# Tutorial Questions

**Exercise 1** Let $R, S$ be rings and $V, W$ be $R$- and $S$-modules respectively.

(i) Suppose that $R$ is commutative. Show that the right $R$-action defined by $v*r := r\cdot v$ defines a right $R$-module.
(ii) Verify that the regular (left) module ${}_RR$ is an $R$-module.
(iii) Let $\alpha : R \to S$ be a ring homomorphism. Prove that $r * w := \alpha(r)\cdot w$ defines an $R$-module.
(iv) Let $\pi : R \to S$ be a surjective ring homomorphism and

$$\mathrm{Ann}(V) = \{r \in R : r\cdot v = 0 \text{ for all } v \in V\}$$

be the annihilator of $V$. Prove that $\mathrm{Ann}(V)$ is an ideal of $R$. Suppose further that $\mathrm{Ann}(V)$ contains $\ker \pi$. Prove that $s * v := r\cdot v$ defines an $S$-module where $\pi(r) = s$.
(v) Let $G$ be an abelian group. Prove that $G$ is an $\mathbb{Z}$-module with the action

$$n\cdot x = \begin{cases} \underbrace{x + x + \cdots + x}_{n \text{ times}} & \text{if } n \geq 0, \\ \underbrace{(-x) + (-x) + \cdots + (-x)}_{-n \text{ times}} & \text{if } n < 0. \end{cases}$$

**Exercise 2** Let $F$ be a field. Prove that every vector space $V$ over $F$ is an $F$-module and subspaces of $V$ are submodules.

**Exercise 3** Let $V$ be an $R$-module. Show that $0\cdot v = 0$ and $(-1)\cdot v = -v$ for all $v \in V$.

**Exercise 4** Let $V$ be an $R$-module.

(i) Show that the sum of two submodules of $V$ is a submodule (the sum $U + W$ is defined as $\{u + w : u \in U,\ w \in W\}$).
(ii) Show that the intersection of any non-empty collection of submodules of $V$ is a submodule.

**Exercise 5** Consider the abelian group $G := C_6 \times C_{10}$ as $\mathbb{Z}$-module. Find the annihilator of $G$.

**Exercise 6** Let $\{V_i\}_{i\in I}$ be a collection of $R$-modules.

(i) The direct product $\prod_{i\in I} V_i$ is the set consisting of all sequences $(v_i)_{i\in I}$ where $v_i \in V_i$ for each $i \in I$ and addition of two sequences is defined componentwise. Define an $R$-action by $r * (v_i)_{i\in I} = (r\cdot v_i)_{i\in I}$. Prove that $\prod_{i\in I} V_i$ is an $R$-module.
(ii) The (external) direct sum $\bigoplus_{i\in I} V_i$ is the subset of $\prod_{i\in I} V_i$ consisting of sequences $(v_i)_{i\in I}$ such that almost all $v_i$'s are zero. Prove that $\bigoplus_{i\in I} V_i$ is a submodule of $\prod_{i\in I} V_i$.

In the case when $I$ is finite, say $I = \{1, \ldots, n\}$, the notions $V_1 \times \cdots \times V_n$ and $V_1 \oplus \cdots \oplus V_n$ are the same.

**Exercise 7** Let $I$ be a left ideal of a ring $R$ and $V$ be an $R$-module. Prove that

$$IV := \{a_1\cdot v_1 + \cdots + a_m\cdot v_m : a_1, \ldots, a_m \in I,\ v_1, \ldots, v_m \in V,\ m \in \mathbb{N}\}$$

is a submodule of $V$.

**Exercise 8** (Group Algebra) Let $G$ be a group and $R$ be a commutative ring. Define $RG$ as the set consisting of formal sum of the form $\sum_{g\in G} r_g g$ where $r_g \in R$ and such that $r_g$'s are almost all zero (finite support). The element $r_g \in R$ is called the coefficient of $g \in G$. Define addition and multiplication on $RG$ as

$$\sum_{g\in G} r_g g + \sum_{g\in G} s_g g = \sum_{g\in G} (r_g + s_g) g,$$
$$\left(\sum_{g\in G} r_g g\right)\left(\sum_{g\in G} s_g g\right) = \sum_{g\in G}\left(\sum_{h\in G} r_{gh} s_{h^{-1}}\right) g$$

Show that $RG$ is an $R$-algebra with the map $\phi : R \to RG$ defined as $\phi(r) = re_G$, that is, $\phi(r)$ is mapped into the formal sum where the coefficients of $g \in G$ are zero if $g \neq e_G$ and $r$ if $g = e_G$.

**Exercise 9**

(i) Let $A$ be an $R$-algebra. Prove that, as $R$-module, we have $r \cdot (ab) = (r \cdot a)b = a(r \cdot b)$ for all $r \in R$ and $a, b \in A$.

(ii) Conversely, suppose that $A$ is a ring and an $R$-module satisfying $r \cdot (ab) = (r \cdot a)b = a(r \cdot b)$ for all $r \in R$ and $a, b \in A$. Prove that $A$ is an $R$-algebra with the map $f : R \to A$ defined by $f(r) = r \cdot 1_A$.

**Exercise 10** Let $\phi : V \to W$ be an $R$-module homomorphism. Prove that $\ker \phi$ and $\operatorname{im} \phi$ are submodules of $V$ and $W$ respectively.

**Exercise 11** Let $\phi : V \to W$ be an $R$-module homomorphism. Prove that $\ker \phi = \{0\}$ if and only if $\phi$ is injective.

**Exercise 12** Consider the group algebra $RG$. Define the $RG$-action on $R$ by, for each $x \in R$,

$$\left(\sum_{g\in G} r_g g\right) * x = \sum_{g\in G} r_g x.$$

Prove that $R$ is an $RG$-module. The module is called the trivial $RG$-module (for the group algebra $RG$).

**Exercise 13** Let $V, W$ be $R$-modules. Prove that the hom-set $\operatorname{Hom}_R(V, W)$ is an abelian group with the binary operation $(\phi + \psi)(v) := \phi(v) + \psi(v)$.

**Exercise 14** Let $X, Y, V$ be $R$-modules and $\beta : X \to Y$ be an $R$-module homomorphism.

(i) Prove that we have a group homomorphism

$$\beta_* : \operatorname{Hom}_R(V, X) \to \operatorname{Hom}_R(V, Y)$$

where $\beta_*(f) = \beta \circ f$.

(ii) Prove that we have a group homomorphism

$$\beta^* : \operatorname{Hom}_R(Y, V) \to \operatorname{Hom}_R(X, V)$$

where $\beta^*(f) = f \circ \beta$.

**Exercise 15** Let $F$ be a field. Show that $F$-module homomorphisms are linear transformations over $F$.

**Exercise 16** Show that $\mathbb{Z}$-module homomorphisms are abelian group homomorphisms.

**Exercise 17** Let $V$ be an $R$-module. Suppose that $r \in Z(R)$. Define $\lambda_r : V \to V$ by $\lambda_r(v) = r \cdot v$. Prove that $\lambda_r$ is an $R$-module homomorphism. Show that the conclusion is false without the assumption that $r \in Z(R)$.

**Exercise 18** Consider the $\mathbb{Z}$-module $V = \mathbb{Z}/2\mathbb{Z}$. Compute $\mathsf{End}_{\mathbb{Z}}(V)$. More generally, prove that

$$\mathsf{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \cong \mathbb{Z}/\gcd(m,n)\mathbb{Z}.$$

**Exercise 19** Let $R$ be a unital ring.

(i) Let $V$ be an $R$-module. Prove that $\mathsf{Hom}_R(R, V) \cong V$ as $R$-modules where the map is given by $\alpha \mapsto \alpha(1)$.
(ii) Prove that $\mathsf{End}_R(R) \cong R^{\mathsf{op}}$ as rings.

**Exercise 20** Fix a positive integer $n$. For each $1 \leq i \leq n$, let $U_i$ be a submodule of an $R$-module $V_i$. Prove that we have the following isomorphism of $R$-modules:

$$(V_1 \times \cdots \times V_n)/(U_1 \times \cdots \times U_n) \cong (V_1/U_1) \times \cdots \times (V_n/U_n).$$

**Exercise 21** Let $I$ be a left ideal of a ring $R$ and consider the free $R$-module $R^n$ of rank $n$. Prove that we have the following isomorphism of $R$-modules:

$$R^n/IR^n \cong \underbrace{(R/IR) \times \cdots \times (R/IR)}_{n \text{ times}}$$

where $IR^n$ has been defined in Exercise 7.

**Exercise 22** Let $V$ be an $R$-module and $A$ be a subset of $V$. Let

$$RA := \{r_1a_1 + \cdots + r_ma_m : r_1, \ldots, r_m \in R,\ a_1, \ldots, a_m \in A,\ m \in \mathbb{N}\}.$$

Recall that $R\emptyset = \{0\}$.

(i) Prove that $RA$ is a submodule of $V$.
(ii) Prove that $RA$ contains the subset $A$.
(iii) Prove that $RA$ is the intersection of the collection of submodules $U$ of $V$ such that $A \subseteq V$. As such, $RA$ is the smallest submodule of $V$ containing $A$.

**Exercise 23** Let $N$ be a submodule of $M$. Prove that $M$ is finitely generated if both $N$ and $M/N$ are finitely generated.

**Exercise 24** Let $R$ be a commutative ring. Show that $R^m \cong R^n$ if and only if $m = n$.
(Hint: Let $I$ be a maximal ideal of $R$. Use Exercise 21.)

**Exercise 25** Let $M$ be the free $\mathbb{Z}$-module $\mathbb{Z} \times \mathbb{Z} \times \cdots$ and let $R = \mathrm{End}_{\mathbb{Z}}(M)$.

(i) Let $\phi(a_1, a_2, \ldots) = (0, a_1, a_2, \ldots)$ and $\psi(a_1, a_2, \ldots) = (a_2, a_3, \ldots)$. Show that $\psi\phi = 1$ but $\phi\psi \neq 1$. Conclude that $R$ is not commutative.

(ii) Let

$$\alpha_1(a_1, a_2, a_3, \ldots) = (a_1, a_3, a_5, \ldots),$$
$$\alpha_2(a_1, a_2, a_3, \ldots) = (a_2, a_4, a_6, \ldots).$$

Prove that $\{\alpha_1, \alpha_2\}$ is a free basis for the regular $R$-module ${}_RR$.
(Hint: Let $\beta_1(a_1, a_2, \ldots) = (a_1, 0, a_2, 0, \ldots)$ and $\beta_2(a_1, a_2, \ldots) = (0, a_1, 0, a_2, \ldots)$. Show that $\alpha_i\beta_i = 1$, $\alpha_1\beta_2 = 0 = \alpha_2\beta_1$ and $\beta_1\alpha_1 + \beta_2\alpha_2 = 1$.)

(iii) Prove that $R \cong R^2$ as $R$-modules.

**Exercise 26** Verify that the module $F(A)$ defined in the Universal Property of Free Modules is an $R$-module and the map $\Phi : F(A) \to M$ is a well-defined $R$-module homomorphism.

**Exercise 27** Let $A$ and $B$ be sets of the same cardinality. Prove that the free modules $F(A), F(B)$ constructed in Universal Property of Free Modules are isomorphic.

**Exercise 28** An $R$-module $M$ is called a torsion module if, for every $m \in M$, there exists a nonzero element $r \in R$ such that $r \cdot m = 0$. Prove that every finite abelian group is a torsion $\mathbb{Z}$-module. Give an example showing the converse of the previous statement is incorrect.

**Exercise 29** An $R$-module $M$ is called irreducible (or simple) if $M \neq 0$ and the only submodules of $M$ are $0$ and $M$. Suppose that $M$ is irreducible and let $0 \neq m \in M$. Prove that $M \cong Rm$.

**Exercise 30** Find all irreducible $\mathbb{Z}$-modules.

**Exercise 31** An element $e \in R$ is called an idempotent if $e^2 = e$. The idempotent is called a central idempotent if it belongs in the center of $R$, i.e., $re = er$ for all $r \in R$. Let $e$ be a central idempotent and $M$ is an $R$-module. Prove that $M$ is equal to the direct sum of the submodules $eM$ and $(1-e)M$ where

$$eM = \{em : m \in M\},$$
$$(1-e)M = \{(1-e)m : m \in M\}.$$

**Exercise 32** Let $M, N$ be right and left $R$-modules respectively and $L$ be an abelian group. Suppose that $\beta : M \times N \to L$ is an $R$-balanced map. Show that the group homomorphism $\xi : F(M \times N) \to L$ obtained via the universal property

$$\begin{tikzcd} M \times N \arrow[r, "\iota"] \arrow[dr, "\beta"'] & F(M \times N) \arrow[d, "\xi"] \\ & L \end{tikzcd}$$

of the free $\mathbb{Z}$-module maps the subgroup $H$ of $F(M \times N)$ generated by

$$\begin{aligned}&(m+m', n) - (m, n) - (m', n),\\ &(m, n+n') - (m, n) - (m, n'),\\ &(mr, n) - (m, rn)\end{aligned}$$

belongs in the kernel of $\xi$ and hence induce a group homomorphism $\Phi : F(M \times N)/H \to L$.

**Exercise 33** Verify the following bimodule structures.

(i) Let $I$ be an ideal of $R$. Prove that $R/I$ is an $(R/I, R)$-bimodule.
(ii) Let $M$ be a left $R$-module and $S \subseteq Z(R)$. Prove that $M$ is an $(R, S)$-bimodule where $m * s := sm$.

**Exercise 34** Let $R, S$ be rings, $Y$ an $(R, S)$-bimodule and $Z$ a right $S$-module. Prove that $\mathrm{Hom}_S(Y, Z)$ is a right $R$-module with the action given by $(\phi * r)(y) = \phi(ry)$ for all $r \in R$ and $y \in Y$.

**Exercise 35**

(i) Show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q}$ are isomorphic as $\mathbb{Q}$-modules.
(ii) Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ and $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C}$ are not isomorphic as $\mathbb{C}$-modules.

**Exercise 36** Let $D$ be a right $\mathbb{Z}$-module and $m \in \mathbb{Z}$. We aim to prove that

$$D \otimes_{\mathbb{Z}} (\mathbb{Z}/m\mathbb{Z}) \cong D/mD$$

where $mD = \{md : d \in D\}$.

(i) Define a map $\beta : D \times (\mathbb{Z}/m\mathbb{Z}) \to D/mD$ by $\beta(d, \overline{a}) = ad + mD$. Prove that $\beta$ is well-defined and $\mathbb{Z}$-balanced.
(ii) By the Universal Property of the Tensor Product, we have a group homomorphism $\Phi : D \otimes_{\mathbb{Z}} (\mathbb{Z}/m\mathbb{Z}) \to D/mD$ such that $\Phi(d \otimes \overline{a}) = ad + mD$. Prove that $\Phi$ is an isomorphism.

**Exercise 37** Complete/Read the proof of the Associativity of the Tensor Product.

**Exercise 38** Let $R$ be a commutative ring and $M, N$ be $R$-modules. Prove that $M \otimes_R N \cong N \otimes_R M$ as $R$-modules.

**Exercise 39** Let $I$ be an indexing set. Let $M$ and $N_i$, one for each $i \in I$, be right and left $R$-modules respectively. Prove that

$$M \otimes_R \bigoplus_{i \in I} N_i \cong \bigoplus_{i \in I} (M \otimes_R N_i).$$

Furthermore, if $M$ is an $(S, R)$-bimodule, then the above is an isomorphism of left $S$-modules.

**Exercise 40** *The conclusion of Exercise 39 is however not true if we replace direct sum with direct product. Let $S = \mathbb{Q}$, $R = \mathbb{Z}$ and $N_i = \mathbb{Z}/(2^i\mathbb{Z})$.

**Exercise 41** Let $N$ be a torsion free $\mathbb{Z}$-module. Let $\mathbb{Z}^\times$ be the set of nonzero integers. Define the relation $\sim$ on $\mathbb{Z}^\times \times N$ by $(m, x) \sim (n, y)$ if and only if $t(my - nx) = 0$ for some $0 \neq t \in \mathbb{Z}$.

(i) Prove that $\sim$ is an equivalence relation and let $\mathbb{Z}^{-1}N$ be the set of equivalence classes.

(ii) Prove that $\mathbb{Z}^{-1}N$ is a $\mathbb{Z}$-module (abelian group) with addition given by

$$[(m, x)] + [(n, y)] = [(mn, my + nx)].$$

(iii) Show that the map $\beta : \mathbb{Q} \times N \to \mathbb{Z}^{-1}N$ defined by

$$\beta(\frac{a}{b}, x) = [(b, ax)]$$

is an $\mathbb{Z}$-balanced map and hence it induces a $\mathbb{Z}$-module homomorphism $f : \mathbb{Q} \otimes_{\mathbb{Z}} N \to \mathbb{Z}^{-1}N$ such that $f(\frac{a}{b} \otimes x) = [(b, ax)]$. (You need to check that the map $\beta$ is well-defined.)

(iv) Define $g : \mathbb{Z}^{-1}N \to \mathbb{Q} \otimes_{\mathbb{Z}} N$ by $g([(m, x)]) = \frac{1}{m} \otimes x$. Prove that $g$ is the inverse of $f$ in part (iii) and hence we have an isomorphism $\mathbb{Q} \otimes_{\mathbb{Z}} N \cong \mathbb{Z}^{-1}N$.

(v) Conclude that, $\frac{1}{m} \otimes x = 0$ in $\mathbb{Q} \otimes_{\mathbb{Z}} N$ if and only if $rx = 0$ for some $0 \neq r \in \mathbb{Z}$.

(For this exercise, you may replace $\mathbb{Z}$ with any integral domain and $\mathbb{Q}$ with its field of fractions.)

**Exercise 42** Prove Corollary 10.16 and Proposition 10.21.

**Exercise 43** Let $(\alpha, \beta, \gamma)$ and $(\alpha', \beta', \gamma')$ be homomorphisms of short exact sequences (SES) of $R$-modules

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, "\alpha"] & Y \arrow[r] \arrow[d, "\beta"] & Z \arrow[r] \arrow[d, "\gamma"] & 0 \\
0 \arrow[r] & X' \arrow[r] \arrow[d, "\alpha'"] & Y' \arrow[r] \arrow[d, "\beta'"] & Z' \arrow[r] \arrow[d, "\gamma'"] & 0 \\
0 \arrow[r] & X'' \arrow[r] & Y'' \arrow[r] & Z'' \arrow[r] & 0
\end{tikzcd}$$

(i) Prove that $(\alpha'\alpha, \beta'\beta, \gamma'\gamma)$ is a homomorphism of short exact sequence.

(ii) Suppose that $(\alpha, \beta, \gamma)$ is an isomorphism (respectively, equivalence) of SES. Prove that $(\alpha^{-1}, \beta^{-1}, \gamma^{-1})$ is an isomorphism (respectively, equivalence) of SES.

**Exercise 44** Prove the second statement of the five lemma: Suppose that the rows are exact, $f_1$ is surjective and both $f_2, f_4$ are injective. Prove that $f_3$ is injective.

$$\begin{tikzcd}
M_1 \arrow[r, "g_1"] \arrow[d, "f_1"] & M_2 \arrow[r, "g_2"] \arrow[d, "f_2"] & M_3 \arrow[r, "g_3"] \arrow[d, "f_3"] & M_4 \arrow[r, "g_4"] \arrow[d, "f_4"] & M_5 \arrow[d, "f_5"] \\
N_1 \arrow[r, "h_1"] & N_2 \arrow[r, "h_2"] & N_3 \arrow[r, "h_3"] & N_4 \arrow[r, "h_4"] & N_5
\end{tikzcd}$$

**Exercise 45** Consider the short exact sequence of groups

$$1 \to A_n \xrightarrow{\iota} S_n \xrightarrow{\mathrm{sgn}} \{\pm 1\} \to 1.$$

(i) Show that there exists a group homomorphism $\gamma : \{\pm 1\} \to S_n$ such that $\mathrm{sgn}\, \gamma = \mathrm{id}_{\{\pm 1\}}$.

(ii) Show that, however, whenever $n \geq 3$, there is no group homomorphism $\delta : S_n \to A_n$ such that $\delta\iota = \mathrm{id}_{A_n}$.

This example shows that the existence of retraction of $\iota$ and section of $\mathrm{sgn}$ for (non-abelian groups) are not equivalent.

**Exercise 46** Consider the exact sequence $\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/2\mathbb{Z} \to 0$ and let $V = \mathbb{Z}/2\mathbb{Z}$. Verify that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})$ is the trivial group and hence the map $\pi^* : \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}) \to \mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z})$ is not surjective.

**Exercise 47** Let $\{F_i : i \in I\}$ be a collection of free $R$-modules.

(i) Prove that the direct sum $\bigoplus_{i\in I} F_i$ is a free module.
(ii) For each $i \in I$, let $P_i$ be a direct summand of $F_i$. Assume that $F_i = P_i \oplus Q_i$ for some submodule $Q_i$ of $F_i$. Prove that $\bigoplus_{i\in I} F_i = \left(\bigoplus_{i\in I} P_i\right) \oplus \left(\bigoplus_{i\in I} Q_i\right)$.
(iii) Conclude that the direct sum of projective modules is projective.

**Exercise 48** Let $R$-mod consist of the objects (left) $R$-modules and the class of morphisms of two objects $X, Y$ is $\mathrm{Hom}_R(X, Y)$. The binary operation of morphisms is given by composition of functions. Prove that $R$-mod is a category. Conclude that $\mathbb{Z}$-modcat is the category of abelian groups which is denoted by Ab.

**Exercise 49** Let $V$ be an $R$-module.

(i) Prove that $\mathrm{Hom}_R(V, -) : R$-mod $\to$ Ab is a covariant functor.
(ii) Prove that $\mathrm{Hom}_R(-, V) : R$-mod $\to$ Ab is a contravariant functor.

**Exercise 50** Suppose that we begin with a category $\mathscr{C}$ and an object $X$ in $\mathscr{C}$. What are the important ingredients so that $\mathscr{F} := \mathrm{Mor}(X, -) : \mathscr{C} \to \mathrm{Ab}$ is a covariant functor?

**Exercise 51** Let $R$ be an integral domain. An $R$-module $M$ is divisible if, for every $0 \neq r \in R$, we have $rM = M$.

(i) Suppose that $Q$ is a nonzero divisible $\mathbb{Z}$-module. Prove that $Q$ is not projective.
(ii) Deduce that $\mathbb{Q}$ is divisible and hence not a projective $\mathbb{Z}$-module.

**Exercise 52** Prove Proposition 10.34, that is, let $I$ be an $R$-module, prove that the following are equivalent statements:

(i) For any $R$-modules $X, Y, Z$, if $0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0$ is exact, then $0 \to \mathrm{Hom}_R(Z, I) \xrightarrow{\beta^*} \mathrm{Hom}_R(Y, I) \xrightarrow{\alpha^*} \mathrm{Hom}_R(X, I) \to 0$ is exact.
(ii) For any $R$-modules $X, Y$, if $0 \to X \xrightarrow{\alpha} Y$ is exact, then, for any $R$-module homomorphism $g : X \to I$, there exists an $R$-module homomorphism $f : Y \to I$ such that $f \circ \alpha = g$.
(iii) If $I$ is isomorphic to a submodule of an $R$-module $Y$, then $I \mid Y$.

(Hint: For (iii) $\Rightarrow$ (i), use the fact that every $R$-module is contained in an injective $R$-module.)

**Exercise 53** Complete the proof of Baer's criterion by verifying that the set

$$\Omega = \{(f', Y') : \mathrm{im}\,\alpha \subseteq Y' \subseteq Y \text{ and } f' : Y' \to Q \text{ such that } f'\alpha = g\},$$

with the partial order $(f', Y') \leq (f'', Y'')$ if and only if $Y' \subseteq Y''$ and $f''|_{Y'} = f'$, satisfies Zorn's lemma.

**Exercise 54** Let $\{Q_i : i \in I\}$ be injective $R$-modules. Prove that $\prod_{i\in I} Q_i$ is injective.

**Exercise 55** (Projective Cover) Let $M$ be an $R$-module. A projective cover $P$ of $M$ is a 'smallest' projective $R$-module such that $P$ projects onto $M$. More precisely, a projective $R$-module $P$ is a projective cover of $M$ if there exists a surjective $R$-module homomorphism $f : P \to M$ such that, for any projective $R$-module $Q$ and $R$-module homomorphism $g : Q \to P$ such that $fg$ is surjective, the map $g$ is surjective. In the literature, $f$ is called an essential map.

(i) If $P, P'$ are projective covers of $M$, prove that $P \cong P'$.
(ii) Show that the $\mathbb{Z}$-module $\mathbb{Z}/2\mathbb{Z}$ has no projective cover.

(However, every $R$-module has injective hull by the Eckmann–Schopf Theorem. Let $M$ be an $R$-module. An injective hull $Q$ of $M$ is a 'smallest' injective $R$-module such that $M$ injects into $Q$. More precisely, an injective $R$-module $Q$ is an injective hull of $M$ if there exists an injective $R$-module homomorphism $f : M \to Q$ such that, for any injective $R$-module $Q'$ and $R$-module homomorphism $g : Q \to Q'$ such that $gf$ is injective, the map $g$ is injective. Injective hulls of a module is unique up to isomorphism.)

**Exercise 56** Let $V, W$ be $R$-modules.

(i) Prove that $V \oplus W$ is projective if and only if both $V, W$ are projective.
(ii) Prove that $V \oplus W$ is injective if and only if both $V, W$ are injective.
(iii) Prove that $V \oplus W$ is flat if and only if both $V, W$ are flat right $R$-modules.

**Exercise 57** Prove the following isomorphisms.

(i) $\mathsf{Hom}_R(\bigoplus_{i\in I} V_i, W) \cong \prod_{i\in I} \mathsf{Hom}_R(V_i, W)$
(ii) $\mathsf{Hom}_R(V, \prod_{j\in J} W_j) \cong \prod_{j\in J} \mathsf{Hom}_R(V, W_j)$

**Exercise 58** Let $G$ be a finite group and $H$ be a subgroup of $G$. Let $k$ be a field. Consider the group algebras $kG$ and $kH$.

(i) Consider $kG$ as right $kH$-module in the natural way. Let $\{x_1, \dots, x_m\}$ be a complete set of left coset representatives of $H$ in $G$. Prove that

$$kG = \bigoplus_{i=1}^{m} \mathsf{span}_k\{x_i h : h \in H\} \cong \bigoplus_{i=1}^{m} kH$$

as right $kH$-modules.
(ii) Conclude that $kG$ is a free right $kH$-module.

For any left $kH$-module $V$, the $kG$-module $\mathsf{Ind}_H^G V := kG \otimes_{kH} V$ is called the induction of $V$ from $H$ to $G$.

(iii) For any exact sequence $0 \to X \to Y \to Z \to 0$ of $kH$-modules, prove that we have an exact sequence of induced modules

$$0 \to \mathsf{Ind}_H^G X \to \mathsf{Ind}_H^G Y \to \mathsf{Ind}_H^G Z \to 0.$$

(iv) Conclude that we have an exact covariant functor $\mathsf{Ind}_H^G : kH\text{-mod} \to kG\text{-mod}$ .

**Exercise 59** Let $Q$ be an injective $R$-module. Recall that the contravariant functor $\mathsf{Hom}_R(-,Q)$ : $R$-mod $\to$ Ab is exact on short exact sequence, i.e., for any exact sequence $0 \to U \to V \to W \to 0$ of $R$-modules, we have an exact sequence $0 \to \mathsf{Hom}_R(W,Q) \to \mathsf{Hom}_R(V,Q) \to \mathsf{Hom}_R(U,Q) \to 0$ of abelian groups. Prove that, if we have an exact chain complex

$$\cdots \to V_{n+1} \xrightarrow{d_{n+1}} V_n \xrightarrow{d_n} V_{n-1} \to \cdots$$

of $R$-modules, then we have an exact cochain complex

$$\cdots \to \mathsf{Hom}_R(V_{n-1},Q) \xrightarrow{d_n^*} \mathsf{Hom}_R(V_n,Q) \xrightarrow{d_{n+1}^*} \mathsf{Hom}_R(V_{n+1},Q) \to \cdots .$$

**Exercise 60** Let $D$ be a flat right $R$-module, i.e., the functor $D \otimes_R -$ is exact (on SES). Prove that, if we have an exact chain complex

$$\cdots \to V_{n+1} \xrightarrow{d_{n+1}} V_n \xrightarrow{d_n} V_{n-1} \to \cdots$$

of left $R$-modules, then we have an exact chain complex

$$\cdots \to D \otimes_R V_{n+1} \xrightarrow{1\otimes d_{n+1}} D \otimes_R V_n \xrightarrow{1\otimes d_n} D \otimes_R V_{n-1} \to \cdots .$$

**Exercise 61** This exercise defines the connecting homomorphism $\delta_n$ in Theorem 17.2 (The Long Exact Sequence in Cohomology). Let $0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0$ be a a short exact sequence of cochain complexes. Let $a \in \mathsf{H}^n(Z)$ and $a = z + \mathsf{im}\, d_n$ where $z \in \ker d_{n+1} : Z^n \to Z^{n+1}$.

(i) Show that there exist $y \in Y^n$ such that $\beta_n(y) = z$ and a unique $x \in \ker d_{n+2} \subseteq X^{n+1}$ such that $\alpha(x) = d(y)$.

(ii) Let $z + \mathsf{im}\, d_n = z' + \mathsf{im}\, d_n$ and $y, y', x, x'$ such that $\beta(y) = z$, $\beta(y') = z'$, $\alpha(x) = d(y)$ and $\alpha(x') = d(y')$. Show that $x + \mathsf{im}\, d_{n+1} = x' + \mathsf{im}\, d_{n+1}$.

(iii) Conclude that we have a map $\delta_n : \mathsf{H}^n(Z) \to \mathsf{H}^{n+1}(X)$ defined by $\delta_n(z + \mathsf{im}\, d_n) = x + \mathsf{im}\, d_{n+1}$.

(iv) Prove that the connecting homomorphism $\delta_n$ is a group homomorphism.

**Exercise 62** (Snake Lemma) The cokernel $\mathsf{coker}\, f$ of an $R$-module homomorphism $f : V \to W$ is defined as $W/\mathsf{im}\, f$. Suppose that we have a commutative diagram below with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "\alpha"] \arrow[d, "f"] & Y \arrow[r, "\beta"] \arrow[d, "g"] & Z \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & X' \arrow[r, "\alpha'"] & Y' \arrow[r, "\beta'"] & Z' \arrow[r] & 0
\end{tikzcd}$$

Use the Long Exact Sequence in Cohomology to prove that we have an exact sequence

$$0 \to \ker f \xrightarrow{\alpha} \ker g \xrightarrow{\beta} \ker h \xrightarrow{\delta} \mathsf{coker}\, f \xrightarrow{\alpha'} \mathsf{coker}\, g \xrightarrow{\beta'} \mathsf{coker}\, h \to 0.$$

(The Snake Lemma has a slightly more general version where $\alpha$ is not necessarily injective nor $\beta'$ is surjective. In this case, we do not have exactness at $\ker f$ nor $\mathsf{coker}\, h$.)

**Exercise 63** Prove Horseshoe Lemma. (You will need the Snake Lemma.)

**Exercise 64** Let $f : V \to V'$ be an $R$-module homomorphism and let $W$ be another $R$-module. Show that, for each $n \geq 0$, we have an induced group homomorphism $\mathscr{F}(f)_n : \mathrm{Ext}^n_R(V', W) \to \mathrm{Ext}^n_R(V, W)$. Furthermore, if $g : V' \to V''$ is another $R$-module homomorphism and $\mathscr{F}(g)_n : \mathrm{Ext}^n_R(V'', W) \to \mathrm{Ext}^n_R(V', W)$ is the induced group homomorphism, then

$$\mathscr{F}(g \circ f)_n = \mathscr{F}(f)_n \circ \mathscr{F}(g)_n.$$

**Exercise 65** Prove Proposition 17.11. That is, given an $R$-module $P$, prove that the following statements are equivalent.

(i) $P$ is projective
(ii) $\mathrm{Ext}^1_R(P, B) = 0$ for all $R$-modules $B$
(iii) $\mathrm{Ext}^n_R(P, B) = 0$ for all $R$-modules $B$ and all $n \geq 1$

**Exercise 66** We have proved that the direct sum of projective modules is projective (in Exercise 47) and direct product of injective is injective (in Exercise 54).

(i) For each $j \in J$, let $(P(j))_\bullet \twoheadrightarrow V_j$ be a projective resolution of $V_j$. Prove that the direct sum of the projective resolutions is a projective resolution of $\bigoplus_{j \in J} V_j$. Use this to show that

$$\mathrm{Ext}^n_R(\bigoplus_{j \in J} V_i, W) \cong \prod_{j \in J} \mathrm{Ext}^n_R(V_j, W).$$

(ii) For each $j \in J$, let $W_j \hookrightarrow (Q(j))_\bullet$ be an injective resolution of $W_j$. Prove that the direct product of the injective resolutions is a injective resolution of $\prod_{j \in J} W_j$. Use this to show that

$$\mathrm{Ext}^n_R(V, \prod_{j \in J} W_i) \cong \prod_{j \in J} \mathrm{Ext}^n_R(V, W_j).$$

(iii) Prove that $\mathrm{Tor}^R_n(V, \bigoplus_{j \in J} W_j) \cong \bigoplus_{j \in J} \mathrm{Tor}^R_n(V, W_j)$.

**Exercise 67** (The mapping cone) Let $f : X \to Y$ be a map of chain complexes $X, Y$ of $R$-modules. The mapping cone $\mathsf{cone}(f)$ is the chain complex with degree $n$ part is $X_{n-1} \oplus Y_n$ and the differential is given by

$$\partial_n(x, y) = (-d_{n-1}(x), d_n(y) - f_{n-1}(x)) = (-d_X(x), d_Y(y) - f(x))$$

for every $x \in X_{n-1}$ and $y \in Y_n$, i.e., the differential $\partial_n$ can be viewed as the following $(2 \times 2)$-matrix:

$$\partial_n = \begin{bmatrix} -d_X & 0 \\ -f & d_Y \end{bmatrix} : \begin{matrix} X_{n-1} \\ \oplus \\ Y_n \end{matrix} \to \begin{matrix} X_n \\ \oplus \\ Y_{n+1} \end{matrix}$$

where the action of the differential is interpreted as matrix multiplication, that is,

$$\partial_n(x, y) = \begin{bmatrix} -d_X & 0 \\ -f & d_Y \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -d_X(x) \\ d_Y(y) - f(x) \end{bmatrix}.$$

Let $X[-1]$ be the chain complex obtained from $X$ by shifting indices where $X[-1]_n = X_{n-1}$ and differential $d[-1]_n : X[-1]_n \to X[-1]_{n+1}$ is given by $d[-1]_n = -(d_X)_n$, that is, the differential for $X[-1]$ is $-d_X$.

(i) Prove that the mapping cone $\mathsf{cone}(f)$ is really a chain complex.

(ii) Prove that we have an short exact sequence of chain complexes

$$0 \to Y \xrightarrow{\gamma} \mathrm{cone}(f) \xrightarrow{\delta} X[-1] \to 0$$

where $\gamma(y) = (0, y)$ and $\delta(x, y) = -x$.

**Exercise 68** Let $D, B$ be right and left $R$-modules respectively. Prove that $\mathrm{Tor}_0^R(D, B) \cong D \otimes_R B$.

**Exercise 69** Prove that the homology groups $\mathrm{Tor}_n^R(D, B)$ is independent of the choice of projective resolution of $B$.

**Exercise 70** Let $X$ be a complex where

$$\cdots \to X^{n-1} \xrightarrow{d_{n-1}} X^n \xrightarrow{d_n} X^{n+1} \to \cdots .$$

Suppose that we have maps $s_n : X^n \to X^{n-1}$ such that $\mathrm{id}_n = d_{n-1}s_n + s_{n+1}d_n$ (the maps $s_n$ are called the chain contraction of the identity). Prove that $X$ is exact.

**Exercise 71** Prove that the bar resolution is really a free resolution of $\mathbb{Z}$ by proving the following statements.

(i) $F_n$ is a free $\mathbb{Z}G$-module.
(ii) $d_n : F_n \to F_{n-1}$ for $n \geq 2$, $d_1$ and $d_0$ are $\mathbb{Z}G$-module homomorphisms.
(iii) Define the maps $s_{-1}, s_0, s_1, \ldots$ as follows:

$$0 \to \mathbb{Z} \xrightarrow{s_{-1}} F_0 \xrightarrow{s_0} F_1 \xrightarrow{s_1} F_2 \to \cdots$$

where $s_{-1}(1) = 1$ and $s_n(g_0|\cdots|g_n) = 1|g_0|\cdots|g_n$. Prove that $d_0s_{-1} = 1$, $d_1s_0 + s_{-1}d_0 = 1$ and $d_{n+1}s_n + s_{n-1}d_n = 1$ for $n \geq 1$.
(iv) The complex

$$\cdots \to F_n \xrightarrow{d_n} F_{n-1} \to \cdots \xrightarrow{d_1} F_0 \xrightarrow{d_0} \mathbb{Z} \to 0$$

is exact.

**Exercise 72** Let $G$ be a finite group and $H$ be a subgroup of $G$. Suppose that $A$ is an $H$-module. Prove that $M_H^G(A) \cong \mathbb{Z}G \otimes_{\mathbb{Z}H} A$.

**Exercise 73** Let $V, V'$ be $G$-modules and $\phi : V \to V'$ be a $G$-module homomorphism. Prove that the induced map $\lambda^* : H^n(G, V) \to H^n(G, V')$ given by, $\lambda^*([f]) = [\phi \circ f]$ for each $f \in Z^n(G, V)$, is precisely the map $\phi^* : \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, V) \to \mathrm{Ext}^n_{\mathbb{Z}G}(\mathbb{Z}, V')$ defined by $\phi^*([\beta]) = [\phi \circ \beta]$ where $\beta \in \ker\,(\delta_n : \mathrm{Hom}_{\mathbb{Z}G}(P_n, V) \to \mathrm{Hom}_{\mathbb{Z}G}(P_{n+1}, V))$. Here, we have used the identification where $C^n(G, V) \cong \mathrm{Hom}_{\mathbb{Z}G}(F_n, V)$.

**Exercise 74** Let $H$ be a subgroup of $G$ such that $[G : H] = m < \infty$ and $A$ be a $G$-module. Choose $G/H = \{g_1, \ldots, g_m\}$. Define $\psi : M_H^G(A) \to A$ by

$$\psi(f) = \sum_{i=1}^{m} g_i f(g_i^{-1}).$$

Show that $\psi$ is a surjective $G$-module homomorphism independent of the choice of the left coset representatives.

# Solutions to the Exercises

**Solution 1** We only check (iv). The rest are easy. Let $r, s \in \mathsf{Ann}(V)$ and $v \in V$. We have $(r+s)\cdot v = r\cdot v + s\cdot v = 0+0 = 0$ and $(rs)\cdot v = r\cdot(s\cdot v) = r\cdot 0 = 0$. So $\mathsf{Ann}(V)$ is a subring of $R$. Let $t \in R$. We have $(tr)\cdot v = t\cdot(r\cdot v) = t\cdot 0 = 0$ and $(rt)\cdot v = r\cdot(t\cdot v) = 0$. So $\mathsf{Ann}(V)$ is an ideal of $R$. We check the $S$-action is well-defined. Let $\pi(r) = s = \pi(r')$. Then $r - r' \in \ker\pi \subseteq \mathsf{Ann}(V)$. So

$$r\cdot v - r'\cdot v = (r-r')\cdot v = 0,$$

i.e., $r\cdot v = r'\cdot v$. This shows that the $S$-action is well-defined. Furthermore, let $\pi(r) = s$, $\pi(r') = s'$ and $v, w \in V$ (so that $\pi(rr') = ss'$ and $\pi(r+r') = s+s'$), we have

$$\begin{aligned} s*(v+w) &= r\cdot(v+w) = r\cdot v + r\cdot w = s*v + s*w,\\ (ss')*v &= (rr')\cdot v = r\cdot(r'\cdot v) = s*(s*v),\\ (s+s')*v &= (r+r')\cdot v = r\cdot v + r'\cdot v = s*v + s'*v. \end{aligned}$$

**Solution 2** Comparing the definitions of module and vector space, we obtain that a vector space over $F$ is an $F$-module. Therefore, subspaces of a vector space are submodules.

**Solution 3** Left as exercise.

**Solution 4** (i) The sum $U+W$ is nonempty because both $U, W$ are nonempty. Let $u, u' \in U$, $w, w' \in W$ and $r \in R$. We have

$$(u+w) + r(u'+w') = (u+ru') + (w+rw') \in U+W.$$

(ii) Let $\{U_i\}_{i\in I}$ be a collection of submodules of $V$. Since $0 \in U_i$, we have $0 \in \bigcap U_i$. Let $r \in R$ and $x, y \in \bigcap U_i$. We have $x + ry \in U_i$ for each $i \in I$ and hence $x + ry \in \bigcap U_i$.

**Solution 5** We claim that $\mathsf{Ann}(G) = 30\mathbb{Z}$ since $30 = \mathsf{lcm}(6, 10)$. Let $C_6 = \langle a\rangle$ and $C_{10} = \langle b\rangle$ and let $n \in \mathbb{Z}$ such that $n \in \mathsf{Ann}(G)$. We have $n\cdot(a, b) = (0, 0)$. So $na = 0$ and $nb = 0$. So $6 \mid n$ and $10 \mid n$. So $\mathsf{lcm}(6, 10) \mid n$, i.e., $30 \mid n$. So $\mathsf{Ann}(G) \subseteq 30\mathbb{Z}$. Conversely, $30m\cdot(ka, \ell b) = m(k30a + \ell 30b) = m(0, 0) = (0, 0)$. So $30\mathbb{Z} \subseteq \mathsf{Ann}(G)$.

**Solution 6** Left as exercise.

**Solution 7** We have $IV$ is not empty because $0 \in IV$. For $\sum a_i v_i, \sum b_j w_j \in IV$ and $r \in R$, we have

$$\sum a_i\cdot v_i + r\sum b_j w_j = \sum a_i\cdot v_i + \sum (rb_j)w_j \in IV$$

because $rb_j \in I$.

**Solution 8** Left as exercise.

**Solution 9** (i) Let $f : R \to A$ be the ring homomorphism defining $A$ as an $R$-algebra. So

$$r \cdot (ab) = f(r)(ab) = (f(r)a)b = (r \cdot a)b$$
$$r \cdot (ab) = f(r)(ab) = (f(r)a)b = (af(r))b = a(f(r)b) = a(r \cdot b).$$

(ii) Let $r, s \in R$ and $a \in A$. We have

$$\begin{aligned} f(r+s) &= (r+s) \cdot 1_A = r \cdot 1_A + s \cdot 1_A = f(r) + f(s), \\ f(rs) &= (rs) \cdot 1_A = 1_A((rs) \cdot 1_A) = 1_A(r \cdot (s \cdot 1_A)) = (r \cdot 1_A)(s \cdot 1_A) = f(r)f(s). \\ f(1_R) &= 1_R \cdot 1_A = 1_A, \\ f(r)a &= (r \cdot 1_A)a = r \cdot (1_A a) = r \cdot (a 1_A) = a(r \cdot 1_A) = af(r) \end{aligned}$$

**Solution 10** Left as exercise.

**Solution 11** Left as exercise.

**Solution 12** Left as exercise.

**Solution 13** Left as exercise.

**Solution 14** We only prove part (i). Let $f : V \to X$. The composition $\beta f : V \to Y$ is an $R$-module homomorphism. Furthermore, for any $v \in V$, we have

$$\begin{aligned} \beta_*(f+g)(v) &= \beta \circ (f+g)(v) = \beta((f+g)(v)) = \beta(f(v)+g(v)) = \beta(f(v)) + \beta(g(v)) \\ &= (\beta \circ f)(v) + (\beta \circ g)(v) = (\beta \circ f + \beta \circ g)(v) = (\beta_*(f) + \beta_*(g))(v). \end{aligned}$$

**Solution 15** Left as exercise.

**Solution 16** Left as exercise.

**Solution 17** Let $v, w \in V$ and $s \in R$. We have

$$\begin{aligned} \lambda_r(v+w) &= r \cdot (v+w) = r \cdot v + r \cdot w = \lambda_r(v) + \lambda_r(w), \\ \lambda_r(s \cdot v) &= r \cdot (s \cdot v) = (rs) \cdot v = (sr) \cdot v = s \cdot (r \cdot v) = s \cdot \lambda_r(v). \end{aligned}$$

Without the assumption that $r \in Z(R)$, we would not have $rs = sr$. Consider the matrix ring $\mathrm{Mat}_2(\mathbb{R})$ and $V$ is the natural module where $V$ is the set of $(2 \times 1)$-matrices over $\mathbb{R}$ where the ring acts by matrix multiplication. Let $r = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, $s = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $v = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. If $\lambda_r$ were an $R$-module homomorphism, we would have

$$\begin{aligned} \lambda_r(s \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix}) &= s \cdot \lambda_r(\begin{pmatrix} 1 \\ 0 \end{pmatrix}) \\ rsv &= srv \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} &= \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \end{aligned}$$

**Solution 18** Since $\mathbb{Z}$-module homomorphisms are just abelian group homomorphisms, given $\phi : V \to V$, we have either $\phi(1) = 0$ or $\phi(1) = 1$. In the first case, $\phi$ is the trivial homomorphism. The other is the identity map.

Suppose now that $\phi \in \mathsf{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z})$. Since $\mathbb{Z}/m\mathbb{Z}$ is cyclic, $\phi$ is determined by its value on $1$. Suppose that $\phi(1) = k \in \mathbb{Z}/n\mathbb{Z}$ where $0 \leq k \leq n-1$. Let $d = \gcd(m, n)$. We claim that $k = \frac{n}{d}j$ where $j = 0, \ldots, d-1$. We have

$$0 = \phi(0) = \phi(m) = m\phi(1) = mk.$$

Therefore, we have $n \mid mk$ and hence $\frac{n}{d} \mid \frac{m}{d}k$. Since $\gcd(\frac{n}{d}, \frac{m}{d}) = 1$, we have $\frac{n}{d} \mid k$. So $k = \frac{n}{d}j$ where $j = 0, \ldots, d-1$ because $k \in [0, n-1]$. Given such $k$, we only need to check that $\phi$ is well-defined. Suppose that $a \equiv b(\mod m)$. We have $n \mid \frac{nm}{d}$ and hence

$$\phi(a) = ka = \frac{n}{d}ja = \frac{n}{d}j(b + \ell m) = kb + \frac{nm}{d}j = kb = \phi(b).$$

We define

$$\Phi : \mathsf{Hom}_{\mathbb{Z}}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}/n\mathbb{Z}) \to \mathbb{Z}/d\mathbb{Z}$$

by $\Phi(\phi) = k$ if $\phi(1) = k$. This is a bijection. We leave it to you to check that $\Phi$ is a group homomorphism.

**Solution 19** (i) We have seen that $\mathsf{Hom}_R(R, V)$ is an abelian group. The action of $R$ on $\mathsf{Hom}_R(R, V)$ is given by $(r * \phi)(s) = \phi(sr)$. We define $\Phi : \mathsf{Hom}_R(R, V) \to V$ by $\Phi(\alpha) := \alpha(1)$ and $\Psi : V \to \mathsf{Hom}_R(R, V)$ by $\Psi(v) := r \mapsto r \cdot v$. For $r, s \in R$, $v \in V$ and $\alpha \in \mathsf{Hom}_R(R, V)$, we have

$$\begin{aligned}
\Phi(r * \alpha) &= (r * \alpha)(1) = \alpha(r) = r \cdot \alpha(1) = r \cdot \Phi(\alpha), \\
\Psi(v)(rs) &= (rs) \cdot v = r \cdot (s \cdot v) = r\dot{\Psi}(v)(s), \\
\Psi(r \cdot v)(s) &= s \cdot (r \cdot v) = (sr) \cdot v = \Psi(v)(sr) = (r * \Psi(v))(s).
\end{aligned}$$

These show that $\Psi$ is well-defined and both $\Phi$ and $\Psi$ are $R$-module homomorphism. Furthermore, we have

$$\begin{aligned}
(\Psi \circ \Phi)(\alpha)(r) &= \Psi(\alpha(1))(r) = r \cdot \alpha(1) = \alpha(r), \\
(\Phi \circ \Psi)(v) &= (\Psi(v))(1) = 1 \cdot v = v.
\end{aligned}$$

So $\Psi$ is the inverse of $\Phi$.

(ii) Let $\Phi : \mathsf{End}_R(R) \to R^{\mathsf{op}}$ be defined by $\Phi(\alpha) = \alpha(1)$. We denote $*$ for the product in $R^{\mathsf{op}}$. We first check that $\Phi$ is surjective. For $r \in R$, we claim that $\alpha(s) := sr$ is an $R$-module endomorphism of $R$. For $t \in R$,

$$\alpha(ts) = (ts)r = t(sr) = t\alpha(s).$$

So $\Phi(\alpha) = \alpha(1) = r$ and hence $\Phi$ is surjective. For $\alpha, \beta \in \mathsf{End}_R(R)$, we have

$$\begin{aligned}
\Phi(\alpha + \beta) &= (\alpha + \beta)(1) = \alpha(1) + \beta(1) = \Phi(\alpha) + \Phi(\beta), \\
\Phi(\alpha\beta) &= (\alpha\beta)(1) = \alpha(\beta(1)) = \beta(1)\alpha(1) = \alpha(1) * \beta(1) = \Phi(\alpha) * \Phi(\beta), \\
\Phi(\mathrm{Id}) &= \mathrm{Id}(1) = 1.
\end{aligned}$$

So $\Phi$ is a ring homomorphism. Let $\alpha \in \ker \Phi$. We have $\alpha(1) = 0$. So $\alpha(r) = r\alpha(1) = 0$. So $\alpha = 0$. This shows that $\Phi$ is injective.

**Solution 20** Define $\phi : V_1 \times \cdots \times V_n \to (V_1/U_1) \times \cdots \times (V_n/U_n)$ by $\phi(v_1, \ldots, v_n) = (v_1 + U_1, \ldots, v_n + U_n)$. Clearly $\phi$ is surjective and

$$\ker \phi = \{(v_1, \ldots, v_n) : v_i \in U_i\} = U_1 \times \cdots \times U_n.$$

We only need to check that $\phi$ is an $R$-module homomorphism and the result then follows using the first isomorphism theorem. For $r \in R$, we have

$$\begin{aligned} \phi(r(v_1, \ldots, v_n)) &= \phi(rv_1, \ldots, rv_n) \\ &= (rv_1 + U_1, \ldots, rv_n + U_n) \\ &= r(v_1 + U_1, \ldots, v_n + U_n) \\ &= r\phi(v_1, \ldots, v_n). \end{aligned}$$

**Solution 21** Let $U_i = IR$. We claim that $U := U_1 \times \cdots \times U_n = IR^n$ as submodule of $R^n$. Notice that

$$IR^n = \{a_1 \cdot v_1 + \cdots + a_m \cdot v_m : a_1, \ldots, a_m \in I,\ v_1, \ldots, v_m \in R^n,\ m \in \mathbb{N}\}.$$

By Exercise 7, $IR$ is a submodule of $R$ and hence $U$ is a submodule of $R^n$. For $u \in U$, we have $u = (u_1, \ldots, u_n)$ where $u_i \in IR$. So $u_i = \sum a_{i,j} r_{i,j}$ (finite sum) with $a_{i,j} \in I$ and $r_{i,j}$. So

$$u = (\sum a_{1,j} r_{1,j}, \ldots, \sum a_{n,j} r_{n,j}) = \sum a_{i,j} e_{i,j} \in IR^n$$

where $e_{i,j}$ denotes the element $(0, \ldots, 0, r_{i,j}, 0, \ldots, 0)$ in $R^n$ where the nonzero component occurs at the $i$th position. Therefore, $U \subseteq IR^n$. Conversely, for $a_1 \cdot v_1 + \cdots + a_m \cdot v_m \in IR^n$, we have

$$a_1 \cdot v_1 + \cdots + a_m \cdot v_m = \sum_{j=1}^{m} (a_j v_{1,j}, \ldots, a_j v_{n,j}) \in U$$

as $a_j v_{i,j} \in U_i$ where $v_j = (v_{1,j}, \ldots, v_{n,j})$. So $IR^n \subseteq U$. Now the desired isomorphism follows using Exercise 20.

**Solution 22** Left as exercise. For part (ii), you need our assumption that $R$ is unital.

**Solution 23** Suppose that $M/N$ and $N$ are generated by the finite sets $A$ and $B$ respectively. Let $A = \{a_1 + N, \ldots, a_m + N\}$ and $B = \{b_1, \ldots, b_n\}$. We claim that $M$ be generated by $\{a_1, \ldots, a_m, b_1, \ldots, b_n\}$. Let $m \in M$. Then $m + N = \sum r_i(a_i + N)$ for some $r_i \in R$. Since $m - \sum r_i a_i \in N$, we have $m - \sum r_i a_i = \sum s_j b_j$ for some $s_j \in R$. So $m = \sum r_i a_i + \sum s_j b_j$.

**Solution 24** Let $I$ be a maximal ideal of $R$. Since $I$ annihilates the module $R^n$, we can view $R^n/IR^n \cong (R/I)^n$ as $R/I$-module. Since $F := R/I$ is a field, it is a vector space over $F$. The vector space $(R/I)^n$ has dimension $n$. Therefore, if $R^m \cong R^n$, then $(R/I)^m \cong (R/I)^n$ as vector spaces over $F$ and hence $m = n$ by linear algebra. The converse is clear.

**Solution 25** Part (i) is easy. For part (ii),

$$\begin{aligned}
\alpha_1\beta_1(a_1,a_2,a_3,\ldots) &= \alpha_1(a_1,0,a_2,0,\ldots) = (a_1,a_2,\ldots),\\
\alpha_2\beta_2(a_1,a_2,a_3,\ldots) &= \alpha_2(0,a_1,0,a_2,\ldots) = (a_1,a_2,\ldots),\\
\alpha_1\beta_2(a_1,a_2,a_3,\ldots) &= \alpha_1(0,a_1,0,a_2,\ldots) = (0,0,\ldots),\\
\alpha_2\beta_1(a_1,a_2,a_3,\ldots) &= \alpha_2(a_1,0,a_2,0,\ldots) = (0,0,\ldots),\\
(\beta_1\alpha_1+\beta_2\alpha_2)(a_1,a_2,a_3,\ldots) &= (\beta_1\alpha_1)(a_1,a_2,a_3,\ldots)+(\beta_2\alpha_2)(a_1,a_2,a_3,\ldots)\\
&= \beta_1(a_1,a_3,a_5,\ldots)+\beta_2(a_2,a_4,a_6,\ldots)\\
&= (a_1,0,a_3,0,a_5,\ldots)+(0,a_2,0,a_4,\ldots)\\
&= (a_1,a_2,a_3,a_4,\ldots).
\end{aligned}$$

For any $x\in R$, we have $x = x\cdot 1 = (x\beta_1)\alpha_1+(x\beta_2)\alpha_2$. Suppose that $x = x_1\alpha_1+x_2\alpha_2$. We have

$$\begin{aligned}
x\beta_2 &= (x_1\alpha_1+x_2\alpha_2)\beta_2 = x_1(\alpha_1\beta_2)+x_2(\alpha_2\beta_2) = 0+x_2 = x_2,\\
x\beta_1 &= (x_1\alpha_1+x_2\alpha_2)\beta_1 = x_1(\alpha_1\beta_1)+x_2(\alpha_2\beta_1) = x_1+0 = x_1.
\end{aligned}$$

Therefore, $\{\alpha_1,\alpha_2\}$ is a free basis for ${}_RR$.

For part (iii), define $\phi : R\to R^2$ by $\phi(x) = (x\beta_1, x\beta_2)$. For $\xi\in R$, we have

$$\begin{aligned}
\phi(x+y) &= ((x+y)\beta_1,(x+y)\beta_2) = (x\beta_1+y\beta_1, x\beta_2+y\beta_2) = (x\beta_1,x\beta_2)+(y\beta_1,y\beta_2)\\
&= \phi(x)+\phi(y),\\
\phi(\xi x) &= (\xi x\beta_1,\xi x\beta_2) = \xi(x\beta_1,x\beta_2) = \xi\phi(x).
\end{aligned}$$

So $\phi$ is an $R$-module homomorphism. For $(y,z)\in R^2$, we have

$$\begin{aligned}
\phi(y\alpha_1+z\alpha_2) &= ((y\alpha_1+z\alpha_2)\beta_1,(y\alpha_1+z\alpha_2)\beta_2)\\
&= (y(\alpha_1\beta_1)+z(\alpha_2\beta_1), y(\alpha_1\beta_2)+z(\alpha_2\beta_2))\\
&= (y+0,0+z) = (y,z).
\end{aligned}$$

So $\phi$ is surjective. Suppose that $\phi(x)=0$. We have $x\beta_1 = 0 = x\beta_2$. So

$$x = x\cdot 1 = (x\beta_1)\alpha_1+(x\beta_2)\alpha_2 = 0+0 = 0.$$

So $\phi$ is injective.

**Solution 26** Left as exercise.

**Solution 27** Let $\phi : A\to B$ be a bijective function. By the universal property, we have

$$\begin{tikzcd}
A \arrow[r, hook, "\iota"] \arrow[d, "\phi"] & F(A) \arrow[d, shift right, "\Psi"'] \\
B \arrow[r, hook, "\jmath"] & F(B) \arrow[u, shift right, "\Phi"']
\end{tikzcd}$$

where $\Psi\iota = \jmath\phi$ and $\Phi\jmath = \iota\phi^{-1}$. We also have $\Phi\Psi\iota = \Phi\jmath\phi = \iota\phi^{-1}\phi = \iota$. Also, $\Psi\Phi\jmath = \jmath$. By the uniqueness, $\Phi\Psi = 1_{F(A)}$ and $\Psi\Phi = 1_{F(B)}$. So $F(A)\cong F(B)$.

**Solution 28** Let $M$ be a finite abelian group. By the classification of the finite abelian group, we may assume that $M = C_{n_1} \times \cdots \times C_{n_k}$ for some positive integers $n_1, \ldots, n_k$. Let $\ell = \mathrm{lcm}(n_1, \ldots, n_k)$. Then $\ell \cdot m = 0$. For the counterexample, let $M$ be the infinite product of $C_2$, i.e., $M = C_2 \times C_2 \times \cdots$. For any $(m_i)_{i\in\mathbb{N}}$, we have $2 \cdot (m_i) = (2 \cdot m_i) = 0$.

**Solution 29** Notice that $Rm$ is a nonzero submodule of $M$ since $R$ contains an identity. So $M = Rm$ by definition.

**Solution 30** They are the cyclic groups of prime orders. Let $M$ be an irreducible $\mathbb{Z}$-module. The subgroups of $M$ are the submodules of $M$. Suppose that $M$ is infinite. Let $0 \neq m \in M$. Then $M = \mathbb{Z}m$. Since $M$ is infinite, there is no nonzero integer $k$ such that $km = 0$. But $\langle 2m \rangle$ is a submodule of $M$. So $m = k(2m)$ for some $k \in \mathbb{N}$, i.e., $(2k-1)m = 0$. This is a contradiction. So $M$ must be finite. By the classification of finite abelian group, $M$ is a direct product of cyclic groups. Since every copy of the direct product is a subgroup of $M$ and $M$ is irreducible, there is only one copy, i.e., $M \cong C_n$. The subgroups of $C_n$ correspond to divisors of $n$. So $n$ must be prime.

**Solution 31** We first check that $eM$ is a submodule of $M$. Clearly, $e0 = 0 \in eM$. So $eM \neq \emptyset$. For $em, em' \in eM$ and $r \in R$, we have

$$em + r(em') = em + (re)m' = em + (erm') = e(m + rm') \in eM.$$

Similarly, $(1-e)M$ is a submodule of $M$. For $m \in M$, we have $m = em + (1-e)m$. So $M = eM + (1-e)M$. Suppose that $x \in eM \cap (1-e)M$. Then $(1-e)m = x = em'$. So

$$\begin{aligned} e(1-e)m &= e^2m' \\ (e-e)m &= em' \\ 0 &= em' = x. \end{aligned}$$

So $eM \cap (1-e)M = \{0\}$. As such, $M = eM \oplus (1-e)M$.

**Solution 32** Left as exercise.

**Solution 33** Left as exercise.

**Solution 34** We first verify that $\phi * r$ belongs in $\mathrm{Hom}_S(Y, Z)$. For any $s \in S$ and $y \in Y$, we have

$$(\phi * r)(ys) = \phi(r(ys)) = \phi((ry)s) = \phi(ry)s = (\phi * r)(y)s.$$

So $\phi * r \in \mathrm{Hom}_S(Y, Z)$. For $r, r' \in R$, we have

$$\begin{aligned} (\phi * (r + r'))(y) &= \phi((r + r')y) = \phi(ry + r'y) = \phi(ry) + \phi(r'y) = (\phi * r)(y) + (\phi * r')(y) \\ &= (\phi * r + \phi * r')(y). \end{aligned}$$

The other axiom is left as an exercise to check.

**Solution 35** In our examples, we have proved that $R \otimes_R N \cong N$ for any $R$-module $N$. So $\mathbb{Q} \otimes_{\mathbb{Q}} \mathbb{Q} \cong \mathbb{Q}$ and $\mathbb{C} \otimes_{\mathbb{C}} \mathbb{C} \cong \mathbb{C}$. Define $\beta : \mathbb{Q} \to \mathbb{Q}$ by $\beta(x) = x$. This is clearly an $\mathbb{Z}$-module homomorphism. By Theorem 10.8, there exists an $\mathbb{Q}$-module homomorphism $\Phi : \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q}$ such that $\Phi(1 \otimes x) = x$. In particular, $\Phi$ is surjective. For any $\frac{r}{s} \otimes x, \frac{a}{b} \otimes y \in \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$, we have

$$\frac{r}{s} \otimes x + \frac{a}{b} \otimes y = \frac{rb}{sb} \otimes x + \frac{sa}{sb} \otimes y = \frac{1}{sb} \otimes (rbx) + \frac{1}{sb} \otimes (say) = \frac{1}{sb} \otimes (rbx + say).$$

So every element in $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ can be expressed as a simple tensor. As such, if $x \otimes y \in \ker \Phi$, we have

$$0 = \Phi(x \otimes y) = \Phi(x(1 \otimes y)) = x\Phi(1 \otimes y) = xy.$$

So either $x = 0$ or $y = 0$. As such, $x \otimes y = 0$. This shows that $\Phi$ is injective. So $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q} \cong \mathbb{Q}$ as $\mathbb{Q}$-modules.

On the other hand, we show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ surjects onto $\mathbb{C} \oplus \mathbb{C}$ as $\mathbb{C}$-modules. Similar as before, let $L = \mathbb{C} \oplus \mathbb{C}$ and $\beta : \mathbb{C} \to \mathbb{C} \oplus \mathbb{C}$ defined by $\beta(a + ib) = (a, ib)$ where $a, b \in \mathbb{R}$. For $r \in \mathbb{R}$ and $a + ib \in \mathbb{C}$, we have

$$\beta(r(a + ib)) = (ra, rib) = r(a, ib) = r\beta(a + ib).$$

So $\beta$ is an $\mathbb{R}$-module homomorphism. By Theorem 10.8, there exists an $\mathbb{C}$-module homomorphism $\Phi : \mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \to \mathbb{C} \oplus \mathbb{C}$ such that $\Phi\iota = \beta$. In particular, $\Phi(1 \otimes (a + ib)) = \Phi\iota(a + ib) = \beta(a + ib) = (a, ib)$. So $(1, 0), (0, i) \in \operatorname{im} \Phi$. Since $\Phi$ is an $\mathbb{C}$-module homomorphism,

$$\Phi(i \otimes i) = \Phi(i(1 \otimes i)) = i(0, i) = (0, -1),$$
$$\Phi(i \otimes 1) = \Phi(i(1 \otimes 1)) = i(1, 0) = (i, 0).$$

So $\Phi$ is surjective. This shows that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ cannot be isomorphic to $\mathbb{C}$ as $\mathbb{C}$-modules.

**Solution 36** (i): Suppose that $\overline{a} = \overline{a'}$, i.e., $a - a' = km$ for some $k \in \mathbb{Z}$. We have $ad + mD = (a' + km)d + mD = a'd + mD$. So $\beta$ is well-defined. Also, for example,

$$\beta(dn, \overline{a}) = a(dn) + mD = (na)d + mD = \beta(d, \overline{na}).$$

So $\beta$ is $\mathbb{Z}$-balanced.

(ii): By Theorem 10.10, we have a group homomorphism $\Phi : D \otimes_{\mathbb{Z}} (\mathbb{Z}/m\mathbb{Z}) \to D/mD$ such that $\Phi(d \otimes \overline{a}) = ad + mD$. Taking $a = 1$, we have that $\Phi$ is surjective. Furthermore,

$$d \otimes \overline{a} = d \otimes a\overline{1} = (da) \otimes \overline{1}.$$

Let $d \otimes 1 \in \ker \Phi$, we have $\Phi(d \otimes \overline{1}) = d + mD = mD$. Then $d \in mD$, i.e., $d = md'$. So $d \otimes 1 = (d'm) \otimes 1 = d' \otimes \overline{m} = d' \otimes 0 = 0$. So $\Phi$ is injective.

**Solution 37** Left as exercise.

**Solution 38** Let $\beta : M \times N \to N \otimes_R M$ be given by $\beta(m, n) = n \otimes m$. This is $R$-bilinear because

$$\begin{aligned}\beta(rm + r'm', n) &= n \otimes (rm + r'm') = n \otimes (rm) + n \otimes (r'm') = (nr) \otimes m + (nr') \otimes m' \\ &= (rn) \otimes m + (r'n) \otimes m' = r(n \otimes m) + r'(n \otimes m') = r\beta(m, n) + r'\beta(m', n).\end{aligned}$$

Similarly, we can prove that $\beta(m, rn + r'n') = r\beta(m, n) + r'\beta(m, n')$. By the universal property, we have an $R$-module homomorphism $\phi : M \otimes_R N \to N \otimes_R M$ such that $\phi(m \otimes n) = n \otimes m$. Similarly, we have an $R$-module homomorphism $\psi : N \otimes_R M \to M \otimes_R N$ such that $\psi(n \otimes m) = m \otimes n$. By the uniqueness of the universal property,

$$\begin{tikzcd} M \times N \arrow[r, "\iota"] \arrow[dr, "\iota"'] & M \otimes_R N \arrow[d, equal] \\ & M \otimes_R N \end{tikzcd}$$

Since $\psi\phi(m \otimes n) = m \otimes n$, we have $\psi\phi = 1$. Similarly, $\phi\psi = 1$. So $\phi$ is an isomorphism.

**Solution 39** Let $\beta : M \times (\bigoplus_{i \in I} N_i) \to \bigoplus_{i \in I}(M \otimes_R N_i)$ be defined as

$$\beta(m, (n_i)) = (m \otimes n_i)_{i \in I}.$$

We have $\beta$ is $R$-balanced. For example,

$$\beta(mr, (n_i)) = ((mr) \otimes n_i)_{i \in I} = (m \otimes (rn_i))_{i \in I} = \beta(m, (rn_i)_{i \in I}) = \beta(m, r(n_i)_{i \in I}).$$

As such, we have a group homomorphism $\Phi : M \otimes_R (\bigoplus_{i \in I} N_i) \to \bigoplus_{i \in I}(M \otimes_R N_i)$ such that $\Phi(m \otimes (n_i)_{i \in I}) = (m \otimes n_i)_{i \in I}$. Similarly, for each $i \in I$, we define $\beta_i : M \times_R N_i \to M \otimes_R (\bigoplus_{i \in I} N_i)$ by $\beta_i(m_i, n_i) = m_i \otimes (n_j^{(i)})_{j \in I}$ where $n_j = 0$ if $j \neq i$. Therefore, there exist group homomorphisms $\Psi_i : M \otimes_R N_i \to M \otimes_R (\bigoplus_{i \in I} N_i)$ such that $\Psi_i(m \otimes n_i) = m_i \otimes (n_j^{(i)})$. Now let $\Psi = \oplus \Psi_i : \bigoplus_{i \in I}(M \otimes_R N_i) \to M \otimes_R (\bigoplus_{i \in I} N_i)$ so that $\Psi(m_i \otimes n_i)_{i \in I} = \sum m \otimes (n_j^{(i)})_{i \in I}$. (This is where the proof fails if we replace direct sum with direct product because the sum $\sum m \otimes (n_j^{(i)})_{i \in I}$ could end up with infinite sum and it does not make sense in $M \otimes_R (\prod_{i \in I} N_i)$). It is now routine to check that $\Psi$ and $\Phi$ are inverses of each other.

Suppose now that ${}_S M_R$. We are left to prove that $\Phi$ is an $S$-module homomorphism. We have

$$\begin{aligned} \Phi(s(m \otimes (n_i)_{i \in I})) &= \Phi((sm) \otimes (n_i)_{i \in I}) \\ &= ((sm) \otimes n_i)_{i \in I} \\ &= s(m \otimes n_i)_{i \in I} \\ &= s\Phi(m \otimes (n_i)_{i \in I}). \end{aligned}$$

**Solution 40** *We have $\mathbb{Q} \otimes_{\mathbb{Z}} N_i = 0$ because

$$x \otimes y = \frac{x}{2^i} \cdot 2^i \otimes y = \frac{x}{2^i} \otimes 2^i \cdot y = \frac{x}{2^i} \otimes 0 = 0.$$

Therefore, $\prod_{i \in I}(\mathbb{Q} \otimes N_i) = 0$. On the other hand, we claim that $\mathbb{Q} \otimes_{\mathbb{Z}} \prod_{i \in I} N_i \neq 0$.

For this, we prove the following statement: Let $N$ be a left nonzero torsion-free $\mathbb{Z}$-module. Then $\mathbb{Q} \otimes_{\mathbb{Z}} N$ is nonzero.

We first define an equivalence relation on $\mathbb{Q} \times N$. Let $(\frac{r}{s}, x) \sim (\frac{a}{b}, y)$ if and only if $(rb)x = (as)y$. Suppose that $\frac{r'}{s'} = \frac{r}{s}$ and $\frac{a'}{b'} = \frac{a}{b}$. We have

$$(sb)(r'b'x - s'a'y) = (rs'bb'x - ss'ab'y) = 0.$$

Since $N$ is torsion-free and $\mathbb{Z} \ni sb \neq 0$, we have $r'b'x = s'a'y$. So the relation $\sim$ is well-defined. Clearly, $(\frac{r}{s}, x) \sim (\frac{r}{s}, x)$ and, $(\frac{r}{s}, x) \sim (\frac{a}{b}, y)$ if and only if $(\frac{a}{b}, y) \sim (\frac{r}{s}, x)$. Furthermore, suppose that $(\frac{r}{s}, x) \sim (\frac{a}{b}, y)$ and $(\frac{a}{b}, y) \sim (\frac{c}{d}, z)$. We have

$$b(rdx - csz) = dasy - sady = 0.$$

Again, since $N$ is torsion-free, we have $rdx = csz$. So $\sim$ is an equivalence relation.

Let $\widetilde{N} = \mathbb{Q} \times N/\sim$ be the set of equivalence classes and we write $[q, x]$ for the equivalence class containing $(q, x)$. It is an $\mathbb{Q}$-module (vector space) where

$$[\frac{a}{b}, x] + [\frac{r}{s}, y] = [\frac{1}{bs}, asx + bry],$$
$$\frac{a}{b}[\frac{r}{s}, y] = [\frac{ar}{bs}, y].$$

The zero element is $[0, 0]$. We claim that $[1, x] \neq [0, 0]$ for $x \neq 0$. If not, we have $x = 1 \cdot 0 = 0$, a contradiction. So $\widetilde{N} \neq 0$ because $N \neq 0$.

We define $\beta : \mathbb{Q} \times N \to \widetilde{N}$ by $\beta(q, x) = [q, x]$. The map $\beta$ is well-defined and surjective. Next, we shall show that $\beta$ is $\mathbb{Z}$-balanced. For $n \in \mathbb{Z}$, $\frac{a}{b}, \frac{a'}{b'} \in \mathbb{Q}$ and $x, x \in N$, we have

$$\begin{aligned}
\beta(\frac{a}{b}n, x) &= [\frac{a}{b}n, x] = [\frac{a}{b}, nx] = \beta(\frac{a}{b}, nx),\\
\beta(\frac{a}{b} + \frac{a'}{b'}, x) &= \beta(\frac{ab' + a'b}{bb'}, x) = [\frac{ab' + a'b}{bb'}, x]\\
&= [\frac{1}{bb'}, ab'x + a'bx] = [\frac{a}{b}, x] + [\frac{a'}{b'}, x] = \beta(\frac{a}{b}, x) + \beta(\frac{a'}{b'}, x)\\
\beta(\frac{a}{b}, x + x') &= [\frac{a}{b}, x + x'] = [\frac{ab}{bb}, x + x'] = [\frac{1}{bb}, abx' + abx]\\
&= [\frac{a}{b}, x] + [\frac{a}{b}, x']\beta(\frac{a}{b}, x) + \beta(\frac{a}{b}, x').
\end{aligned}$$

By Theorem 10.10, there exists a group homomorphism $\Phi : \mathbb{Q} \otimes_{\mathbb{Z}} N \to \widetilde{N}$ such that $\beta = \Phi \circ \iota$ where $\iota : \mathbb{Q} \times N \to \mathbb{Q} \otimes_{\mathbb{Z}} N$. Since $\beta$ is surjective and $\widetilde{N} \neq 0$, we have $\Phi$ is surjective and hence $\mathbb{Q} \otimes_{\mathbb{Z}} N \neq 0$.

For arbitrary $\mathbb{Z}$-module, let

$$N' = \{x \in N : nx = 0 \text{ for some } 0 \neq n \in \mathbb{Z}\}.$$

This is a submodule (the torsion submodule) of $N$. The quotient $N/N'$ is a torsion-free $\mathbb{Z}$-module because, if $n(x + N') = N'$ for some $n \neq 0$, then $nx \in N'$ and hence $x \in N'$. Define the map $\gamma : \mathbb{Q} \times N \to \mathbb{Q} \otimes_{\mathbb{Z}} N'$ by $\gamma(q, x) = q \otimes (x + N')$. We claim that $\gamma$ is a $\mathbb{Z}$-balanced.

$$\begin{aligned}
\gamma(q, x + x') &= q \otimes ((x + x') + N') = q \otimes (x + N') + q \otimes (x' + N') = \gamma(q, x) + \gamma(q, x'),\\
\gamma(q + q', x) &= (q + q') \otimes (x + N') = q \otimes (x + N') + q' \otimes (x + N') = \gamma(q, x) + \gamma(q', x),\\
\gamma(qn, x) &= qn \otimes (x + N') = q \otimes (nx + N') = \gamma(q, nx).
\end{aligned}$$

By Theorem 10.10 again, there exists a group homomorphism $\Phi' : \mathbb{Q} \otimes_{\mathbb{Z}} N \to \mathbb{Q} \otimes_{\mathbb{Z}} (N/N')$. Since $\gamma$ is also surjective, we have $\Phi'$ is surjective. Suppose also that $N/N' \neq 0$. By our statement, $\mathbb{Q} \otimes_{\mathbb{Z}} (N/N') \neq 0$ and hence $\mathbb{Q} \otimes_{\mathbb{Z}} N \neq 0$.

We now apply the statement to our case. Let $N = \prod_{i \in I} N_i$. The submodule $N'$ consists of sequences $(x_i)_{i \in I}$ such that $x_i$'s are almost all zero. Therefore, $N/N'$ is nonzero. For example, it contains $(1)_{i \in I}$. So $\mathbb{Q} \otimes_{\mathbb{Z}} N \neq 0$. This proves our claim.

**Solution 41** (i): It is clear that $(m,x) \sim (n,y)$ if and only if $(n,y) \sim (m,x)$. Also, $(m,x) \sim (m,x)$ because $1(mx - mx) = 0$. Suppose now that $(m,x) \sim (n,y) \sim (k,z)$, i.e., $t(my - nx) = 0$ and $s(nz - ky) = 0$ for some nonzero $s,t \in \mathbb{Z}$. Since $n \neq 0$, we have

$$tsn(mz - kx) = (tm)(snz) - ks(tnx) = tm(sky) - ks(tmy) = 0.$$

So $(m,x) \sim (k,z)$. This shows that $\sim$ is an equivalence relation.

(ii): We only check well-definedness. The rest is left as exercise. Suppose that $(m,x) \sim (m',x')$ and $(n,y) \sim (n',y')$, i.e., $t(mx' - m'x) = 0 = s(ny' - n'y)$ for some nonzero integers $s,t$. We have

$$ts(m'n')(my + nx) = tm'm(sn'y) + sn'n(tm'x) = tm'm(sny') + sn'n(tmx') = ts(mn)(m'y' + n'x').$$

(iii): Again, we only check that the map $\beta$ is well-defined. The rest is left as exercise. Suppose that $\frac{a}{b} = \frac{a'}{b'}$, i.e., $ab' = a'b$. We have

$$b'(ax) = (b'a)x = (a'b)x = b(a'x).$$

So $[(b,ax)] = [(b',a'x)]$. By Theorem 10.10, there exists an abelian group homomorphism $f : \mathbb{Q} \otimes_{\mathbb{Z}} N \to \mathbb{Z}^{-1}N$ such that $f(\frac{a}{b} \otimes x) = [(b,ax)]$.

(iv): Suppose that $(m,x) \sim (m',x')$, i.e., $t(mx' - m'x) = 0$. We have

$$\frac{1}{m} \otimes x = \frac{tm'}{tmm'} \otimes x = \frac{1}{tmm'} \otimes (tm'x) = \frac{1}{mm'} \otimes (tmx') = \frac{tm}{tmm'} \otimes x' = \frac{1}{m'} \otimes x'.$$

So $g$ is well-defined. It is left as an exercise to check that $g$ is a group homomorphism and it is the inverse of $f$.

(v): Suppose that $\frac{1}{m} \otimes x = 0$. By the isomorphism in part (iv), we have

$$0 = f(0) = f(\frac{1}{m} \otimes x) = [(m,x)].$$

The zero element in $\mathbb{Z}^{-1}N$ is $[(1,0)]$. So $(m,x) \sim (1,0)$, i.e., $t(x - 0) = 0$ for some $0 \neq t \in \mathbb{Z}$. Suppose now that $rx = 0$ for some $0 \neq r \in \mathbb{Z}$. We have

$$\frac{1}{m} \otimes x = \frac{r}{rm} \otimes x = \frac{1}{rm} \otimes (rx) = \frac{1}{rm} \otimes 0 = 0.$$

**Solution 42** Self-study.

**Solution 43** Left as exercise.

**Solution 44** Suppose that $f_3(m) = 0$ for some $m \in M_3$. Therefore, $fg(m) = hf(m) = 0$. Since $f_4$ is injective, we have $g(m) = 0$. Since it is exact at $M_2$, there exists $m' \in M_2$ such that $g(m') = m$. We have $hf(m') = fg(m') = f(m) = 0$. So $f(m') \in \ker h_2 = \operatorname{im} h_1$ and hence there exists $n \in N_1$ such that $h(n) = f(m')$. But $f_1$ is surjective. There exists $m'' \in M_1$ such that $f(m'') = n$. We have $fg(m'') = hf(m'') = h(n) = f(m')$. Since $f_2$ is injective, we have $m' = g(m'')$. So $m = g(m') = g^2(m'') = 0$. So $f_3$ is injective.

**Solution 45** (i) Define $\gamma(1) = 1$ and $\gamma(-1) = (1, 2)$. This is a group homomorphism. We have $\operatorname{sgn}\gamma(-1) = \operatorname{sgn}((1, 2)) = -1$. So $\operatorname{sgn}\gamma = \mathrm{id}_{\{\pm 1\}}$. (ii) Suppose that we have a group homomorphism such that $\delta\iota = \mathrm{id}_{A_n}$. In particular, $\delta$ is surjective and there is a normal subgroup $N$ of $S_n$ such that $|N| = |S_n|/|A_n| = 2$. So $N = \langle \tau \rangle$ where $\tau$ is a permutation of order 2, i.e., $\tau = (a_1, b_1)(a_2, b_2) \cdots (a_k, b_k)$ as a product of disjoint cycles where $k \geq 1$. Since $n \geq 3$, there exists $c \notin \{a_1, b_1\}$. Since $N$ is closed under conjugation, we must have $\tau = (a_1, c)\tau(a_1, c) := \sigma$. But $\tau : b_1 \mapsto a_1$ while $\sigma : b_1 \mapsto c$. A contradiction. So there is no such $\delta$.

**Solution 46** Let $f : \mathbb{Z}/2\mathbb{Z} \to \mathbb{Z}$ be a group homomorphism and $f(\overline{1}) = n$. But

$$0 = f(\overline{0}) = f(\overline{1} + \overline{1}) = n + n = 2n.$$

So $n = 0$. This shows that $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z})$ is trivial. On the other hand, $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong C_2$.

**Solution 47** (i) Let $F_i = F(S_i)$ where $F_i$ is free on the subset $S_i$. We claim that $F := \bigoplus_{i \in I} F_i$ is free on the set $\bigcup_{i \in I} S_i$. For any $0 \neq x \in F$, $x = (x_i)_{i \in I} = \sum_{i \in I} x_i$ where $x_i$'s are almost all zero. Notice that $x_i$'s are uniquely determined by $x$. For $x_i \neq 0$, there exist unique nonzero elements $r_{1,i}, \ldots, r_{j_i,i} \in R$ and $s_{1,i}, \ldots, s_{j_i,i} \in S_i$ such that $x_i = r_{1,i}s_{1,i} + \cdots + r_{j_i,i}s_{j_i,i}$. So $x$ is the sum of the $r_{j,i}s_{j,i}$'s in the unique way. So $F$ is free.

(ii) Let $P := \bigoplus_{i \in I} P_i$ and $Q := \bigoplus_{i \in I} Q_i$. Clearly, $P + Q \subseteq F$. Conversely, for $x \in F$, we have $x = (x_i)_{i \in I} = \sum x_i$. Let $x_i = y_i + z_i$ where $y_i \in P_i$ and $z_i \in Q_i$ for each $i \in I$. Notice that $y_i = 0 = z_i$ if $x_i = 0$. Since it is a finite sum, we have $x = (\sum y_i) + (\sum z_i) \in P + Q$. So $F = P + Q$. Let $x \in P \cap Q$. Then $x = (x_i)_{i \in I}$ where $x_i \in P_i \cap Q_i$ for each $i \in I$. Hence $x_i = 0$. So $x = 0$. As such, we obtain a direct sum $P \oplus Q$.

(iii) Combine parts (i) and (ii).

**Solution 48** Left as exercise.

**Solution 49** Left as exercise.

**Solution 50** We require at least that $\mathrm{Mor}(X, Y)$ is an abelian group for every objects $X, Y$, i.e., $f, g : X \to Y$, we have $f + g = g + f \in \mathscr{F}(Y) = \mathrm{Mor}(X, Y)$, there exists $0 \in \mathrm{Mor}(X, Y)$ such that $f + 0 = f$ and there exists $-f \in \mathrm{Mor}(X, Y)$ such that $f + (-f) = 0$. For each $f : Y \to Z$, we have a morphism $\mathscr{F}(f) : \mathscr{F}(Y) \to \mathscr{F}(Z)$ given by $\mathscr{F}(f)(\alpha) = f \circ \alpha$ where $\alpha \in \mathscr{F}(Y) = \mathrm{Mor}(X, Y)$. This morphism must be a group homomorphism, i.e., $\mathscr{F}(f)(\alpha + \beta) = \mathscr{F}(f)(\alpha) + \mathscr{F}(f)(\beta)$, i.e., we require that

$$f \circ (\alpha + \beta) = f \circ \alpha + f \circ \beta.$$

It is straightforward to check that, for $1_Y \in \mathrm{Mor}(Y, Y)$,

$$\mathscr{F}(1_Y)(\alpha) = 1_Y \circ \alpha = \alpha = 1_{\mathscr{F}(Y)}(\alpha).$$

So $\mathscr{F}(1_Y) = 1_{\mathscr{F}(Y)}$. Also,

$$\mathscr{F}(f \circ g)(\alpha) = (f \circ g) \circ \alpha = f \circ (g \circ \alpha) = (\mathscr{F}(f) \circ \mathscr{F}(g))(\alpha).$$

**Solution 51** (i) Let $F$ be a free $\mathbb{Z}$-module with a free basis $S$. We claim that $\bigcap_{n=1}^{\infty} nF = 0$. Let $0 \neq x \in \bigcap_{n=1}^{\infty} nF$. By the assumption, we have unique nonzero elements $n_1, \dots, n_r \in \mathbb{Z}$ and $s_1, \dots, s_r \in S$ such that

$$x = n_1 s_1 + \cdots + n_r s_r.$$

Choose $n \geq 2\max\{|n_1|, \dots, |n_r|\}$. Since $x \in nF$, we have $x = nz$ for some $0 \neq z \in F$. So there exist unique nonzero elements $m_1, \dots, m_k \in \mathbb{Z}$ and $s_1', \dots, s_k' \in S$ such that

$$z = m_1 s_1' + \cdots + m_k s_k'.$$

Combine these equations, we have

$$n_1 s_1 + \cdots + n_r s_r = x = nz = nm_1 s_1' + \cdots + nm_k s_k'.$$

By the uniqueness of the presentation of $x$, we must have $r = k$, without loss of generality, $s_i = s_i'$, and $n_i = nm_i$. But $n_i = nm_i \geq 2|n_i|m_i$. A contradiction. So $x = 0$.

Let $Q$ be a divisible $\mathbb{Z}$-module and suppose on the contrary that $Q$ is projective. Then there exists a free $\mathbb{Z}$-module $F$ such that $Q \mid F$. Without loss of generality, we assume that $Q$ is a submodule of $F$. We claim that $Q \subseteq nF$ for every $n \in \mathbb{Z}_+$. Let $n \in \mathbb{Z}_+$ and $m \in Q$. Since $Q$ is divisible, we have $m' \in Q$ such that $m = nm' \in nF$. So $Q \subseteq \bigcap_{n=1}^{\infty} nF = 0$. This contradicts to our assumption that $Q$ is nonzero. So $Q$ is not projective.

(ii) By part (i), we only need to show that $\mathbb{Q}$ is a divisible $\mathbb{Z}$-module. Let $0 \neq n \in \mathbb{Z}$ and $\frac{r}{s} \in \mathbb{Q}$. Then

$$\frac{r}{s} = n\frac{r}{ns}.$$

So $\mathbb{Q}$ is a divisible $\mathbb{Z}$-module and hence not projective.

**Solution 52** (i) $\Rightarrow$ (ii): We have a SES

$$0 \to X \xrightarrow{\alpha} Y \to Y/\operatorname{im}(\alpha) \to 0.$$

Since $\alpha^*$ is surjective, there exists $f \in \mathsf{Hom}_R(Y, I)$ such that $\alpha^*(f) = g$, i.e., $f\alpha = g$.

(ii) $\Rightarrow$ (iii): We have an exact sequence $0 \to I \xrightarrow{\iota} Y$. Let $g$ be the identity on $I$. Then there exists a homomorphism $f : Y \to I$ such that $f \circ \iota = \mathrm{id}_I$. By Proposition 10.25, we have $Y \cong I \oplus Y/I$. So $I \mid Y$.

(iii) $\Rightarrow$ (i): Suppose that $0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0$ is exact. By Theorem 10.33, we only need to prove the surjectivity of $\alpha^*$. Let $g : X \to I$. Let $J$ be an injective $R$-module such that $I \subseteq J$. By part (iii), there exists $J' \subseteq J$ such that $J = I \oplus J'$. By the injectivity of $J$, there exists $h : Y \to J$ such that $\iota g = h\alpha$. Let $f : Y \to I$ be $f = \pi h$ where $\pi : J \to I$ is the canonical projection. So

$$\alpha^*(f) = f\alpha = \pi h\alpha = \pi\iota g = g.$$

So $\alpha^*$ is surjective.

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r, "\alpha"] \arrow[d, "g"] & Y \arrow[r] \arrow[dl, "f"] \arrow[d, dashed, "h"] & Z \arrow[r] & 0 \\
& I \arrow[r, shift left, "\iota"] & J \arrow[l, shift left, "\pi"] & &
\end{tikzcd}$$

**Solution 53** Let $C$ be a chain in $\Omega$. Let

$$W = \bigcup_{(f',Y')\in C} Y'.$$

We have $\mathrm{im}(\alpha) \subseteq W \subseteq Y$. We claim that $W$ is a submodule of $Y$. Let $w, w' \in W$ and $r \in R$. Suppose that $w \in Y'$ and $w' \in Y''$ where $(f', Y')$ and $(f'', Y'')$ belong in $C$. Without loss of generality, since $C$ is a chain, we assume that $(f', Y') \leq (f'', Y'')$. So $w + w' \in Y''$ and $rw \in Y'$. So $w+w' \in W$ and $rw \in W$. Define $f : W \to Q$ by $f(w) = f'(w)$ if $w \in Y'$ where $(f', Y')$ belongs in $C$. Suppose that $w \in Y'$ and $w' \in Y''$ where $(f', Y')$ and $(f'', Y'')$ belong in $C$. We have $f(w+w') = f''(w+w') = f''(w)+f''(w') = f(w)+f(w')$. Also, $f(rw) = f'(rw) = rf'(w) = rf(w)$. So $(f, W)$ belongs in $\Omega$ and contains all the members in $C$. Therefore, Zorn's lemma applies.

**Solution 54** We use Baer's Criterion. Let $I$ be a left ideal of $R$ and $g : I \to \prod_{i\in I} Q_i =: Q$ be a morphism. Let $\pi_i : Q \to Q_i$ be the canonical morphism into each $Q_i$. The morphism $g_i := \pi_i \circ g : I \to Q_i$ has a lift $f_i : R \to Q_i$. Define $f : R \to Q$ by $f(r) = (f_i(r))_{i\in I} \in Q$. We have

$$f \circ \iota(r) = (f_i(\iota(r)))_{i\in I} = (g_i(r))_{i\in I} = g(r).$$

So $Q$ is injective.
(Notice that the proof fails for direct sum as $(f_i(r))_{i\in I}$ does not belong in $\bigoplus_{i\in I} Q_i$ in general. In fact, Bass–Papp Theorem asserts that, for a ring $R$, any direct sum of any injective $R$-modules is injective is equivalent to ${}_RR$ is Noetherian. So, in general, the direct sum of infinite number of injective $R$-modules needs not be injective.)

**Solution 55** (i): Suppose that $P, P'$ are projective covers of $M$. By definition, there exist surjection $f : P \to M$ and $f' : P' \to M$. Using the projectivity of $P$, we have $g : P \to P'$ such that $f'g = f$. Since $P'$ is a projective cover, $g$ is surjective. Since $P'$ is projective, the map $g$ splits, i.e., we have $g' : P' \to P$ such that $gg' = \mathrm{id}_{P'}$. So $fg' = (f'g)g' = f'(gg') = f'$. Since $f'$ is surjective and $P$ is a projective cover, $g'$ is surjective. For any $x \in \ker g$, let $y \in P'$ such that $g'(y) = x$. We have

$$x = g'(y) = g'(gg')(y) = g'g(x) = g'(0) = 0.$$

So $g$ is also injective. Therefore, $g$ is an isomorphism.

(ii): Let $P$ be a projective cover of $\mathbb{Z}/2\mathbb{Z}$ and let $f : P \to \mathbb{Z}/2\mathbb{Z}$ denote the surjection. Since $\mathbb{Z}/2\mathbb{Z}$ is generated by $\{\overline{1}\}$, there exists a surjection $g : \mathbb{Z} \to \mathbb{Z}/2\mathbb{Z}$. Since $\mathbb{Z}$ is free (and hence projective), there exists $h : \mathbb{Z} \to P$ such that $fh = g$. Since $P$ is a projective cover, $h$ is surjective. So $P$ is a quotient of $\mathbb{Z}$. Notice that $\ker h = n\mathbb{Z}$ for some $n \in \mathbb{Z}$ and hence $P \cong \mathbb{Z}/n\mathbb{Z}$. By one of our example, we have seen that finite abelian group is not projective $\mathbb{Z}$-module. So $n = 0$, i.e., $h$ is an isomorphism. Let $x = h(1)$. So $f(x) = fh(1) = \overline{1}$. Since $P \cong \mathbb{Z}$, $3P$ is also free and hence projective. Consider the inclusion $\iota : 3P \to P$. We have $f\iota(3x) = f(3x) = 3f(x) = \overline{1}$. So $f\iota$ is surjective but $\iota$ is not surjective. This violates our assumption that $P$ is a projective cover. Therefore, $\mathbb{Z}/2\mathbb{Z}$ has no projective cover.

**Solution 56** (i): We have proved the converse in Exercise 47. Suppose that $V \oplus W$ is projective. Then $V \oplus W$ is a direct summand of a free $R$-module $F$. In turns, since $V$ is a direct summand of $V \oplus W$, $V$ is a direct summand of $F$. So $V$ is projective. Similarly, $W$ is projective.

(ii): We have proved the converse in Exercise 54 (by taking the set $I$ consisting of only two elements). Suppose that $V \oplus W$ is injective. Let $0 \to X \xrightarrow{\alpha} Y$ be exact and $g : X \to V$ be a morphism. Since $V \oplus W$ is injective, there exists $f' : Y \to V \oplus W$ such that $f'\alpha = \iota g$. Define $f = \pi f'$ where $\pi\iota = 1_V$. So $f\alpha = \pi f'\alpha = \pi\iota g = g$. This shows that $V$ is injective. For $W$, it is similar.

$$\begin{tikzcd}
0 \arrow[r] & X \arrow[r] \arrow[d, "g"] & Y \arrow[dl, "f"'] \arrow[ddl, dashed, "f'"] \\
& V \arrow[d, hook, "\iota"] & \\
& V \oplus W &
\end{tikzcd}$$

(iii): Suppose that $V, W$ are flat right $R$-modules. Let $0 \to X \xrightarrow{\alpha} Y$ be exact. We have injections $0 \to V \otimes_R X \xrightarrow{1\otimes\alpha} V \otimes_R Y$ and $0 \to W \otimes_R X \xrightarrow{1\otimes\alpha} W \otimes_R Y$. We claim that the map $1\otimes\alpha : (V \oplus W) \otimes_R X \to (V \oplus W) \otimes_R Y$ defined by $(1\otimes\alpha)((v+w)\otimes x) = (v+w)\otimes\alpha(x)$ is injective. Suppose that $(1\otimes\alpha)((v+w)\otimes x) = 0$, i.e., $(v+w)\otimes\alpha(x) = 0$. By the isomorphism $(V \oplus W) \otimes_R Y \cong (V \otimes_R Y) \oplus (W \otimes_R Y)$. The element $(v+w)\otimes\alpha(x)$ is mapped to $v \otimes \alpha(x) + w \otimes \alpha(x) = 0$. Since it is direct sum, we have both $v \otimes \alpha(x) = 0$ and $w \otimes \alpha(x) = 0$. So $v \otimes x = 0$ and $w \otimes x = 0$ by the injectivity. Therefore, $V \oplus W$ is flat.

Conversely, suppose that $V \oplus W$ is flat. Let $0 \to X \xrightarrow{\alpha} Y$ be exact. We have an injection $1 \otimes \alpha : (V \oplus W) \otimes_R X \to (V \oplus W) \otimes_R Y$. We claim that the map $1_V \otimes \alpha : V \otimes_R X \to W \otimes_R Y$ is injective. Notice that $1_V \otimes \alpha = (\pi \otimes 1_Y)(1 \otimes \alpha)\iota$ via the isomorphism $(V \oplus W) \otimes_R Y \cong (V \otimes_R Y) \oplus (W \otimes_R Y)$. Therefore, if $(1_V \otimes \alpha)(v \otimes x) = 0$, we have $(1 \otimes \alpha)((v + 0) \otimes x) = 0$. Since $1 \otimes \alpha$ is injective, we have $(v + 0) \otimes x = 0$ and hence $v \otimes x = 0$.

**Solution 57** (i): Define $\Phi : \mathsf{Hom}_R(\bigoplus_{i\in I} V_i, W) \to \prod_{i\in I} \mathsf{Hom}_R(V_i, W)$ by $\Phi(f) = (f\iota_i)_{i\in I}$ where $\iota_i : V_i \to \bigoplus_{i\in I} V_i$ is the natural inclusion. Clearly $\Phi(f) \in \prod_{i\in I} \mathsf{Hom}_R(V_i, W)$ (and not in $\bigoplus_{i\in I} \mathsf{Hom}_R(V_i, W)$ because $f\iota_i$ could be nonzero for infinite $i$). This is a group homomorphism because

$$\Phi(f+g) = ((f+g)\iota_i) = (f\iota_i + g\iota_i) = (f\iota_i) + (g\iota_i) = \Phi(f) + \Phi(g).$$

Let $\Psi : \prod_{i\in I} \mathsf{Hom}_R(V_i, W) \to \mathsf{Hom}_R(\bigoplus_{i\in I} V_i, W)$ be defined as $\Psi((f_i))((v_i)) = \sum_{i\in I} f_i(v_i)$. The sum makes sense because almost all $v_i$'s (and hence $f_i(v_i)$'s) are zero. Let $g = \Psi((f_i))$. For each $i \in I$, let $v_j = 0$ if $j \neq i$. We have $g\iota_i(v_i) = g((v_j)) = \sum f_j(v_j) = f_i(v_i)$. So $g\iota_i = f_i$. As such, $\Phi\Psi((f_i)) = (g\iota_i) = (f_i)$. On the other hand,

$$\Psi\Phi(f)((v_i)) = \Psi(f\iota_i)((v_i)) = \sum f_i(v_i) = f((v_i)).$$

Therefore, $\Psi$ is the inverse of $\Phi$ and hence $\Phi$ is an isomorphism.

(ii): We leave it as an exercise to check that the map $\Phi : \mathsf{Hom}_R(V, \prod_{j\in J} W_j) \to \prod_{j\in J} \mathsf{Hom}_R(V, W_j)$ defined by $\Phi(f) = (\pi_j f)$ is an isomorphism with the inverse $\Psi : \prod_{j\in J} \mathsf{Hom}_R(V, W_j) \to \mathsf{Hom}_R(V, \prod_{j\in J} W_j)$ defined by $\Psi((f_j))(v) = (f_j(v))$.

**Solution 58** (i): Let $H_i := \mathsf{span}_k\{x_i h : h \in H\}$. The vector space has a basis $G$. Since the distinct left cosets partition $G$, we have a direct sum decomposition of the vector space into subspaces $kG = \bigoplus_{i=1}^m H_i$. For each $x_i H$, we have $(x_i h)h' \in x_i H$ for each $h' \in H$. Therefore, $H_i$ is a right $kH$-module by linearity. As such, we have a direct sum decomposition $kG = \bigoplus H_i$ as right $kH$-modules.

(ii): We claim that $H_i$ is a free right $kH$-module. Let $\phi : H_i \to kH$ be defined as $\phi(x_ih) = h$ and extend $\phi$ linearly. It is clear that $\phi$ is bijective because $H_i$ and $kH$ have bases $x_iH$ and $H$ respectively. For $h' \in H$, we have $\phi((x_ih)h') = \phi(x_i(hh')) = hh' = \phi(x_ih)h'$. So $\phi$ is an $kH$-module isomorphism. As such, $H_i$ is free and hence $kG$ is free using part (i) and Exercise 47(i).

(iii): Since $kG$ is a free right $kH$-module, by Corollary 10.42, it is a flat $kH$-module. As such, the functor $kG \otimes_{kH} -$ is exact by definition. Therefore, we obtain part (iii).

(iv): This follows from Corollary 10.41.

**Solution 59** First of all, $d_{n+1}^* \circ d_n^* = (d_n \circ d_{n+1})^* = 0$. So $\operatorname{im} d_n^* \subseteq \ker d_{n+1}^*$. Suppose now that $\phi \in \ker d_{n+1}^*$, i.e., $\phi : V_n \to Q$ and $\phi d_{n+1} = 0$. So $\ker d_n = \operatorname{im} d_{n+1} \subseteq \ker \phi$. Define $\widetilde{d_n} : V_n/\ker d_n \to V_{n-1}$ by $\widetilde{d_n}(v + \ker d_n) = d_n(v)$. This is well-defined because, if $v \in \ker d_n$, then $d_n(v) = 0$. It is also injective because, if $d_n(v) = 0$, then $v \in \ker d_n$. Define also $\widetilde{\phi} : V_n/\ker d_n \to Q$ by $\widetilde{\phi}(v + \ker d_n) = \phi(v)$. Since $\ker d_n = \operatorname{im} d_{n+1} \subseteq \ker \phi$, the map $\widetilde{\phi}$ is well-defined. By the injectivity of $Q$, there is a map $\gamma : V_{n-1} \to Q$ such that the following diagram commutes:

$$\begin{tikzcd}
0 \arrow[r] & V_n/\ker d_n \arrow[r, "\widetilde{d_n}"] \arrow[d, "\widetilde{\phi}"] & V_{n-1} \arrow[dl, dashed, "\gamma"] \\
& Q &
\end{tikzcd}$$

We have, for each $v \in V_n$,

$$d_n^*(\gamma)(v) = \gamma d_n(v) = \gamma\widetilde{d_n}(v + \ker d_n) = \widetilde{\phi}(v + \ker d_n) = \phi(v).$$

So $v \in \operatorname{im} d_n^*$. Therefore, the cochain complex is exact.

**Solution 60** We have $0 \to \operatorname{im} d_{n+1} \to V_n \xrightarrow{d_n} \operatorname{im} d_n \to 0$ and hence, by the flatness of $D$, we have an injection $0 \to D \otimes_R \operatorname{im} d_{n+1} \xrightarrow{1\otimes\iota} D \otimes_R V_n \xrightarrow{1\otimes d_n} D \otimes_R \operatorname{im} d_n \to 0$. We have

$$\ker (1 \otimes d_n) = \operatorname{im}(1 \otimes \iota) = D \otimes_R \operatorname{im} d_{n+1} = \operatorname{im}(1 \otimes d_{n+1}).$$

Therefore, the chain complex is exact.

**Solution 61** (i): Since $\beta_n : Y^n \to Z^n$ is surjective, there exists $y \in Y^n$ such that $\beta_n(y) = z$. We have

$$\beta(d(y)) = d\beta(y) = d(z) = 0.$$

So $d(y) \in \ker \beta = \operatorname{im} \alpha$. As such, there exists $x \in X^{n+1}$ such that $\alpha(x) = d(y)$. This element $x$ is uniquely determined by $d(y)$ because $\alpha$ is injective. Notice that $\alpha d(x) = d\alpha(x) = d^2(y) = 0$. Since $\alpha$ is injective, $d(x) = 0$, i.e., $x \in \ker d_{n+2}$.

(ii): Since $z - z' \in \operatorname{im} d_n$, we have $z'' \in Z^{n-1}$ such that $d(z'') = z - z'$. Since $\beta$ is surjective, there exists $w \in Y^{n-1}$ such that $\beta(w) = z''$. We have

$$\beta(d(w) - (y - y')) = \beta d(w) - \beta(y - y') = d\beta(w) - (z - z') = d(z'') - (z - z') = 0.$$

So $d(w) - (y - y') \in \ker \beta = \operatorname{im} \alpha$, i.e., there exists $v \in X^n$ such that $\alpha(v) = d(w) - (y - y')$. Notice that

$$\alpha d(v) = d\alpha(v) = d(d(w) - (y - y')) = d^2(w) - d(y - y') = -d(y - y') = \alpha(x' - x).$$

Since $\alpha$ is injective, we have $x = x' - d(v)$. So $x + \operatorname{im} d_{n+1} = x' + \operatorname{im} d_{n+1}$.

(iii): By parts (i) and (ii) above, we see that the map $\delta_n : \mathsf{H}^n(Z) \to \mathsf{H}^{n+1}(X)$ defined by sending $z + \operatorname{im} d_n$ to $x + \operatorname{im} d_{n+1}$ is well-defined.

(iv): Let $z, z' \in \ker d_{n+1} \subseteq Z^n$ and $x, x' \in \ker d_{n+2} \subseteq X^{n+1}$ such that such that $\beta(y) = z$, $\beta(y') = z'$, $\alpha(x) = d(y)$ and $\alpha(x') = d(y')$. We have $\beta(y + y') = z + z'$ and $\alpha(x + x') = d(y + y')$. Therefore,

$$\begin{aligned}\delta_n((z + \operatorname{im} d_n) + (z' + \operatorname{im} d_n)) &= \delta_n((z + z') + \operatorname{im} d_n) = (x + x') + \operatorname{im} d_{n+1} \\ &= (x + \operatorname{im} d_{n+1}) + (x' + \operatorname{im} d_{n+1}) = \delta_n(z + \operatorname{im} d_n) + \delta_n(z' + \operatorname{im} d_n).\end{aligned}$$

**Solution 62** We have cochain complexes $X^\bullet$, $Y^\bullet$ and $Z^\bullet$ where, for example, $X^0 = X$, $X^1 = X'$, $d_1 = f$, $X^n = 0$ and $d_n = 0$ for all $n \geq 2$. By Theorem 17.2, we have a LES

$$0 \to \mathsf{H}^0(X^\bullet) \to \mathsf{H}^0(Y^\bullet) \to \mathsf{H}^0(Z^\bullet) \xrightarrow{\delta_0} \mathsf{H}^1(X^\bullet) \to \mathsf{H}^1(Y^\bullet) \to \mathsf{H}^1(Z^\bullet) \to \mathsf{H}^2(X^\bullet) \to \cdots.$$

We have $\mathsf{H}^2(X^\bullet) = 0$ because $X^2 = 0$. Also, $H^0(X^\bullet) = \ker d_1 = \ker f$. Similarly, $H^0(Y^\bullet) = \ker g$ and $H^0(Z^\bullet) = \ker h$. For the first cohomology groups, we have

$$\mathsf{H}^1(X^\bullet) = \ker d_2 / \operatorname{im} d_1 = X^1 / \operatorname{im} f = X' / \operatorname{im} f = \operatorname{coker} f.$$

Similarly, $H^1(Y^\bullet) = \operatorname{coker} g$ and $H^1(Z^\bullet) = \operatorname{coker} h$. So the LES reduces to

$$0 \to \ker f \xrightarrow{\alpha} \ker g \xrightarrow{\beta} \ker h \xrightarrow{\delta_n} \operatorname{coker} f \xrightarrow{\alpha'} \operatorname{coker} g \xrightarrow{\beta'} \operatorname{coker} h \to 0$$

where the maps are given in Proposition 17.1.

**Solution 63** Let $\gamma : Q_0 \to Y$ be the lift of $\varepsilon''$, i.e., $\beta\gamma = \varepsilon''$. Let $\varepsilon' : P_0 \oplus Q_0$ be the map $\varepsilon'(a + b) = \alpha\varepsilon(a) + \gamma(b)$. It is easy to check that the following diagram commutes:

$$\begin{tikzcd} 0 \arrow[r] & P_0 \arrow[r, "\iota"] \arrow[d, "\varepsilon"] & P_0 \oplus Q_0 \arrow[r, "\pi"] \arrow[d, "\varepsilon'"] & Q_0 \arrow[r] \arrow[d, "\varepsilon''"] & 0 \\ 0 \arrow[r] & X \arrow[r, "\alpha"] & Y \arrow[r, "\beta"] & Z \arrow[r] & 0 \end{tikzcd}$$

By the Snake Lemma, we have an exact sequence

$$0 \to \ker \varepsilon \to \ker \varepsilon' \to \ker \varepsilon'' \to \operatorname{coker} \varepsilon \to \operatorname{coker} \varepsilon' \to \operatorname{coker} \varepsilon'' \to 0.$$

Since $\varepsilon$ and $\varepsilon''$ are surjective, $\operatorname{coker} \varepsilon = 0$ and $\operatorname{coker} \varepsilon'' = 0$. So $\operatorname{coker} \varepsilon' = 0$ and hence $\varepsilon'$ is surjective. Also, the exact sequence reduces to the following SES

$$0 \to \ker \varepsilon \to \ker \varepsilon' \to \ker \varepsilon'' \to 0.$$

Let $d_0 = \varepsilon$, $\partial_0 = \varepsilon''$, $\delta_0 = \varepsilon'$. For inductive step, let $\gamma : Q_n \to \ker \delta_{n-1}$ be the lift of $\partial_n : Q_n \to \operatorname{im} \partial_n$ and let $\delta_n : P_n \oplus Q_n$ be defined as $\delta_n(a + b) = \iota d_n(a) + \gamma(b)$. We have the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & P_n \arrow[r, "\iota"] \arrow[d, "d_n"] & P_n \oplus Q_n \arrow[r, "\pi"] \arrow[d, "\delta_n"] & Q_n \arrow[r] \arrow[d, "\partial_n"] & 0 \\ 0 \arrow[r] & \operatorname{im} d_n = \ker d_{n-1} \arrow[r] & \ker \delta_{n-1} \arrow[r] & \operatorname{im} \partial_n = \ker \partial_{n-1} \arrow[r] & 0 \end{tikzcd}$$

Using the Snake Lemma again, we have an exact sequence

$$0 \to \ker d_n \to \ker \delta_n \to \ker \partial_n \to \operatorname{coker} d_n \to \operatorname{coker} \delta_n \to \operatorname{coker} \partial_n \to 0.$$

Similar as before, since $d_n$ and $\partial_n$ are surjective, we have coker $\partial_n = 0$ and hence $\partial_n$ is surjective. Furthermore, it reduces to a SES

$$0 \to \ker d_n \to \ker \delta_n \to \ker \partial_n \to 0.$$

**Solution 64** Let $P_\bullet$, $P'_\bullet$, $P''_\bullet$ be projective resolutions of $V, V', V''$ respectively. By Proposition 17.4, we have the following commutative diagram:

$$\begin{tikzcd}
\cdots \arrow[r, "d_2"] & P_1 \arrow[r, "d_1"] \arrow[d, "f_1"] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "f_0"] & V \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r, "d'_2"] & P'_1 \arrow[r, "d'_1"] \arrow[d, "g_1"] & P'_0 \arrow[r, "\varepsilon"] \arrow[d, "g_0"] & V' \arrow[r] \arrow[d, "g"] & 0 \\
\cdots \arrow[r, "d''_2"] & P''_1 \arrow[r, "d''_1"] & P''_0 \arrow[r, "\varepsilon"] & V'' \arrow[r] & 0
\end{tikzcd}$$

Taking hom to $W$, we have the following induced maps of cochain complexes

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_R(P_0, W) \arrow[r, "d_1^*"] & \mathrm{Hom}_R(P_1, W) \arrow[r, "d_2^*"] & \mathrm{Hom}_R(P_2, W) \arrow[r] & \cdots \\
0 \arrow[r] & \mathrm{Hom}_R(P'_0, W) \arrow[r, "d_1'^*"] \arrow[u, "f_0^*"] & \mathrm{Hom}_R(P'_1, W) \arrow[r, "d_2'^*"] \arrow[u, "f_1^*"] & \mathrm{Hom}_R(P'_2, W) \arrow[r] \arrow[u, "f_2^*"] & \cdots \\
0 \arrow[r] & \mathrm{Hom}_R(P''_0, W) \arrow[r, "d_1''^*"] \arrow[u, "g_0^*"] & \mathrm{Hom}_R(P''_1, W) \arrow[r, "d_2''^*"] \arrow[u, "g_1^*"] & \mathrm{Hom}_R(P''_2, W) \arrow[r] \arrow[u, "g_2^*"] & \cdots
\end{tikzcd}$$

By Proposition 17.1, we have an induced maps on the cohomology groups $\mathscr{F}(f)_n : \mathrm{Ext}^n_R(V', W) \to \mathrm{Ext}^n_R(V, W)$ and $\mathscr{F}(g)_n : \mathrm{Ext}^n_R(V'', W) \to \mathrm{Ext}^n_R(V', W)$ where $\mathscr{F}(f)_n([\alpha]) = [f_n{*}(\alpha)]$ and $\mathscr{F}(g)_n([\beta]) = [g_n * (\beta)]$. Similarly, we have the induced map $\mathscr{F}(g \circ f)_n : \mathrm{Ext}^n_R(V'', W) \to \mathrm{Ext}^n_R(V, W)$ where $\mathscr{F}(g \circ f)_n([\beta]) = [(g_n \circ f_n) * (\beta)]$. So

$$\mathscr{F}(g \circ f)_n([\beta]) = [(g_n \circ f_n) * (\beta)] = [\beta(g_n f_n)] = [(\beta g_n) f_n] = \mathscr{F}(f)_n([\beta g_n]) = \mathscr{F}(f)_n \mathscr{F}(g)_n([\beta]).$$

**Solution 65** (iii) $\Rightarrow$ (ii) is trivial. For (ii) $\Rightarrow$ (i), we use Theorem 17.10. Now we prove (i) $\Rightarrow$ (iii). Since $P$ is projective, we have a projective resolution for $P$ given by

$$0 \to P \to P \to 0.$$

Hom to $B$, we have

$$0 \to \mathrm{Hom}_R(P, B) \to 0 \to 0 \to \cdots .$$

Therefore, $\mathrm{Ext}^n_R(P, B) = 0$ for all $n \geq 1$.

**Solution 66** (i): Let $V = \bigoplus_{j \in J} V_j$, $P_n := \bigoplus_{j \in J} P(j)_n$ and $d_n : P_{n+1} \to P_n$ be the map defined by $d_n = (d(j)_n)_{j \in J}$. Also, let $\varepsilon : P_0 \to V$ be $\varepsilon = (\varepsilon(j))$. We have

$$\mathrm{im}\, d_n = (\mathrm{im}\, d(j)_n) = (\ker d(j)_{n-1}) = \ker d_{n-1}.$$

For the exactness of $\varepsilon$, let $v = (v_j) \in V$. Since most $v_j$'s are zero, there are $x_j \in P_0$ mostly zero such that $\varepsilon_j(x_j) = (v_j)$. Therefore, $\varepsilon(x_j) = (v_j) = v$. Since $P_n$'s are projective, we have a projective resolution

$$\cdots \to P_2 \to P_1 \to P_0 \xrightarrow{\varepsilon} V \to 0.$$

Hom to $W$, we obtain

$$0 \to \mathsf{Hom}_R(P_0, W) \xrightarrow{d_1^*} \mathsf{Hom}_R(P_1, W) \xrightarrow{d_2^*} \mathsf{Hom}_R(P_2, W) \to \cdots .$$

We have isomorphisms $\Phi : \mathsf{Hom}_R(P_n, W) \rightleftarrows \prod_{j\in J} \mathsf{Hom}_R(P(j)_n, W) : \Psi$ in Exercise 57(i) with $\Phi(f) = (f\iota_i)_{i\in I}$ and $\Psi((f_i))((v_i)) = \sum f_i(v_i)$. Via the isomorphisms, we have

$$\mathsf{Ext}^n_R(V, W) \cong \ker \delta_{n+1} / \operatorname{im} \delta_n$$

where

$$\delta_n = \prod_{j\in J} \delta(j)^*_n : \prod_{j\in J} \mathsf{Hom}_R(P(j)_{n-1}, W) \to \prod_{j\in J} \mathsf{Hom}_R(P(j)_n, W).$$

Therefore,

$$\mathsf{Ext}^n_R(V, W) \cong \prod_{j\in J} \ker d(j)^*_{n+1} / \operatorname{im} d(j)^*_n = \prod_{j\in J} \mathsf{Ext}^n_R(V_i, W).$$

(ii): We leave it as exercise. Use Exercise 57(ii) instead and the fact that the derived functor $\mathsf{Ext}^n_R(V, W)$ constructed using a projective resolution of $V$ is isomorphic with the derived functor constructed using an injective resolution of $W$.

(iii):

**Solution 67** (i): We have

$$\begin{bmatrix} -d_X & 0 \\ -f & d_Y \end{bmatrix} \begin{bmatrix} -d_X & 0 \\ -f & d_Y \end{bmatrix} = \begin{bmatrix} d_X^2 & 0 \\ fd_X - d_Y f & d_Y^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(ii): We need to check that, for each $n$, the sequence

$$0 \to Y_n \xrightarrow{\gamma} X_{n-1} \oplus Y_n \xrightarrow{\delta} X[-1]_n \to 0$$

is exact. We have

$$\ker \delta = 0 \oplus Y_n = \operatorname{im} \gamma.$$

So it is a SES. Next, we need to check that the following diagram commutes:

$$\begin{tikzcd} Y_n \arrow[r, "\gamma"] \arrow[d, "d_Y"] & X_{n-1} \oplus Y_n \arrow[r, "\delta"] \arrow[d, "\partial_n"] & X[-1]_n \arrow[d, "d[-1]_n"] \\ Y_{n-1} \arrow[r, "\gamma"] & X_{n-2} \oplus Y_{n-1} \arrow[r, "\delta"] & X[-1]_{n-1} \end{tikzcd}$$

For $y \in Y_n$, we have

$$\partial_n \gamma(y) = \partial_n(0, y) = (0, d_Y(y)) = \gamma(d_Y(y)).$$

For $(x, y) \in X_n \oplus Y_{n-1}$, we have

$$d[-1]_n \delta(x, y) = d[-1](-x) = -d_X(-x) = d_X(x) = \delta(-d_X(x), y) = \delta\partial_n(x, y).$$

**Solution 68** Let $P_\bullet \twoheadrightarrow B$. We have an exact sequence $P_1 \xrightarrow{d_1} P_0 \xrightarrow{\varepsilon} B \to 0$. By Theorem 10.39, we have an exact sequence

$$D \otimes_R P_1 \xrightarrow{1\otimes d_1} D \otimes_R P_0 \xrightarrow{1\otimes\varepsilon} D \otimes_R B \to 0.$$

Therefore,

$$\mathsf{Tor}_0^R(D,B) \cong D \otimes_R P_0/\operatorname{im}(1 \otimes d_1) = D \otimes_R P_0/\ker\,(1 \otimes \varepsilon) \cong D \otimes_R B.$$

**Solution 69** We shall make use of the statement for chain homotopy equivalent. Let $P_\bullet$ and $P'_\bullet$ be projective resolutions of $B$. Same as in the proof of Theorem 17.6, using Proposition 17.4, we have

$$\begin{tikzcd}
\cdots \arrow[r, "d_3"] & P_2 \arrow[r, "d_2"] \arrow[d, "f_2g_2-1"] \arrow[dl, "h_2"'] & P_1 \arrow[r, "d_1"] \arrow[d, "f_1g_1-1"] \arrow[dl, "h_1"'] & P_0 \arrow[r, "\varepsilon"] \arrow[d, "f_0g_0-1"] \arrow[dl, "h_0"'] & B \arrow[r] \arrow[d, equal] & 0 \\
\cdots \arrow[r, "d_3"'] & P_2 \arrow[r, "d_2"'] & P_1 \arrow[r, "d_1"'] & P_0 \arrow[r, "\varepsilon"'] & B \arrow[r] & 0
\end{tikzcd}$$

This induces

$$\begin{tikzcd}
\cdots \arrow[r, "1\otimes d_3"] & D\otimes_R P_2 \arrow[r, "1\otimes d_2"] \arrow[d, "1\otimes(f_2g_2-1)"] \arrow[dl, "1\otimes h_2"'] & D\otimes_R P_1 \arrow[r, "1\otimes d_1"] \arrow[d, "1\otimes(f_1g_1-1)"] \arrow[dl, "1\otimes h_1"'] & D\otimes_R P_0 \arrow[r] \arrow[d, "1\otimes(f_0g_0-1)"] \arrow[dl, "1\otimes h_0"'] & 0 \\
\cdots \arrow[r, "1\otimes d_3"'] & D\otimes_R P_2 \arrow[r, "1\otimes d_2"'] & D\otimes_R P_1 \arrow[r, "1\otimes d_1"'] & D\otimes_R P_0 \arrow[r] & 0
\end{tikzcd}$$

Notice that $1 \otimes (f_n g_n - 1) = 1 \otimes (f_n g_n) - 1 \otimes 1 = (1 \otimes f_n)(1 \otimes g_n) - 1$. So $(1 \otimes f_n) \circ (1 \otimes g_n) \simeq 1$. Similarly, we have $(1 \otimes g_n) \circ (1 \otimes f_n) \simeq 1$. Since $D \otimes_R P_\bullet$ and $D \otimes_R P'_\bullet$ are chain homotopy equivalent, we have that $\mathsf{Tor}_n^R(D,B)$ is independent of the choice of projective resolution of $B$.

**Solution 70** By assumption, we have

$$\begin{tikzcd}
X : \cdots \arrow[r] & X^{n-1} \arrow[r, "d_{n-1}"] \arrow[d, "1-0"] & X^n \arrow[r, "d_n"] \arrow[d, "1-0"] \arrow[dl, "s_n"'] & X^{n+1} \arrow[r] \arrow[d, "1-0"] \arrow[dl, "s_{n+1}"'] & \cdots \\
X : \cdots \arrow[r] & X^{n-1} \arrow[r, "d_{n-1}"'] & X^n \arrow[r, "d_n"'] & X^{n+1} \arrow[r] & \cdots
\end{tikzcd}$$

Since $1$ and $0$ are chain homotopic, they induces equal maps $1^* = 0^* : H^n(X) \to H^n(X)$. Clearly, $1^*$ is the identity map and $0^*$ is the zero map. The only possibility is that $H^n(X)$ is trivial. So $X$ is exact.

**Solution 71** (i) We write in the bar notation, i.e., $g_0 \otimes \cdots \otimes g_n = g_0|\cdots|g_n$. We first check that

$$F_n = \mathbb{Z}G \otimes_{\mathbb{Z}} \mathbb{Z}G \otimes_{\mathbb{Z}} \cdots \otimes_{\mathbb{Z}} \mathbb{Z}G$$

is a free $\mathbb{Z}G$-module. We claim that $F_n$ is free on the subset

$$A = \{1|g_1|\cdots|g_n : g_1, \dots, g_n \in G\}.$$

As $\mathbb{Z}$-module, $\mathbb{Z}G \cong \bigoplus_{|G|} \mathbb{Z}$. Therefore, by induction and using the fact that direct sum commutes with tensor product, $F_n$ is a free $\mathbb{Z}$-module with a $\mathbb{Z}$-basis $A' := \{g_0|g_1|\cdots|g_n : g_0, g_1, \dots, g_n \in$

$G\}$. To check that $F_n$ is a $\mathbb{Z}G$-module, it suffices to check the following:

$$\begin{aligned}
g\cdot(g_0|g_1|\cdots|g_n) &= gg_0|g_1|\cdots|g_n \in F_n,\\
(gg')\cdot(g_0|g_1|\cdots|g_n) &= (gg')g_0|g_1|\cdots|g_n\\
&= g\cdot(g'g_0|g_1|\cdots|g_n)\\
&= g\cdot(g'\cdot(g_0|g_1|\cdots|g_n)).
\end{aligned}$$

For any $x \in F_n$, there exist unique $n_1,\ldots,n_k \in \mathbb{Z}$ and unique $g^{(1)},\ldots,g^{(k)} \in A'$ such that $x = n_1g^{(1)}+\cdots+n_kg^{(k)}$. For each $g^{(i)}$, we have $g^{(i)} = g_0^{(i)}\cdot(1|g_1^{(i)}|\cdots|g_n^{(i)})$. So $x$ can be written as a finite sum $x = \alpha_1a_1+\cdots+\alpha_\ell a_\ell$ where $a_1,\ldots,a_\ell \in A$ and $\alpha_1,\ldots,\alpha_\ell \in \mathbb{Z}G$. This presentation is clearly unique.

(ii)

$$\begin{aligned}
&d_n(g\cdot(g_0|g_1|\cdots|g_n))\\
&= d_n(gg_0|g_1|\cdots|g_n)\\
&= (gg_0g_1|\cdots|g_{n-1}) + \sum_{i=1}^{n-1}(-1)^i(gg_0|\cdots|g_{i-1}|g_ig_{i+1}|\cdots|g_n) + (-1)^n(gg_0|g_1|\cdots|g_{n-1})\\
&= g\cdot(g_0g_1|\cdots|g_{n-1}) + \sum_{i=1}^{n-1}(-1)^ig\cdot(g_0|\cdots|g_{i-1}|g_ig_{i+1}|\cdots|g_n) + (-1)^ng\cdot(g_0|g_1|\cdots|g_{n-1})\\
&= g\cdot\left((g_0g_1|\cdots|g_{n-1}) + \sum_{i=1}^{n-1}(-1)^i(g_0|\cdots|g_{i-1}|g_ig_{i+1}|\cdots|g_n) + (-1)^n(g_0|g_1|\cdots|g_{n-1})\right)\\
&= g\cdot d_n(g_0|g_1|\cdots|g_n),\\
&d_1(g\cdot(g_0\otimes g_1))\\
&= d_1(gg_0\otimes g_1)\\
&= gg_0(g_1-1)\\
&= g(g_0(g_1-1))\\
&= gd_1(g_0\otimes g_1),\\
&d_0(gg_0) = 1 = g\cdot 1 = g\cdot d_0(g_0).
\end{aligned}$$

(iii)

$$
\begin{aligned}
&d_0 s_{-1}(1) = d_0(1) = 1,\\
&(d_1 s_0 + s_{-1} d_0)(g_0) = d_1(1|g_0) + s_{-1}(1)\\
&= (g_0 - 1) + 1 = g_0,\\
&(d_{n+1} s_n + s_{n-1} d_n)(g_0|g_1|\cdots|g_n)\\
=&d_{n+1}(1|g_0|g_1|\cdots|g_n) + s_{n-1}((g_0g_1|\cdots|g_n)+\\
&\sum_{i=1}^{n-1}(-1)^i(g_0|\cdots|g_{i-1}|g_ig_{i+1}|\cdots|g_n) + (-1)^n(g_0|g_1|\cdots|g_{n-1}))\\
=&(g_0|g_1|\cdots|g_n) + \sum_{i=0}^{n-1}(-1)^{i+1}(1|g_0|\cdots|g_{i-1}|g_ig_{i+1}|\cdots|g_n) + (-1)^{n+1}(1|g_0|g_1|\cdots|g_{n-1}))+\\
&(1|g_0g_1|\cdots|g_n) + \sum_{i=1}^{n-1}(-1)^i(1|g_0|\cdots|g_{i-1}|g_ig_{i+1}|\cdots|g_n) + (-1)^n(1|g_0|g_1|\cdots|g_{n-1}))\\
=&(g_0|g_1|\cdots|g_n) + (-1)^1(1|g_0g_1|\cdots|g_n) + (1|g_0g_1|\cdots|g_n)\\
=&(g_0|g_1|\cdots|g_n).
\end{aligned}
$$

(iv) By the previous exercise, since $s$ is the chain contraction of the identity, we must have $X$ is exact.

**Solution 72** Let $N := \mathbb{Z}G \otimes_{\mathbb{Z}H} A$. Since $G$ is a finite group, we have $[G : H] = n < \infty$ and let $\{g_1, \ldots, g_n\}$ be a complete set of left coset representatives of $H$ in $G$. We have

$$\mathbb{Z}G = \bigoplus_{i=1}^{n} G_i$$

where $G_i = \{\sum c_h(g_ih) : h \in H\}$ and $G_i \cong \mathbb{Z}H$ as right $\mathbb{Z}H$-module. Therefore,

$$N = (\bigoplus_{i=1}^{n} G_i) \otimes_{\mathbb{Z}H} A \bigoplus_{i=1}^{n} A_i$$

where $A_i = \{g_i \otimes a : a \in A\}$. For each $1 \leq i \leq n$ and $a \in A$, define $f_{i,a} : \mathbb{Z}G \to A$ by

$$f_{i,a}(x) = \begin{cases} ha & \text{if } x = hg_i^{-1} \text{ for some } h \in H,\\ 0 & \text{otherwise.}\end{cases}$$

Clearly, for $h' \in H$, we have $h'x = h'hg_i^{-1}$. So $f_{i,a}(h'x) = (h'h)a = h'(ha) = h'f_{i,a}(x)$. Therefore, $f_{i,a} \in M_H^G(A)$. Define the map $\Psi : M_H^G(A) \to N$ by

$$\Psi(f) = \sum_{i=1}^{n} g_i \otimes f(g_i^{-1}).$$

We prove that $\Psi$ is a $G$-module homomorphism. For $g \in G$, we have $gg_i = g_{j_i}h_i$ and hence

$$\begin{aligned}\Psi(g \cdot f) &= \sum_{i=1}^n g_i \otimes (g \cdot f)(g_i^{-1}) \\ &= \sum_{i=1}^n g_i \otimes f(g_i^{-1}g) \\ &= \sum_{i=1}^n g_{j_i} \otimes f(g_{j_i}^{-1}g) \\ &= \sum_{i=1}^n g_{j_i} \otimes f(h_i g_i^{-1}) \\ &= \sum_{i=1}^n g_{j_i} \otimes h_i f(g_i^{-1}) \\ &= \sum_{i=1}^n (g_{j_i}h_i) \otimes f(g_i^{-1}) \\ &= \sum_{i=1}^n (gg_i) \otimes f(g_i^{-1}) \\ &= g(\sum_{i=1}^n g_i \otimes f(g_i^{-1})) \\ &= g\Psi(f)\end{aligned}$$

We now check that $\Psi$ is an isomorphism. Let $\iota : \mathbb{Z}G \times A \to N$ be $\iota(g, a) = g \otimes a$ and $\beta : \mathbb{Z}G \times A \to M_H^G(A)$ be $\beta(g, a) = f_{i,ha}$ if $g = g_i h$. We claim that $\beta$ is $\mathbb{Z}H$-balanced. The only interesting item to check is

$$\beta(gh', a) = f_{i,(hh')a}(x) = f_{i,h(h'a)} = \beta(g, h'a),$$

and the rest follows using linearity property. By the Universal Property of Tensor Product, we have a $G$-module homomorphism $\Phi : N \to M_H^G(A)$ such that $\Phi(g \otimes a) = f_{i,ha}$ where $g = g_i h$. We have

$$\begin{aligned}\Phi\Psi(f) &= \Phi(\sum_{i=1}^n g_i \otimes f(g_i^{-1})) \\ &= \sum_{i=1}^n \Phi(g_i \otimes f(g_i^{-1})) \\ &= \sum_{i=1}^n f_{i,f(g_i^{-1})}\end{aligned}$$

We now check that $\sum_{i=1}^n f_{i,f(g_i^{-1})} = f$. For $x \in G$, let $x = hg_j^{-1}$. So

$$\sum_{i=1}^n f_{i,f(g_i^{-1})}(x) = f_{j,f(g_j^{-1})}(x) = hf(g_j^{-1}) = f(hg_j^{-1}) = f(x).$$

Also, if $g = g_i h$, then

$$\begin{aligned}\Psi\Phi(g \otimes a) = \Psi(f_{i,ha}) &= \sum_{j=1}^n g_j \otimes f_{i,ha}(g_j^{-1}) \\ &= g_i \otimes f_{i,ha}(g_i^{-1}) \\ &= g_i \otimes ha \\ &= g_i h \otimes a \\ &= g \otimes a.\end{aligned}$$

Therefore, we have an isomorphism $M_H^G(A) \cong N$.

**Solution 73** For this, we just need to show that the following diagram commutes:

$$\begin{tikzcd} C^n(G,V) \arrow[r, "\lambda_n"] \arrow[d, "\Phi_n"] & C^n(G,V') \arrow[d, "\Phi_{n+1}"] \\ \mathrm{Hom}_{\mathbb{Z}G}(F_n, V) \arrow[r, "\phi^n"] & \mathrm{Hom}_{\mathbb{Z}G}(F_n, V') \end{tikzcd}$$

Let $f : G^n \to V$ and $(1|x_1|\cdots|x_n) \in F_n$. Then

$$\begin{aligned}(\Phi_{n+1} \circ \lambda_n)(f)(1|x_1|\cdots|x_n) &= \Phi_{n+1}(\phi \circ f)(1|x_1|\cdots|x_n) \\ &= (\phi \circ f)(x_1|\cdots|x_n)) \\ &= \phi(f(x_1|\cdots|x_n)) \\ &= (\phi^n \circ \Phi_n)(f)(1|x_1|\cdots|x_n).\end{aligned}$$

**Solution 74** For each $1 \le i \le m$, let $g_i' = g_i h_i$. We have

$$g_i f(g_i^{-1}) = g_i' h_i^{-1} f(h_i g_i'^{-1}) = g_i' h_i^{-1} h_i f(g_i'^{-1}) = g_i' f(g_i'^{-1}).$$

Let $g \in G$ and, for each $1 \leq i \leq m$, let $gg_i = g_{j_i} h_i$. Notice that $\{j_1, \dots, j_m\} = \{1, \dots, m\}$. So

$$\begin{aligned}\psi(g \cdot f) &= \sum_{i=1}^{m} g_{j_i}(g \cdot f)(g_{j_i}^{-1}) \\ &= \sum_{i=1}^{m} g_{j_i} f(g_{j_i}^{-1} g) \\ &= \sum_{i=1}^{m} g_{j_i} f(h_i g_i^{-1}) \\ &= \sum_{i=1}^{m} g_{j_i} h_i f(g_i^{-1}) \\ &= \sum_{i=1}^{m} g g_i f(g_i^{-1}) \\ &= g\psi(f).\end{aligned}$$

Finally, we check that $\psi$ is surjective. We have seen in the previous question, where we proved $\mathbb{Z}G \otimes_{\mathbb{Z}H} A \cong M_H^G(A)$, that $f_{i, g_i^{-1} a} \in M_H^G(A)$. So

$$\begin{aligned}\psi(f_{i, g_i^{-1}(a)}) &= \sum_{j=1}^{m} g_j f_{i, g_i^{-1}(a)}(g_j^{-1}) \\ &= g_i f_{i, g_i^{-1}(a)}(g_i^{-1}) \\ &= g_i g_i^{-1}(a) = a.\end{aligned}$$

So $\psi$ is surjective.