Lab04-Church's Thesis

CS363-Computability Theory, Xiaofeng Gao, Spring 2016

∗ Please upload your assignment to FTP or submit a paper version on the next class
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1. Suggest the natural definition of computability on domain $\mathbb{Q}$ (rational numbers).

   Solution. Definition:

   - $Z(q) = 0$
   - $q = \begin{cases} \frac{m}{n} & \text{if } q > 0 \text{ and } gcd(m,n) = 1 \\ -\frac{m}{n} & \text{if } q < 0 \text{ and } gcd(m,n) = 1 \end{cases}$
   - $t = \mu x(gcd(\pi_1(\pi(m,n))))$
   - $U_i^n(q_1, q_2, \ldots, q_n) = q_i$

2. Define $f(n)$ as the n-th digit in the decimal expansion of e. Use Church's Thesis to prove that f is computable. (e is the the base of the natural logarithm and can be calculated as the sum of the infinite series: $e = \sum_{n=0}^{\infty} \frac{1}{n!}$)

   Proof. As we can extend the e as the sum of the infinite series. Let $S_k = \sum_{n=0}^{\infty} \frac{1}{n!}$, by theory of infinite series, $S_k < e < S_k + \frac{1}{k!}$. Since S_k is rational, the decimal expansion of S_k can be effectively calculated to any desired number of places using long division. Thus the effective method for calculating f(n) (given a number n) can be described as:
   Find the first $N \leq n+1$ such that the decimal expansion $S_N = a_0 \cdot a_1 \cdot a_2 \cdots a_n a_{n+1} \cdots a_N \cdots$ does not have all of $a_{n+1} a_N$ equal to 9. Then put $f(n) = a_n$.
   To see that this gives the required value, suppose that $a_m \neq 9$ with $n < m \leq N$. Hence $a_0 \cdot a_1 \cdots a_n \cdots a_m \cdots < e < a_0 \cdot a_1 \cdots a_n \cdots (a_m + 1) \cdots$. So the n-th decimal place of e is indeed a_n.
   Thus by Church's Thesis, f is computable. □

3. Suppose there is a two-tape Turing Machine M with alphabet $\Gamma = \{\triangleright, \triangleleft, \Box, 1\}$ and state set $Q = \{q_s, q_1, q_2, q_h\}$. M has the following specifications. Transform M into a single-tape Turing Machine $\widetilde{M}$, and write down the new alphabet and specifications.

$$\begin{aligned} \langle q_s, \triangleright, \triangleright\rangle &\rightarrow \langle q_1, \triangleright, S, R\rangle \\ \langle q_1, \triangleright, \Box\rangle &\rightarrow \langle q_2, 1, R, R\rangle \\ \langle q_2, 1, \Box\rangle &\rightarrow \langle q_2, 1, R, R\rangle \\ \langle q_2, \triangleleft, \Box\rangle &\rightarrow \langle q_h, \triangleleft, S, S\rangle \end{aligned}$$

   Solution. Alphabet $\Gamma = \{\triangleright, \triangleleft, \Box, 1\}$, and the specifications are:

$$\begin{aligned} \langle q_s, \triangleright\rangle &\rightarrow \langle q_1, \triangleright, R\rangle \\ \langle q_1, 1\rangle &\rightarrow \langle q_1, 1, R\rangle \\ \langle q_1, \triangleleft\rangle &\rightarrow \langle q_2, 1, R\rangle \\ \langle q_2, \Box\rangle &\rightarrow \langle q_h, \triangleleft, R\rangle \end{aligned}$$

4. Design a three-tape TM M that computes the function $f(x, y) = x\%y$, where both m and n belong to the natural number set $\mathbb{N}$. The alphabet is $\{1, \Box, \triangleright, \triangleleft\}$, where the input on the first tape is $x + 1$ "1"'s and $y + 1$ "1"'s with a "$\Box$" as the separation. Below is the initial configurations for input (x, y). The result is the number of "1"'s on the output tape with the pattern of $\triangleright 111 \cdots 111 \triangleleft$. First describe your design and then write the specifications of M in the form like $\langle q_S, \triangleright, \triangleright, \triangleright \rangle \to \langle q_1, \triangleright, \triangleright, R, R, R \rangle$ and explain the transition functions in detail (especially the meaning of each state).

Initial Configurations

Tape 1:	$\triangleright$	1	1	$\cdots$	1	1	$\Box$	1	1	$\cdots$	1	1	$\triangleleft$	
	$\uparrow$	$\leftarrow x + 1$ squares $\rightarrow$						$\leftarrow y + 1$ squares $\rightarrow$						
Tape 2:	$\triangleright$	$\Box$	$\Box$	$\cdots$		$\cdots$		$\cdots$			$\Box$	$\Box$	$\Box$	
	$\uparrow$													
Tape 3:	$\triangleright$	$\Box$	$\Box$	$\cdots$		$\cdots$		$\cdots$			$\Box$	$\Box$	$\Box$	
	$\uparrow$													

Solution. Begin.

$$\langle q_s, \triangleright, \triangleright, \triangleright \rangle \quad \to \quad \langle q_1, \triangleright, \triangleright, R, R, S \rangle$$

Copy $x + 1$ '1' to Tape 2

$$\begin{array}{rcl} \langle q_1, 1, \Box, \Box \rangle & \to & \langle q_1, 1, \Box, R, R, S \rangle \\ \langle q_1, \Box, \Box, \Box \rangle & \to & \langle q_2, \triangleleft, \Box, R, S, S \rangle \end{array}$$

Move pointer of Tape 2 to the end.

$$\begin{array}{rcl} \langle q_2, 1, \triangleright, \Box \rangle & \to & \langle q_2, \triangleright, \Box, R, S, S \rangle \\ \langle q_2, \triangleright, \triangleright, \Box \rangle & \to & \langle q_3, \triangleright, \Box, L, L, S \rangle \end{array}$$

Move back pointer of Tape 2 and Tape 3 at the same time and get the result

$$\langle q_3, 1, 1, \Box \rangle \quad \to \quad \langle q_3, 1, \Box, L, L, S \rangle$$

If $x = y$ or $x > y$, calculate $x = x - y$.

$$\begin{array}{rcl} \langle q_3, \Box, \triangleright, \Box \rangle & \to & \langle q_4, \triangleright, \Box, R, R, S \rangle \\ \langle q_3, \Box, 1, \Box \rangle & \to & \langle q_4, \triangleright, \Box, R, R, S \rangle \\ \langle q_3, 1, \triangleright, \Box \rangle & \to & \langle q_7, \triangleright, \Box, S, R, R \rangle \end{array}$$

Get $x = x - y$.

$$\begin{array}{rcl} \langle q_4, 1, 1, \Box \rangle & \to & \langle q_4, 1, \Box, R, R, S \rangle \\ \langle q_4, \triangleleft, \triangleleft, \Box \rangle & \to & \langle q_4, \triangleleft, \Box, L, S, S \rangle \\ \langle q_4, 1, \triangleleft, \Box \rangle & \to & \langle q_5, \Box, \Box, L, L, S \rangle \\ \langle q_5, 1, 1, \Box \rangle & \to & \langle q_5, \Box, \Box, L, L, S \rangle \\ \langle q_5, \Box, 1, \Box \rangle & \to & \langle q_6, \Box, \Box, L, L, S \rangle \\ \langle q_6, \Box, 1, \Box \rangle & \to & \langle q_2, \triangleleft, \Box, R, S, S \rangle \end{array}$$

Else perform copy from Tape 2 to Tape 3 and halt.

$$\begin{aligned}\langle q_7, \Box, 1, \Box\rangle &\rightarrow \langle q_7, \Box, 1, S, R, R\rangle \\ \langle q_7, \Box, \triangleleft, \Box\rangle &\rightarrow \langle q_H, \triangleleft, \triangleleft, S, S, S\rangle\end{aligned}$$