A note on conditional probabilities of multivariate continuous variables with applications to time series analysis

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Abstract

This notes contains derivation of the chain of conditional probabilities for multiple random variables. It is to serve as a reference later on during considerations of MLE estimation of autoregressive processes.

1 Introduction and notation

Let the three-tuple $(\Omega, \mathfrak{F}, \mathbb{P})$ be a probability space, later on referred to briefly as simply Ω. Let the following be continuous variables on Ω: 1) X, Y, Z, 2) $X_1, X_2, ..., X_n$ and 3) $Y_0, Y_1, Y_2, ..., Y_T$.
Let f_X denote the probability density function of the variable X. Let $f_{X|Y}$ denote the probability density function of X with respect to Y.

2 Case of three variables

Let's start with the joint density of variables X, Y, Z, which has joind pdf $f_{X,Y,Z}$. The following holds:

$$f_{X,Y,Z}(x, y, z) = \frac{f_{X,Y,Z}}{1} \cdot \frac{f_{Y,Z}(y, z)}{f_{Y,Z}(y, z)} \cdot \frac{f_Z(z)}{f_Z(z)}$$

matching the fractions numerators and denominators 'diagonally' yields:

$$f_{X,Y,Z}(x, y, z) = \frac{f_{X,Y,Z}}{f_{Y,Z}(y, z)} \cdot \frac{f_{Y,Z}(y, z)}{f_Z(z)} \cdot \frac{f_Z(z)}{1}$$

Using the definition of conditional distribution (for continuous variables):

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

I rewrite one more time to arrive at the well-known result:

$$f_{X,Y,Z}(x,y,z) = f_{X|Y,Z}(x|y,z) \cdot f_{Y|Z}(y|z) \cdot f_Z(z)$$

3 Case of n variables

The three variables case presented in the previous section easily generalized to the case of n variables:

$$\begin{aligned} f_{X_n,X_{n-1},...,X_0}(x_n, x_{n-1}, ..., x_1) =& f_{X_n|X_{n-1},...,X_1}(x_n|x_{n-1}, ..., x_1) \cdot \\ & f_{X_{n-1}|X_{n-2},...,X_1}(x_{n-1}|x_{n-2}, ..., x_1) \cdot ... \cdot \\ & f_{X_2|X_1}(x_2|x_1) \cdot f_{X_1}(x_1) \end{aligned}$$

4 Case of T variables - with time series-like notation

The case below can be used in transformations of the likelihood function for various time series models:

$$\begin{aligned} f_{Y_T,Y_{T-1},...,Y_1,Y_0}(y_T, t_{T-1}, ..., y_1, y_0) =& f_{Y_T|Y_{T-1},...,Y_1,Y_0}(y_T|y_{T-1}, ..., y_1, y_0) \cdot \\ & f_{Y_{T-1}|Y_{T-2},...,Y_1,Y_0}(y_{T-1}|y_{n-2}, ..., y_1) \cdot ... \cdot \\ & f_{Y_1|Y_0}(y_1|y_0) \cdot f_{Y_0}(y_0) \end{aligned}$$