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Exercise 61 Thie exericse defines the connecting homomorphism δ_n in Theorem 17.2 (The Long Exact Sequence in Cohomology). Let $0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0$ be a SES of cochain complexes. Let $\alpha \in H^n(Z)$ and $a = z + \operatorname{im} d_n$ where $z \in \ker d_{n+1} : Z^n \to Z^{n+1}$.

1. Show that there exists $y \in Y^n$ such that $\beta_n(y) = z$ and a unique $x \in \ker d_{n+2} \subseteq X^{n+1}$ such that $\alpha(x) = d(y)$.
2. Let $z = \operatorname{im} d_n = z' + \operatorname{im} d_n$, and y, y', x, x' such that $\beta(y) = z, \beta(y') = z', \alpha(x) = d(y)$ and $\alpha(x') = d(y')$. Show that $x + \operatorname{im} d_{n+1} = x' + \operatorname{im} d_{n+1}$.
3. Conclude that we have a map $\delta_n : H^n(Z) \to H^{n+1}(X)$ defined by $\delta_n(z + \operatorname{im} d_n) = x + \operatorname{im} d_{n+1}$.
4. Prove that the connecting homomorphism δ_n is a group homomorphism.

Proof. $\square$