Linear Algebra: Map of theorems

Tan Wee Han

December 10, 2020

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1 Bilinear Forms and Inner Product

1.1 Vector inequalities

Cauchy-Schwarz Inequality: $\|\langle u, v\rangle\| \leq \|u\| \|v\|$
Triangle Inequality: $\|u + v\| \leq \|u\| + \|v\|$
Pythagoras Theorem: $\|u + v\|^2 = \|u\|^2 + \|v\|^2$

1.2 Orthogonal and Orthonormal basis

Theorem 1.2.1 *An orthogonal set of nonzero vectors is linearly independent.*

Corollary 1.2.1.1 *If V is a finite dimensional inner product space and $n = dim V$, then any orthogonal set of nonzero vectors in V is finite and contains at most n vectors.*

Lemma 1.2.2 *Let $\mathcal{B} = \{v_1, ..., v_n\}$ be an orthonormal basis of V. Then for any $v \in V$,*

$$V = \langle v, v_1\rangle v_1 + \langle v, v_2\rangle v_2 + ... + \langle v, v_n\rangle v_n$$

Corollary 1.2.2.1 *Let $\mathcal{B} = \{v_1, ..., v_n\}$ be an orthonormal basis of V, $T : V \to V$ a linear operator and $[T]_{\mathcal{B}} = (a_{ij})$ Then for all i, j*

$$a_{ij} = \langle T(v_j), v_i\rangle$$

Theorem 1.2.3 *Every finite dimensional inner product space has an orthonormal basis.*

1.3 Orthogonal complement and projection

Theorem 1.3.1 *If W is a subspace of an inner product space V, then its orthogonal complement $W^\perp$ is a subspace of V. In addition, we have*

$$W \cap W^\perp = \{\mathbf{0}\}$$

Theorem 1.3.2 *If W is a finite dimensional subspace of an inner product space V, then*

$$V = W \oplus W^\perp$$

Theorem 1.3.3 *If $\{w_1, ..., w_k\}$ is an orthonormal basis of W then*

$$\mathbf{proj}_W(v) = \Sigma_{j=1}^{k}\langle v, w_j\rangle w_j$$

Theorem 1.3.4 ***Best Approximation:*** *If W a finite dimensional subspace of an inner product space V and $v \in V$, then*

$$\|v - \mathbf{proj}_W(v)\| < \|v - w\|$$

for every vector w in W different from $\mathbf{proj}_W(v)$.

Theorem 1.3.5 ***Least Square Solution:*** *For any real linear system $A\mathbf{x} = \mathbf{b}$ the associated normal system*

$$(A^t A)\mathbf{x} = A^t\mathbf{b}$$

is consistent, and all its solutions are least square solutions of $A\mathbf{x} = \mathbf{b}$

1.4 Adjoint of linear operator

Theorem 1.4.1 *Let V be a finite dimensional inner product space over $\mathbb{F}$. If $f : V \to \mathbb{F}$ is a linear functional, then there exists a unique vector $u \in V$ such that.*

$$f(v) = \langle v, u\rangle \qquad \text{for all } v \in V$$

Theorem 1.4.2 *Let $T : V \to V$ be a linear operator on a finite dimensional inner product space V. Then there exists a unique linear operator $T^* : V \to V$ such that*

$$\langle T(u), v\rangle = \langle u, T^*(v)\rangle$$

T^* is called the adjoint of T.

Theorem 1.4.3 *Let $T : V \to V$ a linear operator on a finite dimensional inner product space V and B be an orthonormal basis of V. Then*

$$[T^*]_{\mathcal{B}} = ([T]_{\mathcal{B}})^*$$

is the conjugate transpose of the matrix $[T]_{\mathcal{B}}$

Theorem 1.4.4 *Let T, T_1 and T_2 be linear operators on a finite dimensional inner product space V. Then:*

$$\begin{array}{ll} (i) & (T_1 + T_2)^* = T_1^* + T_2^* \\ (ii) & c(T)^* = \overline{c}T^* \\ (iii) & (T_1T_2)^* = T_2^*T_1^* \\ (iv) & (T^*)^* = T \end{array}$$

1.5 Unitary Operator

Theorem 1.5.1 *Let V and W be a finite dimensional inner product spaces over the same field. If $f : V \to W$ is a linear transformation, the following are equivalent:*

1. *$\langle T(u), T(v)\rangle = \langle u, v\rangle$ for all $u, v \in V$. i.e. T preserves inner product spaces*
2. *T is an inner product space isomorphism*
3. *T carries every orthonormal basis for V onto an orthonormal basis for W*
4. *$\|T(v)\| = \|v\|$ for all $v \in V$.*

Definition 1.5.1 *A unitary operator on an inner product space is an isomorphism of the space onto itself.*

Theorem 1.5.2 *Let T be a linear operator on an inner product space V. Then T is unitary if and only if the adjoint T^* of T exists and $TT^* = T^*T = I$*

Theorem 1.5.3 *Let V be a finite dimensional inner product space and let $T : V \to V$ be a linear operator. Then T is unitary if and only if the matrix of T in some (or every) ordered basis is a unitary matrix.*

Theorem 1.5.4 *(p.305) For every invertible complex $n \times n$ matrix B, there exists a unique lower-triangular matrix M with positive entries on the main diagonal such that MB is unitary.*

Corollary 1.5.4.1 *(p.307) Let $T^+(n)$ be the set of all complex $n \times n$ lower triangular matrices with positive entries on the main diagonal. Let $U(n)$ be a group of unitary matrices. For each B in $GL(n)$ there exists unique matrices N and U such that $N \in T^+(n)$ and $U \in U(n)$.*