Thesis:

$$I = \int_0^\infty \frac{\ln x}{1+x^2} dx = \frac{5\pi^5}{64}$$

Demonstration. Use substitution $u := \ln x$ to proceed:

$$I = \int_0^\infty \frac{\ln x}{1+x^2} dx = \int_{-\infty}^0 \frac{u^4 e^u}{1+e^{2u}} du$$

Writing $1/(1+e^{2u})$ as geometric series since $e^u \leq 1$:

$$I = \sum_{n=0}^\infty (-1)^n \int_{-\infty}^0 u^4 e^{(2n+1)u} du$$

Substituting $v = -u$ yields:

$$I = \sum_{n=0}^\infty (-1)^n \int_0^\infty v^{(5-1)} e^{-(2n+1)v} dv$$

Since for $\lambda > 0$:

$$\int_0^\infty x^{p-1} e^{-\lambda x} dx = \frac{\Gamma(p)}{\lambda^p}$$

The integral equals:

$$I = \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^5} \Gamma(5) = 4! \sum_{n=0}^\infty \frac{(-1)^n}{(2n+1)^5}$$

The formula can be rewritten using Dirichlet's beta function as:

$$I = 4! \beta(5)$$

It can be proved that the value of Dirichlet beta function at 5 is $5\pi^5/1536$, hence:

$$I = 24 \cdot \frac{\pi^5}{1536} = \frac{5\pi^5}{64}$$