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Robotics assignment 2

The `MATLAB` code of this assignment is provided here in `github`.

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- **Robotics 2023-24**

Part A-1

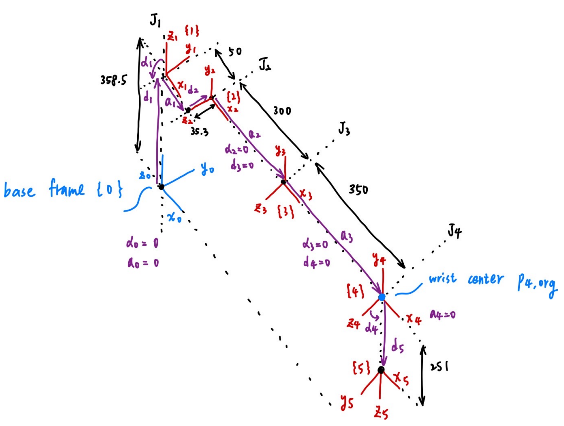

Fig. A

Part A-2

Tab A

J_i (ith joint)	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	358.5	$\theta_1 \in \mathbb{R}$
2	$\frac{\pi}{2}$	50	-35.3	$\theta_2 \in \mathbb{R}$
3	0	300	0	$\theta_3 \in \mathbb{R}$
4	0	350	0	$\theta_4 \in \mathbb{R}$
5	$\frac{\pi}{2}$	0	251	$\theta_5 \in \mathbb{R}$

Part B

$$ {}^{\text{base}}_{5}T = {}^{0}_{5}T = \prod_{i=1}^{5} {}^{i-1}_{i}T = {}^{0}_{1}T\,{}^{1}_{2}T \dots {}^{4}_{5}T \text{ (base as index } 0\text{), where} \tag{1} $$

$$ {}^{i-1}_{i}T = \mathrm{Rot}(\hat{x}_{i-1}, \alpha_{i-1})\mathrm{Trans}(\hat{x}_i, a_{i-1})\mathrm{Trans}(\hat{z}_i, d_i)\mathrm{Rot}(\hat{z}_i, \theta_i) \in SE(3). \tag{2} $$

The symbolic derivation of ${}^{i-1}_{i}T$ in (5) are trivial and can be easily implemented in code, so here only demonstrate the cases as the robot is in zero configuration, *i.e.* $\theta_i = 0,\ \forall i$.

Substituting ***Tab A*** into (2) yields,

${}^{\text{base}}_{5}T$ is,

1	0	0	700
0	-1	0	35.3000
0	0	-1	107.5000
0	0	0	1

where ${}^{\text{base}}_{1}T$ is,

```
1.0000         0         0         0
     0    1.0000         0         0
     0         0    1.0000  358.5000
     0         0         0    1.0000
```

${}^{1}_{2}T$ is,

```
1.0000         0         0   50.0000
     0         0   -1.0000   35.3000
     0    1.0000         0         0
     0         0         0    1.0000
```

${}^{2}_{3}T$ is,

1	0	0	300
0	1	0	0
0	0	1	0
0	0	0	1

${}^3_4T$ is,

1	0	0	350
0	1	0	0
0	0	1	0
0	0	0	1

${}^4_5T$ is,

1	0	0	0
0	0	-1	-251
0	1	0	0
0	0	0	1

Part C-I

Given a robot target pose $(x, y, z, \phi, \theta, \psi)$, where $\{\phi, \theta, \psi\}$ are the $\hat{z} - \hat{y} - \hat{x}$ euler angle w.r.t. the base (world) frame respectively (yall-pitch-roll), the constrained inverse kinematic problem can be decoupled into *position problem* and *orientation problem*.

While before that, since the target position is defined w.r.t. the robot's *wrist center* (frame $\{4\}$ as in ***Fig. A***), we first set the target writst center point (which is also the origin of the frame $\{4\}$, say ${}^0p_{4,\text{org}}$), by shifting the target tip position back by d_5, where,

$$
{}^0p_{4,\text{org}} = {}^0p_{5,\text{org}} + (-d_5)\,{}^0\hat{z}_5 = {}^0p_{5,\text{org}} + (-d_5)\,{}^0_5R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} + {}^0_5R \begin{bmatrix} 0 \\ 0 \\ -d_5 \end{bmatrix} := \begin{bmatrix} x_w \\ y_w \\ z_w \end{bmatrix}, \tag{3}
$$
$$
{}^0_5R = \text{Rot}(\hat{z}, \phi)\text{Rot}(\hat{y}, \theta)\text{Rot}(\hat{x}, \psi). \tag{4}
$$

In the following position problem, the target wrist position $\begin{bmatrix} x_w & y_w & z_w \end{bmatrix}^T$ will be used for simplicity of notations.

1. Position problem (Solve for $\{\theta_1, \theta_2, \theta_3\}$)

> ⚠️ ***Elbow-up configuration***
>
> The the elbow-up configuration can be ensured by imposing the constraint between $\{\theta_1, \theta_2, \theta_3\}$, which is shown in eq. (5).
>
> First, the target tip position is assumed to lie in the ***I*** and ***IV*** quadrant, and it's assumed that there's no offset between the 1st and 2nd joint. In this case, the range of the joint variables can be divided into 2 cases,
>
> $$(x_w, y_w) \in \text{I, IV} \begin{cases} \text{Case 1}: \theta_1 \in [-\frac{\pi}{2}, \frac{\pi}{2}),\ \theta_2 \in [0, \frac{\pi}{2}),\ \theta_3 \text{ s.t. } \theta_2 + \theta_3 \in [-\pi, 0),\ \text{ or} \\ \text{Case 2}: \theta_1 \in [\frac{\pi}{2}, \frac{3\pi}{2}),\ \theta_2 \in [\frac{\pi}{2}, \pi),\ \theta_3 \text{ s.t. } \theta_2 + \theta_3 \in [\pi, 2\pi). \end{cases} \tag{5}$$
>
> From (5) it can be easily seen that there's at most $\#2$ possible solutions of the position problem.
>
> - Due to the presence to offset between the 1st and 2nd joint, the division of the range of θ_1 may need a little bit adjustment, yet the *number of maximally possible solutions of the position problem ($\#2$) stays the same*. Similarly, this holds for the case for $(x_w, y_w) \in \text{II, III}$.

Due to the simple geometry of the robot, the joint variables can be solved intuitively by trigonometry ($\{\theta_1 \to \theta_2 \to \theta_3\}$).

Defining some useful variables,

$$
r := \sqrt{x_w^2 + y_w^2} \in \mathbb{R}^+, \tag{6}
$$
$$
z'_w := z_w - d_1 \in \mathbb{R}. \tag{7}
$$

- ***Case 1*** Suppose $(x, y) \in \text{I, IV},\ \theta_1 \in [-\frac{\pi}{2}, \frac{\pi}{2}),\ \theta_2 \in [0, \frac{\pi}{2}),\ \theta_3 \text{ s.t. } \theta_2 + \theta_3 \in [-\pi, 0)$

  As seen in ***Fig. C.1.1***, θ_1 is solved,

$$\theta_1 = \text{atan2}(y_w, x_w) - \lambda_1, \text{ where} \tag{8}$$
$$\lambda_1 = \text{atan2}(-d_2, \sqrt{r^2 - d_2^2}), \ (d_2 < 0). \tag{9}$$

Also r' is derived for later use,

$$r' = \sqrt{r^2 - d_2^2} - a_1. \tag{10}$$

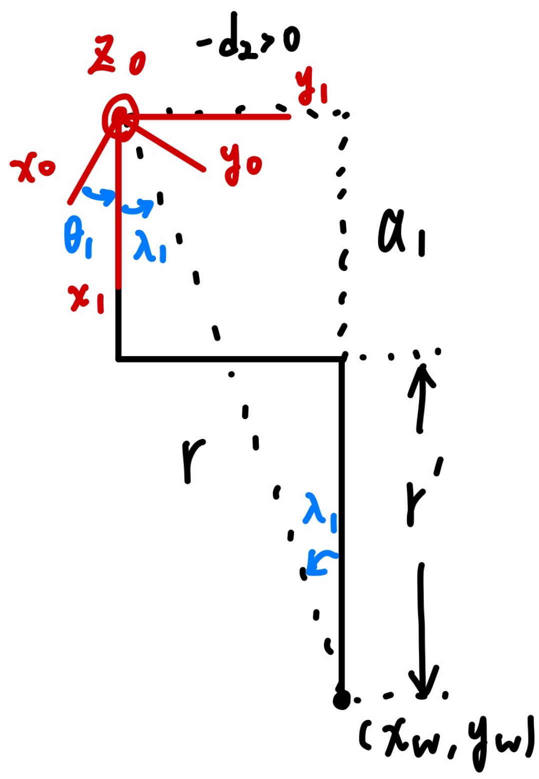

Fig. C.1.1

As seen in ***Fig. C.1.2***, θ_2 and θ_3 can be solved via law of cosine,

$$\cos(\theta_2 - \lambda_2) = \frac{a_2^2 + r'^2 + z_w'^2 - a_3^2}{2a_2\sqrt{r'^2 + z_w'^2}} := D_2 \in \mathbb{R}, \text{ where} \tag{11}$$
$$\lambda_2 = \text{atan2}(z_w', r') \in \mathbb{R}, \tag{12}$$
$$\Rightarrow \theta_2 = \text{atan2}(\pm\sqrt{1 - D_2^2}, D_2) + \lambda_2 \text{ (this holds for both } \lambda_2 < 0 \text{ and } \lambda_2 > 0), \tag{13}$$
$$= \text{atan2}(+\sqrt{1 - D_2^2}, D_2) + \lambda_2 \text{ (for } (x_w, y_w) \in I, IV \text{ in this case).} \tag{14}$$

$$\cos(-\theta_3) = \frac{r'^2 + z_w'^2 - a_2^2 - a_3^2}{2a_2a_3} := D_3 \in \mathbb{R}, \tag{15}$$
$$\Rightarrow \theta_3 = -\text{atan2}(\pm\sqrt{1 - D_3^2}, D_3), \tag{16}$$
$$= -\text{atan2}(+\sqrt{1 - D_3^2}, D_3) \text{ (for } (x_w, y_w) \in I, IV \text{ in this case).} \tag{17}$$

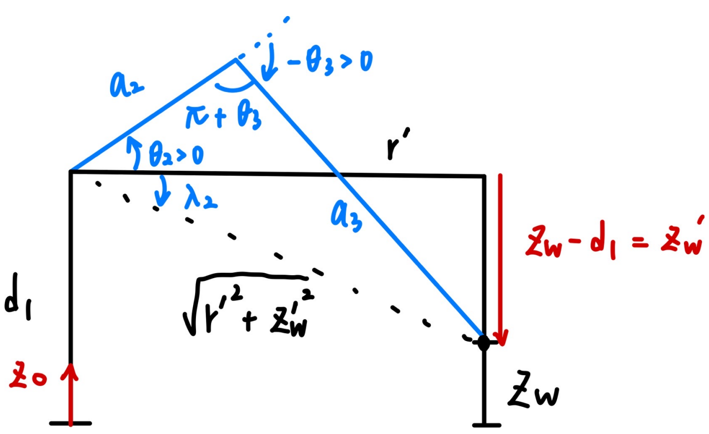

Fig. C.1.2

Eq.s $(8), (13), (16)$ gives the solution of the position problem in this case.■

- ***Case 2*** Suppose $(x, y) \in \text{I, IV}$, $\theta_1 \in [\frac{\pi}{2}, \frac{3\pi}{2})$, $\theta_2 \in [\frac{\pi}{2}, \pi)$, θ_3 s.t. $\theta_2 + \theta_3 \in [\pi, 2\pi)$

  As seen in ***Fig. C.1.3***, θ_1 is solved,

$$\theta_1 = \frac{3\pi}{2} + \lambda_1 - \mu_1, \text{ where} \tag{18}$$
$$\lambda_1 = \text{atan2}(y_w, x_w), \ \mu_1 = \text{atan2}(\sqrt{r^2 - d_2^2}, -d_2), \ (-d_2 > 0). \tag{19}$$

Also r' is derived for later use,

$$r' = \sqrt{r^2 - d_2^2} + a_1. \tag{20}$$

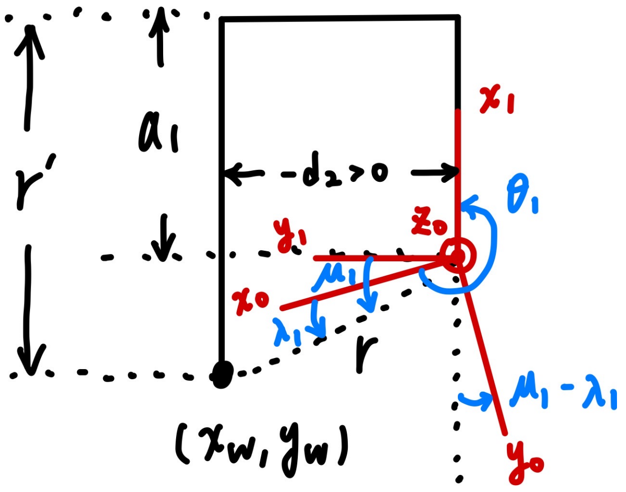

Fig. C.1.3

As seen in ***Fig. C.1.4***, θ_2 and θ_3 can be solved via law of cosine,

$$\cos(\theta_2 + \lambda_2) = \frac{a_3^2 - a_2^2 - (r'^2 + z_w'^2)}{2a_2\sqrt{r'^2 + z_w'^2}} := D_2 \in \mathbb{R}, \text{ where} \tag{21}$$

$$\lambda_2 = \text{atan2}(z_w', r') \in \mathbb{R}, \tag{22}$$

$$\Rightarrow \theta_2 = \text{atan2}(\pm\sqrt{1 - D_2^2}, D_2) - \lambda_2 \text{ (this holds for both } \lambda_2 < 0 \text{ and } \lambda_2 > 0), \tag{23}$$

$$= \text{atan2}(+\sqrt{1 - D_2^2}, D_2) - \lambda_2 \text{ (for } (x_w, y_w) \in I, IV \text{ in this case).} \tag{24}$$

$$\cos(\theta_3) = \frac{r'^2 + z_w'^2 - a_2^2 - a_3^2}{2a_2a_3} := D_3 \in \mathbb{R}, \tag{25}$$

$$\Rightarrow \theta_3 = \text{atan2}(\pm\sqrt{1 - D_3^2}, D_3), \tag{26}$$

$$= \text{atan2}(+\sqrt{1 - D_3^2}, D_3) \text{ (for } (x_w, y_w) \in I, IV \text{ in this case).} \tag{27}$$

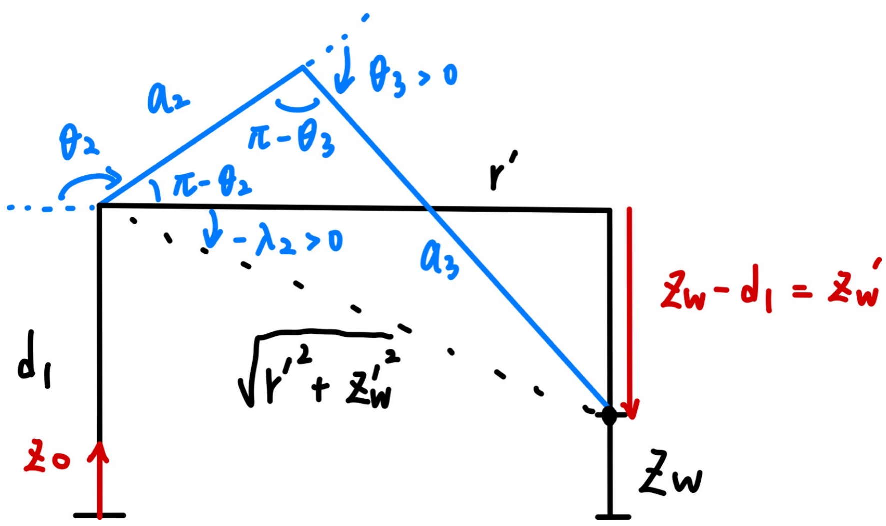

Fig. C.1.4

Eq.s $(18), (23), (26)$ gives the solution of the position problem in this case.■

- Therefore, the above derivations cover all the possible solutions ($\#2$ under the constraints) of the position problem in elbow-up configuration.
- Note that these are the *maximally* possible solutions, which may still be unsolveable if the target wrist position falls outside of the reachable workspace of the first 3 joints (the position problem fails).
- Taking the elbow-up constraint into account, eq. (5) must to be satisfied in the position problem in each case.

2. Orientation problem (Solve for $\{\theta_4, \theta_5\}$)

⚠️ ***Gripper tip pose being vertically downward***

Followed from the results in the position problem, to ensure the gripper tip pose to be vertically downward, we then divide the ranges of joint variables into the same 2 cases again as done in (5), but this time with θ_4, θ_5 involved.

First, the target tip position is assumed to lie in the ***I*** and ***IV*** quadrant, and it's assumed that there's no offset between the 1st and 2nd joint. In this case, the range of the joint variables can be divided into 2 cases (constraints),

$$(x_w, y_w) \in \text{I, IV} \begin{cases} \text{Case 1}: \theta_1 \in [-\frac{\pi}{2}, \frac{\pi}{2}),\ \theta_2 \in [0, \frac{\pi}{2}),\ \theta_3 \text{ s.t. } \theta_2 + \theta_3 \in [-\pi, 0) \Rightarrow \theta_4 \text{ s.t. } \theta_2 + \theta_3 - \frac{\pi}{2} + \theta_4 = -\frac{\pi}{2}, \text{ or} \\ \text{Case 2}: \theta_1 \in [\frac{\pi}{2}, \frac{3\pi}{2}),\ \theta_2 \in [\frac{\pi}{2}, \pi),\ \theta_3 \text{ s.t. } \theta_2 + \theta_3 \in [\pi, 2\pi) \Rightarrow \theta_4 \text{ s.t. } \theta_2 + \theta_3 + \frac{\pi}{2} + \theta_4 = \frac{3\pi}{2}. \end{cases} \tag{28}$$

In both 2 cases, θ_5 is unconstrained.

- As metioned before, due to the presence to offset between the 1st and 2nd joint, the division of the range of θ_1 may need a little bit adjustment, yet the *number of maximally possible solutions of the position problem ($\#2$) stays the same*. Also note that in each case of position problem, the orientation has a corresponding unique solution (or no solution if unreachable). Similarly, this holds for the case for $(x_w, y_w) \in \text{II, III}$.

Now, the problem of solving the last 2 joint variables $\{\theta_4, \theta_5\}$ can be transformed into the problem of parameterizing the rotation matrix, ${}^3_5R$,

$$\underbrace{{}^0_3R^T}_{\text{position problem}} \underbrace{{}^0_5R}_{\text{target pose}} \equiv {}^3_5R = \underbrace{{}^3_4R}_{\theta_4} \underbrace{{}^4_5R}_{\theta_5} = \text{Rot}(\hat{z}, \theta_4)\text{Rot}(\hat{x}, \alpha_4)\text{Rot}(\hat{z}, \theta_5). \tag{29}$$

Set ${}^3_5R$ as,

$${}^3_5R = \left[r_{ij}\right]. \tag{30}$$

Note that since $\alpha_4 = \frac{\pi}{2}$ is already given in the D-H table (it's determined by the robot's nature), by entry-wise matching of the matrices, eq. (26) becomes,

$${}^3_5R = \left[r_{ij}\right] = \begin{bmatrix} \cos\theta_4\cos\theta_5 & -\cos\theta_4\sin\theta_5 & \sin\theta_4 \\ \cos\theta_5\sin\theta_4 & -\sin\theta_4\sin\theta_5 & -\cos\theta_4 \\ \sin\theta_5 & \cos\theta_5 & 0 \end{bmatrix}, \tag{31}$$

Then we can solve for θ_4, and θ_5,

$$\begin{cases} \theta_4 = \text{atan2}(r_{13}, -r_{23}), \\ \theta_5 = \text{atan2}(r_{31}, r_{32}). \end{cases} \tag{32}$$

Eq.s (32) gives the solution of the orientation problem.■

It is easily seen that the orientation problem has either an *unique solution* or *no solution*, where the latter occurs as,

$$r_{33} \neq 0, \tag{33}$$

so that $\{\theta_4, \theta_5\}$ has no solution and the orientaion problem fails.

- Taking into account the constraint that gripper tip should always be vertically downward, eq. (28) must to be satisfied in the orientation problem in each case.

Part C-II

Implement the algorithm provided ***Part C-I*** in `MATLAB` code, we have,

(A)

$$\text{Case 1}: \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{bmatrix} \approx \begin{bmatrix} 0.1071 \\ 0.3634 \\ -1.0218 \\ 0.6584 \\ -0.6783 \end{bmatrix}, \text{ with the elbow and gripper constraints in } (5) \text{ and } (28) \text{ satisfied,} \tag{34}$$

$$\text{Case 2}: \text{No solution (the position problem fails since } 665.984 \approx \sqrt{r'^2 + z_w'^2} > a_2 + a_3 = 650, \text{ unreachable).} \tag{35}$$

(B)

$$
\text{Case 1}: \begin{bmatrix}\theta_1\\\theta_2\\\theta_3\\\theta_4\\\theta_5\end{bmatrix} \approx \begin{bmatrix}0.1071\\0.4762\\-1.0672\\0.5910\\-0.6783\end{bmatrix}, \text{ with the elbow and gripper constraints in (5) and (28) satisfied,} \tag{36}
$$

$$
\text{Case 2}: \text{No solution (the position problem fails since } 659.7615 \approx \sqrt{r'^2 + z_w'^2} > a_2 + a_3 = 650, \text{ unreachable).} \tag{37}
$$

Part C-III

$$
\text{Case 1}: \text{No solution (the position problem fails since } 825.8444 \approx \sqrt{r'^2 + z_w'^2} > a_2 + a_3 = 650, \text{ unreachable),} \tag{38}
$$

$$
\text{Case 2}: \text{No solution (the position problem fails since } 896.3645 \approx \sqrt{r'^2 + z_w'^2} > a_2 + a_3 = 650, \text{ unreachable).} \tag{39}
$$

Both ***Case 1***, and ***Case 2*** failed, and thus the target pose is *unreachable* due to the failure of position problems.

Part D-I

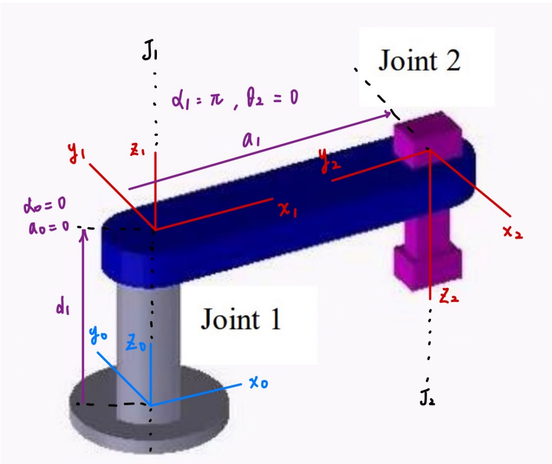

Fig. D.1

Part D-II

Tab D

J_i (ith joint)	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	$d_1 \in \mathbb{R}^+$	$\theta_1 \in \mathbb{R}$
2	π	$a_1 \in \mathbb{R}^+$	$d_2 \in \mathbb{R}$	0

Varying parameters: θ_1 for joint 1, d_2 for joint 2.