Homework No.9

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December 31, 2022

1 Higher Dimensional Sphere

1.1 Problem Description

The interior of a d-dimensional hypersphere of unit radius is defined by the condition $x_1^2 + x_2^2 + ... + x_d^2 \leq 1$. Write a program that finds the volume of a hypersphere using a Monte Carlo method. Test your program for $d = 2$ and $d = 3$ and then calculate the volume for $d = 4$ and $d = 5$, compare your results with the exact results.

1.2 Solution

In fact, we adopted the Hit-and-Miss method under the spirit of the Monte-Carlo approach, The intrgration is straightforward because the volume of a d-dimensional "hypercubic" can be calculated easily with

$$\int_{x_i \in [-\frac{a}{2}, \frac{a}{2})} d^d x_i = a^d \tag{1}$$

where $i \in 1, 2, ..., d$ and d is the number of dimension.

The volume of a hypersphere is therefore

$$\frac{n}{N} a^d \tag{2}$$

assuming there are N random number generated and n of them has a norm less than a

1.3 Output and Analysis

We first use the 3D and 2D cases to check the validity of our MC method. Sampling at 10^4 points, the volume is evaluated according to 2. We can get a theoretical result for spheres in dimensions. Just integrate in the domain where $\|x\| < a$ and the integral is

$$V_n = \frac{\pi^{n/2} R^n}{\Gamma(1 + \frac{n}{2})} \tag{3}$$

where half factorial is defined as $(\frac{n}{2})! = \frac{n}{2} \cdot (\frac{n}{2}-1)...\frac{1}{2}\sqrt{\pi}$ Our M-C results and the theoretical results are compared in the table below:

d	V: MC	V: theory	relative error
2	3.1416	3.1416	2.5×10^{-6}
3	4.1752	4.1888	3.2×10^{-3}
4	4.8496	4.9348	1.7×10^{-2}
5	5.1200	5.2638	2.6×10^{-2}

It can be seen that the error increases as dimension increases. I further calculate more volumes and there is a maximum in the sphere volume at $d = 5$, as is shown in 1. When the dimension is rather high, our method produces noticable error and I believe this can be relieved by using the sample mean method.

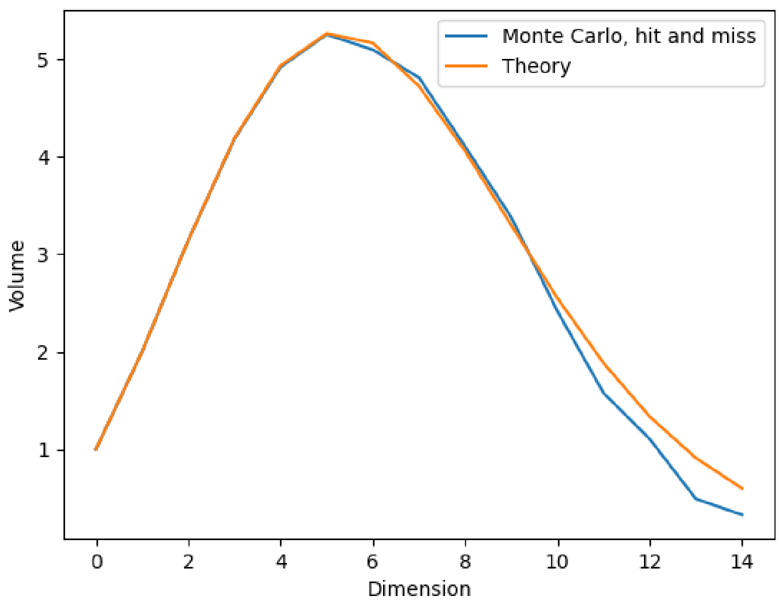

Figure 1: More hyperspheres, volume to dimension. When d larger than 10, the error is noticable.

1.4 Pseudocode

Algorithm 1 Volume of hypersphere

Input: dimension d

Output: volume of the hypersphere calculated by the Monte-Carlo method.

$N \leftarrow 1 \times 10^4$
$n \leftarrow 0$
for i in range(N) **do**
    generate a random d-dimensional vector $\vec{x}$, each component distributed linearly in $[-1/2, 1/2)$
    if $\|\vec{x}\| < 1$ **then**
        $n \leftarrow n + 1$
    end if
end for

Return: n/N

2 3D Classical Heisenburg Model