List of Lemmas

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Contents

1 Value of integral $\int_0^\infty x^\eta/(1+x^\theta)dx$ - formula

Thesis. The following holds for $\eta, \theta > 0$ and $\eta + 1 < \theta$:

$$I = \int_0^\infty \frac{x^\eta}{1+x^\theta}dx = \frac{\pi}{\theta \sin\left(\pi \frac{1+\eta}{\theta}\right)}$$

Demonstration. The key idea - transform the integral into a beta function. Using the substitution $u = x^\theta$ to get rid of the the power in the denominator:

$$\int_0^\infty \frac{x^\eta}{1+x^\theta}dx = \frac{1}{\theta}\int_0^\infty \frac{u^{(\eta+1)/\theta-1}}{1+u}dx$$

Proceeding to obtain $u/(1+u)$, $1/(1+u)$ in the integrand:

$$\frac{1}{\theta}\int_0^\infty \frac{u^{(\eta+1)/\theta-1}}{1+u}du = \frac{1}{\theta}\int_0^\infty \left(\frac{u}{1+u}\right)^{1-\frac{\eta+1}{\theta}}\left(\frac{1}{1+u}\right)^{(3-\frac{\eta+1}{\theta})-1} du$$

Moving further towards beta function - using substitution: $v = u/(1+u)$ (remember about changing the limits of integration):

$$\begin{aligned}\frac{1}{\theta}\int_0^\infty \left(\frac{u}{1+u}\right)^{1-\frac{\eta+1}{\theta}}\left(\frac{1}{1+u}\right)^{(3-\frac{\eta+1}{\theta})-1} du &= \frac{1}{\theta}\int_0^1 v^{\frac{\eta+1}{\theta}-1}(1-v)^{(1-\frac{\eta+1}{\theta})-1}du \\ &= \frac{1}{\theta}B(\frac{\eta+1}{\theta}, 1-\frac{\eta+1}{\theta})\end{aligned}$$

Thus, by the basic property of the beta function:

$$I = \frac{1}{\theta}B(1-\frac{\eta+1}{\theta}, 1-\frac{\eta+1}{\theta}) \quad = \quad \frac{1}{\beta}\frac{\Gamma(\frac{\eta+1}{\theta})\Gamma(1-\frac{\eta+1}{\theta})}{\Gamma(1)}$$

The Euler reflection formula states that for $z \notin \mathbb{Z}$:

$$\Gamma(x)\Gamma(1-x) = \frac{\pi}{\sin(\pi x)}$$

This can be appied in the case at hand (since by assumption $\eta + 1 < \theta$) to obtain:

$$I = \frac{\pi}{\theta}\frac{1}{\sin(\frac{1+\eta}{\theta}\pi)}$$