Tutorial Questions

Exercise 1 Let R, S be rings and V, W be R- and S-modules respectively.

(i) Suppose that R is commutative. Show that the right R-action defined by $v*r := r\cdot v$ defines a right R-module.
(ii) Verify that the regular (left) module ${}_RR$ is an R-module.
(iii) Let $\alpha : R \to S$ be a ring homomorphism. Prove that $r * w := \alpha(r)\cdot w$ defines an R-module.
(iv) Let $\pi : R \to S$ be a surjective ring homomorphism and

$$\mathsf{Ann}(V) = \{r \in R : r\cdot v = 0 \text{ for all } v \in V\}$$

be the annihilator of V. Prove that $\mathsf{Ann}(V)$ is an ideal of R. Suppose further that $\mathsf{Ann}(V)$ contains $\ker \pi$. Prove that $s * v := r\cdot v$ defines an S-module where $\pi(r) = s$.
(v) Let G be an abelian group. Prove that G is an $\mathbb{Z}$-module with the action

$$n\cdot x = \begin{cases} \underbrace{x + x + \cdots + x}_{n \text{ times}} & \text{if } n \geq 0, \\ \underbrace{(-x) + (-x) + \cdots + (-x)}_{-n \text{ times}} & \text{if } n < 0. \end{cases}$$

Exercise 2 Let F be a field. Prove that every vector space V over F is an F-module and subspaces of V are submodules.

Exercise 3 Let V be an R-module. Show that $0\cdot v = 0$ and $(-1)\cdot v = -v$ for all $v \in V$.

Exercise 4 Let V be an R-module.

(i) Show that the sum of two submodules of V is a submodule (the sum $U + W$ is defined as $\{u + w : u \in U,\ w \in W\}$).
(ii) Show that the intersection of any non-empty collection of submodules of V is a submodule.

Exercise 5 Consider the abelian group $G := C_6 \times C_{10}$ as $\mathbb{Z}$-module. Find the annihilator of G.

Exercise 6 Let $\{V_i\}_{i\in I}$ be a collection of R-modules.

(i) The direct product $\prod_{i\in I} V_i$ is the set consisting of all sequences $(v_i)_{i\in I}$ where $v_i \in V_i$ for each $i \in I$ and addition of two sequences is defined componentwise. Define an R-action by $r * (v_i)_{i\in I} = (r\cdot v_i)_{i\in I}$. Prove that $\prod_{i\in I} V_i$ is an R-module.
(ii) The (external) direct sum $\bigoplus_{i\in I} V_i$ is the subset of $\prod_{i\in I} V_i$ consisting of sequences $(v_i)_{i\in I}$ such that almost all v_i's are zero. Prove that $\bigoplus_{i\in I} V_i$ is a submodule of $\prod_{i\in I} V_i$.

In the case when I is finite, say $I = \{1, \dots, n\}$, the notions $V_1 \times \cdots \times V_n$ and $V_1 \oplus \cdots \oplus V_n$ are the same.

Exercise 7 Let I be a left ideal of a ring R and V be an R-module. Prove that

$$IV := \{a_1\cdot v_1 + \cdots + a_m\cdot v_m : a_1, \dots, a_m \in I,\ v_1, \dots, v_m \in V,\ m \in \mathbb{N}\}$$

is a submodule of V.

Exercise 8 (Group Algebra) Let G be a group and R be a commutative ring. Define RG as the set consisting of formal sum of the form $\sum_{g\in G} r_g g$ where $r_g \in R$ and such that r_g's are almost all zero (finite support). The element $r_g \in R$ is called the coefficient of $g \in G$. Define addition and multiplication on RG as

$$\sum_{g\in G} r_g g + \sum_{g\in G} s_g g = \sum_{g\in G} (r_g + s_g) g,$$

$$\left(\sum_{g\in G} r_g g\right)\left(\sum_{g\in G} s_g g\right) = \sum_{g\in G}\left(\sum_{h\in G} r_{gh} s_{h^{-1}}\right) g$$

Show that RG is an R-algebra with the map $\phi : R \to RG$ defined as $\phi(r) = re_G$, that is, $\phi(r)$ is mapped into the formal sum where the coefficients of $g \in G$ are zero if $g \neq e_G$ and r if $g = e_G$.

Exercise 9

(i) Let A be an R-algebra. Prove that, as R-module, we have $r \cdot (ab) = (r \cdot a)b = a(r \cdot b)$ for all $r \in R$ and $a, b \in A$.

(ii) Conversely, suppose that A is a ring and an R-module satisfying $r \cdot (ab) = (r \cdot a)b = a(r \cdot b)$ for all $r \in R$ and $a, b \in A$. Prove that A is an R-algebra with the map $f : R \to A$ defined by $f(r) = r \cdot 1_A$.

Exercise 10 Let $\phi : V \to W$ be an R-module homomorphism. Prove that $\ker \phi$ and $\operatorname{im} \phi$ are submodules of V and W respectively.

Exercise 11 Let $\phi : V \to W$ be an R-module homomorphism. Prove that $\ker \phi = \{0\}$ if and only if ϕ is injective.

Exercise 12 Consider the group algebra RG. Define the RG-action on R by, for each $x \in R$,

$$\left(\sum_{g\in G} r_g g\right) * x = \sum_{g\in G} r_g x.$$

Prove that R is an RG-module. The module is called the trivial RG-module (for the group algebra RG).

Exercise 13 Let V, W be R-modules. Prove that the hom-set $\mathrm{Hom}_R(V, W)$ is an abelian group with the binary operation $(\phi + \psi)(v) := \phi(v) + \psi(v)$.

Exercise 14 Let X, Y, V be R-modules and $\beta : X \to Y$ be an R-module homomorphism.

(i) Prove that we have a group homomorphism

$$\beta_* : \mathrm{Hom}_R(V, X) \to \mathrm{Hom}_R(V, Y)$$

where $\beta_*(f) = \beta \circ f$.

(ii) Prove that we have a group homomorphism

$$\beta^* : \mathrm{Hom}_R(Y, V) \to \mathrm{Hom}_R(X, V)$$

where $\beta^*(f) = f \circ \beta$.

Exercise 15 Let F be a field. Show that F-module homomorphisms are linear transformations over F.

Exercise 16 Show that $\mathbb{Z}$-module homomorphisms are abelian group homomorphisms.

Exercise 17 Let V be an R-module. Suppose that $r \in Z(R)$. Define $\lambda_r : V \to V$ by $\lambda_r(v) = r \cdot v$. Prove that λ_r is an R-module homomorphism. Show that the conclusion is false without the assumption that $r \in Z(R)$.

Exercise 18 Consider the $\mathbb{Z}$-module $V = \mathbb{Z}/2\mathbb{Z}$. Compute $\mathsf{End}_{\mathbb{Z}}(V)$. More generally, prove that

$$\mathsf{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/\gcd(m,n)\mathbb{Z}.$$

Exercise 19 Let R be a commutative ring.

(i) Let V be an R-module. Prove that $\mathsf{Hom}_R(R, V) \cong V$ as R-modules where the map is given by $\alpha \mapsto \alpha(1)$.
(ii) Prove that $\mathsf{End}_R(R) \cong R$ as rings.

Exercise 20 Fix a positive integer n. For each $1 \leq i \leq n$, let U_i be a submodule of an R-module V_i. Prove that we have the following isomorphism of R-modules:

$$(V_1 \times \cdots \times V_n)/(U_1 \times \cdots \times U_n) \cong (V_1/U_1) \times \cdots \times (V_n/U_n).$$

Exercise 21 Let I be a left ideal of a ring R and consider the free R-module R^n of rank n. Prove that we have the following isomorphism of R-modules:

$$R^n/IR^n \cong \underbrace{(R/IR) \times \cdots \times (R/IR)}_{n \text{ times}}$$

where IR^n has been defined in Exercise 7.

Exercise 22 Let V be an R-module and A be a subset of V. Let

$$RA := \{r_1a_1 + \cdots + r_ma_m : r_1, \ldots, r_m \in R,\ a_1, \ldots, a_m \in A,\ m \in \mathbb{N}\}.$$

Recall that $R\emptyset = \{0\}$.

(i) Prove that RA is a submodule of V.
(ii) Prove that RA contains the subset A.
(iii) Prove that RA is the intersection of the collection of submodules U of V such that $A \subseteq V$. As such, RA is the smallest submodule of V containing A.

Exercise 23 Let N be a submodule of M. Prove that M is finitely generated if both N and M/N are finitely generated.

Exercise 24 Let R be a commutative ring. Show that $R^m \cong R^n$ if and only if $m = n$.
(Hint: Let I be a maximal ideal of R. Use Exercise 21.)

Exercise 25 Let M be the free $\mathbb{Z}$-module $\mathbb{Z} \times \mathbb{Z} \times \cdots$ and let $R = \mathsf{End}_{\mathbb{Z}}(M)$.

(i) Let $\phi(a_1, a_2, \ldots) = (0, a_1, a_2, \ldots)$ and $\psi(a_1, a_2, \ldots) = (a_2, a_3, \ldots)$. Show that $\psi\phi = 1$ but $\phi\psi \neq 1$. Conclude that R is not commutative.

(ii) Let

$$\alpha_1(a_1, a_2, a_3, \ldots) = (a_1, a_3, a_5, \ldots),$$
$$\alpha_2(a_1, a_2, a_3, \ldots) = (a_2, a_4, a_6, \ldots).$$

Prove that $\{\alpha_1, \alpha_2\}$ is a free basis for the regular R-module ${}_RR$.
(Hint: Let $\beta_1(a_1, a_2, \ldots) = (a_1, 0, a_2, 0, \ldots)$ and $\beta_2(a_1, a_2, \ldots) = (0, a_1, 0, a_2, \ldots)$. Show that $\alpha_i\beta_i = 1$, $\alpha_1\beta_2 = 0 = \alpha_2\beta_1$ and $\beta_1\alpha_1 + \beta_2\alpha_2 = 1$.)

(iii) Prove that $R \cong R^2$ as R-modules.

Exercise 26 Verify that the module $F(A)$ defined in the Universal Property of Free Modules is an R-module and the map $\Phi : F(A) \to M$ is a well-defined R-module homomorphism.

Exercise 27 Let A and B be sets of the same cardinality. Prove that the free modules $F(A), F(B)$ constructed in Universal Property of Free Modules are isomorphic.

Exercise 28 An R-module M is called a torsion module if, for every $m \in M$, there exists a nonzero element $r \in R$ such that $r \cdot m = 0$. Prove that every finite abelian group is a torsion $\mathbb{Z}$-module. Give an example showing the converse of the previous statement is incorrect.

Exercise 29 An R-module M is called irreducible (or simple) if $M \neq 0$ and the only submodules of M are 0 and M. Suppose that M is irreducible and let $0 \neq m \in M$. Prove that $M \cong Rm$.

Exercise 30 Find all irreducible $\mathbb{Z}$-modules.

Exercise 31 An element $e \in R$ is called an idempotent if $e^2 = e$. The idempotent is called a central idempotent if it belongs in the center of R, i.e., $re = er$ for all $r \in R$. Let e be a central idempotent and M is an R-module. Prove that M is equal to the direct sum of the submodules eM and $(1-e)M$ where

$$eM = \{em : m \in M\},$$
$$(1-e)M = \{(1-e)m : m \in M\}.$$

Exercise 32 Let M, N be right and left R-modules respectively and L be an abelian group. Suppose that $\beta : M \times N \to L$ is an R-balanced map. Show that the group homomorphism $\xi : F(M \times N) \to L$ obtained via the universal property

$$\begin{tikzcd} M \times N \arrow[r, "\iota"] \arrow[dr, "\beta"'] & F(M \times N) \arrow[d, "\xi"] \\ & L \end{tikzcd}$$

of the free $\mathbb{Z}$-module maps the subgroup H of $F(M \times N)$ generated by

$$(m+m', n) - (m,n) - (m',n),$$
$$(m, n+n') - (m,n) - (m,n'),$$
$$(mr, n) - (m, rn)$$

belongs in the kernel of ξ and hence induce a group homomorphism $\Phi : F(M \times N)/H \to L$.

Exercise 33 Verify the following bimodule structures.

(i) Let I be an ideal of R. Prove that R/I is an $(R/I, R)$-bimodule.
(ii) Let M be a left R-module and $S \subseteq Z(R)$. Prove that M is an (R, S)-bimodule where $m * s := sm$.

Exercise 34 Let R, S be rings, Y an (R, S)-bimodule and Z a right S-module. Prove that $\mathrm{Hom}_S(Y, Z)$ is a right R-module with the action given by $(\phi * r)(y) = \phi(ry)$ for all $r \in R$ and $y \in Y$.

Exercise 35

(i) Show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$ are isomorphic as $\mathbb{Q}$-modules.
(ii) Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ are not isomorphic as $\mathbb{C}$-modules.

Exercise 36 Let D be a right $\mathbb{Z}$-module and $m \in \mathbb{Z}$. We aim to prove that

$$D \otimes_{\mathbb{Z}} (\mathbb{Z}/m\mathbb{Z}) \cong D/mD$$

where $mD = \{md : d \in D\}$.

(i) Define a map $\beta : D \times (\mathbb{Z}/m\mathbb{Z}) \to D/mD$ by $\beta(d, \overline{a}) = ad + mD$. Prove that β is well-defined and $\mathbb{Z}$-balanced.
(ii) By the Universal Property of the Tensor Product, we have a group homomorphism $\Phi : D \otimes_{\mathbb{Z}} (\mathbb{Z}/m\mathbb{Z}) \to D/mD$ such that $\Phi(d \otimes \overline{a}) = ad + mD$. Prove that Φ is an isomorphism.

Exercise 37 Complete/Read the proof of the Associativity of the Tensor Product.

Exercise 38 Let R be a commutative ring and M, N be R-modules. Prove that $M \otimes_R N \cong N \otimes_R M$ as R-modules.

Exercise 39 Let I be an indexing set. Let M and N_i, one for each $i \in I$, be right and left R-modules respectively. Prove that

$$M \otimes_R \bigoplus_{i \in I} N_i \cong \bigoplus_{i \in I} (M \otimes_R N_i).$$

Exercise 40 *The conclusion of Exercise 39 is however not true if we replace direct sum with direct product. Let $S = \mathbb{Q}$, $R = \mathbb{Z}$ and $N_i = \mathbb{Z}/(2^i\mathbb{Z})$.

Exercise 41 Let N be an $\mathbb{Z}$-module. Let $\mathbb{Z}^\times$ be the set of nonzero integers. Define the relation $\sim$ on $\mathbb{Z}^\times \times N$ by $(m,x) \sim (n,y)$ if and only if $my = nx$.

(i) Prove that $\sim$ is an equivalence relation and let $\mathbb{Z}^{-1}N$ be the set of equivalence classes.

(ii) Prove that $\mathbb{Z}^{-1}N$ is a $\mathbb{Z}$-module (abelian group) with addition given by

$$[(m,x)] + [(n,y)] = [(mn, my+nx)].$$

(iii) Show that the map $\beta : \mathbb{Q} \times N \to \mathbb{Z}^{-1}N$ defined by

$$\beta(\frac{a}{b}, x) = [(b, ax)]$$

is an $\mathbb{Z}$-balanced map and hence it induces a $\mathbb{Z}$-module homomorphism $f : \mathbb{Q} \otimes_{\mathbb{Z}} N \to \mathbb{Z}^{-1}N$ such that $f(\frac{a}{b} \otimes x) = [(b, ax)]$. (You need to check that the map β is well-defined.)

(iv) Define $g : \mathbb{Z}^{-1}N \to \mathbb{Q} \otimes_{\mathbb{Z}} N$ by $g([(m,x)]) = \frac{1}{m} \otimes x$. Prove that g is the inverse of f in part (iii) and hence we have an isomorphism $\mathbb{Q} \otimes_{\mathbb{Z}} N \cong \mathbb{Z}^{-1}N$.

(v) Conclude that, $\frac{1}{m} \otimes x = 0$ in $\mathbb{Q} \otimes_{\mathbb{Z}} N$ if and only if $x = 0$.

(For this exercise, you may replace $\mathbb{Z}$ with any integral domain and $\mathbb{Q}$ with its field of fractions.)

Exercise 42 Prove Corollary 10.16 and Proposition 10.21.

Exercise 43 Let (α, β, γ) and $(\alpha', \beta', \gamma')$ be homomorphisms of short exact sequences (SES) of R-modules

$$\begin{array}{ccccccccc}
0 & \longrightarrow & X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & 0\ . \\
 & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} & & \\
0 & \longrightarrow & X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle\alpha'} & & \downarrow{\scriptstyle\beta'} & & \downarrow{\scriptstyle\gamma'} & & \\
0 & \longrightarrow & X'' & \longrightarrow & Y'' & \longrightarrow & Z'' & \longrightarrow & 0
\end{array}$$

(i) Prove that $(\alpha'\alpha, \beta'\beta, \gamma'\gamma)$ is a homomorphism of short exact sequence.

(ii) Suppose that (α, β, γ) is an isomorphism (respectively, equivalence) of SES. Prove that $(\alpha^{-1}, \beta^{-1}, \gamma^{-1})$ is an isomorphism (respectively, equivalence) of SES.

Exercise 44 Prove the second statement of the five lemma: Suppose that the rows are exact, f_1 is surjective and both f_2, f_4 are injective. Prove that f_3 is injective.

$$\begin{array}{ccccccccc}
M_1 & \xrightarrow{g_1} & M_2 & \xrightarrow{g_2} & M_3 & \xrightarrow{g_3} & M_4 & \xrightarrow{g_4} & M_5 \\
\downarrow{\scriptstyle f_1} & & \downarrow{\scriptstyle f_2} & & \downarrow{\scriptstyle f_3} & & \downarrow{\scriptstyle f_4} & & \downarrow{\scriptstyle f_5} \\
N_1 & \xrightarrow{h_1} & N_2 & \xrightarrow{h_2} & N_3 & \xrightarrow{h_3} & N_4 & \xrightarrow{h_4} & N_5
\end{array}$$

Exercise 45 Consider the short exact sequence of groups

$$1 \to A_n \xrightarrow{\iota} S_n \xrightarrow{\mathrm{sgn}} \{\pm 1\} \to 1.$$

(i) Show that there exists a group homomorphism $\gamma : \{\pm 1\} \to S_n$ such that $\mathrm{sgn}\,\gamma = \mathrm{id}_{\{\pm 1\}}$.

(ii) Show that, however, whenever $n \geq 3$, there is no group homomorphism $\delta : S_n \to A_n$ such that $\delta\iota = \mathrm{id}_{A_n}$.

This example shows that the existence of retraction of ι and section of sgn for (non-abelian groups) are not equivalent.

Exercise 46 Consider the exact sequence $\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \to 0$ and let $V = \mathbb{Z}/2\mathbb{Z}$. Verify that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})$ is the trivial group and hence the map $\pi^* : \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ is not surjective.

Exercise 47 Let $\{F_i : i \in I\}$ be a collection of free R-modules.

(i) Prove that the direct sum $\bigoplus_{i\in I} F_i$ is a free module.
(ii) For each $i \in I$, let P_i be a direct summand of F_i. Assume that $F_i = P_i \oplus Q_i$ for some submodule Q_i of F_i. Prove that $\bigoplus_{i\in I} F_i = \left(\bigoplus_{i\in I} P_i\right) \oplus \left(\bigoplus_{i\in I} Q_i\right)$.
(iii) Conclude that the direct sum of projective modules is projective.

Exercise 48 Let R-mod consist of the objects (left) R-modules and the class of morphisms of two objects X, Y is $\mathrm{Hom}_R(X, Y)$. The binary operation of morphisms is given by composition of functions. Prove that R-mod is a category. Conclude that $\mathbb{Z}$-modcat is the category of abelian groups which is denoted by Ab.

Exercise 49 Let V be an R-module.

(i) Prove that $\mathrm{Hom}_R(V, -) : R$-mod $\to$ Ab is a covariant functor.
(ii) Prove that $\mathrm{Hom}_R(-, V) : R$-mod $\to$ Ab is a contravariant functor.

Exercise 50 Suppose that we begin with a category $\mathscr{C}$ and an object X in $\mathscr{C}$. What are the important ingredients so that $\mathscr{F} := \mathrm{Mor}(X, -) : \mathscr{C} \to \mathrm{Ab}$ is a covariant functor?

Exercise 51 Let R be an integral domain. An R-module M is divisible if, for every $0 \neq r \in R$, we have $rM = M$.

(i) Suppose that Q is a nonzero divisible $\mathbb{Z}$-module. Prove that Q is not projective.
(ii) Deduce that $\mathbb{Q}$ is divisible and hence not a projective $\mathbb{Z}$-module.

Exercise 52 Prove Proposition 10.34, that is, let I be an R-module, prove that the following are equivalent statements:

(i) For any R-modules X, Y, Z, if $0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0$ is exact, then $0 \to \mathrm{Hom}_R(Z, I) \xrightarrow{\beta^*} \mathrm{Hom}_R(Y, I) \xrightarrow{\alpha^*} \mathrm{Hom}_R(X, I) \to 0$ is exact.
(ii) For any R-modules X, Y, if $0 \to X \xrightarrow{\alpha} Y$ is exact, then, for any R-module homomorphism $g : X \to I$, there exists an R-module homomorphism $f : Y \to I$ such that $f \circ \alpha = g$.
(iii) If I is isomorphic to a submodule of an R-module Y, then $I \mid Y$.

(Hint: For (iii) $\Rightarrow$ (i), use the fact that every R-module is contained in an injective R-module.)

Exercise 53 Complete the proof of Baer's criterion by verifying that the set

$$\Omega = \{(f', Y') : \mathrm{im}\,\alpha \subseteq Y' \subseteq Y \text{ and } f' : Y' \to Q \text{ such that } f'\alpha = g\},$$

with the partial order $(f', Y') \leq (f'', Y'')$ if and only if $Y' \subseteq Y''$ and $f''|_{Y'} = f'$, satisfies Zorn's lemma.

Exercise 54 Let $\{Q_i : i \in I\}$ be injective R-modules. Prove that $\prod_{i\in I} Q_i$ is injective.

Exercise 55 (Projective Cover) Let M be an R-module. A projective cover P of M is a 'smallest' projective R-module such that P projects onto M. More precisely, a projective R-module P is a projective cover of M if there exists a surjective R-module homomorphism $f : P \to M$ such that, for any projective R-module Q and R-module homomorphism $g : Q \to P$ such that fg is surjective, the map g is surjective. In the literature, f is called an essential map.

(i) If P, P' are projective covers of M, prove that $P \cong P'$.
(ii) Show that the $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$ has no projective cover.

(However, every R-module has injective hull by the Eckmann–Schopf Theorem. Let M be an R-module. An injective hull Q of M is a 'smallest' injective R-module such that M injects into Q. More precisely, an injective R-module Q is an injective hull of M if there exists an injective R-module homomorphism $f : M \to Q$ such that, for any injective R-module Q' and R-module homomorphism $g : Q \to Q'$ such that gf is injective, the map g is injective. Injective hulls of a module is unique up to isomorphism.)

Exercise 56 Let V, W be R-modules.

(i) Prove that $V \oplus W$ is projective if and only if both V, W are projective.
(ii) Prove that $V \oplus W$ is injective if and only if both V, W are injective.
(iii) Prove that $V \oplus W$ is flat if and only if both V, W are flat right R-modules.

Exercise 57 Prove the following isomorphisms.

(i) $\mathsf{Hom}_R(\bigoplus_{i\in I} V_i, W) \cong \prod_{i\in I} \mathsf{Hom}_R(V_i, W)$
(ii) $\mathsf{Hom}_R(V, \prod_{j\in J} W_j) \cong \prod_{j\in J} \mathsf{Hom}_R(V, W_j)$

Exercise 58 Let G be a finite group and H be a subgroup of G. Let k be a field. Consider the group algebras kG and kH.

(i) Consider kG as right kH-module in the natural way. Let $\{x_1, \dots, x_m\}$ be a complete set of left coset representatives of H in G. Prove that

$$kG = \bigoplus_{i=1}^{m} \mathsf{span}_k\{x_i h : h \in H\} \cong \bigoplus_{i=1}^{m} kH$$

as right kH-modules.
(ii) Conclude that kG is a free right kH-module.

For any left kH-module V, the kG-module $\mathsf{Ind}_H^G V := kG \otimes_{kH} V$ is called the induction of V from H to G.

(iii) For any exact sequence $0 \to X \to Y \to Z \to 0$ of kH-modules, prove that we have an exact sequence of induced modules

$$0 \to \mathsf{Ind}_H^G X \to \mathsf{Ind}_H^G Y \to \mathsf{Ind}_H^G Z \to 0.$$

(iv) Conclude that we have an exact covariant functor $\mathsf{Ind}_H^G : kH\text{-mod} \to kG\text{-mod}$.

Exercise 59 Let Q be an injective R-module. Recall that the contravariant functor $\mathsf{Hom}_R(-, Q) : R\text{-mod} \to \mathsf{Ab}$ is exact on short exact sequence, i.e., for any exact sequence $0 \to U \to V \to W \to 0$ of R-modules, we have an exact sequence $0 \to \mathsf{Hom}_R(W, Q) \to \mathsf{Hom}_R(V, Q) \to \mathsf{Hom}_R(U, Q) \to 0$ of abelian groups. Prove that, if we have an exact chain complex

$$\cdots \to V_{n+1} \xrightarrow{d_{n+1}} V_n \xrightarrow{d_n} V_{n-1} \to \cdots$$

of R-modules, then we have an exact cochain complex

$$\cdots \to \mathsf{Hom}_R(V_{n-1}, Q) \xrightarrow{d_n^*} \mathsf{Hom}_R(V_n, Q) \xrightarrow{d_{n+1}^*} \mathsf{Hom}_R(V_{n+1}, Q) \to \cdots .$$

Exercise 60 Let D be a flat right R-module, i.e., the functor $D \otimes_R -$ is exact (on SES). Prove that, if we have an exact chain complex

$$\cdots \to V_{n+1} \xrightarrow{d_{n+1}} V_n \xrightarrow{d_n} V_{n-1} \to \cdots$$

of left R-modules, then we have an exact chain complex

$$\cdots \to D \otimes_R V_{n+1} \xrightarrow{1 \otimes d_{n+1}} D \otimes_R V_n \xrightarrow{1 \otimes d_n} D \otimes_R V_{n-1} \to \cdots .$$

Exercise 61 This exercise defines the connecting homomorphism δ_n in Theorem 17.2 (The Long Exact Sequence in Cohomology). Let $0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0$ be a a short exact sequence of cochain complexes. Let $a \in \mathsf{H}^n(Z)$ and $a = z + \mathsf{im}\, d_n$ where $z \in \mathsf{ker}\, d_{n+1} : Z^n \to Z^{n+1}$.

- (i) Show that there exist $y \in Y^n$ such that $\beta_n(y) = z$ and a unique $x \in \mathsf{ker}\, d_{n+2} \subseteq X^{n+1}$ such that $\alpha(x) = d(y)$.
- (ii) Let $z + \mathsf{im}\, d_n = z' + \mathsf{im}\, d_n$ and y, y', x, x' such that $\beta(y) = z$, $\beta(y') = z'$, $\alpha(x) = d(y)$ and $\alpha(x') = d(y')$. Show that $x + \mathsf{im}\, d_{n+1} = x' + \mathsf{im}\, d_{n+1}$.
- (iii) Conclude that we have a map $\delta_n : \mathsf{H}^n(Z) \to \mathsf{H}^{n+1}(X)$ defined by $\delta_n(z + \mathsf{im}\, d_n) = x + \mathsf{im}\, d_{n+1}$.
- (iv) Prove that the connecting homomorphism δ_n is a group homomorphism.

Exercise 62 (Snake Lemma) The cokernel $\mathsf{coker}\, f$ of an R-module homomorphism $f : V \to W$ is defined as $W/\, \mathsf{im}\, f$. Suppose that we have a commutative diagram below with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "\alpha"] \arrow[d, "f"] & Y \arrow[r, "\beta"] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & X' \arrow[r, "\alpha'"] & Y' \arrow[r, "\beta'"] & Z' \arrow[r] & 0
\end{tikzcd}$$

Use the Long Exact Sequence in Cohomology to prove that we have an exact sequence

$$0 \to \mathsf{ker}\, f \xrightarrow{\alpha} \mathsf{ker}\, g \xrightarrow{\beta} \mathsf{ker}\, h \xrightarrow{\delta} \mathsf{coker}\, f \xrightarrow{\alpha'} \mathsf{coker}\, g \xrightarrow{\beta'} \mathsf{coker}\, h \to 0.$$

(The Snake Lemma has a slightly more general version where α is not necessarily injective nor β' is surjective. In this case, we do not have exactness at $\mathsf{ker}\, f$ nor $\mathsf{coker}\, h$.)

Exercise 63 Prove Horseshoe Lemma. (You will need the Snake Lemma.)

Exercise 64 Let $f : V \to V'$ be an R-module homomorphism and let W be another R-module. Show that, for each $n \geq 0$, we have an induced group homomorphism $\mathscr{F}(f)_n : \mathsf{Ext}^n_R(V', W) \to \mathsf{Ext}^n_R(V, W)$. Furthermore, if $g : V' \to V''$ is another R-module homomorphism and $\mathscr{F}(g)_n : \mathsf{Ext}^n_R(V'', W) \to \mathsf{Ext}^n_R(V', W)$ is the induced group homomorphism, then

$$\mathscr{F}(g \circ f)_n = \mathscr{F}(f)_n \circ \mathscr{F}(g)_n.$$

Exercise 65 Prove Proposition 17.11. That is, given an R-module P, prove that the following statements are equivalent.

(i) P is projective
(ii) $\mathsf{Ext}^1_R(P, B) = 0$ for all R-modules B
(iii) $\mathsf{Ext}^n_R(P, B) = 0$ for all R-modules B and all $n \geq 1$

Exercise 66 We have proved that the direct sum of projective modules is projective (in Exercise 47) and direct product of injective is injective (in Exercise 54).

(i) For each $j \in J$, let $(P(j))_\bullet \twoheadrightarrow V_j$ be a projective resolution of V_j. Prove that the direct sum of the projective resolutions is a projective resolution of $\bigoplus_{j\in J} V_j$. Use this to show that

$$\mathsf{Ext}^n_R(\bigoplus_{j\in J} V_i, W) \cong \prod_{j\in J} \mathsf{Ext}^n_R(V_j, W).$$

(ii) For each $j \in J$, let $W_j \hookrightarrow (Q(j))_\bullet$ be an injective resolution of W_j. Prove that the direct product of the injective resolutions is a injective resolution of $\prod_{j\in J} W_j$. Use this to show that

$$\mathsf{Ext}^n_R(V, \prod_{j\in J} W_i) \cong \prod_{j\in J} \mathsf{Ext}^n_R(V, W_j).$$

(iii) Prove that $\mathsf{Tor}^R_n(V, \bigoplus_{j\in J} W_j) \cong \bigoplus_{j\in J} \mathsf{Tor}^R_n(V, W_j)$.

Exercise 67 (The mapping cone) Let $f : X \to Y$ be a map of chain complexes X, Y of R-modules. The mapping cone $\mathsf{cone}(f)$ is the chain complex with degree n part is $X_{n-1} \oplus Y_n$ and the differential is given by

$$\partial_n(x, y) = (-d_{n-1}(x), d_n(y) - f_{n-1}(x)) = (-d_X(x), d_Y(y) - f(x))$$

for every $x \in X_{n-1}$ and $y \in Y_n$, i.e., the differential ∂_n can be viewed as the following (2×2)-matrix:

$$\partial_n = \begin{bmatrix} -d_X & 0 \\ -f & d_Y \end{bmatrix} : \begin{matrix} X_{n-1} \\ \oplus \\ Y_n \end{matrix} \to \begin{matrix} X_n \\ \oplus \\ Y_{n+1} \end{matrix}$$

where the action of the differential is interpreted as matrix multiplication, that is,

$$\partial_n(x, y) = \begin{bmatrix} -d_X & 0 \\ -f & d_Y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -d_X(x) \\ d_Y(y) - f(x) \end{bmatrix}.$$

Let $X[-1]$ be the chain complex obtained from X by shifting indices where $X[-1]_n = X_{n-1}$ and differential $d[-1]_n : X[-1]_n \to X[-1]_{n+1}$ is given by $d[-1]_n = -(d_X)_n$, that is, the differential for $X[-1]$ is $-d_X$.

(i) Prove that the mapping cone $\mathsf{cone}(f)$ is really a chain complex.

(ii) Prove that we have an short exact sequence of chain complexes

$$0 \to Y \xrightarrow{\gamma} \mathsf{cone}(f) \xrightarrow{\delta} X[-1] \to 0$$

where $\gamma(y) = (0, y)$ and $\delta(x, y) = -x$.

Exercise 68 Let D, B be right and left R-modules respectively. Prove that $\mathsf{Tor}_0^R(D, B) \cong D \otimes_R B$.

Exercise 69 Prove that the homology groups $\mathsf{Tor}_n^R(D, B)$ is independent of the choice of projective resolution of B.