1. STRUCTURAL BEHAVIOUR OF BEAMS

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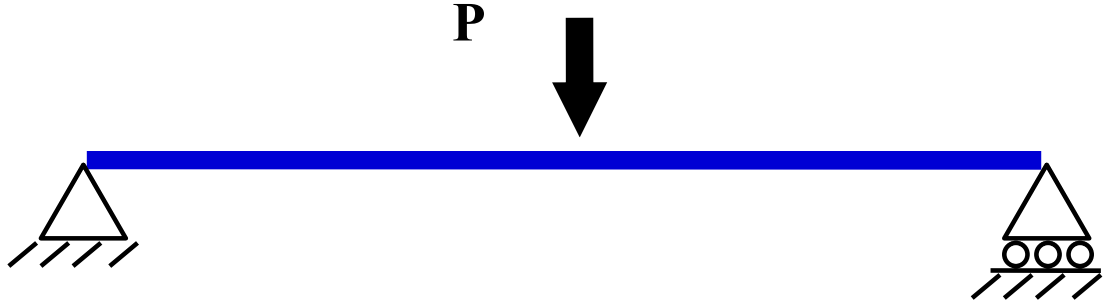
1.1 Introduction

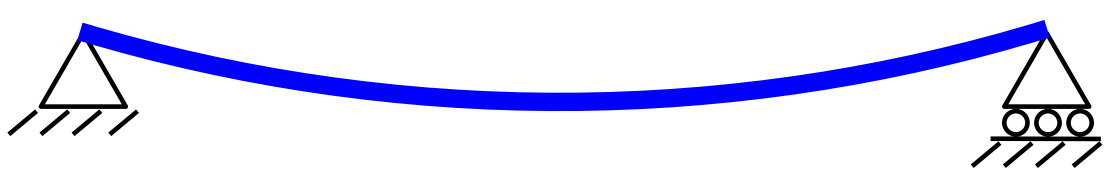
1.1.1 What is a beam?

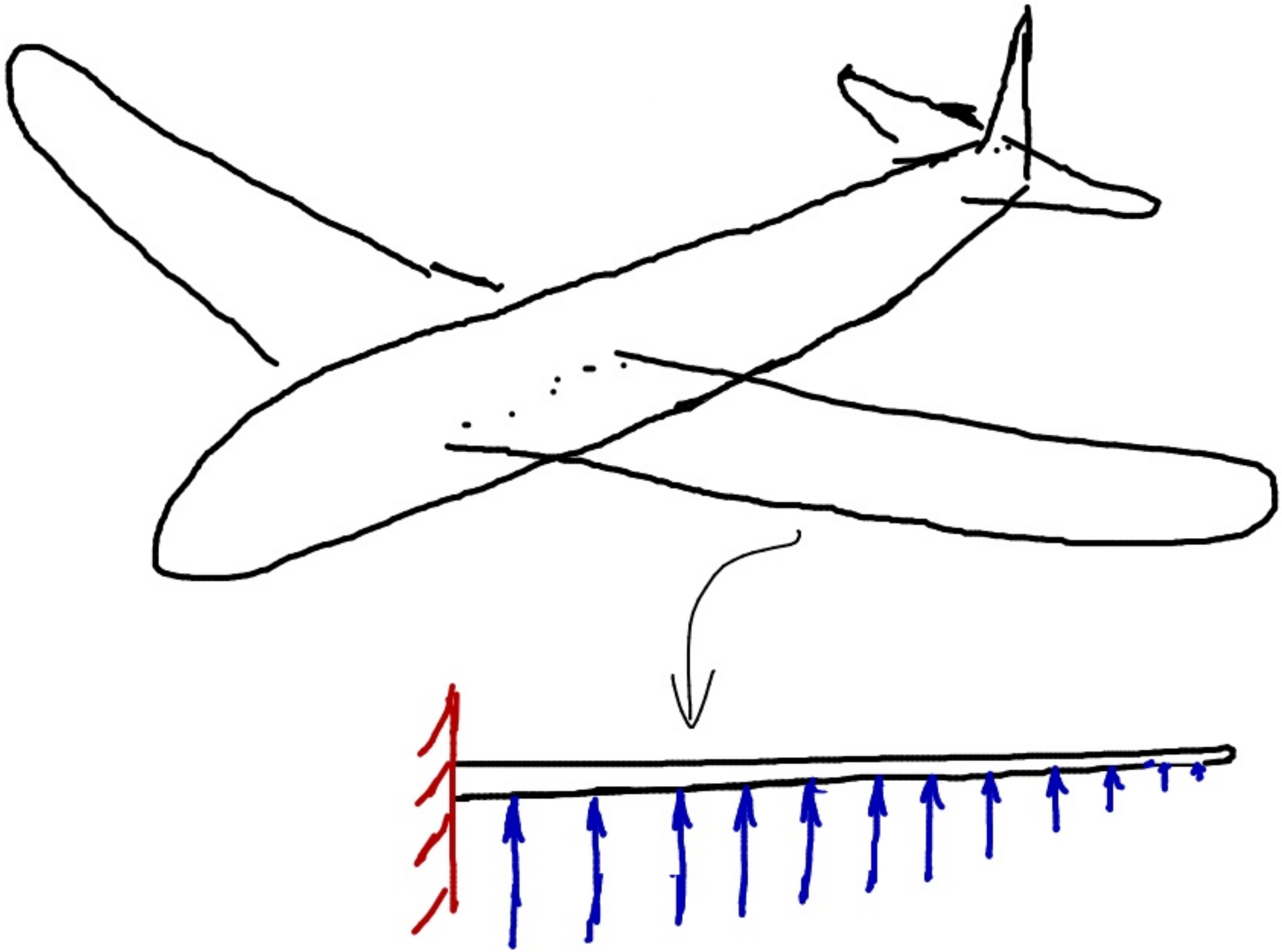
A bar is a beam if it is subjected to a transverse loading which results in bending of the bar.

Example:



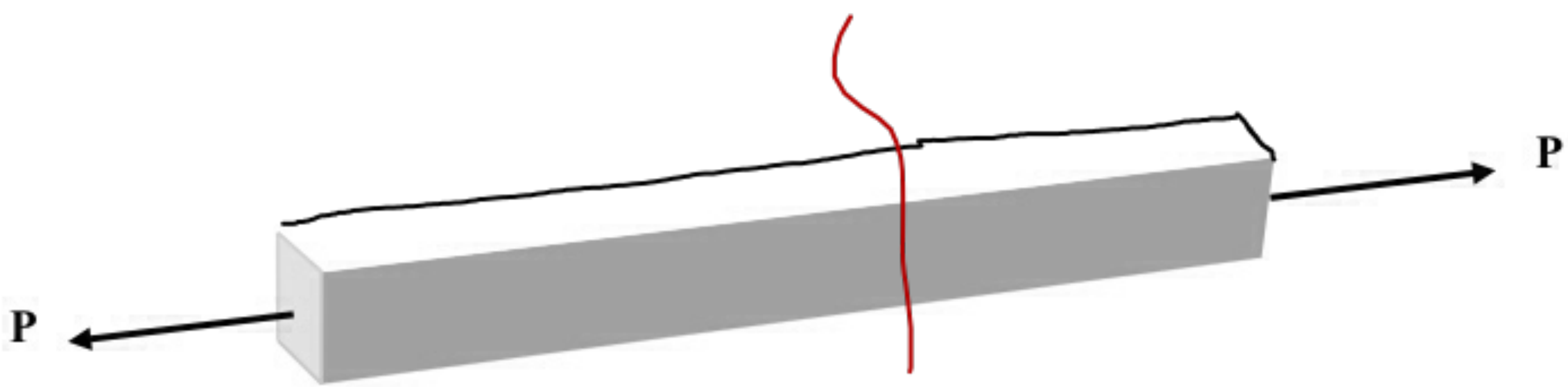
deflected shape:





1.1.2 What stresses will develop in a beam?

We will start by recalling the stresses that develop in an axially loaded bar:

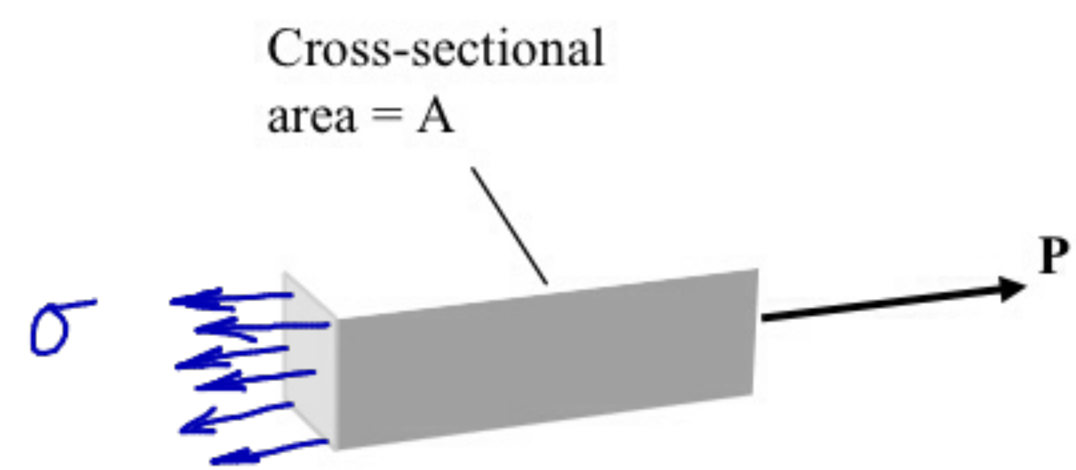


If we cut the bar and consider the free body diagram of, say, the right part

then it is clear that the resultant of the stresses acting on the cut face must be equal & opposite to the applied load P to satisfy longitudinal equilibrium.

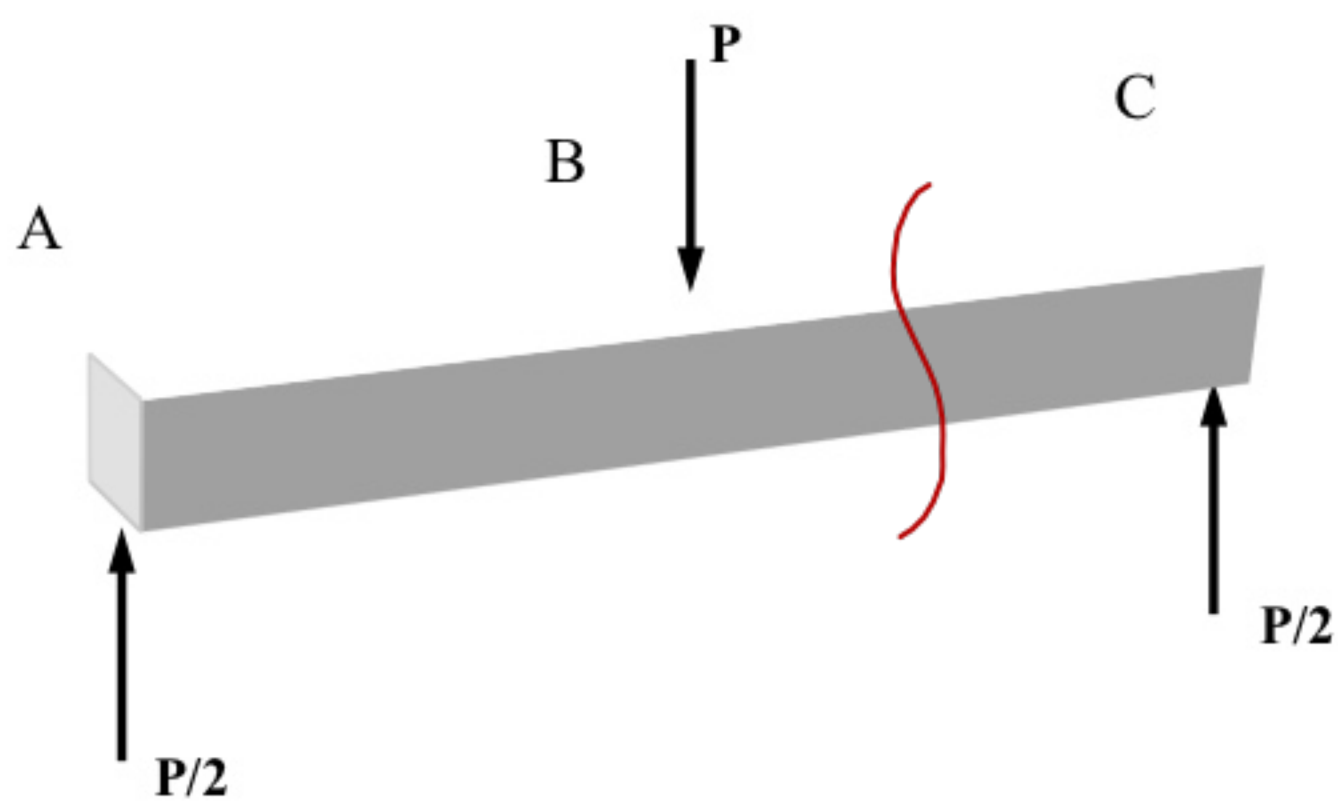
i.e. $R = P$

The stresses acting on the cut face to produce this resultant are direct stresses σ

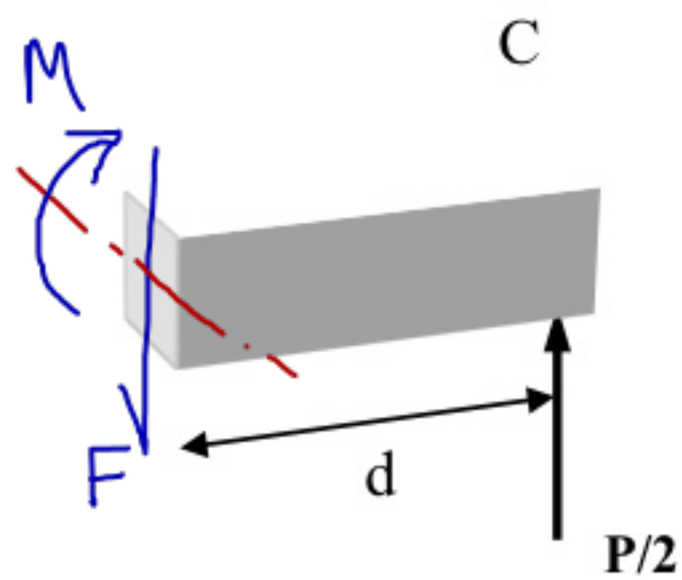


where $\sigma = P/A$ (uniform for P applied at the centroid of the cross section.)

So what happens in a beam? The free body diagram for the centrally loaded, simply supported beam, which we considered earlier, is shown below.



Now cutting the beam between B and C, and considering the free body diagram of the right hand part:



It is evident that

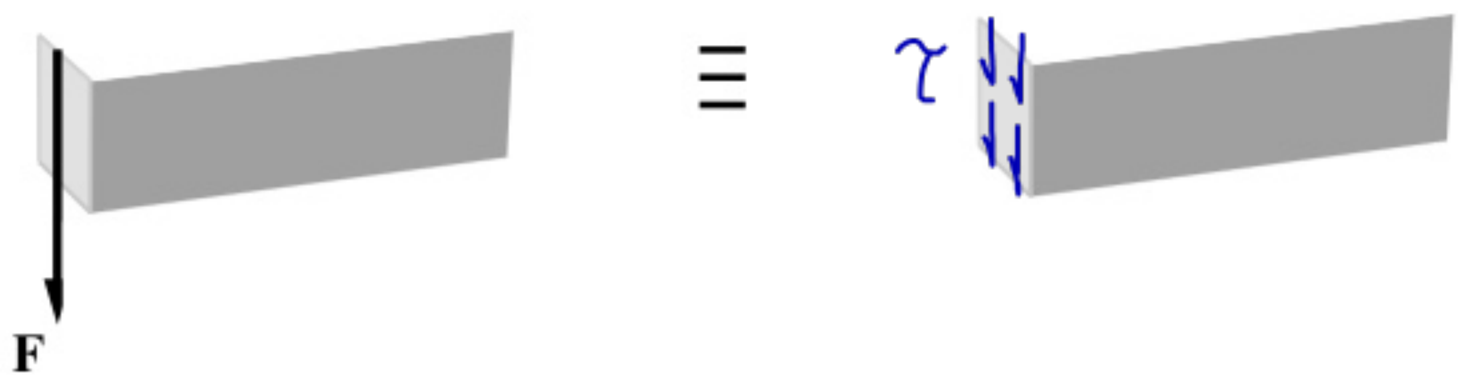
a) for vertical equilibrium the stresses on the cut must have a vertical resultant F which in this case $F = P/2$

b) for horizontal equilibrium the stresses on the cut must have a zero horizontal resultant

c) for moments equilibrium the stresses on the cut must have a moment resultant M where, in this case, $M = \frac{P}{2} . d$

What stresses acting on the cut face give rise to the resultants **F** and **M**?

The vertical force **F** will be the resultant of vertical shear stresses acting on the cut face



F is called the SHEAR FORCE acting in the beam at this section

The moment **M** will be the resultant of direct stresses, σ, acting on the cut face

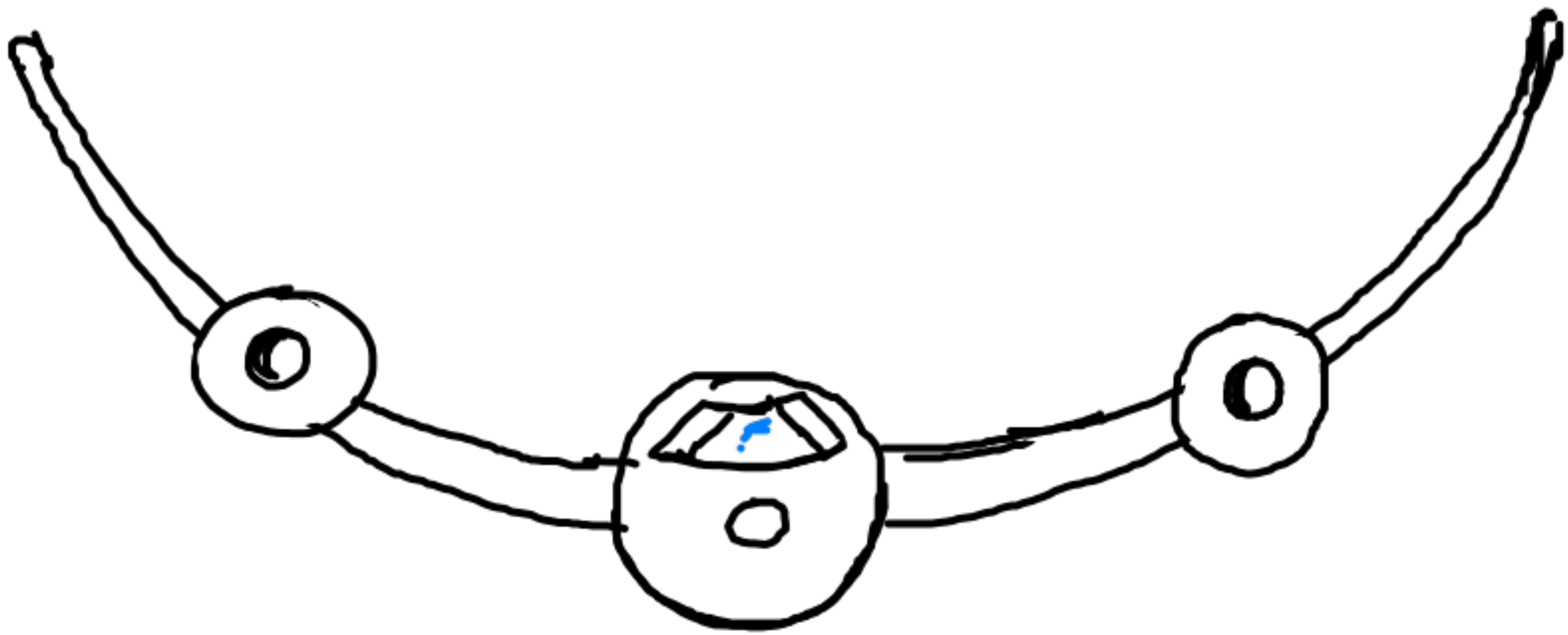
M is called the BENDING MOMENT acting in the beam at this section

1.13 What do engineers need to know about beam structures?

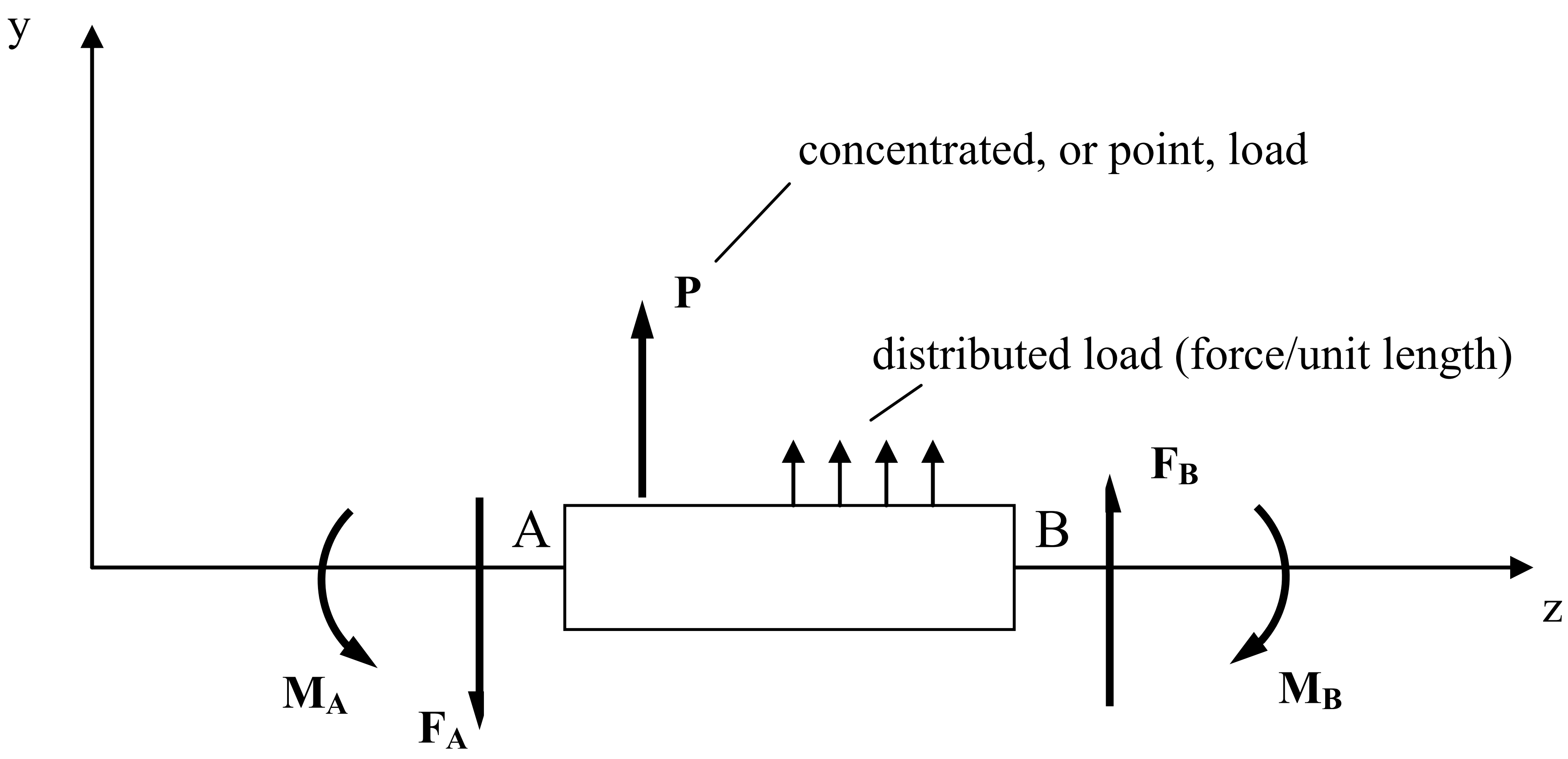
i) Is it strong enough?
For this we need to be able to evaluate the stresses in the beam and check to see if these cause failure of the beam material.

ii) Is it stiff enough?
For this we need to be able to determine the deformed shape of the beam structure.

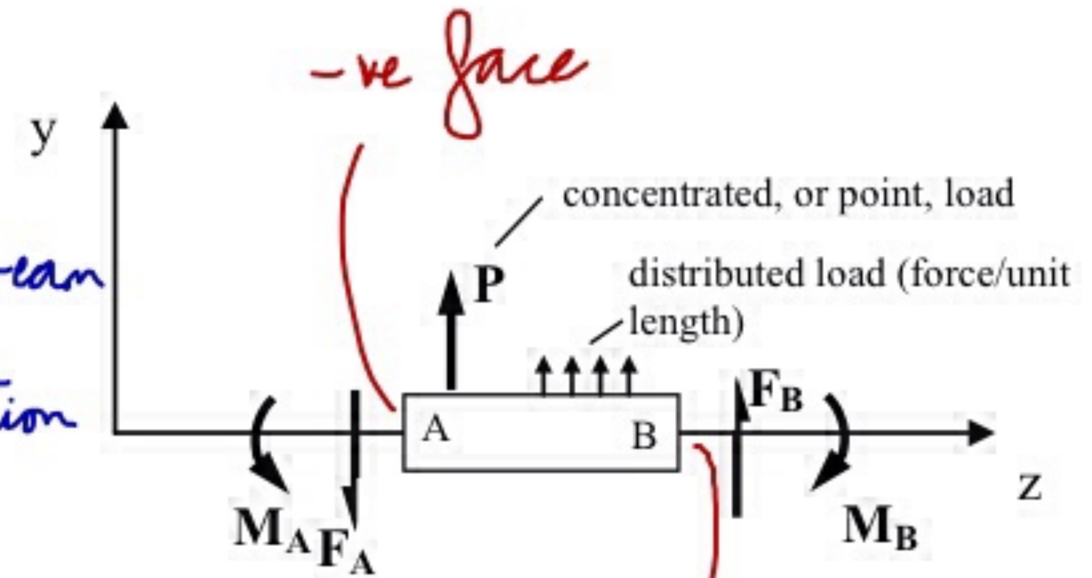
Using Engineers' Theory of Bending (ETB) will enable us to answer these two questions.



Sign Convention for applied load, shear force and bending moment



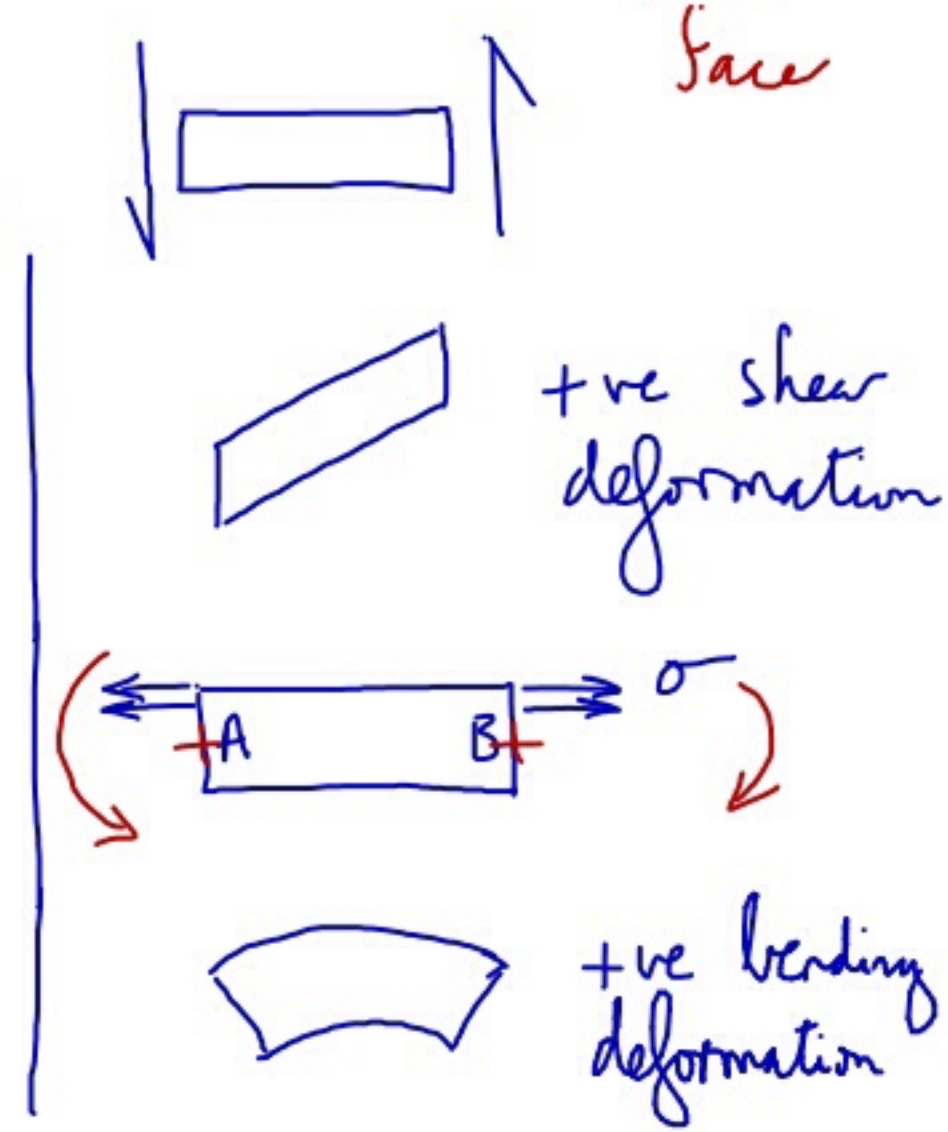
Axes : z-axis runs longitudinally along the beam
y-axis runs upwards in transverse direction



Stress resultants :

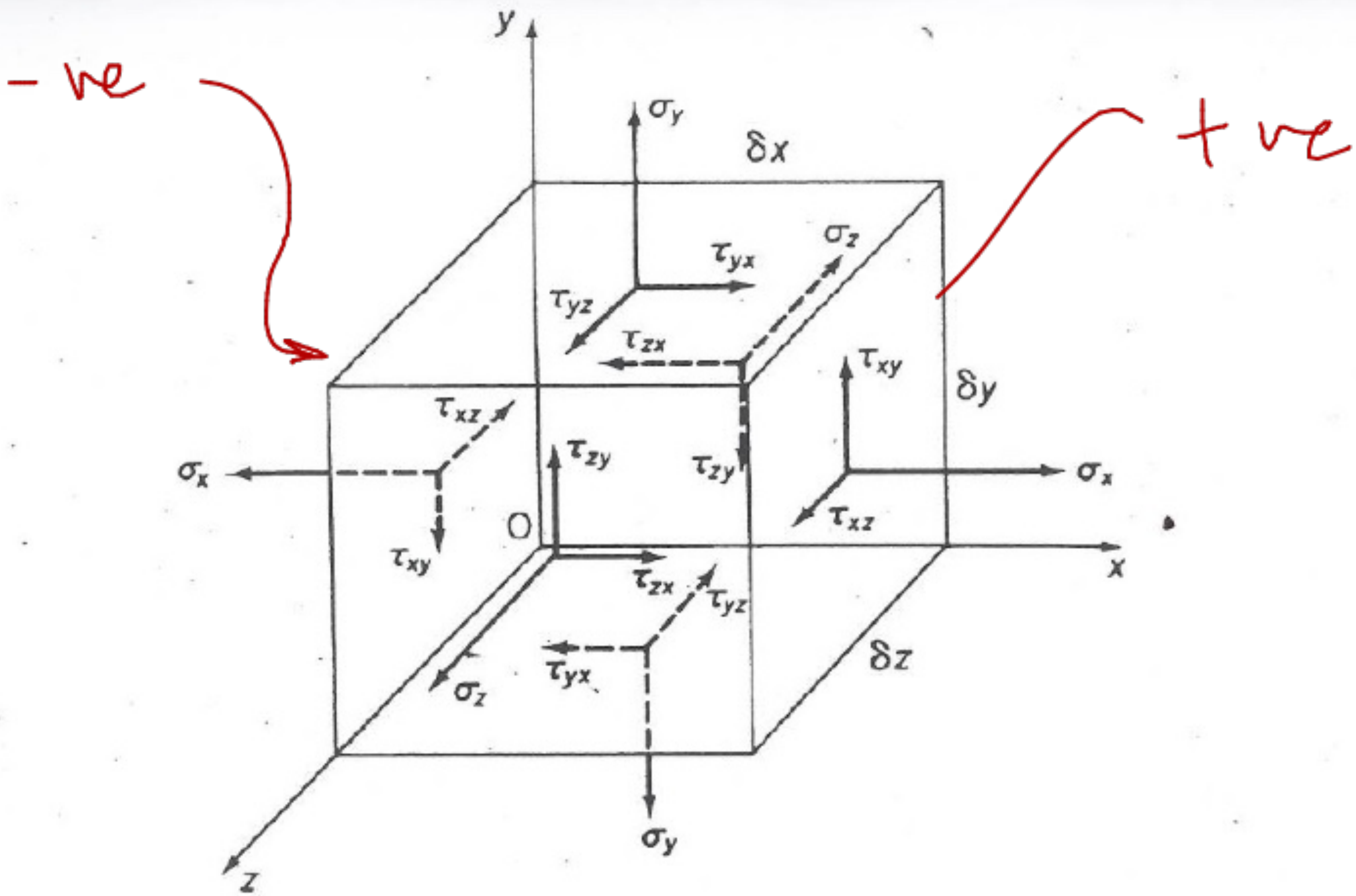
Shear force : +ve in the direction of +ve shear stresses

Bending moment : +ve in the direction of the moment caused by +ve direct stresses acting in the upper part of the beam

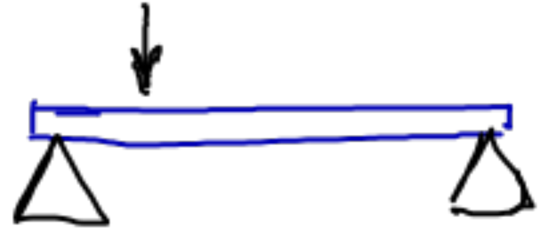


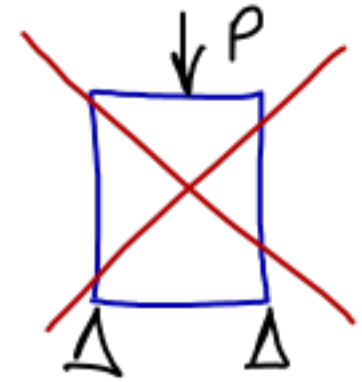
Applied loads :

+ve if acting in +ve y direction



<u>Sign convention</u>: The normal stresses directed away from their related surfaces are tensile and positive, and likewise compressive stresses are negative. Shear stresses are positive when they act in the positive direction of the relevant axis in a plane on which the direct tensile stress is in the positive direction of the axis. All the stresses in figure 6 are positive.





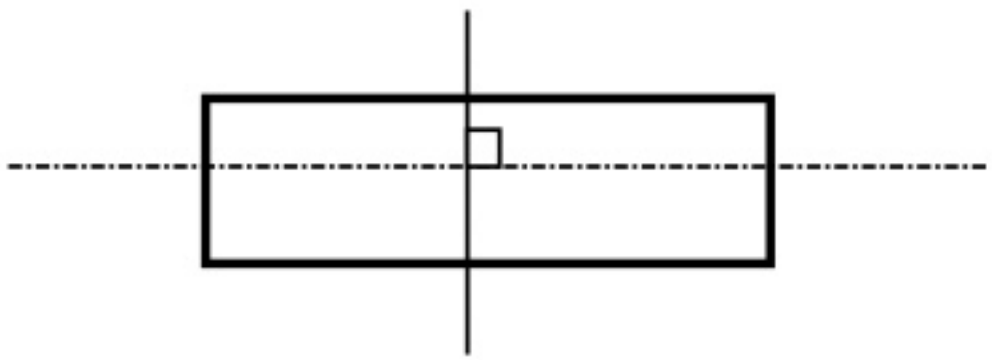
1.3 Engineers' Theory of Bending

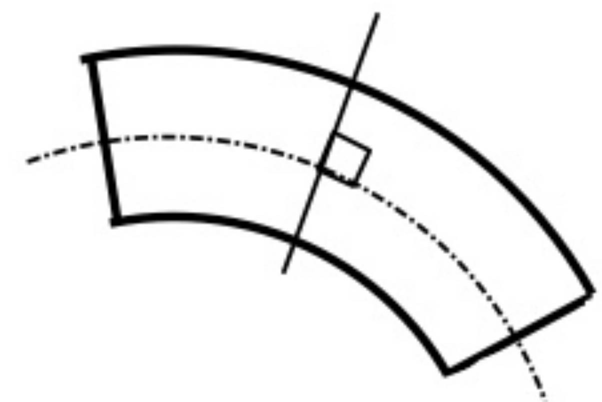
1.3.1 Assumptions

i) The beam is slender i.e. lateral dimensions of the beam are small compared to its length.

ii) Plane sections normal to the longitudinal axis of the beam remain plane and normal to the axis after bending.

i.e.





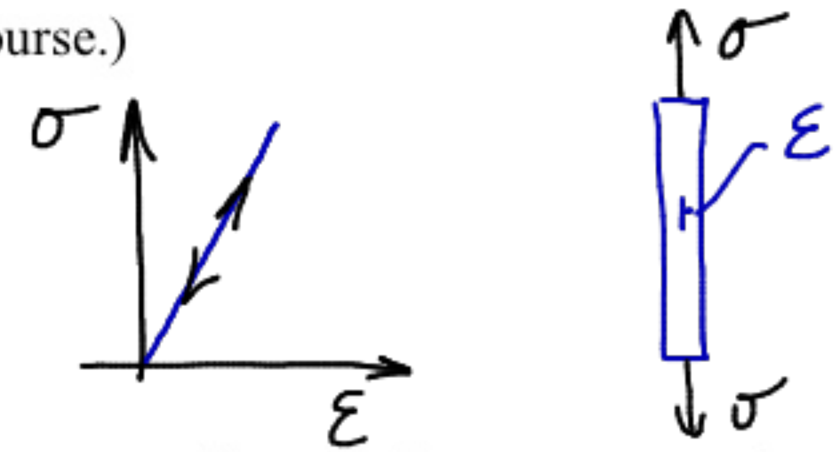
before bending

after bending

Note : this implies that the deformation due to shear is neglected.

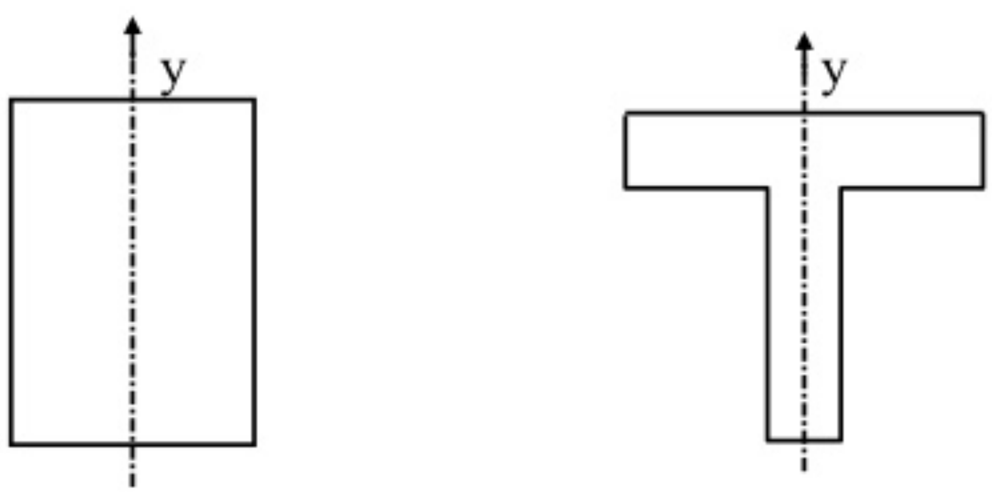
We will show later that for slender beams the deformation due to bending is normally much greater than that due to shear. (We will also not concern ourselves here with the distribution of shear stresses - this will be dealt with later in the 2nd year structures course.)

iii) The beam material is linear elastic

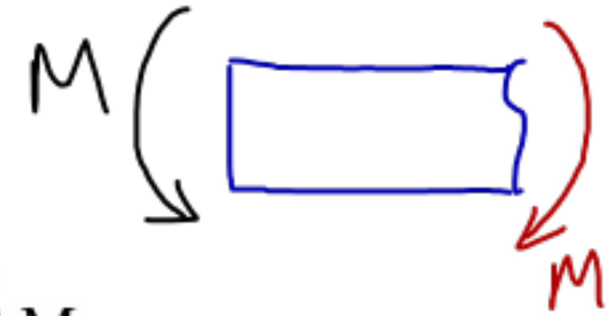


iv) In this course we will restrict ourselves to cases where the beam cross section is symmetric about the y-axis .

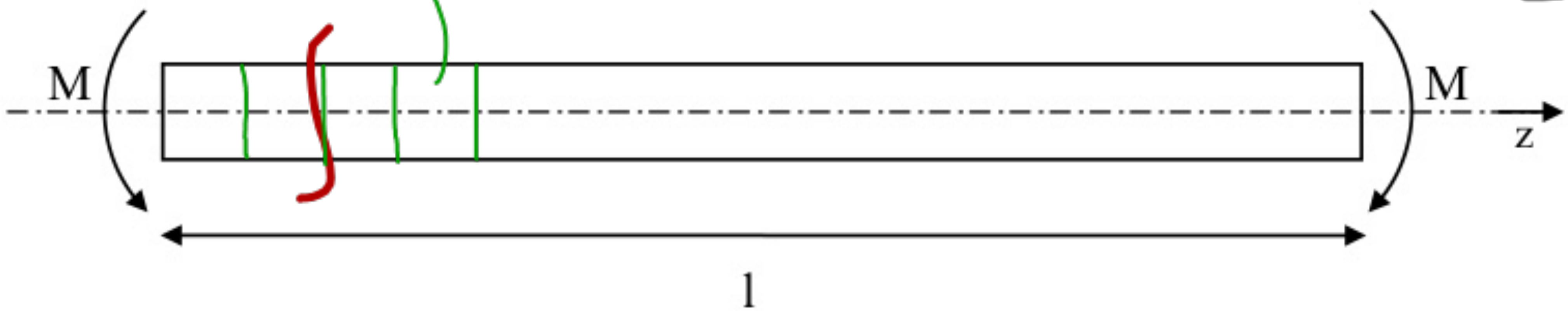
e.g.



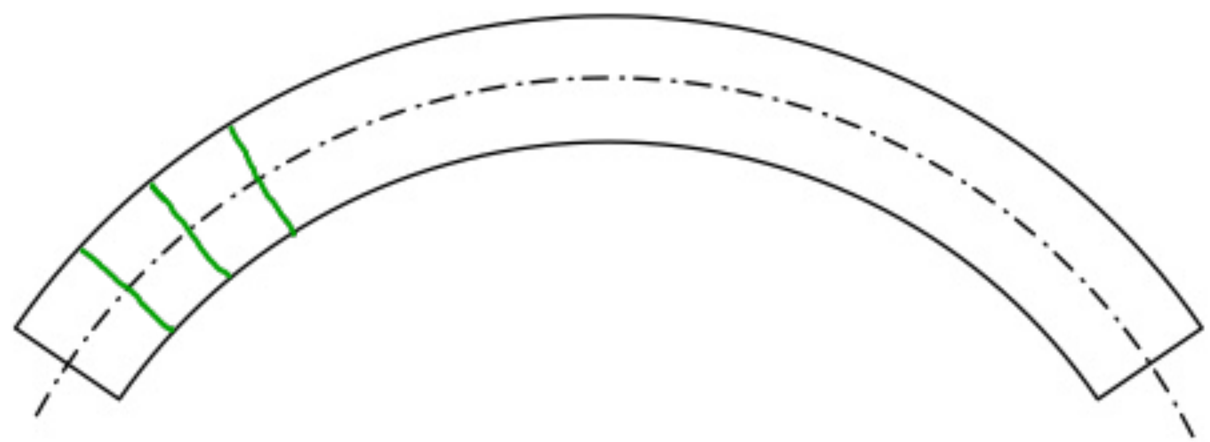
(Non-symmetric sections will be examined in the 2nd year.)



Consider a beam with a constant bending moment M acting over a length l.



Deformed shape

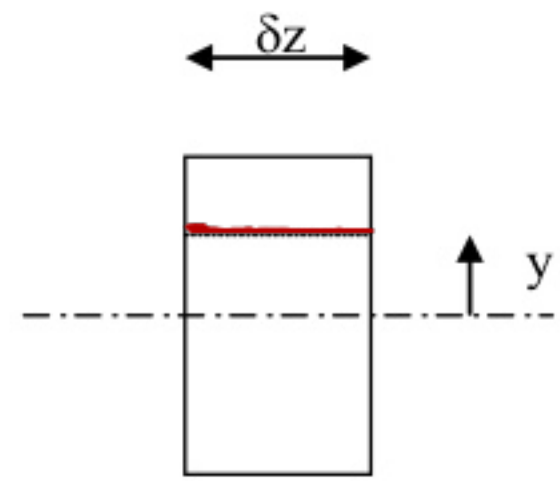


If we subdivide this length into n equal lengths, each of these short lengths of beam will be subjected to a constant bending moment M and so will have the same deformed shape – regardless of where within the original length ‘l’ the small length was taken. To satisfy this condition the beam must have deformed into an arc of a circle.

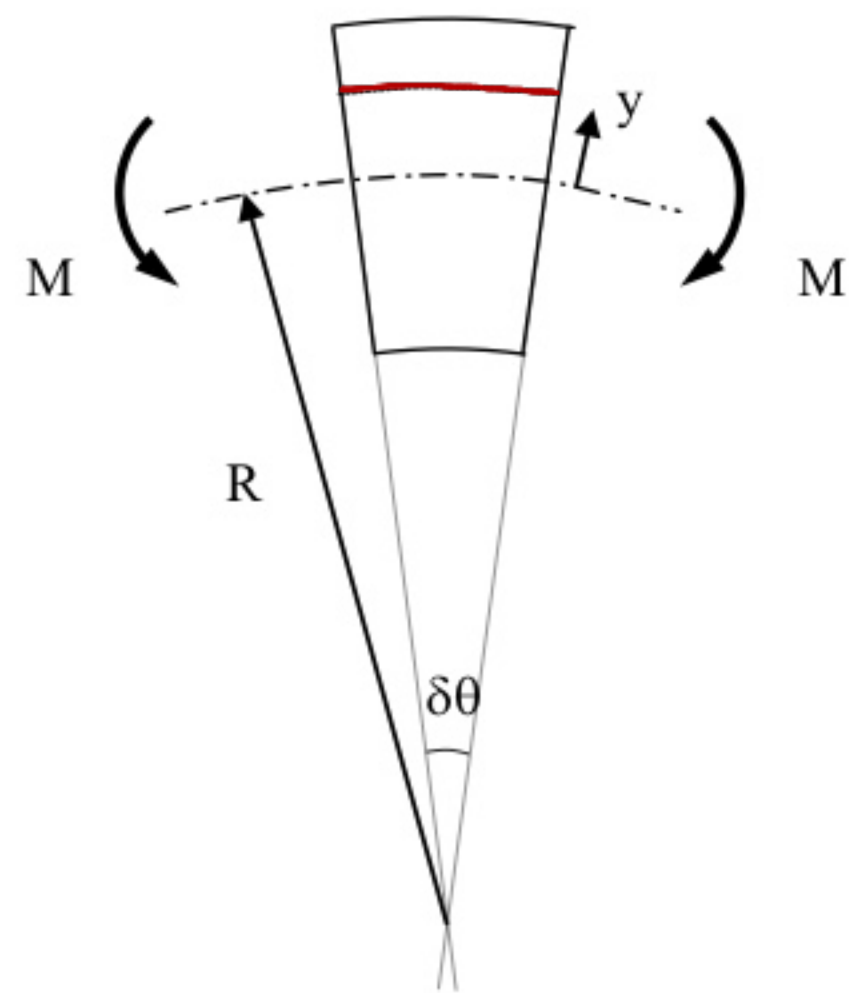
In a real beam the bending moment will not, in general, be constant along the beam. However over a sufficiently small length of the beam the bending moment can be considered constant.

Consider a small length δz in the initially straight beam

before bending:

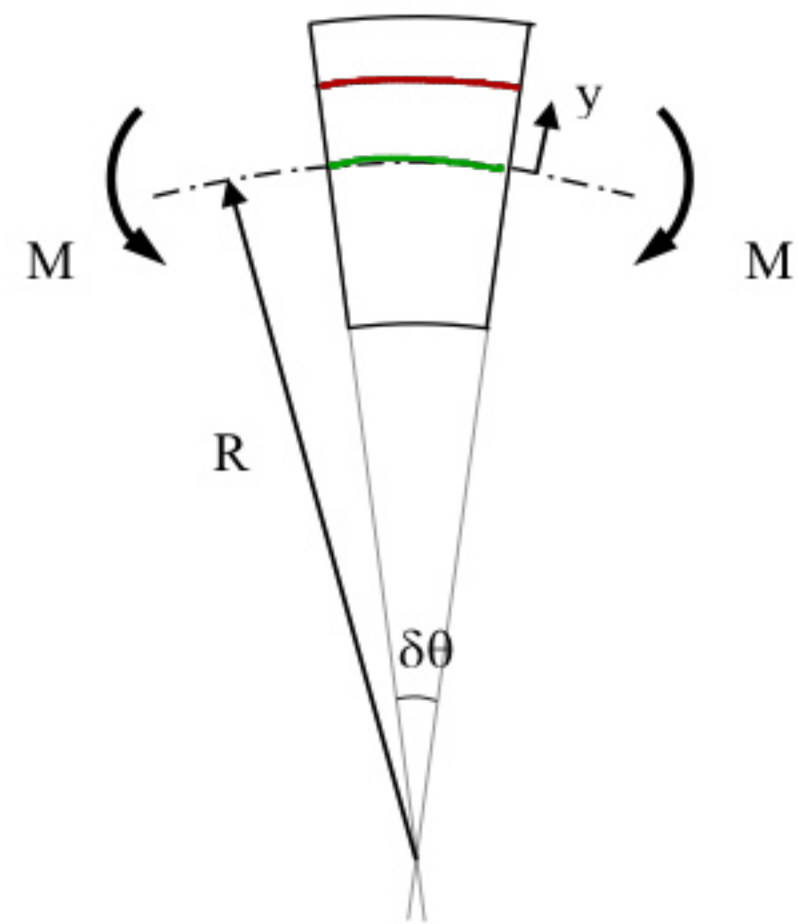


and after bending:



Assuming the longitudinal axis has deformed to an arc of radius R and that along this axis the beam has not been extended or compressed due to the applied load then this short element subtends an angle

$$\delta\theta = \frac{\delta z}{R}$$



[Note 1/R is called the curvature; it is positive in the direction of positive moment].

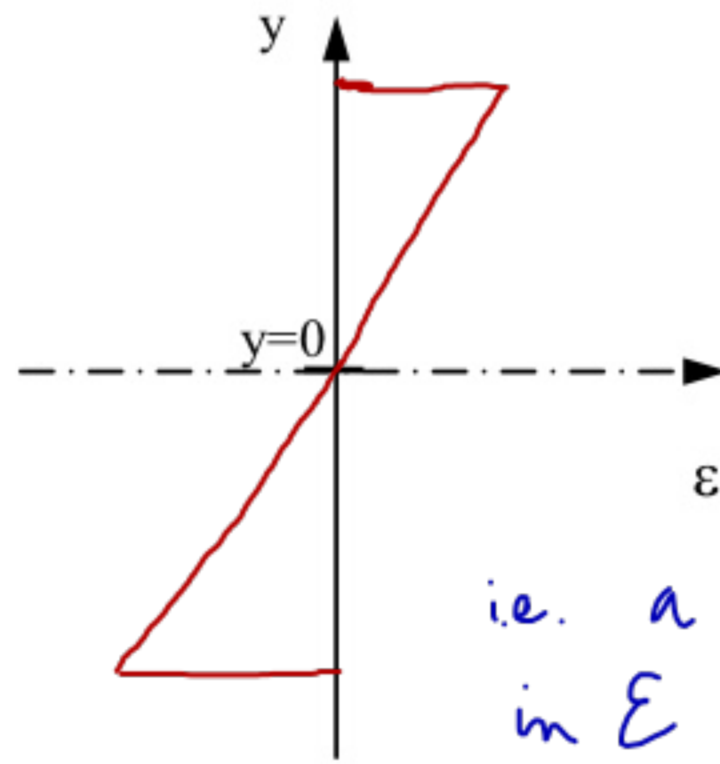
At some position y above the longitudinal axis the length of the arc in the deformed beam element is given by

$$(R+y)\cdot\delta\theta = (R+y)\frac{\delta z}{R}$$

And so the longitudinal direct strain, ε, at this position y above the longitudinal axis is given by

$$\varepsilon = \frac{(R+y)\cdot\frac{\delta z}{R} - \delta z}{\delta z} = \frac{y}{R}$$

strain distribution:



i.e. a linear variation
in ε due to bending

1.3.3 Stress distribution and its relationship to the bending moment

The relationship between strains and stresses comes from Hooke's law for a linear elastic material

i.e. $$\varepsilon = \frac{\sigma}{E} \quad \text{or} \quad \sigma = E\varepsilon$$

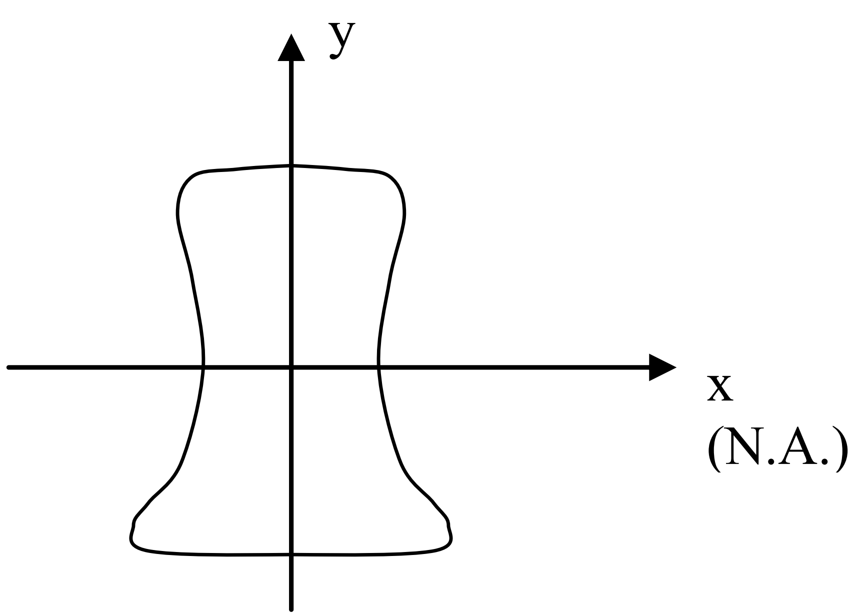
We have already shown that due to bending

$$\varepsilon = \frac{y}{R}$$

and so $$\sigma = \frac{Ey}{R}$$

i.e. linear variation in direct stress due to bending.

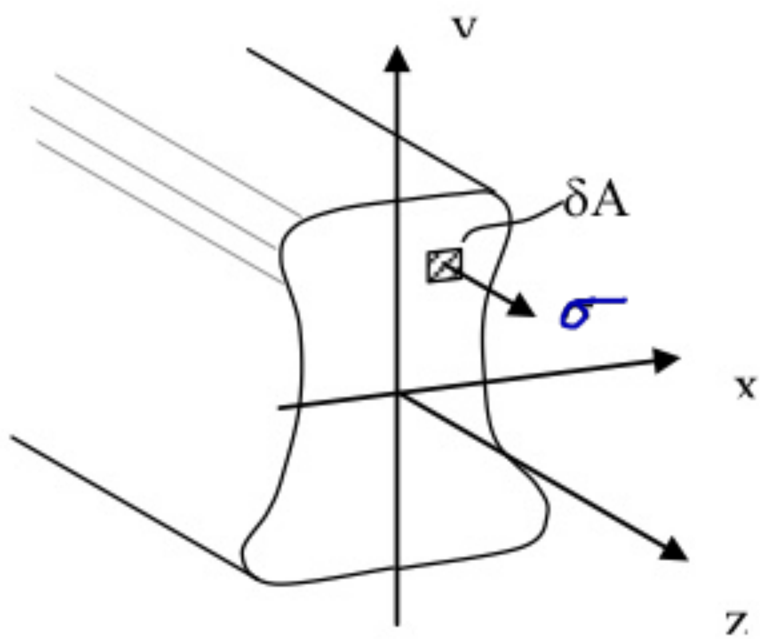
Note that at $y = 0$, $\sigma = 0$ so there is a line, the x-axis, across the beam cross section on which the stresses due to bending are zero. This line is called the Neutral Axis (N.A).



Now, recall that the direct stress due to bending must have

i) zero longitudinal (i.e. z-direction) resultant force

ii) a moment resultant (about the x-axis) equal to the bending moment, M, at that section so as to be in equilibrium with the loading on the beam.



Consider a small area δA on the cross section. The z-direction resultant force acting on this element due to direct stress σ is σδA and so total z-direction resultant for whole section is given by

$$\int_{\text{section}} \sigma \, dA$$

But for ETB we have shown

$$\sigma = \frac{Ey}{R}$$

and so for zero z-direction resultant we require

$$\int_{\text{section}} \frac{Ey}{R} \cdot dA = 0$$

Now E is the Young's modulus of the beam material and R is the radius of curvature of the longitudinal axis of the beam, neither of which vary with the position of our element δA. So the above equation becomes

$$\frac{E}{R}\int_{\text{section}} y\,dA = 0 \quad \text{i.e. we require} \quad \int_{\text{section}} y\,dA = 0$$

Now $\int_{\text{section}} y\,dA$ is the first moment of area of the section about the x-axis. You will recall you used this to evaluate the y co-ordinate of the centroid of a section of area A:

$$\bar{y}A = \int_{\text{section}} y\,dA$$

then $\bar{y}$ must be zero i.e. the line y=0 (the x-axis) must pass through the centroid of the cross section for the z-direction stress resultant to be zero.

Now considering the moment resultant about the x-axis we can see that due to the element δA there is a moment equal to

$$\sigma y \delta A$$

And so the moment resultant for the whole section is

$$\int_{\text{section}} y \sigma \, dA$$

and this must equal M, the bending moment at that section.

Substituting for σ as $\frac{Ey}{R}$ we have

$$\frac{E}{R} \int_{\text{section}} y^2 \, dA = M$$

Now we have met $\int_{\text{section}} y^2 dA$ in the Foundation Mechanics course. It is the second moment of area, I, of the cross section about the x-axis. Substituting $I = \int_{\text{section}} y^2 dA$ the equation for the moment resultant becomes

$$\frac{E}{R} \cdot I = M$$

Now we have already shown $\sigma = \dfrac{Ey}{R}$

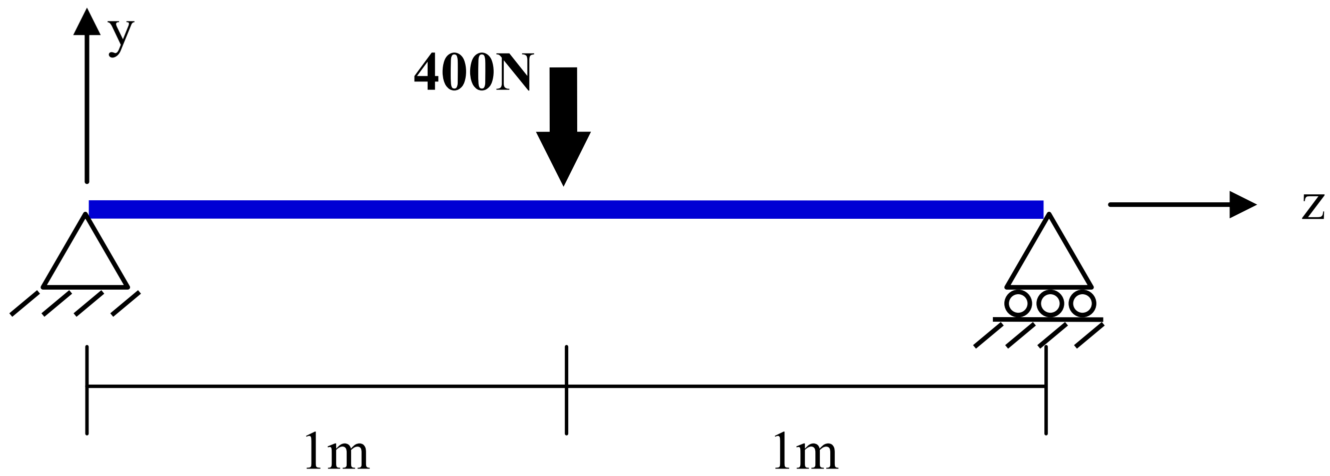
and so $$\sigma = \frac{M}{I} \cdot y$$

We can also write

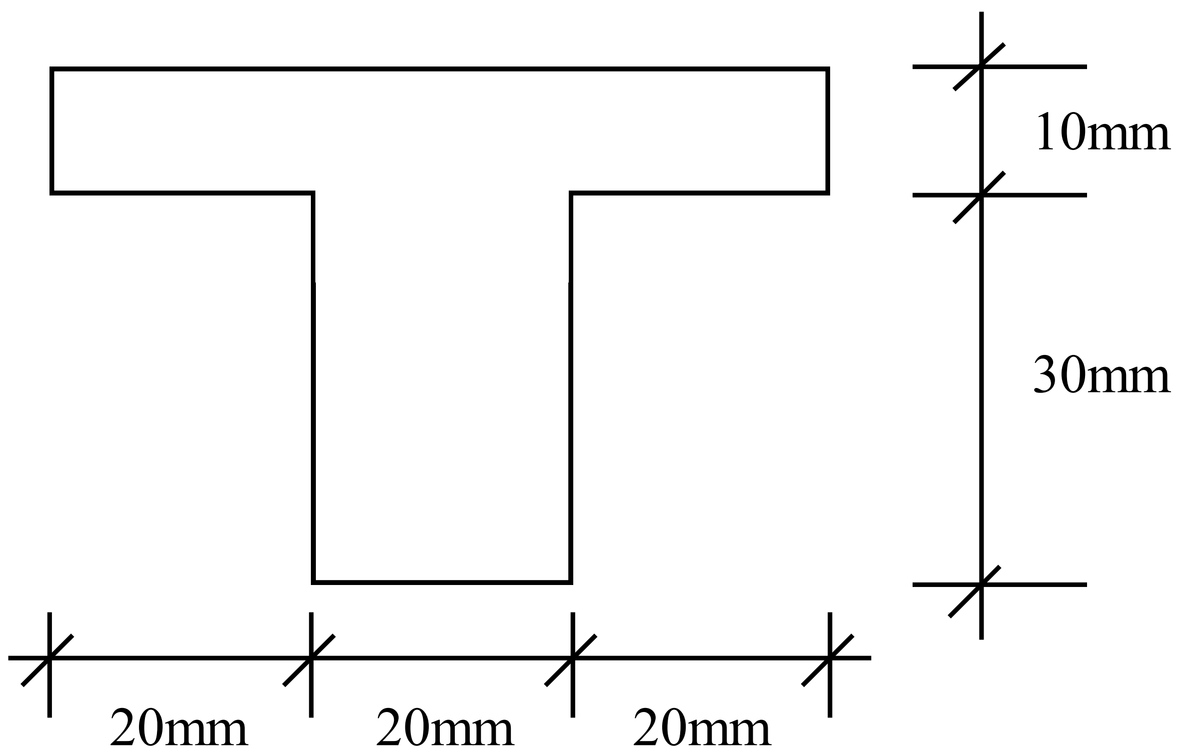
$$\frac{M}{I} = \frac{E}{R} = \frac{\sigma}{y}$$

(Probably the most commonly used structures equation after $\sigma = P/A$ for axially loaded bars.)

Example



Beam cross-section :



Evaluate the direct stresses due to bending at z = 1.5m

Outline of solution method:

i) calculate M

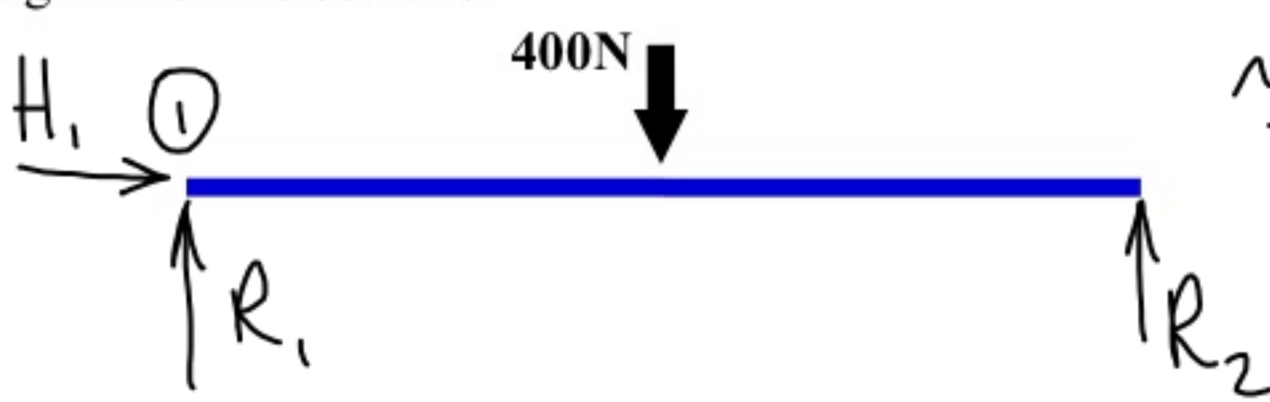
ii) determine the position of the centroid

iii) calculate I about the centroidal x-axis

iv) calculate σ using $\dfrac{My}{I}$

i) Calculate M

The free body diagram for the beam is

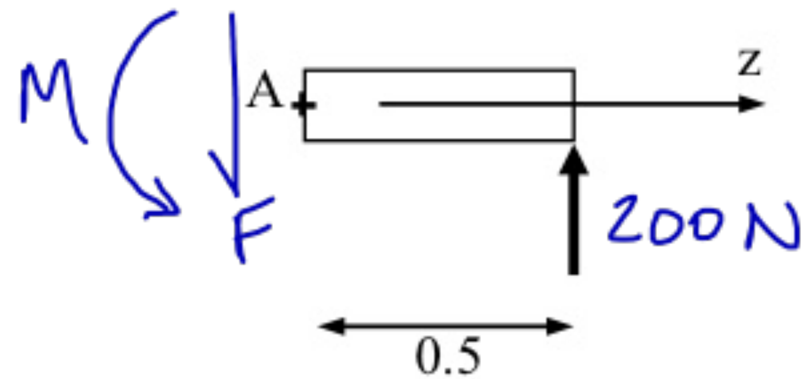


Horiz. equilib. $\Rightarrow H_1 = 0$

From symmetry & vertical equilib. $\Rightarrow R_1 = R_2 = 200\,N$

[Check can get same answer by moments equilib about 1 or 2]

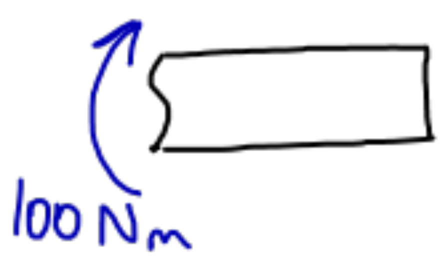
We require the bending moment, M, at z = 1.5m, so cut the beam at this position and draw the free body diagram for, say, the right hand part. Show positive M & F (bending moment and shear force) acting on the cut face.



Writing an equation for moment equilibrium about point A on the cut face gives

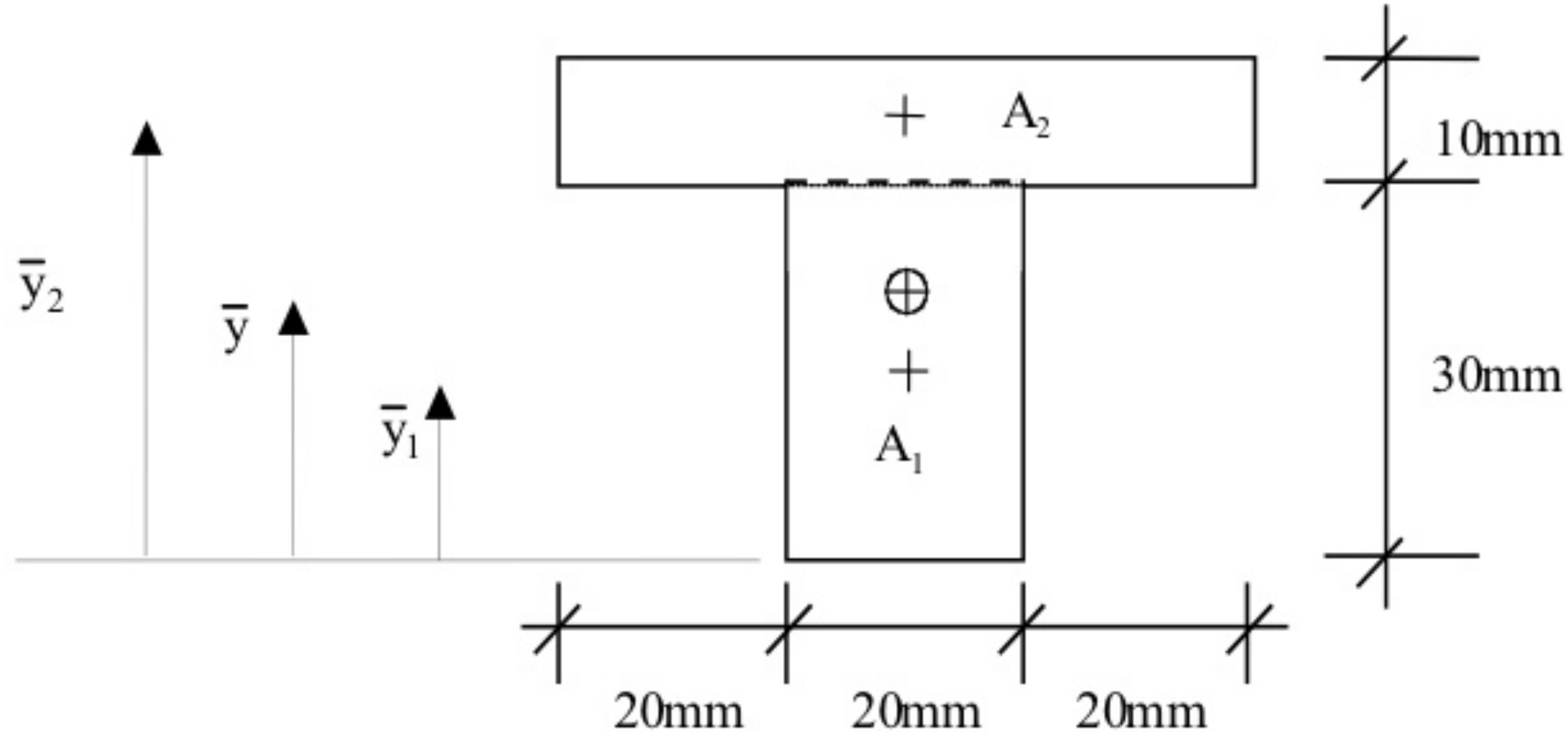
+ve $200 \times 0.5 + M = 0$

Therefore M = -100 Nm



(Check that you get the same answer if you use the free body diagram for the left hand part.)

ii) Determine the position of the centroid (see Appendix 1)



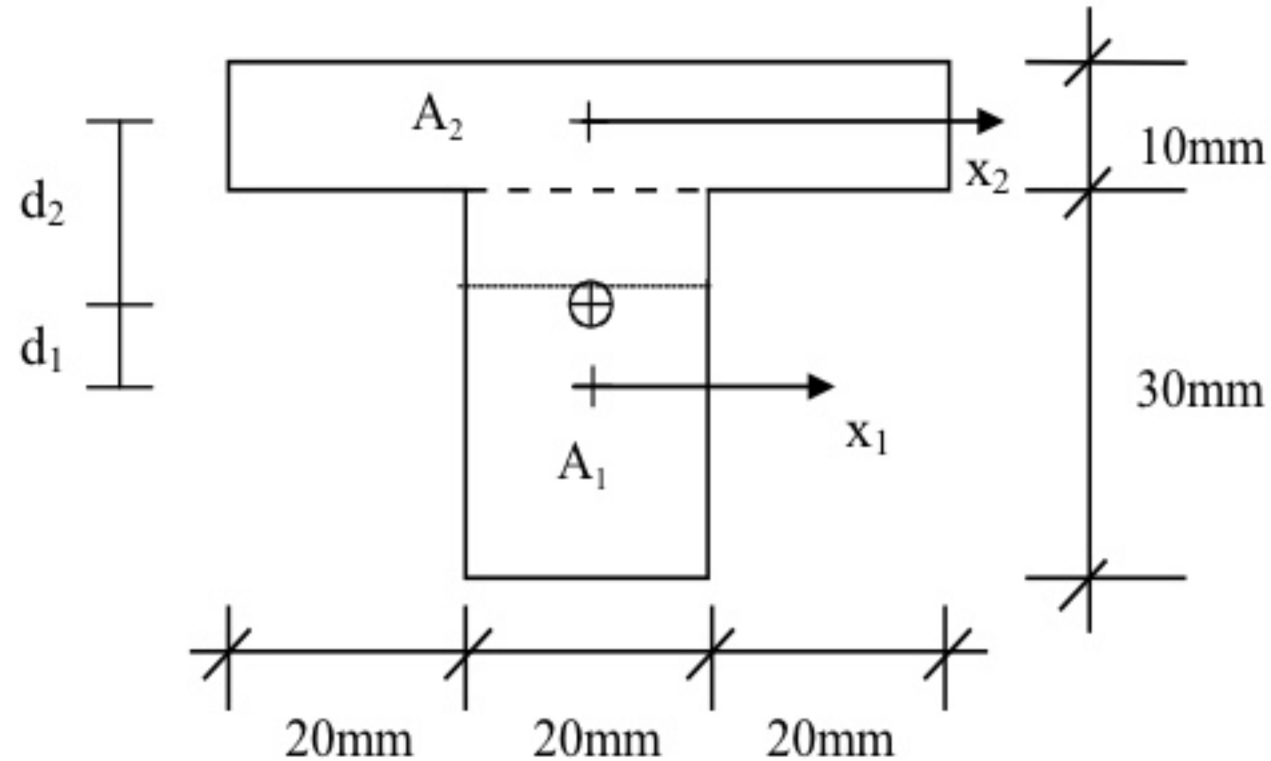
$$\bar{y}A = \int_{\text{section}} y\,dA$$

$$= \bar{y}_1 A_1 + \bar{y}_2 A_2$$

Therefore $\bar{y}(30\times 20 + 10\times 60) = 15(30\times 20) + 35(10\times 60)$

which gives $\bar{y} = 25$ mm

iii) Calculate I about the centroidal x-axis (see Appendices 2 &3)



$$I_x = \int_{\text{section}} y^2 dA$$

$$= I_{x_1}\Big|_{A_1} + d_1^2 A_1 + I_{x_2}\Big|_{A_2} + d_2^2 A_2$$

$$= \frac{30^3 \times 20}{12} + (25-15)^2 (30 \times 20) + \frac{10^3 \times 60}{12} + (35-25)^2 \times 60 \times 10$$

$$= \underline{170\,000 \text{ mm}^4}$$

iv) Calculate σ

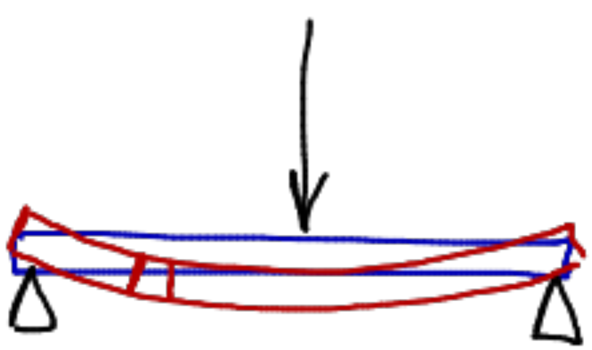
Use $\sigma = \dfrac{My}{I}$.   At z = 1.5 m we have shown M = -100N-m

At the upper edge of the section ($y = y_{max} = 15$mm)

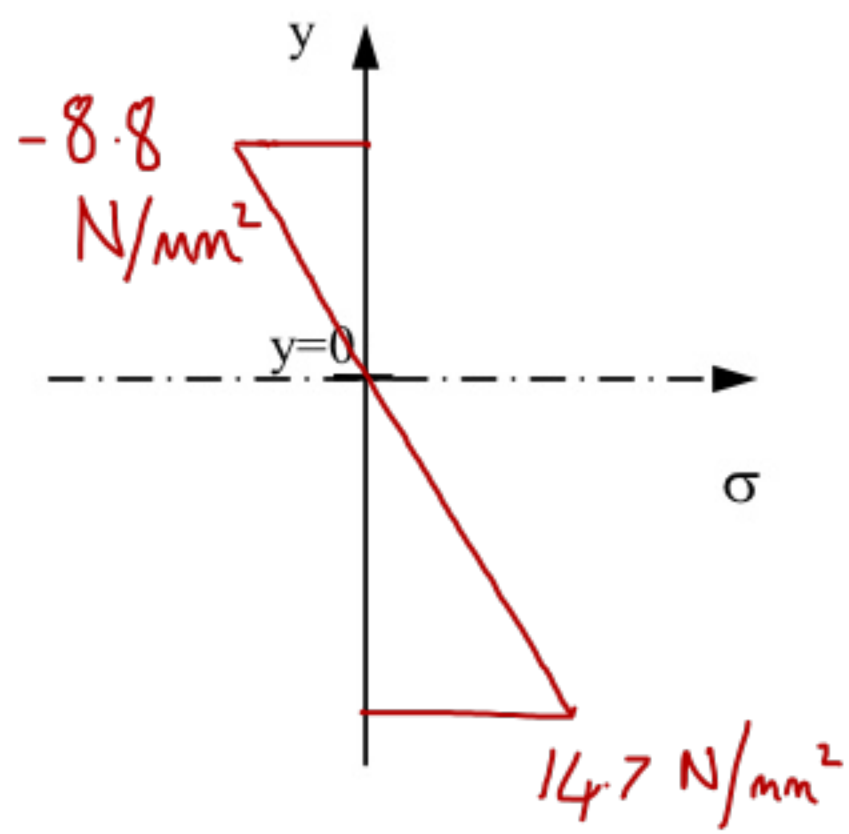
$$\sigma = \frac{-100 \times 10^3 \times 15}{170\,000} = \underline{-8.8 \text{ N/mm}^2}$$

At the lowest edge of the section ($y = y_{min} = -25$mm)

$$\sigma = \frac{-100 \times 10^3 \times (-25)}{170\,000} = \underline{14.7 \text{ N/mm}^2}$$



Distribution of the direct stress due to bending over the depth of the cross section:



1.3.4 The significance of I

Recall

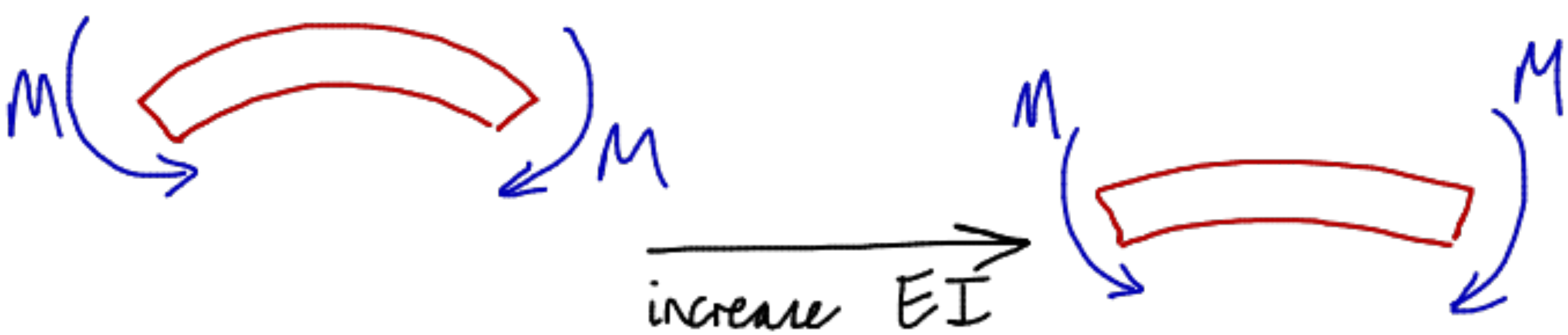
$$\frac{\mathbf{M}}{\mathbf{I}} = \frac{\mathbf{E}}{\mathbf{R}} = \frac{\sigma}{\mathbf{y}}$$

Considering first the deformation of the beam we have

$$1/R = \frac{M}{EI}$$

flexural stiffness

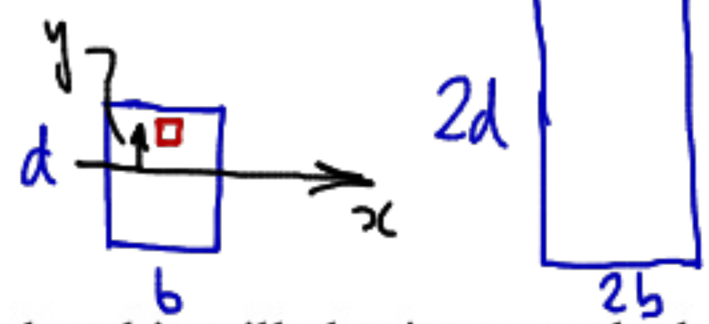
So increasing EI gives a greater radius of curvature R, i.e the beam becomes stiffer.



E, the Young's modulus, is the material property which contributes to the flexural stiffness.

I is the geometric property which contributes to the flexural stiffness.

How do we increase I?



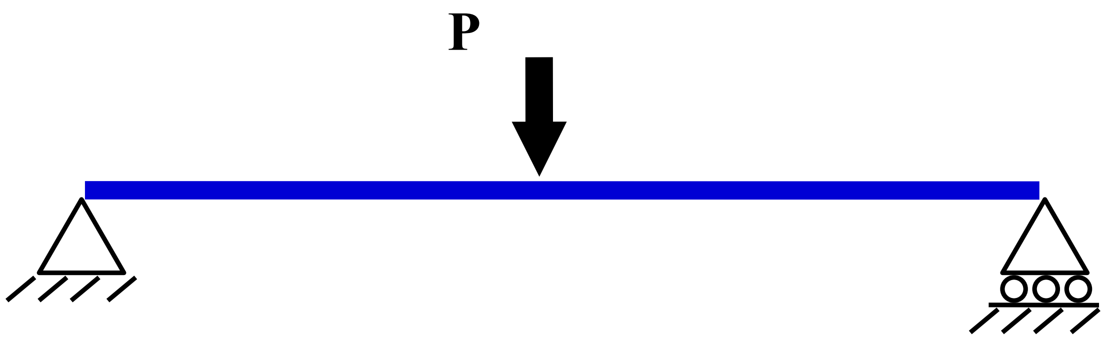
Clearly we could simply scale up the cross-section but this will also increase the beam weight – an important consideration, especially for aerospace structures. If we do not want to increase the weight we will need to increase I without increasing the cross-sectional area.

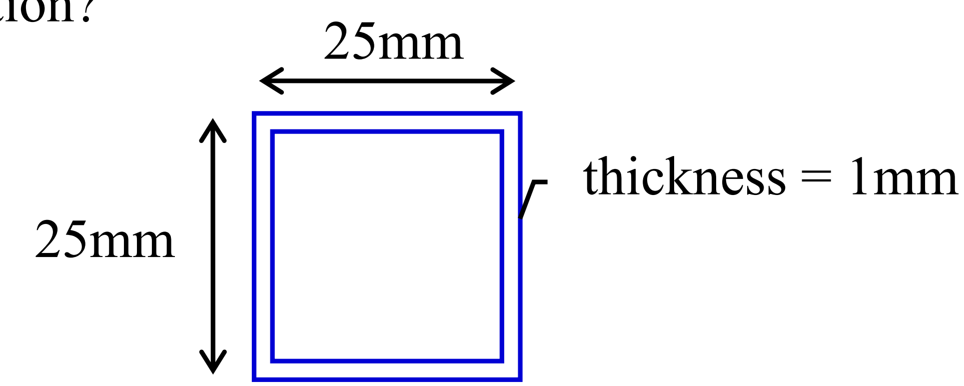
Recalling $$I = \int_{\text{section}} y^2 dA$$

then we can see that I can be increased by redistributing the cross sectional area further from the x-axis.

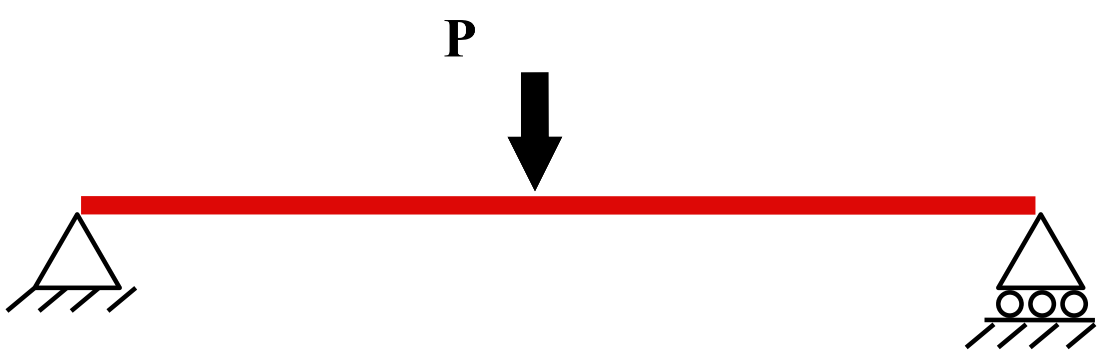
An example of the sigificance of I

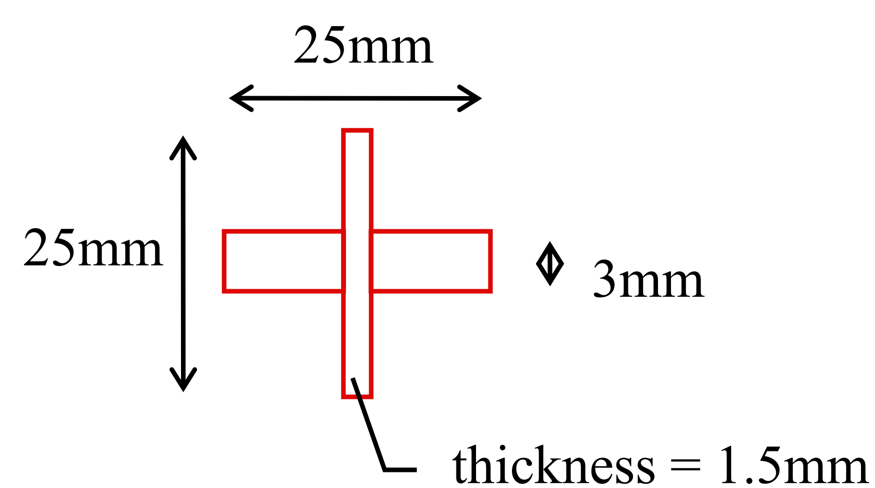
Consider the two beams shown below, both of which have the same span and are subjected to the same centre-point load. Which beam will have the greatest deflection?





beam cross section (area = $96mm^2$)

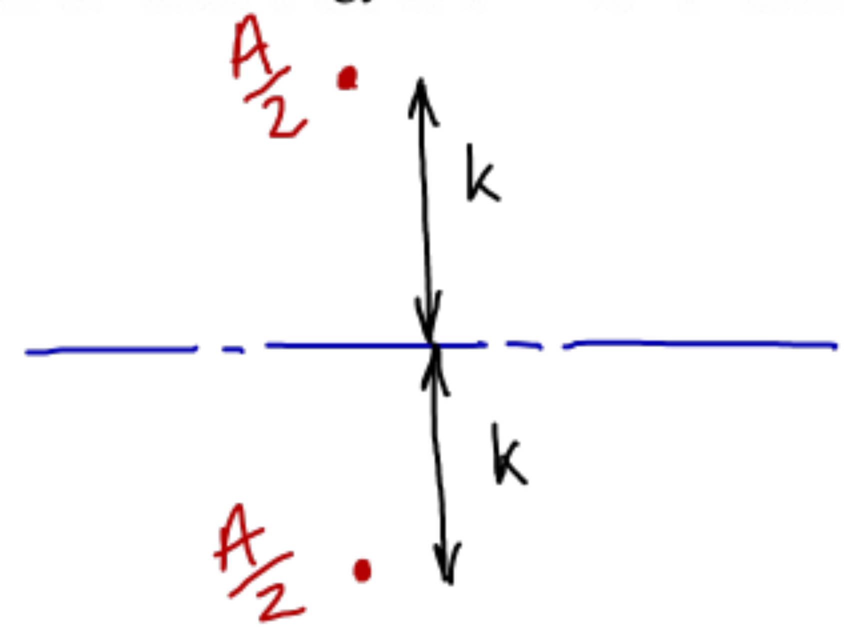




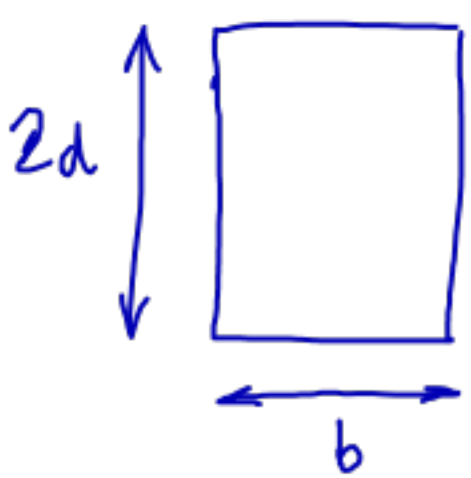
beam cross section (area = $108mm^2$)

A measure of the efficiency of a section shape is the 'radius of gyration' k which is defined by

$$I = Ak^2$$



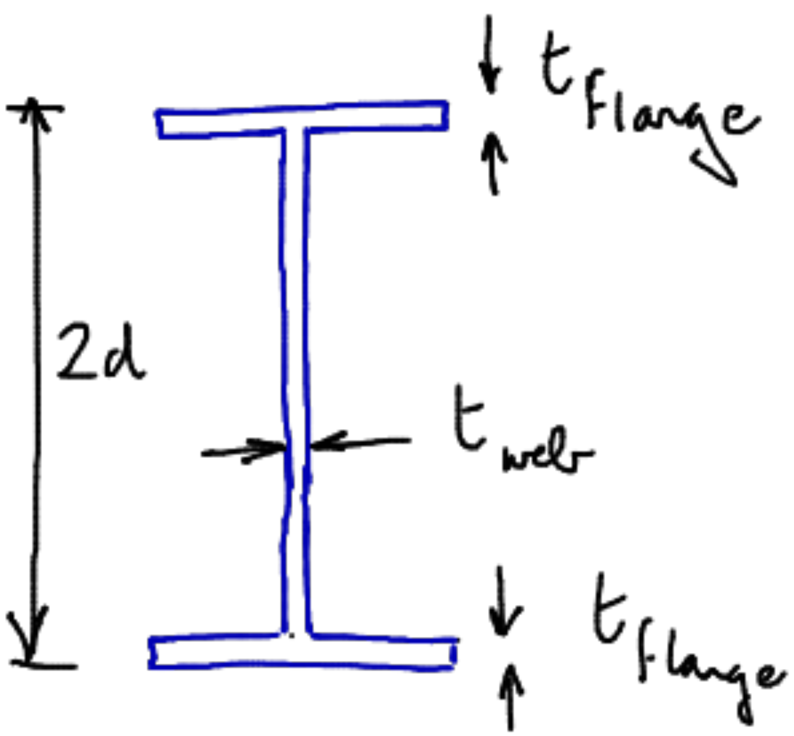
For a rectangular section



$$I = \frac{(2d)^3 \cdot b}{12}, \quad A = 2db$$

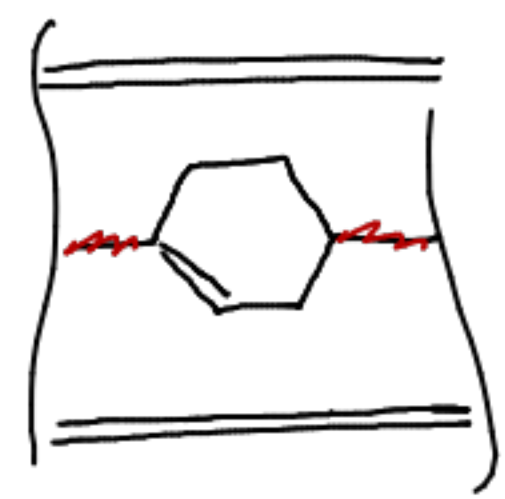
$$\therefore k = \sqrt{\frac{(2d)^3 b}{12 \cdot 2d \cdot b}} = \frac{d}{\sqrt{3}} = \underline{0.58\,d}$$

For an 'I' – beam



if $t_{flange} \gg t_{web}$ & $2d \gg t_{flange}$

then $\underline{k \simeq d}$



So far we've only considered the effect of I on the stiffness of the beam. The peak stresses due to bendng are given by

$$\sigma = \frac{M y_{max}}{I} \quad \& \quad \sigma = \frac{M}{I} y_{min}$$

So for a given y_{max} , y_{min} then to reduce the direct stresses we should increase I.

1.4 Determining Shear Force and Bending Moment Diagrams for Statically Determinate Beams

These diagrams indicate the value of the shear force and bending moment at any point along the beam. We will discuss two approaches to determining these diagrams; one uses the method of sections and the other uses the differential relationship between loading, shear force and bending moment.

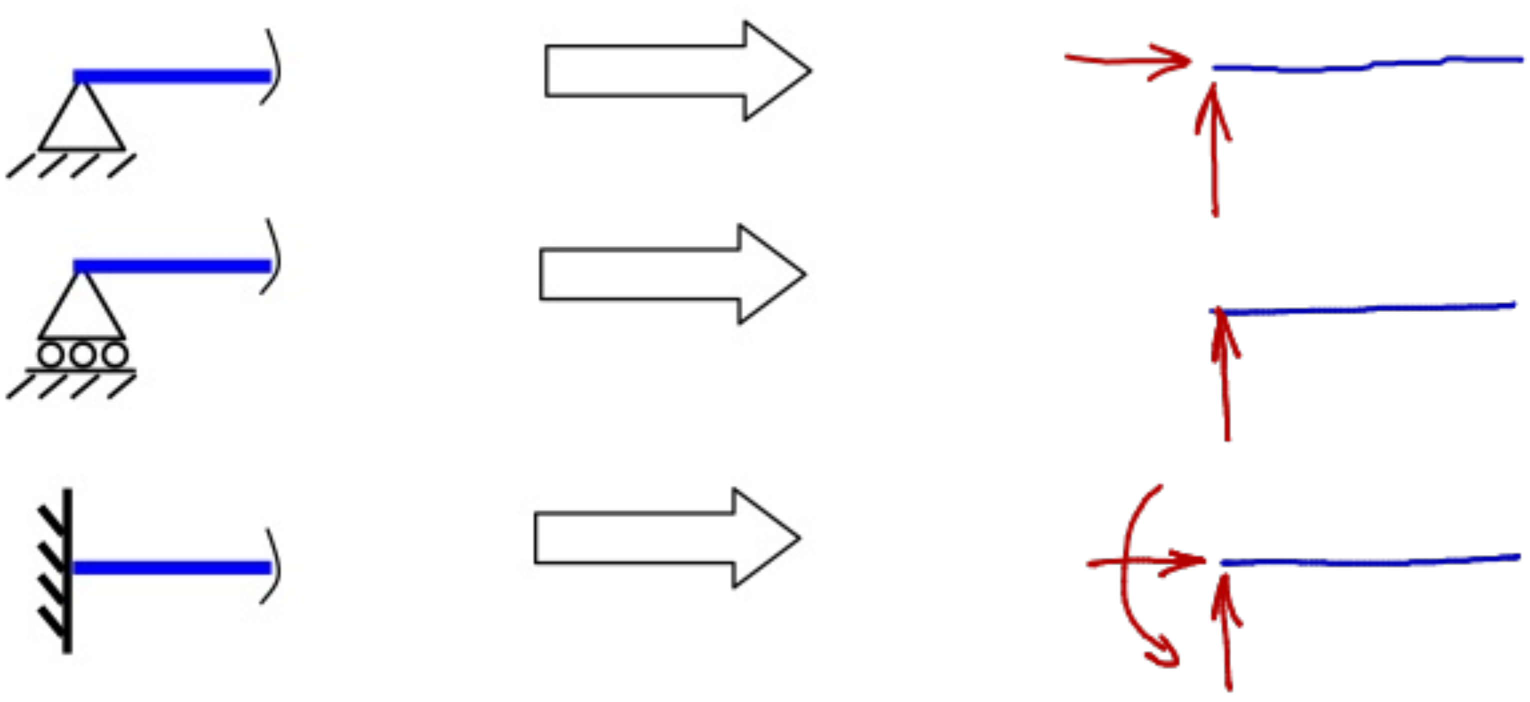
1.4.1 Method of Sections

We have already met this method in the examples we have considered so far. The steps we perform are:

i) Draw the free body diagram of the whole beam. To do this you will need to replace the supports with reaction forces and/or moments as appropriate.

E.g.

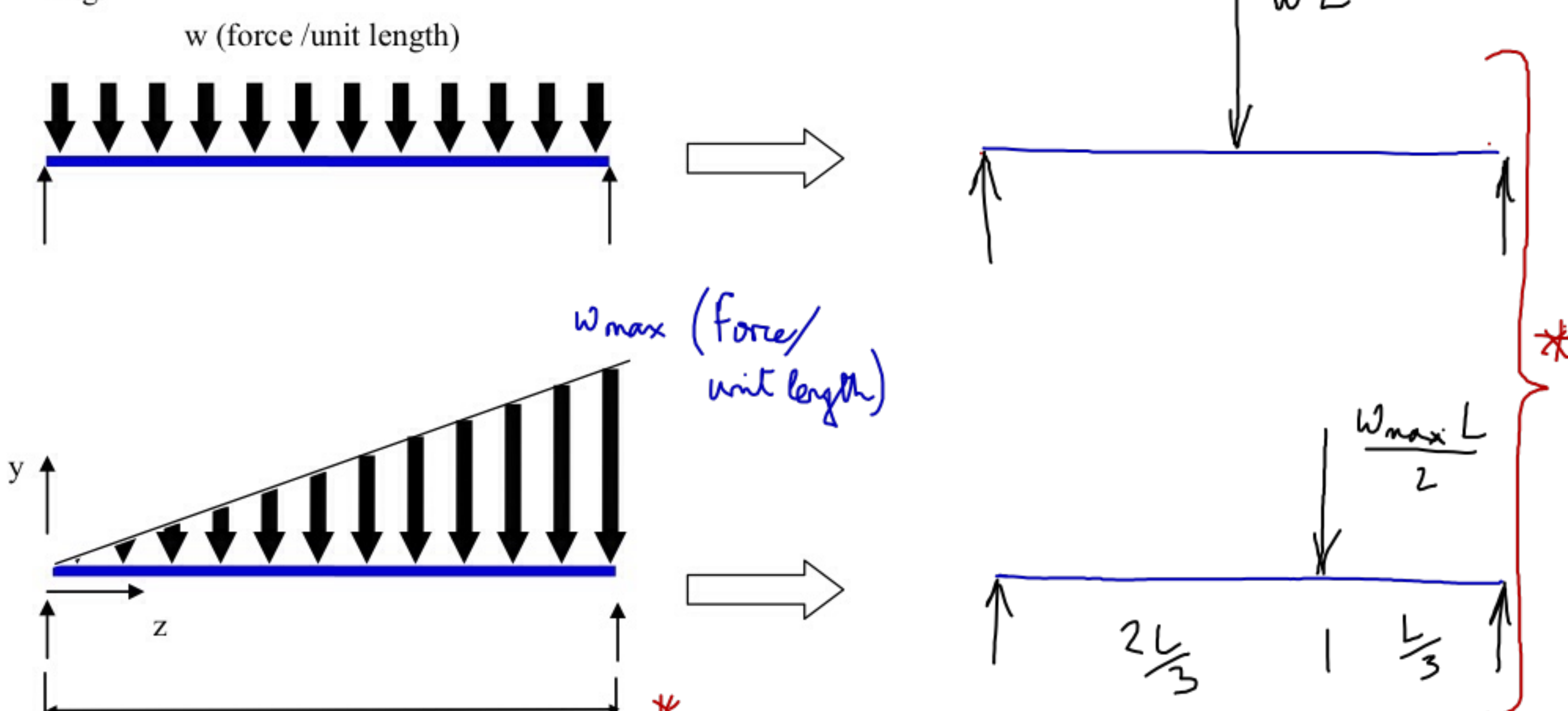
Support detail



ii) Write equations of equilibrium to determine the reactions.

It may be convenient to replace any distributed load by its resultant force acting at the appropriate position.

E.g.

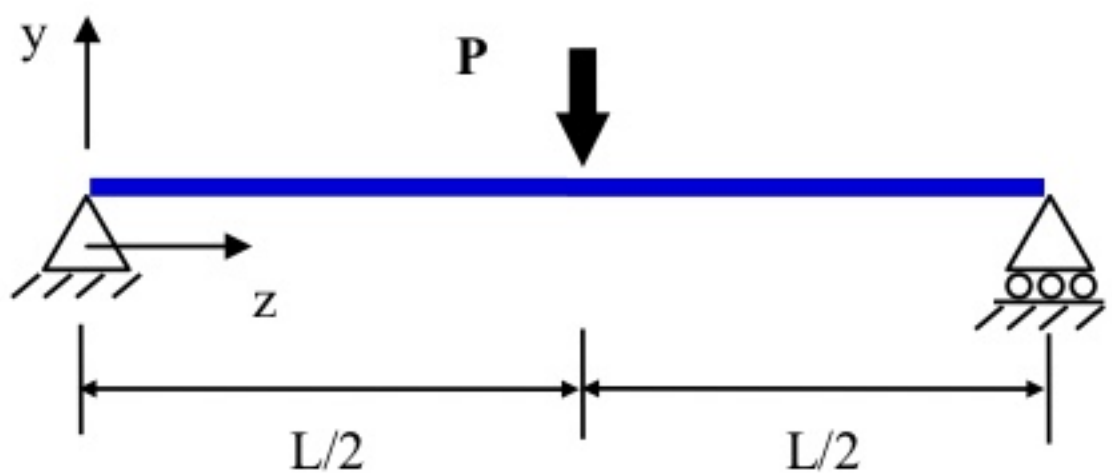


* ONLY FOR OVERALL EQUILIBRIUM

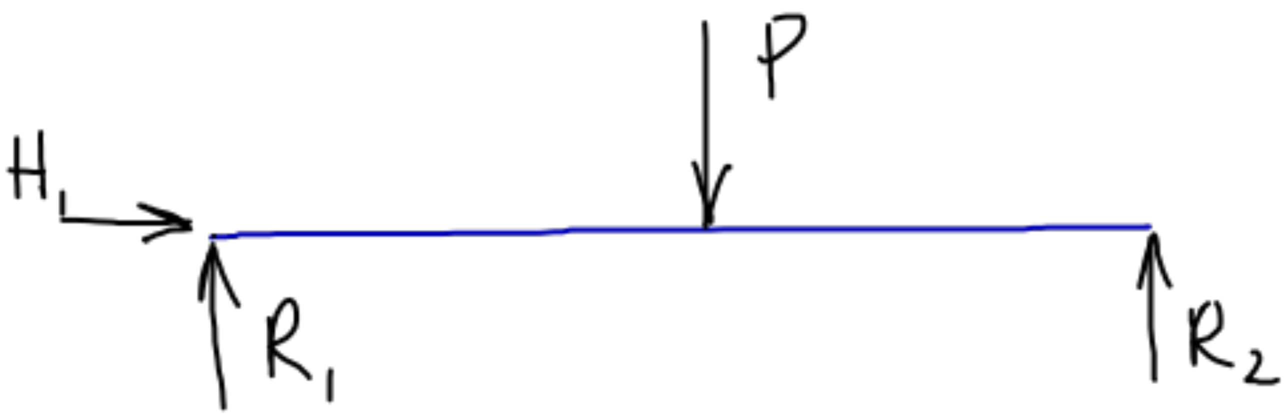
iii) Cut the beam at the position where the stress resultants are required and mark on the positive bending moment, M, and shear force, F, acting on the cut faces.

iv) Choose whichever part is most convenient and write equations of equilibrium to evaluate M & F. Again, it may be convenient to replace any distributed load acting on this part by its resultant force acting at the correct position.

Example I



free body diagram (fbd)

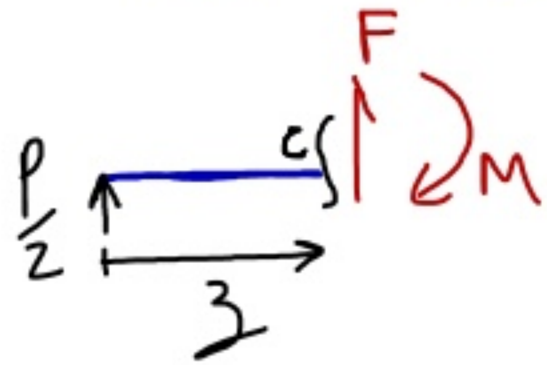


Horiz. eq. $\Rightarrow H_1 = 0$

Moments eq. $\Rightarrow R_1 = R_2 = \frac{P}{2}$

Now cut the beam at some position z & draw fbd for the left part. Then mark +ve M & F on the cut face. Two cases must be considered

<u>$0 < z < L/2$</u>



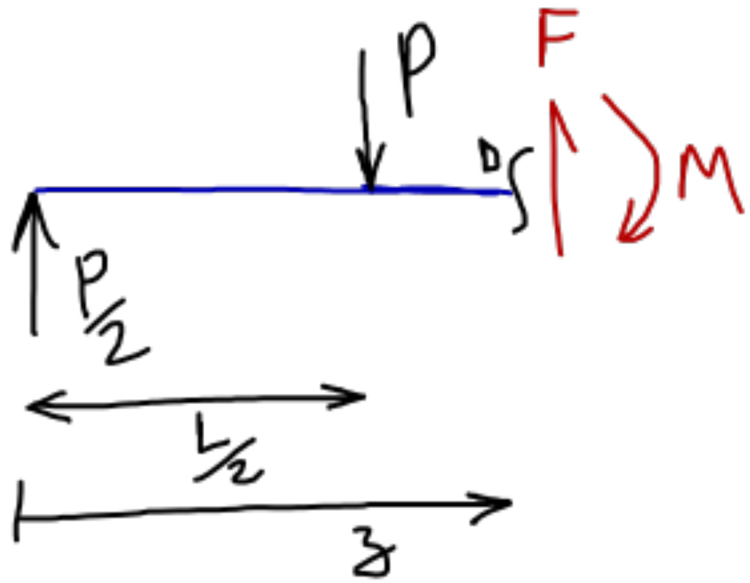
Vert. eq. $\Rightarrow \frac{P}{2} + F = 0 \implies F = \frac{-P}{2}$

Moments eq. about c +ve

$$\frac{P}{2} \cdot z + M = 0$$

$$\therefore M = \frac{-P}{2} z$$

$\frac{L}{2} < z < L$



Vert. eq. $\Rightarrow$ $\frac{P}{2} - P + F = 0$ $\therefore$ $\underline{F = \frac{P}{2}}$

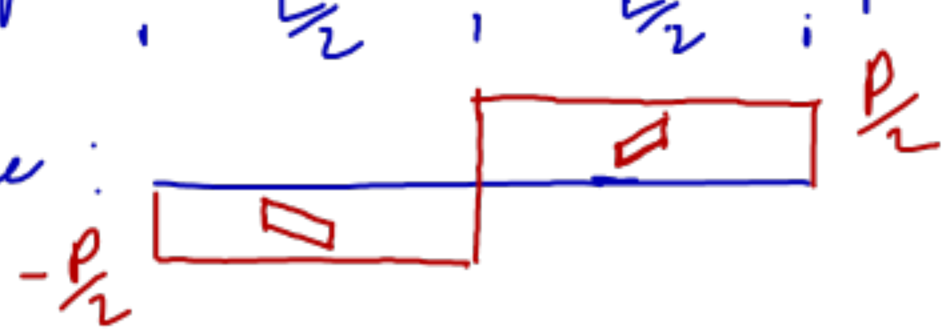
Moments eq about D +ve

$$\frac{P}{2} z - P.\left(z - \frac{L}{2}\right) + M = 0 \quad \therefore \quad \underline{M = \frac{P}{2}\left(z - L\right)}$$

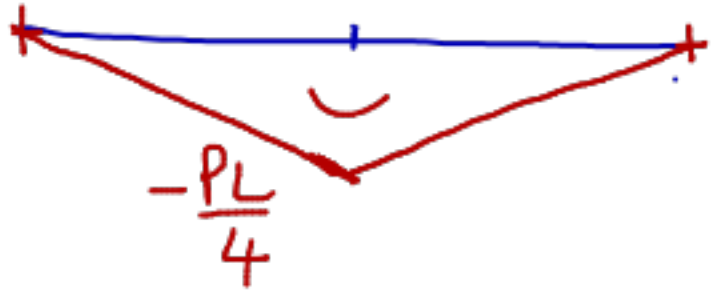
((Check using right hand fbds))

These equations can be used to plot the shear force & bending moment diagrams

Shear force :

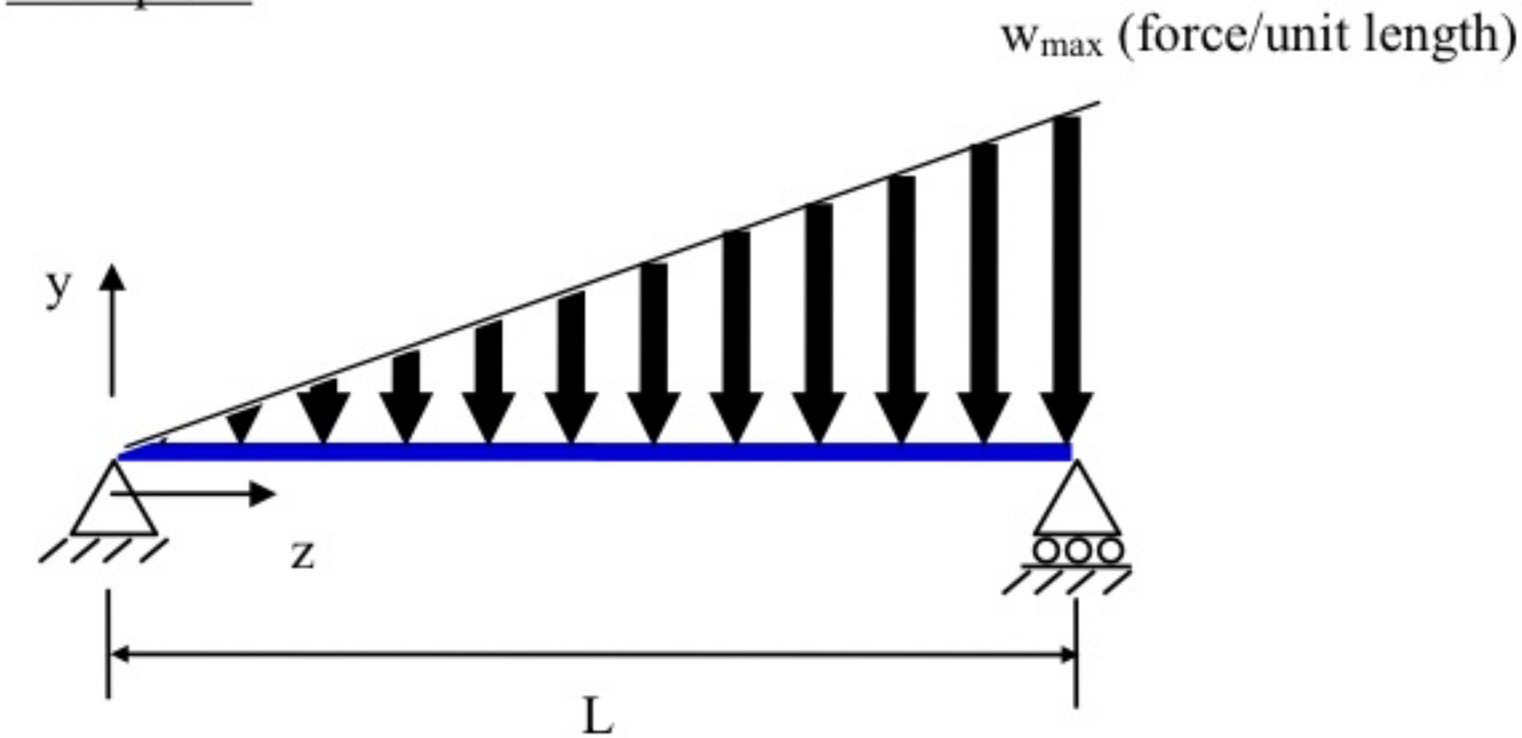


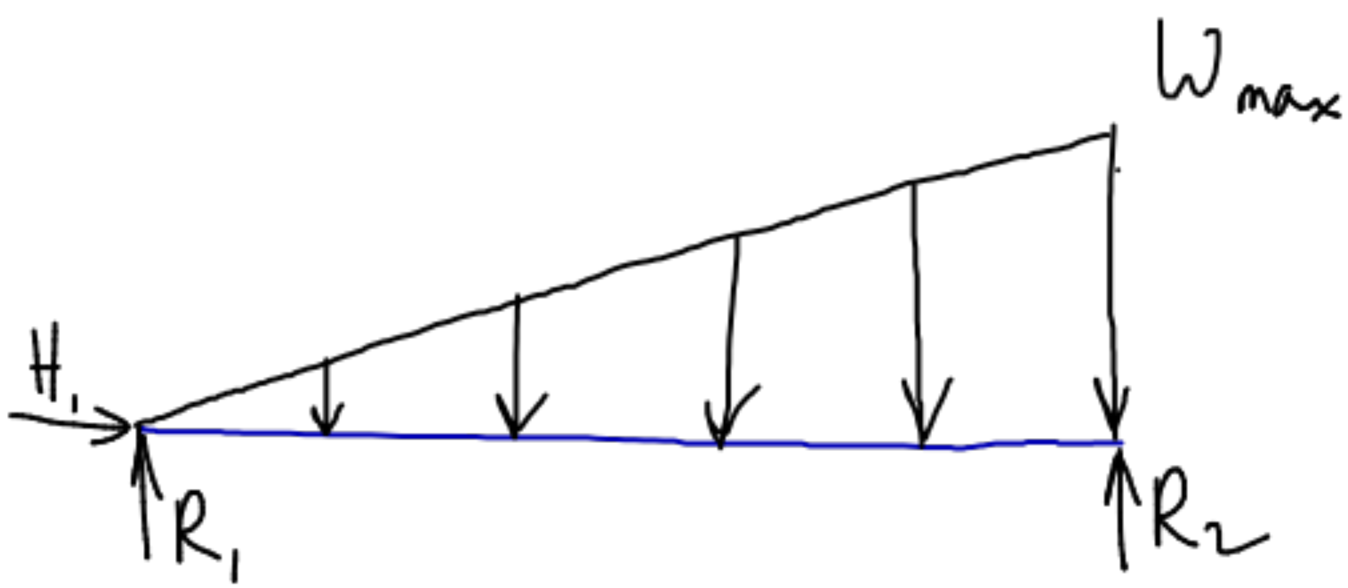
Bending moment :



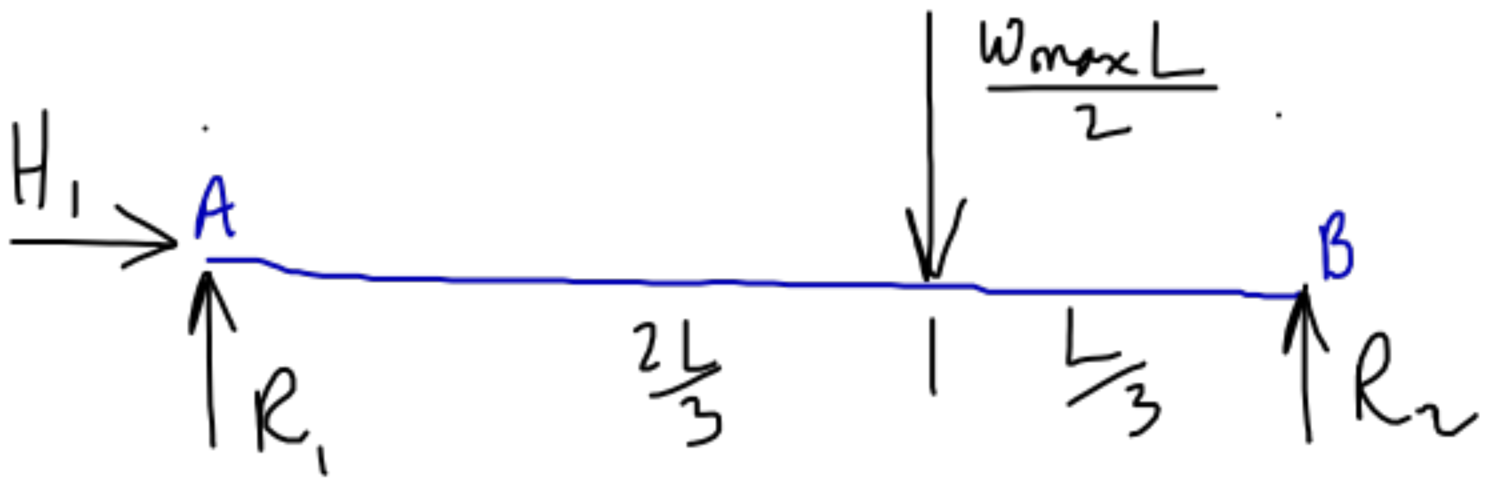
+ve M :

Example II





for determining the reactions this can be simplified as:



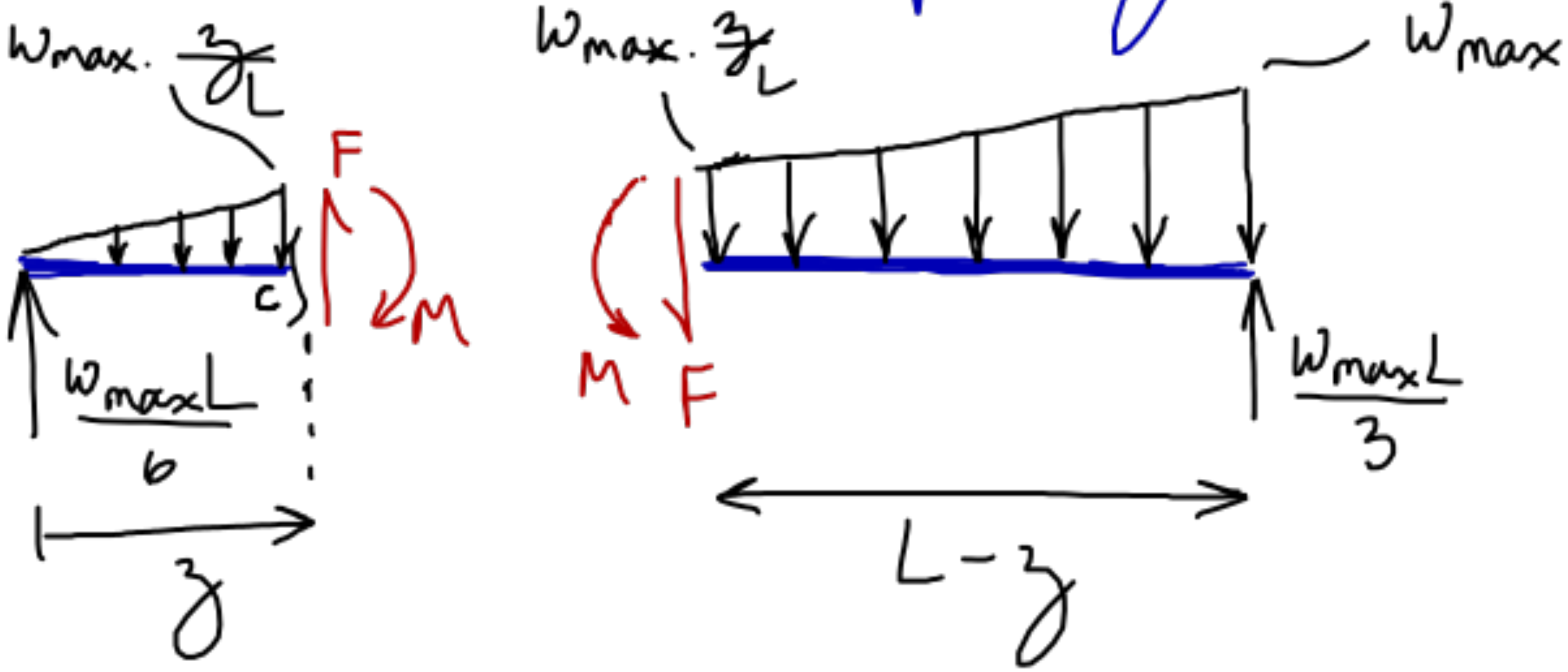
Horiz. eq. $\Rightarrow$ $H_1 = 0$

Moments eq about A:

$$+\circlearrowleft \quad \frac{w_{max} L}{2} \cdot \frac{2L}{3} - R_2 L = 0 \quad \therefore \quad R_2 = \frac{w_{max} L}{3}$$

Taking moments about B (or vert. eq.) $\Rightarrow R_1 = \frac{w_{max} L}{6}$

Now cut the beam at some position z & mark +ve F & M



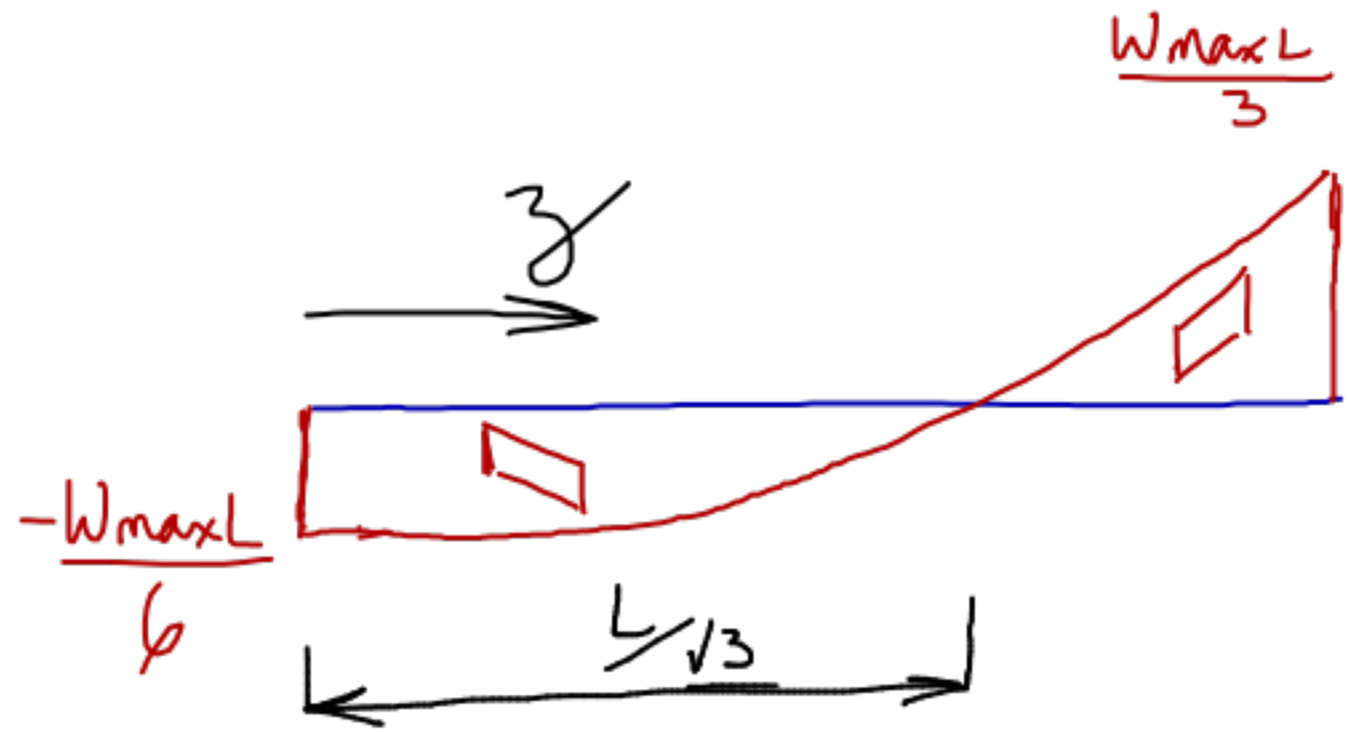
[Note: we have shown the distributed load here. We are NOT analysing a beam subjected to $\frac{w_{max}L}{2}$

Using the left fbd

Vert. eq.

$$\frac{w_{max}L}{6} + F - w_{max} \cdot \frac{z}{L} \cdot \frac{z}{2} = 0$$

$$\Rightarrow \quad F = w_{max}\left(\frac{z^2}{2L} - \frac{L}{6}\right)$$



Moments eq about c :

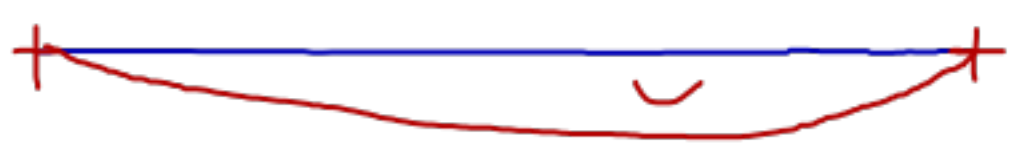
+ve

$$\frac{w_{max}L}{6}\cdot z - w_{max}\frac{z}{L}\cdot\frac{z}{2}\cdot\frac{z}{3} + M = 0$$

$$\therefore\ M = w_{max}\frac{z^3}{6L} - w_{max}\frac{Lz}{6}$$

$$M = \frac{w_{max}L^2}{6}\left(\left(\frac{z}{L}\right)^3 - \left(\frac{z}{L}\right)\right)$$

Bending moment diagram :



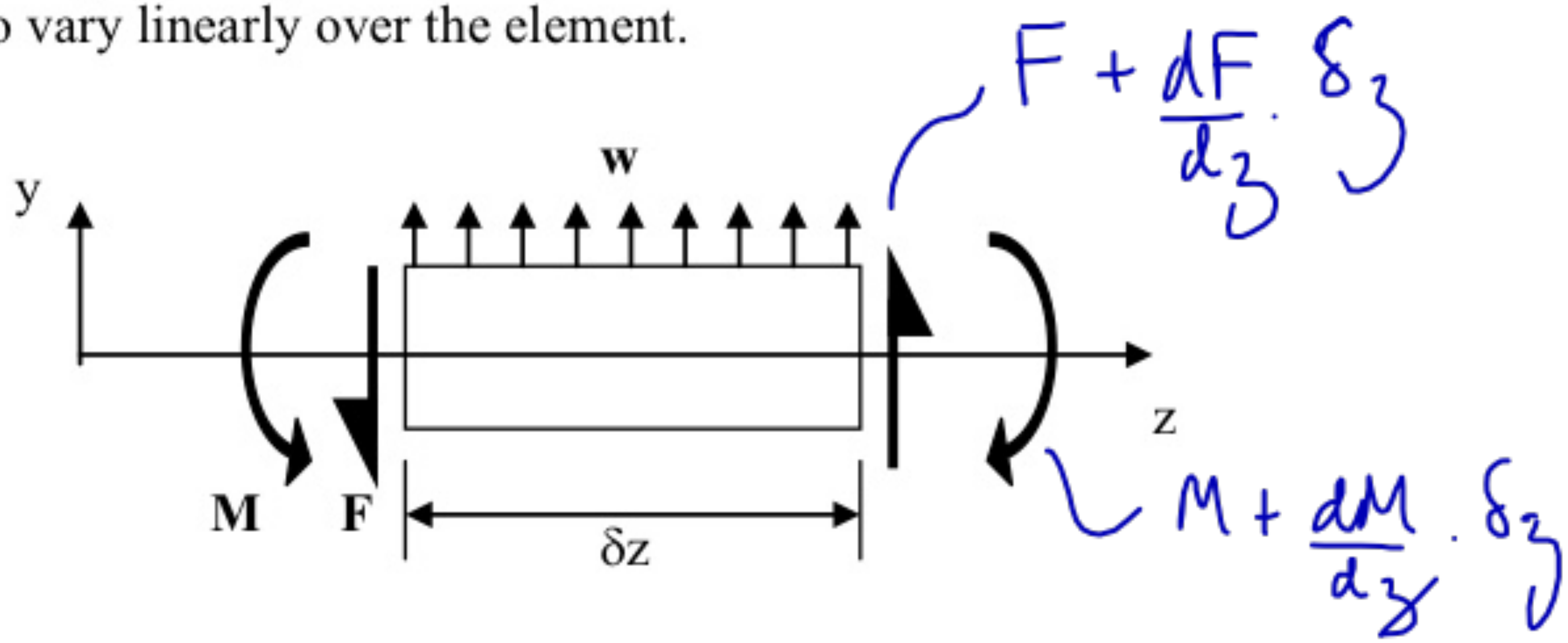
use $\frac{dM}{dz} = 0$ to find M_{max}

$$M_{max} = -\frac{w_{max}L^2}{9\sqrt{3}} \text{ at } z = \frac{L}{\sqrt{3}}$$

Determining Shear Force and Bending Moment Diagrams for Statically Determinate Beams (cont.)

1.4.2 Using the differential relationship between loading, shear force and bending moment.

Consider a short length of beam that is subjected to an arbitrary load. We will examine an element so short that the loading intensity can be considered as uniform and the shear force and bending moment can be assumed to vary linearly over the element.



Note:
w, F & M
all shown in
+ve directions

Considering vertical equilibrium first:

$$F + \frac{dF}{dz} \cdot \delta z - F + w \delta z = 0$$

$$\therefore \quad \frac{dF}{dz} = -w$$

Now writing an equation for moments equilibrium about the centre of the element:

+ve

$$M + \frac{dM}{dz}\cdot\delta z - M - \left(F + \frac{dF}{dz}\cdot\delta z\right)\cdot\frac{\delta z}{2} - F\frac{\delta z}{2} = 0$$

$$\frac{dM}{dz}\cdot\delta z - F\delta z - \frac{dF}{dz}\cdot\frac{\delta z^2}{2} = 0$$

$$\therefore \frac{dM}{dz} - F - \frac{dF}{dz}\cdot\frac{\delta z}{2} = 0$$

As $\delta z \to 0$

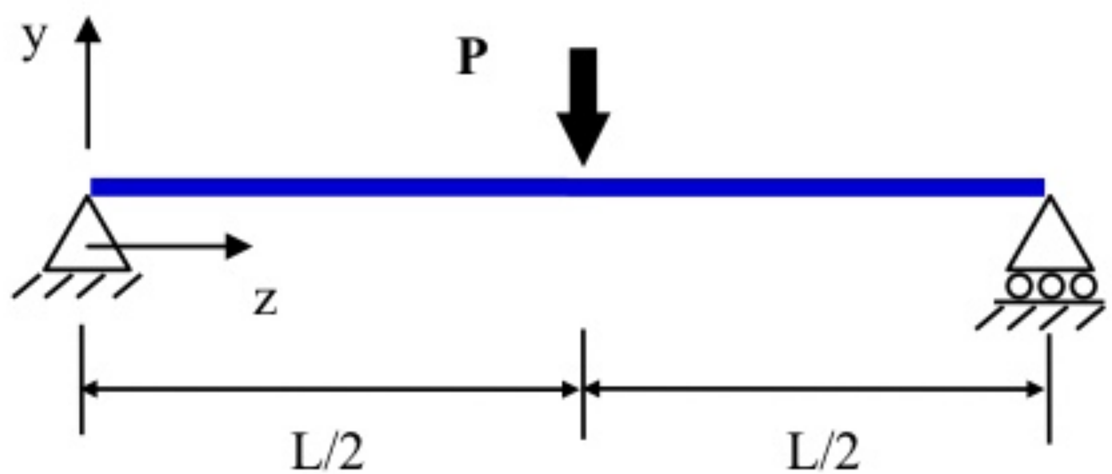
$$\therefore \frac{dM}{dz} = F$$

We can use these equations to determine the shear force and bending moment distribution from the load intensity on a beam.

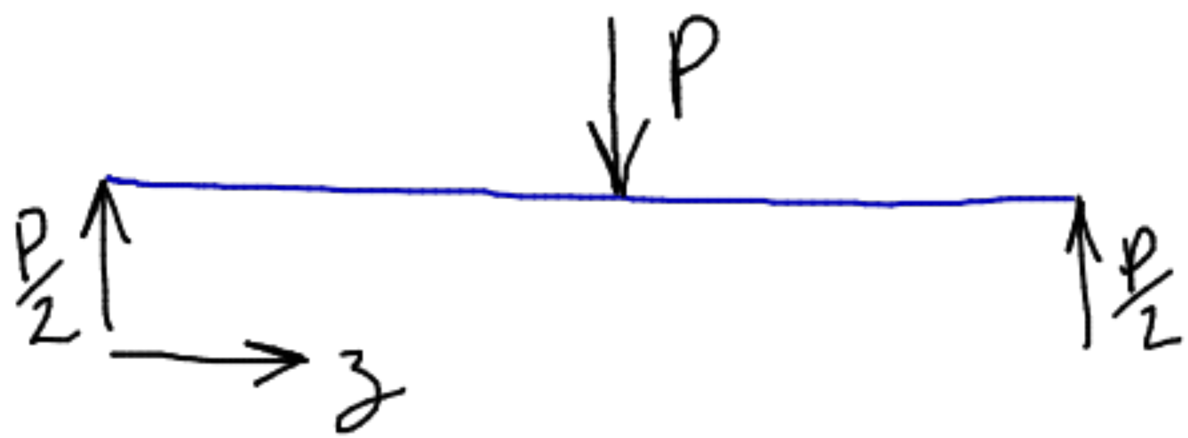
$$\frac{dF}{dz} = -w$$

$$\frac{dM}{dz} = F$$

<u>Example I</u>



Free body diagram:



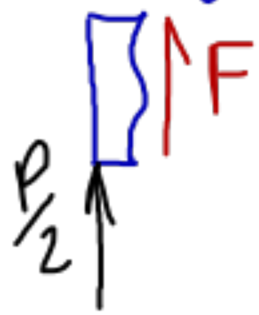
Noting that the loading is discontinuous we consider the two halves.

For $0 < z < L/2$ $\quad w = 0$

$$\therefore \frac{dF}{dz} = 0$$

$$\therefore F = C_1$$

at $z = 0$



$$\frac{P}{2} + F = 0$$

$$\therefore F = -\frac{P}{2} \qquad \therefore C_1 = -\frac{P}{2}$$

For $L/2 < z < L$

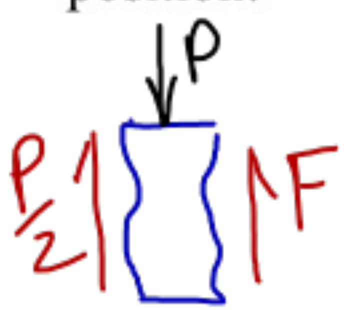
$$\frac{dF}{dz} = 0 \Rightarrow F = C_2$$

$$F = \frac{P}{2}$$

We can evaluate C_2

<u>either</u> by noting at $z = L$, $F = P/2$ therefore $C_2 = P/2$

<u>or</u> by examining the free body diagram of an element of the beam at the loading position:

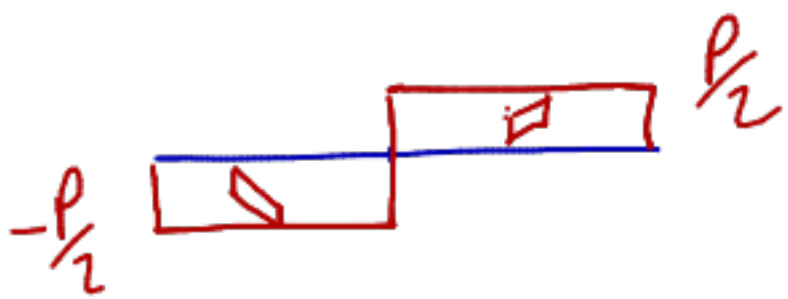


$$\therefore F + \frac{P}{2} - P = 0$$

$$\therefore F = \frac{P}{2} \quad \left(\text{ie } C_2 = \frac{P}{2}\right)$$

So we have for $0 < z < L/2$ $F = -\frac{P}{2}$

and for $L/2 < z < L$ $F = \frac{P}{2}$



Continuing for the left half ($0 < z < L/2$) we have

$$\frac{dM}{dz} = F = -\frac{P}{2} \quad \therefore M = -\frac{P}{2}z + D_1$$

Noting at $z = 0$, $M = 0$ (simple support and no applied moment) then $D_1 = 0$.

$$\therefore \underline{M = -\frac{P}{2}z}$$

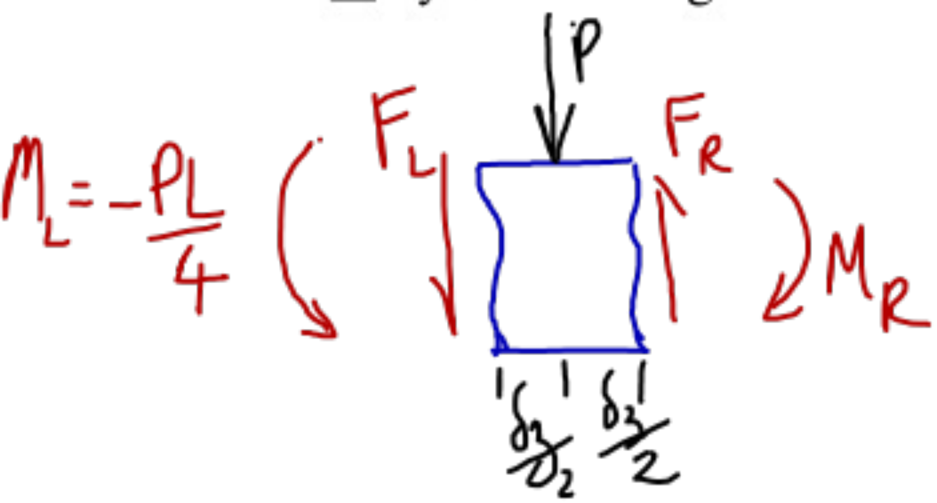
For the right half (L/2 < z < L)

$$\frac{dM}{dz} = \frac{P}{2} \quad \therefore \quad M = \frac{P}{2} z + D_2$$

We can determine D_2 by

either by noting at z = L $\quad M = 0 \quad \therefore \quad 0 = \frac{P}{2} L + D_2 \quad \therefore \quad D_2 \;\; -\frac{PL}{2}$

or by considering moments equilibrium of an element at the loading point



As $\delta z \to 0$

$$M_R - M_L = 0$$

$$\therefore \quad M_R = M_L = -\frac{PL}{4} \quad \therefore \quad -\frac{PL}{4} = \frac{P}{2} \cdot \frac{L}{2} + D_2$$

i.e. the boundary condition is

$$\Rightarrow D_2 = -\frac{PL}{2}$$

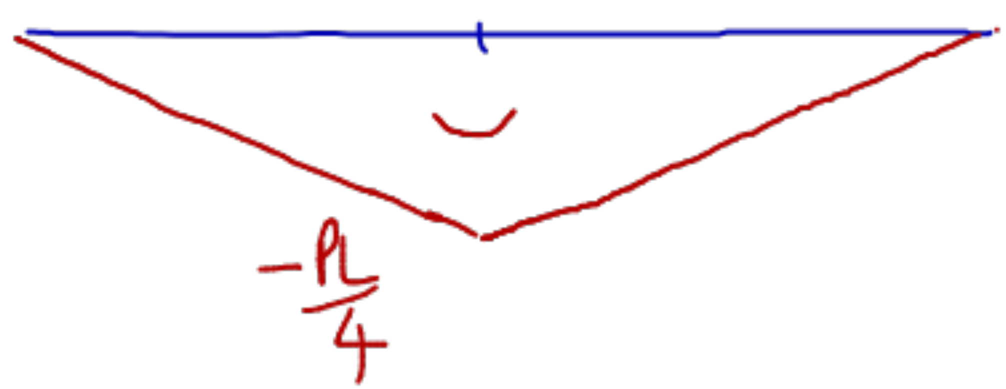
$$\therefore \quad \underline{M = \frac{P}{2} z - \frac{PL}{2}}$$

Therefore summarising for the bending moment we have

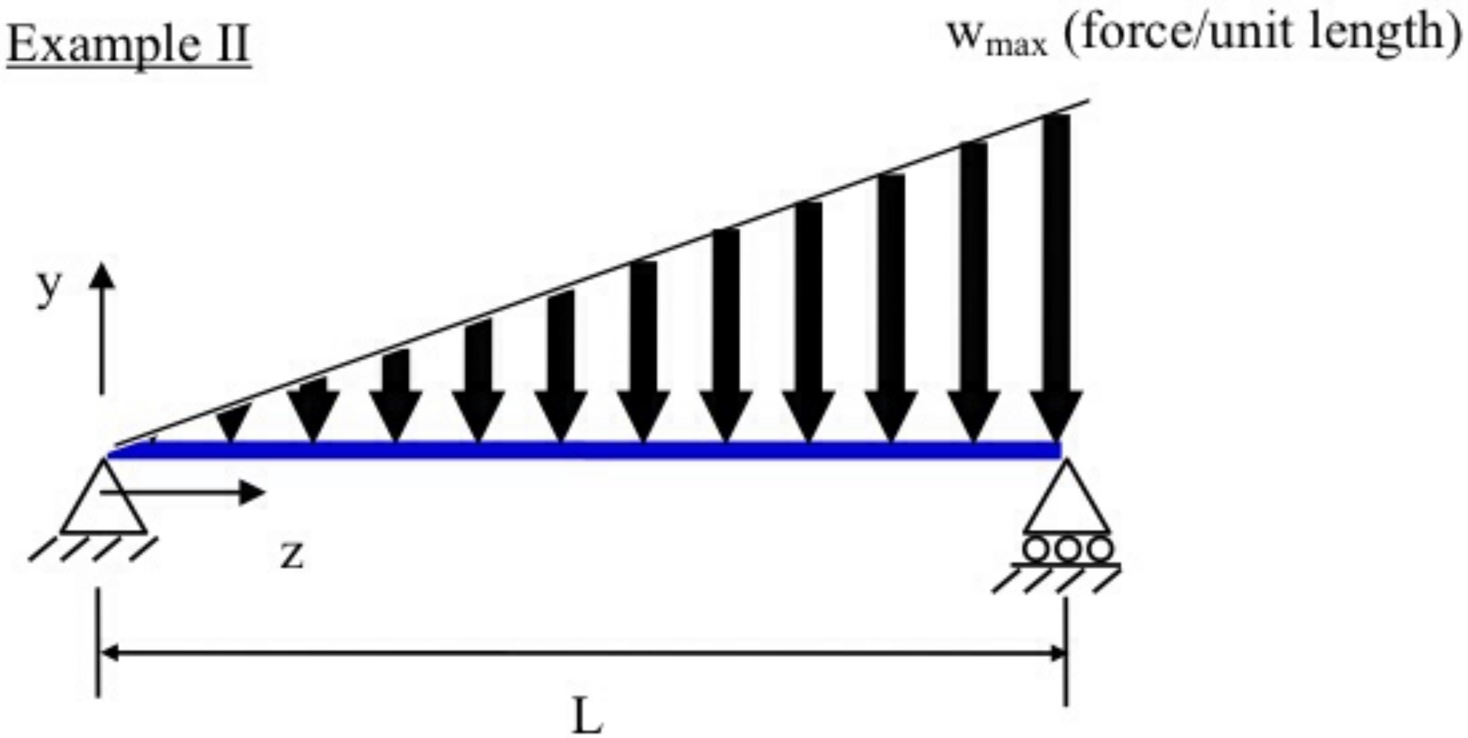
for $0 < z < L/2$ $\quad M = -\frac{P}{2} z$

and for $L/2 < z < L$ $\quad M = \frac{P}{2} z - \frac{PL}{2}$

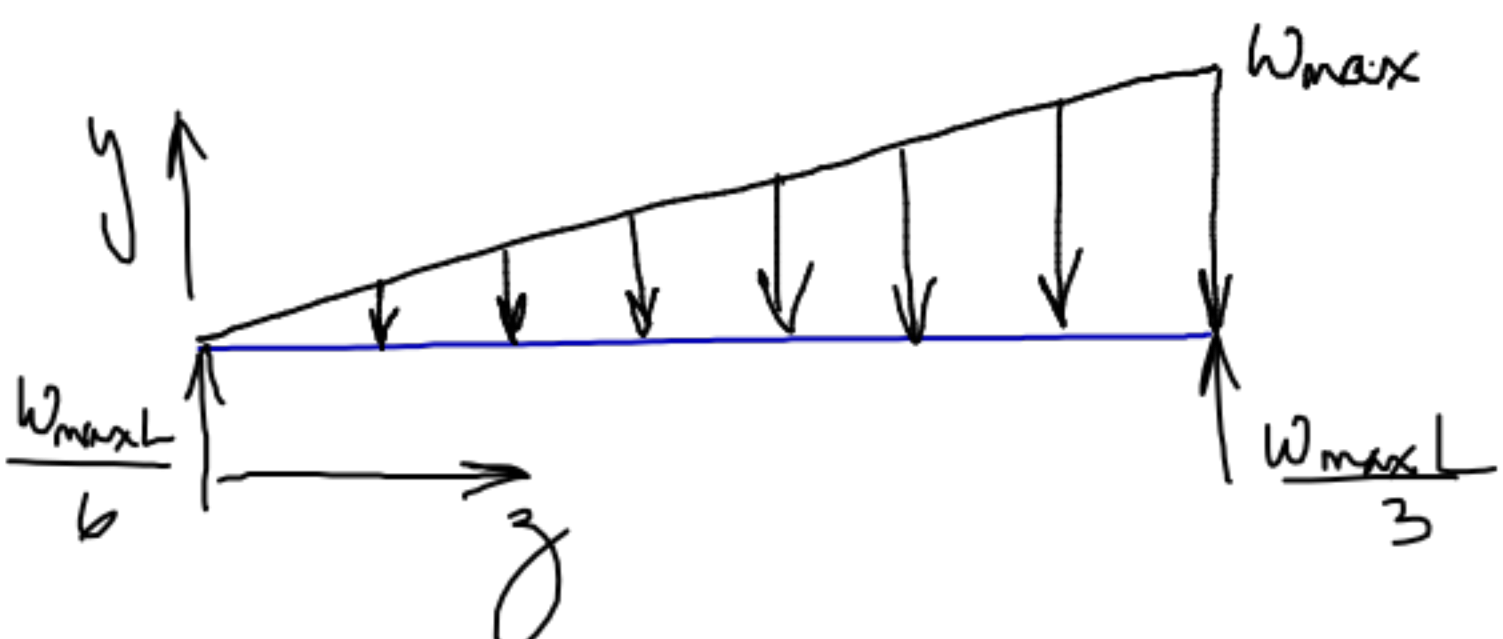
as from method of sections



Example II



Free body diagram and reaction forces can be determined as before:



Now $\dfrac{dF}{dz} = -w$

Recalling positive w is upward then in this case

$$w = -w_{max}\frac{z}{L}, \quad \therefore \frac{dF}{dz} = \frac{w_{max}z}{L} \quad \therefore F = \frac{w_{max}z^2}{2L} + C$$

Noting at z = 0

at $z=0$ $\quad F = \dfrac{-w_{max}L}{6}$

$$\therefore \frac{-w_{max}L}{6} = 0 + C \quad \therefore C = \frac{-w_{max}L}{6}$$

$$\therefore \underline{F = \frac{w_{max}z^2}{2L} - \frac{w_{max}L}{6}}$$

(as from method of sections)

Now $\dfrac{dM}{dz} = F = \dfrac{w_{max} z^2}{2L} - \dfrac{w_{max} L}{6}$

$$\therefore \quad M = \frac{w_{max} z^3}{6L} - \frac{w_{max} L}{6} z + D$$

Noting at z = 0 $M = 0$ (simple support & no applied moment)

$\therefore$ $D = 0$ &

$$\underline{M = \frac{w_{max} z^3}{6L} - \frac{w_{max} L}{6} z}$$

(as before for method of sections)

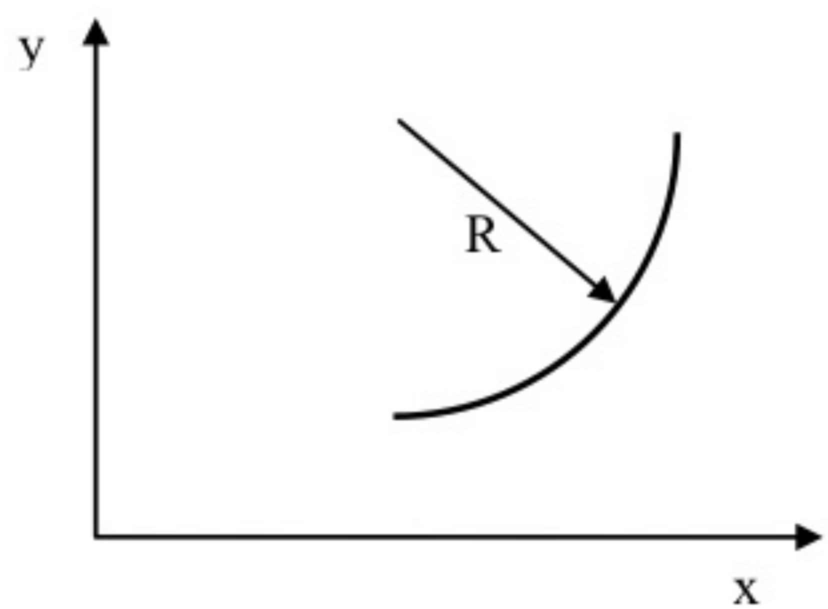
1.5 Deflection of beams

1.5.1 Beam displacement by integration from curvature

We have already shown that

$$\frac{M}{I} = \frac{E}{R} = \frac{\sigma}{y} \qquad \text{ie. } \frac{1}{R} = \frac{M}{EI}$$

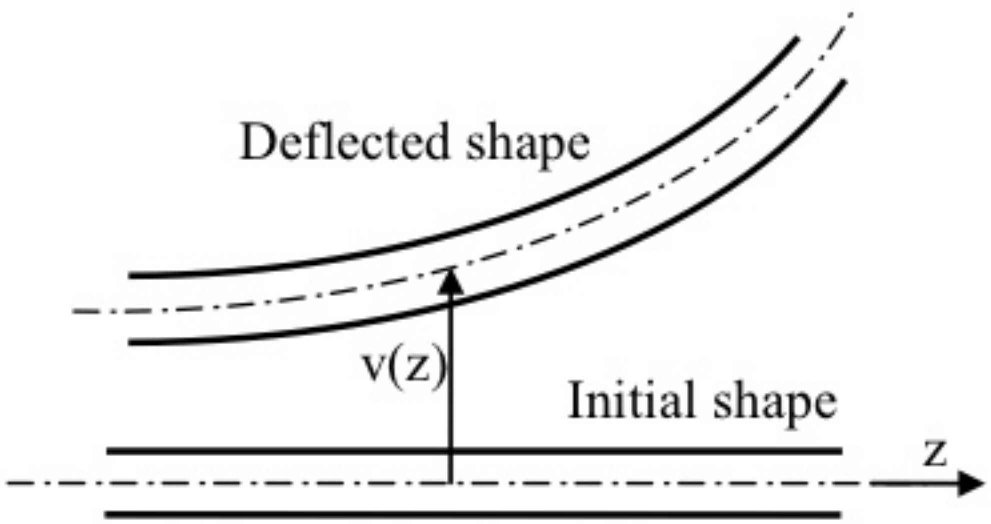
So from the bending moment diagram we can evaluate the curvature (1/R) at any position. In Appendix 4 we show that the curvature, 1/r, of any curve y(x) is given by



$$\frac{1}{R} = \frac{\frac{d^2y}{dx^2}}{\left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}}}$$

(But note that, according to our sign convention this negative curvature – positive curvature is in the direction of positive moment.)

If we denote the deflection of the beam by v(z) (positive in the direction of positive y):



then the curvature (using our sign convention) can be expressed in terms of v as

$$\frac{1}{R} = -\frac{\frac{d^2v}{dz^2}}{\left[1+\left(\frac{dv}{dz}\right)^2\right]^{3/2}}$$

Assuming small deflections and slopes

$$\frac{dv}{dz} \ll 1$$

then we can write

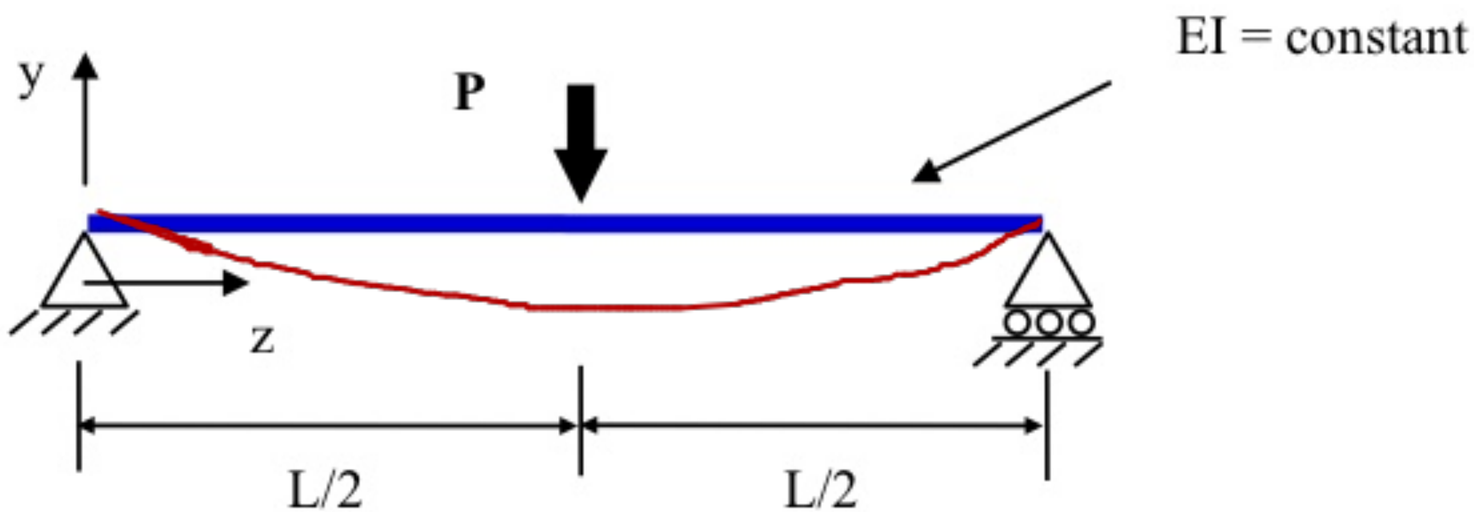
$$1/R = -\frac{d^2v}{dz^2}$$

and so using $\frac{1}{R} = \frac{M}{EI}$ we have

$$\frac{d^2v}{dz^2} = \frac{-M}{EI}$$

therefore knowing M(z) we can integrate to get the deflected shape v(z).

Example 1



We have shown

for $0 < z < L/2$ $M = -Pz/2$

& for $L/2 < z < L$ $M = P(z-L)/2$

Therefore $0 < z < L/2$

we have $$\frac{d^2v}{dz^2} = \frac{-M}{EI} = \frac{Pz}{2EI}$$

$$\frac{dv}{dz} = \frac{P}{2EI} \cdot \frac{z^2}{2} + C$$

from symmetry $\frac{dv}{dz} = 0$ at $z = L/2$

$$0 = \frac{P}{2EI}\left(\frac{L}{2}\right)^2 \frac{1}{2} + C$$

$$\therefore C = -\frac{PL^2}{16EI}$$

$$\therefore \frac{dv}{dz} = \frac{P}{4EI}z^2 - \frac{PL^2}{16EI}$$

$$\therefore v = \frac{Pz^3}{12EI} - \frac{PL^2 z}{16EI} + D$$

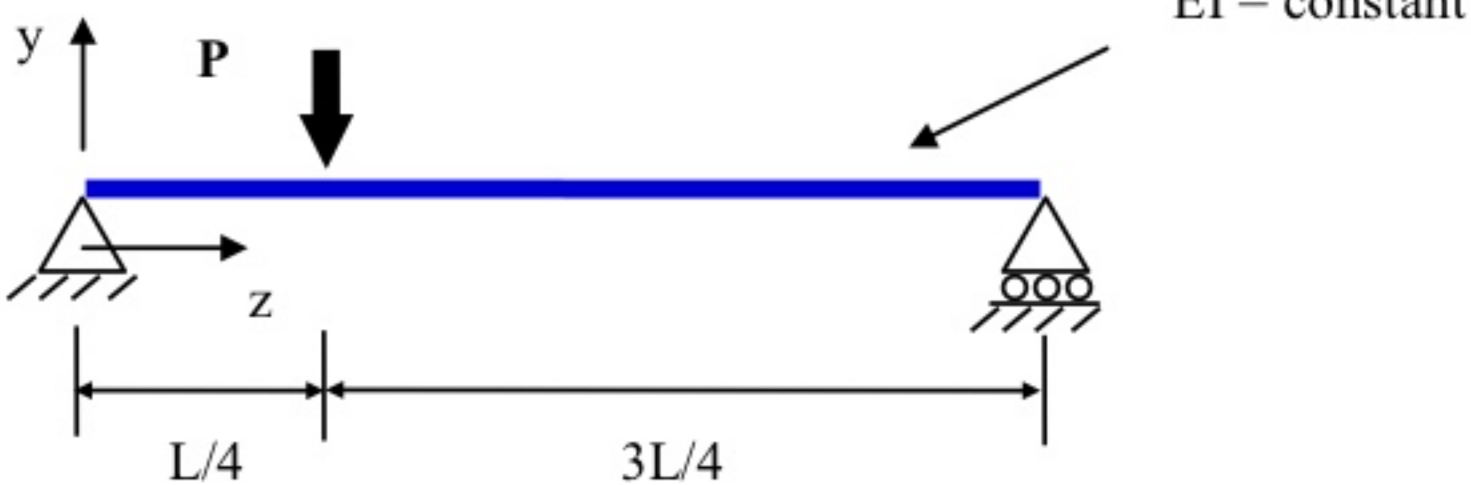
at $z = 0$, $v = 0$

$$\therefore 0 = 0 + 0 + D \Rightarrow D = 0$$

$$\underline{v = \frac{Pz^3}{12EI} - \frac{PL^2}{16EI}z}$$

[at $z = L/2$
$\delta = -\frac{PL^3}{48EI}$

Example 2



We can show

for $0 < z < L/4$ $M = -3Pz/4$

& for $L/4 < z < L$ $M = P(z-L)/4$

and so for $0 < z < L/4$

$$\frac{d^2v}{dz^2} = -\frac{M}{EI} \qquad \left.\frac{d^2v}{dz^2}\right|_{left} = \frac{3P}{4EI}z$$

$$\left.\frac{dv}{dz}\right|_{left} = \frac{3Pz^2}{8EI} + C_1$$

$$\left.v\right|_{left} = \frac{3Pz^3}{24EI} + C_1 z + D_1$$

Similarly for L/4 < z < L

$$\left.\frac{d^2v}{dz^2}\right|_{right} = \frac{-P(z-L)}{4EI}$$

$$\left.\frac{dv}{dz}\right|_{right} = \frac{-P}{4EI}\left(\frac{z^2}{2} - Lz\right) + C_2 \qquad \left.v\right|_{right} = \frac{-P}{4EI}\left(\frac{z^3}{6} - \frac{Lz^2}{2}\right) + C_2 z + D_2$$

We have four unknowns : C_1, C_2, D_1, D_2

Boundary conditions? – we need to solve for 4 unknowns

i) at z = 0, v = 0 – for $v|_{left}$
ii) at z = L, v = 0 – for $v|_{right}$
iii) at z = L/4

$$\left.v\right|_{left} = \left.v\right|_{right}$$

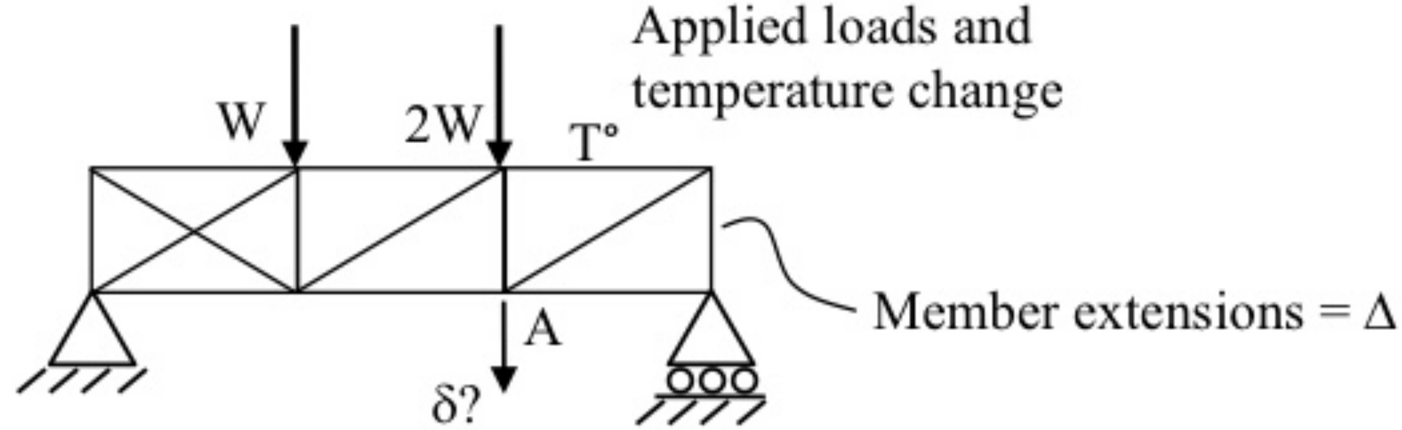
iv) at z = L/4

$$\left.\frac{dv}{dz}\right|_{left} = \left.\frac{dv}{dz}\right|_{right}$$

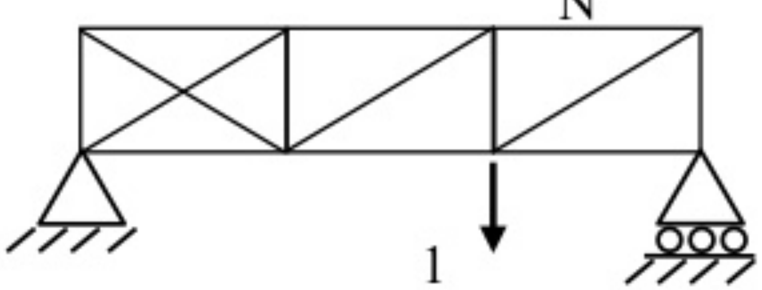
1.5.2 Beam displacements by the virtual work method

Re-cap of the virtual work method for pin-jointed frameworks.

Consider a pin-jointed framework for which the member extensions, Δ are known. If the vertical deflection, δ, is required at node A, then apply a vertical, virtual unit load and determine the bar forces, $\overline{N}$, in equilibrium with the unit load.



Member forces in equilibrium with the unit virtual load $\overline{N}$



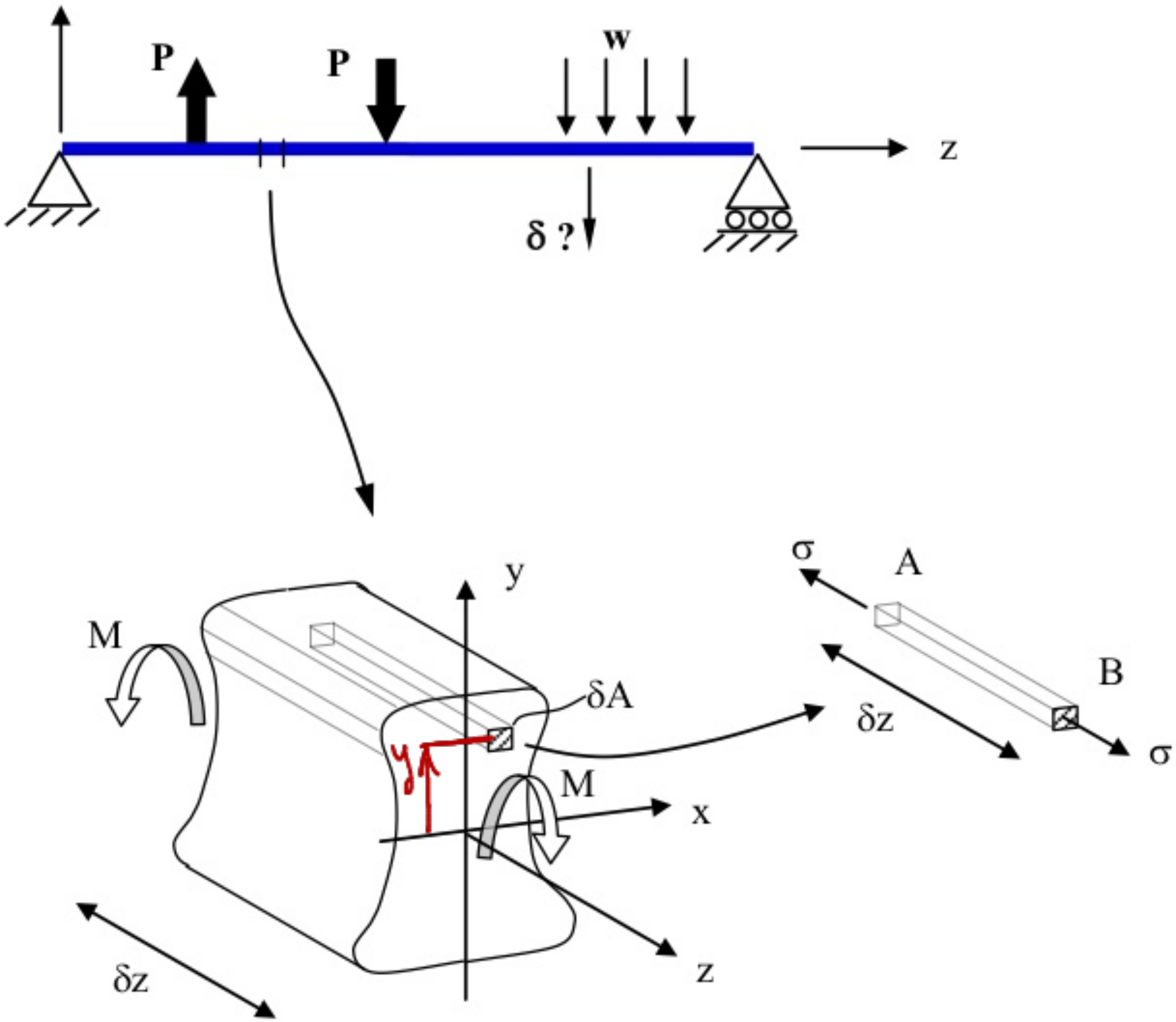
The Principle of Virtual Work states

in equilibrium

$$1 . \delta = \sum_{\text{all bars}} \Delta . \overline{N}$$

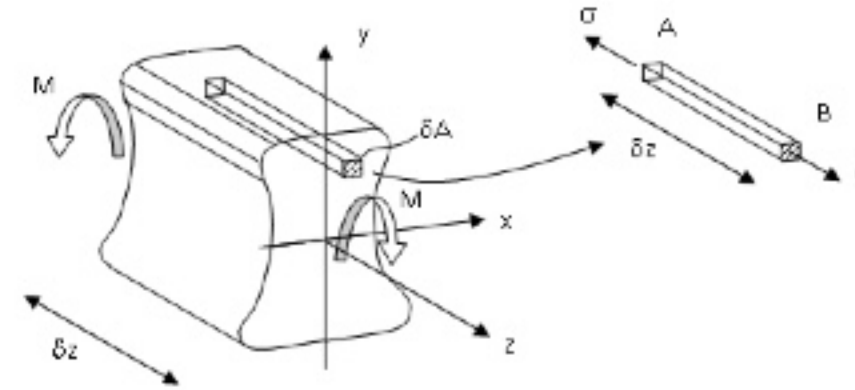
compatible

This same method can be applied to beams. Consider a short element of length δz of a beam subjected to an arbitrary loading. If the element length is sufficiently short then the bending moment can be treated as constant over the length δz. Now consider a small element of area δA in the cross section:



Now σ can be considered constant over δA. So it is as if we have a bar AB of X-area δA subjected to an axial stress σ of

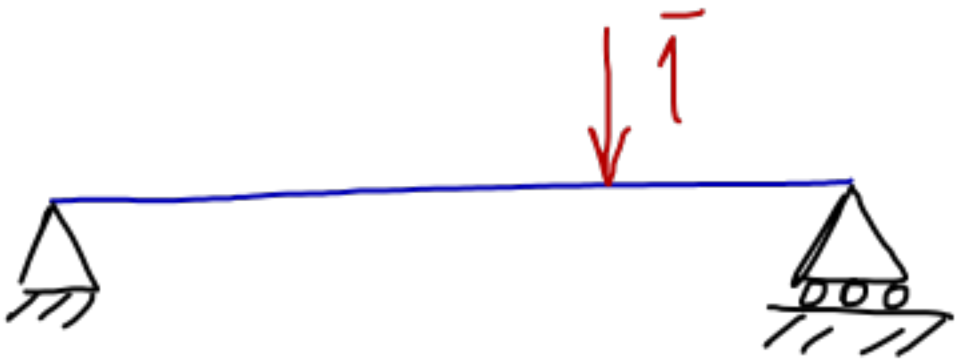
$$\sigma = \frac{My}{I}$$



The extension Δ of this bar is

$$\Delta = \varepsilon . \delta z = \frac{My}{EI} . \delta z$$

If we require the vertical displacement at some point, we apply a a vertical unit virtual force at that point:



We find the virtual bending moment in equilibrium with the unit load ($\bar{M}$) and from this we can determine the virtual bar force, $\bar{N}$, in AB:

$$\bar{N} = \frac{\bar{M} y}{I} \cdot \delta A$$

We can now set up the virtual work equation:

$$\bar{1}.\delta = \sum_{\text{all bars}} \Delta . \bar{N}$$

$$= \sum_{\substack{\text{all bars}\\ \text{like AB}}} \frac{M}{EI} y \frac{\bar{M} y}{I} . \delta A \, \delta z$$

Note M & $\bar{M}$ will vary along the beam and so we will need to integrate

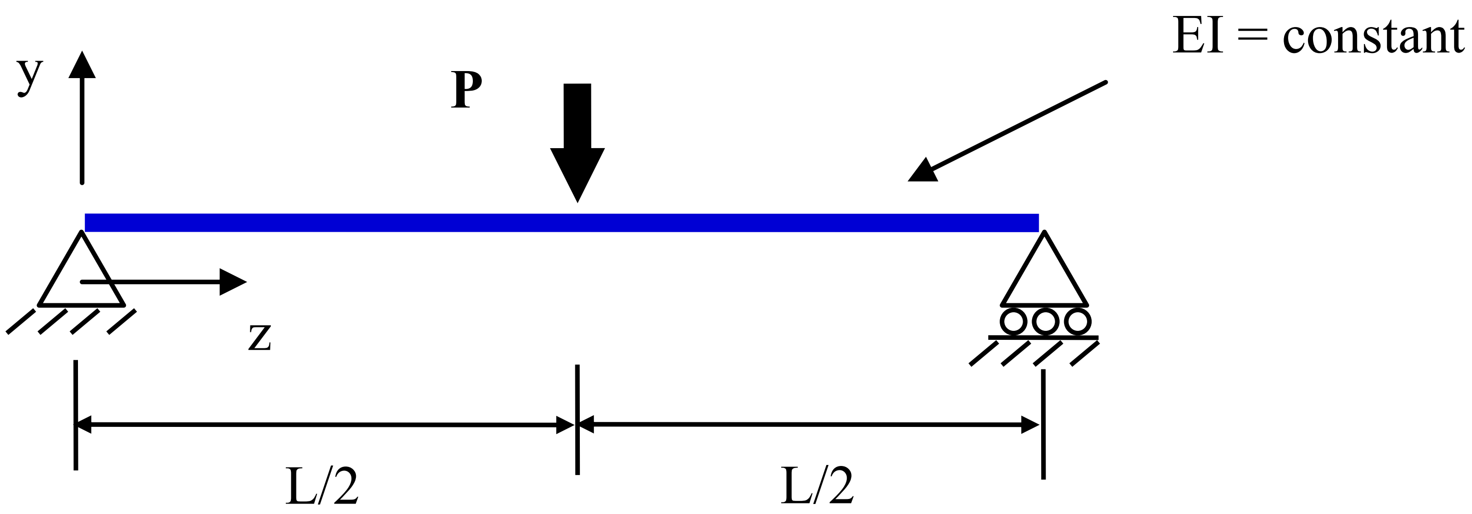
$$= \int_{\substack{\text{over}\\ \text{beam}\\ \text{length}}} \int_A \frac{M}{EI} y \frac{\bar{M} y}{I} \, dA . dz$$

$$= \int_{\substack{\text{beam}\\ \text{length}}} \frac{M}{EI} \frac{\bar{M}}{I} \int_A y^2 \, dA . dz \quad \text{but} \int_A y^2 dA = I$$

$$\therefore \underline{\bar{1}.\delta = \int_{\text{beam length}} \frac{M}{EI} \bar{M} . dz}$$

1.5.2 Beam displacements by the virtual work method (cont.)

<u>Example</u>

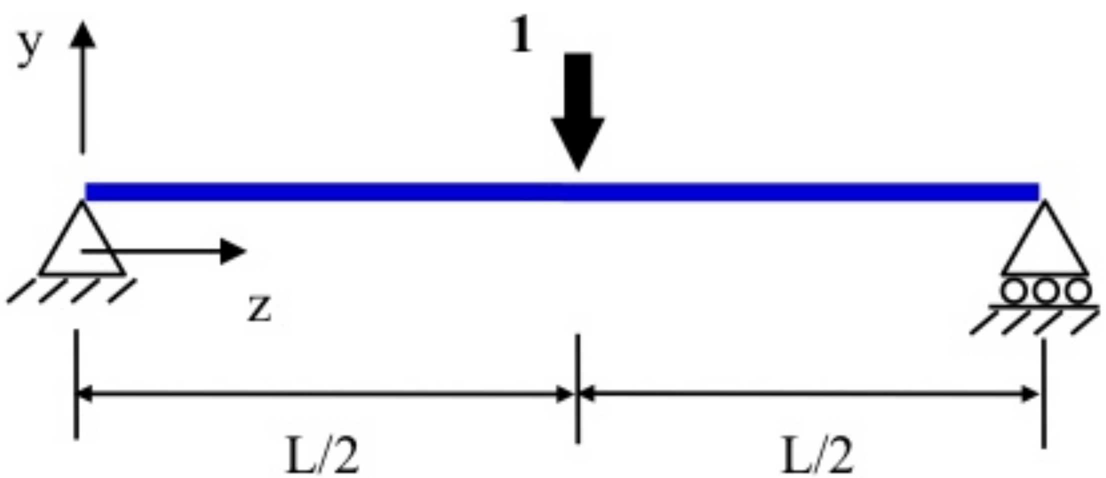


We have shown

for $0 < z < L/2$ $M = -Pz/2$

& for $L/2 < z < L$ $M = P(z-L)/2$

If we require the vertical deflection at the mid-span then we apply a unit virtual load at this position:



And so $\overline{M}$ is given by

for $0 < z < L/2$ $\quad \overline{M} = -\frac{z}{2}$

& for $L/2 < z < L$ $\quad \overline{M} = \frac{z - L}{2}$

$$1 \cdot \delta = \int_{\text{beam length}} \frac{M}{EI} \cdot \overline{M} \cdot dz$$

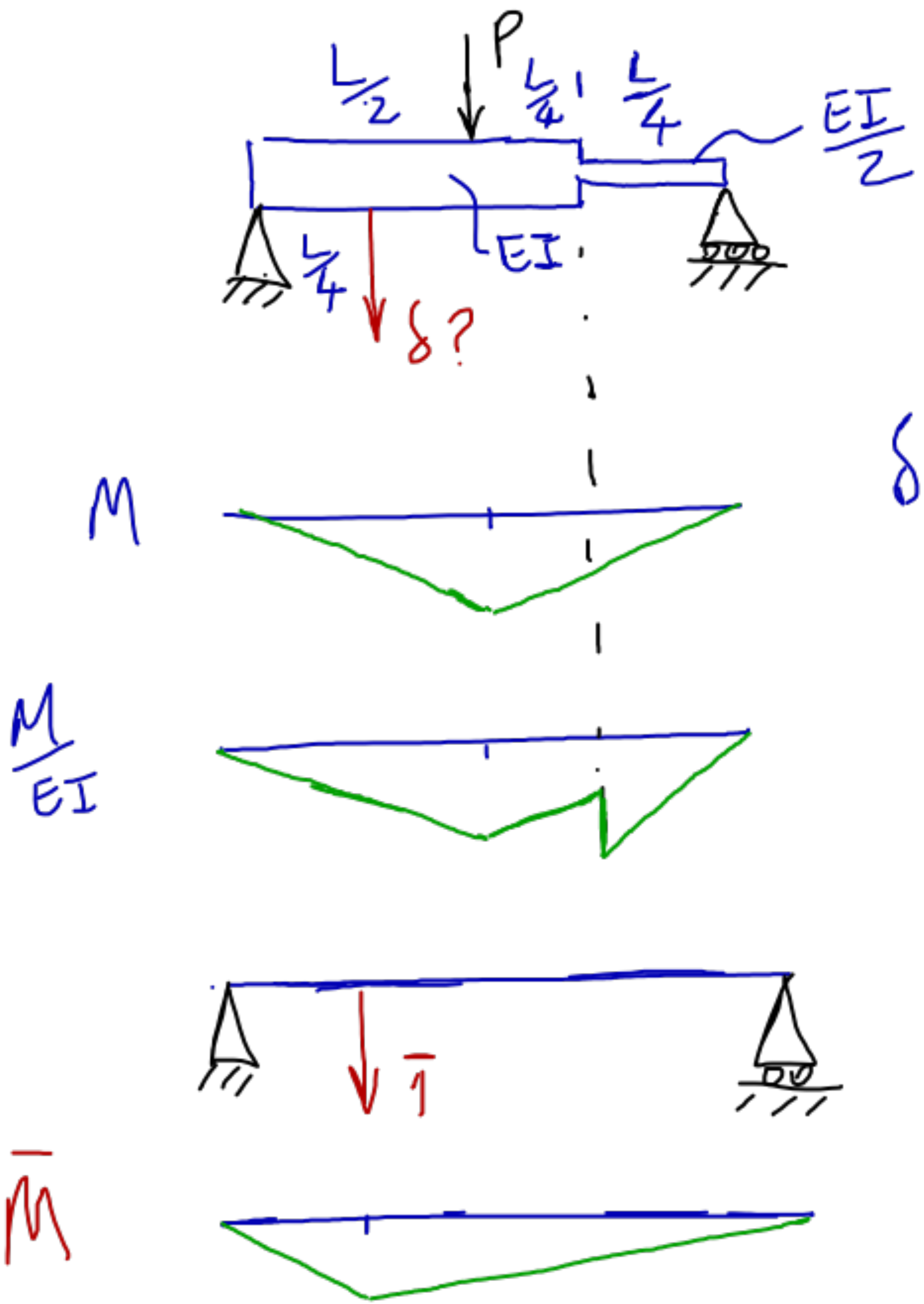
$$= \int_0^{L/2} -\frac{Pz}{2EI} \cdot \left(-\frac{z}{2}\right) \cdot dz + \int_{L/2}^{L} \frac{P(z-L)}{2EI} \cdot \frac{(z-L)}{2} \cdot dz$$

$$= \frac{P}{12}\left[z^3\right]_0^{L/2} + \frac{P}{12EI}\left[(z-L)^3\right]_{L/2}^{L}$$

$$= 2 \times \frac{P}{12EI} \cdot \frac{L^3}{8}$$

$$\underline{\delta = \frac{PL^3}{48EI}} \quad \text{(as before)}$$

Note: +ve δ is in the direction of the unit virtual load.



$$\delta = \int_0^L \frac{M}{EI}\cdot \bar{M}\, dz$$

$$= \int_0^{L/4} \cdots + \int_{L/4}^{L/2} \cdots + \int_{L/2}^{3L/4} \cdots + \int_{3L/4}^{L} \cdots$$

Additional notes on the virtual work method for beams

i) In the example considered EI was constant along the length of the beam. If EI varies along the length then this would simply be included in the integral. If EI was discontinuous the integral would be performed over each continuous part and summed.

See earlier page

ii) The unit virtual load method is particularly useful when we want the deflection at a single point rather than at all points along the beam.

iii) We may want to calculate the rotation, θ, at a point rather than the deflection, δ. Note that in the expression

$$1.\delta = \int \frac{M}{EI} \overline{M}\, dz$$

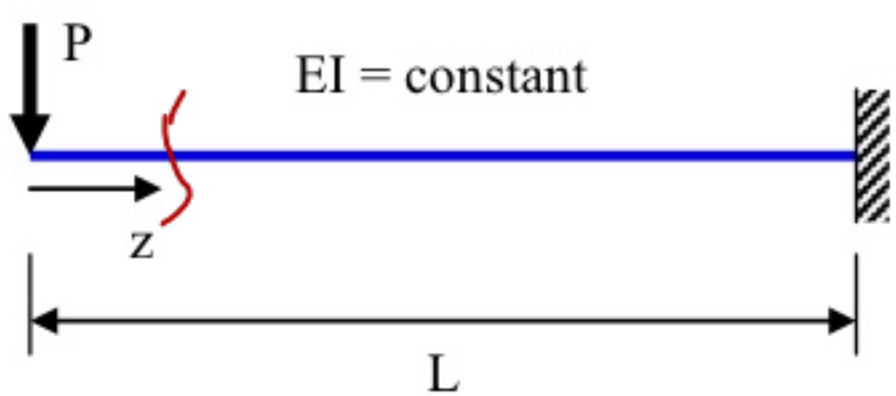
1.δ is virtual work (unit virtual force x real displacement).

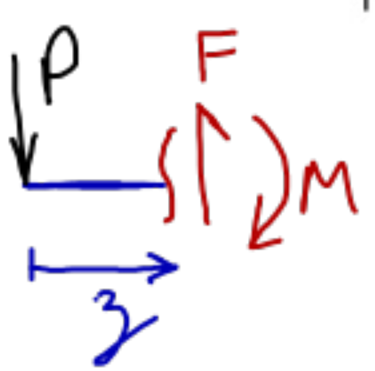
If we need θ then the equivalent virtual work expression would be : unit virtual moment $\times \theta$

So to determine a rotation at a point we apply a unit virtual moment at that point.

Example

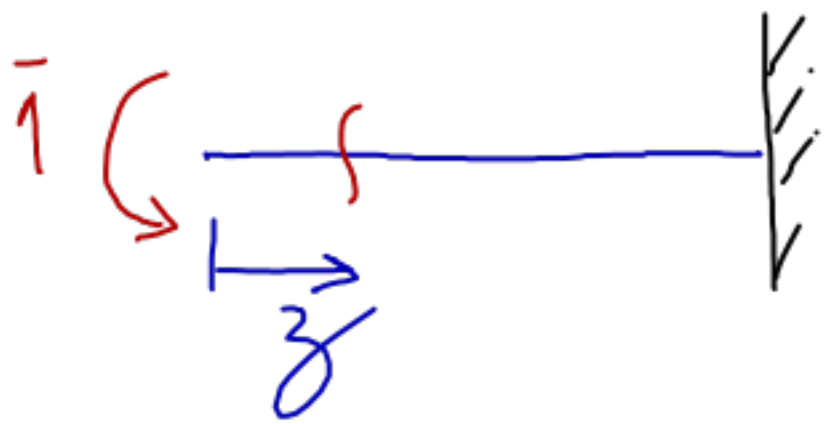
Calculate the rotation at the tip of the cantilever beam shown below.

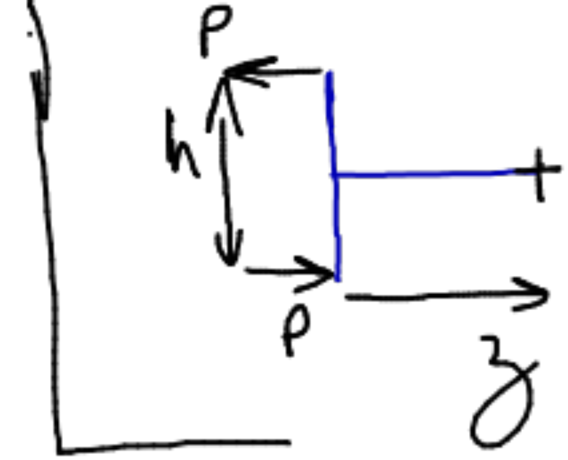


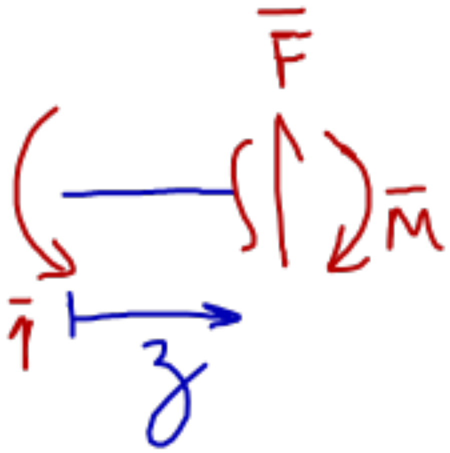


Moments equilib. about cut (+ve) $\Rightarrow -Pz + M = 0$

$\Rightarrow \underline{M = Pz}$







Moments equilib. about cut +ve ↻ : $\bar{M} - 1 = 0$

$$\Rightarrow \bar{M} = 1$$

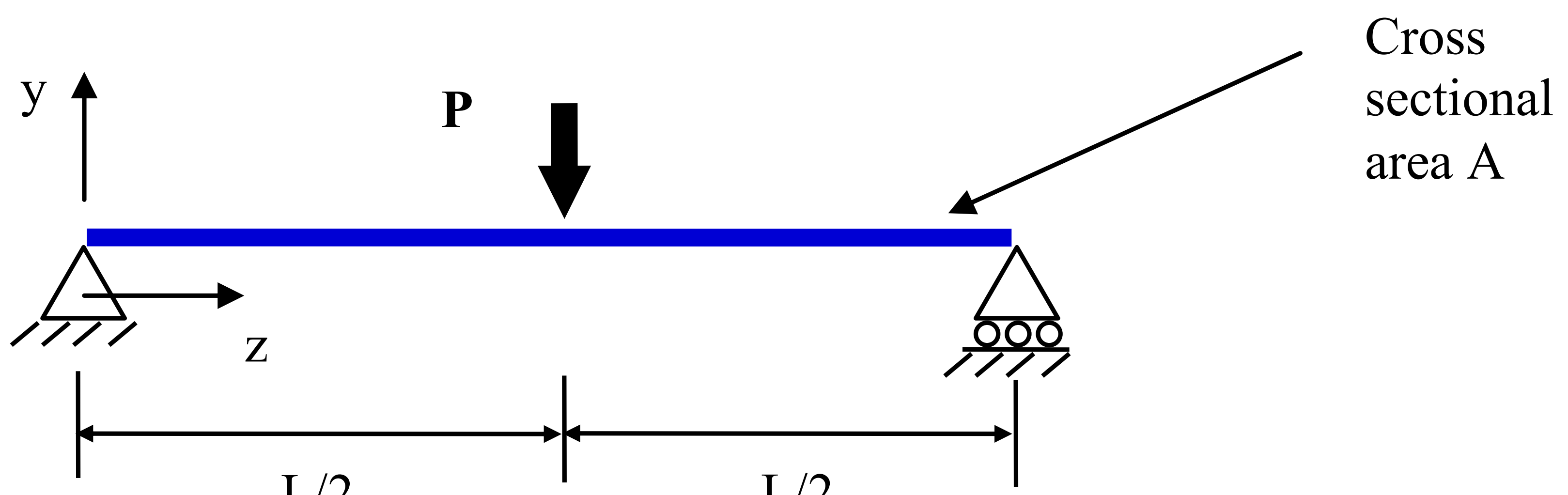
$$\therefore \bar{1} \cdot \theta = \int \frac{M}{EI} \cdot \bar{M} \cdot dz = \int_0^L \frac{Pz}{EI} \cdot 1 \cdot dz$$

$$= \frac{P}{EI} \left[\frac{z^2}{2} \right]_0^L = \underline{\frac{PL^2}{2EI}} \text{ (radians)}$$

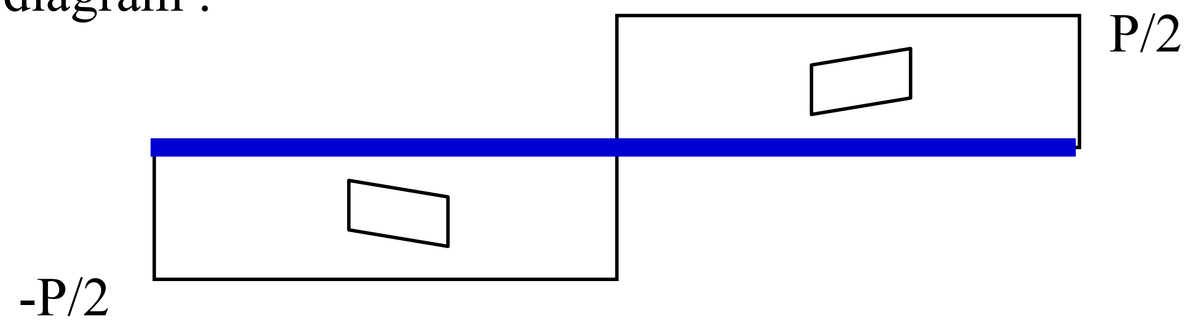
1.5.3 The significance of shear deflections

You will not learn to calculate shear deflections accurately until the third year. However we can perform some approximate calculations which indicate the relative significance of bending and shear deflections.

Continuing with the centrally loaded, simply supported beam example :

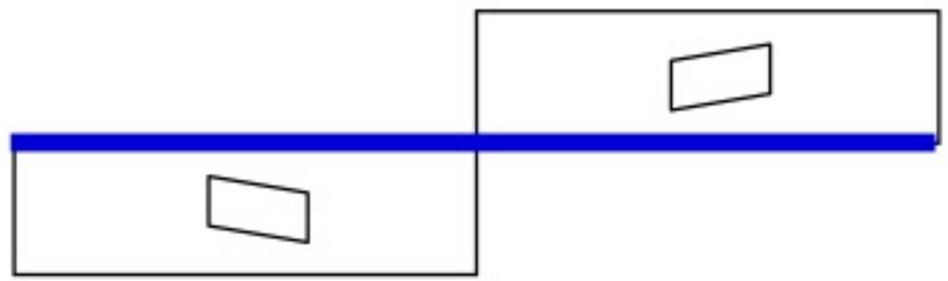


Shear force diagram :



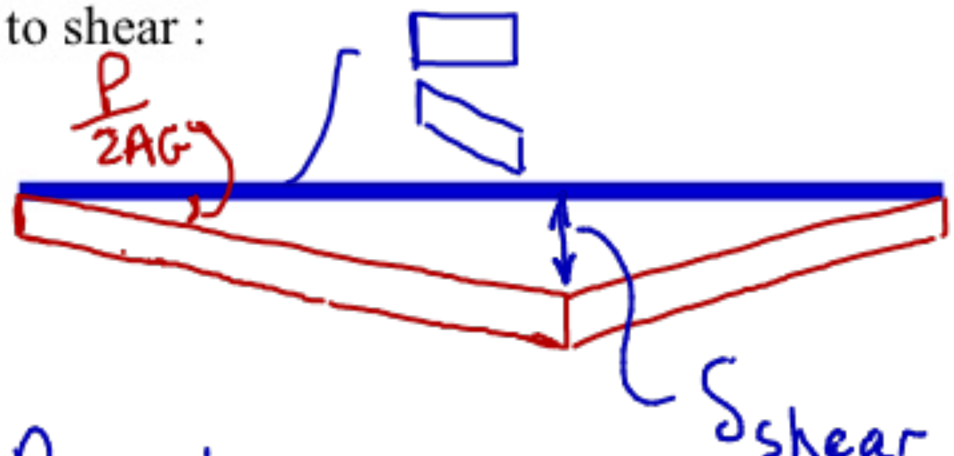
Approximate shear strain, γ, :

$\gamma \simeq -\frac{P}{2AG}$



$\gamma \simeq \frac{P}{2AG}$

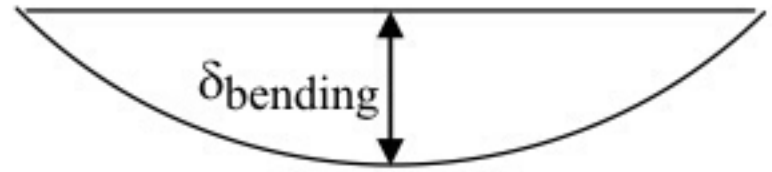
$\therefore$ Deflection due to shear :



$$\therefore \ \delta_{shear} \simeq \frac{P}{2AG} \cdot \frac{L}{2}$$

We have already shown

$\delta_{bending} = PL^3/48EI$



$$\therefore \frac{\delta_{shear}}{\delta_{bending}} \approx \frac{48EI}{PL^3} \cdot \frac{P.L}{4AG} = \frac{12}{L^2} \cdot \frac{I}{A} \cdot \frac{E}{G}$$

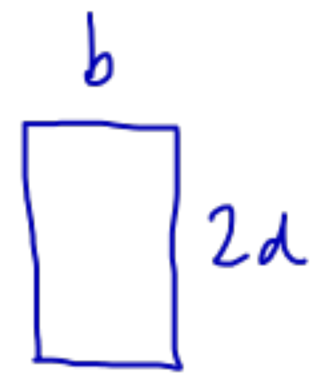
Now $\frac{I}{A} = k^2$ where k is the radius of gyration of the cross section

$G = \frac{E}{2(1+\nu)}$ where ν is the Poissons ratio

taking $\nu = 1/3 \Rightarrow \frac{E}{G} = 8/3$

$$\frac{\delta_{shear}}{\delta_{bending}} = 12\left(\frac{k}{L}\right)^2 \cdot \frac{8}{3} = 32\left(\frac{k}{L}\right)^2$$

Now for a rectangular section



$\Rightarrow k = 0.58d$

and recalling we are using slender beams, say $\frac{L}{2d} \gg 10$

Then $$\frac{\delta_{shear}}{\delta_{bending}} < 32\left(\frac{0.58d}{20d}\right)^2$$

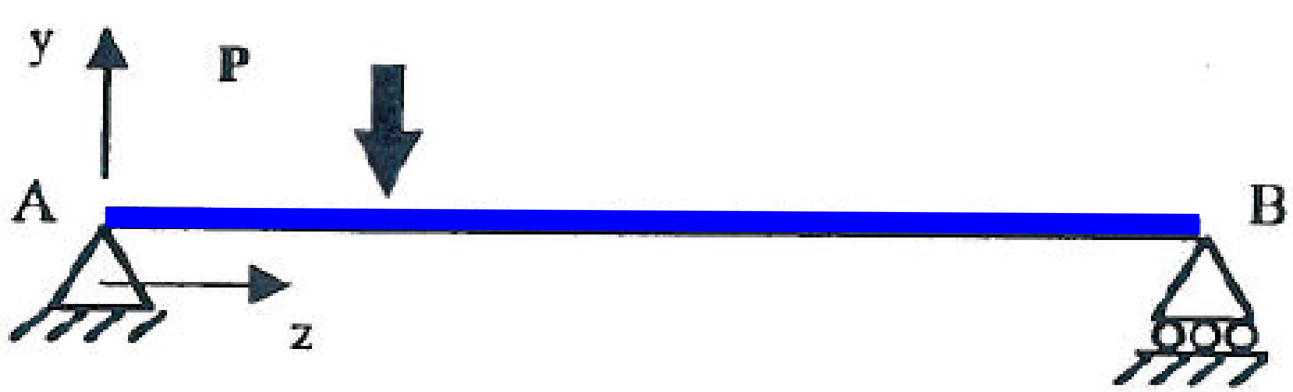
$$< 0.027$$ i.e. the shear displacement is $< 3\%$ of the bending displacement.

1.6 Statically Indeterminate Beams

1.6.1 Introduction

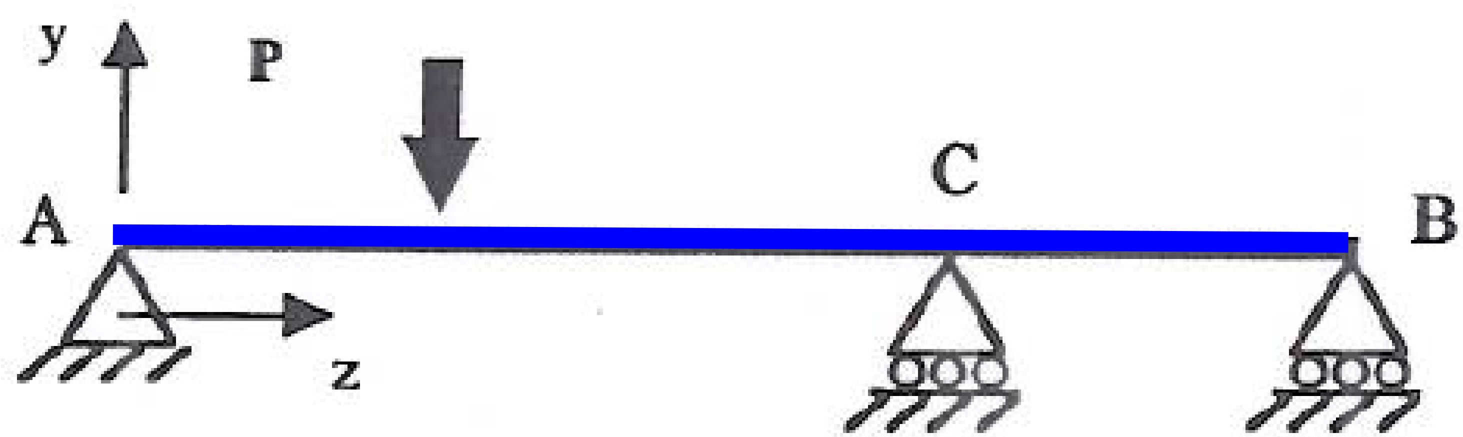
So far we have examined beams which are statically determinate i.e we could determine the reaction forces and reaction moments (and the shear force and bending moments) from consideration of equilibrium alone. In this section we will examine how to analyse statically indeterminate beams (sometimes referred to as redundant beams). We can identify the degree of indeterminacy (or the number of redundancies) by considering the number of 'cuts' required to make the beam statically determinate.

e.g.



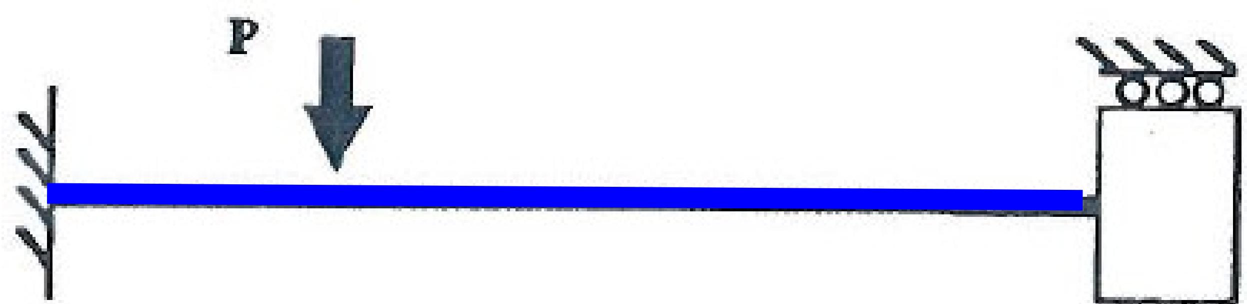
We have seen this beam is statically determinate

Therefore in a beam like

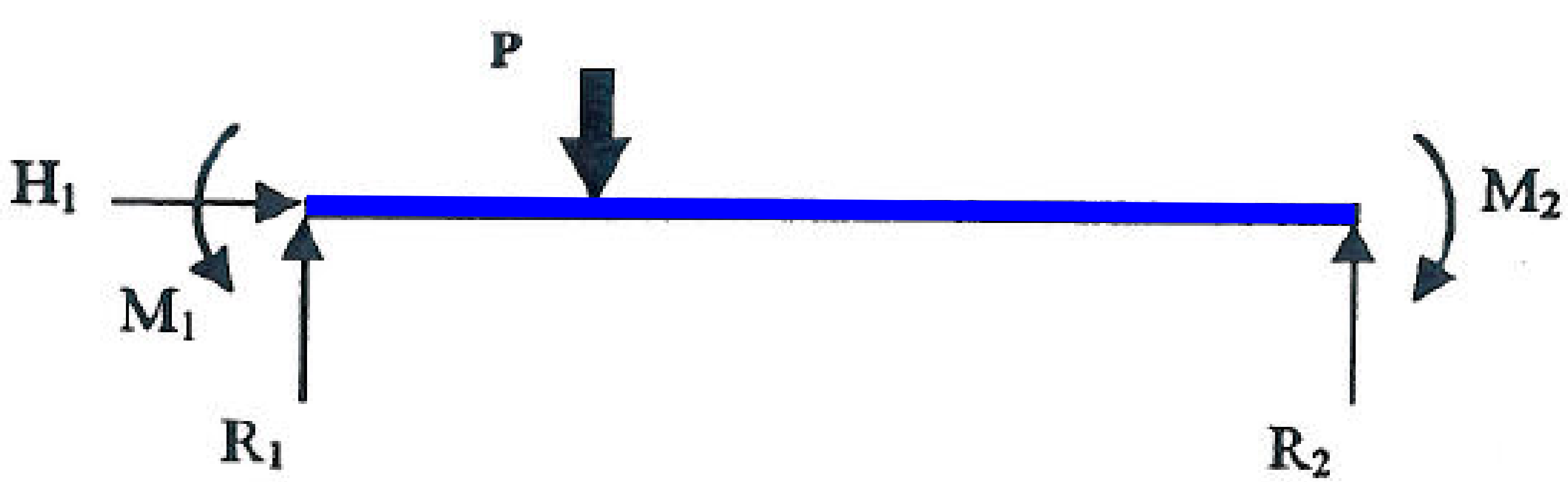


we would have to remove one of the vertical reactions (i.e. make one 'cut') to make the beam statically determinate. This beam therefore has one degree of indeterminacy (or is singly redundant, has one redundancy).

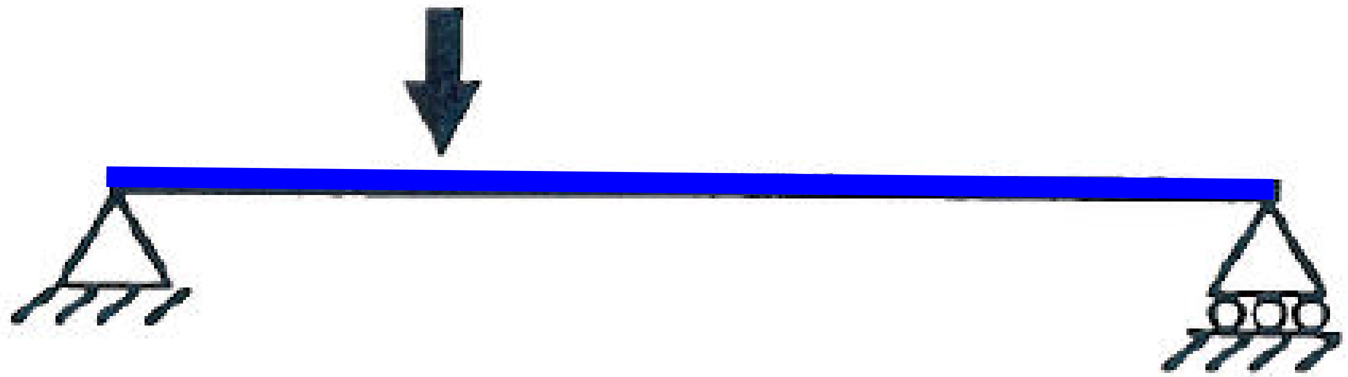
In this beam



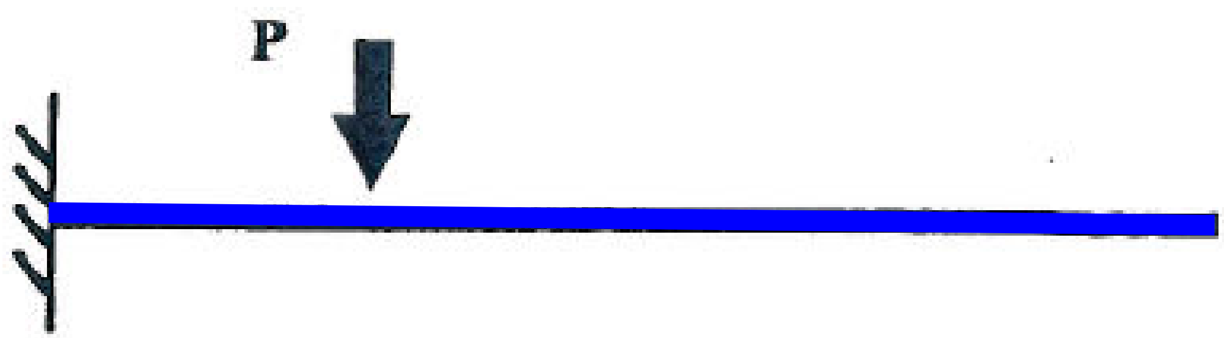
we can see the free body diagram would be



To make this beam statically determinate we could 'cut' the two reaction moments i.e. the beam becomes:



or we could 'cut' both reactions at the right hand end:

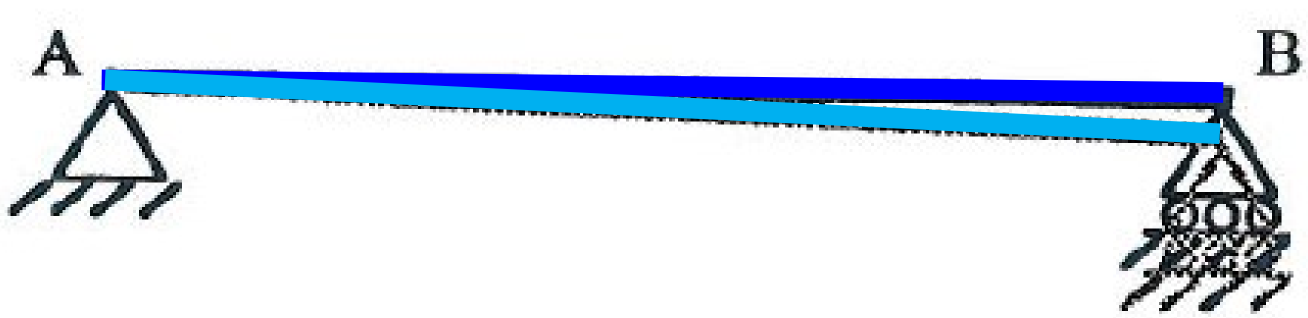


i.e. this beam has two degrees of indeterminacy.

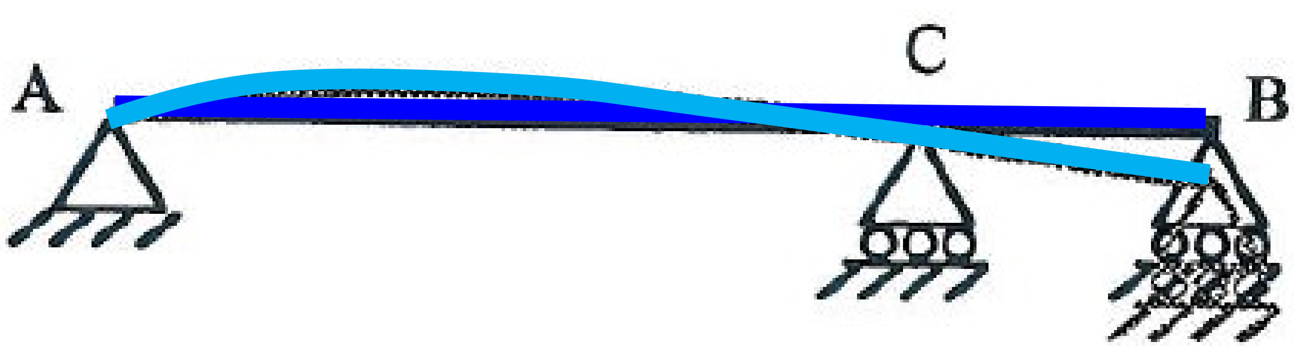
This year we will only be considering cases that have one degree of indeterminacy – in the second year you will deal with more general cases.

Why does statical indeterminacy matter?

- if the beam is statically indeterminate then the structural analysis is more complicated but this analysis will often by performed using a computer program so the complexity of the analysis is normally not significant.
- it is the effect of the indeterminacy on the structural behaviour that is important. Returning to the first example we can see that

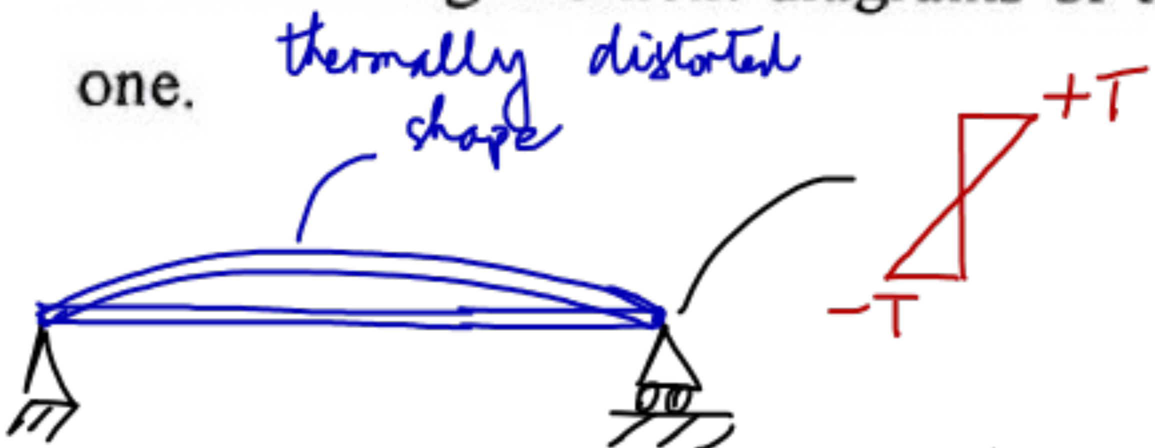


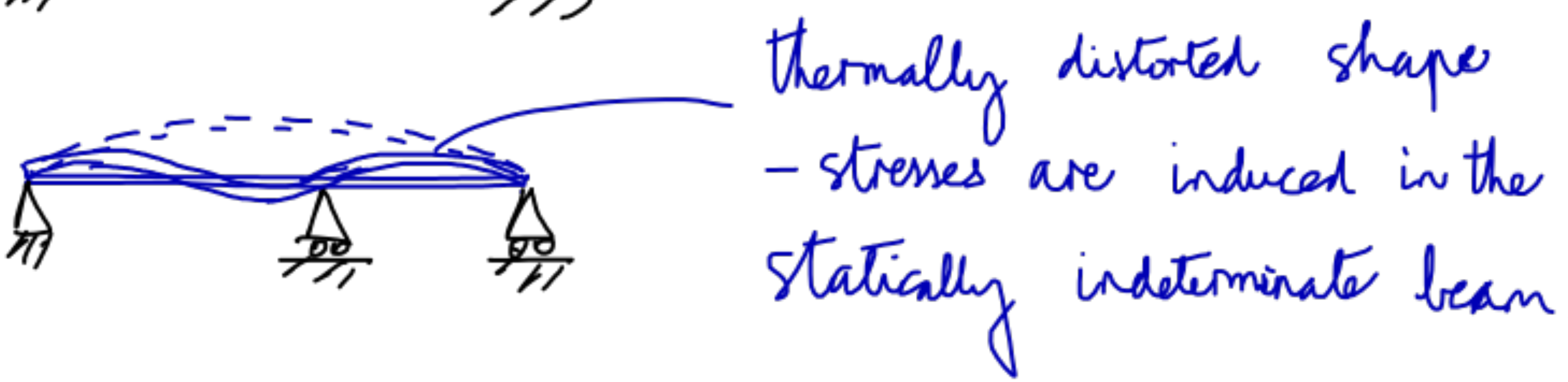
for the statically determinate case if the support at B is moved down a small distance no additional stresses will be induced in the beam.



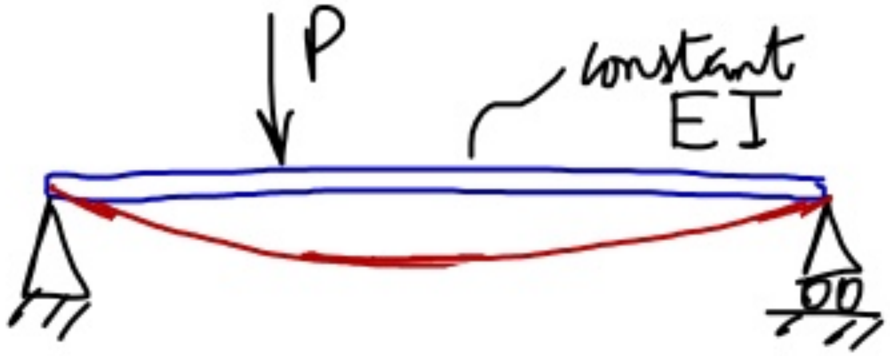
but for the statically indeterminate case a similar movement at B will result in bending of the beam.

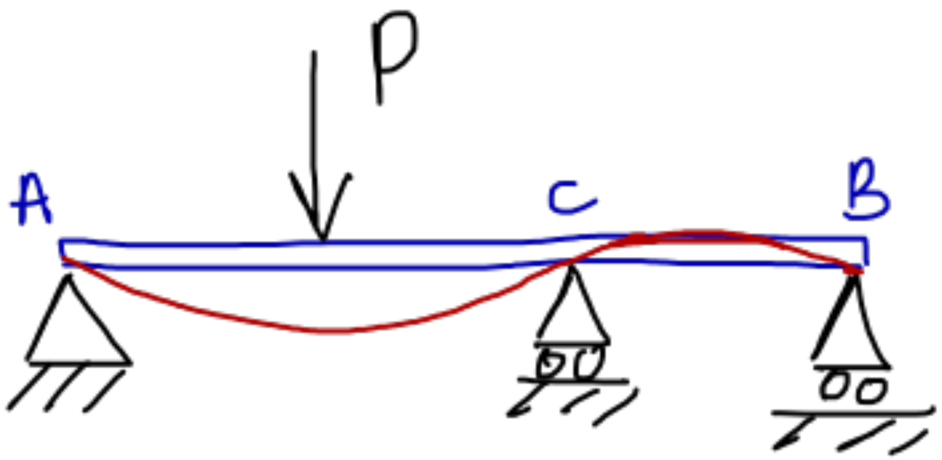
Similarly thermally induced bending would result in stresses in the statically indeterminate beam but none in the statically determinate one. Also changing the beam material or cross section in one part of the beam will alter the shear force/bending moment diagrams of the indeterminate beam but not of the statically one.



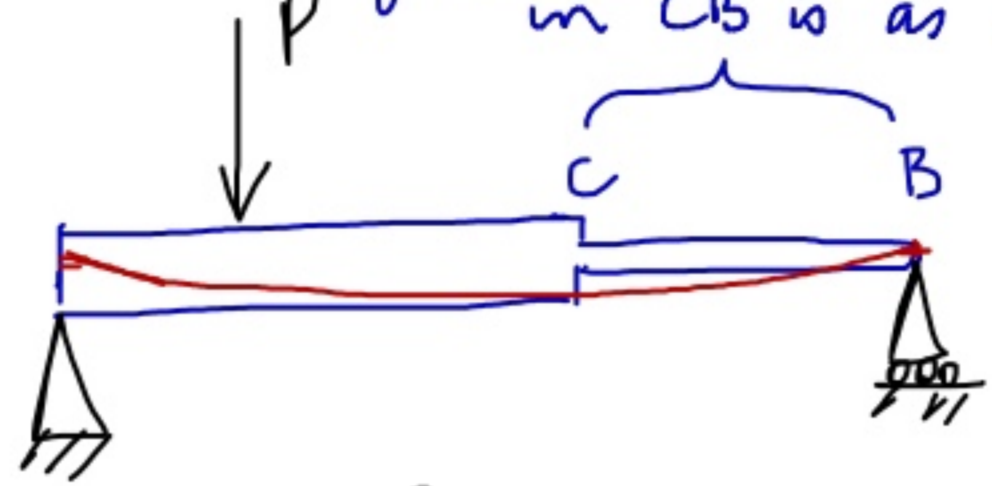


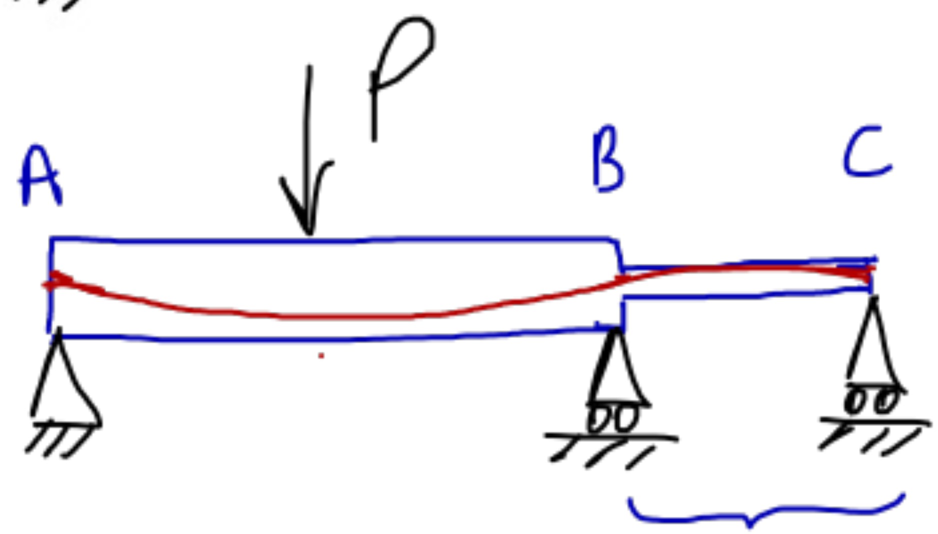
Change in EI





Bending moment remains unchanged throughout the beam & curvature in CB is as original

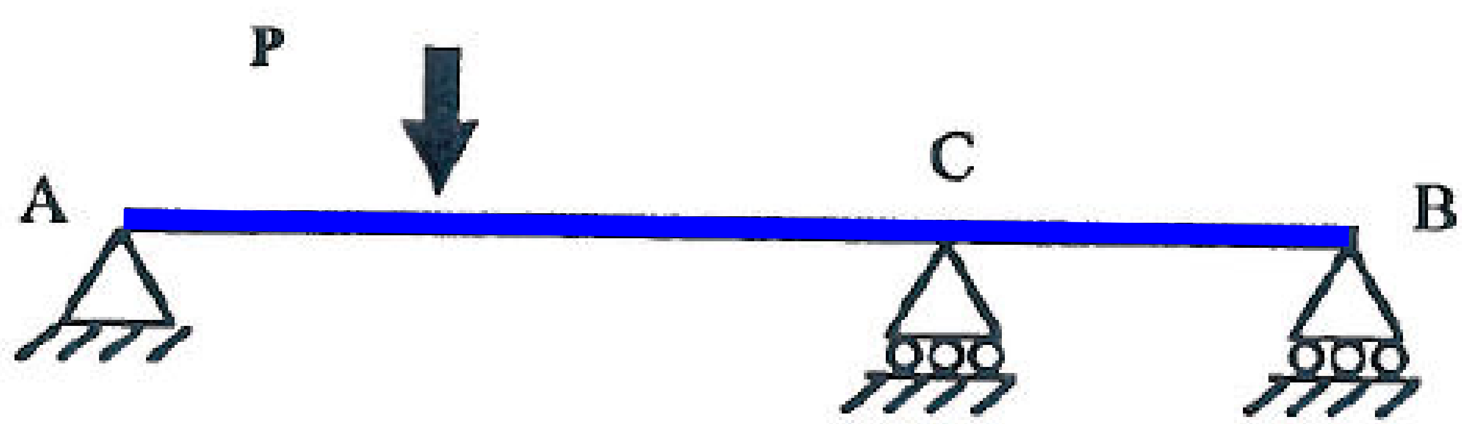




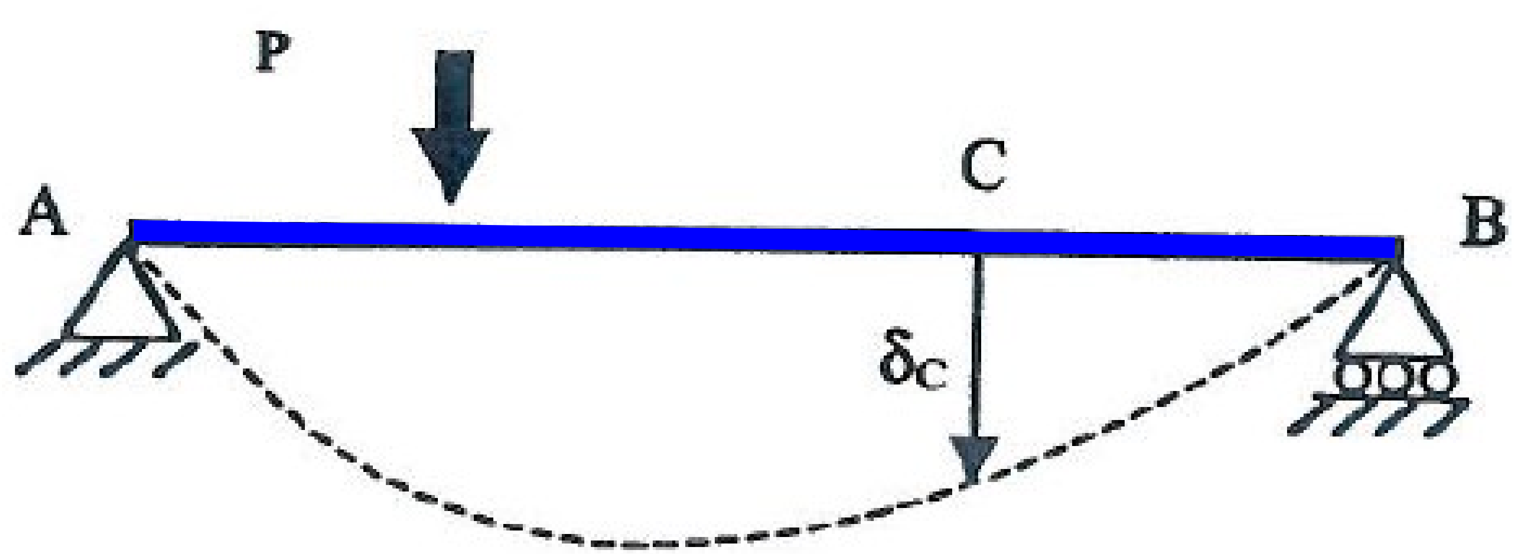
deflections & curvature are reduced in CB and so the BM is also reduced.

1.6.2 Analysing statically indeterminate beams

We have seen that we can not analyse these beams using equilibrium alone. If we consider the following beam again



we know that we can make this determinate by 'cutting' one vertical reaction e.g.



But in this 'cut' beam there is now a deflection, δ_C, which in the original structure is zero. We need to find the vertical reaction force which applied at C will return this deflection to zero (ie satisfy the compatability requirement). Therefore to analyse a statically indeterminate beam we are going to need equations for deflected shapes as well as equations of equilibrium.

1.6.2.1 Using differential relationships

We have already established

equations of equilibrium:

$$\frac{dF}{dz} = -w \quad \text{(i)}$$

$$\frac{dM}{dz} = F \quad \text{(ii)}$$

and equations for deflection:

$$\frac{d^2v}{dz^2} = -\frac{M}{EI} \quad \text{(iii)}$$

combining (i) and (ii) gives

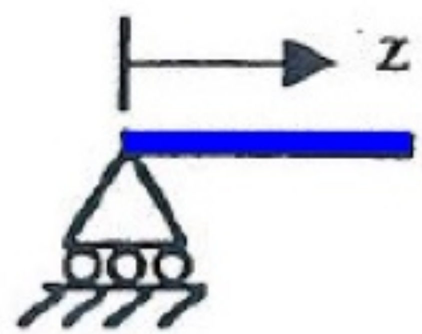
$$\frac{d^2M}{dz^2} = -w$$

and combining this with (iii) gives

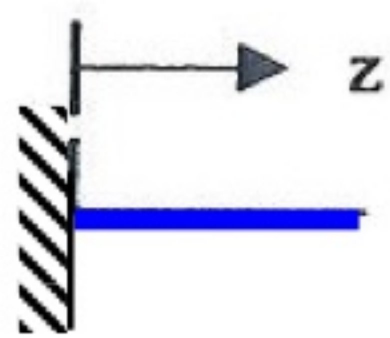
$$\frac{d^2}{dz^2}\left(EI\,\frac{d^2v}{dz^2}\right) = w \quad \text{(iv)}$$

When using (iv) we will generate constants of integration which we will need to evaluate by examining boundary conditions:

Kinematic boundary conditions (ie on deflected shape) e.g.



$$v\big|_{z=0} = 0$$



$$v\big|_{z=0} = 0$$

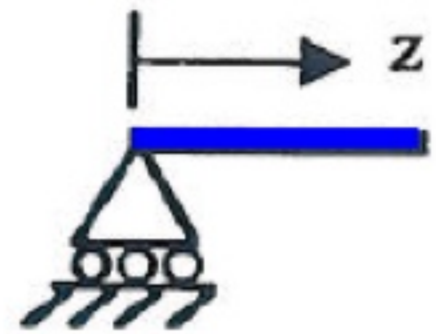
$$\frac{dv}{dz}\bigg|_{z=0} = 0$$

and equilibrium boundary conditions e.g.

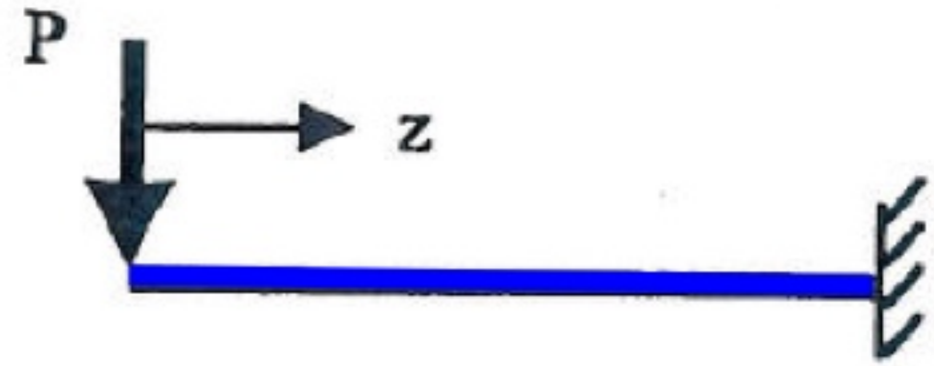
$$M = -EI\frac{d^2v}{dz^2}$$

$$\therefore F = \frac{d}{dz}\left(-EI\frac{d^2v}{dz^2}\right)$$

$$= -EI\frac{d^3v}{dz^3} \text{ for constant } EI$$

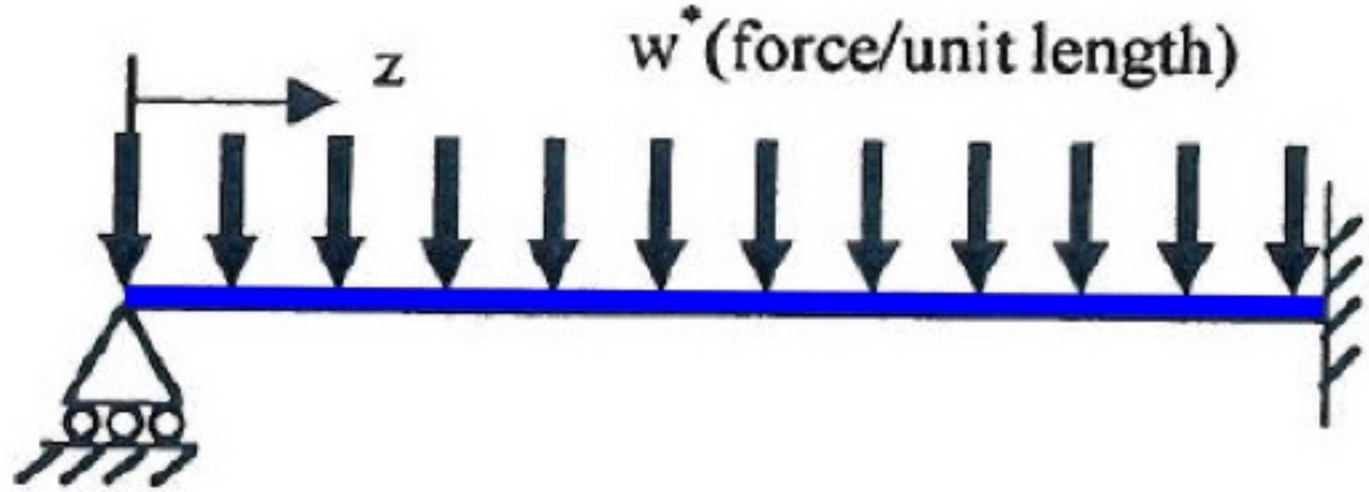


$$M\Big|_{z=0} = 0$$



$$M\Big|_{z=0} = 0 \qquad F\Big|_{z=0} = P$$

Example



EI constant throughout

$$(EIv'')'' = w \qquad (\text{+ve } w \text{ is upwards})$$

$$EIv'''' = -w^*$$

$$EIv''' = -w^* z + C_1$$

In this case there are no boundary conditions on F $(= -EIv''')$

$$\therefore \; EIv'' = -\frac{w^* z^2}{2} + C_1 z + C_2$$

at $z=0$, $M(= -EIv'') = 0 \Rightarrow C_2 = 0$

$$EIv' = -\frac{w^* z^3}{6} + \frac{C_1 z^2}{2} + C_3$$

at $z = L$, $v' = 0$ $\therefore$ $0 = -\frac{w^* L^3}{6} + \frac{C_1 L^2}{2} + C_3$

$$\Rightarrow C_3 = \frac{w^* L^3}{6} - \frac{C_1 L^2}{2}$$

$$\therefore EIv = -\frac{w^* z^4}{24} + \frac{C_1 z^3}{6} + \left(\frac{w^* L^3}{6} - \frac{C_1 L^2}{2}\right) z + C_4$$

at $z = 0$, $v = 0$ $\Rightarrow C_4 = 0$

at $z = L$, $v = 0$ $\Rightarrow$ $0 = -\frac{w^* L^4}{24} + \frac{C_1 L^3}{6} + \frac{w^* L^4}{6} - \frac{C_1 L^3}{2}$

$$\Rightarrow C_1 = \frac{3w^* L}{8}$$

$$\therefore EIv = w^* \left(-\frac{z^4}{24} + \frac{3z^3 L}{48} + \frac{zL^3}{6} - \frac{3L^3 z}{16} \right)$$

$$\therefore EIv = \frac{-w^* L^4}{48}\left(2\left(\frac{z}{L}\right)^4 - 3\left(\frac{z}{L}\right)^3 + \frac{z}{L}\right)$$

Having determined v we can now evaluate F & M using

$F = -EIv'''$ & $M = -EIv''$ (try this)

and so determine the reactions.

e.g. reaction at $z=0$

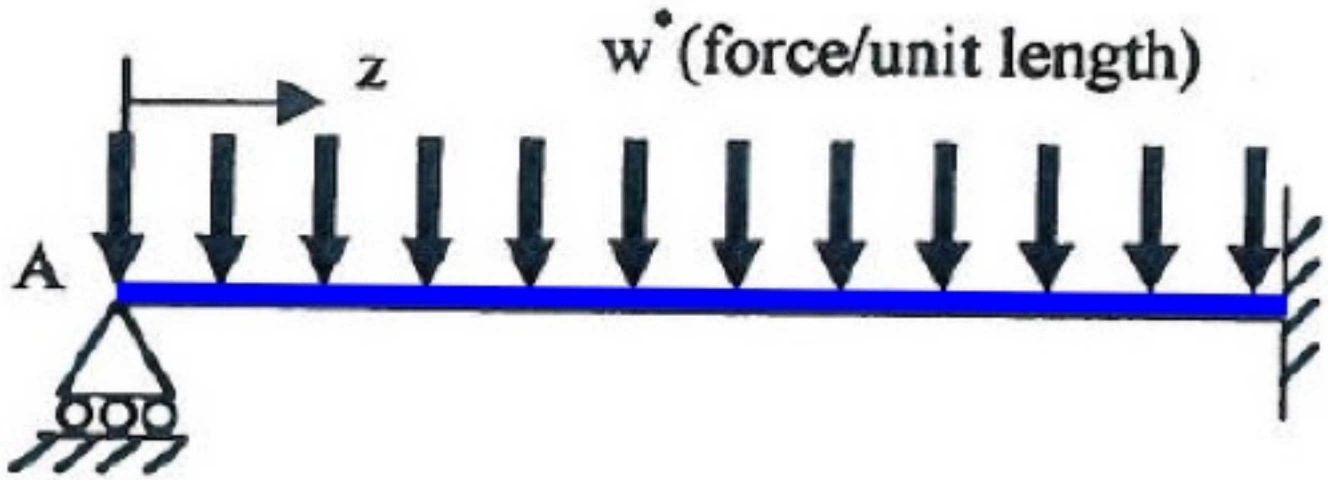
$$\therefore R_1 = -F|_{z=0}$$

$$= \frac{-w^* L^4}{48}\left(\frac{-3.3.2.1}{L^3}\right) = \frac{3w^* L}{8}$$

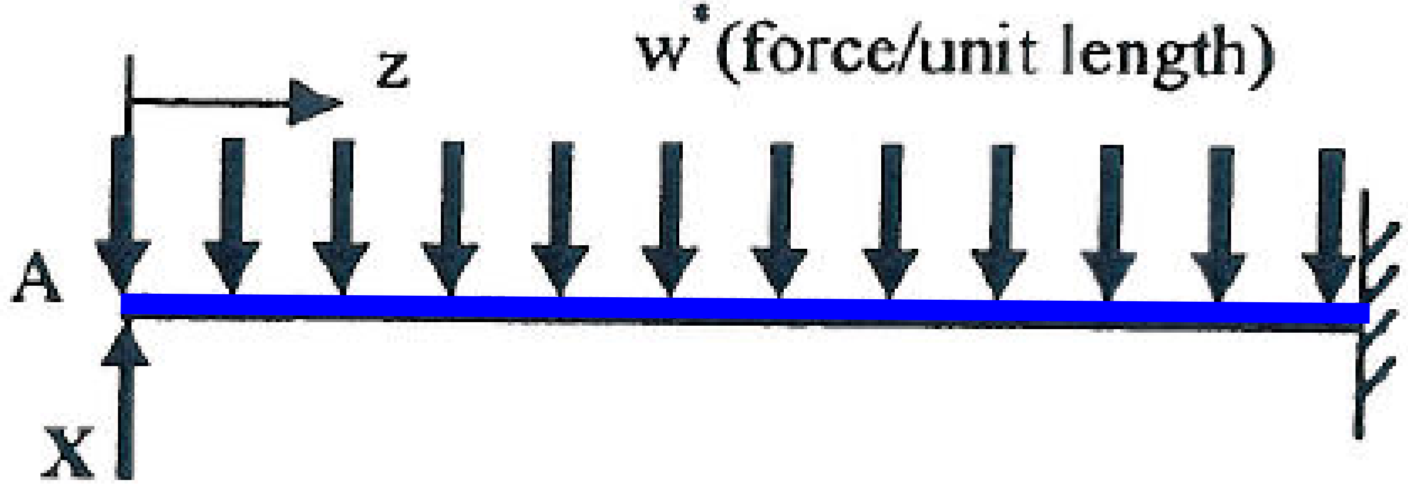
1.6.2.2 Using the unit load method

In this approach we consider one of the unknown reactions in our singly redundant beam as an applied force (or moment). This is a reaction that we could cut to make the beam statically determinate.

Example The beam examined in the previous section



can be considered as



We then note that we can consider this as two separate load cases by linear superpositon:

(Note that the deflections are taken as positive in the direction of X.)

We now note that in the real structure the deflection at A is zero i.e. we require that X is sufficiently large so that

$$\delta_x + \delta_o = 0$$

Noting that $\delta_x = X\delta_1$ where δ_1 is the deflection due to an upward unit load applied at A i.e.

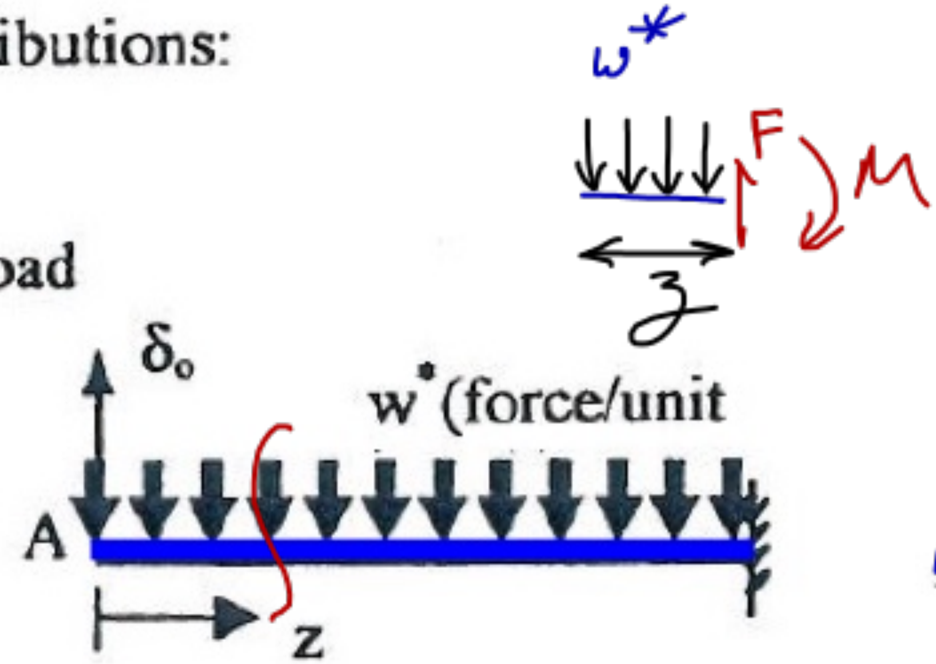
then we require

$$X.\delta_1 + \delta_0 = 0$$

We could determine the deflections δ_o & δ_1 by direct integration but the ~~unit load~~ virtual work method is more convenient.

Moment distributions:

real applied load

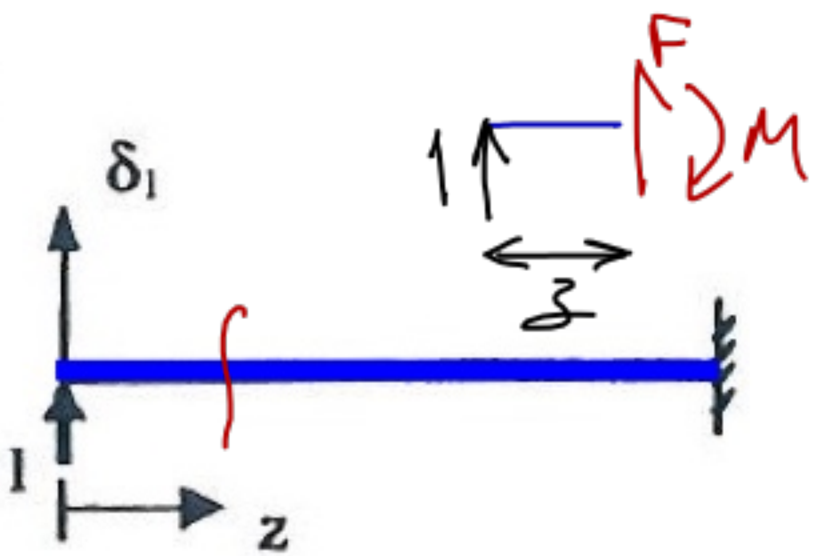


Moments equilib about cut:

$$M - \frac{w^* z^2}{2} = 0$$

$$M_o = \frac{w^* z^2}{2}$$

unit load case



$$M_1 = -z$$

?

$$\frac{-z}{1\delta} \mid z$$

To determine the deflection we will use

$$1.\delta = \int \frac{M}{EI} . \overline{M} . dz$$

where $\overline{M}$ is the bending moment due to a unit upward virtual force at A.

Noting $\bar{M} = M_1$

we have $\delta_0 = \int_0^L \frac{M_0}{EI} \cdot M_1 \, dz$

$$= \frac{-w^*}{2EI} \int_0^L z^3 \, dz$$

$$= \frac{-w^*}{2EI} \left[\frac{z^4}{4} \right]_0^L = \underline{\frac{-w^* L^4}{8EI}}$$

$$\delta_1 = \int_0^L \frac{M_1}{EI} \cdot \bar{M} \, dz = \int_0^L \frac{M_1^2}{EI} \cdot dz$$

$$= \frac{1}{EI} \int_0^L z^2 \cdot dz = \frac{1}{EI} \left[\frac{z^3}{3} \right]_0^L$$

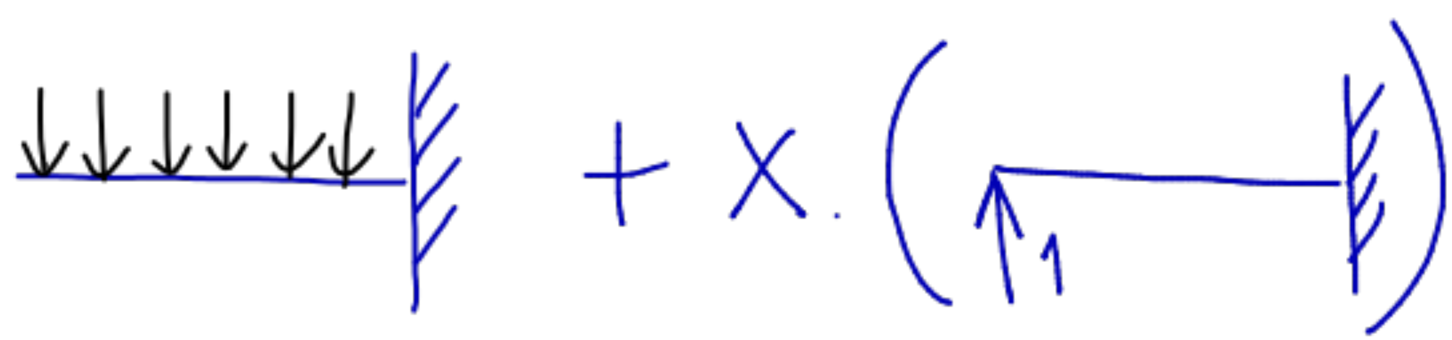
$$= \underline{\frac{L^3}{3EI}}$$

Now $\delta_0 + X\delta_1 = 0$

$$\therefore \frac{-w^* L^4}{8EI} + \frac{X L^3}{3EI} = 0$$

$$\therefore X = \frac{3}{8} w^* L$$

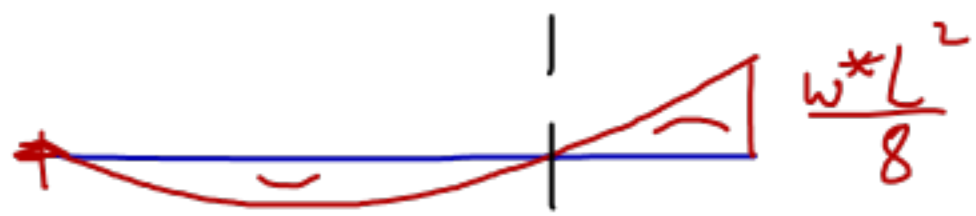
Bending moment, shear force & reactions can be determined by adding the two load cases:



So, for example, M is given by:

$$M = \frac{w^* z^2}{2} + \frac{3w^* L}{8}(-z)$$

$\therefore$ M diagram:

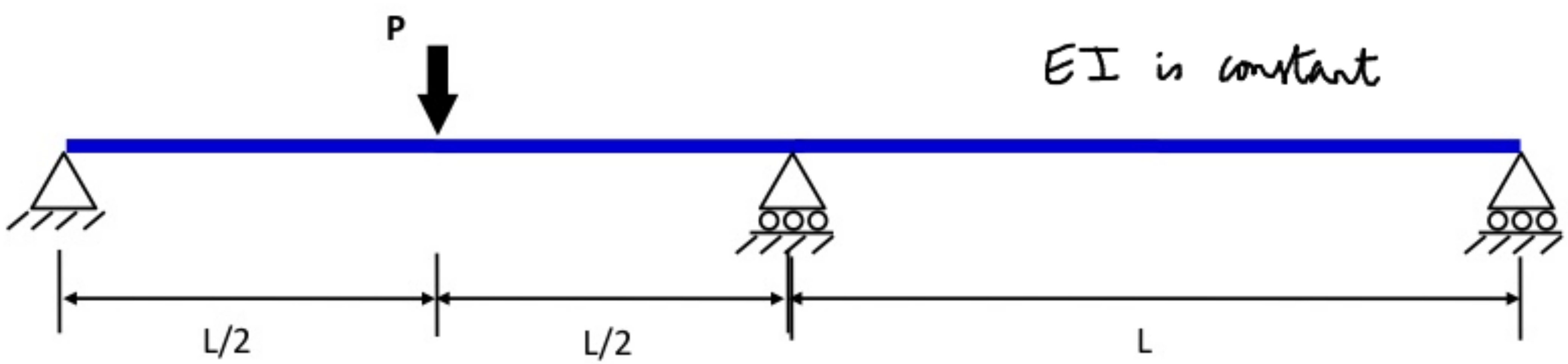


Deflected shape:

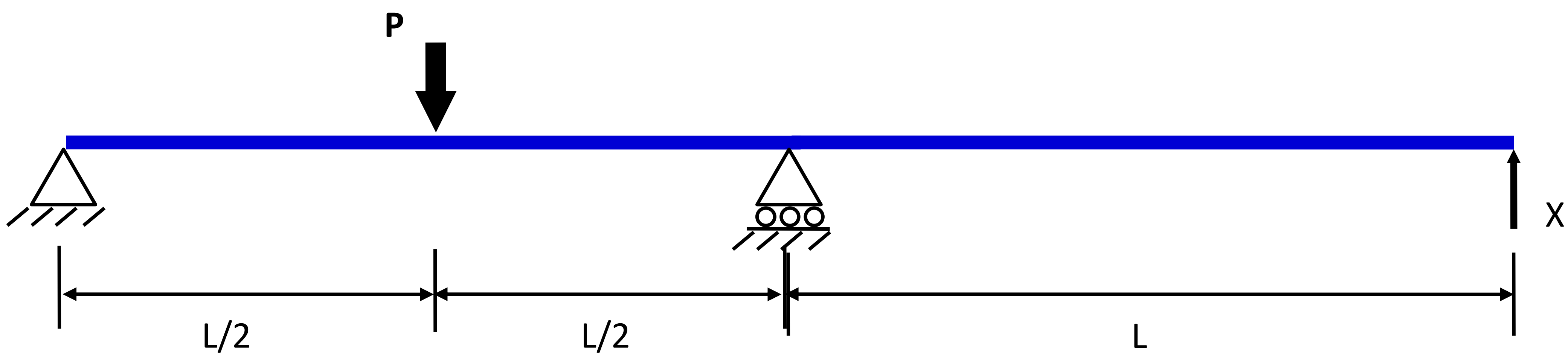
Section 1.6.2.2 (cont.) Unit load method for statically indeterminate beams

Second example

Determine the bending moment diagram for the statically indeterminate beam shown below.

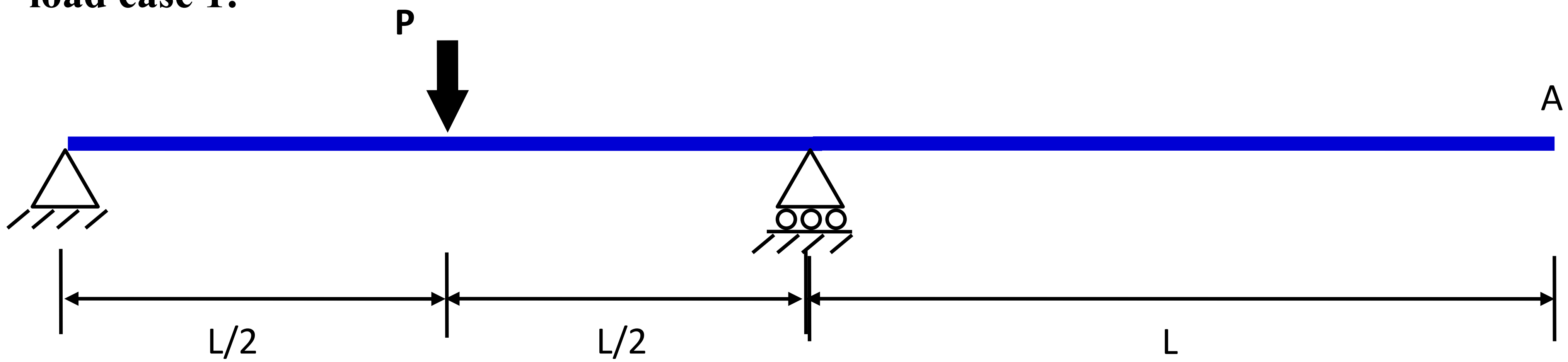


We will consider the vertical reaction at the right hand support as an applied force X (ie we are ‘cutting’ the right hand support):



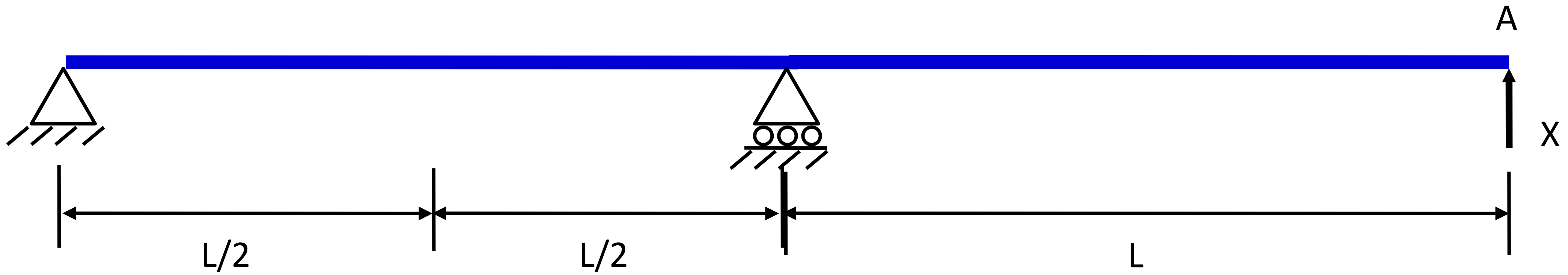
Now this can be considered as a superposition of two load cases :

load case 1:



and

load case 2:



If the upward deflection at A due to the real applied load (with X=0) is δ_o

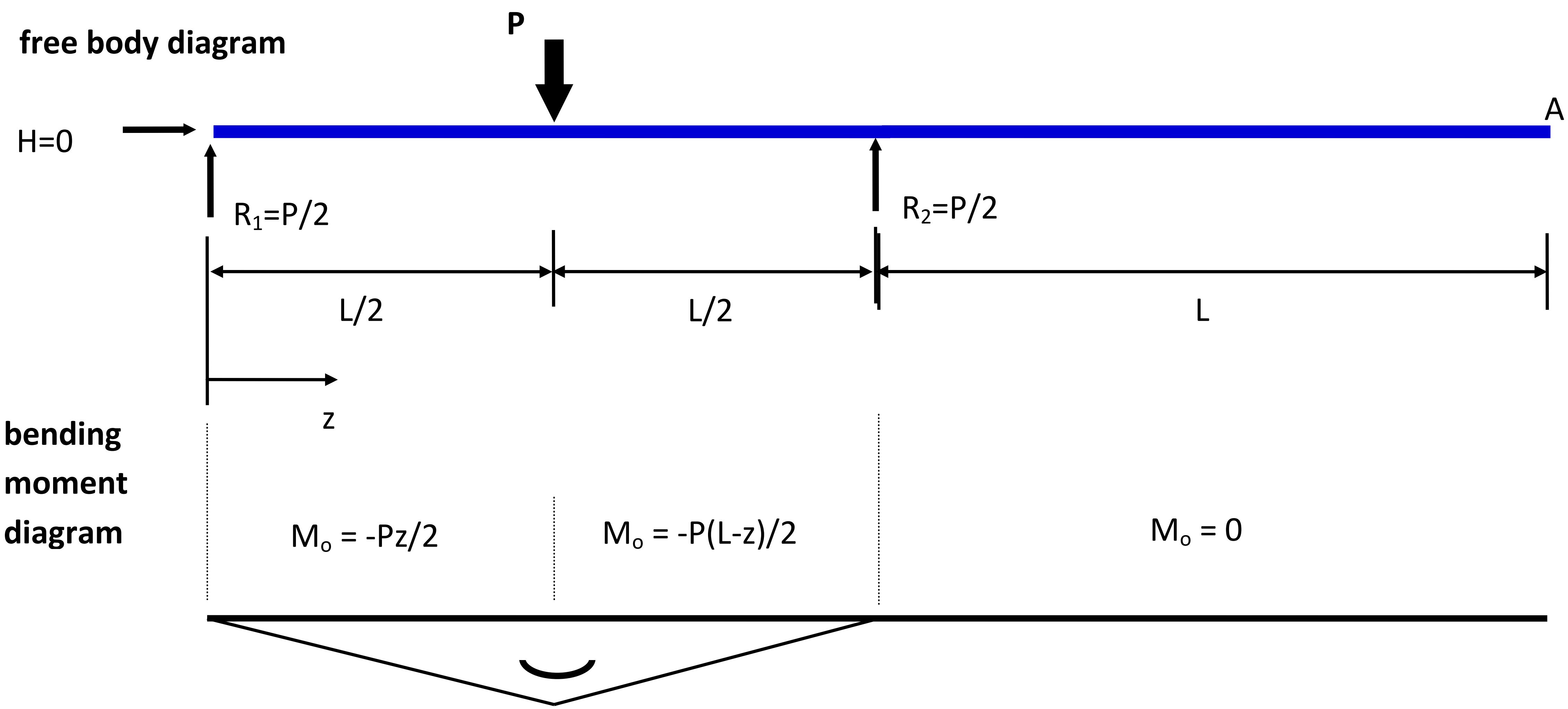
and the upward deflection at A due to X=1 is δ_1

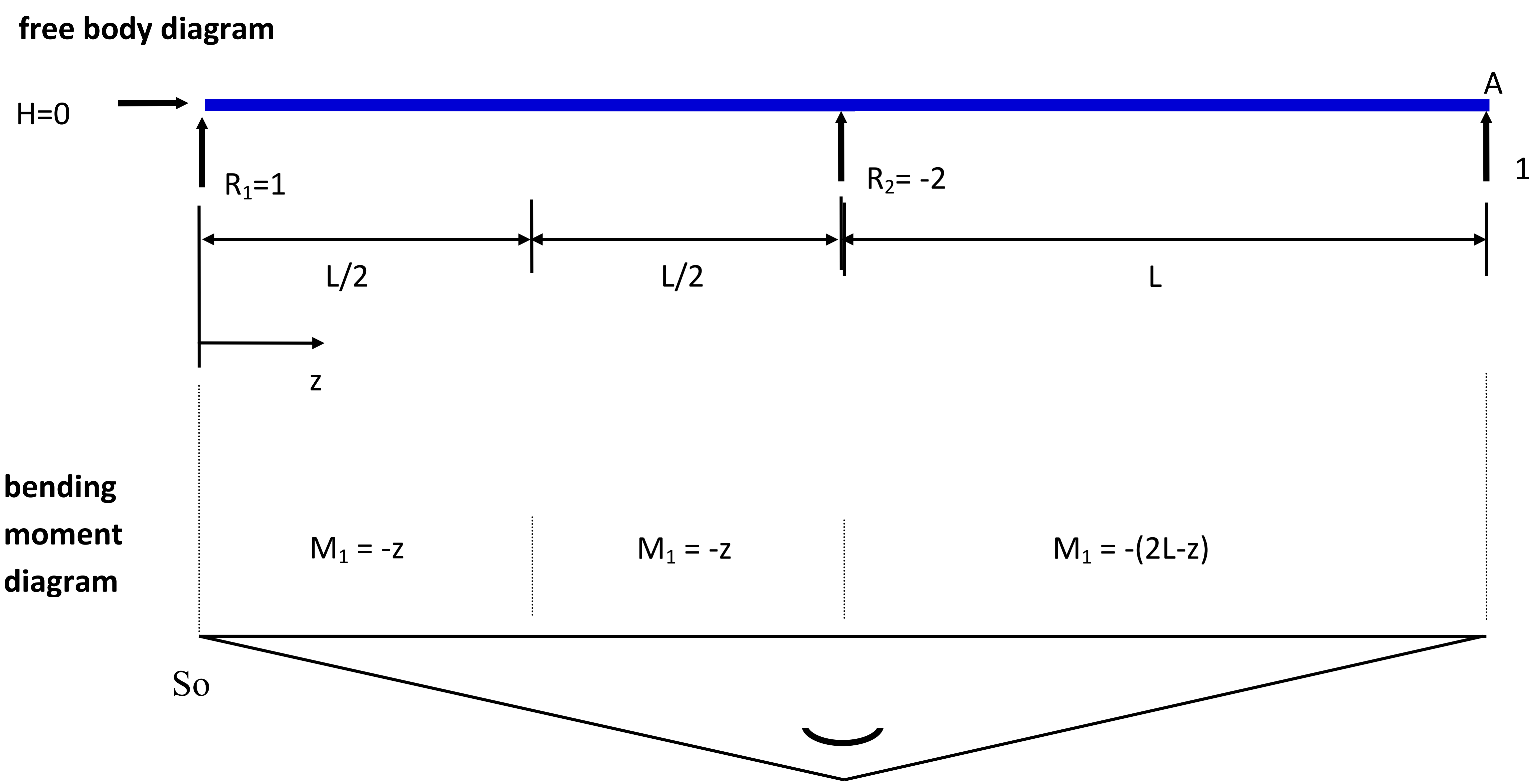
then, since in the actual structure the deflection at A is zero, we require

$$\delta_o + X.\ \delta_1 = 0$$

Using the virtual work method to calculate the deflections δ_o and δ_1 (ie $\delta = \int \frac{M}{EI} \overline{M} dz$) we require the moment due to the real applied load (M_o) and the moment due to an upward unit load applied at A ($M_1 = \overline{M}$).

You should be able to show :


free body diagram
P
A
H=0
R1=P/2
R2=P/2
L/2
L/2
L
z
bending
moment
diagram
Mo = -Pz/2
Mo = -P(L-z)/2
Mo = 0


free body diagram
H=0
A
R1=1
R2= -2
1
L/2
L/2
L
z
bending
moment
diagram
M1 = -z
M1 = -z
M1 = -(2L-z)
So

$$\delta_o = \int_0^{2L} \frac{M_o}{EI} M_1 dz$$

$$\delta_o = \int_0^{L/2} \frac{M_o}{EI} M_1 dz + \int_{L/2}^{L} \frac{M_o}{EI} M_1 dz + \int_{L}^{2L} \frac{M_o}{EI} M_1 dz$$

$$\delta_o = \int_0^{L/2} \frac{-Pz}{2EI}.(-z)dz + \int_{L/2}^{L} \frac{-P(L-z)}{2EI}.(-z)dz + 0$$

$$\underline{\delta_o = \frac{PL^3}{48EI} + \frac{PL^3}{24EI} = \frac{PL^3}{16EI}}$$

and

$$\delta_1 = \int_0^{2L} \frac{M_1^{\ 2}}{EI} dz$$

$$\delta_1 = \int_0^{L} \frac{M_1^{\ 2}}{EI} dz + \int_L^{2L} \frac{M_1^{\ 2}}{EI} dz$$

$$\delta_1 = \int_0^{L} \frac{z^2}{EI} dz + \int_L^{2L} \frac{(2L - z)^2}{EI} dz + 0$$

$$\underline{\delta_1 = \frac{L^3}{3EI} + \frac{L^3}{3EI} = \frac{2L^3}{3EI}}$$

Recalling that we require

$$\delta_o + X.\ \delta_1 = 0$$

then

$$X = -\ \delta_o / \ \delta_1 = -\frac{3P}{32}$$

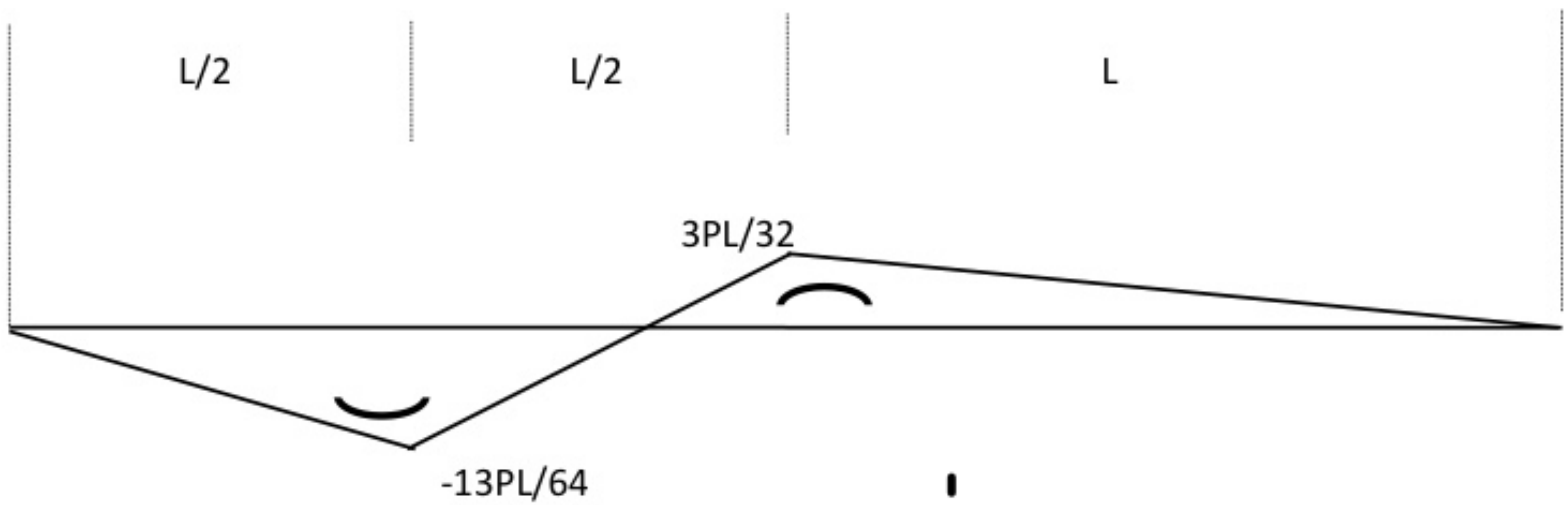
We can now use linear superposition to determine the reactions, bending moments etc in the real structure. Eg. the reaction at the central support is given by

reaction at central support $+ \left(-\frac{3P}{32}\right)$ x (reaction at cental support

due to load case 1 due to load case 2)

$$= P/2 + \left(-\frac{3P}{32}\right)(-2) = \underline{\left(\frac{11P}{16}\right)}$$

You should be able to confirm that the bending moment diagram for the statically indeterminate beam is as shown below.

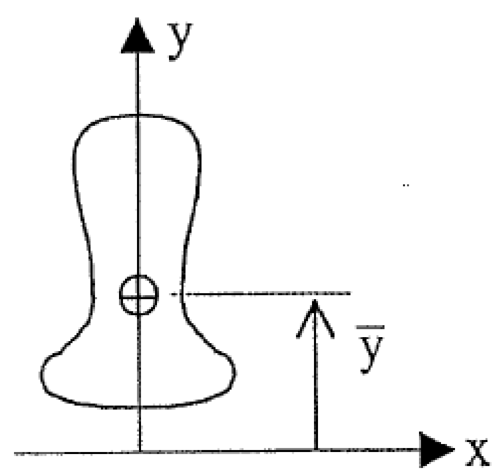


APPENDIX 1

Centroid Determination and First Moments of Area

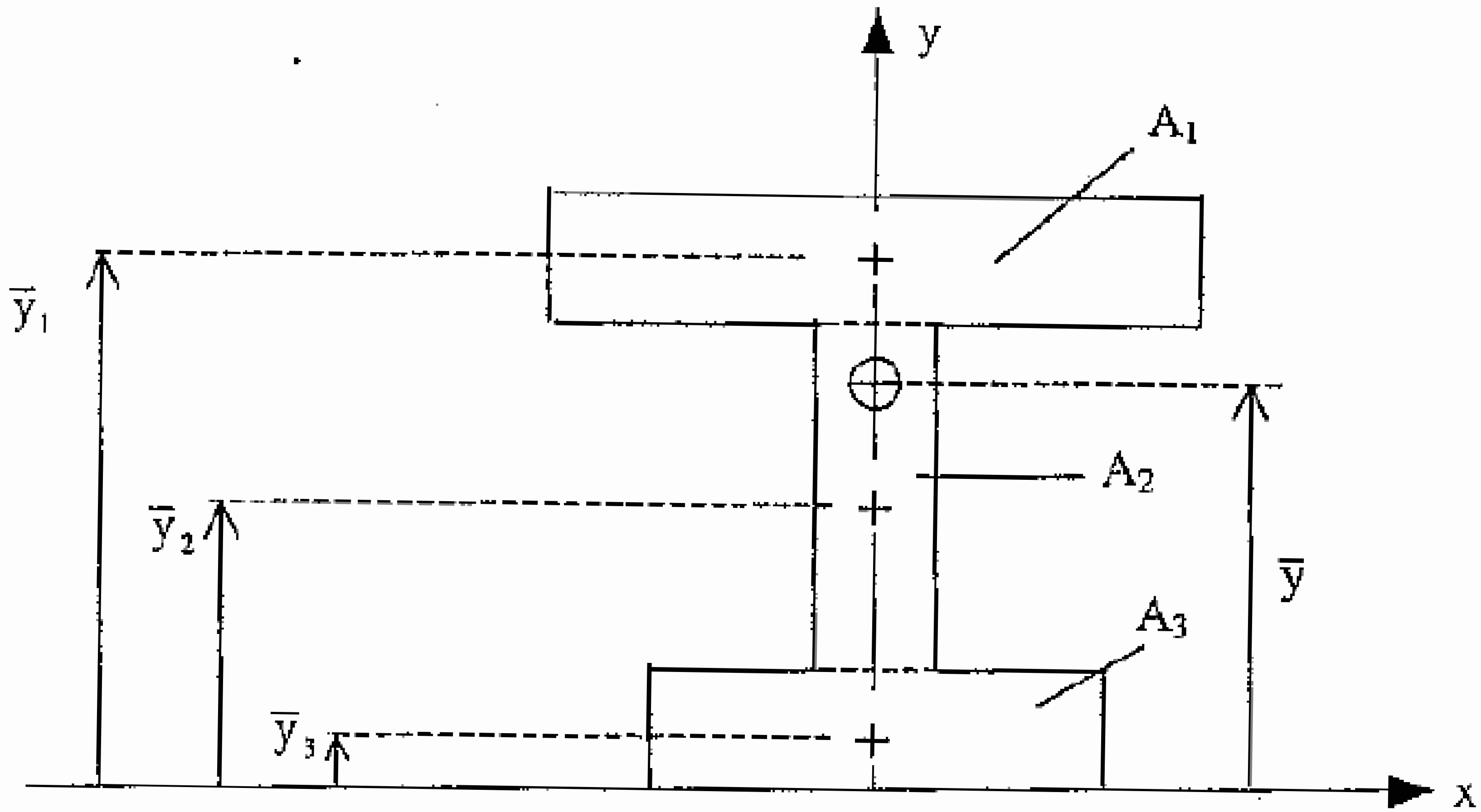
We can determine the y co-ordinate, $\overline{y}$, of the centroid of a section by using the following equation

$$\overline{y}A = \int_{section} y\,dA \qquad \text{(i)}$$



Note: ⊕ indicates the centroid of the section

If the section can be considered as a collection of simple shapes with known centroids



Note: + indicates the centroid of each rectangle

where $\bar{y}_1, \bar{y}_2, \bar{y}_3$ locate the centroids of each of the rectangles of area A_1, A_2, A_3

then equation (i) can be re-written as

$$\bar{y}A = \int_{A_1} y\,dA + \int_{A_2} y\,dA + \int_{A_3} y\,dA \qquad \text{(ii)}$$

But each of the integrals in equation (ii) can be re-written using equation (i) e.g.

$$\int_{A_1} y \, dA = \bar{y}_1 A_1 \qquad \textbf{(iii)}$$

So equation (ii) becomes

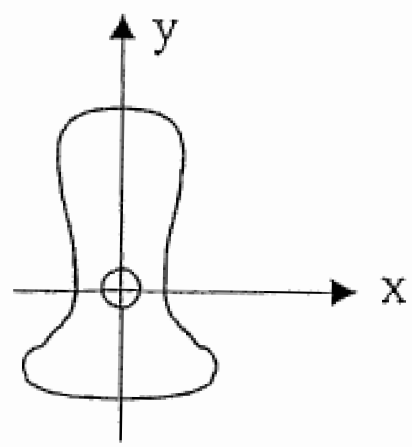
$$\bar{y}A = \bar{y}_1 A_1 + \bar{y}_2 A_2 + \bar{y}_3 A_3 \qquad \textbf{(iv)}$$

i.e.

$$\bar{y} = \frac{\bar{y}_1 A_1 + \bar{y}_2 A_2 + \bar{y}_3 A_3}{A} \qquad \textbf{(v)}$$

Second Moments of Area, I

For ETB we require the second moment of area about an axis passing through the centroid of the section

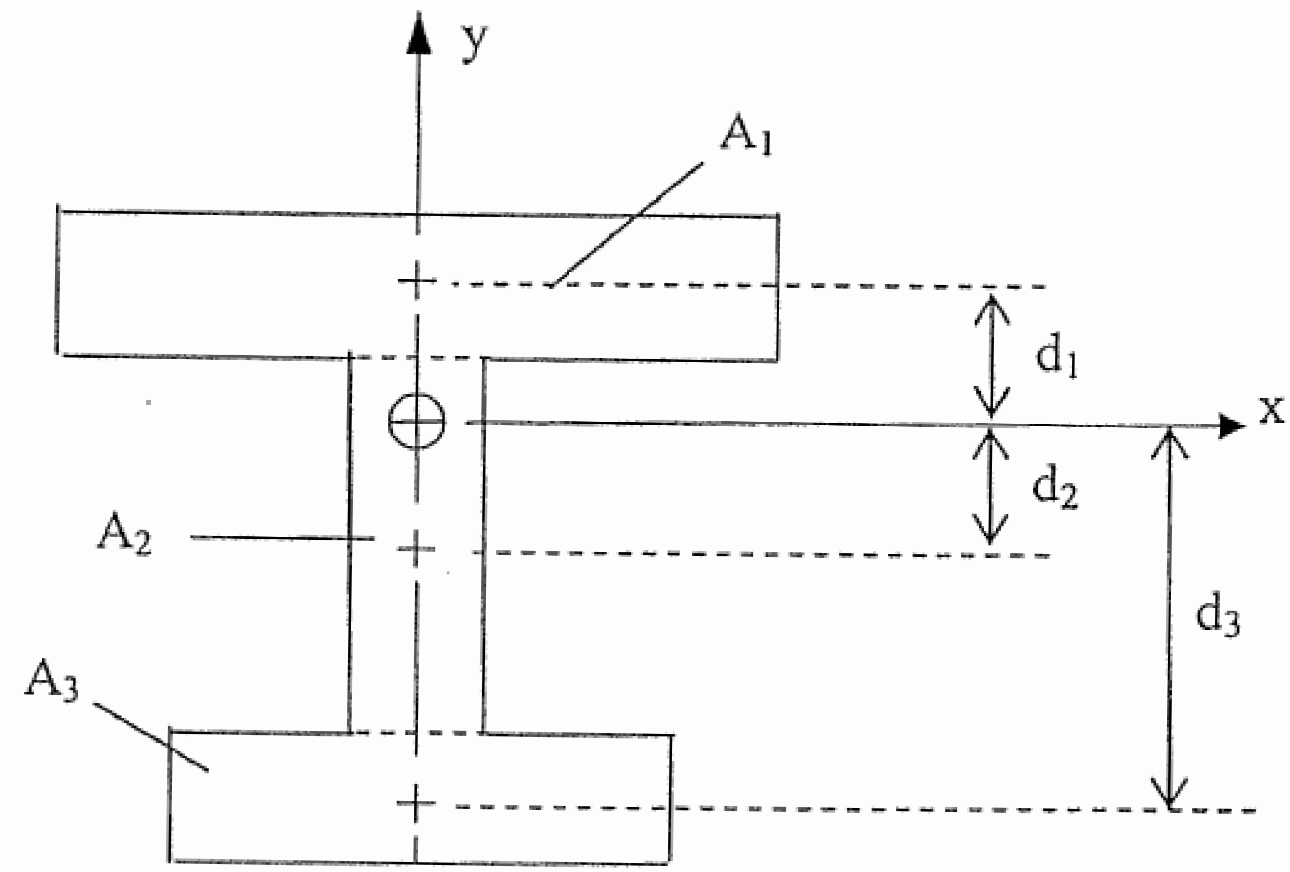


Note: $\oplus$ indicates the centroid of the section

I about the x-axis is given by

$$I = \int_{\text{Section}} y^2 dA \qquad \textbf{(i)}$$

If the section can be considered as a collection of simple shapes



Note: + indicates the centroid of each rectangle

then the integral in equation (i) can be performed over each of the simple shapes:

$$I = \int_{A_1} y^2\, dA + \int_{A_2} y^2\, dA + \int_{A_3} y^2\, dA = I_x\Big|_{A_1} + I_x\Big|_{A_2} + I_x\Big|_{A_3} \qquad \text{(ii)}$$

This equation can be re-written using the parallel axis theorem, see Appendix 3, which for area A_1 can be expressed as

$$I_x\Big|_{A_1} = I_{x_1}\Big|_{A_1} + d_1^{\,2} A_1 \qquad \textbf{(iii)}$$

$$\begin{bmatrix} \text{I about some} & & \text{I about its} & & \\ \text{parallel axis} & = & \text{centroid axis} & + & \text{Area x (distance between axes)}^2 \end{bmatrix}$$

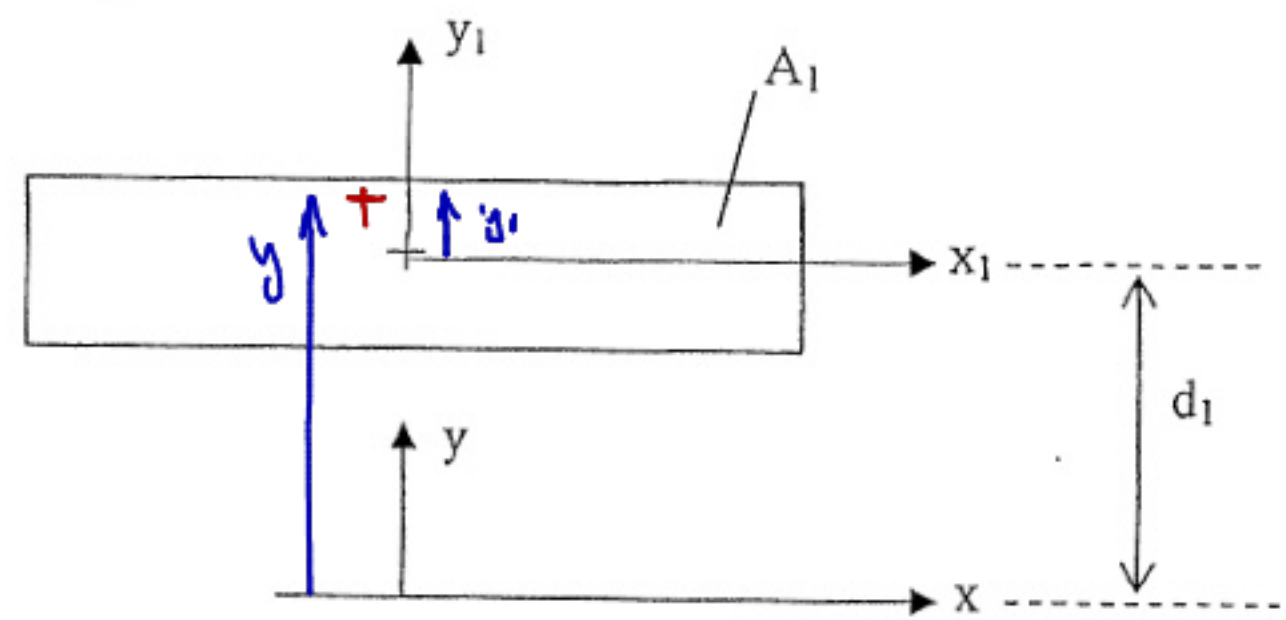
so using (iii), equation (ii) becomes

$$I = I_{x_1}\Big|_{A_1} + d_1^{\,2}A_1 + I_{x_2}\Big|_{A_2} + d_2^{\,2}A_2 + I_{x_3}\Big|_{A_3} + d_3^{\,2}A_3 \qquad \textbf{(iv)}$$

where x_1, x_2, x_3 are the centroidal axes for A_1, A_2, A_3 and d_1, d_2, d_3 are the distances of these axes from the centroidal x - axis of the whole section.

Parallel Axis Theorem

Consider an area A_1, as shown below where x_1, y_1 are the axes with their origin at the centroid of the area and x, y are some parallel set as shown.



Note: $y = y_1 + d_1$

The I value about the x axis, I_x, is given by

$$I_x = \int_{A_1} y^2 \, dA = \int_{A_1} (y_1 + d_1)^2 dA = \int_{A_1} y_1^2 \, dA + 2d_1 \int_{A_1} y_1 \, dA + d_1^2 \int_{A_1} dA$$

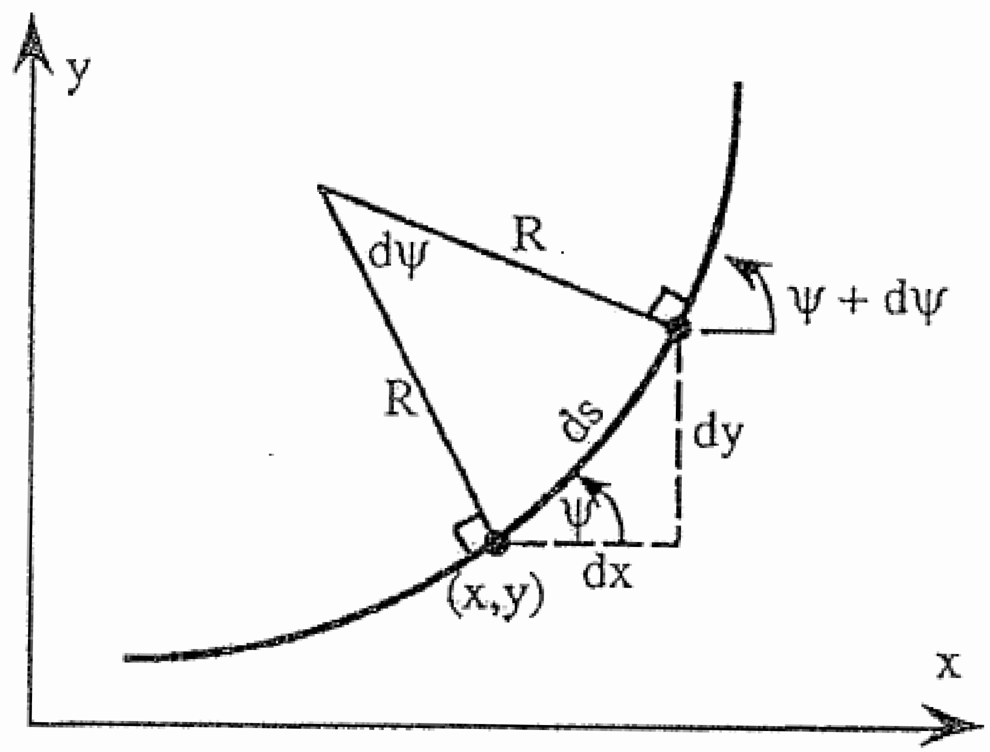
Now $\int_{A_1} y_1 \, dA = 0$, since x_1 is a centroidal axis and $\int_{A_1} dA = A_1$

$$\text{Therefore } \int_{A_1} y^2 \, dA = \int_{A_1} y_1^{\,2} \, dA + d_1^{\,2} A_1$$

$$\text{i.e. } I_x \Big|_{A_1} = I_{x_1} \Big|_{A_1} + d_1^{\,2} A_1$$

$$\left[\begin{array}{c} \text{I about some} \\ \text{parallel axis} \end{array} = \begin{array}{c} \text{I about its} \\ \text{centroid axis} \end{array} + \text{Area x (distance between axes)}^2 \right]$$

Radius of Curvature of any Curve y(x)



Consider a small arc ds extending from the point (x,y) to (x + dx, y + dy) at which the slope ψ changes to ψ + dψ.

The slope $\psi = \frac{dy}{dx}$ is not considered small.

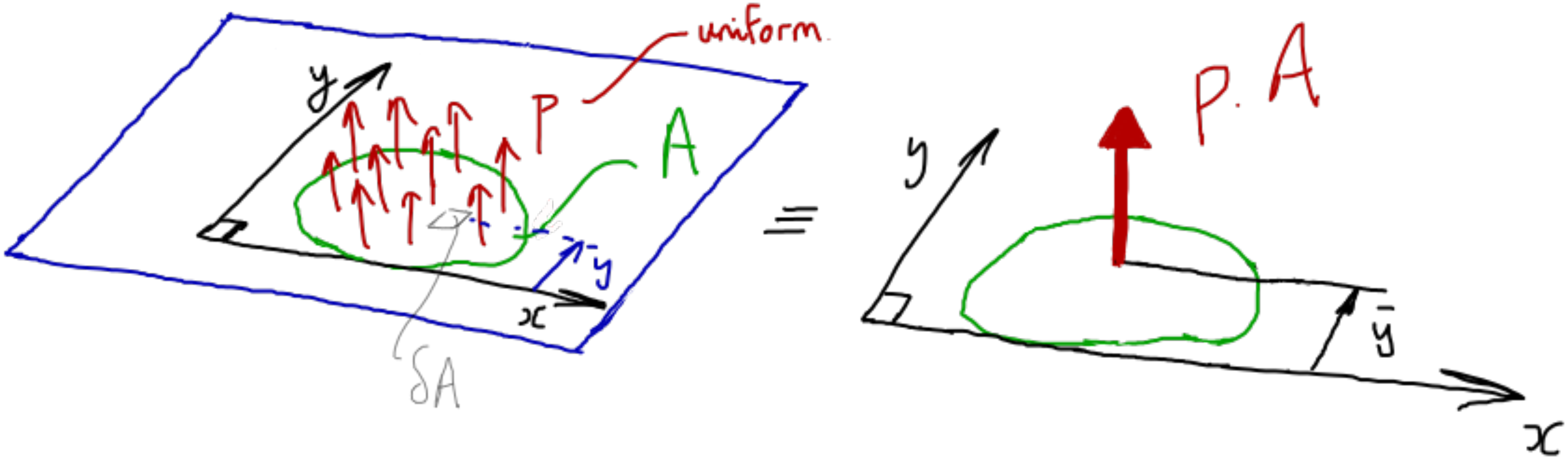
Clearly $ds = R d\psi$, $\therefore \frac{1}{R} = \frac{d\psi}{ds}$

Now $\psi = \tan^{-1}\frac{dy}{dx} = \tan^{-1} y'(x)$ say.

$$\therefore \frac{d\psi}{dx} = \frac{d\psi}{d(y')} \cdot \frac{d(y')}{dx} = \frac{1}{1 + (y')^2} \cdot \frac{d^2y}{dx^2} \qquad \textbf{(a)}$$

Also $ds^2 = dy^2 + dx^2 = dx^2(1 + (y')^2)$ **(b)**

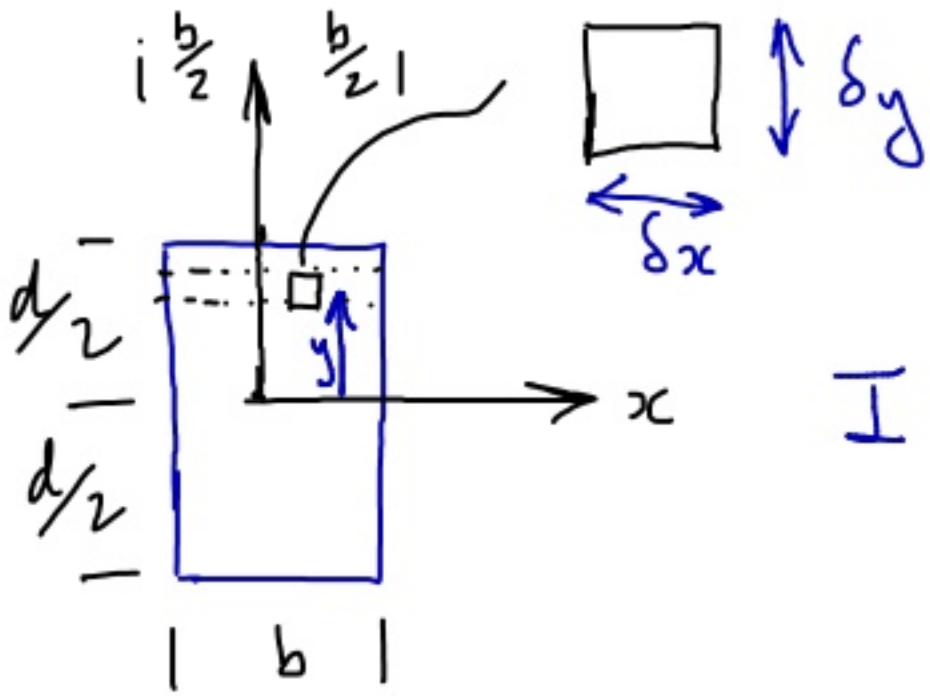
$$\therefore \frac{1}{R} = \frac{d\psi}{ds} = \frac{d\psi}{dx} \cdot \frac{dx}{ds} = \frac{d^2y}{dx^2} \bigg/ \left[1 + \left(\frac{dy}{dx}\right)^2\right]^{\frac{3}{2}} \text{ using (a) and (b).}$$



$$\int_A p y \, dA = pA \cdot \bar{y}$$

$$\therefore \bar{y} = \frac{\int_A y \, dA}{A}$$

This is the y-coordinate of the centroid



$$I = \int_{\text{section}} y^2 \, dA = \int_{-d/2}^{d/2} \int_{-b/2}^{b/2} y^2 \cdot dx \cdot dy$$

$$= \int_{-d/2}^{d/2} \left[y^2 x \right]_{-b/2}^{b/2} \cdot dy$$

$$= \left[\frac{y^3}{3} \cdot b \right]_{-d/2}^{d/2}$$

$$= \underline{\frac{d^3 b}{12}}$$