Tutorial #9 Date No.

65) Prove prop 17.11, that is given ${}_R P$ TFAE: ① P is proj

② $\mathrm{Ext}^1_R(P,B)=0$ $\forall_R B$ ③ $\mathrm{Ext}^n_R(P,B)=0$ $\forall_R B$, $\forall n\geq 1$

③ => ② is Trivial

② => ① $\boxed{T17.10}$ Tells us that if $0\to X\to Y\to Z\to 0$ is exact, then $\exists$ LES

$0\to \mathrm{Hom}(P,X)\to \mathrm{Hom}(P,Y)\to \mathrm{Hom}(P,Z)\to \mathrm{Ext}^1_R(P,X)\to \cdots$

But by ② $\mathrm{Ext}^1_R(P,X)=0$, hence by $\boxed{PS.30.1}$ P is Proj

① => ③ Take the trivial proj res $\cdots 0\to P\xrightarrow{Id} P\to 0$, Now taking $\mathrm{Hom}(P_\bullet,B)$ gives

$0\to \mathrm{Hom}(P,B)\to 0\to 0$, which by def implies $\mathrm{Ext}^n_R(P,B)=0$ $\forall n\geq 1$

66) Direct sum of proj is proj $\boxed{Ex47}$, Direct product of inj is inj $\boxed{Ex54}$

① $\forall j\in J$ $(P(j))_\bullet \twoheadrightarrow V_j$ is proj res of V_j. Prove direct sum of proj res is proj res of $\oplus V_j$, show $\mathrm{Ext}^n_R(\oplus V_j, W)\cong \prod \mathrm{Ext}^n_R(V_j,W)$

Let $V=\oplus V_j$ and let $P_n=\bigoplus P(j)_n$, that is direct sum of all n-th terms in $P(j)$ and $d_n: P_{n+1}\to P_n$ defined by $d_n=\oplus d(j)_n$, $\varepsilon=\oplus\varepsilon(j)$

For any j $\cdots\xrightarrow{d(j)_2} P(j)_1 \xrightarrow{d(j)_1} P(j)_0 \xrightarrow{\varepsilon(j)} V_j\to 0$ Picture to help

$\boxed{\text{WTS:}}$ $\cdots\xrightarrow{\oplus d(j)_2}\oplus P(j)_1\xrightarrow{\oplus d(j)_1}\oplus P(j)_0\xrightarrow{\oplus\varepsilon(j)}\oplus V_j\to 0$ Visualize the problem.

$\cdots\xrightarrow{d_2} P_1\xrightarrow{d_1} P_0\xrightarrow{\varepsilon} V\to 0$ Show row 2's proj res

① We know P_i is proj by $\boxed{Ex47}$

② $\mathrm{Im}(d_n)=\mathrm{Im}(\oplus d(j)_n)=\oplus \mathrm{Im}(d(j)_n)\overset{*}{=}\oplus \ker(d(j)_{n-1})=\ker(d_{n-1})$

$*$By exactness of $(P(j))_\bullet\to V_j$, we are left to show exactness of ε that is ε is surj. Let $v=(v_j)\in V$, since $\varepsilon(j)$ is surj $\exists x_j\in P(j)_0$ s.t. $\varepsilon(j)(x_j)=v_j$, since most $v_j=0$ => most $x_j=0$ and $(x_j)\in P_0$ s.t. $\varepsilon((x_j))=(\varepsilon(j)(x_j))=(v_j)=v$

This shows direct sum of proj res is proj res of V

Now take $\mathrm{Hom}(-,W)$ to get $0\to \mathrm{Hom}(P_0,W)\xrightarrow{d_1^*}\mathrm{Hom}(P_1,W)\xrightarrow{d_2^*}\cdots$

By $\boxed{Ex57}$ $\mathrm{Hom}(P_n,W)=\mathrm{Hom}(\oplus P(j)_n,W)\cong\prod \mathrm{Hom}(P(j)_n,W)$, say via Φ_n

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}(P_0,W) \arrow[r, "d_1^*"] \arrow[d, "\Phi_1"] & \mathrm{Hom}(P_1,W) \arrow[r, "d_2^*"] \arrow[d, "\Phi_2"] & \cdots \\
0 \arrow[r] & \prod \mathrm{Hom}(P(j)_0,W) \arrow[r, "\delta_1"] & \prod \mathrm{Hom}(P(j)_1,W) \arrow[r, "\delta_2"] & \cdots
\end{tikzcd}$$

Hence $\mathrm{Ext}^n_R(V,W)=\ker(d^*_{n+1})/\mathrm{Im}(d^*_n)$

$\cong \ker(\delta_{n+1})/\mathrm{Im}(\delta_n)$

$=\prod \ker(d^*_{n+1}(j))/\mathrm{Im}(d^*_n(j))\cong\prod \mathrm{Ext}^n_R(V_j,W)$

(ii) For each $j\in J$, $W_j\hookrightarrow (Q(j))_\bullet$ is inj res of W_j. Prove direct product of inj res is inj res of $\prod W_j$. Show $\mathrm{Ext}^n_R(V,\prod W_j)\cong\prod\mathrm{Ext}^n_R(V,W_j)$

This is just the dual argument of ① Use $\boxed{Ex54}$ and $\boxed{Ex57.2}$

$\partial_{n+1}(\partial_n(x,y)) = \partial_{n+1}(-d^X_{n-1}(x), d^Y_n(y) - f_{n-1}(x))$

$= (-d^X_n(-d^X_{n-1}(x)), d^Y_{n+1}(d^Y_n(y) - f_{n-1}(x)) - f_n(-d^X_{n-1}(x)))$

$= (d^X_n d^X_{n-1}(x), d^Y_{n+1} d^Y_n(y) - d^Y_{n+1} f_{n-1}(x) + f_n d^X_{n-1}(x)) \in (X_{n-3}, /$

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(iii) Prove $\mathrm{Tor}^R_n(V, \oplus W_j) \cong \oplus \mathrm{Tor}^R_n(V, W_j)$

Let $(Q(j))_\bullet \to W_j$ be proj res on W_j, then by (i) we have $Q_\bullet = \oplus Q(j)_\bullet$ is proj res for $W = \oplus W_j$. Now $\mathrm{Tor}^R_n(V, W) = H_n(V \otimes_R Q_\bullet)$.

By $\boxed{\text{Ex39}}$ $V \otimes_R Q_\bullet = V \otimes_R (\oplus Q(j)_\bullet) \cong \oplus (V \otimes_R (Q(j)_\bullet))$

Kernels and images are computed component wise hence Homology commuts with direct sums, that is $H_n(V \otimes_R Q_\bullet) \cong H_n(\oplus(V \otimes_R (Q(j)_\bullet)))$

$\cong \oplus H_n(V \otimes_R (Q(j)_\bullet)) \cong \oplus \mathrm{Tor}^R_n(V, W_j)$ as desired.

67) $f: X_\bullet \to Y_\bullet$ map of chain complexes $\mathrm{cone}(f)$ is chain complex where $\mathrm{cone}(f)_n = X_{n-1} \oplus Y_n$

$\partial_n(x,y) = (-d^X_{n-1}(x), d^Y_n(y) - f_{n-1}(x)) = (-dx(x), dy(y) - f(x))$ $\forall x \in X_{n-1}, y \in Y_n$

Can be viewed as $\partial_n = \begin{bmatrix} -dx & 0 \\ -f & dy \end{bmatrix} : \begin{matrix} X_{n-1} \\ \oplus \\ Y_n \end{matrix} \to \begin{matrix} X_{n-2} \\ \oplus \\ Y_{n-1} \end{matrix}$

$$\begin{tikzcd} X_n \arrow[r, "d^X_n"] \arrow[d, "f_n"] & X_{n-1} \arrow[r, "d^X_{n-1}"] \arrow[d, "f_{n-1}"] & X_{n-2} \arrow[d, "f_{n-2}"] \\ Y_n \arrow[r, "d^Y_n"'] & Y_{n-1} \arrow[r, "d^Y_{n-1}"'] & Y_{n-2} \end{tikzcd}$$

where $\partial_n(x,y) = \begin{bmatrix} -dx & 0 \\ -f & dy \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -dx(x) \\ dy(y) - f(x) \end{bmatrix}$

(i) Prove $\mathrm{cone}(f)$ is a chain complex. $\boxed{\text{WTS: } \partial_{n-1}\partial_n = 0}$

$\partial_{n-1}\partial_n(x,y) = \partial_{n-1}(-d^X_{n-1}(x), d^Y_n(y) - f_{n-1}(x)) = (-d^X_{n-2}(-d^X_{n-1}(x)), d^Y_{n-1}(d^Y_n(y) - f_{n-1}(x)) - f_{n-2}(-d^X_{n-1}(x)))$

$= (d^X_{n-2} d^X_{n-1}(x), d^Y_{n-1} d^Y_n(y) - d^Y_{n-1} f_{n-1}(x) + f_{n-2} d^X_{n-1}(x)) = (0,0)$ as desired since $d^X_{n-2} d^X_{n-1} = 0$,

$d^Y_{n-1} d^Y_n = 0$, $f_{n-2} d^X_{n-1} = d^Y_{n-1} f_{n-1}$.

Let $X[-1]$ be chain complex obtained from X by shifting indices $X[-1]_n = X_{n-1}$

and $d[-1]_n : X[-1]_n \to X[-1]_{n-1}$ is $d[-1]_n(x) = -d^X_{n-1}$

(ii) Prove we have SES of complexes $0 \to Y \xrightarrow{\gamma} \mathrm{cone}(f) \xrightarrow{\delta} X[-1] \to 0$

$\gamma(y) = (0,y)$, $\delta(x,y) = -x$

$\boxed{\text{WTS: } 0 \to Y_n \xrightarrow{\gamma} X_{n-1} \oplus Y_n \xrightarrow{\delta} X[-1]_n \to 0 \text{ is exact}}$

$\mathrm{Im}(\gamma) = 0 \oplus Y_n = \ker(\delta)$, so sequence is exact. Now we show following commutes

$$\begin{tikzcd} Y_n \arrow[r, "\gamma_n"] \arrow[d, "d^Y_n"'] & X_{n-1} \oplus Y_n \arrow[r, "\delta_n"] \arrow[d, "\partial_n"] & X[-1]_n = X_{n-1} \arrow[d, "d[-1]_n = -d^X_{n-1}"] \\ Y_{n-1} \arrow[r, "\gamma_{n-1}"] & X_{n-2} \oplus Y_{n-1} \arrow[r, "\delta_{n-1}"] & X[-1]_{n-1} = X_{n-2} \end{tikzcd}$$

(1) $\partial_n \gamma_n(y) = \partial_n(0,y) = (0, d^Y_n(y)) = \gamma_{n-1}(d^Y_n(y))$

(2) $d[-1]_n \delta_n(x,y) = -d^X_{n-1}(-x) = d^X_{n-1}(x) = \gamma_{n-1}(-d^X_{n-1}(x), *) = \gamma_{n-1}\partial(x,y)$

Hence the diagram commutes

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68) $D_R, {}_RB$ Prove $\mathrm{Tor}_0^R(D,B) \cong D\otimes_R B$)

Let $P_\bullet \twoheadrightarrow B$, that is $\to P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} B \to 0$ is exact, which induces exact

$D\otimes_R P_1 \xrightarrow{1\otimes d_1} D\otimes_R P_0 \xrightarrow{1\otimes\varepsilon} D\otimes_R B \to 0 \Rightarrow \mathrm{Im}(1\otimes d_1) = \ker(1\otimes\varepsilon)$, then

$\mathrm{Tor}_0^R(D,B) \cong D\otimes_R P_0 / \mathrm{Im}(1\otimes d_1) \cong D\otimes_R P_0/\ker(1\otimes\varepsilon) \cong D\otimes_R B$ by 1st iso

69) Prove prop 14.1: Homology groups $\mathrm{Tor}_n^R(D,B)$ are indep of proj res

$g: P_\bullet \to P_\bullet'$

$f: P_\bullet' \to P_\bullet$

Let $P_\bullet, P_\bullet'$ be 2 proj res of B, following same proof as $\boxed{76}$ we get

$$\begin{tikzcd}
{} \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] \arrow[d, "f_1g_1-1"'] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "f_0g_0-1"] \arrow[dl, "s_0"] & B \arrow[r] \arrow[d, "Id_B"] & 0 \\
{} \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"'] & P_0 \arrow[r, "\varepsilon"'] & B \arrow[r] & 0
\end{tikzcd}$$

where $f_ig_i - 1 = d_i s_{i-1} + s_i d_{i+2}$

Now appy $D\otimes_R -$ which preserves arrows

$\boxed{\text{WTS: } D\otimes P_\bullet, D\otimes P_\bullet' \text{ are homotopy Equiv}}$

That is $(1\otimes f_n)(1\otimes g_n) \simeq Id_{D\otimes P_\bullet}$, and

$(1\otimes g_n)(1\otimes f_n) \simeq Id_{D\otimes P_\bullet'}$. Now we have

$$\begin{tikzcd}
{} \arrow[r] & D\otimes P_2 \arrow[r, "1\otimes d_2"] \arrow[d, "P\otimes(f_2g_2-1)"'] & D\otimes P_1 \arrow[r, "1\otimes d_1"] \arrow[d, "1\otimes(f_1g_1-1)"] \arrow[dl, "1\otimes s_1"'] & D\otimes P_0 \arrow[r] \arrow[d, "P\otimes(f_0g_0-1)"] \arrow[dl, "1\otimes s_0"'] & 0 \\
{} \arrow[r] & D\otimes P_2 \arrow[r] & D\otimes P_1 \arrow[r] & D\otimes P_0 \arrow[r] & 0
\end{tikzcd}$$

$1_D\otimes(f_ng_n - 1_{P_n}) = 1_D\otimes f_ng_n - 1_D\otimes 1_{P_n}$

$\qquad \| \qquad = (1_D\otimes f_n)(1_D\otimes g_n) - 1_{D\otimes P_n}$

$(1\otimes d_n)(1\otimes s_{n-1}) + (1\otimes s_n)(1\otimes d_2)$

That is $(1\otimes f)(1\otimes g_\bullet) \simeq Id_{D\otimes P_\bullet}$, similarly $(1\otimes g)(1\otimes f) \simeq ID_{D\otimes P_\bullet'}$.

Finally we know that If $D\otimes P_\bullet$ is homotopy equiv $D\otimes P_\bullet'$, then the homology groups are isomorphic, that is $\mathrm{Tor}_n^R(D,B) = H_n(D\otimes P_\bullet) \cong H_n(D\otimes P_\bullet') = \mathrm{Tor}_n^R(D,B)$.