A note on conditional maximum likelihood estimation of gaussian AR(1) process [WiP!!!]

Artur Wegrzyn

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Abstract

This note contains derivation of conditional maximum likelihood estimators of parameters of AR(1) gaussian process. This note is intended as a future reference for the author, hence not too much intention is given at ensuring that propounding of all steps is sufficient to satisfy the tastes of more fastidious readers. Also, notation is at times quite sloppy and is assumed to be self-explanatory.

1 AR(1) process - overview

Let $\{Y_t\}_{t=-\infty}^{\infty}$ be a weakly stationary stochastic process that obeys the equation:

$$Y_t = \phi Y_{t-1} + c + \epsilon_t \tag{1}$$

with gaussian error term, i.e.:

$$\epsilon_t \sim N(0, \sigma^2) \tag{2}$$

and all the error terms being uncorrelated. Furthermore, suppose that $T+1$ consecutive realizations of the process $\{Y_t\}$ are observed for times $t = 0, 1, ..., T$ and that they are denoted $y_0, y_1, y_2, ..., y_T$.

2 Likelihood function

I start by looking at the conditional distribution of Y_t with respect to Y_{t-1}. To this end, suppose that the value of Y_{t-1} is known and equal to y_{t-1}. Then:

$$Y_t = c + \phi y_{t-1} + \epsilon_t$$

Unsurprisingly, then:

$$E(Y_t|Y_{t-1} = y_{t-1}) = c + \phi y_{t-1}$$

$$Var(Y_t|Y_{t-1} = y_{t-1}) = \sigma^2$$

As $\{\epsilon_t\}$ has been assumed to be gaussian, it follows that the conditional probability density function of Y_t with respect to $Y_{t-1} = y_{t-1}$ is given by:

$$f_{Y_t|Y_{t-1}}(y_t|y_{t-1}) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{1}{2}\frac{(y_t - c - \phi y_{t-1})^2}{\sigma^2}\right) \tag{3}$$

This formula will be of great use in the next step which is rewriting of joint probability density function of $(Y_T, Y_{T-1}, ..., Y_1, Y_0)$.

Consequently, let's deal with the joint density. Using the simple fact presented in [1], I have:

$$\begin{aligned} f_{Y_T,Y_{T-1},...,,Y_1,Y_0}(y_T, y_{T-1}, ..., y_1, y_0) =& f_{Y_T|Y_{T-1},...,Y_1,Y_0}(y_T|y_{T-1}, ..., y_1, y_0)\cdot \\ & f_{Y_{T-1}|Y_{T-2},...,Y_1,Y_0}(y_{T-1}|y_{n-2}, ..., y_1, y_0) \cdot ...\cdot \\ & f_{Y_1|Y_0}(y_1|y_0) \cdot f_{Y_0}(y_0) \end{aligned}$$

Written more succintly:

$$f_{Y_T,Y_{T-1},...,,Y_1,Y_0}(y_T, y_{T-1}, ..., y_1, y_0) = f_{Y_0}(y_0)\cdot \prod_{t=1}^{T} f_{Y_t|Y_{t-1},...,y_0}(y_t|y_{t-1}, ..., y_0)$$

Since I aim for *conditional* likelihood, I devide both sided by $f_{Y_0}(y_0)$ and obtain:

$$f_{Y_T,Y_{T-1},...,,Y_1|Y_0}(y_T, y_{T-1}, ..., y_1|y_0) = \prod_{t=1}^{T} f_{Y_t|Y_{t-1},...,y_0}(y_t|y_{t-1}, ..., y_0)$$

Now, by equation 3 one can easily see that since the value of the process at time t is conditional only on its value at time $t-1$ the conditional densities in the product in the equation above reduce as follows:

3 Derivation of maximum likelihood estimators of the parameters

4 Literature

The exposition in this note follows closely the one in [2], chapter 5.

References

[1] A. Wegrzyn, "A note on conditional probabilities of multivariate continuous variables with applications to time series analysis." `https://github.com/wegar-2/latex_files/blob/master/probability_calculus/conditional_probabilities_for_time_series.pdf`, 2021.

[2] J. D. Hamilton, *Time Series Analysis*. Princeton University Press, 1994.