COMPSCI 575: Combinatorics and Graph Theory

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Lecture 7

Ordinary generating functions — OGFs

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- Example: $1, 1, 1, \cdots$ has OGF $1 + z + z^2 + \cdots = 1/(1 - z)$.
- Example: for fixed nonnegative integer n, the sequence $\binom{n}{0}, \dots, \binom{n}{n}, \dots$ has OGF $(1 + z)^n$.

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- Basic operations on sequences coming from combinatorics turn out to correspond to common algebra/calculus operations on OGFs.
- The equality $\sum_n z^n = 1/(1-z)$ is purely formal at this stage but also makes sense for $|z| < 1$, and that allows us to use methods from calculus/mathematical analysis.

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- We can use these to compute some important special formulae.

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- Define the convolution $f \star g$ of sequences $(a_0, a_1, \ldots)$ and $(b_0, b_1, \ldots)$ by
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Then $f \star g \leftrightarrow F(z)G(z)$.
- In particular, using $G(z) = (1 - z)^{-1}$ we see that partial sums $\sum_{k=0}^{n} a_k$ correspond to $F(z)/(1 - z)$.

Some important OGFs

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- Thus

$$\sum_{n\geq 1} H_n z^n = -\log(1-z)/(1-z)$$

where $H_n := \sum_{k=1}^{n} 1/k$ is the nth harmonic number.

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- Note that F is a rational function.

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- Note that a problem involving natural numbers leads to an OGF with irrational numbers!

Another nice use: simplifying sums ("Snake Oil method")

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- Change variable by $l = n - k$ to get

$$F(z) = \sum_{k \geq 0} z^k \sum_{l \geq 0} z^l \binom{k}{l} = \sum_{k \geq 0} z^k (1+z)^k = \frac{1}{1 - z(1+z)}.$$