Complementing Büchi Automata with a Subset-tuple Construction

Joel Allred
Department of Informatics
University of Fribourg, Switzerland
E-Mail: joel.allred@unifr.ch

Ulrich Ultes-Nitsche
Department of Informatics
University of Fribourg, Switzerland
E-Mail: uun@unifr.ch

***Abstract*—Complementation of Büchi automata is well known for being difficult. In the worst case, a state-space growth of $(0.76n)^n$ is unavoidable. Recent studies suggest that "simpler" algorithms perform better than more involved ones on practical cases. In this paper, we present a simple "direct" algorithm for complementing Büchi automata. It involves a structured subset construction (using tuples of subsets of states) that produces a deterministic automaton. This construction leads to a complementation procedure that resembles the straightforward complementation algorithm for deterministic Büchi automata, the latter algorithm actually being a special case of our construction.**

I. Introduction

Checking ω-language containment is important in linear-time temporal verification. For ω-languages B and P, testing $B \subseteq P$ is done algorithmically by testing $B \cap \overline{P} = \emptyset$, where $\overline{P}$ is the complement of P. If P is regular and represented by a Büchi-automaton, a Büchi-automaton representing $\overline{P}$ can be constructed effectively. In the worst case, the automaton for $\overline{P}$ has $(0.76n)^n$ many states [1], where n is the number of states of the automaton for P.

Complementation of Büchi automata is difficult because in general, Büchi automata cannot be made deterministic. Complementation of a deterministic Büchi automaton $\mathcal{A}$ is, however, quite simple: construct a copy of $\mathcal{A}$, remove all accepting states in the copy, make the remaining states in the copy accepting states, make all states in $\mathcal{A}$ non-accepting, and finally allow nondeterministically moving from $\mathcal{A}$ to corresponding states in its copy. So the complement automaton to a deterministic Büchi automaton consists of two deterministic automata (let's call them an upper and a lower automaton, if we assume that one is drawn below the other) with transitions from upper to the lower automaton but no transitions back from the lower to the upper one. We adapt this construction to nondeterministic automata by constructing a deterministic version of a given automaton that then Büchi-accepts a different ω-language but can still be used in a construction similar to the one for deterministic Büchi-automata to produce a complement automaton. It results again in an automaton consisting basically of two deterministic automata that are then combined by allowing to move nondeterministically from the upper to the lower one but not vice versa.

The run analysis that we developed independently leads to similar results as the analysis by Fogarty, Kupferman, Vardi, and Wilke [2] who elegantly translate the slice-based approach in [3] to a rank-based approach. Instead of unifiying interim constructions, we directly construct a complement automaton using the sole analysis of the runs of the original automaton, in a similar way deterministic Büchi automata are complemented.

When developing our construction, the key notion will be that of a *greedy* accepting run that we introduce in this paper. In short, a greedy run is a run that always aims at reaching an accepting state as quickly as possible. Our construction will consider only greedy runs. By doing so, visits of sets of accepting states in runs of the constructed automaton allow faithfully to identify whether or not they can also occur in a (greedy) run of the given automaton. To consider only greedy runs in a deterministic version of a given Büchi-automaton, we basically conduct a subset construction to which we add sufficient structure to identify greedy runs: the states in the construction are tuples of sets of states of the given automaton, sets of non-accepting states and sets of accepting states are never mixed (to avoid losing too much information [4]).

From the so constructed deterministic automaton, by identifying a particular property of state sets occurring in the state-set tuples (they are either *continued* or *discontinued*), we can construct a nondeterministic one accepting the complement ω-language of the given Büchi automaton in a similar way as for deterministic Büchi automata. The usual complementation procedure for deterministic Büchi automata (Section 3 in [5]) will be a special case of our construction.

II. Preliminaries

Let Σ be an alphabet (set of finite cardinality). The set of all ω-words (infinitely long words) over Σ is designated by Σ^ω. Subsets of Σ^ω are called ω-languages over Σ.

Büchi automaton $\mathcal{A} = (Q, \Sigma, \delta, q_{in}, F)$ consists of finite set Q of states, input alphabet Σ, transition function $\delta : Q \times \Sigma \to 2^Q$, initial state $q_{in} \in Q$, and set $F \subseteq Q$ of accepting states.

δ is extended in the usual way to finite words (instead of only single symbols) and sets of state (instead of only single states).

Let $x = x(0)x(1)x(2)\cdots \in \Sigma^\omega$. A run r of $\mathcal{A}$ on x is a sequence $r(0)r(1)r(2)\cdots$ of states such that $r(0) = q_{in}$ and $r(i+1) \in \delta(r(i), x(i))$, for all $i \geq 0$. Runs on finite words are defined similarly.

Let $\omega(r)$ be the set of all states that recur infinitely often in r. Run r is accepting if and only if $\omega(r) \cap F \neq \emptyset$. Automaton $\mathcal{A}$ Büchi-accepts x if and only if there exists an accepting run of $\mathcal{A}$ on x [6], [7]. The set of all ω-words accepted by $\mathcal{A}$ is the ω-language that $\mathcal{A}$ accepts.

If, for all $q \in Q$ and $a \in \Sigma$, $\delta(q, a)$ is a singleton set or the empty set, then $\mathcal{A}$ is called deterministic. Otherwise it is called nondeterministic. For ω-languages, the language classes accepted by deterministic and nondeterministic automata differ: There exist ω-languages that can only be represented by nondeterministic finite automata [6], [7].

We call a (finite) run $r = r(0)r(1)r(2)\cdots r(n)$ on some finite word *greedy* if it always tries to reach, on its way to $r(n)$, accepting states as quickly as possible. To define greediness formally, let $\alpha : Q \to \{0, 1\}$ be a function that assigns 1 to a state if and only if it is accepting. α can be extended in the usual way to a function $\alpha : Q^* \to \{0, 1\}^*$ on sequences of states by its stepwise application.

We interpret words over $\{0, 1\}$ as binary numbers with the most significant bit to the left. We define that $r = r(0)r(1)r(2)\cdots r(n)$ on finite word w is *greedy* if and only if there does not exist a finite run $r' = r(0)r'(1)r'(2)\cdots r'(n-1)r(n)$ on w (note that r and r' coincide in their first and last state) such that $\alpha(r) < \alpha(r')$.

The definition of greediness can be extended to infinite runs r by defining that infinite run r is greedy if and only if all of its finite prefixes are greedy.

III. The Subset-Tuple Construction

A. Definitions

Let $\mathcal{A} = (Q, \Sigma, \delta, q_{in}, F)$ be a Büchi automaton. We construct an interim deterministic Büchi-automaton $\mathcal{A}' = (Q', \Sigma, \delta', (\{q_{in}\}), \emptyset)$ from which we will later derive the complement automaton of $\mathcal{A}$. The state set of $\mathcal{A}'$ contains non-empty disjoint sets of $\mathcal{A}$-states:[1]

$$Q' = \bigcup_{m=1}^{|Q|} \{(S_1, \ldots, S_m) \in (2^Q \setminus \{\emptyset\})^m \mid (\forall 1 \leq j < k \leq m : S_j \cap S_k = \emptyset)\}.$$

[1]By "$\mathcal{A}$-states" we refer to the states of automaton $\mathcal{A}$.

$\mathcal{A}'$ does not have accepting states as $\mathcal{A}$'s acceptance condition will not influence the construction of the complement automaton. We will use some other observation about runs of $\mathcal{A}'$ to define the acceptance condition when finally constructing the complement automaton.

We define transition function $\delta' : Q' \times \Sigma \to Q'$ subsequently. Let $p' \in Q'$ and let $m = |p'|$. Let $p'(j)$ be the jth component of p' (note that $p'(j)$ is a set of $\mathcal{A}$-states). For $a \in \Sigma$, we define the a-successor of the jth component of p' to be

$$\sigma(p', j, a) = \delta(p'(j), a) \setminus \bigcup_{k=j+1}^{m} \delta(p'(k), a).$$

$\sigma(p', j, a)$ contains all $\mathcal{A}$-states that are a-successors of $\mathcal{A}$-states in $p'(j)$ and not already contained in $\sigma(p', k, a)$, for $k > j$. From this definition, we get immediately that sets $\sigma(p', j, a)$ and $\sigma(p', k, a)$ are disjoint.

The definition of $\sigma(p', j, a)$ guarantees that if an $\mathcal{A}$-state could occur in multiple components of a tuple, we will keep it only in the rightmost component. Note that even though $\delta(p'(j), a)$ may not be empty, $\sigma(p', j, a)$ still can be empty. We partition each set $\sigma(p', j, a)$ into non-accepting $\mathcal{A}$-states and accepting $\mathcal{A}$-states:

$$\sigma_n(p', j, a) = \sigma(p', j, a) \cap (Q \setminus F),$$
$$\sigma_a(p', j, a) = \sigma(p', j, a) \cap F.$$

As we just partitioned the sets $\sigma(p', j, a)$, the resulting sets are still pairwise disjoint.

We put $\sigma_a(p', j, a)$ to the right of $\sigma_n(p', j, a)$ and remove all empty sets. We define the transition function of $\mathcal{A}'$:

$$\delta'(p', a) = q',$$

where q' is obtained by removing all empty sets in

$$(\sigma_n(p', 1, a), \sigma_a(p', 1, a), \ldots, \sigma_n(p', m, a), \sigma_a(p', m, a))$$

but otherwise keeping the order of the non-empty components.

Lemma 1. *For all reachable states $p' \in Q'$ in $\mathcal{A}'$ and all $1 \leq j < k \leq |p'|$, $p'(j)$ and $p'(k)$ are nonempty and disjoint.*

B. Example

We take the automaton in Figure 1 as an example.

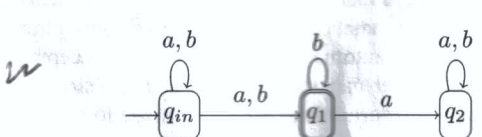
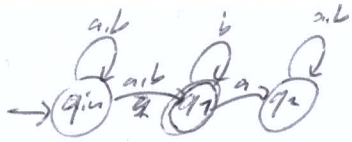


Fig. 1. Büchi automaton $\mathcal{A}$ accepting $\{a, b\}^* \cdot \{b\}^\omega$.



Automaton $\mathcal{A}$ in Figure 1 Büchi-accepts all ω-words over $\{a, b\}$ that contain finitely many occurrences of symbol a. From $\mathcal{A}$, we construct $\mathcal{A}'$ stepwise. The initial state of $\mathcal{A}'$ is the 1-tuple that contains $\mathcal{A}$-state set $\{q_{in}\}$ (see Figure 2).

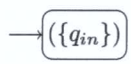


Fig. 2. Initial fragment of $\mathcal{A}'$.

With respect to $\mathcal{A}$'s transition relation, we get

$$\sigma((\{q_{in}\}), 1, a) = \delta(\{q_{in}\}, a) = \{q_{in}, q_1\},$$

$$\sigma((\{q_{in}\}), 1, b) = \delta(\{q_{in}\}, b) = \{q_{in}, q_1\}.$$

Because $\{q_{in}, q_1\}$ contains non-accepting $\mathcal{A}$-state q_{in} as well as accepting $\mathcal{A}$-state q_1, $\{q_{in}, q_1\}$ is partitioned into two sets:

$$\sigma_n((\{q_{in}\}), 1, a) = \sigma_n((\{q_{in}\}), 1, b) = \{q_{in}\},$$

$$\sigma_a((\{q_{in}\}), 1, a) = \sigma_a((\{q_{in}\}), 1, b) = \{q_1\},$$

and we get $(\{q_{in}\}, \{q_1\})$ as the successor of $(\{q_{in}\})$ when $\mathcal{A}'$ reads a or b (see Figure 3).

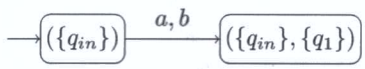


Fig. 3. Initial fragment of $\mathcal{A}'$.

In the next step, we calculate for symbol a:

$$\sigma((\{q_{in}\}, \{q_1\}), 2, a) = \delta(\{q_1\}, a) = \{q_2\}$$

and

$$\begin{aligned}\sigma((\{q_{in}\}, \{q_1\}), 1, a) &= \delta(\{q_{in}\}, a) \setminus \delta(\{q_1\}, a) \\ &= \{q_{in}, q_1\} \setminus \{q_2\} = \{q_{in}, q_1\}.\end{aligned}$$

As in the previous step, $\{q_{in}, q_1\}$ is partitioned into the two sets $\{q_{in}\}$ and $\{q_1\}$, and we get $(\{q_{in}\}, \{q_1\}, \{q_2\})$ as the a-successor of $(\{q_{in}\}, \{q_1\})$.

Similarly, for symbol b, we calculate

$$\sigma((\{q_{in}\}, \{q_1\}), 2, b) = \delta(\{q_1\}, b) = \{q_1\}$$

and

$$\begin{aligned}\sigma((\{q_{in}\}, \{q_1\}), 1, b) &= \delta(\{q_{in}\}, b) \setminus \delta(\{q_1\}, b) \\ &= \{q_{in}, q_1\} \setminus \{q_1\} = \{q_{in}\}.\end{aligned}$$

No sets need to be partitioned and we get $(\{q_{in}\}, \{q_1\})$ as the b-successor of $(\{q_{in}\}, \{q_1\})$. (see Figure 4).

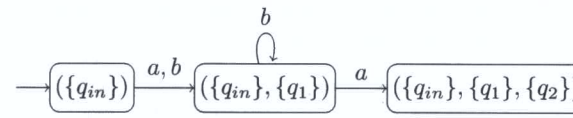


Fig. 4. Initial fragment of $\mathcal{A}'$.

Now we calculate for symbol a:

$$\begin{aligned}\sigma((\{q_{in}\}, \{q_1\}, \{q_2\}), 3, a) &= \delta(\{q_2\}, a) = \{q_2\}, \\ \sigma((\{q_{in}\}, \{q_1\}, \{q_2\}), 2, a) &= \delta(\{q_1\}, a) \setminus \delta(\{q_2\}, a) \\ &= \{q_2\} \setminus \{q_2\} = \emptyset,\end{aligned}$$

and

$$\begin{aligned}&\sigma((\{q_{in}\}, \{q_1\}, \{q_2\}), 1, a) \\ &= \delta(\{q_{in}\}, a) \setminus (\delta(\{q_1\}, a) \cup \delta(\{q_2\}, a)) \\ &= \{q_{in}, q_1\} \setminus \{q_2\} = \{q_{in}, q_1\}.\end{aligned}$$

As previously, $\{q_{in}, q_1\}$ is partitioned into the two sets $\{q_{in}\}$ and $\{q_1\}$, the empty set is removed, and we get $(\{q_{in}\}, \{q_1\}, \{q_2\})$ as the a-successor of $(\{q_{in}\}, \{q_1\}, \{q_2\})$.

Similarly, for symbol b, we calculate

$$\begin{aligned}\sigma((\{q_{in}\}, \{q_1\}, \{q_2\}), 3, b) &= \delta(\{q_2\}, b) = \{q_2\}, \\ \sigma((\{q_{in}\}, \{q_1\}, \{q_2\}), 2, b) &= \delta(\{q_1\}, b) \setminus \delta(\{q_2\}, b) \\ &= \{q_1\} \setminus \{q_2\} = \{q_1\}.\end{aligned}$$

and

$$\begin{aligned}&\sigma((\{q_{in}\}, \{q_1\}, \{q_2\}), 1, b) \\ &= \delta(\{q_{in}\}, b) \setminus (\delta(\{q_1\}, b) \cup \delta(\{q_2\}, b)) \\ &= \{q_{in}, q_1\} \setminus \{q_1, q_2\} = \{q_{in}\}.\end{aligned}$$

No sets need to be partitioned and we get $(\{q_{in}\}, \{q_1\}, \{q_2\})$ as the b-successor of $(\{q_{in}\}, \{q_1\}, \{q_2\})$, completing the construction (see Figure 5).

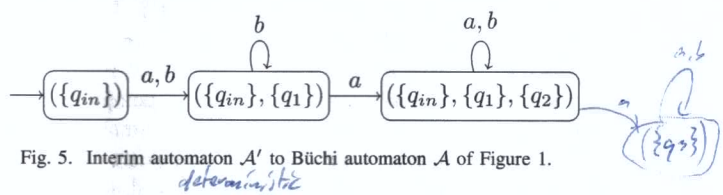


Fig. 5. Interim automaton $\mathcal{A}'$ to Büchi automaton $\mathcal{A}$ of Figure 1.

C. Additional Notation

As pointed out in the previous section, it is important to identify for transitions in $\mathcal{A}'$ which component in a successor state results from which component in the predecessor state. In addition, it will be important to identify which component

will eventually have no successor component any more (the component disappears eventually, it is *discontinued*).

Let a be a symbol in Σ. Let p' and q' be two $\mathcal{A}'$-states such that $\delta'(p', a) = q'$. Let $1 \leq j \leq |p'|$ and $1 \leq k \leq |q'|$.

If $q'(k) \subseteq \sigma(p', j, a)$, then we write

$$p'(j) \overset{a}{\mapsto} q'(k),$$

indicating that p''s a-successor q' contains component k *because* p' contains component j. We extend this definition to finite words $w = a_0 a_1 \dots a_l \in \Sigma^*$ in the usual way:

$$q'_0(j_0) \overset{w}{\mapsto} q'_{l+1}(j_{l+1})$$

if and only if there exist states $q'_1, q'_2, \dots, q'_l$ and indices $j_1, j_2, \dots, j_l$ such that for all i, $0 \leq i \leq l$, we have $q'_i(j_i) \overset{a_i}{\mapsto} q'_{i+1}(j_{i+1})$.

If we take the example from Figure 5, we have, for instance,

$$(\{q_{in}\}, \{q_1\}, \{q_2\})(1) \overset{a}{\mapsto} (\{q_{in}\}, \{q_1\}, \{q_2\})(2),$$

because when reading a in state $(\{q_{in}\}, \{q_1\}, \{q_2\})$, the successor state that is also $(\{q_{in}\}, \{q_1\}, \{q_2\})$ contains $\{q_1\}$ *because* $\{q_1\} \subseteq \sigma((\{q_{in}\}, \{q_1\}, \{q_2\}), 1, a)$. If we change however the symbol, then

$$(\{q_{in}\}, \{q_1\}, \{q_2\})(2) \overset{b}{\mapsto} (\{q_{in}\}, \{q_1\}, \{q_2\})(2),$$

because when reading b in state $(\{q_{in}\}, \{q_1\}, \{q_2\})$, the successor state that is also $(\{q_{in}\}, \{q_1\}, \{q_2\})$ contains $\{q_1\}$ *because* $\{q_1\} \subseteq \sigma((\{q_{in}\}, \{q_1\}, \{q_2\}), 2, b)$.

For the remainder of this paper, we always assume that $\mathcal{A}'$ is complete. This can be achieved, for instance, by making $\mathcal{A}$ complete before constructing $\mathcal{A}'$. It guarantees that for each ω-word x, a unique ($\mathcal{A}'$ is deterministic) run of $\mathcal{A}'$ on x does always exist.

Let $x = x(0)x(1)x(2) \dots \in \Sigma^\omega$. Let $r' = r'(0)r'(1)r'(2) \dots$ be the run of $\mathcal{A}'$ on x.

Let $i \geq 0$ and let $1 \leq j \leq |r'(i)|$. We write

$$r'(i)(j)_\perp$$

to indicate that either there does not exist a k such that $r'(i)(j) \overset{x(i)}{\mapsto} r'(i+1)(k)$, or for each k such that $r'(i)(j) \overset{x(i)}{\mapsto} r'(i+1)(k)$ we have $r'(i+1)(k)_\perp$. The notation $r'(i)(j)_\perp$ is used to indicate that in the run r' of $\mathcal{A}'$ on x, component j of state $r'(i)$ will disappear. Either it disappears immediately (it does not have an $x(i)$-successor) or it disappears eventually (all its $x(i)$-successors will disappear). We will say that the component is *discontinued* in $\mathcal{A}'$'s run on x.

Conversely, we write

$$r'(i)(j)^\top$$

if and only if $r'(i)(j)_\perp$ does not hold (we then say that $r'(i)(j)$ is *continued* in $\mathcal{A}'$'s run on x). In addition, we write

$$r'(i)(j)^\top_F$$

if and only if $r'(i)(j)^\top$ and $q'(i)(j) \subseteq F$, and

if and only if there exists j such that $r'(i)(j)^\top_F$.

If $r'(i)(j)^\top$, then $r'(i)(j)$ has an $x(i)$-successor $r'(i+1)(k)$ such that $r'(i+1)(k)^\top$, because otherwise $r'(i)(j)_\perp$ would hold. Therefore, and because of the pairwise disjointness of components in $\mathcal{A}'$-states (Lemma 1), the number of continued components cannot decrease from one state to the next in $\mathcal{A}'$'s run on x:

Lemma 2. *Let k be the number of different components $r'(i)(j)$ of state $r'(i)$ such that $r'(i)(j)^\top$, and let l be the number of different components $r'(i+1)(j)$ of state $r'(i+1)$ such that $r'(i+1)(j)^\top$. Then $k \leq l$.*

As an example, let $y = bababa\dots$. The run r' of the automaton in Figure 5 on ω-word y is then:

$$(\{q_{in}\})(\{q_{in}\}, \{q_1\})(\{q_{in}\}, \{q_1\}, \{q_2\})(\{q_{in}\}, \{q_1\}, \{q_2\})\dots$$

In this run, we now label all components of $\mathcal{A}'$-states either with $^\top$ or $_\perp$, depending on whether they are continued or not:

$$(\{q_{in}\}^\top)(\{q_{in}\}^\top, \{q_1\}^\top)(\{q_{in}\}^\top, \{q_1\}_\perp, \{q_2\}^\top)$$
$$(\{q_{in}\}^\top, \{q_1\}_\perp, \{q_2\}^\top)(\{q_{in}\}^\top, \{q_1\}_\perp, \{q_2\}^\top)\dots$$

As we could see in the construction of the automaton in Figure 5, $(\{q_{in}\}, \{q_1\}, \{q_2\})(2)$ does not have a successor (i.e. the successor set is the empty set) when reading symbol a, because

$$\begin{aligned}\sigma((\{q_{in}\}, \{q_1\}, \{q_2\}), 2, a) &= \delta(\{q_1\}, a) \setminus \delta(\{q_2\}, a) \\ &= \{q_2\} \setminus \{q_2\} = \emptyset.\end{aligned}$$

Therefore $(\{q_{in}\}, \{q_1\}, \{q_2\})(2)$ is labelled with $_\perp$ in the run, as the run is on an ω-word that contains infinitely many times symbol a, and $(\{q_{in}\}, \{q_1\}, \{q_2\})(2)$ never persists.

The situation changes entirely, when we consider ω-word $z = aabbbb\dots$ (two as followed by exclusively bs). Run r' of $\mathcal{A}'$ on z is again:

$$(\{q_{in}\})(\{q_{in}\}, \{q_1\})(\{q_{in}\}, \{q_1\}, \{q_2\})(\{q_{in}\}, \{q_1\}, \{q_2\})\dots$$

However, labelling all components of $\mathcal{A}'$-states either with $^\top$ or $_\perp$, depending on whether they are continued or not, leads now to:

$$(\{q_{in}\}^\top)(\{q_{in}\}^\top, \{q_1\}^\top)(\{q_{in}\}^\top, \{q_1\}^\top, \{q_2\}^\top)$$
$$(\{q_{in}\}^\top, \{q_1\}^\top, \{q_2\}^\top)(\{q_{in}\}^\top, \{q_1\}^\top, \{q_2\}^\top)\dots$$

Now $(\{q_{in}\}, \{q_1\}, \{q_2\})(2)$ is always labelled with $^\top$ because the absence of symbol a in z results in $(\{q_{in}\}, \{q_1\}, \{q_2\})(2)$

always having a successor:

$$\sigma((\{q_{in}\},\{q_1\},\{q_2\}),2,b) = \delta(\{q_1\},b) \setminus \delta(\{q_2\},b)$$
$$= \{q_1\} \setminus \{q_2\} = \{q_1\}.$$

Because q_1 is an accepting $\mathcal{A}$-state, the run can even be labelled as:

$$(\{q_{in}\}^\top)(\{q_{in}\}^\top,\{q_1\}_F^\top)(\{q_{in}\}^\top,\{q_1\}_F^\top,\{q_2\}^\top)$$
$$(\{q_{in}\}^\top,\{q_1\}_F^\top,\{q_2\}^\top)(\{q_{in}\}^\top,\{q_1\}_F^\top,\{q_2\}^\top)\dots$$

With our new notation, we will show in the next section that run r' of $\mathcal{A}'$ on ω-word x will contain infinitely many states with component labels $\{\}_F^\top$ if and only if the original automaton $\mathcal{A}$ Büchi-accepts x.

D. Some Properties of the Construction

Let $q' \in Q'$. We will write "$\bigcup q'$" to designate $\bigcup_{i=1}^{|q'|} q'(i)$, i.e. the set of all $\mathcal{A}$-states that occur in q'. From the definition of the transition function δ' we get immediately:

Lemma 3. $\delta(\bigcup q', a) = \bigcup \delta(q', a)$.

For the remainder of this section, let $a \in \Sigma$, let $w \in \Sigma^*$, let $x \in \Sigma^\omega$, and let r' be the run of $\mathcal{A}'$ on x.

Lemma 4. $\delta(\{q_{in}\}, w) = \bigcup \delta'(q'_{in}, w)$.

Proof. We prove the lemma by an induction on the length $|w|$ of w. If $|w| = 0$, then $\mathcal{A}$ reaches $\{q_{in}\} = \bigcup q'_{in}$ by reading w. If $|w| > 0$, then $w = va$ with $v \in \Sigma^*$ and $a \in \Sigma$. Assuming $\delta(\{q_{in}\}, v) = \bigcup \delta'(q'_{in}, v)$ we get:

$$\delta(\{q_{in}\}, w) = \delta(\delta(\{q_{in}\}, v), a) = \delta(\bigcup \delta'(q'_{in}, v), a)$$
$$= \bigcup \delta'(\delta'(q'_{in}, v), a) = \bigcup \delta'(q'_{in}, w),$$

where the second equality holds by induction, and the third applies Lemma 3. □

Lemma 5. *If there are infinitely many j such that $r'(j)$ contains a continued set of accepting states (written: $r'(j)_F^\top$), then $\mathcal{A}$ accepts x.*

Proof. From Lemma 2 we know that the number of continued components cannot decrease when moving forward in a run. Because each state in $\mathcal{A}'$ can have at most m components, where m is the number of states in $\mathcal{A}$, the number of continued components must remain constant after some position k in run r'. Let this number be l where $1 \le l \le m$. So after position k in r', each state will contain l continued components, and therefore each continued component will have exactly one continued component as its successor when reading the respective symbol.

So after position k, we can identify l sequences of consecutive continued components in r'. Because there are infinitely many continued components in r' that are sets of accepting $\mathcal{A}$-states, at least one of these l sequences of consecutive continued components must contain infinitely many continued components that are sets of accepting $\mathcal{A}$-states (because l is finite).

Therefore there must be sequences $i_1, i_2, i_3 \dots$ of run positions and $j_1, j_2, j_3 \dots$ of component indices such that, for all $n \ge 1$ and $w_n = x(i_n) \dots x(i_{n+1} - 1)$:

$$r'(i_n)(j_n)_F^\top$$

and

$$r'(i_n)(j_n) \overset{w_n}{\mapsto} r'(i_{n+1})(j_{n+1}).$$

Then there exists to each $\mathcal{A}$-state $q_{n+1} \in r'(i_{n+1})(j_{n+1})$ an $\mathcal{A}$-state $q_n \in r'(i_n)(j_n)$ such that $q_{n+1} \in \delta(q_n, w_n)$. Applying Koenig's Lemma [8], there exists a sequence $p_1, p_2, p_3, \dots$ of $\mathcal{A}$-states such that $p_{n+1} \in \delta(p_n, w_n)$ and $p_1 \in \delta(q_{in}, x(0) \dots x(i_1 - 1))$. Therefore there exists a run r of $\mathcal{A}$ on x that contains the states p_n. Because all p_n are accepting $\mathcal{A}$-states, r is an accepting run and $\mathcal{A}$ accepts x. □

Lemma 6. *If $\mathcal{A}$ accepts x, then there are infinitely many i such that $r'(i)$ contains a continued set of accepting states (written: $r'(i)_F^\top$).*

Proof. Let r be a greedy accepting run of $\mathcal{A}$ on x and let r' be the run of $\mathcal{A}'$ on x. We can then prove (see below) that there exists a sequence $j_0, j_1, j_2 \dots$ of component indices, $1 \le j_0, j_1, j_2 \dots \le m$, such that for all $i \ge 0$:

$$r(i) \in r'(i)(j_i) \quad (1)$$

and

$$r'(i)(j_i) \overset{x(i)}{\mapsto} r'(i+1)(j_{i+1}) \quad (2)$$

Therefore all $r'(i)(j_i)$ are continued. In addition, because r is accepting, infinitely many $r(i)$ are accepting and so are infinitely many $r'(i)(j_i)$, because they contain $r(i)$ and are therefore sets of accepting $\mathcal{A}$-states. Then infinitely many $r'(i)(j_i)$ are continued and accepting, proving the lemma.

What we still need to prove to complete the proof of the lemma is that $r(i) \in r'(i)(j_i)$ and $r'(i)(j_i) \overset{x(i)}{\mapsto} r'(i+1)(j_{i+1})$. We prove this fact by contradiction: we assume it is not true and prove that then r cannot be greedy.

Because of Lemma 4, there is always a component $r'(i)(j_i)$ in $\mathcal{A}'$-state $r'(i)$ that contains $r(i)$. If the statement is not true, then the sequence $j_0, j_1, j_2, \dots$ of component indices such that $r(i) \in r'(i)(j_i)$ must contain a position $k \ge 0$ such that

$$r'(k)(j_k) \overset{x(k)}{\not\mapsto} r'(k+1)(j_{k+1}).$$

Let k be the smallest such position. According to the definition of $\mathcal{A}'$'s transition function δ', there must be a $j'_k > j_k$ such that

$$r'(k)(j'_k) \overset{x(k)}{\mapsto} r'(k+1)(j_{k+1}).$$

Then there is a sequence $j'_0, j'_1, j'_2, \ldots, j'_k$ of component positions such that, for all $1 \leq n \leq k-1$ such that

$$r'(n)(j'_n) \overset{x(n)}{\mapsto} r'(n+1)(j'_{n+1}).$$

$j_0, j_1, j_2, \ldots, j_k$ and $j'_0, j'_1, j'_2, \ldots, j'_k$ will coincide in a few first indices.[2] Let l be the first position at which $j_0, j_1, j_2, \ldots, j_k$ and $j'_0, j'_1, j'_2, \ldots, j'_k$ start to differ. Because $j'_k > j_k$, also $j'_l > j_l$. So we have

$$r'(l-1)(j_{l-1}) \overset{x(l-1)}{\mapsto} r'(l)(j_l),$$

$$r'(l-1)(j_{l-1}) \overset{x(l-1)}{\mapsto} r'(l)(j'_l),$$

and

$$j'_l > j_l.$$

From the definition of $\mathcal{A}'$'s transition function δ' we get that $r'(l)(j_l)$ contains non-accepting $\mathcal{A}$-states, but $r'(l)(j'_l)$ contains accepting $\mathcal{A}$-states. Let $\tilde{r}(0)\tilde{r}(1)\ldots\tilde{r}(k+1)$ be the run of $\mathcal{A}$ on $x(0)x(1)\ldots x(k)$ such that for all $1 \leq n \leq k+1$

$$\tilde{r}(n) \in r'(n)(j'_n)$$

and

$$\tilde{r}(k+1) = r(k+1).$$

Then $\tilde{r}(0)\tilde{r}(1)\ldots\tilde{r}(k+1)$ and $r(0)r(1)\ldots r(k+1)$ are runs of $\mathcal{A}$ on $x(0)x(1)\ldots x(k)$, and $\tilde{r}(0)\tilde{r}(1)\ldots\tilde{r}(k+1)$ visits an accepting state earlier than $r(0)r(1)\ldots r(k+1)$ does. So $r(0)r(1)\ldots r(k+1)$ cannot be greedy, and consequently r cannot be greedy as $r(0)r(1)\ldots r(k+1)$ is a prefix of r, contradicting the choice of r and completing the proof of the lemma. □

Putting Lemmas 5 and 6 together, we obtain:

Corollary 1. *$\mathcal{A}$ accepts x if and only if r' contains infinitely many $\mathcal{A}'$-states that contain continued sets of accepting $\mathcal{A}$-states.*

To complete this section, let us briefly look at the size of the construction: we calculate the number of states in $\mathcal{A}'$. These states are n-tuples of disjoint sets of $\mathcal{A}$-states, where n ranges from 1 to m and m is the number of states of $\mathcal{A}$. Let $f(m,n)$ be the function calculating the number of states of $\mathcal{A}'$ containing n components.

Let us start with $f(m,1)$, i.e. the number of $\mathcal{A}'$-states with one component. Each of the m $\mathcal{A}$-states is either in the component or it is not, leading to two possibilities each. Therefore there are 2^m possibilities altogether. However, there is one possibility that represents the component being the empty set, which does not represent an $\mathcal{A}'$-state an consequently must be subtracted (all other possibilities represent a non-empty component). Therefore

$$f(m,1) = 2^m - 1.$$

Because there exists only one 0-tuple "()", $f(m,0) = 1$ and $f(m,1)$ can be calculated by

$$f(m,1) = 2^m - f(m,0).$$

So for an arbitrary n-tuple, each of the m states of $\mathcal{A}$ can be in one of the n components of the tuple or do not occur at all. So there are $n+1$ possibilities for each of the m states, leading to $(n+1)^m$ possibilities to form such n-tuples. However, we must remove the n-tuples that contain empty sets as components. An n-tuple in which exactly i components are non-empty can be calculated by counting the number of i tuples that are $\mathcal{A}'$-states (that guarantees that the i components are indeed not empty), i.e. $f(m,i)$, multiplied by the different possibilities to distribute i non-empty sets over n components, i.e. $\binom{n}{i}$, where $\binom{n}{i}$ is the binomial coefficient $\frac{n!}{i!(n-i)!}$. So

$$f(m,n) = (n+1)^m - \sum_{i=0}^{n-1} \left(\binom{n}{i} f(m,i)\right),$$

and the number of states in $\mathcal{A}'$ is

$$|Q'| = \sum_{i=1}^{m} f(m,i).$$

First calculations show that this figure is between $m!$ and m^m. A more precise approximation is still to be investigated.

At this stage, we have a deterministic automaton, which is not a Büchi automaton, that is equivalent to the original non-deterministic Büchi automaton.

IV. The Complementation Construction

A. Additional Indices

With the use of a colouring of the tuple components, we can now extend automaton $\mathcal{A}'$ described in Section III, to build an automaton $\mathcal{A}_c$ complement to $\mathcal{A}$. In the process, we create an *upper* (non-accepting) automaton $\hat{\mathcal{A}}$ and a *lower* (accepting) automaton $\check{\mathcal{A}}$ and put them together into automaton $\mathcal{A}_c$. This colouring takes the form of an index that can take integer values in $[-1, 2]$. The meaning of the colours will be presented further down.

Whereas a state q' of $\mathcal{A}'$ is written $(S_1, S_2, \ldots, S_m)$, a state q of $\hat{\mathcal{A}}$, $\check{\mathcal{A}}$ or $\mathcal{A}_c$ is written as

$$q = \big((S_1, c_1), (S_2, c_2), \ldots, (S_m, c_m)\big),$$

where $S_j \in 2^Q \setminus \{\emptyset\}$ and $c_j \in [-1, 2]$, $\forall 1 \leq j \leq m$.

[2] They coincide at least in $j'_0 = j_0 = 1$, because $r'(0)$ is the initial state of $\mathcal{A}'$ that contains a single component (containing the initial state of $\mathcal{A}$).

B. Upper (Non-accepting) Part

The upper part $\hat{\mathcal{A}}$ represents the initial part of $\mathcal{A}_c$ and contains no accepting states. The denomination "upper" comes from the fact that the authors generally picture the accepting part of the complement under the non-accepting one. The structure is inspired by the complementation construction for deterministic Büchi automata, where the non-accepting part is simply a non-accepting copy of the original automaton representing everything that can happen in a finite prefix of an ω-word. Accepting runs of $\mathcal{A}_c$ then jump to the lower (accepting) part.

In the upper automaton $\hat{\mathcal{A}} = (\hat{Q}, \Sigma, \hat{\delta}, \hat{q}_{in}, \hat{F})$ we simply append colour -1 to all components of each $\hat{\mathcal{A}}$-state. States (written $(S_1, S_2, \ldots, S_m)$) of the interim automaton $\mathcal{A}'$ are thus rewritten $\big((S_1, -1), (S_2, -1), \ldots (S_m, -1)\big)$ in $\hat{\mathcal{A}}$. The transition function should be rewritten accordingly as follows. Let $a \in \Sigma$, $p' := (S_1, \ldots, S_m) \in Q', q' := (S'_1, \ldots, S'_{m'}) \in Q'$. If

$$\delta'(p, a) = q$$

where δ' is as in Section III, then for

$$\hat{p} := \big((S_1, -1), \ldots, (S_m, -1)\big) \in \hat{Q} \text{ and}$$
$$\hat{q} := \big((S'_1, -1), \ldots, (S'_{m'}, -1)\big) \in \hat{Q},$$

$\hat{\delta}$ is defined by:

$$\hat{\delta}(\hat{p}, a) := \hat{q}.$$

As for the other elements:

- $\hat{q}_{in} := \big((\{q_{in}\}, -1)\big)$
- $\hat{F} := \emptyset$

Only $\hat{\mathcal{A}}$-states will have components with colour -1.

C. Lower (Accepting) Part

The lower automaton $\check{\mathcal{A}} := (\check{Q}, \Sigma, \check{\delta}, \check{q}_{in}, \check{F})$ can be defined in a similar fashion, but with values $0, 1, 2$ for the colouring indices and a non-empty accepting set $\check{F}$. The set of accepting states F_c is then defined according to the colouring of the components of the states.

In a nutshell, for a an ω-word $x \in \Sigma^\omega$ we want the run r_c of $\mathcal{A}_c$ on x to be accepting if and only if each greedy run r of the original automaton $\mathcal{A}$ on x either:

- eventually stops visiting states of F, or
- is discontinued (i.e. is finite or becomes non-greedy).

Translated to $\check{\mathcal{A}}$ (where any finite behaviour has already been taken care of in $\hat{\mathcal{A}}$, these constraints become:

- the run does not visit states of F, or
- it is discontinued (i.e. is finite or becomes non-greedy).

We now give some insight into the meaning of the colours. Let i' be the point where run r_c jumps to the lower part, i.e. $i \geq i' \Leftrightarrow r_c(i) \in \check{\mathcal{A}}$. The colours of the components of the states of $\check{\mathcal{A}}$ hold some information on what happens in the greedy runs of $\mathcal{A}$, "after" i' .

- Colour $c = 0$:
  If a tuple component (S_j, c_j) is 0-coloured in $r_c(i)$, i.e. $(S_j, 0) \in r_c(i)$, for an $i \geq i'$, then for each state q in S_j, the greedy run r of $\mathcal{A}$ s.t. $r(i) = q$ has not yet visited an accepting $\mathcal{A}$-state since i'. An $\check{\mathcal{A}}$-state containing only 0-coloured components can be set accepting, because all greedy runs of $\mathcal{A}$ that may have visited an accepting $\mathcal{A}$-state between i and i' have disappeared.
- Colour $c = 2$:
  If a tuple component (S_j, c_j) is 2-coloured in $r_c(i)$, i.e. $(S_j, 2) \in r_c(i)$, for an $i \geq i'$, then for each state q in S_j, the greedy run r of $\mathcal{A}$ s.t. $r(i) = q$ has visited an accepting $\mathcal{A}$-state since i'. $\mathcal{A}_c$-states containing 2-coloured components are kept non-accepting, because that greedy run r has visited accepting $\mathcal{A}$-states and has not yet disappeared. If it never disappears, then r_c only visits non-accepting states and it is correct to reject r_c in $\mathcal{A}_c$.
- Colour $c = 1$:
  If a tuple component (S_j, c_j) is 1-coloured in $r_c(i)$, i.e. $(S_j, 1) \in r_c(i)$, for an $i \geq i'$, then for each state q in S_j, the greedy run r of $\mathcal{A}$ s.t. $r(i) = q$ has visited an accepting $\mathcal{A}$-state since i', but there also exist 2-coloured components that have not yet disappeared. We say that 1-coloured components are *on hold*, meaning that for the moment, we wait until the 2-coloured components disappear. When this happens, 1-coloured components become 2-coloured in the following state. $\mathcal{A}_c$-states that contain only 0-coloured or 1-coloured components are set accepting, because all greedy runs of $\mathcal{A}$ that have visited an accepting $\mathcal{A}$-state between i and i' have disappeared. If this happens infinitely often, then no greedy run of $\mathcal{A}$ can visit infinitely many accepting $\mathcal{A}$-states, and it is correct to accept r_c in $\mathcal{A}_c$.

We define $\check{\mathcal{A}}$ formally.

First, let us write the set of possible states:

$$\check{Q} := \bigcup_{m=1}^{|Q|} \big\{\big((S_1, c_1), \ldots, (S_m, c_m)\big) \in (2^Q \setminus \{\emptyset\} \times [0, 2])^m \big| \\ \forall\, 1 \leq j < k \leq m, S_j \cap S_k = \emptyset\big\}.$$

The transition function $\check{\delta}$ is defined as follows:

$$\check{\delta} : \check{Q} \times \Sigma \to \check{Q}$$

As earlier, we extend the transition function δ' of Section III to define $\check{\delta}$. Let $\check{p} := \big((S_1, c_1), \ldots, (S_m, c_m)\big) \in \check{Q}$ be a state of $\check{\mathcal{A}}$.

Let $p' := (S_1, \ldots, S_m)$ be the corresponding state of $\mathcal{A}'$ obtained by removing the c_j's. Let $a \in \Sigma$ and let $q' = (S'_1, \ldots, S'_{m'})$ be the unique state of $\mathcal{A}'$ s.t. $\delta'(p', a) = q'$.

We now define the values of the indices $c'_{j'}$ of the a-successor of $\check{p}$:

$$\forall 1 \leq j' \leq m', \text{ let } 1 \leq j_{pred} \leq m \text{ be (unique) s.t. } \check{p}(j_{pred}) \xrightarrow{a} \check{q}(j'),$$

$$\begin{cases} \text{if } \forall 1 \leq j \leq m : c_j \neq 2 \quad \text{then} \begin{cases} c'_j := 0 & \text{if } c_{j_{pred}} = 0 \\ & \quad \wedge\ S'_{j'} \nsubseteq F \\ c'_j := 2 & \text{otherwise} \end{cases} \\ \text{if } \exists 1 \leq j \leq m : c_j = 2 \quad \text{then} \begin{cases} c'_j := 0 & \text{if } c_{j_{pred}} = 0 \\ & \quad \wedge\ S'_{j'} \nsubseteq F \\ c'_j := 2 & \text{if } c_{j_{pred}} = 2 \\ c'_j := 1 & \text{otherwise} \end{cases} \end{cases}$$

For $\check{q} := \big((S'_1, c'_1) \ldots, (S'_{m'}, c'_{m'})\big)$ we define:

$$\check{\delta}(\check{p}, a) := \check{q}.$$

Even though it will not be used, the initial state can be defined as :

$$\check{q}_{in} := \big((\{q_{in}\}, 0)\big).$$

The set $\check{F}$ of accepting states holds all states that do not contain a 2-coloured component :

$$\check{F} := \bigcup \{(S_1, c_1), \ldots, (S_m, c_m)) \in \check{Q} \mid \forall 1 \leq j \leq m, c_j < 2\}.$$

D. The Complement Automaton

Applying the construction below, we can join automata $\hat{\mathcal{A}}$ and $\check{\mathcal{A}}$ to build a new automaton $\mathcal{A}_c$ representing a language which is complement to the language of the original automaton $\mathcal{A}$. The set of states Q_c is the union of the sets of states of both automata and the nondeterministic transition function δ_c is the union of the functions of the two automata plus some extra transitions which unidirectionally link the upper automaton to the lower one. As mentioned earlier, the accepting states are only located in the lower deterministic part of $\mathcal{A}_c$.

We create the automaton $\mathcal{A}_c = (Q_c, \Sigma, \delta_c, q_c, F_c)$ complement to $\mathcal{A}$ by joining automata $\hat{\mathcal{A}}$ and $\check{\mathcal{A}}$ in the following way:

- $Q_c = \hat{Q} \cup \check{Q}$
- Let $p = \big((S_1, c_1), \ldots, (S_m, c_m)\big) \in Q_c$ and $a \in \Sigma$. The nondeterministic transition function

$$\delta_c : Q_c \times \Sigma \to 2^{Q_c}$$

is defined as follows [3]:

$$\begin{cases} \delta_c(p, a) := & \hat{\delta}(p, a) \cup \check{\delta}\Big(\big((S_1, 0), \ldots, (S_m, 0)\big), a\Big) \\ & \text{if } p \in \hat{\mathcal{A}}, \\ \delta_c(p, a) := \check{\delta}(p, a) & \text{otherwise} \end{cases}$$

- $q_c := \hat{q}_{in}$
- $F_c := \check{F}$

The proof of correctness of the construction is done in detail in Appendix A. Adding the colouring to $\mathcal{A}'$, does not introduce more than 3^i, for each $\mathcal{A}'$-state with i components. So the size of the entire construction is not more than

$$|Q_c| = \big(\sum_{i=1}^{m} f(m,i)\big) + \big(\sum_{i=1}^{m} 3^i f(m,i)\big).^4$$

We resume our previous example and construct the complement of the automaton of Figure 1.

To avoid cumbersome notation in the following example, a component (S_j, c_j) will be denoted as:

$$\begin{array}{llll} \widehat{S_j} & \text{if } c_j = -1 & \overline{S_j} & \text{if } c_j = 1 \\ S_j & \text{if } c_j = 0 & \overline{\overline{S_j}} & \text{if } c_j = 2 \end{array}$$

The upper part $\hat{\mathcal{A}}$ of the complement automaton is just the automaton $\mathcal{A}'$ where we append colour -1 to all components (Figure 6).

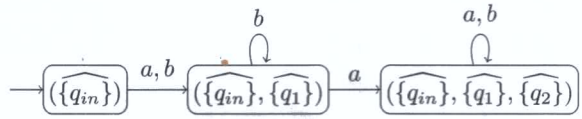


Fig. 6. Upper part $\hat{\mathcal{A}}$ of Büchi automaton $\mathcal{A}_c$ complement to $\mathcal{A}$ of Figure 1.

The lower part $\check{\mathcal{A}}$ is constructed in a similar way, but with the addition of colours 0, 1 and 2. Let's first look at the a-successor in $\check{\mathcal{A}}$ of $(\widehat{\{q_{in}\}})$. The two successor sets are $\{q_{in}\}$ and $\{q_1\}$. By definition of the transition function $\check{\delta}$, since $\{q_{in}\} \cap F = \emptyset$, the colour (initially 0) remains 0. For the set $\{q_1\}$, since $\{q_1\} \subseteq F$, the new colour is 2. So the a-successor of $(\widehat{\{q_{in}\}})$ in $\check{\mathcal{A}}$ is $(\{q_{in}\}, \overline{\overline{\{q_1\}}})$. The same holds for symbol b and we can draw the transition (Figure 7):

Let's now look at the a-successor of this newly created state. The a-successor of $\{q_1\}$ is $\{q_2\}$ and colour 2 remains. The a-successors of $\{q_{in}\}$ are $\{q_{in}\}$ (left successor) and $\{q_1\}$ (right-successor). Since $\{q_{in}\} \cap F = \emptyset$, the colour of $\{q_{in}\}$ remains 0. The colour of $\{q_1\}$ is set to 1 because $\{q_1\} \subseteq F$ and colour 2 already exists in the predecessor state. So the a-successor of

[3] Transitions from the upper to the lower part are as if originating from entirely 0-coloured states.

[4] Size of the upper part (which is equal to the size of $\mathcal{A}'$) plus the size of the lower part.

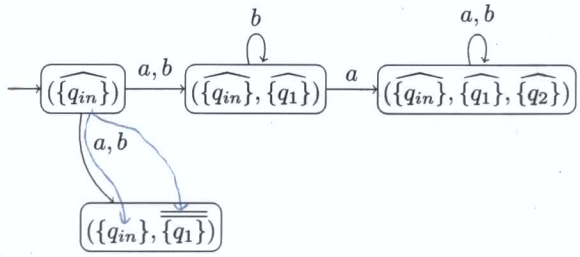


Fig. 7. Interim lower part $\check{A}$ of Büchi automaton $\mathcal{A}_c$.

$(\{q_{in}\}, \overline{\{q_1\}})$ is finally $(\{q_{in}\}, \overline{\{q_1\}}, \overline{\{q_2\}})$ and we can draw the transition in our automaton (Figure 8).

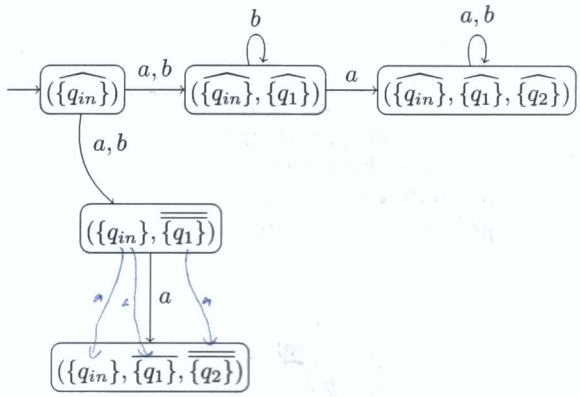


Fig. 8. Interim lower part $\check{A}$ of Büchi automaton $\mathcal{A}_c$.

By applying this simple method for all reachable $\mathcal{A}_c$-states, we get the automaton of Figure 9, where the accepting states are the states of the lower part which do not contain a 2-coloured component. Here it is only the case for state $(\{q_{in}\}, \overline{\{q_1\}}, \{q_2\})$.

E. Possibilities for Optimization

Without further proof here, it is easy to see that there is some room for optimization. For instance in the case where automaton $\mathcal{A}$ is complete, we observe that $\mathcal{A}_c$-states of which the rightmost component has colour 2 can be ignored, as well as all their successors. This is because the completeness enforces that "branches" of the execution tree can only "die" if at some point, the deletion process described in section III removes the component. Since the deletion process is done right-to-left, this is only possible if there exists another component on the right, which is not the case here. Therefore a rightmost branch is persistent and if its colour is 2, this colour never disappears and future states can never become accepting.

Taking our previous example, since $\mathcal{A}$ is complete, the optimization directly leads to the automaton of Figure 10 (applied at construction time and not as a reduction from Figure 9).

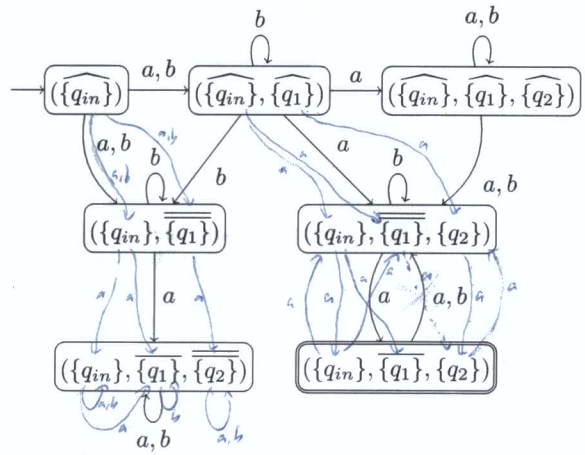


Fig. 9. Automaton $\mathcal{A}_c$ complement to $\mathcal{A}$.

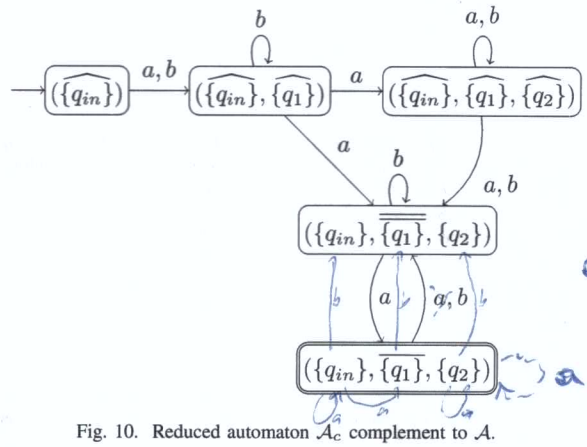


Fig. 10. Reduced automaton $\mathcal{A}_c$ complement to $\mathcal{A}$.

V. Conclusion

We have presented here a direct complementation construction for Büchi automata. The goal was not to find an efficient algorithm, such a search would have been vain because of the theoretical limits of complementation. We however strongly believe that some optimizations can stem from a better comprehension of the internal mechanism of the complement operation on Büchi automata. A complete complexity study is yet to be done, but early performance tests on random samples [9] show that despite not beating existing algorithms with respect to the number of generated states, execution times are reduced compared to [10], which appears to be a consequence of not having to manage extra formalisms such that they exist in other constructions.

Constructed as a concatenation of two deterministic automata, the resulting complement automaton has the property of being deterministic in the limit. As a side note, the nondeterminism

degree of the complement is 2. Another interesting property is the fact that the simple complementation algorithm for deterministic Büchi automata is a special case of our construction.

Future prospects are two-fold. First, we would like to achieve a thorough study of the complexity of our construction, including the search for bad cases; finding "bad" automata where the state explosion is unavoidable is very instructive as it allows to better understand under what conditions the complementation can effectively be done. We have already experimented with the automata used by Michel [11] (cited after [12]) when showing that a state explosion of $2^{O(n \log n)}$ cannot be avoided.

The second prospect is the empirical study of performance. As it is suggested above, optimizations can be derived simply by looking at the unfolding of the process, which appears to be very promising.

References

[1] S. Schewe, "Büchi complementation made tight," in *STACS*, ser. LIPIcs, S. Albers and J.-Y. Marion, Eds., vol. 3. Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik, Germany, 2009, pp. 661–672.

[2] S. Fogarty, O. Kupferman, T. Wilke, and M. Y. Vardi, "Unifying büchi complementation constructions," *Logical Methods in Computer Science*, vol. 9, no. 1, 2013.

[3] D. Kähler and T. Wilke, "Complementation, disambiguation, and determinization of büchi automata unified," in *ICALP (1)*, ser. Lecture Notes in Computer Science, L. Aceto, I. Damgård, L. A. Goldberg, M. M. Halldórsson, A. Ingólfsdóttir, and I. Walukiewicz, Eds., vol. 5125. Springer, 2008, pp. 724–735.

[4] U. Ultes-Nitsche, "A power-set construction for reducing Büchi automata to non-determinism degree two," *Information Processing Letters (IPL)*, vol. 101, no. 3, pp. 107–111, February 2007.

[5] F. Nießner, U. Nitsche, and P. Ochsenschläger, "Deterministic ω-regular liveness properties," in *Proceedings of the 3rd International Conference on Developments in Language Theory (DLT'97)*, S. Bozapalidis, Ed., Thessaloniki, Greece, 1998, pp. 237–247.

[6] J. R. Büchi, "On a decision method in restricted second order arithmetic," in *Proceedings of the International Congress on Logic, Methodology and Philosophy of Science 1960*, E. Nagel *et al.*, Eds. Stanford University Press, 1962, pp. 1–11.

[7] W. Thomas, "Automata on infinite objects," in *Formal Models and Semantics*, ser. Handbook of Theoretical Computer Science, J. van Leeuwen, Ed., vol. B. Elsevier, 1990, pp. 133–191.

[8] H. Hoogeboom and G. Rozenberg, "Infinitary languages: Basic theory and applications to concurrent systems," in *Current Trends in Concurrency*, ser. Lecture Notes in Computer Science, J. de Bakker, W.-P. de Roever, and G. Rozenberg, Eds., vol. 224. Springer Verlag, 1986, pp. 266–342.

[9] C. Göttel, "Implementation of an Algorithm for Büchi Complementation," Bachelor thesis, University of Fribourg, Switzerland, 2013.

[10] M.-H. Tsai, S. Fogarty, M. Y. Vardi, and Y.-K. Tsay, "State of Büchi complementation." in *CIAA'10*, 2010, pp. 261–271.

[11] M. Michel, "Complementation is more difficult with automata on infinite words," CNET, Paris, Tech. Rep., 1988.

[12] W. Thomas, "Handbook of formal languages, vol. 3," G. Rozenberg and A. Salomaa, Eds. New York, NY, USA: Springer-Verlag New York, Inc., 1997, ch. Languages, Automata, and Logic, pp. 389–455. [Online]. Available: http://dl.acm.org/citation.cfm?id=267871.267878

Appendix A
Proof of Language Equivalence

In order to prove the language equivalence between $L(\mathcal{A})$ and the complement of $L(\mathcal{A}_c)$, we first show the left-to-right direction, i.e. if an ω-word x is accepted by $\mathcal{A}$ then it will not be accepted by $\mathcal{A}_c$, i.e. all runs of $\mathcal{A}_c$ on x will be rejected.

Lemma 7. *Let $x = x(0)x(1)x(2)\ldots \in L(\mathcal{A})$. Then: $\forall$ run r_c of $\mathcal{A}_c$ on x, r_c is non-accepting.*

Proof. Let r' be the unique run of $\mathcal{A}'$ on x. Let r_c be a run of $\mathcal{A}_c$ on x. We show that r_c cannot be accepting. Consider two cases:

1) $r_c \subseteq \hat{\mathcal{A}}$, i.e. the run remains in the upper part. Then r_c never visits an accepting state and is thus non-accepting.
2) $r_c \cap \check{\mathcal{A}} \neq \emptyset$, i.e. $\exists l : r(l) \in \check{Q}$. Since $x \in L(A)$ Lemma 6 gives us: $\exists i' \geq l : r(i') \overset{F}{\top}$, so $\exists j' : 1 \leq j' \leq |r'(i')|$ s.t. $r'(i')(j') \in F$. Thus, the definition of the transition function δ_c (stemming from $\check{\delta}$) ensures that $r_c(i')(j')$ is 1 or 2-coloured. We consider those two cases:
   a) $r_c(i')(j')$ is 2-coloured: The transition function δ_c gives us that there exists a sequence of components $(S_{i'}, c_{i'})(S_{i'+1}, c_{i'+1}) \cdots \in (2^Q \times [-1, 2])^\omega$ s.t.
   - $(S_{i'}, c_{i'}) = r_c(i')(j')$
   - $\forall i \geq i'$, $(S_i, c_i) \in r_c(i)$
   - $\forall i \geq i'$, $(S_i, c_i) \overset{x(i)}{\mapsto} (S_{i+1}, c_{i+1})$
   - $\forall i \geq i'$, $c_i = 2$ (the colour cannot decrease).

   Since each state in $r_c(i)$ for $i \geq i'$ contains a 2-coloured component, none of these states can be accepting and r_c is thus a non-accepting run.
   b) $r_c(i')(j')$ is 1-coloured: Again we consider the sequence of components starting at $r_c(i')(j')$: $(S_{i'}, c_{i'})(S_{i'+1}, c_{i'+1}) \cdots \in (2^Q \times [-1, 2])^\omega$ s.t.
   - $(S_{i'}, c_{i'}) = r_c(i')(j')$
   - $\forall i \geq i'$, $(S_i, c_i) \in r_c(i)$
   - $\forall i \geq i'$, $(S_i, c_i) \overset{x(i)}{\mapsto} (S_{i+1}, c_{i+1})$

   Additionally, the definition of δ_c ensures that the colour c cannot decrease, so

   $$\forall i \geq i',\ c_i \geq 1.$$

   There are now two possibilities, either the colour eventually turns to 2, either it remains 1 forever:
   i) $\exists k' > i' : c_{k'} = 2$: Since $r(i')$ is continued, and the colour cannot decrease, we have $\forall i \geq k'$, $(S_i, 2) \in r(i)$ and thus $r(i) \notin F_c$. It follows that r_c is a non-accepting run.
   ii) $\forall k \geq i' : c_k = 1$: Here, the colour of the sequence remains 1 forever. By definition of the transition function δ_c, if a 1-coloured component exists at level k in the run, it implies that there exists a 2-coloured component at level $k-1$. Since $c_k = 1$ is true $\forall k \geq i'$,

then $\forall k \geq i' - 1$, there exists a 2-coloured component in $r_c(k)$, making all the states of r_c non-accepting for $k \geq i' - 1$.

So there exists no accepting run of x on $\mathcal{A}_c$. x is therefore in $\overline{L(\mathcal{A}_c)}$. □

We now show the opposite direction of the equivalence statement, by proving that an ω-word not accepted by $\mathcal{A}$ can produce an accepting run of $\mathcal{A}_c$.

Lemma 8. *Let $x = x(0)x(1)x(2)\ldots \in \Sigma^\omega \setminus L(\mathcal{A})$. Then $\exists$ run r_c of $\mathcal{A}_c$ on x s.t. r_c is accepting.*

Proof. Let r' be the unique run of $\mathcal{A}'$ on x (such a run exists under the assumption that $\mathcal{A}'$ is complete). By Lemma 5, there exist only finitely many i s.t. $r'(i)\overline{\in}F$. This means that $\exists i' \geq 0$ s.t. $\forall i \geq i'$, $!r'(i)\overline{\in}F$. We now construct an accepting run r_c of $\mathcal{A}_c$ on x which jumps to the lower part $\check{\mathcal{A}}$ exactly at that point i' where r' will have ceased holding sets that are continued and accepting.

Let $r_c = r_c(0)r_c(1)r_c(2)\cdots \in Q_c^\omega$ be a run of $\mathcal{A}_c$ on x s.t. $r_c(i) \in \check{Q}$ iff $i \geq i'$.

We show, by contradiction, that r_c is accepting. Suppose it is not. Then $\exists i'' \geq i'$ s.t. $\forall i \geq i''$, $r_c(i) \notin F_c$, i.e. r_c eventually visits only non-accepting states.

By definition of the acceptance condition F_c of $\mathcal{A}_c$, at least one of components of $r_c(i'')$ is 2-coloured. Formally: $\exists 1 \leq j'' \leq |r_c(i'')|$ s.t. $r_c(i'')(j'')$ is 2-coloured.

We now define the unique greedy run of $\mathcal{A}$ containing $r_c(i'')(j'')$:

Let $(S_0, c_0)(S_1, c_1)(S_2, c_2)\cdots \in (2^Q \times [-1, 2])^\omega$ be a sequence s.t.:

- $\forall k \geq 0, (S_k, c_k) \overset{x(k)}{\mapsto} (S_{k+1}, c_{k+1})$
- $(S_{i'}, c_{i'}) = r_c(i')(j')$ for a unique $j' : 1 \leq j' \leq |r_c(i')|$
- $(S_{i''}, c_{i''}) = r_c(i'')(j'')$.

Since $r_c(i'')(j'')$ is 2-coloured, the definition of δ_c implies that there must exist an $i''' : i' \leq i''' \leq i''$ and a $j''' : 1 \leq j''' \leq |r_c(i''')|$ s.t. $r_c(i''')(j''') \subseteq F$ [5], which contradicts the fact that $\forall i \geq i'$, $!r'(i)\overline{\in}F$. Therefore our assumption that r_c is not an accepting run is contradicted and $x \in L(\mathcal{A}_c)$.

This proves the correctness of the construction. □

As a consequence of lemmas 7 and 8, we have the following:

Corollary 2. *For $x \in \Sigma^\omega$, $x \in L(\mathcal{A}) \Leftrightarrow x \in \overline{L(\mathcal{A}_c)}$.*

[5] In words, this means that if we find 2-coloured component, then at some point in the past of this sequence, an accepting state of $\mathcal{A}$ was visited.