Linear Algebra: Map of theorems

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1.1 Vector inequalities

Cauchy-Schwarz Inequality: $\|\langle u, v\rangle\| \leq \|u\| \|v\|$
Triangle Inequality: $\|u+v\| \leq \|u\| + \|v\|$
Pythagoras Theorem: $\|u+v\|^2 = \|u\|^2 + \|v\|^2$

1.2 Orthogonal and Orthonormal basis

Theorem 1.2.1 *An orthogonal set of nonzero vectors is linearly independent.*

Corollary 1.2.1.1 *If V is a finite dimensional inner product space and $n = dimV$, then any orthogonal set of nonzero vectors in V is finite and contains at most n vectors.*

Lemma 1.2.2 *Let $\mathcal{B} = \{v_1, ..., v_n\}$ be an orthonormal basis of V. Then for any $v \in V$,*

$$V = \langle v, v_1\rangle v_1 + \langle v, v_2\rangle v_2 + ... + \langle v, v_n\rangle v_n$$

Corollary 1.2.2.1 *Let $\mathcal{B} = \{v_1, ..., v_n\}$ be an orthonormal basis of V, $T : V \to V$ a linear operator and $[T]_{\mathcal{B}} = (a_{ij})$ Then for all i, j*

$$a_{ij} = \langle T(v_j), v_i\rangle$$

Theorem 1.2.3 *Every finite dimensional inner product space has an orthonormal basis.*

1.3 Orthogonal complement and projection

Theorem 1.3.1 *If W is a subspace of an inner product space V, then its orthogonal complement $W^{\perp}$ is a subspace of V. In addition, we have*

$$W \cap W^{\perp} = \{\mathbf{0}\}$$

Theorem 1.3.2 *If W is a finite dimensional subspace of an inner product space V, then*

$$V = W \oplus W^{\perp}$$

Theorem 1.3.3 *If $\{w_1, ..., w_k\}$ is an orthonormal basis of W then*

$$\mathbf{proj}_W(v) = \Sigma_{j=1}^{k} \langle v, w_j \rangle w_j$$

Theorem 1.3.4 ***Best Approximation:*** *If W a finite dimensional subspace of an inner product space V and $v \in V$, then*

$$\|v - \mathbf{proj}_W(v)\| < \|v - w\|$$

for every vector w in W different from $\mathbf{proj}_W(v)$.

Theorem 1.3.5 ***Least Square Solution:*** *For any real linear system $A\mathbf{x} = \mathbf{b}$ the associated normal system*

$$(A^t A)\mathbf{x} = A^t \mathbf{b}$$

is consistent, and all its solutions are least square solutions of $A\mathbf{x} = \mathbf{b}$

1.4 Adjoint of linear operator

Theorem 1.4.1 *Let V be a finite dimensional inner product space over $\mathbb{F}$. If $f : V \to \mathbb{F}$ is a linear functional, then there exists a unique vector $u \in V$ such that.*

$$f(v) = \langle v, u \rangle \qquad \text{for all } v \in V$$

Theorem 1.4.2 *Let $T : V \to V$ be a linear operator on a finite dimensional inner product space V. Then there exists a unique linear operator $T^* : V \to V$ such that*

$$\langle T(u), v \rangle = \langle u, T^*(v) \rangle$$

T^* is called the adjoint of T.

Theorem 1.4.3 *Let $T : V \to V$ a linear operator on a finite dimensional inner product space V and $\mathcal{B}$ be an orthonormal basis of V. Then*

$$[T^*]_{\mathcal{B}} = ([T]_{\mathcal{B}})^*$$

is the conjugate transpose of the matrix $[T]_{\mathcal{B}}$

Theorem 1.4.4 *Let T, T_1 and T_2 be linear operators on a finite dimensional inner product space V. Then:*

$$\begin{aligned}
&(i) \quad (T_1 + T_2)^* = T_1^* + T_2^* \\
&(ii) \quad c(T)^* = \overline{c} T^* \\
&(iii) \quad (T_1 T_2)^* = T_2^* T_1^* \\
&(iv) \quad (T^*)^* = T
\end{aligned}$$

1.5 Self-adjoint and Normal Operator

Main motivation for this section is to find out under what conditions does V have an orthonomal basis of eigenvectors for T.

Definition 1.5.1 *A linear operator T on a finite dimensional inner product space V is called **self-adjoint** if $T = T^*$, i.e. it satisfies the equation:*

$$\langle T(u), v \rangle = \langle u, T(v) \rangle$$

Theorem 1.5.1 *(p. 312) A self-adjoint linear operator T on a finite dimensional inner product space V.Then all eigenvalues of T is real. And the eigenvectors associated with the distinct eigenvalues are orthogonal.*

Theorem 1.5.2 *(p. 313) On a finite dimensional inner product space of positive dimension, every self-adjoint operator has at least one eigenvalue which also means it has at least one eigenvector.*

(note proof of this in complex vector space do not require T to be self-adjoint, but in real vector space it does)

Lemma 1.5.3 *Let V be a finite dimensional inner product space. Let $T : V \to V$ be any linear operator. Suppose W is a subspace of V which is invariant under T. Then $W^{\perp}$ is invariant under T^**

Theorem 1.5.4 *Let V be a finite dimensional inner product space (complex or real). Let $T : V \to V$ be a self adjoint linear operator. T is orthgonally diagonalizable if and only if it is self-adjoint.*
(proof uses Lemma 1.5.3)

Corollary 1.5.4.1 *Let A be an $n \times n$ Hermitian matrix. Then there is a unitary matrix P such that $P^{-1}AP$ is diagonal. (i.e. A is unitariliy equivalent to a diagonal matrix). If A is a real symmetric matrix, there is a real orthogonal matrix P such that that $P^{-1}AP$ is diagonal.*

Definition 1.5.2 *Let V be a finite dimensional inner product space. Let $T : V \to V$ be a linear operator on V. We say that T is normal if it commutes with its adjoint:*

$$TT^* = T^*T$$

Lemma 1.5.5 *Let V be a finite dimensional complex inner product space and $T : V \to V$ a normal linear operator. Then if $v \in V$ is an eigenvector of T corresponding to the eigenvalue λ, then it is also an eigenvector of T^* corresponding to the eigenvalue $\overline{\lambda}$. That is for $v \in V$,*

$$T(v) = \lambda v \Rightarrow T^*(v) = \overline{\lambda} v$$

Lemma 1.5.6 *Let V be a finite dimensional complex inner product space and $T : V \to V$ any linear operator. Then V has an orthonormal basis $\mathcal{B}$ such that matrix $[T]_{\mathcal{B}}$ is upper triangular.*

Theorem 1.5.7 *A linear operator on a finite dimensional complex inner product space is orthogonally diagonalizable if and only if it is normal.*
(proof uses lemma 1.5.5 and 1.5.6)

1.6 Unitary Operator

Theorem 1.6.1 *Let V and W be a finite dimensional inner product spaces over the same field. If $f : V \to W$ is a linear transformation, the following are equivalent:*

1. *$\langle T(u), T(v)\rangle = \langle u, v\rangle$ for all $u, v \in V$. i.e. T preserves inner product spaces*
2. *T is an inner product space isomorphism*
3. *T carries every orthonormal basis for V onto an orthonormal basis for W*
4. *$\|T(v)\| = \|v\|$ for all $v \in V$.*

Definition 1.6.1 *A unitary operator on an inner product space is an isomorphism of the space onto itself.*

Theorem 1.6.2 *Let T be a linear operator on an inner product space V. Then T is unitary if and only if the adjoint T^* of T exists and $TT^* = T^*T = I$*

Theorem 1.6.3 *Let V be a finite dimensional inner product space and let $T : V \to V$ be a linear operator. Then T is unitary if and only if the matrix of T in some (or every) ordered basis is a unitary matrix.*

Theorem 1.6.4 *(p.305) For every invertible complex $n \times n$ matrix B, there exists a unique lower-triangular matrix M with positive entries on the main diagonal such that MB is unitary.*

Corollary 1.6.4.1 *(p.307) Let $T^+(n)$ be the set of all complex $n \times n$ lower triangular matrices with positive entries on the main diagonal. Let $U(n)$ be a group of unitary matrices. For each B in $GL(n)$ there exists unique matrices N and U such that $N \in T^+(n)$ and $U \in U(n)$.*

2 Operators on Inner Product Spaces

Let $\mathcal{L}(V;V)$ denote the space of all linear operators from V to V.
Let $\mathcal{F}(V,V,\mathbb{F})$ denote the space of all forms on V.

2.1 Forms on Inner Product Spaces

Definition 2.1.1 *A sesquilinear form on a vector space V over $\mathbb{F}$ is a function $f : V \times V \to \mathbb{F}$ such that*

$$\begin{aligned} (a) \quad & f(c\alpha+\beta,\gamma) = cf(\alpha,\gamma) + f(\beta,\gamma) \\ (b) \quad & f(\alpha, c\beta+\gamma) = \bar{c}f(\alpha,\beta) + f(\alpha,\gamma) \end{aligned}$$

Theorem 2.1.1 *(p. 320) Let V be a finite dimensional inner product space and f a form on V. Then*

1. *If $f \in \mathcal{F}(V,V,\mathbb{F})$, then there exists a unique linear operator $T_f : V \to V$ such that*

$$f(\alpha,\beta) = \langle T_f\alpha, \beta\rangle \quad \forall \alpha,\beta \in V$$

2. *The map $\phi : \mathcal{F}(V,V,\mathbb{F}) \to \mathcal{L}(V;V)$ defined by $\phi(f) := T_f$ for all $f \in \mathcal{F}(V,V,\mathbb{F})$ is an isomorphism of vector spaces*

Definition 2.1.2 *If f is a form and $\mathcal{B} = \{\alpha_1, ..., \alpha_n\}$ an arbitrary ordered basis of V, the matrix A with entries*

$$A_{ij} = f(\alpha_j, \alpha_i)$$

is called the matrix of f in the ordered basis of $\mathcal{B}$

Proposition 2.1.1 *let V be a finite dimensional inner product space over field $\mathbb{F}$ and f is a form. If $\mathcal{B}$ is an orthonormal basis of V, then $[f]_{\mathcal{B}} = [T_f]_{\mathcal{B}}$.*
(note that this lemma does not work if basis is not orthonormal)

Theorem 2.1.2 *Let f be a form on a finite dimensional complex inner product space V. Then there is an orthonormal basis for V in which the matrix is upper-triangular.*

(note that this theorem uses the fact that $\mathcal{F}(V,V,\mathbb{F})$ is isomorphic to $\mathcal{L}(V;V)$, which allows us to transfer the result of lemma 1.5.6)

Definition 2.1.3 *A form f on a real or complex vector space V is called **Hermitian** if*

$$f(\alpha,\beta) = \overline{f(\beta,\alpha)}$$

Proposition 2.1.2 *Let f be a form on complex inner product space. f is Hermitian if and only if T_f is self-adjoint.*

Theorem 2.1.3 *Let V be a complex vector space and f a form on V such that $f(\alpha,\alpha)$ is real for every $\alpha \in V$. Then f is Hermitian.*

Corollary 2.1.3.1 *Let T be a linear operator on a complex finite dimensional inner product space V. Then T is self-adjoint if and only if $\langle T\alpha, \alpha\rangle$ is real for every $\alpha \in V$.*

Theorem 2.1.4 ***Principle Axis Theorem:*** *For every Hermitian form f on a finite dimensional inner product space V, there is an orthonormal basis in which f is represented by a diagonal matrix with real entries.*

(Proof of this theorem uses results from theorem 1.5.4 and 1.5.1)

3 Bilinear Forms

Let $L(V, V; \mathbb{F})$ denote the space of all bilinear forms.

3.1 Bilinear Forms Introduction

Definition 3.1.1 *let V be a vector space over the field F. A bilinear form on V is a function f, which assigns to each ordered pair of vectors α, β in V a scalar $f(\alpha, \beta)$ in F, and which satisfies*

$$\begin{aligned} (a) \quad & f(c\alpha + \beta, \gamma) = cf(\alpha, \gamma) + f(\beta, \gamma) \\ (b) \quad & f(\alpha, c\beta + \gamma) = cf(\alpha, \beta) + f(\alpha, \gamma) \end{aligned}$$

Definition 3.1.2 *let V be a finite-dimensional vector space and let $\mathcal{B} = \alpha_1, ..., \alpha_2$ be an ordered basis for V. If f is a bilinear form on V, the **matrix of f in the ordered basis** $\mathcal{B}$ is the $n \times n$ matrix A with entries $A_{ij} = f(\alpha_i, \alpha_j)$. We will denote this matrix by $[f]_{\mathcal{B}}$*

Theorem 3.1.1 *let V be a finite-dimensional vector space over the field $\mathbb{F}$. For each ordered basis $\mathcal{B}$ of V, the function which associates each bilinear form on V its matrix in the ordered basis $\mathcal{B}$ is an isomorphism. i.e. The map*

$$\phi : L(V, V, \mathbb{F}) \to M_n(\mathbb{F})$$

is an isomorphism.

Corollary 3.1.1.1 *If $\mathcal{B} = \{\alpha_1, ...\alpha_n\}$ is an ordered basis for V, and $B* = \{L_1, ..., L_n\}$ is the dual basis for V^*, then the n^2 bilinear forms*

$$f_{ij}(\alpha, \beta) = L_i(\alpha)L_j(\beta)$$

form a basis for the space $L(V, V, \mathbb{F})$ and $dim(L(V, V, \mathbb{F})) = n^2$.

Theorem 3.1.2 *We denote $L_f(\alpha)$ as the bilinear form that fixes α hence becoming a linear functional on any β in V. We denote $R_f(\beta)$ similarly but fixing β.*

Let f be a bilinear form on the finite-dimensional vector space V. Let L_f and R_f be the linear transformation from V to V^* defined by $(L_f\alpha)(\beta) = f(\alpha, \beta) = (R_f\beta)(\alpha)$. Then rank $(R_f) =$ rank (L_f).

Definition 3.1.3 *if f is a bilinear form on the finite-dimensional space V, the **rank** of f is the integer $r =$ rank $(L_f) =$ rank (R_f).*

Corollary 3.1.2.1 *If f is a bilinear form on the n-dimensional vector space V, the following are equivalent:*

1. *rank $(f) = n$.*
2. *For each non zero α in V, there is a β in V such that $f(\alpha, \beta) \neq 0$*
3. *For each non zero β in V, there is a α in V such that $f(\alpha, \beta) \neq 0$*

Definition 3.1.4 *A biliear form f on a vector space V is called **non-degenerate** (or **non-singular**) if it satisfies (2) and (3) of corollary 3.1.2.1.*