Lab09-Recursively Enumerable Set(2)

CS363-Computability Theory, Xiaofeng Gao, Spring 2016

∗ Please upload your assignment to FTP or submit a paper version on the next class

∗ If there is any problem, please contact: steinsgate@sjtu.edu.cn

∗ Name:Wenhao Zhu StudentId: 5130309717 Email: weehowe.z@gmail.com

1. Suppose A is an r.e. set. Prove the following statements.

   (a) Show that the sets $\bigcup_{x \in A} W_x$ and $\bigcup_{x \in A} E_x$ are both r.e.

   Proof. As $x \in W_x$is partially decidable.So there $\exists x \in A$ such that $z \in W_x$ is partially decidable.$\exists x \in A$ that $z \in W_x$ is equal to $x \in \bigcup_{x \in A} W_x$,so it is also partially decidable.Then the set $\bigcup_{x \in A} W_x$ is r.e. The same as $\bigcup_{x \in A} E_x$, we can prove in the same way. □

   (b) Show that $\bigcap_{x \in A} W_x$ is not necessarily r.e. (*Hint*: $\forall t \in \mathbb{N}$ let $K_t = \{x : P_x(x) \downarrow \text{ in t steps}\}$. Show that for any t, K_t is recursive; moreover $K = \bigcup_{t \in \mathbb{N}} K_t$ and $\overline{K} = \bigcup_{t \in \mathbb{N}} \overline{K}_t$.)

   Proof. As we konw $z \in \bigcap_{x \in A} W_x$ means $\forall x \in A$, $z \in W_x$, $z \in W_x$ is partially decidable but $\forall x \in A$, however $z \in W_x$ is partially decidable, so the set is not necessarily decidable. □

2. Prove that $A \subseteq \mathbb{N}^n$ is r.e. iff $A = \varnothing$ or there is a total computable function $f : \mathbb{N} \to \mathbb{N}^n$ such that $A = Ran(\boldsymbol{f})$. (A *computable function* $\boldsymbol{f}$ from $\mathbb{N}$ to $\mathbb{N}^n$ is an n-tuple $\boldsymbol{f} = (f_1, \ldots, f_n)$ where each f_i is a unary computable function and $\boldsymbol{f}(x) = (f_1(x), \ldots, f_n(x))$.)

   Proof. if $A = \varnothing$ then $x \in A$ is p.d. so A is r.e. If there is a total computable function $f : \mathbb{N} \to \mathbb{N}^n$ such that $A = Ran(\boldsymbol{f})$,then $A = (f_1(x), \ldots, f_n(x))$.So N is m-reducable to A. Since N is r.e.,A is also r.e. □

3. Suppose that f is a total computable function, A is a recursive set and B is an r.e.set. Show that $f^{-1}(A)$ is recursive and that $f(A)$, $f(B)$ and $f^{-1}(B)$ are r.e. but not necessarily recursive. What extra information about these sets can be obtained if f is a bijection?

   Proof. $x \in f^{-1}(A)$ is equal to $f(x) \in A$ and since A is recursive, then $f(x) \in A$ is decidable so $x \in f^{-1}(A)$ is decidable so $f^{-1}(A)$ is recursive.$x \in f(A)$ is equal to $\exists y \in N, y \in A$ and $f(y) = x$ and since $y \in A$ and $f(y) = x$ is decidable,it is partially decidable so $f(A)$ is r.e.$x \in f(B)$ is equal to $\exists y \in N, y \in B$ and $f(y) = x$,it is p.d. so $f(B)$ is r.e. Similarly we can prove $f^{-1}(B)$ is r.e. As f is a total computable function, if f is bijective, then $f(A)$ is recursive. □

4. A set D is the difference of r.e. sets (*d.r.e.*) iff $D = A - B$ where A, B are both *r.e.*.

   (a) Show that the set of all *d.r.e.* sets is closed under the formation of intersection.

   Proof. For $\forall d.r.e$ set $M = A - B$ and $N = C - D$ where A,B,C,D are all r.e. $M \bigcap N = (A-B) \bigcap (C-D) = A \bigcap C + B \bigcap D - A \bigcap D - B \bigcap C$.so it is also *d.r.e.* Thus *d.r.e* sets is closed under the formation of intersection. □

   (b) Show that if $C_n = \{x \mid |W_x| = n\}$, then C_n is *d.r.e.* for all $n \geq 0$.

   Proof. We can define a program that for numbers in N, if number $\in W_x$ more than n, then it will halt, otherwise we let it undefined. Thus it is d.r.e □