Solving generic parametric linear matrix inequalities

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Parametric linear matrices

Let $\boldsymbol{y} = (y_1, \dots, y_t)$, $\boldsymbol{x} = (x_1, \dots, x_n)$.

$f \in \mathbb{Q}[\boldsymbol{y}][\boldsymbol{x}]_{\leq 1}$ = linear polynomial in $\boldsymbol{x}$ parametric in $\boldsymbol{y}$

$A \in \mathbb{S}_m(\mathbb{Q}[\boldsymbol{y}][\boldsymbol{x}]_{\leq 1})$ = parametric linear matrix

$$A(y,x) = \begin{bmatrix} y_1y_2 & x_1 & y_2^3x_2 \\ x_1 & y_2+y_3 & y_1x_3 \\ y_2^3x_2 & y_1x_3 & y_1^2+y_3 \end{bmatrix}$$

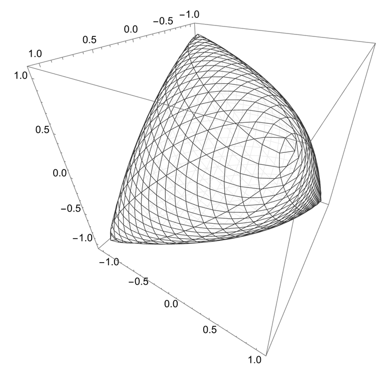

Figure 1. Spectrahedron $\{x \in \mathbb{R}^3 : A(y,x) \succeq 0\}$, with $y = (1,1,0)$

Problem

Let $A \in \mathbb{S}_m(\mathbb{Q}[\boldsymbol{y}][\boldsymbol{x}]_{\leq 1})$, and $\mathcal{P}$ be the set of parameters $y \in \mathbb{R}^t$ such that the parametric **linear matrix inequality** (LMI)

$$A(y,\cdot) \succeq 0$$

is feasible, i.e. for some $x \in \mathbb{R}^n$, $A(y,x)$ is positive semidefinite / only has nonnegative eigenvalues.

Goal: extract a formula Φ describing (a dense subset of) $\mathcal{P}$

$$\begin{array}{ccc} A \succeq 0 & \xrightarrow{\text{[Naldi-Safey El Din-Taylor-W 2025]}} & \Phi \\ \downarrow{\scriptstyle\text{specialize}} & & \downarrow{\scriptstyle\text{specialize}} \\ A(y,\cdot) \succeq 0 & \xrightarrow[\text{[Henrion-Naldi-Safey El Din 2016]}]{} & \Phi(y) \end{array}$$

Applications

- Parametric sum-of-squares problem
  [Motzkin 1967] [Robinson 1973]
- Convergence analyses of first-order optimization methods
  [Drori-Teboulle 2014] [Taylor-Hendrickx-Glineur 2018] [Kim-Fessler 2021] [Lieder 2021] [Drori-Taylor 2023]

Quantifier Elimination?

Let $g_0, \dots, g_m$ be the coefficients of $\det(A + \lambda I_m)$ in λ. Then,

$$A(y,x) \succeq 0 \iff g_0(y,x) \geq 0 \wedge \ldots \wedge g_m(y,x) \geq 0.$$

Compute Φ = eliminate $\exists x$ in the formula (QE)

Contributions

Input
Parametric LMI

Quantifier elimination

Output
Feasibility formula

Parametric SolveLMI
[Naldi-Safey El Din-Taylor-W 2025]

Real root classification
[Gaillard-Safey El Din 2024]

0-dim systems in $\mathbb{Q}(\boldsymbol{y})[\boldsymbol{x}, \dots]$

Figure 2. Algorithmic pipeline

State of the Art

- CAD: doubly exponential in n [Collins 1975]
- Border polynomials
  [Yang-Xia 2005] [Liang-Jeffrey-Maza 2008] [Moroz 2006] [Lazard-Rouillier 2007] [Le-Safey El Din 2022]

Parametric SolveLMI

Parametric LMI — Parametric SolveLMI → 0-dim systems in $\mathbb{Q}(\boldsymbol{y})[\boldsymbol{x}, \dots]$

rank stratification

parametrization

Parametric determinantal varieties

Parametric sample points in each connected component

specialize

specialize

Determinantal varieties — SolveLMI → Sample points in each connected component

[Henrion-Naldi-Safey El Din 2016]

Figure 3. Reduction to zero-dimensional systems

Arithmetic Complexity

For generic $A \in \mathbb{S}_m(\mathbb{Q}[\boldsymbol{y}]_{\leq d}[\boldsymbol{x}]_{\leq 1})$, # of operations in $\mathbb{Q}$:

$$2^{O(mt)} n^{O(1)} (md)^{O(t)} (\delta\Delta)^{O(t)}$$

where

$$\delta \in n^{O(m^2)} \quad \text{and} \quad \Delta \in e^{O(m^2 \log m)} n^{O(1)} d^{O(m^2+t)} t^{O(1)}.$$

Complexity: polynomial in n when m is fixed

Benchmark

- First implementation in Maple, with calls to msolve
  [Berthomieu-Eder-Safey El Din 2021]

	RRC	RRC sig	QE/Maple	QE/Wolfram
MKN11	5.0 s	1.5 s	5.7 s	0.06 s
RBN11	5.0 s	1.6 s	7.1 s	0.04 s
GRD12	1.0 s	3.7 s	∞	0.5 s
GRD13	19 s	17 s	∞	∞
GRD14	∞	∞	∞	∞
GRD21	0.5 s	1.7 s	1.3 s	0.1 s
GRD22	5.8 s	2 min	∞	42 min
GRD23	∞	∞	∞	∞
PPM21	0.3 s	0.3 s	0.3 s	0.005 s
PPM31	0.3 s	0.4 s	0.4 s	0.007 s
DRS32	2.2 s	8 h	∞	∞
DRS33	18 min	∞	∞	∞
DRS42	52 s	∞	∞	∞
DRS43	∞	∞	∞	∞

Table 1. Benchmark results