Notes On the Concepts of Relation and Function [WiP]

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Abstract

The purpose of this text is to store a number of facts related to the concepts of *relation* and *function*. This note is primarily intended for author's own use.

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1 Existence of Cartesian product in ZFC

Fundamentally, I work with ZFC (cf. [2], Chapter X for more details; I am not going to introduce the full formalism here).

1.1 Existence of Ordered Pair

Consider two non-empty sets X and Y and let x, y be such that $x \in X, y \in Y$.

1. By the Axiom of Pair it is possible to form the set P given x, y:

$$P = \{x, y\}$$

2. Take Kuratowski's definition of ordered pair (x, y):

$$(x, y) = \{\{x\}, \{x, y\}\} \tag{1}$$

But how can we justify existance of the set (x, y)?

3. Showing that (1) is allowed. Consider x and apply the Axiom of Pair to two sets a, b such that $a = b = x$. This yields the set P_x:

$$P_x = \{x, x\} = \{x\}$$

4. Now it has been established that the sets $\{x\}$ and $\{x, y\}$ exist for $x \in X, y \in Y$. Hence, applying Axiom of Pair to $\{x\}$ and $\{x, y\}$ one more time yields a set Π:

$$\Pi = \{\{x\}, \{x, y\}\}$$

which is just (1).

Thus, given non-empty sets X and Y and some of their members $x \in X$ and $y \in Y$, it is permitted to form the ordered pair (x, y).

1.2 Existence of Cartesian product

It remains to be demonstrated that the set of all ordered pairs of members of X and Y also exists.

1. Take any $x \in X, y \in Y$ and form the ordered pair (x, y).

2. Apply Axiom of Pair to a and b such that $a = b = (x, y)$ to get the set $\{a, b\}$, i.e. the set $\{(x, y)\}$.
3. Use Axiom of Union to Form the following set U_y:

$$U_y = \bigcup_{x \in X} \{(x, y)\}$$

4. Use Axiom of union again, this time - to form the following set C:

$$C := \bigcup_{y \in Y} \bigcup_{x \in X} \{(x, y)\} = \bigcup_{y \in Y} U_y \tag{2}$$

It remains to be shown that the set C indeed contains all the ordered pairs that can be formed from the members of sets X and Y. To this end, pick any $x_0 \in X$ and $y_0 \in Y$ and form the orderd pair (x_0, y_0).

The question still stands - is it the case that $(x_0, y_0) \in C$? By definition:

$$(x_0, y_0) \in C \Leftrightarrow \exists y \in Y : (x_0, y_0) \in U_y$$

Consider $y = y_0$. Since:

$$U_{y_0} = \bigcup_{x \in X} \{(x, y_0)\}$$

it does indeed hold that $(x_0, y_0) \in C$.

It is now established that the set C of all ordered pairs of the members of the sets X and Y exists. This set is called *Cartesian product* and is denoted $X \times Y$. To summarize:

Definition 1. (Cartesian product of two sets) Let X, Y be nonempty sets. Then, the set of all ordered pairs (x, y) such that $x \in X, y \in Y$ is denoted $X \times Y$ and it is called the Cartesian product of sets X and Y.

2 Relations

Let's keep the sets X and Y nonempty. It has been established that it is permitted to form the Cartesian product $X \times Y$.

Definition 2. (Relation) Given a Cartesian product $X \times Y$ of two nonempty sets X and Y any subset R of $X \times Y$ (i.e. $R \subset X \times Y$) is a relation.

If $R \subset X \times Y$ then relation R is called a relation *over* sets X and Y. If $R \subset X \times X$ then relation R is called a relation *in* set X.

Suppose that $(x, y) \in R$. Then it is said that x is in relation R with y. Notation xRy is also used to denote the fact that $(x, y) \in R$.

Definition 3. (Properties of relations) Let R be a relation in set X. Relation R is:

1. reflexive $:\Leftrightarrow \forall x \in X : xRx$
2. anti-reflexive (or irreflexive) $:\Leftrightarrow \forall x \in X : \neg xRx$
2. symmetric $:\Leftrightarrow \forall x, y \in X : xRy \to yRx$
3. anti-symmetric $:\Leftrightarrow \forall x, y \in X : xRy \to \neg yRx$
4. asymmetric
5. transitive $:\Leftrightarrow \forall x, y, z \in X : (xRy \wedge yRz) \to xRz$
6. total (or connected)

3 Orders

3.1 Definitions of Various Kinds of Orders

3.2 Strict Order and Weak Order

4 Function - Definition

5 Inverse Function - Definition 1

6 Inverse Function - Definition 2

7 Composition of Functions

8 Important Application: Equipollence of Sets

9 Bibliography

The concept of function is a fundamental one for mathematics in general, so it is discussed in most instrodoctory handbooks that cover foundations of mathematics, algebra and analysis.

I would like to share subjective selection of bibliography in English and Polish here. In my view, the book [1] is an especially friendly reference for those looking for both: general introduction to foundations of mathematics and friendly exposition of the concept of function.

References

[1] R. Z. (ed.) et al. *Sets, Logic, Computation. An Open Introduction*. Fall 2019.

[2] P. Suppes. *Axiomatic Set Theory*. Dover, 1972.