1.2 Procrustes Problem

$$R = \underset{\Omega}{\operatorname{argmin}} \|\Omega A - B\|_F^2$$

$\Omega^T \Omega = I$, Ω is an unitary matrix

$$\|X\|_F^2 = \langle X, X \rangle_F = tr(X^T X)$$

$$R = \underset{\Omega}{\operatorname{argmin}} \|\Omega A - B\|_F^2$$

$$= \underset{\Omega}{\operatorname{argmin}} \langle \Omega A - B, \Omega A - B \rangle_F$$

$$= \underset{\Omega}{\operatorname{argmin}}\ tr\left((\Omega A - B)^T (\Omega A - B)\right)$$

$$= \underset{\Omega}{\operatorname{argmin}}\ tr\left((\Omega A)^T \Omega A - (\Omega A)^T B - B^T(\Omega A) + B^T B\right)$$

$$\because tr((\Omega A)^T B) = tr(((\Omega A)^T B)^T) = tr(B^T(\Omega A))$$

$$= \underset{\Omega}{\operatorname{argmin}} \left(tr((\Omega A)^T \Omega A) + tr(B^T B) - 2\langle \Omega A, B \rangle_F\right)$$

$\because \Omega$ is unitary matrix

$$= \underset{\Omega}{\operatorname{argmin}} \left(tr(A^T A) + tr(B^T B) - 2\langle \Omega A, B \rangle_F\right)$$

$$= \underset{\Omega}{\operatorname{argmax}} \langle \Omega A, B \rangle_F$$

$$= \underset{\Omega}{\operatorname{argmax}}\ tr((\Omega A)^T B)$$

$$= \operatorname{argmax}\ tr(A^T B \Omega)$$

$$\because tr(AB) = tr(BA)$$

$$\sim_{\Omega}$$

$$= \underset{\Omega}{\operatorname{argmax}} \langle \Omega, BA^T \rangle$$

$$= \underset{\Omega}{\operatorname{argmax}} \langle \Omega, U\Sigma V^T \rangle$$

$$= \underset{\Omega}{\operatorname{argmax}} \, tr(\Omega^T U \Sigma V^T)$$

$$= \underset{\Omega}{\operatorname{argmax}} \, tr(U\Omega^T V^T \Sigma)$$

$$= \underset{\Omega}{\operatorname{argmax}} \langle U^T \Omega V, \Sigma \rangle$$ ← diagonal matrix

$\because (U^T\Omega V)^T (U^T \Omega V)$

$= V^T \Omega^T U U^T \Omega V = I$

$\therefore$ $U^T \Omega V$ is orthogonal matrix

$\langle U^T\Omega V, \Sigma \rangle_F$ is only maximal when all components of $U^T\Omega V$ is on diagonal axis, also because it is unitary, it need to be identity matrix

$$U^T \Omega V = I$$

$$\Omega = V^T U$$

$\because \det(\Omega) = 1$ (rotation matrix)

$$\therefore \quad \Omega = V^T \Sigma' U$$

$$\Sigma' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{sign}(\det(V^T U)) \end{bmatrix}$$