Notes On the Diagonal Argument

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Abstract

The purpose of this text is to store a number of facts concerning the diagonal argument. This note is primarily intended for author's own use.

Contents

1 Introduction to the Diagonal Argument

This section discusses at more length the version of the diagonal argument that seems (at least to the author) the most insightful for those looking to understand the underlying *idea*. This exposition is intended to be just an extended rephrasing of the diagonal argument presented in the enlightening book by Smith [1].

Consider the problem of equipollence of the set of natural numbers $\mathbb{N}$ and the set of infinite sequences of ones and zeros - i.e. the set $\{0,1\}^{\mathbb{N}}$. Assume that the two sets are equipollent, i.e. that there exists a bijection[1] $f : \mathbb{N} \longrightarrow \{0,1\}^{\mathbb{N}}$. This means that the members of the set $\{0,1\}^{\mathbb{N}}$ can be enumerated. In other words, if we wrote down the values of f for consecutive natural numbers:

$$f(0), f(1), f(2), ..., f(n-1), f(n), f(n+1), ... \tag{1}$$

we could be sure to find a certain $(x_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{N}}$ somewhere in the list (1).

As the next step, consider writing down the list of infinite binary sequences in a tabular format - as demonstrated in the table 1. More specifically, suppose that each row of the table corresponds to a value of f for some natural number. The value is an infinite binary sequence, which is a mapping from the set of natural numbers into the set $\{0,1\}$. It is therefore tempting to number the columns of the table with consecutive natural numbers $0, 1, 2, ...$ The body of cells thus obtained can be considered to be an infinite $\mathbb{N} \times \mathbb{N}$ matrix A:

$$A = [a_{m,n}]_{\mathbb{N}\times\mathbb{N}}$$

Where m points at m-th row of the matrix and n points at n-th column of the matrix (both indexes m and n start at zero).

It stands to reason that the m-th row of the matrix A is the value of the function f at m, i.e. an infinite binary sequence $f(m)$. Taking:

$$f(m) = (a_{m,n})_{n\in\mathbb{N}}$$

it is clear that the element $a_{m,n}$ is the n-th binary digit of the infinte binary sequence $f(m)$.

[1]Note: the symbol $\rightarrow$ is taken to have different denotation than the symbol $\longrightarrow$. The former is a logical connective representing logical conditional, whereas the latter is used in the definition of functions to represent the transition from the domain into the codomain of a function.

n:	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**...**
f(0):	**1**	1	0	0	0	1	0	1	1	1	0	...
f(1):	1	**0**	1	0	1	1	1	1	0	1	1	...
f(2):	1	1	**0**	0	1	1	0	0	1	0	0	...
f(3):	1	1	0	**0**	0	0	1	1	1	0	0	...
...	...	...	...	...	...	...	...	...	...	...	...	...

Table 1: Putative bijection $f : \mathbb{N} \longrightarrow \{0,1\}^{\mathbb{N}}$ written down. Only first four values $f(0), f(1), f(2)$ and $f(3)$ are (partially) shown.

2 Generalized Form of the Diagonal Argument

3 Applications of Diagonal Argument to Select Equipollence Problems

4 Application of the Diagonal Argument to the Halting Problem

5 Bibliography

References

[1] P. Smith. *An Introduction to Gödel Theorems.* Logic Matters, 2020.