# Optimal Bounds for Transformations of $\omega$-Automata

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**Abstract.** In this paper we settle the complexity of some basic constructions of $\omega$-automata theory, concerning transformations of automata characterizing the set of $\omega$-regular languages. In particular we consider Safra's construction (for the conversion of nondeterministic Büchi automata into deterministic Rabin automata) and the appearance record constructions (for the transformation between different models of deterministic automata with various acceptance conditions). Extending results of Michel (1988) and Dziembowski, Jurdziński, and Walukiewicz (1997), we obtain sharp lower bounds on the size of the constructed automata.

## 1 Introduction

The theory of $\omega$-automata offers interesting transformation constructions, allowing to pass from nondeterministic to deterministic automata and from one acceptance condition to another. The automaton models considered in this paper are nondeterministic Büchi automata Büchi, 1962 [1], and deterministic automata of acceptance types Muller Muller, 1963 [12], Rabin Rabin, 1969 [13], Streett Streett, 1982 [17], and parity Mostowski, 1984 [11]. There are two fundamental constructions to achieve transformations between these models. The first is based on the data structure of *Safra trees* [14] (for the transformation from nondeterministic to deterministic automata), and the second on the data structure of *appearance records* [2,3,5] (for the transformations between deterministic Muller, Rabin, Streett, and parity automata). In this paper, we show that for most of the transformations, these constructions are optimal, sharpening previous results from the literature. This requires an analysis and extension of examples as proposed by Michel [10] and Dziembowski, Jurdziński, and Walukiewicz [4].

The first construction of deterministic Rabin automata from nondeterministic Büchi automata is due to McNaughton [9]. Safra's construction [14] generalizes the classical subset construction by introducing trees of states (Safra trees) instead of sets of states, yielding a complexity of $n^{\mathcal{O}(n)}$ (where $n$ is the number of states of the Büchi automaton). Using an example of Michel [10] one obtains the optimality of Safra's construction in the sense that there is no conversion of nondeterministic Büchi automata with $n$ states into deterministic Rabin automata with $2^{\mathcal{O}(n)}$ states and $\mathcal{O}(n)$ pairs in the acceptance condition (see the

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survey [18]). A drawback is the restriction to Rabin automata with $\mathcal{O}(n)$ pairs. In the present paper we eliminate this restriction. This is shown in Section 3. Also, using results of Section 4, we obtain an optimal bound for the transformation of nondeterministic Büchi automata into deterministic Streett automata.

For the transformations between deterministic models, the construction of appearance records, introduced by Büchi [2,3] and Gurevich, Harrington [5] in the context of infinite games, is useful. To transform Muller automata into other deterministic automata one uses *state appearance records* (SAR). The main component of an SAR is a permutation of states, representing the order of the last visits of the states in a run. This leads to a size of $(\mathcal{O}(n))!$ of the resulting automaton, where $n$ is the number of states of the original automaton.

For the transformation of Rabin or Streett automata into other deterministic models one uses *index appearance records* (IAR). The main component of an IAR is a permutation of the indices of the pairs (of state sets) in the acceptance condition, representing the order of the last visits of the first components of these pairs. This leads to a size of $(\mathcal{O}(r))!$ of the resulting automaton, where $r$ is the number of pairs of the original automaton.

Dziembowski, Jurdziński, and Walukiewicz [4] have studied the state appearance records as memory entries in automata which execute winning strategies in infinite games. They presented an example of an infinite game over a graph with $2n$ vertices where each winning strategy requires a memory of size $n!$. Starting from this example we introduce families of languages which yield optimal lower bounds for all automata transformations which involve appearance record constructions (either in the form of SAR or IAR).

Table 1 at the end of the paper lists the different transformations considered. In this paper we show that all these transformations involve a factorial blow up.

The lower bounds as exposed in this paper show that in $\omega$-automata theory the single exponential lower bound $2^{\Omega(n)}$ as known from the subset construction from the classical theory of automata on finite words has been extended to $2^{\Omega(n \cdot \log n)}$. So we see in which sense it is necessary to pass from sets of states (classical case) to sequences or even trees of states.

We leave open the question of an optimal lower bound for the transformation of nondeterministic Büchi automata into deterministic Muller automata.

The present results are from the author's diploma thesis [8]. Thanks to Wolfgang Thomas for his advice in this research.

## 2 Notations and Definitions

For an arbitrary set $X$ we denote the set of infinite sequences (or infinite words) over $X$ by $X^\omega$ and the set of finite words over $X$ by $X^*$. For a sequence $\sigma \in X^\omega$ and for an $i \in \mathbb{N}$ the element on the $i$th position in $\sigma$ is denoted by $\sigma(i)$, i.e., $\sigma = \sigma(0)\sigma(1)\sigma(2)\cdots$. The infix of $\sigma$ from position $i$ to position $j$ is denoted by $\sigma[i, j]$. We define $In(\sigma)$, the *infinity set* of $\sigma$, to be the set of elements from $X$ that appear infinitely often in $\sigma$. The length of a word $w \in X^*$ is denoted by $|w|$.

An *$\omega$-automaton* $\mathcal{A}$ is a tuple $(Q, \Sigma, q_0, \delta, Acc)$. The tuple $(Q, \Sigma, q_0, \delta)$ is called the *transition structure* of $\mathcal{A}$, where $Q \neq \emptyset$ is a finite set of states, $\Sigma$ is a finite alphabet, and $q_0$ is the initial state. The transition function $\delta$ is a function $\delta : Q \times \Sigma \to Q$ for deterministic automata and $\delta : Q \times \Sigma \to 2^Q$ for nondeterministic automata. The last component $Acc$ is the *acceptance condition*.

A *run* of $\mathcal{A}$ on a word $\alpha \in \Sigma^\omega$ is an infinite state sequence $\sigma \in Q^\omega$ such that $\sigma(0) = q_0$ and for all $i \in \mathbb{N}$ one has $\sigma(i+1) = \delta(\sigma(i), \alpha(i))$ for deterministic automata, and $\sigma(i+1) \in \delta(\sigma(i), \alpha(i))$ for nondeterministic automata.

A run is called *accepting* iff it satisfies the acceptance condition. We will specify this below for the different forms of acceptance conditions. The language $L(\mathcal{A})$ that is *accepted* or *recognized* by the automaton is defined as $L(\mathcal{A}) = \{\alpha \in \Sigma^\omega \mid \text{ there is an accepting run of } \mathcal{A} \text{ on } \alpha\}$.

In this paper we consider acceptance conditions of type Büchi, Muller, Rabin, Streett, and parity.

Let $\mathcal{A} = (Q, \Sigma, q_0, \delta, Acc)$ be an $\omega$-automaton. For the different acceptance types mentioned above the acceptance condition $Acc$ is given in different forms.

A Büchi condition [1] refers to a set $F \subset Q$, and a run $\sigma$ of a Büchi automaton is defined to be accepting iff $In(\sigma) \cap F \neq \emptyset$.

A Muller condition [12] refers to a system of sets $\mathcal{F} \subset 2^Q$; a run $\sigma$ of a Muller automaton is called accepting iff $In(\sigma) \in \mathcal{F}$.

A Rabin condition [13] refers to a list of pairs $\Omega = \{(E_1, F_1), \ldots, (E_r, F_r)\}$ with $E_i, F_i \subseteq Q$ for $i = 1, \ldots r$. A run $\sigma$ of a Rabin automaton is called accepting iff there exists an $i \in \{1, \ldots, r\}$ such that $In(\sigma) \cap F_i \neq \emptyset$ and $In(\sigma) \cap E_i = \emptyset$.

A Streett condition [17] also refers to such a list $\Omega$ but it is used in the dual way. A run $\sigma$ of a Streett automaton is called accepting iff for every $i \in \{1, \ldots, r\}$ one has $In(\sigma) \cap F_i = \emptyset$ or $In(\sigma) \cap E_i \neq \emptyset$.

A parity condition [11] refers to a mapping $c : Q \to \{0, \ldots, k\}$ with $k \in \mathbb{N}$. A run $\sigma$ of a parity automaton is called accepting iff $\min\{c(q) \mid q \in In(\sigma)\}$ is even. The numbers $0, \ldots, k$ are called *colors* in this context.

Obviously the Muller condition is the most general form of acceptance condition. This means every automaton $\mathcal{A}$ of the form above can be transformed into an equivalent Muller automaton just by collecting the sets of states that satisfy the acceptance condition of $\mathcal{A}$.

Let us note that Parity conditions can also be represented as Rabin and as Streett conditions:

**Proposition 1.** Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, c)$ with $c : Q \to \{0, \ldots, k\}$ be a parity automaton and let $r = \lfloor \frac{k}{2} \rfloor$. Let $\Omega = \{(E_0, F_0), \ldots, (E_r, F_r)\}$ with $E_i = \{q \in Q \mid c(q) < 2i\}$ and $F_i = \{q \in Q \mid c(q) = 2i\}$ for $i = 0, \ldots, r$. Furthermore let $\Omega' = \{(E'_0, F'_0), \ldots, (E'_r, F'_r)\}$ with $E'_i = \{q \in Q \mid c(q) < 2i + 1\}$ and $F'_i = \{q \in Q \mid c(q) = 2i + 1\}$ for $i = 0, \ldots, r$. Then the Rabin automaton $\mathcal{A}_1 = (Q, \Sigma, \delta, q_0, \Omega)$ and the Streett automaton $\mathcal{A}_2 = (Q, \Sigma, \delta, q_0, \Omega')$ are equivalent to $\mathcal{A}$.

Complementation of deterministic Muller, Rabin Streett, and parity automata

For a deterministic automaton $\mathcal{A}$ with a Muller, Rabin, Streett, or parity condition we give a deterministic automaton recognizing the complementary language. This automaton is called the dual of $\mathcal{A}$. For a Muller or a parity automaton the dual automaton is of the same type. For a Rabin automaton the dual automaton is a Streett automaton and vice versa.

Complementation of deterministic Muller, Rabin, Streett, and parity automata

**Proposition 2.** (i) Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, \mathcal{F})$ be a deterministic Muller automaton. The deterministic Muller automaton $\mathcal{A}' = (Q, \Sigma, \delta, q_0, 2^Q \setminus \mathcal{F})$ recognizes $\Sigma^\omega \setminus L(\mathcal{A})$.

(ii) Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, \Omega)$ be a deterministic Rabin (Streett) automaton. The deterministic Streett (Rabin) automaton $\mathcal{A}' = (Q, \Sigma, \delta, q_0, \Omega)$ recognizes $\Sigma^\omega \setminus L(\mathcal{A})$.

(iii) Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, c)$ be a deterministic parity automaton. The deterministic parity automaton $\mathcal{A}' = (Q, \Sigma, \delta, q_0, c')$ with $c'(q) = c(q) + 1$ for every $q \in Q$ recognizes $\Sigma^\omega \setminus L(\mathcal{A})$.

Because of the special structure of Rabin conditions on the one hand and Streett conditions of the onther hand, we can state the following about the union of infinity sets.

**Proposition 3.** (i) Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, \Omega)$ be a Streett automaton and let $R, S \subseteq Q$ be two infinity sets of possible runs satisfying the acceptance condition of $\mathcal{A}$. Then a run with infinity set $R \cup S$ also satisfies the acceptance condition of $\mathcal{A}$.

(ii) Let $\mathcal{A} = (Q, \Sigma, \delta, q_0, \Omega)$ be a Rabin automaton and let $R, S \subseteq Q$ be two infinity sets of possible runs not satisfying the acceptance condition of $\mathcal{A}$. Then a run with infinity set $R \cup S$ also does not satisfy the acceptance condition of $\mathcal{A}$.

## 3 Optimality of Safra's Construction

In this section we show the optimality of Safra's construction ([14]) which transforms a nondeterministic Büchi automaton with $n$ states into a deterministic Rabin automaton with $2^{\mathcal{O}(n \cdot \log n)}$ states. The main part of the proof consists of Lemma 5, which states that there exists a family $(L_n)_{n \geq 2}$ of languages, such that $L_n$ can be recognized by a nondeterministic Büchi automaton with $\mathcal{O}(n)$ states, but the complement of $L_n$ can not be recognized by a nondeterministic Streett automaton with less than $n!$ states.

This lemma is essentially due to Michel, who proved that there is a family $(L_n)_{n \geq 2}$ of languages, such that $L_n$ can be recognized by a nondeterministic Büchi automaton with $\mathcal{O}(n)$ states, but the complement of $L_n$ can not be recognized by a nondeterministic Büchi automaton with less than $n!$ states. Here we use the same family of languages as Michel but show the stronger result that there is no nondeterministic Streett automaton with less than $n!$ states recognizing the complement of $L_n$.

We define the languages $L_n$ via Büchi automata $\mathcal{A}_n$ over the alphabet $\Sigma_n = \{1, \ldots, n, \#\}$. Later we adapt the idea for languages over a constant alphabet. For a technical reason we use a set of initial states instead of one initial state, but recall that we can reduce the automata to the usual format by adding one extra state.

Define the automaton $\mathcal{A}_n = (Q_n, \Sigma_n, Q_0^n, \delta_n, F_n)$ as follows.

- $Q_n = \{q_0, q_1, \ldots, q_n\}$, $Q_0^n = \{q_1, \ldots, q_n\}$, and $F_n = \{q_0\}$.
- The transition function $\delta_n$ is defined by

$$\begin{array}{lll} \delta_n(q_0, a) & = \{q_a\} & \text{for } a \in \{1, \ldots, n\}, \\ \delta_n(q_0, \#) & = \emptyset, & \\ \delta_n(q_i, a) & = \{q_i\} & \text{for } a \in \Sigma_n, i \in \{1, \ldots, n\}, a \neq i, \\ \delta_n(q_i, i) & = \{q_i, q_0\} & \text{for } i \in \{1, \ldots, n\}. \end{array}$$

The automaton is shown in Figure 1. The idea can be adjusted to automata with the constant alphabet $\{a, b, \#\}$ by coding $i \in \{1, \ldots, n-1\}$ with $a^i b$, $n$ with $a^n a^* b$ and $\#$ with $\#$. The resulting automaton is shown on the right hand side of Figure 1 and still has $\mathcal{O}(n)$ states.

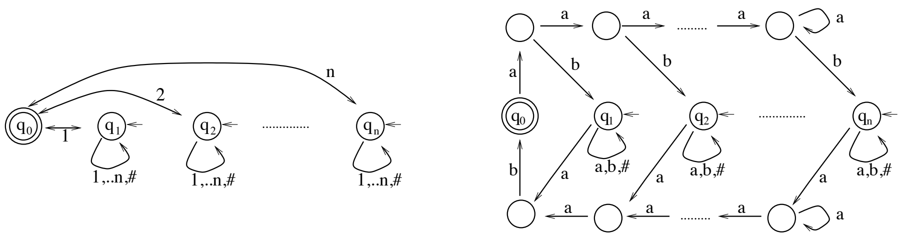


**Fig. 1.** The transition structure of the Büchi automaton $\mathcal{A}_n$. On the left hand over the alphabet $\{1, \ldots, n, \#\}$ and on the right hand over the alphabet $\{a, b, \#\}$. A nondeterministic Streett automaton for the complementary language needs at least $n!$ states.

As an abbreviation we define $L_n = L(\mathcal{A}_n)$. Before we prove the main lemma, we first give a characterization of the languages $L_n$, which is not difficult to prove.

**Lemma 4.** *Let $n \in \mathbb{N}$ and $\alpha \in \Sigma_n^\omega$. Then the following two statements are equivalent.*

(i) $\alpha \in L_n$.
(ii) *There exist $i_1, \ldots, i_k \in \{1, \ldots, n\}$ such that each pair $i_1 i_2, \ldots, i_{k-1} i_k, i_k i_1$ appears infinitely often in $\alpha$.*

(Our definition corrects an inaccuracy of [18], where the automata are defined in the same way but with $q_0$ as only initial state. For the languages defined in this way condition (ii) has to be sharpened.)

**Lemma 5.** *Let $n \geq 2$. The complement $\Sigma_n^\omega \setminus L_n$ of $L_n$ can not be recognized by a nondeterministic Streett automaton with less than $n!$ states.*

*Proof.* Let $n \in \mathbb{N}$ and let $\mathcal{A}' = (Q', \Sigma_n, q_0', \delta', \Omega)$ be a Streett automaton with $L' := L(\mathcal{A}') = \Sigma_n^\omega \setminus L_n$ and let $(i_1 \dots i_n)$, $(j_1 \dots j_n)$ be different permutations of $(1 \dots n)$. Define $\alpha = (i_1 \dots i_n \#)^\omega$ and $\beta = (j_1 \dots j_n \#)^\omega$. From Lemma 4 we obtain $\alpha, \beta \notin L'$. Thus we have two successful runs $r_\alpha, r_\beta \in Q'^\omega$ of $\mathcal{A}'$ on $\alpha, \beta$. Define $R = In(r_\alpha)$ and $S = In(r_\beta)$. If we can show $R \cap S = \emptyset$, then we are done because there are $n!$ permutations of $(1 \dots n)$.

Assume $R \cap S \neq \emptyset$. Under this assumption we can construct $\gamma \in \Sigma_n^\omega$ with $\gamma \in L_n \cap L'$. This is a contradiction, since $L'$ is the complement of $L_n$.

Let $q \in R \cap S$. The we can choose a prefix $u$ of $\alpha$ leading from $q_0$ to $q$, an infix $v$ of $\alpha$ containing the word $i_1 \dots i_n$, leading from $q$ to $q$ through all states of $R$, and an infix $w$ of $\beta$ containing the word $j_1 \dots j_n$, leading from $q$ to $q$ through all states of $S$.

Now let $\gamma = u(vw)^\omega$. A run $r_\gamma$ of $\mathcal{A}'$ on $\gamma$ first moves from $q_0$ to $q$ (while reading $u$), and then cycles alternatingly through $R$ (while reading $v$) and $S$ (while reading $w$). Therefore $r_\gamma$ has the infinity set $R \cup S$. Because $R$ and $S$ satisfy the Streett condition, $R \cup S$ also does (Proposition 3) and we have $\gamma \in L'$.

To show $\gamma \in L_n$ we first note that, if $k$ is the lowest index with $i_k \neq j_k$, then there exist $l, m > k$ with $j_k = i_l$ and $i_k = j_m$. By the choice of the words $v$ and $w$ one can see that $\gamma$ contains infinitely often the segments $i_1 \dots i_n$ and $j_1 \dots j_n$. Thus $\gamma$ also contains the segments $i_k i_{k+1}, \dots, i_{l-1} i_l, j_k j_{k+1}, \dots, j_{m-1} j_m$. Now, using Lemma 4, we can conclude $\gamma \in L_n$.

**Theorem 6.** *There exists a family $(L_n)_{n \geq 2}$ of languages such that for every $n$ the language $L_n$ can be recognized by a nondeterministic Büchi automaton with $\mathcal{O}(n)$ states but can not be recognized by a deterministic Rabin automaton with less than $n!$ states.*

*Proof.* Consider the family of languages from Lemma 5. Let $n \in \mathbb{N}$. Assume there exists a deterministic Rabin automaton with less than $n!$ states recognizing $L_n$. The dual of this automaton is a deterministic Streett automaton with less than $n!$ states recognizing $\Sigma_n^\omega \setminus L_n$. This contradicts Lemma 5.

Since parity conditions are special cases of the Rabin conditions, Theorem 6 also holds for parity automata instead of Rabin automata.

The theorem sharpens previous results of literature (see the survey [18]), where it is shown that there is no conversion of Büchi automata with $\mathcal{O}(n)$ states into deterministic Rabin automata with $2^{\mathcal{O}(n)}$ states and $\mathcal{O}(n)$ pairs. We point out that our proof is almost the same as in [18]. Only a few changes where needed to get this slightly stronger result.

The example demonstrates the optimality of Safra's construction for the transformation of nondeterministic Büchi automata into deterministic Rabin automata. For the transformation of nondeterministic Büchi automata into deterministic Muller automata this question is open. The known lower bound for this transformation is $2^{\Omega(n)}$ ([16]). In the following we will see that the example from above can not be used to show an optimal lower bound for Muller automata: We construct Muller automata $\mathcal{M}_n$, $n \in \mathbb{N}$, with $\mathcal{O}(n^2)$ states recognizing the language $L_n$.

For $n \in \mathbb{N}$ define the Muller automaton $\mathcal{M}_n = (Q'_n, \Sigma_n, q'_0, \delta'_n, \mathcal{F}_n)$ by $Q'_n = \Sigma_n \times \Sigma_n$, $q'_0 = (\#, \#)$, $\delta'_n((i,j), a) = (j, a)$, and $F \in \mathcal{F}_n$ iff there exist $i_1, \ldots, i_k \in \{1, \ldots, n\}$ such that $(i_1, i_2)$,. . .,$(i_{k-1}, i_k)$, $(i_k, i_1) \in F$.

The automaton just collects all pairs of letters occurring in the input word and then decides, using the Muller acceptance condition, if the property from Lemma 4 is satisfied. Thus we have $L(\mathcal{M}_n) = L_n$.

In the Muller automata $\mathcal{M}_n$ every superset of an accepting set is accepting too. Therefore the languages considered may even be recognized by deterministic Büchi automata ([7]). Thus, we can restrict the domain of the regular $\omega$-languages, and get a sharpened version of Theorem 6: The factorial blow up in the transformation of Büchi automata into deterministic Rabin automata already occurs over the class $G_\delta$ of those languages which are acceptable by a deterministic Büchi automaton.

One may also ask for a lower bound for the transformation of nondeterministic Büchi automata into deterministic Streett automata. The result we obtain belongs to this section and therefore we mention it here: There exists a family $(L_n)_{n \geq 2}$ of languages ($L_n$ over an alphabet of size $n$) such that for every $n$ the language $L_n$ can be recognized by a nondeterministic Büchi automaton with $\mathcal{O}(n)$ states but can not be recognized by a deterministic Streett automaton with less than $n!$ states.

The proof uses results from the next section and thus the claim will be restated there (Theorem 8).

## 4 Optimality of the Appearance Record Constructions

Appearance records [2,3,5], abbreviated AR, serve to transform Muller, Rabin, and Streett automata into parity automata. For these constructions two different forms of AR's are used, namely state appearance records (SAR) and index appearance records (IAR).

The SAR construction (see e.g. [18]) serves to transform deterministic Muller automata with $n$ states into an equivalent deterministic parity automata with $(\mathcal{O}(n))!$ states and $\mathcal{O}(n)$ colors. Since parity automata are special kinds of Rabin and Streett automata (Proposition 1), this construction also transforms Muller automata into Rabin or Streett automata.

The IAR construction (see e.g. [15]) transforms a deterministic Streett automaton with $n$ states and $r$ pairs into an equivalent parity automaton with $n \cdot (\mathcal{O}(r))!$ states and $\mathcal{O}(r)$ colors. Because of the duality of Rabin and Streett conditions and the self duality of parity conditions (Proposition 2), the IAR construction can be used for all nontrivial transformations between Rabin, Streett, and parity automata.

In this section we will show that all the AR constructions are of optimal complexity. The idea for the proof originates in [4], where the optimality of the SAR as memory for winning strategies in Muller games was shown. Just to avoid confusion we would like to point out that the family of automata we use in our proof is not just an adaption of the games from [4]. The winning condition of the games and the acceptance condition of the automata are not related. Our proof also does not generalize the one from [4], because the used family of automata can not be adapted to games requiring memory $n!$.

We first give a theorem showing the optimality of the IAR construction for the transformation of Streett into Rabin automata and then explain how to apply the theorem to get the optimality of all other AR transformations.

**Theorem 7.** *There exists a family $(L_n)_{n\geq 2}$ of languages such that for every $n$ the language $L_n$ can be recognized by a deterministic Streett automaton with $\mathcal{O}(n)$ states and $\mathcal{O}(n)$ pairs but can not be recognized by a deterministic Rabin automaton with less than $n!$ states.*

*Proof.* We define the languages $L_n$ via deterministic Streett automata $\mathcal{A}_n$ over the alphabet $\{1,\ldots,n\}$. Later we will explain how we can adapt the proof for an alphabet of constant size. The transition structure of $\mathcal{A}_n$ is shown schematically in Figure 2. Formally, for $n \geq 2$, we define the Streett automaton $\mathcal{A}_n = (Q_n, \Sigma_n, q_0^n, \delta_n, \Omega_n)$ as follows.

- $Q_n = \{-n,\ldots,-1,1,\ldots,n\}$ and $q_0^n = -1$.
- For $i,j \in \{1,\ldots,n\}$ let $\delta_n(i,j) = -j$ and $\delta_n(-i,j) = j$.
- $\Omega_n = \{(E_1,F_1),\ldots,(E_n,F_n)\}$ with $E_i = \{i\}$ and $F_i = \{-i\}$.

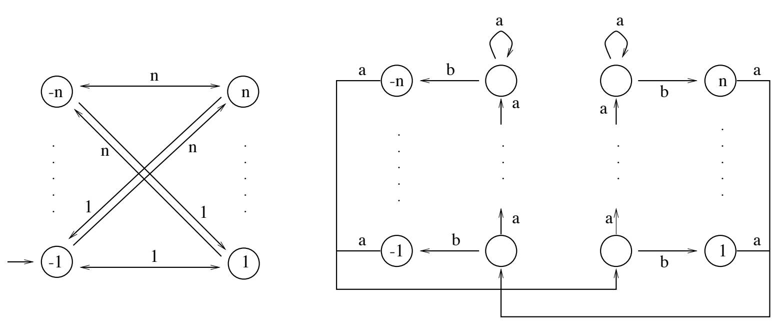


**Fig. 2.** The transition structure of the Streett automaton $\mathcal{A}_n$. On the left hand over the alphabet $\{1,\ldots,n\}$ and on the right hand over the alphabet $\{a,b\}$. An equivalent deterministic Rabin automaton needs at least $n!$ states.

To characterize the words in $L_n$ we use the following notation. For a word $\alpha \in \Sigma_n^\omega$ let $even(\alpha)$ be the set containing the letters that infinitely often occur on an even position in $\alpha$ and let $odd(\alpha)$ be the set containing the letters that infinitely often occur on an odd position in $\alpha$. This means $even(\alpha) = In(\alpha(0)\alpha(2)\alpha(4)\cdots)$ and $odd(\alpha) = In(\alpha(1)\alpha(3)\alpha(5)\cdots)$. From the definition of $\mathcal{A}_n$ follows that a word $\alpha \in \Sigma_n^\omega$ is in $L_n$ if and only if $odd(\alpha) \subseteq even(\alpha)$. As a consequence of this, for $\alpha \in \Sigma_n^\omega$ and $u \in \Sigma_n^*$ with $|u|$ even, the word $u\alpha$ is in $L_n$ if and only if $\alpha$ is in $L_n$. Therefore, in a deterministic automaton recognizing $L_n$, every state that can be reached by reading a prefix of even length can be used as initial state without changing the accepted language.

We will prove by induction that every deterministic Rabin automaton recognizing $L_n$ needs at least $n!$ states.

We show the base case of the induction for $n = 2$. An automaton recognizing a nonempty proper subset of $\Sigma_2^\omega$ needs at least 2 states. Therefore the base case of the induction holds.

Now let $n > 2$ and let $\mathcal{B} = (Q, \Sigma_n, q_0, \delta, \Omega)$ be a deterministic Rabin automaton with $L(\mathcal{B}) = L_n$. Let $Q_{\text{even}}$ be the states that can be reached from $q_0$ by reading a prefix of even length.

For every $i \in \{1, \ldots, n\}$ and every $q \in Q_{\text{even}}$ we construct a deterministic Rabin automaton $\mathcal{B}_i^q$ over $\Sigma_n \setminus \{i\}$ by removing all $i$-transitions from $\mathcal{B}$. Furthermore $q$ is the initial state of $\mathcal{B}_i^q$. Since $q$ can be reached in $\mathcal{B}$ after having read a prefix of even length, the language recognized by $\mathcal{B}_i^q$ is $L_{n-1}$ (if $i \neq n$ then the names of the letters are different but the language essentially equals $L_{n-1}$). Thus, by the induction hypothesis, $\mathcal{B}_i^q$ has at least $(n-1)!$ states.

We can strengthen this statement as follows. In every $\mathcal{B}_i^q$ ($i \in \{1, \ldots, n\}$ and $q \in Q_{\text{even}}$) is a strongly connected component with at least $(n-1)!$ states. Just take a strongly connected component $S$ in $\mathcal{B}_i^q$ such that there is no other strongly connected component reachable from $S$ in $\mathcal{B}_i^q$. Let $p \in S$ be a state that is reachable from $q$ in $\mathcal{B}_i^q$ by reading a prefix of even length. As we have seen above, we can use $p$ as initial state in $\mathcal{B}_i^q$ without changing the accepted language. Therefore, by the induction hypothesis, $S$ must contain at least $(n-1)!$ states.

Now, for $i \in \{1, \ldots, n\}$, we construct words $\alpha_i \in \Sigma_n^\omega$ with runs $\sigma_i$ of $\mathcal{B}$ such that $|In(\sigma_i)| \geq (n-1)!$ and $In(\sigma_i) \cap In(\sigma_j) = \emptyset$ for $i \neq j$. Then we are done because $|Q'| \geq \sum_{i=1}^n |In(\sigma_i)| \geq n \cdot (n-1)! = n!$.

For $i \in \{1, \ldots, n\}$ construct the word $\alpha_i$ as follows. First take a $u_0 \in (\Sigma_n \setminus \{i\})^*$ such that $u_0$ has even length and contains every letter from $\Sigma_n \setminus \{i\}$ on an even and on an odd position. Furthermore $\mathcal{B}_i^{q_0}$ should visit at least $(n-1)!$ states while reading $u_0$. This is possible since $\mathcal{B}_i^{q_0}$ contains a strongly connected component with $\geq (n-1)!$ states. Let $q_1$ be the state reached by $\mathcal{B}_i^{q_0}$ after having read the word $u_0 ik$, where $k \in \{1, \ldots, n\} \setminus \{i\}$. Then we choose a word $u_1 \in (\Sigma_n \setminus \{i\})^*$ with the same properties as $u_0$, using $\mathcal{B}_i^{q_1}$ instead of $\mathcal{B}_i^{q_0}$. This means $u_1$ has even length, contains every letter from $\Sigma_n \setminus \{i\}$ on an even and on an odd position, and $\mathcal{B}_i^{q_1}$ visits at least $(n-1)!$ states while reading $u_1$.

Repeating this procedure we get a word $\alpha_i = u_0 ik u_1 ik u_2 ik \cdots$ with $even(\alpha_i) = \{1, \ldots, n\} \setminus \{i\}$, $odd(\alpha_i) = \{1, \ldots, n\}$, and therefore $\alpha_i \notin L_n$. For the run $\sigma_i$ of $\mathcal{A}'$ on $\alpha_i$ we have $|In(\sigma_i)| \geq (n-1)!$. Hence it remains to show $In(\sigma_i) \cap In(\sigma_j) = \emptyset$ for $i \neq j$.

Assume by contradiction that there exist $i \neq j$ with $In(\sigma_i) \cap In(\sigma_j) \neq \emptyset$. Then we can construct a word $\alpha$ with run $\sigma$ such that $even(\alpha) = even(\alpha_i) \cup even(\alpha_j) = \{1, \ldots, n\}$, $odd(\alpha) = odd(\alpha_i) \cup odd(\alpha_j) = \{1, \ldots, n\}$ and $In(\sigma) = In(\sigma_i) \cup In(\sigma_j)$, by cycling alternatingly through the infinity sets of $\sigma_i$ and $\sigma_j$ (as in the proof of Lemma 5). This is a contradiction since in Rabin automata the union of rejecting cycles is rejecting (Proposition 3), but $\alpha$ is in $L_n$.

To adapt the proof for an alphabet of constant size we can code every letter $i \in \{1, \ldots, n-1\}$ with $a^i b$ and $n$ with $a^n a^* b$. The resulting automaton looks like shown on the right hand side of Figure 2 and still has $\mathcal{O}(n)$ states.

Theorem 7 shows the optimality of the IAR construction for the transformation of deterministic Streett automata into deterministic Rabin automata. The duality of these two types of automata (Prop. 2) gives us an analogue theorem, with the roles of Rabin automata and Streett automata exchanged. Parity automata are special cases of Rabin automata and of Streett automata. Therefore we also get analogue theorems for the transformation of Rabin automata into parity automata and for the transformation of Streett automata into parity automata. Furthermore the property of the example automata to have $\mathcal{O}(n)$ states and $\mathcal{O}(n)$ pairs also gives us analogue theorems, when starting with Muller automata instead of Rabin or Streett automata. Thus, Theorem 7 shows the optimality of all AR constructions listed in Table 1.

A different construction for the conversion between Rabin and Streett automata is given in [6]. It converts a deterministic Streett automaton with $n$ states and $r$ pairs into a deterministic Rabin automaton with $\mathcal{O}(n \cdot r^k)$ states and $l$ pairs, where $k$ is the Streett index of the language and $l$ is the Rabin index of the language. The Rabin (Streett) index of a language is the number of pairs needed in the acceptance condition to describe the language with a deterministic Rabin (Streett) automaton. The languages from the family $(L_n)_{n \geq 2}$ have Rabin and Streett index $\mathcal{O}(n)$ and therefore the complexity of the construction is of order $n^{\mathcal{O}(n)}$ for our example automata. Hence, as a result of this section, the transformation from [6] is also optimal.

At the end of Section 3 we stated a lower bound for the transformation of nondeterministic Büchi automata into deterministic Streett automata. For that aim we show that the languages $(\Sigma_n^\omega \setminus L_n)_{n \geq 2}$ of the present section can be recognized by Büchi automata with $\mathcal{O}(n)$ states. Then we are done because every deterministic Streett automaton recognizing $\Sigma_n^\omega \setminus L_n$ needs at least $n!$ states (Theorem 7 and Prop. 2).

**Theorem 8.** *There exists a family $(L_n)_{n \geq 2}$ of languages ($L_n$ over an alphabet of $n$ letters) such that for every $n$ the language $L_n$ can be recognized by a nondeterministic Büchi automaton with $\mathcal{O}(n)$ states but can not be recognized by a deterministic Streett automaton with less than $n!$ states.*

*Proof.* As mentioned above it suffices to show that there is a family $(\mathcal{B}_n)_{n \geq 2}$ of Büchi automata such that $\mathcal{B}_n$ has $\mathcal{O}(n)$ states and recognizes $\Sigma_n^\omega \setminus L_n$. From the characterization of $L_n$ in the proof of Theorem 7 we know that $\alpha \in L_n$ iff $odd(\alpha) \subseteq even(\alpha)$ and therefore $\alpha \notin L_n$ iff there exists an $i \in \{1, \ldots, n\}$ with

$i \in odd(\alpha)$ and $i \notin even(\alpha)$. Intuitively the Büchi automaton guesses the $i$ and then verifies that it appears infinitely often on an odd position and from some point on never on an even position. Formally $\mathcal{B}_n = (Q_n, \Sigma_n, q_0^n, \delta_n, F_n)$ is defined as follows.

- $Q_n = \{q_o, q_e, q_o^1, q_e^1, q_f^1, \dots q_o^n, q_e^n, q_f^n\}$, $q_0^n = q_o$, and $F_n = \{q_f^1, \dots, q_f^n\}$.
- For $i \in \Sigma_n$ and $j \in \{1, \dots, n\}$ let

$$\delta_n(q_o, i) = \{q_e\}, \qquad \delta_n(q_e, i) = \{q_o, q_o^1, \dots, q_o^n\},$$
$$\delta_n(q_o^j, i) = \begin{cases} \{q_e^j\} \text{ if } i \neq j \\ \emptyset \text{ if } i = j, \end{cases} \qquad \delta_n(q_e^j, i) = \begin{cases} \{q_o^j\} \text{ if } i \neq j \\ \{q_f^j\} \text{ if } i = j, \end{cases}$$
$$\delta_n(q_f^j, i) = \begin{cases} \{q_e^j\} \text{ if } i \neq j \\ \emptyset \text{ if } i = j. \end{cases}$$

The automaton is built in such a way that it is in one of the states from $\{q_e, q_e^1, \dots, q_e^n\}$ iff the last letter was on an even position.

Let $\alpha \in \Sigma_n^\omega \setminus L_n$ and let $j \in odd(\alpha) \setminus even(\alpha)$. A successful run of $\mathcal{B}_n$ stays in the states $q_o$ and $q_e$ up to the point where $j$ does not appear on an even position anymore. Then it moves to $q_o^j$. Always when a $j$ appears on an odd position in $\alpha$, the automaton is in $q_e^j$ and then moves to $q_f^j$. Since there does not appear a $j$ on an even position anymore, the automaton can continue to infinity and accepts $\alpha$ because it visits $q_f^j$ infinitely often. Therefore we have $\Sigma_n^\omega \setminus L_n \subseteq L(\mathcal{B}_n)$.

Now let $\alpha \in L(\mathcal{B}_n)$. There exists a $j$ such that in an accepting run $\mathcal{B}_n$ from some point on only visits states from $\{q_o^j, q_e^j, q_f^j\}$ and infinitely often visits $q_f^j$. If the last read letter was on an odd position, then $\mathcal{B}_n$ is in $q_o^j$ or in $q_f^j$ and therefore $j$ may only appear on an even position before $\mathcal{B}_n$ moves to the states $\{q_o^j, q_e^j, q_f^j\}$. But since $\mathcal{B}_n$ infinitely often visits $q_f^j$, there must be a $j$ on an odd position infinitely often and therefore we have $L(\mathcal{B}_n) \subseteq \Sigma_n^\omega \setminus L_n$.

**Table 1.** Synopsis of automaton transformations and pointers to optimality results. (The transformation $\star$ is the only one that is not known to be optimal.)

| From \ To | Muller det. | Rabin det. | Streett det. | Parity det. |
|---|---|---|---|---|
| Büchi ndet. | Safra trees $\star$ | Safra trees Thm 6 | 1.Safra trees 2.IAR Thm 8 | 1.Safra trees 2.IAR Thm 6 |
| Muller det. | — | SAR Thm 7 | SAR Thm 7 | SAR Thm 7 |
| Rabin det. | trivial | — | IAR Thm 7 | IAR Thm 7 |
| Streett det. | trivial | IAR Thm 7 | — | IAR Thm 7 |

## 5 Conclusion

For several different transformations of $\omega$-automata we have seen that the lower bound is $2^{\Omega(n\cdot\log n)}$. The two basic constructions considered in this paper (Safra trees and appearance records) meet these lower bounds and therefore are of optimal complexity. In comparison to the theory of $*$-automata, where determinization is exponential too, but with a linear exponent, we get an additional factor of $\log n$ in the exponent for transformations of $\omega$-automata.

An unsolved problem is the lower bound for the transformation of nondeterministic Büchi automata into deterministic Muller automata. The known lower bound is $2^{\Omega(n)}$, which can be proven by a simple pumping argument as for $*$-automata.

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