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Proof that

$$\frac{1}{2^{2\mathrm{k}}}\sum_{i=0}^{2^k-1}\sum_{j=0}^{2^k-1}(i-j)^2 \;=\; \frac{2^{2k}\;-\;1}{6}$$

Proof:

$$\frac{1}{2^{2k}}\sum_{i=0}^{2^k-1}\sum_{j=0}^{2^k-1}(i-j)^2$$

$$=\frac{1}{2^{2k}}\sum_{i=0}^{2^k-1}\sum_{j=0}^{2^k-1}(i^2-2ij+j^2)$$

$$=\frac{1}{2^{2k}}(\sum_{i=0}^{2^k-1}\sum_{j=0}^{2^k-1}i^2-\;2\sum_{i=0}^{2^k-1}\sum_{j=0}^{2^k-1}ij+\sum_{i=0}^{2^k-1}\sum_{j=0}^{2^k-1}j^2)$$

其中令

$$\mathrm{a}\;=\sum_{i=0}^{2^k-1}\sum_{j=0}^{2^k-1}i^2$$

$$=\sum_{i=0}^{2^k-1}2^k i^2$$

$$=\;2^k\cdot\;\frac{(2^k-1)(2^k)(2^{k+1}-1)}{6}$$

$$b\;=2\sum_{i=0}^{2^k-1}\sum_{j=0}^{2^k-1}ij$$

$$=2\sum_{i=0}^{2^k-1}\frac{(2^k-1)2^k}{2}i$$

$$=\;(2^k-1)2^k\cdot\frac{(2^k-1)2^k}{2}$$

$$c=\sum_{i=0}^{2^k-1}\sum_{j=0}^{2^k-1}j^2$$

$$=\sum_{i=0}^{2^k-1}\frac{(2^k-1)2^k(2^{k+1}-1)}{6}$$

$$= 2^k \cdot \frac{(2^k-1)(2^k)(2^{k+1}-1)}{6}$$

故

$$\therefore \frac{1}{2^{2k}}\left(\sum_{i=0}^{2^k-1}\sum_{j=0}^{2^k-1} i^2 - 2\sum_{i=0}^{2^k-1}\sum_{j=0}^{2^k-1} ij + \sum_{i=0}^{2^k-1}\sum_{j=0}^{2^k-1} j^2\right)$$

$$= \frac{1}{2^{2k}}\left(2^k \cdot \frac{(2^k-1)(2^k)(2^{k+1}-1)}{6} - (2^k-1)2^k \cdot \frac{(2^k-1)2^k}{2}\right.$$

$$\left. + 2^k \cdot \frac{(2^k-1)(2^k)(2^{k+1}-1)}{6}\right)$$

$$= (2^k-1)\left[\frac{2\cdot(2^{k+1}-1)}{6} - \frac{2^k-1}{2}\right]$$

$$= (2^k-1)\cdot\frac{4\cdot 2^k - 2 - 3\cdot 2^k + 3}{6}$$

$$= \frac{1}{6}(2^k - 1)(2^k + 1)$$

$$= \frac{2^{2k}-1}{6}$$