Tutorial #7 Date No.

49) ${}_R V$

i) $Hom_R(V,-) : R\text{-mod} \to Ab$ is covariant functor

① Let $F: R\text{-mod} \to Ab$ where for R-mod W $F(W) = Hom_R(V,W)$

② If $f: W \to W'$, then $F(f) : Hom_R(V,W) \to Hom_R(V,W')$ where $F(f)(\alpha) = f\circ\alpha$

We've already proved well definedness and group homo before $\boxed{\text{Prop 27}}$ for example

ⓐ $F(Id_W)(\alpha) = Id_W \circ \alpha = \alpha \Rightarrow F(Id_W) = Id_{Hom_R(V,W)}$

ⓑ $g: W' \to W''$ then $F(g\circ f)(\alpha) = g\circ f\circ\alpha = g\circ(f\circ\alpha) = F(g)(f\circ\alpha) = F(g)F(f)(\alpha)$

$\Rightarrow F(g\circ f) = F(g)F(f)$

ii) $Hom_R(-,V) : R\text{-mod} \to Ab$ is contravariant functor

① Let $G: R\text{-mod} \to Ab$ where for R-mod W, $G(W) = Hom_R(W,V)$

② If $f: W \to W'$, then $G(f) : Hom_R(W',V) \to Hom_R(W,V)$ where $G(f)(\alpha) = \alpha\circ f$

Well definedness and group homo have been proved before $\boxed{\text{Theo 33}}$

ⓐ $G(Id_W)(\alpha) = \alpha\circ Id_W = \alpha \Rightarrow G(Id_W) = Id_{Hom_R(W,V)}$

ⓑ $g: W' \to W''$ then $G(g\circ f)(\alpha) = \alpha\circ g\circ f = (\alpha\circ g)\circ f = G(f)(\alpha\circ g) = G(f)G(g)(\alpha)$

$\Rightarrow G(g\circ f) = G(f)G(g)$

50) Suppose $X \in \mathcal{C}$, what is needed $\mathcal{F}: Mor(X,-) : \mathcal{C} \to Ab$ is a covariant functor

① We require that $Mor(X,Y)$ is abelian group, that is if $f,g \in Mor(X,Y)$

$f+g = g+f \in Mor(X,Y)$, $0 \in Mor(X,Y)$ s.t. $f+0 = f$ and $-f \in Mor(X,Y)$

② For each $f: Y \to Z$ in $\mathcal{C}$, $\mathcal{F}(f) : Mor(X,Y) \to Mor(X,Z)$ is a group homo

where $\mathcal{F}(f)(\alpha) = f\circ\alpha$, in order to be a group homo, that is

$\mathcal{F}(f)(\alpha+\beta) = \mathcal{F}(f)(\alpha) + \mathcal{F}(f)(\beta)$ we require $f\circ(\alpha+\beta) = f\circ\alpha + f\circ\beta$

The rest follows

51) Let R be ID. ${}_R M$ is divisible if $\forall r \neq 0$ $rM = M$

ⓘ Suppose $Q \neq 0$ divisible $\mathbb{Z}$-mod. Prove Q is not proj

Suppose Q is proj, then by $\boxed{\text{P30.4}}$ it is direct summand of free $\mathbb{Z}$-mod F, $F = Q \oplus F'$

where Q is a submod of F. $\boxed{\text{Claim: } Q \subseteq nF \ \forall n \in \mathbb{Z}^+}$, let $m \in Q$, since Q is divisible

$Q = nQ$ $\exists m' \in Q$ s.t. $m = nm' \in nF$ ∎

$\boxed{\text{Claim: } \bigcap_{n=1}^{\infty} nF = 0}$ Let F have free basis S, let $0 \neq x \in \bigcap_{n=1}^{\infty} nF$, by assumption

$\exists n_1, \dots, n_r, s_1, \dots, s_r$ s.t. $x = \sum n_i s_i$, now take $n \geq 2\max\{|n_i|\}$, $x \in nF$

$\Rightarrow x = nz$ where $z = \sum^{k} m_i s_i' \in F$ $m_i \in \mathbb{Z}$, $s_i' \in S$, hence

$\sum_{i=1}^{r} n_i s_i = x = nz = \sum_{i=1}^{k} n m_i s_i' \Rightarrow r = k$, $s_i = s_i' \Rightarrow n_i = nm_i$ but $n_i = nm_i \geq 2|n_i| m_i$

$\to\leftarrow$ hence $x = 0$ ∎

Since $Q \subseteq nF$ $\forall n \in \mathbb{Z}^+ \Rightarrow Q \subseteq \bigcap nF = 0$ $\to\leftarrow$ Q is non zero

Date No.

(ii) Deduce $\mathbb{Q}$ is divisible and hence not a proj $\mathbb{Z}$-mod

Clearly for $n \in \mathbb{Z}^+$ $\mathbb{Q} = n\mathbb{Q}$ since $\frac{r}{s} = n\frac{r}{ns}$, by (i) $\mathbb{Q}$ is not proj $\mathbb{Z}$-mod

52) Prove $\boxed{\text{Prop } 10.34}$

(i) => (ii) Take $0 \to X \xrightarrow{\alpha} Y \to Y/Im(\alpha) \to 0$, this is exact, hence by (i)
$0 \to Hom(Y/Im(\alpha), I) \to Hom(Y, I) \xrightarrow{\alpha^*} Hom(X, I) \to 0$ is exact
hence α^* is surj $\Rightarrow \exists f \in Hom(Y, I)$ s.t. $\alpha^*(f) = g = f\circ\alpha$

(ii) => (iii) We have exact seq $0 \to I \xrightarrow{i} Y$, hence $\exists f \in Hom(Y, I)$ s.t.
$f\circ\alpha = Id_I$, by $\boxed{\text{Prop}25}$ the seq splits and hence $Y = Im(i) \oplus Y'$
$= I \oplus Y'$

(iii) => (i) Suppose $0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0$ is exact, by $\boxed{\text{Theo } 33}$ we are only
left to show $Hom(Y, I) \xrightarrow{\alpha^*} Hom(X, I) \to 0$ is exact iff α^* is surj.

$$\begin{tikzcd} X \arrow[r, hook, "\alpha"] \arrow[d, "g"'] & Y \arrow[dl, "f"'] \arrow[d, "h"] \\ I \arrow[r, shift left, "i"] & J \arrow[l, shift left, "\pi"] \end{tikzcd}$$

Let $g \in Hom(X, I)$ $\boxed{\text{WTS: } \exists f \in Hom(Y, I) \text{ s.t. } \alpha^*(f) = f\circ\alpha = g}$
Using Hint Let J be injective R-mod s.t. $I \subseteq J$, by (iii) $J = I \oplus J'$
by inj of J, $\exists h: Y \to J$ s.t. $h\alpha = ig$. Now take $f: Y \to I$ s.t.
$f = \pi h$, then $\alpha^*(f) = f\alpha = \pi h \alpha = \pi i g = g$ ∎

53) $\Omega = \{(f', Y') : Im(\alpha) \subseteq Y' \subseteq Y$ and $f': Y' \to \mathbb{Q}$ s.t. $f'\alpha = g\}$ with partial order
$(f', Y') \leq (f'', Y'')$ iff $Y' \subseteq Y''$ and $f''|_{Y'} = f'$ satisfies Zorn lemma

(1) Ω is non-empty was showed

(2) Every chain has a maximal element: Let C be a chain in Ω
then take $W = \bigcup_{(f', Y') \in C} Y'$, then $Im(\alpha) \subseteq W \subseteq Y$ $\boxed{\text{Claim: } W \text{ is submod of } Y}$
Let $w, w' \in W$, $r \in R$, suppose $w \in Y'$, $w' \in Y''$, $(f', Y'), (f'', Y'') \in C$
Assume $(f', Y') \leq (f'', Y'')$ then $w + w' \in Y'' \subseteq W$ and $rw \in Y' \subseteq W$ ∎
Define $f: W \to \mathbb{Q}$ by $f(w) = f'(w)$ if $w \in Y'$, then $f(w+w') = f''(w+w')$
$= f''(w) + f''(w') = f(w) + f(w')$ and $f(rw) = f'(rw) = rf'(w) = rf(w)$
$\Rightarrow (W, f) \in \Omega$ and it is the maximal element of C, hence Zorn Lemma applies ∎

54) Let $\{Q_i : i \in I\}$ be inj Rmods. Prove $\prod Q_i$ is inj. $i: R \to I$
Let I be left ideal of R, $g: I \to Q = \prod Q_i$, let $\pi_i: Q \to Q_i$ be canonical
map, then $g_i = \pi_i \circ g: I \to Q_i$ has lift $f_i: R \to Q_i$ by Baer
Take $f: R \to Q$ where $f(r) = (f_i(r)) \in Q$, Now we check $f|_I = g$
Let $a \in I$ $f(a) = (f_i(a)) \overset{*}{=} (g_i(a)) = g(a)$ (* This is true since $f_i|_I = g_i$)

55) ${}_RM$. A $\boxed{\text{proj cover}}$ P of M is a "smallest" proj R-mod s.t. P projects onto M.
Proj ${}_RP$ is Proj cover of M if there exists a surj R-mod homo $f: P \to M$ s.t. for any proj ${}_RQ$, R-mod homo $g: Q \to P$ s.t. fg is surj then g is surj. f is called $\boxed{\text{essential map}}$

(i) If P, P' are proj covers of M $\Rightarrow P \cong P'$

$$\begin{tikzcd} & P' \arrow[dl, shift left, "g'"] \arrow[dr, "f'"] & \\ P \arrow[ur, shift left, "g"] \arrow[rr, two heads, "f"'] & & M \end{tikzcd}$$

By def there exists surj R-mod homo $f: P \to M$, $f': P' \to M$. By $\boxed{P30.2}$ there is a lift $g: P \to P'$ s.t. $f'g = f$. Now f is surj hence by Proj cover of P', g is surj. Take SES $0 \to Ker(g) \to P \xrightarrow{g} Im(g) = P' \to 0$, since P' is proj the SES splits and hence by $\boxed{P25.2}$ $\exists g': P' \to P$ s.t. $gg' = Id_{P'}$. Now $fg' = f'gg' = f'$, hence by Proj cover of P, g' is surj. Now we show $ker(g) = \{0\} \Rightarrow g$ is inj: Let $x \in ker(g)$ $x \in P$, hence by surj of g', $\exists y \in P'$ s.t. $x = g'(y) = g'(g(g'(y))) = g'(g(x)) = g'(0) = 0$.
Hence g is isomorphism ∎

(ii) Show $\mathbb{Z}$-mod $\mathbb{Z}/2\mathbb{Z}$ has no proj cover

$$\begin{tikzcd} & P \arrow[dr, "f"] & \\ \mathbb{Z} \arrow[ur, "h"] \arrow[rr, two heads, "g"'] & & \mathbb{Z}/2\mathbb{Z} \end{tikzcd}$$

Suppose it has a proj cover P. Let $f: P \to \mathbb{Z}/2\mathbb{Z}$ be surj, now $\mathbb{Z}/2\mathbb{Z}$ is generated by $\bar{1}$, take canonical surj $g: \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$. Since $\mathbb{Z}$ is free $\Rightarrow$ proj, there is a lift $h: \mathbb{Z} \to P$, $fh = g$, which by proj cover is surj. By 1st iso $\mathbb{Z}/Ker(h) \cong Im(h) = P$, hence $ker(h) = n\mathbb{Z}$.
We have seen in $\boxed{\text{pg 392, Example 4}}$ that $\mathbb{Z}/n\mathbb{Z}$ is not proj for $n \geq 2$, if $n = 1$ $P = \{0\}$ →←
Hence it must be that $n = 0 \Rightarrow h$ is an iso. Let $x = h(1) \Rightarrow f(x) = fh(1) = g(1) = \bar{1}$, now take $3P = \{3p : p \in P\} \cong 3\mathbb{Z}$ is also free, hence proj, now consider $i: 3P \to P$, then $fi(3x) = f(3x) = 3f(x) = \bar{1}$, hence fi is surj and f is surj but i is clearly not →← proj cover of $\mathbb{Z}/2\mathbb{Z}$ ∎

56) ${}_RV$, ${}_RW$

(i) Show $V \oplus W$ is proj iff V, W are both proj

($\Leftarrow$) $\boxed{\text{Ex 47}}$ ($\Rightarrow$) Suppose $V \oplus W$ is proj, then it is direct sum of free mod F
$F = V \oplus W \oplus F'$ by associativity and commutativity of $\oplus$ $F = V \oplus (W \oplus F') = W \oplus (V \oplus F')$ making V, W direct summands of F and hence Proj ∎

(ii) Show $V \oplus W$ is inj iff V, W are both inj

($\Leftarrow$) $\boxed{\text{Ex 54}}$ since finite direct prod = direct sum ($\Rightarrow$) Suppose $V \oplus W$ is inj, We show V is inj

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "\psi"] \arrow[d, "g"] & Y \arrow[dl, dashed, "f"] \arrow[ddl, bend left, "f'"] \\ & V \arrow[d, shift left, "i"] & \\ & V \oplus W \arrow[u, shift left, "\pi"] & \end{tikzcd}$$

$0 \to X \xrightarrow{\psi} Y$ is exact, ψ is inj and $ig: X \to V \oplus W$, hence by inj of $V \oplus W$ $\exists f': Y \to V \oplus W$ s.t. $f'\psi = ig$ $\boxed{\text{WTS: } \exists f: Y \to V \text{ s.t. } f\psi = g}$ Define $\pi: Im(i) \to V$ which is iso then define $f = \pi f'$, then $f\psi = \pi f' \psi = \pi i g = g$. Hence, V is inj, similarly W is inj ∎

$(\pi_V \otimes 1_Y)(1 \otimes \alpha) = (\pi_V 1) \otimes (1_Y \circ \alpha) \in \otimes Y$

$(\pi_V \otimes 1_Y) \circ (1 \otimes \alpha) \circ (i_V \otimes 1_X) = (\pi_V \circ 1 \circ i_V) \otimes (1_Y \circ \alpha \circ 1_X) \in V \otimes_R Y$

Date No.

(iii) Prove $V \oplus W$ is flat iff both V, W are both flat right mods

($\Leftarrow$) Suppose V, W are both flat right R-mods, we use $\boxed{\text{Prop 10.2}}$.

Let $0 \to X \xrightarrow{\alpha} Y$ be exact, then by flatness $0 \to V \otimes_R X \xrightarrow{1_V \otimes \alpha} V \otimes_R Y$, $0 \to W \otimes_R X \xrightarrow{1_W \otimes \alpha} W \otimes_R Y$ are exact. $\boxed{\text{WTS: } F = V \oplus W,\ 0 \to F \otimes_R X \xrightarrow{1 \otimes \alpha} F \otimes_R Y \text{ is exact}}$

That is show $1 \otimes \alpha : F \otimes_R X \to F \otimes_R Y$ defined by $(1 \otimes \alpha)((v+w) \otimes x) = (v+w) \otimes \alpha(x)$ is inj. Suppose $(v+w) \otimes x \in \ker(1 \otimes \alpha)$, that is $(v+w) \otimes \alpha(x) = 0$, now by $\boxed{\text{Theo 17}}$ $(V \oplus W) \otimes_R Y \cong (V \otimes_R Y) \oplus (W \otimes_R Y)$, hence $(v+w) \otimes \alpha(x) = (v \otimes \alpha(x)) + (w \otimes \alpha(x)) = 0$ since it is in a direct sum, $v \otimes \alpha(x) = 0$, $w \otimes \alpha(x) = 0$, which by inj of $1 \otimes \alpha$ for V, W $\Rightarrow v \otimes x = 0$, $w \otimes x = 0$ $\Rightarrow (v+w) \otimes x = 0$ $\Rightarrow \ker(1 \otimes \alpha) = \{0\}$ ∎

($\Rightarrow$) Suppose $V \oplus W = F$ is flat, let $0 \to X \xrightarrow{\alpha} Y$ be exact, we get injection $1 \otimes \alpha : F \otimes_R X \to F \otimes_R Y$, Now we show V is flat by showing $1_V \otimes \alpha : V \otimes_R X \to V \otimes_R Y$ is inj, now let $i_V : V \to V \oplus W$, $\pi_V : V \oplus W \to V$ be the canonical inclusion, projection maps. then $1_V \otimes \alpha = (\pi_V \otimes 1_Y) \circ (1 \otimes \alpha) \circ (i_V \otimes 1_X)$ hence if $(1_V \otimes \alpha)(v \otimes x) = 0 \Rightarrow (\pi_V \otimes 1_Y) \circ (1 \otimes \alpha) \circ (i_V \otimes 1_X)(v \otimes x) = 0$

Now $(\pi_V \circ 1 \circ i_V)(v) = (v+0) \Rightarrow (1 \otimes \alpha)((v+0) \otimes x) = 0$, by inj of $1 \otimes \alpha$ $\Rightarrow (v+0) \otimes x = v \otimes x = 0$, hence $\ker(1_V \otimes \alpha) = \{0\}$ and V is flat, similarly W is flat ∎