Lab08-Recursively Enumerable Set

CS363-Computability Theory, Xiaofeng Gao, Spring 2016

∗ Please upload your assignment to FTP or submit a paper version on the next class
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1. Let A, B be subsets of $\mathbb{N}$. Define sets $A \oplus B$ and $A \otimes B$ by

$$A \oplus B = \{2x \mid x \in A\} \cup \{2x+1 \mid x \in B\},$$
$$A \otimes B = \{\pi(x,y) \mid x \in A \text{ and } y \in B\},$$

where π is the pairing function $\pi(x,y) = 2^x(2y+1) - 1$. Prove that

(a) $A \oplus B$ is recursive iff A and B are both recursive.

(b) If $A, B \neq \varnothing$, then $A \otimes B$ is recursive iff A and B are both recursive.

Solution.

(a) (if) $c_{A\oplus B}(x) = \begin{cases} 1, & \text{if } x \text{ is even and } x/2 \in A, \text{or if } x \text{ is odd and } (x-1)/2 \in B, \\ 0, & \text{otherwise.} \end{cases}$ Since A and B are both recursive, $c_{A\oplus B}(x)$ is computable, so $A \oplus B$ is recursive.

(only if) $c_A(x) = \begin{cases} 1, & \text{if } 2x \in A \oplus B, \\ 0, & \text{otherwise.} \end{cases}$ Since $A \oplus B$ is recursive, $c_A(x)$ is computable, so A is recursive. Similarly, B is recursive.

(b) (if) $c_{A\otimes B}(x) = \begin{cases} 1, & \text{if } \pi_1(x) \in A \text{ and } \pi_2(x) \in B, \\ 0, & \text{otherwise.} \end{cases}$ Since A and B are both recursive, $c_{A\otimes B}(x)$ is computable, so $A \otimes B$ is recursive.

(only if) Since $A, B \neq \emptyset$, we can find an element a in A and b in B. Then $c_A(x) = c_{A\otimes B}(\pi(x,b))$, $c_B(x) = c_{A\otimes B}(\pi(a,x))$. By substitution, $c_A(x)$ and $c_B(x)$ are computable, so A and B are recursive.

□

2. Which of the following sets are recursive? Which are r.e.? Which have r.e. complement? Prove your judgements.

(a) $\{x \mid P_m(x)\downarrow \text{ in } t \text{ or fewer steps }\}$ (m, t are fixed).

(b) $\{x \mid x \text{ is a power of } 2\}$;

(c) $\{x \mid \phi_x \text{ is injective}\}$;

(d) $\{x \mid y \in E_x\}$ (y is fixed);

Solution.

(a) $c_A(x) = \begin{cases} 1, & \text{if } \mathrm{j}(m,x,t) = 0, \\ 0, & \text{if } \mathrm{j}(m,x,t) \neq 0. \end{cases}$ $\mathrm{j}(m,x,t)$ is primitive recursive, so $c_A(x)$ is computable, so A is recursive. So it is r.e. and has r.e. complement.

(b) $c_A(x) = \begin{cases} 1, & \text{if } 2^{(x)_1} = x, \\ 0, & \text{otherwise.} \end{cases}$ $c_A(x)$ is computable, so A is recursive. So it is r.e. and has r.e. complement.

(c) Define $f(x,y) = \begin{cases} y, & \text{if } x \in W_x, \\ \uparrow, & \text{otherwise.} \end{cases}$ $f(x,y)$ is computable. According to s-m-n theorem there exists a total computable function k such that $\phi_{k(x)}(y) \simeq f(x,y)$. Clearly, $k : K \leq_m \{x \mid \phi_x \text{ is injective}\}$, so it is not recursive.
It has r.e. compliment because ϕ_x is not injective $\Leftrightarrow \exists a \exists b.(\phi_x(a) = \phi_x(b) \wedge a \neq b)$. The right side is a partial decidable predicate thus $\{x \mid \phi_x \text{ is not injective}\}$ is r.e.
$\{x \mid \phi_x \text{ is injective}\}$ is not r.e. because if it is r.e., $\{x \mid \phi_x \text{ is injective}\}$ would be recursive.

(d) Define $f(x,z) = \begin{cases} y, & \text{if } x \in W_x, \\ \uparrow, & \text{otherwise.} \end{cases}$ $f(x,z)$ is computable. According to s-m-n theorem there is a total computable function k such that $\phi_{k(x)}(z) = f(x,z)$. Clearly, $k : \{x \mid x \in W_x\} \leq_m \{x \mid y \in E_x\}$, so it is not recursive.
It is r.e. because $y \in E_x \Leftrightarrow \exists a \exists t (P_x(a) \downarrow y \text{ in } t \text{ steps})$. The right side is partially decidable thus $\{x \mid y \in E_x\}$ is r.e.
It doesn't have r.e. compliment because if so, $\{x \mid y \in E_x\}$ would be recursive.

$\square$

3. Prove following statements.

(a) Let $B \subseteq \mathbb{N}$ and $n > 1$; prove that B is r.e. then the predicate $M(x_1, \ldots, x_n)$ given by "$M(x_1, \ldots, x_n) \equiv 2^{x_1} 3^{x_2} \ldots p_n^{x_n} \in B$" is partially decidable.

(b) Prove that $A \subseteq \mathbb{N}^n$ is r.e. iff $\{2^{x_1} 3^{x_2} \ldots p_n^{x_n} \mid (x_1, \ldots, x_n) \in A\}$ is r.e..

Solution.

(a) As we know that B is r.e., so $\chi_B(x)$ is computable. Define $f(x_1, x_2, \ldots, x_n) = 2^{x_1} 3^{x_2} \ldots p_n^{x_n}$, then
$\chi_M(x_1, x_2, \ldots, x_n) = \chi_B(f(x_1, x_2, \ldots, x_n))$. By substitution, χ_M is computable, so M is partially decidable.

(b)

$\square$