

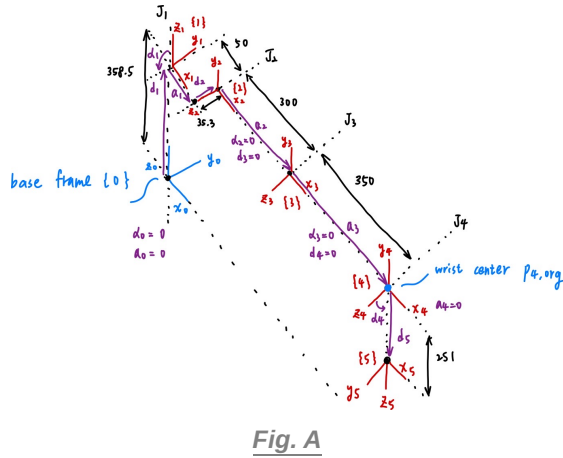
Robotics assignment 2

• Robotics 2023-24

• R11631045 Wei-Hsuan Cheng 程偉軒

In this assignment, the **MATLAB** package [NxRLab](#) / [ModernRobotics](#) is used, and the **MATLAB** code is provided here in [github](#).

Part A-1



Part A-2

Tab A

J_i (i th joint)	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	358.5	$\theta_1 \in \mathbb{R}$
2	$\frac{\pi}{2}$	50	-35.3	$\theta_2 \in \mathbb{R}$
3	0	300	0	$\theta_3 \in \mathbb{R}$
4	0	350	0	$\theta_4 \in \mathbb{R}$
5	$\frac{\pi}{2}$	0	251	$\theta_5 \in \mathbb{R}$

Part B

$${}^{\text{base}}_5T = {}^0_5T = \prod_{i=1}^5 {}^{i-1}_iT = {}^0_1T {}^1_2T \dots {}^4_5T \text{ (base as index 0), where} \quad (1)$$

$${}^{i-1}_iT = \text{Rot}(\hat{x}_{i-1}, \alpha_{i-1}) \text{Trans}(\hat{x}_i, a_{i-1}) \text{Trans}(\hat{z}_i, d_i) \text{Rot}(\hat{z}_i, \theta_i) \in SE(3). \quad (2)$$

The symbolic derivation of ${}^{i-1}_iT$ in (5) are trivial and can be easily implemented in code, so here only demonstrate the cases as the robot is in zero configuration, i.e. $\theta_i = 0, \forall i$.

Substituting **Tab A** into (2) yields,

${}^{\text{base}}_5T$ is,

1	0	0	700
0	-1	0	35.3000
0	0	-1	107.5000
0	0	0	1

where ${}^{\text{base}}_1T$ is,

1.0000	0	0	0
0	1.0000	0	0
0	0	1.0000	358.5000
0	0	0	1.0000

1_2T is,

1.0000	0	0	50.0000
0	0	-1.0000	35.3000
0	1.0000	0	0
0	0	0	1.0000

2_3T is,

$$\begin{bmatrix} 1 & 0 & 0 & 300 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

3_4T is,

$$\begin{bmatrix} 1 & 0 & 0 & 350 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

4_5T is,

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & -251 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Part C-I

Given a robot target pose $(x, y, z, \phi, \theta, \psi)$, where $\{\phi, \theta, \psi\}$ are the $\hat{z} - \hat{y} - \hat{x}$ euler angle w.r.t. the base (world) frame respectively (yall-pitch-roll), the constrained inverse kinematic problem can be decoupled into *position problem* and *orientation problem*.

While before that, since the target position is defined w.r.t. the robot's *wrist center* (frame $\{4\}$ as in **Fig. A**), we first set the target wrist center point (which is also the origin of the frame $\{4\}$, say ${}^0p_{4,\text{org}}$), by shifting the target tip position back by d_5 , where,

$${}^0p_{4,\text{org}} = {}^0p_{5,\text{org}} + (-d_5) {}^0\hat{z}_5 = {}^0p_{5,\text{org}} + (-d_5) {}^0_5R \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} + {}^0_5R \begin{bmatrix} 0 \\ 0 \\ -d_5 \end{bmatrix} := \begin{bmatrix} x_w \\ y_w \\ z_w \end{bmatrix}, \quad (3)$$

$${}^0_5R = \text{Rot}(\hat{z}, \phi) \text{Rot}(\hat{y}, \theta) \text{Rot}(\hat{x}, \psi). \quad (4)$$

In the following position problem, the target wrist position $\begin{bmatrix} x_w & y_w & z_w \end{bmatrix}^T$ will be used for simplicity of notations.

1. Position problem (Solve for $\{\theta_1, \theta_2, \theta_3\}$)



Elbow-up configuration

The the elbow-up configuration can be ensured by imposing the constraint between $\{\theta_1, \theta_2, \theta_3\}$, which is shown in eq. (5).

First, the target tip position is assumed to lie in the I and IV quadrant, and it's assumed that there's no offset between the 1st and 2nd joint. In this case, the range of the joint variables can be divided into 2 cases,

$$(x_w, y_w) \in \text{I, IV} \begin{cases} \text{Case 1 : } \theta_1 \in [-\frac{\pi}{2}, \frac{\pi}{2}), \theta_2 \in [0, \frac{\pi}{2}), \theta_3 \text{ s.t. } \theta_2 + \theta_3 \in [-\pi, 0), \text{ or} \\ \text{Case 2 : } \theta_1 \in [\frac{\pi}{2}, \frac{3\pi}{2}), \theta_2 \in [\frac{\pi}{2}, \pi), \theta_3 \text{ s.t. } \theta_2 + \theta_3 \in [\pi, 2\pi). \end{cases} \quad (5)$$

From (5) it can be easily seen that there's at most #2 possible solutions of the position problem.

- Due to the presence to offset between the 1st and 2nd joint, the division of the range of θ_1 may need a little bit adjustment, yet the *number of maximally possible solutions of the position problem (#2) stays the same*. Similarly, this holds for the case for $(x_w, y_w) \in \text{II, III}$.

Due to the simple geometry of the robot, the joint variables can be solved intuitively by trigonometry ($\{\theta_1 \rightarrow \theta_2 \rightarrow \theta_3\}$).

Defining some useful variables,

$$r := \sqrt{x_w^2 + y_w^2} \in \mathbb{R}^+, \quad (6)$$

$$z'_w := z_w - d_1 \in \mathbb{R}. \quad (7)$$

- **Case 1** Suppose $(x, y) \in \text{I, IV}$, $\theta_1 \in [-\frac{\pi}{2}, \frac{\pi}{2})$, $\theta_2 \in [0, \frac{\pi}{2})$, θ_3 s.t. $\theta_2 + \theta_3 \in [-\pi, 0)$

As seen in **Fig. C.1.1.1**, θ_1 is solved,

$$\theta_1 = \text{atan2}(y_w, x_w) - \lambda_1, \text{ where} \quad (8)$$

$$\lambda_1 = \text{atan2}(-d_2, \sqrt{r^2 - d_2^2}), (d_2 < 0). \quad (9)$$

Also r' is derived for later use,

$$r' = \sqrt{r^2 - d_2^2} - a_1. \quad (10)$$

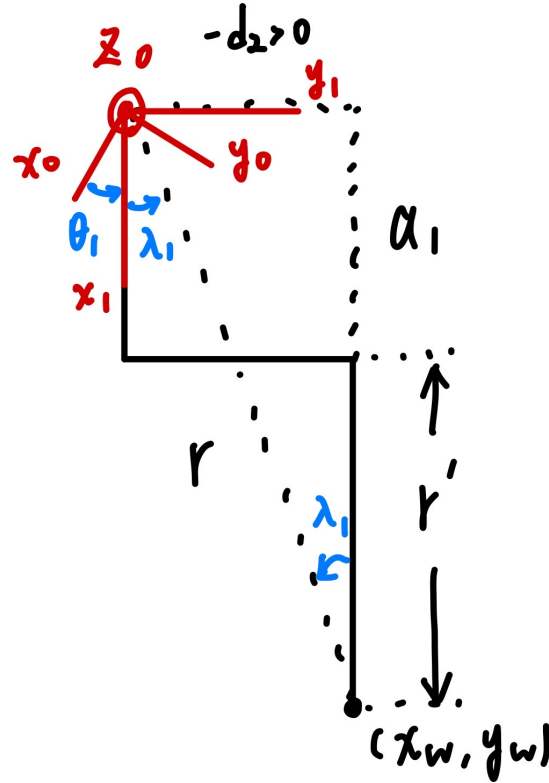


Fig. C.1.1

As seen in **Fig. C.1.2**, θ_2 and θ_3 can be solved via law of cosine,

$$\cos(\theta_2 - \lambda_2) = \frac{a_2^2 + r'^2 + z_w'^2 - a_3^2}{2a_2\sqrt{r'^2 + z_w'^2}} := D_2 \in \mathbb{R}, \text{ where} \quad (11)$$

$$\lambda_2 = \text{atan2}(z_w', r') \in \mathbb{R}, \quad (12)$$

$$\Rightarrow \theta_2 = \text{atan2}(\pm\sqrt{1 - D_2^2}, D_2) + \lambda_2 \text{ (this holds for both } \lambda_2 < 0 \text{ and } \lambda_2 > 0), \quad (13)$$

$$= \text{atan2}(+\sqrt{1 - D_2^2}, D_2) + \lambda_2 \text{ (for } (x_w, y_w) \in I, IV \text{ in this case).} \quad (14)$$

$$\cos(-\theta_3) = \frac{r'^2 + z_w'^2 - a_2^2 - a_3^2}{2a_2a_3} := D_3 \in \mathbb{R}, \quad (15)$$

$$\Rightarrow \theta_3 = -\text{atan2}(\pm\sqrt{1 - D_3^2}, D_3), \quad (16)$$

$$= -\text{atan2}(+\sqrt{1 - D_3^2}, D_3) \text{ (for } (x_w, y_w) \in I, IV \text{ in this case).} \quad (17)$$

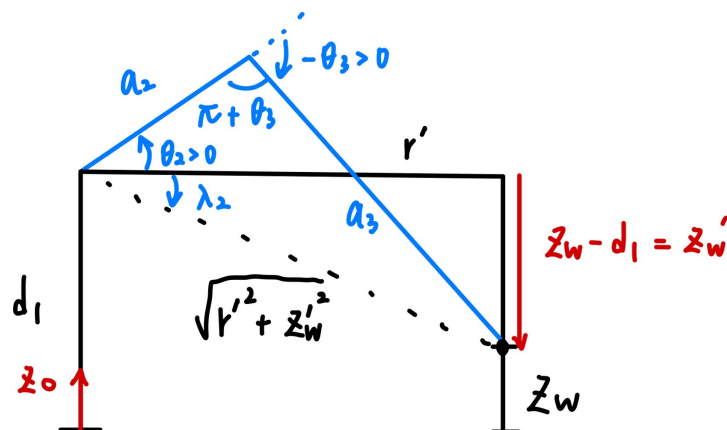


Fig. C.1.2

Eq.s (8), (13), (15) gives the solution of the position problem in this case. ■

- **Case 2** Suppose $(x, y) \in I, IV$, $\theta_1 \in [\frac{\pi}{2}, \frac{3\pi}{2})$, $\theta_2 \in [\frac{\pi}{2}, \pi)$, θ_3 s.t. $\theta_2 + \theta_3 \in [\pi, 2\pi)$

As seen in **Fig. C.1.3**, θ_1 is solved,

$$\theta_1 = \frac{3\pi}{2} + \lambda_1 - \mu_1, \text{ where} \quad (18)$$

$$\lambda_1 = \text{atan2}(y_w, x_w), \mu_1 = \text{atan2}(\sqrt{r^2 - d_2^2}, -d_2), (-d_2 > 0). \quad (19)$$

Also r' is derived for later use,

$$r' = \sqrt{r^2 - d_2^2} + a_1. \quad (20)$$

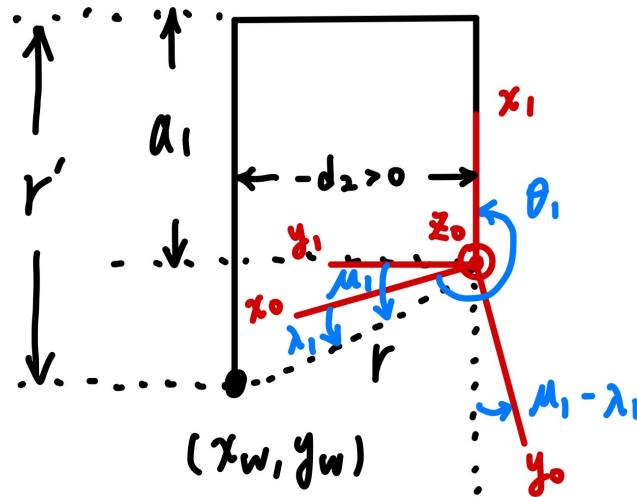


Fig. C.1.3

As seen in **Fig. C.1.4**, θ_2 and θ_3 can be solved via law of cosine,

$$\cos(\theta_2 + \lambda_2) = \frac{a_3^2 - a_2^2 - (r'^2 + z_w'^2)}{2a_2\sqrt{r'^2 + z_w'^2}} := D_2 \in \mathbb{R}, \text{ where} \quad (21)$$

$$\lambda_2 = \text{atan2}(z_w', r') \in \mathbb{R}, \quad (22)$$

$$\Rightarrow \theta_2 = \text{atan2}(\pm\sqrt{1 - D_2^2}, D_2) - \lambda_2 \text{ (this holds for both } \lambda_2 < 0 \text{ and } \lambda_2 > 0), \quad (23)$$

$$= \text{atan2}(+\sqrt{1 - D_2^2}, D_2) - \lambda_2 \text{ (for } (x_w, y_w) \in I, IV \text{ in this case)}. \quad (24)$$

$$\cos(\theta_3) = \frac{r'^2 + z_w'^2 - a_2^2 - a_3^2}{2a_2a_3} := D_3 \in \mathbb{R}, \quad (25)$$

$$\Rightarrow \theta_3 = \text{atan2}(\pm\sqrt{1 - D_3^2}, D_3), \quad (26)$$

$$= \text{atan2}(+\sqrt{1 - D_3^2}, D_3) \text{ (for } (x_w, y_w) \in I, IV \text{ in this case)}. \quad (27)$$

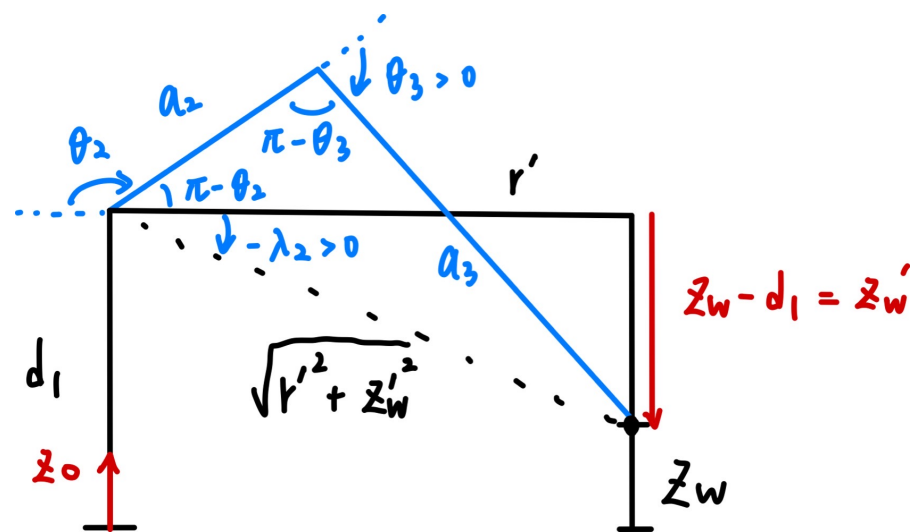


Fig. C.1.4

Eq.s (16), (21), (23) gives the solution of the position problem in this case. ■

- Therefore, the above derivations cover all the possible solutions (#2 under the constraints) of the position problem in elbow-up configuration.
- Note that these are the *maximally* possible solutions, which may still be unsolvable if the target wrist position falls outside of the reachable workspace of the first 3 joints (the position problem fails).
- Taking the elbow-up constraint into account, eq. (5) must to be satisfied in the position problem in each case.

2. Orientation problem (Solve for $\{\theta_4, \theta_5\}$)



Gripper tip pose being vertically downward

Followed from the results in the position problem, to ensure the gripper tip pose to be vertically downward, we then divide the ranges of joint variables into the same 2 cases again as done in (5), but this time with θ_4, θ_5 involved.

First, the target tip position is assumed to lie in the **I** and **IV** quadrant, and it's assumed that there's no offset between the 1st and 2nd joint. In this case, the range of the joint variables can be divided into 2 cases (constraints),

$$(x, y) \in \text{I, IV} \begin{cases} \text{Case 1 : } \theta_1 \in [-\frac{\pi}{2}, \frac{\pi}{2}), \theta_2 \in [0, \frac{\pi}{2}), \theta_3 \text{ s.t. } \theta_2 + \theta_3 \in [-\pi, 0) \Rightarrow \theta_4 \text{ s.t. } \theta_2 + \theta_3 - \frac{\pi}{2} + \theta_4 = -\frac{\pi}{2}, \text{ or} \\ \text{Case 2 : } \theta_1 \in [\frac{\pi}{2}, \frac{3\pi}{2}), \theta_2 \in [\frac{\pi}{2}, \pi), \theta_3 \text{ s.t. } \theta_2 + \theta_3 \in [\pi, 2\pi) \Rightarrow \theta_4 \text{ s.t. } \theta_2 + \theta_3 + \frac{\pi}{2} + \theta_4 = \frac{3\pi}{2}. \end{cases} \quad (28)$$

In both 2 cases, θ_5 is unconstrained.

- As mentioned before, due to the presence of offset between the 1st and 2nd joint, the division of the range of θ_1 may need a little bit adjustment, yet the *number of maximally possible solutions of the position problem (#2) stays the same*. Also note that in each case of position problem, the orientation has a corresponding unique solution (or no solution if unreachable). Similarly, this holds for the case for $(x_w, y_w) \in \text{II, III}$.

Now, the problem of solving the last 2 joint variables $\{\theta_4, \theta_5\}$ can be transformed into the problem of parameterizing the rotation matrix, 3_5R ,

$$\underbrace{{}^0_3R^T}_{\text{position problem}} \underbrace{{}^0_5R}_{\text{target pose}} \equiv {}^3_5R = \underbrace{{}^3_4R}_{\theta_4} \underbrace{{}^4_5R}_{\theta_5} = \text{Rot}(\hat{z}, \theta_4) \text{Rot}(\hat{x}, \alpha_4) \text{Rot}(\hat{z}, \theta_5). \quad (29)$$

Set 3_5R as,

$${}^3_5R = [r_{ij}]. \quad (30)$$

Note that since $\alpha_4 = \frac{\pi}{2}$ is already given in the D-H table (it's determined by the robot's nature), by entry-wise matching of the matrices, eq. (26) becomes,

$${}^3_5R = [r_{ij}] = \begin{bmatrix} \cos \theta_4 \cos \theta_5 & -\cos \theta_4 \sin \theta_5 & \sin \theta_4 \\ \cos \theta_5 \sin \theta_4 & -\sin \theta_4 \sin \theta_5 & -\cos \theta_4 \\ \sin \theta_5 & \cos \theta_5 & 0 \end{bmatrix}, \quad (31)$$

Then we can solve for θ_4 , and θ_5 ,

$$\begin{cases} \theta_4 = \text{atan2}(r_{13}, -r_{23}), \\ \theta_5 = \text{atan2}(r_{31}, r_{32}). \end{cases} \quad (32)$$

It is easily seen that the orientation problem has either a *unique solution* or *no solution*, where the latter occurs as,

$$r_{33} \neq 0, \quad (33)$$

so that $\{\theta_4, \theta_5\}$ has no solution and the orientation problem fails.

- Taking into account the constraint that gripper tip should always be vertically downward, eq. (28) must be satisfied in the orientation problem in each case.

Part C-II

Implement the algorithm provided **Part C-I** in MATLAB code, we have,

(A)

$$\text{Case 1 : } \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{bmatrix} \approx \begin{bmatrix} 0.1071 \\ 0.3634 \\ -1.0218 \\ 0.6584 \\ -0.6783 \end{bmatrix}, \text{ with the elbow and gripper constraints in (5) and (28) satisfied,} \quad (34)$$

$$\text{Case 2 : No solution (the position problem fails since } 665.984 \approx \sqrt{r'^2 + z'_w{}^2} > a_2 + a_3 = 650, \text{ unreachable).} \quad (35)$$

(B)

- - - -

$$\text{Case 1 : } \begin{bmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \\ \theta_5 \end{bmatrix} \approx \begin{bmatrix} 0.1071 \\ 0.4762 \\ -1.0672 \\ 0.5910 \\ -0.6783 \end{bmatrix}, \text{ with the elbow and gripper constraints in (5) and (28) satisfied,} \quad (36)$$

$$\text{Case 2 : No solution (the position problem fails since } 659.7615 \approx \sqrt{r'^2 + z_w'^2} > a_2 + a_3 = 650, \text{ unreachable).} \quad (37)$$

Part C-III

$$\text{Case 1 : No solution (the position problem fails since } 825.8444 \approx \sqrt{r'^2 + z_w'^2} > a_2 + a_3 = 650, \text{ unreachable),} \quad (38)$$

$$\text{Case 2 : No solution (the position problem fails since } 896.3645 \approx \sqrt{r'^2 + z_w'^2} > a_2 + a_3 = 650, \text{ unreachable).} \quad (39)$$

Both **Case 1**, and **Case 2** failed, and thus the target pose is *unreachable* due to the failure of position problems.

Part D-I

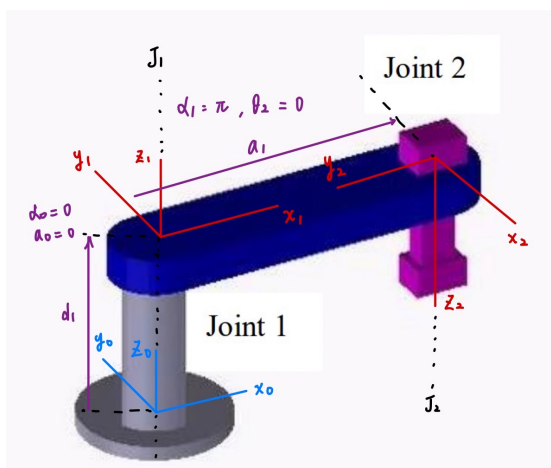


Fig. D.1

Part D-II

Tab D

J_i (i th joint)	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	$d_1 \in \mathbb{R}^+$	$\theta_1 \in \mathbb{R}$
2	π	$a_1 \in \mathbb{R}^+$	$d_2 \in \mathbb{R}$	0

Varying parameters: θ_1 for joint 1, d_2 for joint 2.