Lab06-Universal Program

CS363-Computability Theory, Xiaofeng Gao, Spring 2016

∗ Please upload your assignment to FTP or submit a paper version on the next class

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1. (a) Show that there is a decidable predicate $Q(x, y, z)$ such that
   i. $y \in E_x$ if and only if $\exists z.Q(x, y, z)$
   ii. if $y \in E_x$ and $Q(x, y, z)$, then $\phi_x((z)_1) = y$.

   Proof. Note that $y \in E_x$ iff there exists some s such that $\phi_x(s) = y$, that is, iff the computation $P_x(s)$ halts with output y after a infinite some t steps. Thus under the effective coding $z = 2^s 3^t$ we can define: $Q(x, y, z) = s_1(x, s, y, t) = s_1(x, (z)_1, y, (z)_2)$ Now Q is decidable by substitution. □

   (b) Deduce that there is a computable function $g(x, y)$ such that
   i. $g(x, y)$ is defined if and only if $y \in E_x$.
   ii. if $y \in E_x$, then $g(x, y) \in W_x$ and $\phi_x(g(x, y)) = y$; i.e. $g(x, y) \in \phi_x^{-1}(\{y\})$.

   Proof. Define $g(x, y) \simeq (\mu z Q(x, y, z))_1$ □

   (c) Deduce that if f is a computable injective function (not necessarily total or surjective) then f^{-1} is computable. (cf. exercise 2-5.4(1)).

   Proof. f is computable,accroding to unbounded minimization $\mu y(f(x) = y)$ is computable .Since $f^{-1}(y) = \mu y(f(x) = y)$ so $f^{-1}$is computable. □

2. (cf. example 3-7.1(b)) Suppose that f and g are unary computable functions; assuming that T_1 has been formally proved to be decidable, prove formally that the function $h(x)$ defined by $h(x) = \begin{cases} 1 & \text{if } x \in Dom(f) \text{ or } x \in Dom(g), \\ \uparrow & \text{otherwise,} \end{cases}$ is computable.

   Proof. Because f and g are unary computable functions, there exists e_1, $e_2 \in N$ such that $f \simeq \phi_{e_1}$, $g \simeq \phi_{e_2}$. So, $h(x) \simeq sg(t(T1(e_1, x, t) \text{ OR } T1(e_1, x, t)) + 1)$. $h(x)$ is defined if and only if $\exists t, T1(e_1, x, t)$ or $T1(e_2, x, t)$. And when it is defined,it outputs 1. $h(x)$ uses only one operation and is composed of several primitive recursive functions and predicates, so, it is computable. □

3. Show that there is a total computable function $k(e_1, e_2)$ such that $\phi_{k(e_1,e_2)}(x)$ is the characteristic function for predicate "$M_1(x)$ and $M_2(x)$", where M_1 and M_2 are both decidable predicate and $\phi_{e_1} = c_{M_1}$, $\phi_{e_2} = c_{M_2}$.

   Proof. Define $f(e_1, e_2, x) = \phi_{e_1}(x)\phi_{e_2}(x) = \psi_U(e_1, x)\psi_U(e_2, x)$.
   We can know that $f(e_1, e_2, x)$ is also a characteristic function for "$M_1(x)$ and $M_2(x)$". Because "$M_1(x)$ and $M_2(x)$" are both decidable predicate, $f(e_1, e_2, x)$ must be a computable function. By s-m-n theorem,there is a total computable function $k(e_1, e_2)$ such that $\phi_{k(e_1,e_2)}(x) \simeq f(e_1, e_2, x)$. □

4. Show that there is a total computable function $s(x, y)$ such that for all x, y, $E_{s(x,y)} = W_x \cup E_y$.

Proof. For pair x,y,we can define such a function: $f(x,y,z) = \begin{cases} \frac{z}{2} & \text{z is even} \\ \phi_x(\frac{z+1}{2}) & \text{z is odd} \end{cases}$
By the s-n-m theory there exists a total computable function $s(x,y)$ such that for all x and y, $\phi_{s(x,y)}(z) = f(x,y,z)$. So it is obvious that $E_s(x,y) = W_X \cup E_y$. $\square$

5. Suppose that $f(x)$ is computable; show that there is a total computable function $k(x)$ such that for all x, $W_{k(x)} = f^{-1}(W_x)$.

Proof. Define $g(x,y) = \begin{cases} 1 & \text{if } f(y) \in W_x \\ undefined & \text{otherwise} \end{cases}$,
As we know that $f(x)$ is computable, so $g(x,y)$ is also compuatble. By s-m-n theorem, there is a total computable function $k(x)$ such that $\phi_{k(x)}(y) \simeq g(x,y)$. And it is obvious that for any fixed x, the domain of $\phi_{k(x)}(y)$ is $f^{-1}(W_x)$ by definition of $g(x,y)$. $\square$