2. 群的性质

$\{\mathbb{Z},+\}$是群, 满足群的四个性质:

1. 整数加整数仍然是整数
2. 加法结合律
3. 幺元为0
4. 逆为本身取负

$\{\mathbb{N},+\}$不是群, 对于正整数, 其逆不是自然数.

3. 向量叉乘的李代数性质

三维空间中的两个向量$\mathbf{a},\mathbf{b}$, 有:

$$\begin{aligned}
&\mathbf{a}\times\mathbf{b}=-b\times a\\
&\mathbf{a}\times\mathbf{b}=\mathbf{a}^{\wedge}\mathbf{b}=\begin{bmatrix}0&-a_3&a_2\\a_3&0&-a_1\\-a_2&a_1&0\end{bmatrix}\mathbf{b}=-\begin{bmatrix}0&-b_3&b_2\\b_3&0&-b_1\\-b_2&b_1&0\end{bmatrix}\mathbf{a}
\end{aligned}$$

显然, $\mathfrak{g}=\left(\mathbb{R}^3,\mathbb{R},\times\right)$

性质①显然满足, 叉乘的结果任然是三维向量.
将叉乘转换为矩阵相乘, 容易证明性质②也满足.
性质③, 根据叉乘的定义也可知满足.
性质④,

$$\begin{aligned}
[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]&=X\times(Y\times Z)+Y\times(Z\times X)+Z\times(X\times Y)\\
&=X^{\wedge}Y^{\wedge}Z-Y^{\wedge}X^{\wedge}Z-X\times Y\times Z\\
&=(X^{\wedge}Y^{\wedge}-Y^{\wedge}X^{\wedge})Z-(X^{\wedge}Y)^{\wedge}Z\\
&=(X^{\wedge}Y^{\wedge}-Y^{\wedge}X^{\wedge}-(X^{\wedge}Y)^{\wedge})Z
\end{aligned}$$

将括号内的矩阵展开:

$$\begin{aligned}
X^{\wedge}Y^{\wedge}&=\begin{bmatrix}-x_3y_3-x_2y_2&x_2y_1&x_3y_1\\x_1y_2&-x_3y_3-x_1y_1&x_3y_2\\x_1y_3&x_2y_3&-x_2y_2-x_1y_1\end{bmatrix}\\
Y^{\wedge}X^{\wedge}&=\begin{bmatrix}-x_3y_3-x_2y_2&x_1y_2&x_1y_3\\x_2y_1&-x_3y_3-x_1y_1&x_2y_3\\x_3y_1&x_3y_2&-x_2y_2-x_1y_1\end{bmatrix}\\
(X^{\wedge}Y)^{\wedge}&=\begin{bmatrix}0&[x_2y_1-x_1y_2]&[x_3y_1-x_1y_3]\\ [x_1y_2-x_2y_1]&0&[x_3y_2-x_2y_3]\\ [x_1y_3-x_3y_1]&[x_2y_3-x_3y_2]&0\end{bmatrix}
\end{aligned}$$

可得$X^{\wedge}Y^{\wedge}-Y^{\wedge}X^{\wedge}-(X^{\wedge}Y)^{\wedge}=0$, 性质④得证.

4. SE(3)指数映射推导

根据李代数se(3)的定义:

$$\mathfrak{se}(3)=\left\{\boldsymbol{\xi}=\begin{bmatrix}\rho\\\phi\end{bmatrix}\in\mathbb{R}^6,\boldsymbol{\rho}\in\mathbb{R}^3,\boldsymbol{\phi}\in\mathfrak{so}(3),\boldsymbol{\xi}^{\wedge}=\begin{bmatrix}\phi^{\wedge}&\rho\\\mathbf{0}^T&0\end{bmatrix}\in\mathbb{R}^{4\times4}\right\}$$

分析乘积:

$$\begin{aligned}
&\left(\begin{bmatrix}\rho\\\phi\end{bmatrix}^{\wedge}\right)^2=\begin{bmatrix}\phi^{\wedge}&\rho\\\mathbf{0}^T&0\end{bmatrix}\begin{bmatrix}\phi^{\wedge}&\rho\\\mathbf{0}^T&0\end{bmatrix}=\begin{bmatrix}(\phi^{\wedge})^2&\phi^{\wedge}\rho\\\mathbf{0}^T&0\end{bmatrix}\\
\Rightarrow&\left(\begin{bmatrix}\rho\\\phi\end{bmatrix}^{\wedge}\right)^n=\begin{bmatrix}(\phi^{\wedge})^n&(\phi^{\wedge})^{n-1}\rho\\\mathbf{0}^T&0\end{bmatrix}
\end{aligned}$$

指数Taylor展开:

$$
\begin{aligned}
\exp\left(\left[\begin{array}{l}\rho \\ \phi\end{array}\right]^{\wedge}\right) &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\left[\begin{array}{l}\rho \\ \phi\end{array}\right]^{\wedge}\right)^{n} = \frac{1}{0!}\left(\left[\begin{array}{l}\rho \\ \phi\end{array}\right]^{\wedge}\right)^{0} + \sum_{n=1}^{\infty} \frac{1}{n!}\left(\left[\begin{array}{cc}\left(\phi^{\wedge}\right)^{n} & \left(\phi^{\wedge}\right)^{n-1} \rho \\ \mathbf{0}^{T} & 0\end{array}\right]\right) \\
&= \mathbf{I} + \left[\begin{array}{cc}\sum_{n=1}^{\infty} \frac{1}{n!}\left(\phi^{\wedge}\right)^{n} & \sum_{n=1}^{\infty} \frac{1}{n!}\left(\phi^{\wedge}\right)^{n-1} \rho \\ \mathbf{0}^{T} & 0\end{array}\right] \\
&= \left[\begin{array}{cc}\sum_{n=0}^{\infty} \frac{1}{n!}\left(\boldsymbol{\phi}^{\wedge}\right)^{n} & \sum_{n=0}^{\infty} \frac{1}{(n+1)!}\left(\boldsymbol{\phi}^{\wedge}\right)^{n} \boldsymbol{\rho} \\ \mathbf{0}^{T} & 1\end{array}\right]
\end{aligned}
$$

由此, 得到Jacobian矩阵. 这里, 利用了矩阵的0次幂是$\mathbf{I}$. 参考SO(3)指数映射的推导, 对Jacobian矩阵做进一步化简:

$$
\begin{aligned}
\mathbf{J} &= \sum_{n=0}^{\infty} \frac{1}{(n+1)!}\left(\boldsymbol{\phi}^{\wedge}\right)^{n} = \sum_{n=0}^{\infty} \frac{1}{(n+1)!}\left(\theta \mathbf{a}^{\wedge}\right)^{n} \\
&= \mathbf{I} + \frac{1}{2!} \theta \mathbf{a}^{\wedge} + \frac{1}{3!} \theta^{2} \mathbf{a}^{\wedge} \mathbf{a}^{\wedge} + \frac{1}{4!} \theta^{3} \mathbf{a}^{\wedge} \mathbf{a}^{\wedge} \mathbf{a}^{\wedge} + \ldots \\
\Rightarrow \theta \mathbf{J} &= \left(\theta + \frac{1}{2!} \theta^{2} \mathbf{a}^{\wedge} + \frac{1}{3!} \theta^{3} \mathbf{a}^{\wedge} \mathbf{a}^{\wedge} + \frac{1}{4!} \theta^{4} \mathbf{a}^{\wedge} \mathbf{a}^{\wedge} \mathbf{a}^{\wedge} + \ldots\right) \\
&= -\left(\theta - \frac{1}{3!} \theta^{3} + \frac{1}{5!} \theta^{5} - \ldots\right) \mathbf{a}^{\wedge} \mathbf{a}^{\wedge} - \left(1 - \frac{1}{2!} \theta^{2} + \frac{1}{4!} \theta^{4} - \ldots\right) \mathbf{a}^{\wedge} + \theta + \theta \mathbf{a}^{\wedge} \mathbf{a}^{\wedge} + \mathbf{a}^{\wedge} \\
\Rightarrow J &= \frac{\sin \theta}{\theta} I + \left(1 - \frac{\sin \theta}{\theta}\right) \boldsymbol{a} \boldsymbol{a}^{T} + \frac{1 - \cos \theta}{\theta} \boldsymbol{a}^{\wedge}
\end{aligned}
$$

5.伴随

首先, 证明$\boldsymbol{R} \boldsymbol{a}^{\wedge} \boldsymbol{R}^{\mathrm{T}} = (\boldsymbol{R} \boldsymbol{a})^{\wedge}$:
直观理解为, 对于任意向量$\mathbf{v}$, $\boldsymbol{R} \boldsymbol{a}^{\wedge} \boldsymbol{R}^{\mathrm{T}} \mathbf{v}$可以解释为, 先根据$\mathbf{R}$做一个逆旋转, 然后与$\mathbf{a}$做叉乘, 再根据$\mathbf{R}$旋转回来. 这个等价于对$\mathbf{a}$做完旋转后再叉乘. 又可理解为, 向量先旋转到一个局部坐标系中做叉乘再旋转回全局坐标系, 与先旋转到全局坐标系, 再叉乘等价.

$$
(\boldsymbol{R} \boldsymbol{a}) \times \mathbf{v} = (\boldsymbol{R} \boldsymbol{a}) \times \left(\boldsymbol{R} \boldsymbol{R}^{T} \mathbf{v}\right) = \boldsymbol{R}\left[\mathbf{a} \times \left(\boldsymbol{R}^{T} \mathbf{v}\right)\right]
$$

对任意向量$\mathbf{v}$均成立, 故而$\boldsymbol{R} \boldsymbol{a}^{\wedge} \boldsymbol{R}^{\mathrm{T}} = (\boldsymbol{R} \boldsymbol{a})^{\wedge}$.

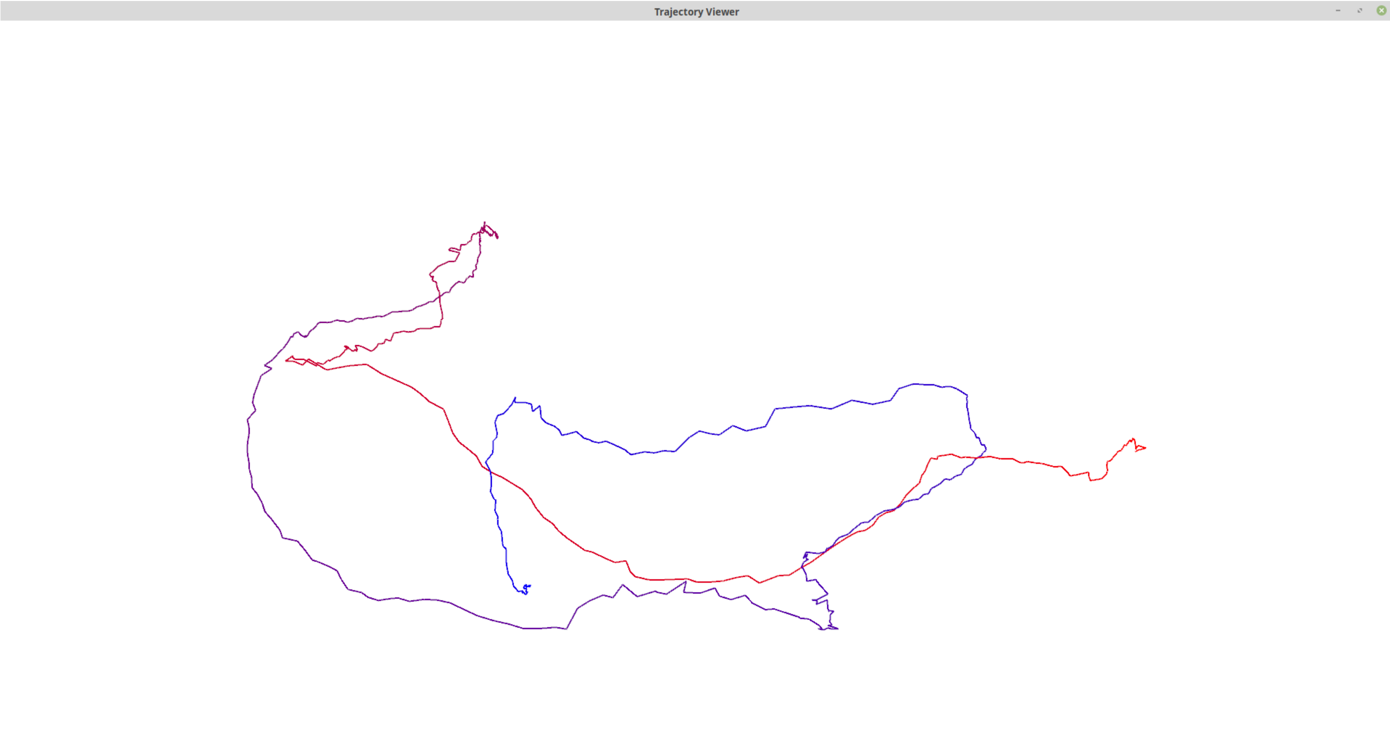
接下来证明$\boldsymbol{R} \cdot \boldsymbol{q}^{\wedge} \cdot \boldsymbol{R}^{T} = (\boldsymbol{R} \cdot \boldsymbol{q})^{\wedge} \Rightarrow \boldsymbol{R} \exp \left(\boldsymbol{p}^{\wedge}\right) \boldsymbol{R}^{\mathrm{T}} = \exp \left((\boldsymbol{R} \boldsymbol{p})^{\wedge}\right)$

令$\boldsymbol{p} = t \cdot \boldsymbol{q}, t \in \mathbb{R}$, 在$t = 0$处对$t$求导,有

$$
\begin{aligned}
\left.\frac{d}{d t}\right|_{t=0}\left[\boldsymbol{R} \cdot \exp (t \cdot \boldsymbol{q}) \cdot \boldsymbol{R}^{T}\right] &= \left.\frac{d}{d t}\right|_{t=0} \exp \left((\boldsymbol{R} \cdot \mathbf{t} \cdot \boldsymbol{q})_{\times}\right) \\
\boldsymbol{R} \cdot \left.\frac{d}{d t}\right|_{t=0}\left[\mathbf{I} + (t \cdot \boldsymbol{q})_{\times} + O\left(t^{2}\right)\right] \cdot \boldsymbol{R}^{T} &= \left.\frac{d}{d t}\right|_{t=0}\left[\mathbf{I} + (\boldsymbol{R} \cdot t \cdot \boldsymbol{q})_{\times} + O\left(t^{2}\right)\right] \\
\boldsymbol{R} \cdot (t \cdot \boldsymbol{q})_{\times} \cdot \boldsymbol{R}^{T} &= (\boldsymbol{R} \cdot t \cdot \boldsymbol{q})_{\times} \\
\boldsymbol{R} \cdot \boldsymbol{q}^{\wedge} \cdot \boldsymbol{R}^{T} &= (\boldsymbol{R} \cdot \boldsymbol{q})^{\wedge}
\end{aligned}
$$

6. 轨迹的描绘

1. $\boldsymbol{T}_{WC}$的平移部分, 物理意义是机器人在世界坐标系中的位置, 因此, 画出平移部分就画出了机器人运动的轨迹.
2. 结果如下图, 代码见附件.



7. 轨迹的误差

我的运算结果答案为: `2.20548` 与参考答案 `2.207` 有些相差, 若有错误, 望指正, 代码见附件.