d_n) \to 0$ is exact, hence since D is flat $0 \to D \otimes_R \mathrm{Im}(d_{n+1}) \xrightarrow{1 \otimes i} D \otimes_R V_n \xrightarrow{1 \otimes d_n} D \otimes_R \mathrm{Im}(d_n) \to 0$ is exact
That is $\ker(1 \otimes d_n) = \mathrm{Im}(1 \otimes i) = D \otimes_R \mathrm{Im}(d_{n+1}) = \mathrm{Im}(1 \otimes d_{n+1})$, hence chain is exact

61) Let $0 \to X^\bullet \xrightarrow{\alpha} Y^\bullet \xrightarrow{\beta} Z^\bullet \to 0$ be SES of complexes. Let $a \in H^n(Z^\bullet)$
$a = z + \mathrm{Im}(d_n)$ where $z \in \ker(d^Z_{n+1}) : Z^n \to Z^{n+1}$

(i) Show $\exists y \in Y^n$ s.t. $\beta_n(y) = z$, and $\exists!\, x \in \ker(d^X_{n+2}) \subseteq X^{n+1}$ s.t. $\alpha_{n+1}(x) = d^Y_{n+1}(y)$
$\beta_n : Y^n \to Z^n$ is surj by exactness, $\exists y \in Y^n$ s.t. $\beta_n(y) = z$. We have
$d^Z_{n+1}(\beta_n(y)) = \beta_{n+1}(d^Y_{n+1}(y)) = d^Z_{n+1}(z) = 0 \Rightarrow d^Y_{n+1}(y) \in \ker(\beta_{n+1}) = \mathrm{Im}(\alpha_{n+1})$
hence $\exists x \in X^{n+1}$ s.t. $\alpha_{n+1}(x) = d^Y_{n+1}(y)$. x is unique since α is inj,
and $x \in \ker(d^X_{n+2})$ since $\alpha_{n+2}(d^X_{n+2}(x)) = d^Y_{n+2}(\alpha_{n+1}(x)) = d^Y_{n+2} d^Y_{n+1}(y) = 0$
hence by inj of α_{n+1} $d^X_{n+2}(x) = 0$

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(ii) $z + Im(d_n^z) = z' + Im(d_n^z)$, $\beta_n(y) = z$, $\beta_n(y') = z'$, $\alpha_{n+1}(x) = d_{n+1}^y(y)$, $\alpha_{n+1}(x') = d_{n+1}^y(y')$

Show $x + Im(d_{n+1}^x) = x' + Im(d_{n+1}^x)$

Since $z - z' \in Im(d_n^z) \Rightarrow \exists z'' \in Z^{n-1}$ s.t. $d_n^z(z'') = z - z'$, since β_{n-1} is surj

$\exists w \in Y^{n-1}$ s.t. $\beta_{n-1}(w) = z''$ we have $\beta_n(d_n^y(w) - (y-y')) = \beta_n(d_n^y(w)) - \beta_n(y-y')$

$= d_n^z \beta_{n-1}(w) - (z-z') = d_n^z(z'') - (z-z') = 0$ hence $d_n^y(w) - (y-y') \in ker(\beta_n) = Im(\alpha_n)$

$\Rightarrow \exists v \in X^n$ s.t. $\alpha_n(v) = d_n^y(w) - (y-y')$, hence

$\alpha_{n+1}(d_{n+1}^x(v)) = d_{n+1}^y(\alpha_n(v)) = d_{n+1}^y d_n^y w - d_{n+1}^y(y-y') = -d_{n+1}^y(y-y') = \alpha_{n+1}(x'-x)$

Since α_{n+1} is inj $\Rightarrow d_{n+1}^x(v) = x' - x \Rightarrow x' = x + d_{n+1}^x(v)$ ■

(iii) Conclude $\delta_n : H^n(Z^\bullet) \to H^{n+1}(X)$ where $\delta(z + Im(d_n)) = x + Im(d_{n+1})$

By (i) (ii) the map is well defined

(iv) Prove δ_n is group homo

Let $z, z' \in ker(d_{n+1}^z)$, $x, x' \in ker(d_{n+2}^x)$ s.t. $\beta_n(y) = z$, $\beta_n(y') = z'$

$\alpha_{n+1}(x) = d_{n+1}^y(y)$, $\alpha_{n+1}(x') = d_{n+1}^y(y')$. $\beta_n(y+y') = z + z'$, $\alpha_{n+1}(x+x') = d_{n+1}^y(y+y')$

$\delta_n(z + z' + Im(d_n)) = x + x' + Im(d_{n+1}) = \delta_n(z + Im(d_n)) + \delta_n(z' + Im(d_n))$

62) $coker(f) = W/Im(f)$, $f: V \to W$

$$\begin{array}{ccccccccc} & & X^\bullet & & Y^\bullet & & Z^\bullet & & \\ 0 & \to & X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z & \to & 0 \\ & & d_1^x \downarrow f & & d_1^y \downarrow g & & d_1^z \downarrow h & & \\ 0 & \to & X' & \xrightarrow{\alpha'} & Y' & \xrightarrow{\beta'} & Z' & \to & 0 \end{array}$$

Use LES to prove $0 \to ker(f) \xrightarrow{\alpha} ker(g) \xrightarrow{\beta} ker(h) \xrightarrow{\delta} coker(f) \xrightarrow{\alpha^*} coker(g) \xrightarrow{\beta^*} coker(h) \to 0$

By $\boxed{T17.2}$ we get LES $0 \to H^0(X^\bullet) \to H^0(Y^\bullet) \to H^0(Z^\bullet) \to H^1(X^\bullet) \to H^1(Y^\bullet) \to H^1(Z^\bullet) \to \cdots$

And $H^2(X^\bullet) = 0$ since $X^2 = 0$. $H^0(X^\bullet) = ker(f)$, $H^0(Y^\bullet) = ker(g)$, $H^0(Z^\bullet) = ker(h)$

$H^1(X^\bullet) = ker(d_2)/Im(d_1) = X'/Im(f) = coker(f)$ similarly for others we get desired LES

63) Prove Horseshoe Lemma. Let $0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0$ be SES and $P_\bullet \to X$, $Q_\bullet \to Z$

Now we prove by induction on the columns

<u>Base Case:</u>

$$\begin{array}{ccccccccc} 0 & \to & P_0 & \xrightarrow{i_0} & P_0 \oplus Q_0 & \xrightarrow{\pi_0} & Q_0 & \to & 0 \\ & & d_0 \downarrow & & \downarrow \phi_0 & \overset{r}{\swarrow} & \downarrow \delta_0 & & \\ 0 & \to & X & \xrightarrow{\alpha} & Y & \xrightarrow{\beta} & Z & \to & 0 \end{array}$$

Since Q_0 is proj $\exists r : Q_0 \to Y$ st. $\beta r = \delta_0$

Let $\phi_0 : P_0 \oplus Q_0 \to Y$ where $\phi_0(a+b) = \alpha d_0(a) + r(b)$

Then it is clear diagram commutes

By Snake Lemma we get $0 \to ker(d_0) \to ker(\phi_0) \to ker(\delta_0) \to coker(d_0) \to coker(\phi_0) \to coker(\delta_0) \to 0$

Since d_0, δ_0 are surj $\Rightarrow coker(d_0) = 0 = coker(\delta_0) \Rightarrow coker(\phi_0) = 0 \Rightarrow \phi_0$ is surj

as desired, and $0 \to ker(d_0) \to ker(\phi_0) \to ker(\delta_0) \to 0$ is Exact ■

<u>I.H:</u> For $k < n$, ϕ_k is surjection onto $ker(\phi_{k-1}) \subseteq P_{k-2} \oplus Q_{k-2}$

$0 \to ker(d_k) \to ker(\phi_k) \to ker(\delta_k) \to 0$ is exact

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Final case:

$$\begin{tikzcd} 0 \arrow[r] & P_n' \arrow[r, "i_n"] \arrow[d, "d_n"] & P_n \oplus Q_n \arrow[r, "\pi_n"] \arrow[d, "\phi_n"] & Q_n \arrow[r] \arrow[d, "\delta_n"] \arrow[dl, dashed, "\gamma"'] & 0 \\ 0 \arrow[r] & \ker(d_{n-1}) \arrow[r, "i_{n-1}"] & \ker(\phi_{n-1}) \arrow[r, "\pi_{n-1}"] & \ker(\delta_{n-1}) \arrow[r] & 0 \end{tikzcd}$$

The second row is exact by IH. Now let $\gamma : Q_n \to \ker(\phi_{n-1})$ be lift of δ_n, $\phi_n(a+b) = i_{n-1}d_n(a) + \gamma(b)$. By Snake Lemma we get $0 \to \ker(d_n) \to \ker(\phi_n) \to \ker(\delta_n) \to \operatorname{coker}(d_n) \to \operatorname{coker}(\phi_n) \to \operatorname{coker}(\delta_n) \to 0$

Now since $\ker(d_{n-1}) = \operatorname{Im}(d_n)$, $\ker(\delta_n) = \operatorname{Im}(\delta_n)$ by Proj res $\Rightarrow \operatorname{coker}(d_n) = \operatorname{coker}(\delta_n) = 0$
$\Rightarrow \operatorname{coker}(\phi_n) = 0 \Rightarrow \phi_n$ is surj and we get SES $0 \to \ker(d_n) \to \ker(\phi_n) \to \ker(\delta_n) \to 0$ as desired.

64) Let $f : V \to V'$ be R-mod homo, RW. Show that for $n \geq 0$ we have induced group homo $\mathcal{F}(f)_n : \operatorname{Ext}_R^n(V', W) \to \operatorname{Ext}_R^n(V, W)$. Furthermore if $g : V' \to V''$, $\mathcal{F}(g)_n : \operatorname{Ext}_R^n(V'', W) \to \operatorname{Ext}_R^n(V', W)$ then $\mathcal{F}(g \circ f)_n = \mathcal{F}(f)_n \circ \mathcal{F}(g)_n$

Let $P_\cdot, P_\cdot', P_\cdot''$ be proj Res of V, V', V'' resp, by $\boxed{\text{Prop 4}}$ we get $\star$, then $\operatorname{Hom}(-,W)$ gives us $\star\star$

$$\star \quad \begin{tikzcd} \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] \arrow[d, "f_1"] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "f_0"] & V \arrow[d, "f"] \\ \arrow[r, "d_2'"] & P_1' \arrow[r, "d_1'"] \arrow[d, "g_1"] & P_0' \arrow[r, "\varepsilon'"] \arrow[d, "g_0"] & V' \arrow[d, "g"] \\ \arrow[r, "d_2''"] & P_1'' \arrow[r, "d_1''"] & P_0'' \arrow[r, "\varepsilon''"] & V'' \end{tikzcd}$$

$$\star\star \quad \begin{tikzcd} 0 \arrow[r] & \operatorname{Hom}(P_0, W) \arrow[r, "d_1^*"] & \operatorname{Hom}(P_1, W) \arrow[r, "d_2^*"] & X^\cdot \\ 0 \arrow[r] & \operatorname{Hom}(P_0', W) \arrow[r, "d_1'^*"] \arrow[u, "f_0^*"] & \operatorname{Hom}(P_1', W) \arrow[r, "d_2'^*"] \arrow[u, "f_1^*"] & Y^\cdot \\ 0 \arrow[r] & \operatorname{Hom}(P_0'', W) \arrow[r, "d_1''^*"] \arrow[u, "g_0^*"] & \operatorname{Hom}(P_0'', W) \arrow[r, "d_2''^*"] \arrow[u, "g_1^*"] & Z^\cdot \end{tikzcd}$$

Now by $\boxed{\text{Prop 1}}$ we get induced group homo $\mathcal{F}(f)_n : H^n(Y^\cdot) = \operatorname{Ext}_R^n(V', W) \to H^n(X^\cdot) = \operatorname{Ext}_R^n(V, W)$, $\mathcal{F}(g)_n : \operatorname{Ext}_R^n(V'', W) \to \operatorname{Ext}_R^n(V', W)$ where $\mathcal{F}(f)_n(\bar{\alpha}) = \overline{f_n^*(\alpha)} = \overline{\alpha f_n}$, $\mathcal{F}(g)_n(\bar{\beta}) = \overline{g_n^*(\beta)} = \overline{\beta g_n}$. Similarly omitting middle we get $\mathcal{F}(g \circ f)_n : \operatorname{Ext}_R^n(V'', W) \to \operatorname{Ext}_R^n(V, W)$ where
$\mathcal{F}(g \circ f)_n(\bar{\alpha}) = \overline{(g_n f_n)^*(\alpha)} = \overline{\alpha g_n f_n} = \overline{f_n^*(g_n^*(\alpha))} = f_n^*(\overline{g_n^*(\alpha)}) = \mathcal{F}(f)_n(g_n^*(\bar{\alpha}))$
$= \mathcal{F}(f)_n \mathcal{F}(g)_n(\bar{\alpha})$