Linear Algebra: Map of theorems

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1 Bilinear Forms and Inner Product

1.1 Vector inequalities

Cauchy-Schwarz Inequality: $\|\langle u, v\rangle\| \leq \|u\| \, \|v\|$
Triangle Inequality: $\|u + v\| \leq \|u\| + \|v\|$
Pythagoras Theorem: $\|u + v\|^2 = \|u\|^2 + \|v\|^2$

1.2 Orthogonal and Orthonormal basis

Theorem 1.2.1 *An orthogonal set of nonzero vectors is linearly independent.*

Corollary 1.2.1.1 *If V is a finite dimensional inner product space and $n = dimV$, then any orthogonal set of nonzero vectors in V is finite and contains at most n vectors.*

Lemma 1.2.2 *Let $\mathcal{B} = v_1, ..., v_n$ be an orthonormal basis of V. Then for any $v \in V$,*

$$V = \langle v, v_1\rangle v_1 + \langle v, v_2\rangle v_2 + ... + \langle v, v_n\rangle v_n$$

Corollary 1.2.2.1 *Let $\mathcal{B} = v_1, ..., v_n$ be an orthonormal basis of V, $T : V \to V$ a linear operator and $[T]_\mathcal{B} = (a_{ij})$ Then for all i, j*

$$a_{ij} = \langle T(v_j), v_i\rangle$$

Theorem 1.2.3 *Every finite dimensional inner product space has an orthonormal basis.*

1.3 Orthogonal complement and projection

Theorem 1.3.1 *If W is a subspace of an inner product space V, then its orthogonal complement $W^\perp$ is a subspace of V. In addition, we have*

$$W \cap W^\perp = \{\mathbf{0}\}$$

Theorem 1.3.2 *If W is a finite dimensional subspace of an inner product space V, then*

$$V = W \oplus W^\perp$$

Theorem 1.3.3 *If $\{w_1, ..., w_k\}$ is an orthonormal basis of W then*

$$\mathbf{proj}_W(v) = \Sigma_{j=1}^k \langle v, w_j\rangle w_j$$

Theorem 1.3.4 ***Best Approximation:*** *If W a finite dimensional subspace of an inner product space V and $v \in V$, then*

$$\|v - \mathbf{proj}_W(v)\| < \|v - w\|$$

for every vector w in W different from $\mathbf{proj}_W(v)$.

Theorem 1.3.5 ***Least Square Solution:*** *For any real linear system $A\mathbf{x} = \mathbf{b}$ the associated normal system*

$$(A^t A)\mathbf{x} = A^t\mathbf{b}$$

is consistent, and all its solutions are least square solutions of $A\mathbf{x} = \mathbf{b}$

1.4 Adjoint of linear operator