# Lab05-Numbering Programs

CS363-Computability Theory, Xiaofeng Gao, Spring 2016

* Please upload your assignment to FTP or submit a paper version on the next class
* If there is any problem, please contact: nongeek.zv@gmail.com

* Name:Wenhao Zhu StudentId: 5130309717 Email: weehowe.z@gmail.com

1. Show that there is a total computable function $k$ such that for each $n$,

   (a) $k(n)$ is an index of the function $\lfloor \sqrt[n]{x} \rfloor$.

   (b) $W_{k(n)}^{(m)} = \{(y_1, \ldots, y_m) : y_1 + y_2 + \ldots + y_m = n\}$ $(m \geq 1)$.

   (c) $E_{k(n)} = W_n$.

   **Solution.** (a)Let $P_m$ be the program corresponding to $f_{P_m}(x) = \lfloor \sqrt[n]{x} \rfloor$ then the $P_m$ can be describe as 1)keep calculating $i^n$, where i grow up 2)when the first $i^n > x$, stop 3)the result is $(i - 1)$. So apparently we can decide the $P_m$ therefore by the Church's Thesis, $k(n)$ is computable.

   (b) $k(n)$ is the index of the function that will terminate $\iff \sum_{i=0}^{m} y_i = n$

   So the corresponding $P_{k(n)}$ can check the input satisfy the condition. If not it will turn into a endless loop.

   By the Church's Thesis, the $k(n)$ is computable.

   (c) $f_{P_{k(n)}}(x)$ is the inverse function of the $f_{Pn}$, $k(n)$ is apparently computable.

2. (a) Find $P_{1028}$. Distinguish what are $\phi_{1028}(x)$ and $\phi_{1028}^{(n)}(x_1, \cdots, x_n)$ and their corresponding $W_{1028}(x)$, $E_{1028}(x)$ and $W_{1028}^{(n)}(x)$, $E_{1028}^{(n)}(x)$;

   **Solution.**

   $P_{1028} = 2^{10} + 2^2$, so we can know that $a_1 = 2, a_2 = 7$. Therefore, the instructions are T(1,1) and J(1,1,2). $\phi_{1028}(x) = 1, W_{1028}(x) = N, E_{1028}(x) = 1$.

   (b) Let $P$ be the program J(1,2,4), Z(1), S(1). Calculate $\gamma(P)$.

   **Solution.**

   $\beta(J(1,2,4)) = 4\zeta(1,2,4) + 3 = 4\pi(\pi(0,1),3) + 3 = 111$

   $\beta(Z(1)) = 0$

   $\beta(S(1)) = 1$

   Thus, $\gamma(P) = 2^{111} + 2^{112} + 2^{114} - 1$

3. (a) (Cantor) Show that the set of all functions from $\mathbb{N}$ to $\mathbb{N}$ is not denumerable.

   *Proof.* Define $\mathcal{F}$ as all the functions maps $\mathbb{N}$ to $\mathbb{N}$. Let $f_0$, $f_1$, $f_2$, . . . be any sequence of elements of $\mathcal{F}$ and by Cantor's Diagonal Method, we can define $f \in \mathcal{F}$ by:

   $$f(i) = \begin{cases} f_i(i) + 1 & \text{if } f_i(n) \text{ is defined} \\ 0 & \text{otherwise} \end{cases}$$

   By construction $f(i) \neq f_i(n)$, so $f \neq f_i$ ( for any $i \in \mathbb{N}$ ), thus the set of all functions from $\mathbb{N}$ to $\mathbb{N}$ is not denumerable. □

   (b) Show that the set of all non-computable total functions from $\mathbb{N}$ to $\mathbb{N}$ is not denumerable.

*Proof.* Define $\mathcal{F}$ as all non-computable total functions maps $\mathbb{N}$ to $\mathbb{N}$. Let $f_0$, $f_1$, $f_2$, . . . be any sequence of elements of $\mathcal{F}$, and these functions can be derived by previous problem, by Cantor's Diagonal Method, we can define $f \in \mathcal{F}$ by:

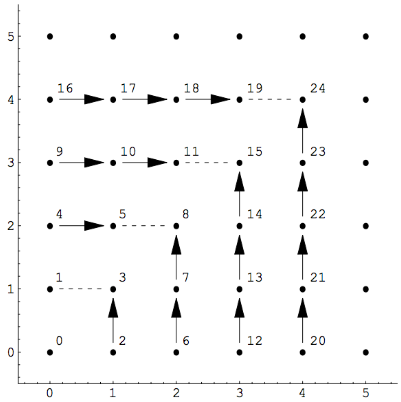
$$f(i) = \begin{cases} f_i(i) + a(i) + 1 & \text{if } f_i(n) \text{ is defined} \\ a(i) & \text{otherwise} \end{cases}$$

By construction $f(i) \neq f_i(n)$, so $f \neq f_i$ ( for any $i \in \mathbb{N}$ ), thus the set of all non-computable total functions from $\mathbb{N}$ to $\mathbb{N}$ is not denumerable. □

4. Alternative Selection of $\pi$

The $\pi$ function where $\pi(x, y) = 2^x(2y + 1) - 1$ can enumerate linearly all pairs of natural numbers $(x, y) \in \mathbb{N} \times \mathbb{N}$. However, it does not generate a trace in the first quadrant of the plane. Correspondingly, instead of applying this $\pi$ function, we can define an alternative bijection $\pi'$, such that $\pi' : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ and it grows horizontally and vertically according to the right figure. Thus we have:

$\pi'(0, 0) = 0$, $\pi'(0, 1) = 1$, $\pi'(1, 0) = 2$,
$\pi'(1, 1) = 3$, $\pi'(0, 2) = 4$, $\pi'(1, 2) = 5$,
$\pi'(2, 0) = 6$, $\pi'(2, 1) = 7$, $\pi'(2, 2) = 8$, etc.




Now please develop a mathematical formula for $\pi'$, (like the notation of original $\pi$), and prove the correctness of your design.

**Solution.**

$$\pi' = \begin{cases} x + y^2 & \text{if } x < y \\ x^2 + x + y & \text{otherwise} \end{cases}$$

Now we begin to prove that $\pi'$ is bijective.
**Injective** For $\forall(x_1, y_1), (x_2, y_2)$, with $x_1 \neq x_2$ or $y_1 \neq y_2$. We can define $R = \pi'(x_1, y_1) - \pi'(x_2, y_2)$

(a) $x_1 < y_1$ and $x_2 < y_2$: $R = (x_1 - x_2) + (y_1 - y_2)(y_1 + y_2)$

(b) $x_1 \leq y_1$ and $x_2 \leq y_2$: $R = (x_1 + x_2 + 1)(x_1 - x_2) + (y_1 - y_2)$

(c) $x_1 < y_1$ and $x_2 \leq y_2$: $R = x_1 + y_1^2 - x_2^2 - x_2 - y_2$.

We can infer that none of $R = 0$, thus it is injective.
**Surjective** For $\forall n \in \mathbb{N}$, $\exists a, b \in \mathbb{N}$, such that $n = a + b^2$. If $a < b$, then we let $x = a$, $y = b$, else if $a \geq b$, we can let $x = b$, $y = a - b$. That means we fin That means $\forall n \in \mathbb{N}$, we can find $x$,$y$, makes $\pi'(x, y) = n$. So it is surjective.

Therefore the function we desgin is a bijective map form $\mathbb{N}$ to $\mathbb{N}$.