Lab03-Recursive Function

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1. Show that the following functions are primitive recursive:

   (a) $half(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even,} \\ \frac{n-1}{2}, & \text{if } n \text{ is odd.} \end{cases}$

   (b) $\max\{x_1, x_2, \cdots, x_n\}$ = the maximum of x_1, x_2, $\cdots$, x_n.

   (c) $f(x)$ = the sum of all prime divisors of x.

   (d) $g(x) = x^x$.

   Solution. (a) $half(0) = 0$, $half(x+1) = half(x) + rm(2, x)$

   (b) $max(x, y) \backsimeq x \dot{-} (x \dot{-} y)$ $max\{x, y\} = max(x, y)$
   $\max\{x_1, x_2, \cdots, x_n\} = max(max\{x_1, x_2, \cdots, x_{n-1}\}, x_n)$

   (c) $f(x) = \sum_{z \leq x} Pr(div(z, x) \cdot z)$

   (d) $g(x) = power(x, x)$

2. Show the computability of the following functions by minimalisation.

   (a) $f^{-1}(x)$, if $f(x)$ is a total injective computable function.

   (b) $f(a) = \begin{cases} \text{the least non-negative integral root of } p(x) - a \ (a \in \mathbb{N}), \\ \text{undefined if there's no such root,} \end{cases}$

   where $p(x)$ is a polynomial with integer coefficients.

   (c) $f(x, y) = \begin{cases} x/y & \text{if } y \neq 0 \text{ and } y|x, \\ \text{undefined} & \text{otherwise.} \end{cases}$

   Solution. (a) $f^{-1}(y) = \mu z(f(z) = y)$

   (b) $f(y) = \mu z(z^2 = p(x) - y, (z \in \mathbb{N}))$

   (c) $f(x, y) = \mu z(|mult(z, y) - x| = 0)$

3. Let $\pi(x, y) = 2^x(2y + 1) - 1$. Show that π is a computable bijection from $\mathbb{N}^2$ to $\mathbb{N}$, and that the functions π_1, π_2 such that $\pi(\pi_1(z), \pi_2(z)) = z$ for all z are computable.

   Solution. Injective. Given that $\pi(p, q) = \pi(x, y)$, then $2^p(2q + 1) = 2^x(2y + 1)$. If $p = 0$ or $x = 0$, suppose $p = 0$, then $2q + 1$ is odd and $2^x(2y + 1)$ must be odd, so $x = 0$. If $p \neq 0$ and $x \neq 0$, then $2^p | 2^x(2y + 1)$, so $p = x$. Now that $p = x$, apparently $q = y$. It is injective.

   Surjective. n is arbitrary. let 2^x be the highest power of 2 dividing $n+1$. Then the quotient must be odd of the form $2y + 1$. So $n = 2^x(2y + 1) - 1 = \pi(x, y)$. It is surjective.

   $\pi_1(n) = \mu z \leq (n+1)(2^z > n+1) - 1$ is computable. $\pi_2(n) = qt(2, qt(2^{\pi_1(n)}, n+1) - 1)$ is also computable. It is easy to see that $\pi(\pi_1(n), \pi_2(n)) = n$ are computable.

4. Show that the following function is primitive recursive (with the help of $\pi(x, y)$, perhaps):

$$\begin{aligned} f(0) &= 1, \\ f(1) &= 1, \\ f(n+2) &= f(n) + f(n+1). \end{aligned}$$

Solution. Define $g(x)$ as below:

$$g(0) = \pi(1,1)$$
$$g(x+1) = \pi(\pi_2(g(x)), \pi_1(g(x)) + \pi_2(g(x)))$$

Then $f(x) = \pi_1(g(x))$. Therefore, we can know $f(x)$ is primitive recursive.

5. Coding Technology.

   Any number $x \in \mathbb{N}$ has a unique expression as

   (1) $x = \sum_{i=0}^{\infty} \alpha_i 2^i$, with $\alpha_i = 0$ or 1, for all i.

   Hence, if $x > 0$, there are unique expressions for x in the forms

   (2) $x = 2^{b_1} + 2^{b_2} + \ldots + 2^{b_l}$, with $0 \le b_1 < b_2 < \ldots < b_l$ and $l \ge 1$, and

   (3) $x = 2^{a_1} + 2^{a_1+a_2+1} + \ldots + 2^{a_1+a_2+\ldots+a_k+k-1}$. (The expression (3) is a way of regarding x as coding the sequence $(a_1, a_2, \ldots, a_l)$ of numbers)

   Show that each of the functions α, l, b, a defined below is computable.

   (a) $\alpha(i,x) = \alpha_i$ as in the expression (1);

   (b) $l(x) = \begin{cases} l \text{ as in (2)}, & \text{if } x > 0, \\ 0 & \text{otherwise;} \end{cases}$

   (c) $b(i,x) = \begin{cases} b_i \text{ as in (2)}, & \text{if } x > 0 \text{ and } 1 \le i \le l, \\ 0 & \text{otherwise;} \end{cases}$

   (d) $a(i,x) = \begin{cases} a_i \text{ as in (3)}, & \text{if } x > 0 \text{ and } 1 \le i \le l, \\ 0 & \text{otherwise;} \end{cases}$

   Solution. (a) $\alpha(i,x) = rm(2, qt(2^i, x))$ is computable.

   (b) $l(x) = \sum_{i \le x} \alpha(i,x)$ is computable.

   (c) $b(i,x) = \begin{cases} \mu z \le x(\sum_{j \le z} \alpha(j,x) - i) & \text{if } x > 0 \text{ and } 1 \le i \le l(x) \\ 0 & \text{otherwise} \end{cases}$

   (d) a_i means the number of 0's between the $i - 1_{th}$ 1 and i_{th} 1 between the binary x. For $x > 0$ and $1 \le i \le l(x)$, define

   $$a(1,x) = b(1,x)$$
   $$a(i+1,x) = b(i+1,x) \dot{-} b(i,x) \dot{-} 1$$