Empirical Performance Investigation of a Büchi Complementation Construction

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Abstract

This will be the abstract.

Acknowledgements

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Background

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1.1 Büchi Automata and Other ω-Automata

Formal definitions for example in [39][40][53]

1.1.1 Büchi Automata

Büchi automata are a type of the so-called ω-automata ("omega"-automata). ω-automata are finite state automata that process infinite words. Thus, an ω-automaton never "stops" reading a word, because the word it is reading has an infinite number of symbols. But still, ω-automata can accept or reject the words they read by the means of special *acceptance conditions*. In fact, the only difference between classical finite state automata on finite words and ω-automata is the acceptance condition.

For the case of Büchi automata, this is the Büchi acceptance condition that we describe below.

Descriptions: [39][53]

Büchi Acceptance Condition

A Büchi automaton A is defined by the 5-tuple $A = (Q, \Sigma, q_0, \delta, F)$ with the following components:

- Q: a finite set of states
- Σ: a finite alphabet
- q_0: an initial state, $q_0 \in Q$
- δ: a transition function, $\delta : Q \times \Sigma \to 2^Q$
- F: a set of accepting states, $F \in 2^Q$

We denote by Σ^ω the set of all words of infinite length over the alphabet Σ. A Büchi automaton runs on the elements of Σ^ω. In the following, we define the acceptance behaviour of a Büchi automaton A on a word $\alpha \in \Sigma^\omega$.

- A *run* of Büchi automaton A on a word $\alpha \in \Sigma^\omega$ is a sequence of states $q_0 q_1 q_2 \ldots$ such that q_0 is A's initial state and $\forall i \geq 0 : q_{i+1} \in \delta(q_i, \alpha_i)$
- $\inf(\rho) \in 2^Q$ is the set of states that occur infinitely often in a run ρ
- A run ρ is accepting if an only if $\inf(\rho) \cap F \neq \varnothing$
- A Büchi automaton A accepts a word $\alpha \in \Sigma^\omega$ if and only if there is an accepting run of A on α

Expressivity

Büchi automata are expressively equivalent to the ω-regular languages. This means that every language recognised by a Büchi automaton is an ω-regular language, and that for every ω-regular language there exists a Büchi automaton that recognises it. This property has been proved by Büchi himself in his initial publication in 1962 [4].

However, this equivalence with the ω-regular languages does only hold for *non-deterministic* Büchi automata. Deterministic Büchi automata are less expressive than non-deterministic Büchi automata. In particular, the class of languages represented by deterministic Büchi automata is a strict subset of the class of languages represented by non-deterministic Büchi automata. This property has also been proved by Büchi [4].

This means that there exist languages that can be recognised by a non-deterministic Büchi automaton, but not by a deterministic one. A typical example is the language $(0+1)^*1^\omega$. This is the language of all words consisting of 0 and 1 with a finite number of 0 and an infinite number of 1. It is proved in various publications that this language can be recognised by a non-deterministic Büchi automaton, but not by a deterministic Büchi automaton [49][32].

The most important consequence of this fact is that Büchi automata can, in general, not be determinised. This means that it is not possible to turn *every* non-deterministic Büchi automaton into a deterministic

one. This contrasts with the case of the classical finite state automata on finite words, where *every* non-deterministic automata (NFA) can be turned into a deterministic automaton (DFA), by the means of, for example, the subset constrution introduced by Rabin and Scott in 1959 [29].

It has been stated that the fact that Büchi automata can in general not be determinised is the main reason that Büchi complementation is such a hard problem [25]. We will see why this is the case below.

Complementation

Büchi automata are closed under complementation. This means that the complement of every Büchi automaton (non-deterministic or deterministic) is in turn a Büchi automaton. This result has been proved by Büchi in his introducing paper from 1962 [4].

The difficulty of the concrete complementation problem does however strongly depend on whether the Büchi automaton is deterministic or non-deterministic. For deterministic Büchi automata, the complementation is "easy" and regarded as a "solved problem". There is a well-known construction introduced by Kurshan in 1987 that complements a deterministic Büchi automaton in polynomial time [17]. The resulting complement is a non-deterministic Büchi automaton and has a size that is at most the double of the input automaton.

For non-deterministic Büchi automata, on the other hand, the complementation problem is much more difficult. The main reason is, as mentioned, the fact that non-deterministic Büchi automata cannot be determinised. If they could be determinised, then a non-deterministic Büchi automata could be complemented by first determinising it, and then complementing the deterministic automaton with the Kurshan construction. If the determinisation construction would also be efficient (that is, having polynomial complexity), then we would have an efficient complementation procedure for non-deterministic Büchi automata. In this case, "Büchi complementation" would probably be no active research topic but rather a solved problem.

However, non-deterministic Büchi automata cannot be determinised, and hence this straightforward complementation approch is not possible. Consequently, different ways for complementing non-deterministic Büchi automata have to be found, and these ways turn out to be very complex. It is this complexity that makes Büchi complementation an active research topic as, regarding the concrete usages of Büchi complementation in, for example, model checking, it is of great importance to find more and more efficient ways to complement non-deterministic Büchi automata.

1.1.2 Other ω-Automata

After the introduction of Büchi automata in 1962, several other types of ω-automata have been proposed. The most notable ones are Muller automata (Muller, 1963 [22]), Rabin automata (Rabin, 1969 [30]), Streett automata (Streett, 1982 [38]), and parity automata (Mostowski, 1985 [21]).

Description in [53]

These automata differ from Büchi automata only in their acceptance condition, that is, the condition that a run ρ is accepted. Table 1.1 gives a formal definition of the acceptance conditions of these types of ω-automata.

Type	Definitions	Acceptance condition
Muller	$F \subseteq 2^Q$	$\inf(\rho) \in F$
Rabin	$\{(E_1, F_1), \ldots, (E_r, F_r)\}$, $E_i, F_i \subseteq Q$	$\exists i : \inf(\rho) \cap E_i = \varnothing \wedge \inf(\rho) \cap F_i \neq \varnothing$
Streett	$\{(E_1, F_1), \ldots, (E_r, F_r)\}$, $E_i, F_i \subseteq Q$	$\forall i : \inf(\rho) \cap E_i \neq \varnothing \vee \inf(\rho) \cap F_i = \varnothing$
Parity	$c : Q \to \{1, \ldots, k\}$, $k \in \mathbb{N}$	$\min\{c(q) \mid q \in \inf(\rho)\} \bmod 2 = 0$
Büchi	$F \subseteq Q$	$\inf(\rho) \cap F \neq \varnothing$

Table 1.1: Acceptance conditions of Muller, Rabin, Streett, parity, and Büchi automata.

For the Muller acceptance condition, the set of infinitely occuring states of a run ($\inf(\rho)$) must match one of several predefined set of states. The Muller acceptance condition is the most general one, and all the other acceptance conditions in Table 1.1 can be expressed by the Muller condition [18].

The Rabin and Streett acceptance conditions are the negations of each other. This means that a run satisfies the Rabin acceptance condition, if and only if it does not satisfy the Streett acceptance condition. They both use a list of pairs of state sets. A run is accepted if there is a pair for which the first element contains an infinitely occuring state and the second element does not (Rabin condition), or if for all pairs the first elements do not contain an infinitely occuring state or all the second elements do contain an infinitely occuring state (Streett condition).

The parity condition assigns a number (color) to each state. A run is accpted if and only if the infinitely often occuring state with the smallest number has an even number.

At this point we will start using a notation for the different types of ω-automata that has been used in [27] and [42]. It consists of a three-letter acronymes of the form $\{D, W\} \times \{B, M, R, S, P\} \times W$. The first letter, D or N specifies whether the automaton is deterministic or non-deterministc. The second letter is the initial letter of the automaton type, that is, B for Büchi, M for Muller, R for Rabin, S for Streett, and P for parity automata. The third letter specifies on which the automaton runs, and is in our case always W meaning "words". Thus, throughout this thesis we will use DBW for deterministic Büchi automata, NBW for non-deterministic Büchi automata, DMW for deterministic Muller automata, and so on.

Regarding the expressivity of Muller, Rabin, Streett, and parity automata, it turned out that, like non-deterministic Büchi automata, they are equivalent to the ω-regular languages [39]. However, unlike Büchi automata, for Muller, Rabin, Streett, and parity automata this equivalence holds for deterministic *and* non-deterministic automata. That is, unlike Büchi automata, these automata *can* be determinised. In summary, there is thus an equivalence between NBW, DMW, NMW, DRW, NRW, DSW, NSW, DPW, NPW, and the ω-regular languages. Only the DBW, as a special case, has a different expressivity, which is a strict subset of the expressivities of the other automata types.

1.2 Run Analysis in Non-Deterministic Automata

In a deterministic automaton, every input word has exactly one run. In a non-deterministic automaton, however, an input word may have a large number of different runs. It is this fact that generally makes the complementation of non-deterministic automata much harder than the complementation of deterministic automata. Because for the complementation of an automaton A to its complement automaton B, there is the following principle:

$$\textbf{\textit{All} the runs of } A \textbf{ on word } \alpha \textbf{ are non-accepting} \iff B \textbf{ accepts the word } \alpha$$

For deterministic automata, there is only a single run of A on α. Thus, for concluding that the complement B must accept α, it is enough to verify that the corresponding run of A on α is non-accepting. However, if A is a non-deterministic automaton, then there are potentially infinitely many (for ω-automata) runs of A on α. To conclude that the complement B must accept α requires thus to verify that *all* these runs of A are non-accepting.

Complementation of non-deterministic automata thus requires

There are two main structures that are used to investigate all the runs of a non-deterministic ω-automaton on a specific word, directed acyclic graphs (DAGs) and trees [53].

1.2.1 Run DAGs

Run DAGs arrange all the runs of a non-deterministic automaton on a specific word in a directed acyclic graph. This graph is structured into levels, and on each level there is a vertex for every state of the automaton. The width of a run DAG is the number of states on a level, and is thus n if the automaton has n states. The number of levels is infinite for an ω-word.

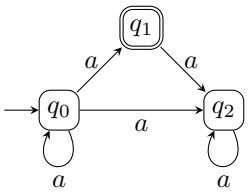


Figure 1.1: Non-deterministic Büchi automaton A.

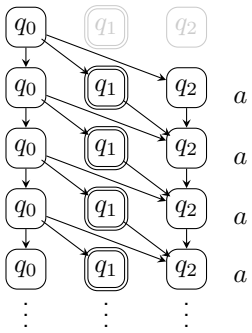


Figure 1.2: First few levels of the run DAG for the runs of automaton A (Figure 1.1 on the word a^ω.

Figure 1.1 shows an example non-deterministic Büchi automaton A that we will use to demonstrate the different run analysis techniques in this and the following sections. Figure 1.16 shows the first few levels of the run DAG of the automaton A on the word a^ω.

Descriptions: [5] (runs are *paths*) [53]

1.2.2 Run Trees

Trees are the second structure that is used for analysing all the runs of a non-deterministic automaton on a specific word. Run trees are the simplest form of the different tree variants. A run tree is basically a direct unwinding of all the runs of an automaton on a word in tree form. Each node in the tree represents a state of the automaton, and each non-deterministic transition in the automaton is representd in the tree as a child of a node.

Figure 1.12 shows the first few levels of the run tree of the example automaton A (see Figure 1.1) on the word a^ω. A note on notation: during this thesis we will adopt the convention to use rectangles with sharp corners for nodes of a tree. Furthermore, we will use double-lined rectangles for nodes that correspond to a accepting states of the automaton.

Naturally, the number of levels of a run tree is infinite for an ω-word. More importantly, however, the width of a level (number of nodes on a level) is not bounded and may become infinite as well.

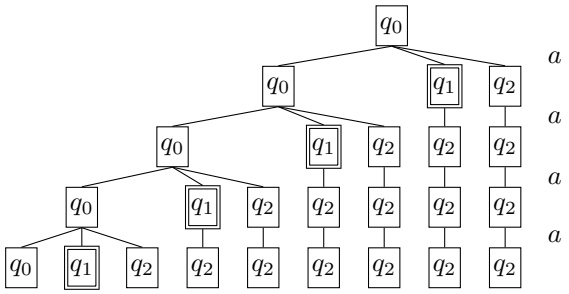


Figure 1.3: Automaton A and the first five levels of the run tree of the runs of A on the word a^ω.

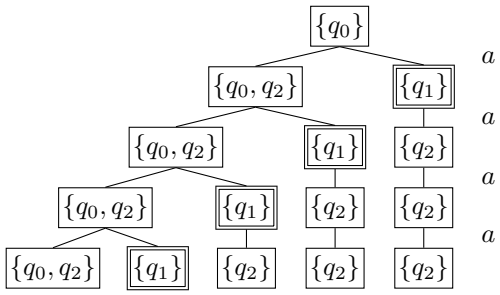


Figure 1.4: Automaton A and the first five levels of the split tree of the runs of A on the word a^ω.

1.2.3 Split Trees

A split tree is an enhancement of a run tree where all the non-accepting and accepting children of a node are aggregated so that each node has at most two children, a non-accepting one and an accepting one. This makes a split tree a binary tree. Furthermore, a node of a split tree does not correspond to a single state of the automaton, but to a set of states. In Figure 1.13, we show the first few levels of the split tree of the example automaton A (see Figure 1.1) on the word a^ω.

Descriptions: [52]

Compared to run trees, split trees give up information on individual runs. By looking at the split tree in Figure 1.13, we can see that, for example, there must be a run from q_0 in the root to q_0 at level 4, but we cannot see which states this run visits on its way. What we can however see is that this run for sure does not visit any accepting states, but only non-accepting states. It turns out that with regard to Büchi complementation, this is the actually essential information that we need to know about runs.

Split trees in fact embody a modified subset construction where accepting and non-accepting successor input-states are not mixed and added as two separate states to the output automaton. Applying this construction to an NBW results in an equivalent NBW whose degree of non-determinism is however reduced to two. Such a construction has been described in [48].

In our split tree in Figure 1.13 we always put the accepting child to the right of the non-accepting child. This convention is necessary for a property of split trees that leads to *reduced* split trees (see next section), namely the presence or "greedy" rightmost runs. Note that the reverse convention, putting the accepting child to the left of the non-accepting child, is also possible. We refer to the former case (accepting right, non-accepting left) as the *right-to-left* version, and to the latter (accepting left, non-accepting right) as the *left-to-right* version.

1.2.4 Reduced Split Trees

Reduced split trees are split trees where only one occurrence of a state on each level is kept and the others are removed. The state that is kept is either the rightmost one, if right-to-left split trees ar used, or the leftmost one, if left-to-right split trees are used. While both versions are used in the literature, in this this thesis, we will use the right-to-left version. Figure 1.10 shows the first few levels of the (right-to-left) reduced split tree of the automaton A (Figure 1.1) on the word a^ω.

Comparing the reduced split tree with the corresponding split tree (Figure 1.13) reveals which states have been removed in the reduced split tree. The root and level 1 are similar in both trees. On level 2, the state q_2 is removed from the leftmost node, because q_2 already appears farther to the right on this level. On level 3, the state q_2 in the second node from the right is removed because there is already a q_2 to the right of it. As q_2 was the only state of this node, this causes the entire node to disappear. On the same level, q_2 of the leftmost node is also removed for the same reason. On level 4, following the same pattern, three occurrences of q_2 are removed.

A possible procedure for constructing a new level of a reduced split tree is to start creating children at the rightmost node and then proceed from right to left. In the level under construction, a specific state

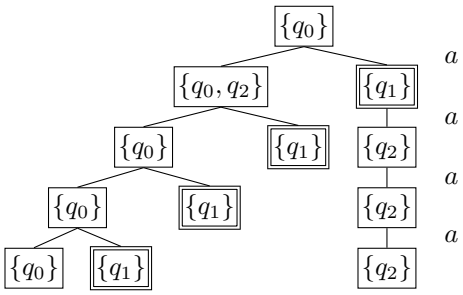


Figure 1.5: Automaton A and the first five levels of the reduced split tree of the runs of A on the word a^ω.

is only added if it does *not already* occur *to the right* of it. This processing from right to left gives the name to right-to-left reduced split trees.

This omission of states (and thus entire runs) is "tailored" to the Büchi complementation problem. It in fact removes all but one of the runs that, after a certain number of steps, end up in the same state. We will call these runs *re-joining* runs. For example, in the split tree in Figure 1.13, there are four runs that, after reading four symbols, "re-join" in q_2. In the reduced split tree (Figure 1.10), three of these re-joining runs are removed and only one is kept.

The run that is kept is always the one that is farthest at the right in the tree, and we call it the *rightmost* run. An alternative name for the rightmost run is *greedy* run. This comes from the fact that this run visits an accepting state earlier than the other re-joining runs. For example, in our previous example with the runs re-joining in q_2, the run that is kept in the reduced split tree visits an accepting state (q_1) already after the first symbol, whereas the other runs visit an accepting state only after the second or third symbol, respectively, or not at all.

Note that visiting an accepting state corresponds to a *right-turn* in a right-to-left split tree. This is because the accepting states are always in the right child of the current node. Thus, the greedy run has actually the earliest right-turn, which in turn is the reason that it is farthest at the right, and thus the rightmost run of all the re-joining runs. In the following, we will use the terms greedy and rightmost interchangeably[1]

Summarising, a reduced split tree is thus a split tree where we keep only the rightmost one of a couple of rejoining runs. The base of this rule is the following observation:

Any re-joining run is accepting $\Longleftarrow$ the rightmost re-joining run is accepting

That is, if any of the re-joining runs is accepting, then the rightmost run is accepting too, and on the other hand, if the rightmost run is not accepting, then none of the re-joining runs is accepting. A proof of this claim can be found in [52][2].

Coming back to our problem of Büchi complementation, the main information we need from the run analysis of the input automaton is whether *all* the runs on a specific word are non-accepting or not. By considering only greedy runs, we actually test a couple of runs for this property at once. If the greedy run is non-accepting, then we know instantly that all its related re-joining runs are also non-accepting. If on the other hand the greedy run is accepting, then there is at least one accepting run anyway, and the complement automaton must reject the word. Thus, reduced split trees are a clever way to reduce the problem of analysing the large number of runs of a non-deterministic Büchi automaton on a specific word, by keeping only the information that is essential for the purpose of Büchi complementation.

The most important property of reduced split trees is that their width (the number of states on a level) is bounded by the number of states of the corresponding automaton, and thus finite. This does not hold for split trees. For example, the width of the split tree in Figure 1.13 becomes infinite with an infinite number

[1] Note that for left-to-right split trees, "rightmost" must be substituted by "leftmost".

[2] In this work, the authors use left-to-right split trees, in which case the greedy run is the leftmost rather than the rightmost run.

of levels. The width of the reduced split tree in Figure 1.10, however, never exceeds three, because three is the number of states of the input automaton.

The bounded width of reduced split trees is actually what makes them usable for Büchi complementation constructions, in particular the slice-based construction, including the Fribourg construction. Because in these constructions, levels of reduced split trees define states of the output automaton, and this is only possible if there is a finite number of distinct levels, so that there is a finite number of possible states in the output automaton.

Formal descriptions of reduced split trees can be found in [52] and [53]. Note that these works use left-to-right split trees, which exchanges the roles of "right" and "left" with respect to our description.

1.3 Büchi Complementation Constructions

Since the introduction of Büchi automata in 1962, many constructions for complementing non-deterministic Büchi automata have been proposed.

1.3.1 Ramsey-Based Approach

The Ramsey-based approach has its name from a Ramsey-based combinatorial argument that is used in the complementation constructions. Ramsey was a British mathematician who lived at the beginning of the 20th century and founded a branch of combinatorics called the Ramsey theory [9].

Common to the Ramsey-based complementation constructions is that they stay completely within in the framework of Büchi automata. That is, they do not include intermediate automata of different types, as for example the determinization-based constructions. Rather, Ramsey-based constructions construct the complement automata directly by combinatorial operations on the states and transitions.

Büchi, 1962

The first Büchi complementation construction at all was described by Büchi himself, along with the introduction of Büchi automata in 1962 [4]. This complemenation construction is a Ramsey-based construction. It involves a combinatorial argument based on work by Ramsey [31]. The construction is complicated, and has a doubly-exponential worst-case state complexity of $2^{2^{O(n)}}$ [51]. This means that if we assume, for example, the concrete complexity to be 2^{2^n}, then an automaton with 6 states may result in a complement with at most 2^{2^6} states, which is more than 18 quintillions (18 billion billions).

The complexity of this worst-case is very high, and it would probably be impossible to complement such a worst-case automaton in practice. This is why all the subsequent complementation constructions, until today, have the goal to reduce this worst-case complexity. In this way, the worst-case state complexity became the main measure of performance for Büchi complementation constructions.

Sistla, Vardi, and Wolper, 1985

Another Ramsey-based construction has been introduced by Sistla, Vardi, and Wolper in 1987 [37] (first published in 1985 [36]). It is an improvement of Büchi's construction and the first one that involves only an exponential, instead of a doubly-exponential, worst-case state complexity. The complexity of this construction has been calculated to be $O\left(2^{4n^2}\right)$ (see [33][26]).

The Ramsey-based approach is the oldest of the four approaches and it was particularly

1.3.2 Determinization-Based Approach

The determinization-based complementation constructions proceed by converting an NBW to a deterministic automaton, complementing the deterministic automaton, and finally converting the complement automaton back to an NBW. The deterministic automaton cannot be a DBW (because NBW and DBW are not equivalent), however it can be a DMW, DRW, DSW, or DPW.

The idea behind this approach is that the complementation of deterministic ω-automata is easier than the complementation of non-deterministic ω-automata. The complementation problem is then in fact reduced to conversions between different types of automata. From these conversions, the conversion from the initial NBW to a deterministic ω-automaton is the most difficult and crucial one.

Safra, 1988

The first determinisation-based complementation construction has been described by Safra in 1988 [33]. Safra's main work was actually a determinisation construction for converting an NBW to a DRW. This is what today is known as *Safra's construction*. Safra then describes complementation as a possible application of his determinisation construction. He also presents the additional conversions that are needed for the entire complementation construction. The conversion steps of Safra's complementation procedure are as follows.

1. NBW $\longrightarrow$ DRW (Safra's construction)
2. DRW $\longrightarrow$ $\overline{\text{DSW}}$ (complementation)
3. $\overline{\text{DSW}}$ $\longrightarrow$ $\overline{\text{DRW}}$
4. $\overline{\text{DRW}}$ $\longrightarrow$ $\overline{\text{NBW}}$

The complementation step from a DRW to a DSW that accepts the complement language can be trivially done by interpreting the Rabin acceptance condition as a Streett acceptance condition. This is possible, because these two acceptance conditions are the negations of each other (see Section 1.1.2. The conversions from DSW to DRW, and from DRW to NBW are not of major difficulty or complexity, and are described by Safra in [33] (Lemma 3 and Lemma 5).

The core is the conversion from NBW to DRW (Safra's construction). This construction is basically a modified subset construction. That is, the output automaton is built up from an initial state step-by-step by adding new states and transitions. The main difference to the subset construction is that in Safra's construction, the output-states consist of trees of subsets of input-states, rather than just of subsets of input-states. These trees of subsets of states are called *Safra trees*. The details of the construction are rather intricate, but well described in [33]. The deterministic automaton that results from Safra's construction can then be interpreted as a Rabin automaton.

Description of Safra trees/construction: [2] [32]

The state growth of Safra's construction is $2^{O(n \log n)}$, where n is the number of states of the input automaton. The additional conversions (DSW to DRW, and DRW to NBW) have a lower state complexity than this, so that the overall complexity of the entire complementation procedure is still $2^{O(n \log n}$.

Muller and Schupp, 1995

Most other determinisation-based complementation constructions are based on improvements of Safra's construction. One of them is the construction for converting NBW to DRW proposed in 1995 by Muller an Schupp. This construction is said to be simpler and more intuitive than Safra's construction [32], however, often produces larger output automata in practice [2]. The theoretically caluclated state complexity of the Muller-Schupp construction is $2^{O(n \log n)}$, that is, similar to Safra's construction. A comparison of the Muller-Schupp construction and Safra's construction can be found in [2].

Descriptionof Muller-Schupp trees: [2]

Piterman, 2007

Another improvement of Safra's construction has been proposed in 2007 by Piterman from EPF Lausanne [28] (first presented at a conference in 2006 [27]). This construction converts a NBW to a DPW, rather than a DRW. Piterman's construction uses a more compact version of Safra trees, which allows it to produce smaller output automata. The concrete worst-case state growth of Piterman's construction is $2n^n n!$, opposed to $12^n n^{2n}$ of Safra's construction [28]. Complementation with Piterman's construction is done in the following steps.

1. NBW $\longrightarrow$ DPW (Piterman's construction)
2. DPW $\longrightarrow$ $\overline{\text{DPW}}$ (complementation)
3. $\overline{\text{DPW}}$ $\longrightarrow$ $\overline{\text{NBW}}$

The complementation step from a DPW to a DPW accepting the complement language can be trivially done by, for example, increasing the number of each state by 1. The conversion from a DPW to an NBW can alse be done without major complexity [42].

1.3.3 Rank-Based Approach

The rank-based approach was the third of the four proposed main complementation approaches. It does neither include Ramsey theroy, nor determinisation. Rather, it is based on run analysis with run DAGs. The link of run analysis with run DAGs to complementation is as follows. A run DAG allows to summarise all the possible runs of an automaton on a specific word. If all these runs are rejecting, then we say that the entire run DAG is rejecting. In this case, the automaton does not accept the word, and consequently, the complement automaton must accept this word. Conversely, if one or more runs in the run DAG are not rejecting, then the entire run DAG is not rejecting. In this case, the automaton accepts the word, and consequently, the complement automaton must no accept this word.

The information of whether a run DAG is rejecting or not is expressed with so-called ranks. These are numbers that are assigned to the vertices of a run DAG, one rank per vertex. These ranks are assigned in a way that each run of a run DAG eventually gets trapped in a rank. From this information it is then possible to deduce whether the run DAG is rejecting or not. This in turn determines whether the complement automaton must accept the given word, or not.

This entire analysis of run DAGs with ranks is included in a subset construction. This means that the individual run DAGs are not constructed explicitly for each word, but rather implicitly "on-the-fly" within the complement automaton under construction. From a practical point of view, this means that rank-based constructions proceed in a subset construction based fashion. That is, the construction of the complement automaton is started with an initial state, and then step-by-step, successor states are added. Each output state consists of subsets of input-states.

Klarlund, 1991

The first rank-based construction has been proposed in 1991 by Klarlund [12]. However, Klarlund used the term *progress measure* instead of *rank*. This is because he looked at the ranks as a measure for the "progress" of a run towards the satisfaction of a certain property. The term *rank* has, to the best of our knowledge, been introduced by Thomas in 1999 [41]. Klarlund also did not mention run DAGs, but they are implicit in his description of the construction. The construction works as described above by performing a modified subset construction.

Kupferman and Vardi, 1997/2001

This construction by Kupferman and Vardi has been published as a preliminary conference version in 1997 [15], and as a journal version in 2001 [16]. Both publications are entitled "Weak Alternating Automata Are Not That Weak". The idea of the construction described by Kupferman and Vardi is the

same as Klarlund's construction from 1991 [12]. However, Kupferman and Vardi provide two different descriptions for this idea.

The first description does not use run DAGs and ranks, but rather converst the input automaton to a weak alternating automaton, which is complemented, and then converted back to a non-deterministic Büchi automata. Weak alternating automata (WAA) have been introduced in 1986 by Muller, Saoudi, and Schupp [23]. Kupferman and Vardi state that this construction is conceptually simpler and easier implementable than Klarlund's construction [12]. This first version of Kupferman and Vardi's construction is described in both, the publications from 1997 [15] and 2001 [16].

Description of alternating automata: [49] (Section 2.5)

The second description in turn is rank-based, as described above, and works in the subset construction fashion without intermediate automata. Kupferman and Vardi point out that this version of the construction is identical to Klarlund's construction. What changes is just the terminology, for example "ranks" instead of "progress measure". This second version of Kupferman and Vardi's construction is to the best of our knowledge only described in the publication from 2001 [16], however, we are not sure, because we could not access the publication from 1997[15].

There is an *odd ranking* if and only if all the runs of the run DAG are rejecting. Odd ranking: all the paths get trapped in an odd rank. Only non-accepting states have odd ranks.

Description in [5] [50]

The automata produced by the two versions of Kupferman and Vardi's construction are identical. The worst-case state complexity has been calculated to be approximately $(6n)^n$ [35][50].

Thomas, 1999

This construction by Thomas [41] is based on the WAA construction by Kupferman and Vardi from 1997 [15]. It uses the concept of ranks, but does not proceed in the subset construction manner, as Klarlund's construction [12] and Kupferman and Vardi's second version [16]. Rather, it transforms the input NBW to an intermediate automaton, complements it, and converts the result back to an NBW. That is, it proceeds in a similar fashion as Kupferman and Vardi's first version [15]. The type of the intermediate automaton is a weak alternating parity automaton (WAPA), that is, a weak alternating automaton with the parity acceptance condition.

Friedgut, Kupferman, and Vardi, 2006

In 2006, Friedgut, Kupferman, and Vardi published a paper entitled "Büchi Complementation Made Tighter" [7] (a preliminary version of the paper has appeared in 2004 [6]). There, they describe an improvement to the second (rank-based) version of Kupferman and Vardi's construction from 2001 [16]. The improvement consists in the so-called *tight ranking*, a more sophisticated ranking function. It allows to massively reduce the worst-case state complexity of the construction to $(0.96n)^n$.

Tight rankings: description in [5] [50]

Schewe, 2009

In 2009, Schewe presented another improvement to the construction by Friedgut, Kupferman, and Vardi from 2006 [35]. His paper is entitled "Büchi Complementation Made Tight", which hints at the relation to the paper by Friedgut, Kupferman, and Vardi [7]. Schewe's improvement consists in a further refinement of the construction, in particular the use of turn-wise tests in the cut-point construction step. This improvement allows to further reduce the worst-case state complexity of the construction to $(0.76\,(n+1))^{n+1}$. This coincides, modulo a polynomial factor, with the lower bound for the state complexity of Büchi complementation of $(0.76n)^n$ that has been previously established by Yan in 2006 [54][55].

This result narrows down the possible range for the real worst-case state complexity of Büchi complementation considerably. It cannot be lower than the lower bound of $(0.76n)^n$ by Yan, and it cannot

be higher than the complexity of Schwewe's construction of $(0.76\,(n+1))^{n+1}$. For this reason, we say that the proven worst-case complexity of a specific construction serves as an upper bound for the actual complexity of the problem.

1.3.4 Slice-Based Approach

The slice-based approach was the last approach that has been proposed. Its idea is very similar to the rank-based approach, but the main difference is the use of reduced split trees instead of run DAGs. The basic idea is to look at a state of the output automaton under construction as a horizontal level of a reduced split tree. Based on this, for each alphabet symbol, the succeeding level of the reduced split tree is determined, which results in a new state in the output automaton. These levels of reduced split trees are called *slices*, hence the name slice-based approach.

Like rank-based constructions, slice-based construction are essentially enhanced subset constructions. The slice-based constructions, however, include two runs of a subset construction, where the second one is typically more sophisticated than the first one.

Vardi and Wilke, 2007

The first slice-based Büchi complementation construction has been proposed in 2007 by Vardi and Wilke [52]. In this work, the authors review translations from various logics, including monadic second order logic of one successor (S1S), to ω-automata. They devise the slice-based complementation construction as a by-product of a determinisation construction for Büchi automata that they also introduce in this work.

Vardi and Wilke use left-to-right reduced split trees for their construction. That means, accepting states are put to the left of non-accepting states, and only the left most occurrence of each state is kept. The construction works by two passes of the enhanced subset construction. The first one (initial phase) is as described above. The second one (repetition phase), does additionally include decorations of the vertices of the reduced split trees (subsets) consisting of the three labels *inf*, *die*, and *new*. These decoration serves to keep track of the criterion that a word is rejected if and only if all of the branches of the corresponding reduced split tree contain only a finite number of left-turns. The worst-case state complexity of Vardi and Wilke's construction is $(3n)^n$ [52].

The slice-based construction by Vardi and Wilke is very similar to the Fribourg construction that we describe in Chapter 2. An obvious difference is that the Fribourg construction uses right-to-left, rather than left-to-right, reduced split trees. However, this is an arbitrary choice, and has no influence on the result of the constructions. Another difference is that the transition from the initial phase to the repetition phase is handled quite differently by Vardi and Wilke, than for the corresponding automata parts in the Fribourg construction.

Kähler and Wilke, 2008

The slice-based construction by Kähler and Wilke from 2008 [11] is a generalisation of the construction by Vardi and Wilke from 2007 [52]. Kähler and Wilke proposed a construction idea that can be used for both, complementation and disambiguation. Consequently, this construction is less efficient than Vardi and Wilke's construction. It has a worst-case state complexity of $4(3n)^n$ [42].

A comparison of the rank-based and slice-based complementation approaches has been done by Fogarty, Kupferman, Wilke, and Vardi [5]. In this work, the authors also describe a translation of the slice-based construction by Kähler and Wilke [11] to a rank-based construction.

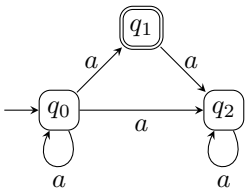


Figure 1.6: Example NBW A that will be used in different places throughout this thesis. The alphabet of A consists of the single symbol a, consequently, A can only process the single ω-word a^ω. This word is rejected by A, so the automaton is empty.

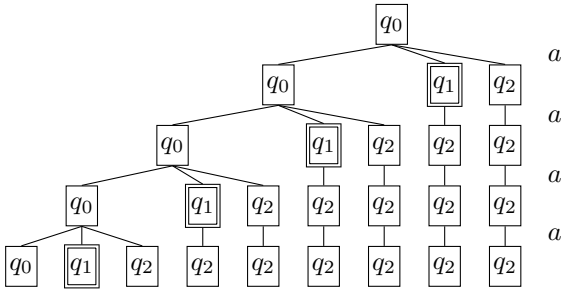


Figure 1.7: Automaton A and the first five levels of the run tree of the runs of A on the word a^ω.

1.4 Run Analysis

A deterministic automaton has exactly one run on every word. A non-deterministic automaton, on the other hand, may have multiple runs on a given word. The analysis of all runs of a word, in some form or another, an integral part of Büchi complemenation constructions. Remember that a non-deterministic automaton accepts a word if there is *at least one* accepting run. Consequently, a word is rejected if only if *all* the runs are rejecting. That is, if B is the complement Büchi automaton of A, then B has to accept a word w if and only if *all* the runs of A on w are rejecting. For constructing the complement B, we have thus to consider all the possible runs of A on every word.

There are two main data structures that are used for analysing the runs of a non-deterministic automaton on a word. These are trees and DAGs (directed acyclic graphs) [53]. In this section, we present both of them. We put however emphasis on trees, as they are used by the subset-tuple construction presented in Chapter **??**.

1.4.1 From Run Trees to Split Trees

The one tree data-structure that truly represents *all* the runs of an automaton on a word are run trees. The other variants of trees that we present in this section are basically derivations of run trees that sacrifice information about individual runs, by merging or discarding some of them, at the benefit of becoming more concise. Figure 1.12 shows the first few levels of the run tree of the example automaton A from Figure 1.6 on the word a^ω.

In a run tree, every vertex represents a single state and has a descendant for every a-successor of this state, if a is the current symbol of the word. A run is thus represented as a branch of the run tree. In particular, there is a one-to-one mapping between branches of a run tree and runs of the automaton on the given word.

We mentioned that the other tree variants that we talk about in this section, split trees and reduced split trees, make run trees more compact by not keeping information about individual runs anymore. They thereby relinquish the one-to-one mapping between branches of the tree and runs. Let us look at one extreme of this aggregation of runs which is done by the subset construction. This will motivate the definition of split trees, and at the same time shows why the subset construction fails for determinising

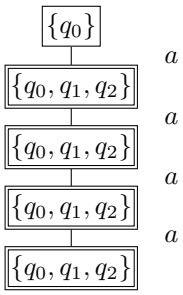


Figure 1.8: Automaton A and the first five levels of the run tree of the runs of A on the word a^ω.

NBW [3].

For determinising an automaton A, the subset construction in effect merges all the diverse runs of A on word w to one single run by merging all the states on a level of the corresponding run tree to one single state. This state will be a state of the output automaton B, and is labelled with the set of A-states it includes. Figure 1.8 shows this effect with our example automaton from Figure 1.6 and the word a^ω.

Clearly, this form of tree created by the subset construction is the most concise form a run tree can be brought to. However, almost all information about individual runs in A has been lost. All that can be said by looking at the structure in Figure 1.8 is that there must be at least one continued A-run on a^ω(all the other runs visiting the other A-states on each level, might be discontinued). But which states a possible continued run visits cannot be deduced.

This lack of identification A-runs is the reason why the subset construction fails for determinising Büchi automata. Note that a B-state of the subset construction is accepting if the set of A-states it represents contains at least one accepting A-state. For our example, this means that the state $\{q_0, q_1, q_2\}$ is accepting (this is also indicated in Figure 1.8). This state is visited infinitely often by the unified run on a^ω. Hence, the DBW B, resulting from applying the subset construction to the NBW A, accepts a^ωwhile A does not accept it.

By looking closer at the trees in Figure 1.8 and 1.12, the reason for this problem becomes apparent. If we look for example at the second level of the subset-construction tree we can deduce that there must be an A-run that visits the accepting A-state q_1. Let us call this run r_{q_1}. However, at the third level, we cannot say anything about r_{q_1} anymore, whether it visits one of the non-accepting states or again q_1 on the third level, or whether it even ended at the second level. In turn, what we know on the third level in our example is that there is again an A-run, r'_{q_1}, that visits q_1. However, whether r'_{q_1} is r_{q_1}, and in turn the future of r'_{q_1} cannot be deduced. In our example we end up with the situation that there are infintiely many visits to q_1 in the unifed B-run, but we don't know if the reaon for this are one or more A-runs that visit q_1 infinitely often, or infinitely man A-runs where each one visits q_1 only finitely often (the way it is in our example). In the first case, it would be correct ot accept the B-run, in the second case however it would be wrong as the input automaton A does not accept the word. The subset construction does not distinguish these two cases and hence the determinised automaton B may accept words that the input automaton A rejects. In general, the language of an output DBW of the subset construction is a superset of the language of the input NBW.

This raises the question how the subset construction can be minimally modified such that the output automaton is equivalent to the input automaton. One solution is to not mix accepting and non-accepting A-states in the B-states. That is, instead of creating one B-state that contains all the A-states, as in the subset construction, one creates two B-states where one contains the accepting A-states and the other the non-accepting A-states. Such a construction has been formalised in [48]. The output automaton B is then not deterministic, but it is equivalent to A. The type of run analysis trees that correspond to this refined subset construction are split trees. Figure 1.13 shows the first five levels of the split tree of our example automaton A on the word a^ω.

Let us see why the splitted subset construction produces output automata that are equivalent to the input automata. For this equivalence to hold, a branch of a reduced split tree must include infinitely

[3]The NBW that *can* be turned into DBW.

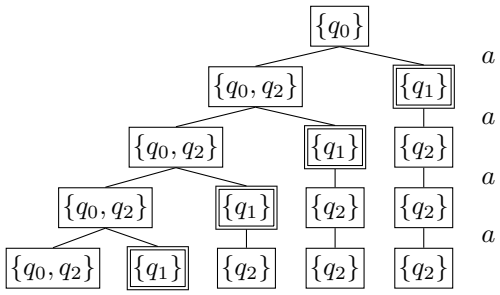


Figure 1.9: Automaton A and the first five levels of the split tree of the runs of A on the word a^ω.

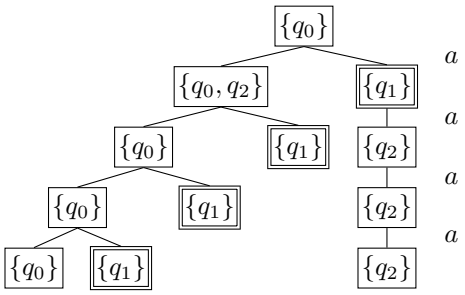


Figure 1.10: Automaton A and the first five levels of the reduced split tree of the runs of A on the word a^ω.

many accepting vertices if and only if there is an A-run that visits at least one accepting A-state infinitely often. For an infinite branch of a split tree, there must be at least one continued A-run. If this infinite branch includes infinitely many accepting vertices, then this A-run must infinitely many times go trough an accepting A-state. This is certain, because an accepting vertex in a split tree contains *only* accepting A-states. Since there are only finitely many accepting A-states, the A-run must visit at least one of them infinitely often. On the other hand, if an A-run includes infinite visits to an accepting state, then this results in a branch of the split tree with infinitely many accepting vertices, since every A-run must be "contained" in a branch of the split tree.

Split trees can be seen as run trees where some of the branches are contracted to unified branches. In particular, a split tree unifies as many branches as possible, such that the resulting tree still correctly represents the Büchi acceptance of all the runs included in a unified branch. This can form the basis for constructions that transform an NBW to another equivalent NBW. Split trees are for example the basis for Muller-Schupp trees in Muller and Schupp's Büchi determinisation construction [24], cf. [2].

1.4.2 Reduced Split Trees

It turns out that split trees can be compacted even more. The resulting kind of tree is called reduced split tree. In a reduced split tree, each A-state occurs at most once on every level. Figure 1.10 shows the reduced split tree corresponding to the split tree in Figure 1.13. As can be seen, only one occurrence of each A-state on each level is kept, the other are discarded. To allow this, however, the order of the accepting and non-accepting siblings in the tree matters. Either the accepting child is always put to the right of the non-accepting child (as in our example in Figure 1.10, or vice versa. We call the former variant a right-to-left reduced split tree, and the latter a left-to-right reduced split tree. In this thesis, we will mainly adopt the right-to-left version.

A reduced split tree is constructed like a split tree, with the following restrictions.

- For determining the vertices on level $n+1$, the parent vertices on level n have to be processed from right to left
- From every child vertex on level $n+1$, subtract the A-states that occur in some vertex to the right of it on level $n+1$

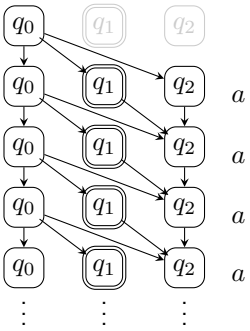


Figure 1.11: Automaton A and the first five levels of the run DAG of the runs of A on the word a^ω.

- Put the accepting child to the right of the non-accepting child on level $n + 1$

A very important property of reduced split trees is that they have a fixed width. The width of a tree is the maximal number of vertices on a level. For reduced split trees, this is the number of states of the input automaton A. As we will see, the subset-tuple construction (like other slice-based constructions) uses levels of a reduced split tree as states othe output automaton, and the limited size of these levels ensures an upper bound on the number of states these constructions can create.

By deleting A-states from a level of a reduced split tree, we actually delete A-runs that reach the same A-state on the same substring of the input word. For example, in the split tree in Figure 1.13 we see that there are at least for A-runs on the string $aaaa$ from the inital state q_0 to q_2. The reduced split tre in Figure 1.10, however, contains only one run on $aaaa$ from q_0 to q_2, namely the rightmost branch of the tree. The information about all the other runs is lost. This single run that is kept is very special and, as we will see shorty, it represents the deleted runs. We will call this run the *greedy run*. The reason for calling it greedy is that it visits an accepting state earlier than any of the deleted runs. In a right-to-left reduced split tree, the greedy run is always the rightmost of the runs from the root to a certain A-state on a certain level. In left-to-right reduced split tree, the greedy run would in turn be the leftmost of these runs.

We mentioned that the greedy run somehow represents the deleted runs. More precisely, the relation is as follows and has been proved in [52]: if any of the deleted runs is a prefix of a run that is Büchi-accepted (that is, an infinite run visiting infinitely many accepting A-states), then the greedy run is so too. That means that if the greedy cannot be expanded to a Büchi-accepting run, then none of the deleted runs could be either. Conversely, if any of the deleted runs could become Büchi-accepting, then the greedy run can so too. So, the greedy run is sufficient to indicate the existence or non-existence of a Büchi-accepting run with this prefix, and it is safe to delete all the other runs.

1.4.3 Run DAGs

DAGs (directed acyclic graphs) are, after trees, the second form for analysing the runs of a non-deterministic automaton on a given word. A run DAG has the form of a matrix with one column for each A-state and a row for each position in the word. The directed edges go from the vertices on one row to the vertices on the next row (drawn below) according to the transitions in the automaton on the current input symbol. Figure 1.16 shows the first five rows of the run DAG of the example automaton in Figure 1.6 on the word a^ω.

Like run trees, run DAGs represent all the runs of an automaton on a given word. However, run DAGs are more compact than run trees. The rank-based complementation constructions are based on run DAGs.

1.5 Run Analysis

In a deterministic automaton every word has exactly one run. In a non-deterministic automaton, however, a given word may have multiple runs. The analysis of the different runs of a given word on

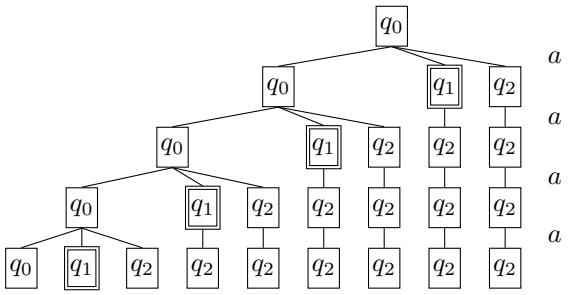


Figure 1.12: Automaton A and the first five levels of the run tree of the runs of A on the word a^ω.

an automaton plays an important role in the complementation of Büchi automata. There are several techniques for analysing the runs of a word that we present in this section.

1.5.1 Run Trees

The simplest of run analyis technique is the run tree. A run tree is a direct unfolding of all the possible runs of an automaton A on a word w. Each vertex v in the tree represents a state of A that we denote by $\sigma(v)$. The descendants of a vetex v on level i are vertices representing the successor states of $\sigma(n)$ on the symbol $w(i+1)$ in A. In this way, every branch of the run tree originating in the root represents a possible run of automaton A on word w.

Figure 1.12 shows an example automaton A and the first five levels of the run tree for the word $w = a^\omega$ (infinite repetitions of the symbol a). Each branch from the root to one of the leaves represents a possible way for reading the first four positions of w. On the right, as a label for all the edges on the corresponding level, is the symbol that causes the depicted transitions.

(A does not accept any word, it is empty. The only word it could accept is a^ω which it does not accept.)

We define by the width of a tree the maximum number of vertices occurring at any level [24]. Clearly, for ω-words the width of a run tree may become infinite, because there may be an infinite number of levels and each level may have more vertices than the previous one.

1.5.2 Failure of the Subset-Construction for Büchi Automata

Run trees allow to conveniently reveal the cause why the subset construction does not work for determinising Büchi automata, which in turn motivates the basic idea of the next run analysis technique, split trees.

Applying the subset construction to the same NBW A used in the previous example, we get the automaton A' shown in Figure **??**. Automaton A' is indeed a DBW but it accepts the word a^ω which A does not accept. If we look at the run tree of A on word a^ω, the subset construction merges the individual states occuring at level i of the tree to one single state s_i, which is accepting if at least one of its components is accepting. Equally, the individual transitions leading to and leaving from the individual components of s_i are merged to a unified transition. The effect of this is that we lose all the information about these indiviudal transitions. This fact is depicted in Figure **??**. For the NFA acceptance condition this does not matter, but for NBW it is crucial because the acceptance condition depends on the history of specific runs. In the example in Figure **??**, a run ρ of A visiting the accepting state q_1 can never visit an accepting state anymore even though the unified run of which ρ is part visits q_1 infinitely often. But the latter is achieved by infinitely many different runs each visiting q_1 just once.

It turns out that enough information about individual runs to ensure the Büchi acceptance condition could be kept, if accepting and non-accepting state are not mixed in the subset construction. Such a constructio has bee proposed in [48]. Generally, the idea of treating accepting and non-accepting states separately is important in the run analyis of Büchi automata.

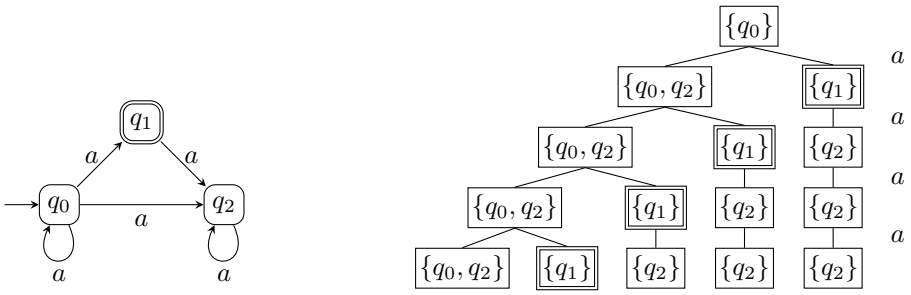


Figure 1.13: Automaton A and the first five levels of the split tree of the runs of A on the word a^ω.

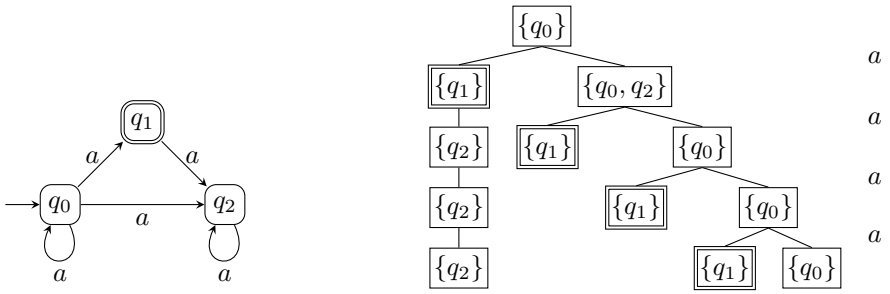


Figure 1.14: Automaton A and the first five levels of the left-to-right reduced split tree of the runs of A on the word a^ω.

1.5.3 Split Trees

Split trees can be seen as run trees where the accepting and non-accepting descendants of a node n are aggregated in two nodes. We will call the former the *accepting child* and the latter the *non-accepting child* of n. Thus in a split tree, every node has at most two descendants (if either the accepting or the non-accepting child is empty, it is not added to the tree), and the nodes represent sets of states rather than individual states. Figure 1.13 shows the first five levels of the split tree of automaton A on the word a^ω.

The order in which the accepting and non-accepting child are

The notion of split trees (and reduced split trees, see next section) has been introduced by Kähler and Wilke in 2008 for their slice-based complementation construction [11], cf. [5]. However, the idea of separating accepting from non-accepting states has already been used earlier, for example in Muller and Schupp's determinisation-based complementation construction from 1995 [24]. Formal definitions os split trees can be found in [11][5].

1.5.4 Reduced Split Trees

The width of a split tree can still become infinitely large. A reduced split tree limits this width to a finite number with the restriction that on any level a given state may occur at most once. This is in effect the same as saying that if in a split tree there are multiple ways of going from the root to state q, then we keep only one of them.

1.5.5 Run DAGs

A run DAG (DAG stands for directed acyclic graph) can be seen as a graph in matrix form with one column for every state of A and one row for every position of word w. The edges are defined similarly than in run trees. Figure 1.16 shows the run DAG of automaton A on the word $w = a^\omega$.

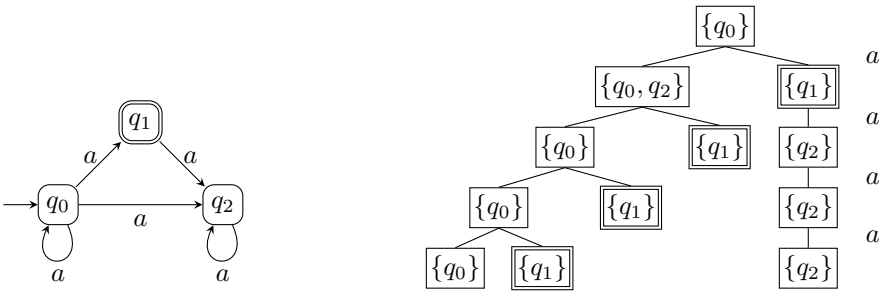


Figure 1.15: Automaton A and the first five levels of the left-to-right reduced split tree of the runs of A on the word a^ω.

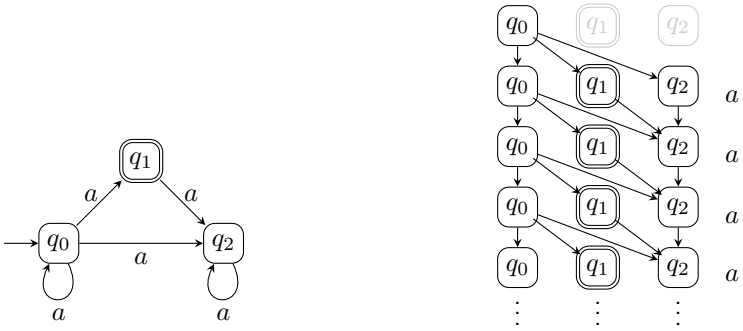


Figure 1.16: Automaton A and the first five levels of the run DAG of the runs of A on the word a^ω.

1.6 Empirical Performance Investigations

1.7 Preliminaries

1.7.1 Büchi Automata

Büchi automata have been introduced in 1962 by Büchi [4] in order to show the decidability of monadic second order logic; over the successor structure of the natural numbers [3].

he had proved the decidability of the monadic-second order theory of the natural numbers with successor func- tion by translating formulas into finite automata [52] (p. 1)

Büchi needed to create a complementation construction (proof the closure under complementation of Büchi automata) in order to prove Büchi's Theorem.

Büchi's Theorem: S1S formulas and Büchi automata are expressively equivalent (there is a NBW for every S1S formula, and there is a S1S formula for every NBW).

Definitions

Informally speaking, a Büchi automaton is a finite state automaton running on input words of infinite length. That is, once started reading a word, a Büchi automaton never stops. A word is accepted if it results in a run (sequence of states) of the Büchi automaton that includes infinitely many occurrences of at least one accepting state.

More formally, a Büchi automaton A is defined by the 5-tuple $A = (Q, \Sigma, q_0, \delta, F)$ with the following components.

- Q: a finite set of states
- Σ: a finite alphabet

- q_0: an initial state, $q_0 \in Q$
- δ: a transition function, $\delta : Q \times \Sigma \to 2^Q$
- F: a set of accepting states, $F \in 2^Q$

We denote by Σ^ω the set of all words of infinite length over the alphabet Σ. A Büchi automaton runs on the elements of Σ^ω. In the following, we define the acceptance behaviour of a Büchi automaton A on a word $\alpha \in \Sigma^\omega$.

- A *run* of Büchi automaton A on a word $\alpha \in \Sigma^\omega$ is a sequence of states $q_0 q_1 q_2 \dots$ such that q_0 is A's initial state and $\forall i \geq 0 : q_{i+1} \in \delta(q_i, \alpha_i)$
- $\inf(\rho) \in 2^Q$ is the set of states that occur infinitely often in a run ρ
- A run ρ is accepting if an only if $\inf(\rho) \cap F \neq \varnothing$
- A Büchi automaton A accepts a word $\alpha \in \Sigma^\omega$ if and only if there is an accepting run of A on α

The set of all the words that are accepted by a Büchi automaton A is called the *language* $L(A)$ of A. Thus, $L(A) \subseteq \Sigma^\omega$. On the other hand, the set of all words of Σ^ω that are rejected by A is called the *complement language* $\overline{L(A)}$ of A. The complement language can be defined as $\overline{L(A)} = \Sigma^\omega \setminus L(A)$.

Büchi automata are closed under union, intersection, concatenation, and complementation [49].

Continued/discontinued runs

A deterministic Büchi automaton (DBW) is a special case of a non-determnistic Büchi automaton (NBW). A Büchi automaton is a DBW if $|\delta(q, \alpha)| = 1$, $\forall q \in Q, \forall \alpha \in \Sigma$. That is, every state has for every alphabet symbol exactly one successor state. A DBW can also be defined directly by replacing the transition function $\delta : Q \times \Sigma \to 2^Q$ with $\delta : Q \times \Sigma \to Q$ in the above definition.

Expressiveness

It has been showed by Büchi that NBW are expressively equivalent the ω-regular languages [4]. That means that every language that is recognised by a NBW is a ω-regular language, and on the other hand, for every ω-regular language there exists a NBW recognising it.

However, this equivalence does not hold for DBW (Büchi showed it too). There are ω-regular languages that cannot be recognised by any DBW. A typical example is the language $(0 + 1)^* 1^\omega$. This is the language of all infinite words of 0 and 1 with only finitely many 0. It can be shown that this language can be recognised by a NBW (it is thus a ω-regular language) but not by a DBW [49][32]. The class of languages recognised by DBW is thus a strict subset of ω-regular languages recognised by NBW. We say that DBW are less expressive than NBW.

An implication of this is that there are NBW for which no DBW recognising the same language exists. Or in other words, there are NBW that cannot be converted to DBW. Such an inequivalence is not the case, for example, for finite state automata on finite words, where every NFA can be converted to a DFA with the subset construction [10][29]. In the case of Büchi automata, this inequivalence is the main cause that Büchi complementation problem is such a hard problem [25] and until today regarded as unsolved.

1.7.2 Other ω-Automata

After the introduction of Büchi automata in 1962, several other types of ω-automata have been proposed. The best-known ones are by Muller (Muller automata, 1963) [22], Rabin (Rabin automata, 1969) [30], Streett (Streett automata, 1982) [38], and Mostowski (parity automata, 1985) [21].

All these automata differ from Büchi automata, and among each other, only in their acceptance condition, that is, the condition for accepting or rejecting a run ρ. We can write a general definition of ω-automata that covers all of these types as $(Q, \Sigma, q_0, \delta, Acc)$. The only difference to the 5-tuple defining Büchi automata is the last element, Acc, which is a general acceptance condition. We list the acceptance condition of all the different ω-automata types below [18]. Note that again a run ρ is a sequence of states, and $\inf(\rho)$ is the set of states that occur infinitely often in run ρ.

Type	**Definitions**	**Run ρ accepted if and only if...**
Büchi	$F \subseteq Q$	$\inf(\rho) \cap F \neq \varnothing$
Muller	$F \subseteq 2^Q$	$\inf(\rho) \in F$
Rabin	$\{(E_1, F_1), \ldots, (E_r, F_r)\}$, $E_i, F_i \subseteq Q$	$\exists i : \inf(\rho) \cap E_i = \varnothing \wedge \inf(\rho) \cap F_i \neq \varnothing$
Streett	$\{(E_1, F_1), \ldots, (E_r, F_r)\}$, $E_i, F_i \subseteq Q$	$\forall i : \inf(\rho) \cap E_i \neq \varnothing \vee \inf(\rho) \cap F_i = \varnothing$
Parity	$c : Q \to \{1, \ldots, k\}$, $k \in \mathbb{N}$	$\min\{c(q) \mid q \in \inf(\rho)\} \bmod 2 = 0$

In the Muller acceptance condition, the set of infinitely occuring states of a run ($\inf(\rho)$) must match a predefined set of states. The Rabin and Streett conditions use pairs of state sets, so-called accepting pairs. The Rabin and Streett conditions are the negations of each other. This allows for easy complementation of deterministic Rabin and Streett automata [18], which will be used for certain Büchi complementation construction, as we will see in Section1.8. The parity condition assigns a number (color) to each state and accepts a run if the smallest-numbered of the infinitely often occuring states has an even number. For all of these automata there exist non-deterministic and deterministic versions, and we will refer to them as NMW, DMW (for non-deterministic and deterministic Muller automata), and so on.

In 1966, McNaughton made an important proposition, known as *McNaughton's Theorem* [19]. Another proof given in [39]. It states that the class of languages recognised by deterministic Muller automata are the ω-regular languages. This means that non-deterministic Büchi automata and deterministic Muller automata are equivalent, and consequently every NBW can be turned into a DMW. This result is the base for the determinisation-based Büchi complementation constructions, as we will see in Section 1.8.2.

It turned out that also all the other types of the just introduced ω-automata, non-deterministic and determinstic, are equivalent among each other [32][14][13][18][39]. This means that all the ω-automata mentioned in this thesis, with the exception of DBW, are equivalent and recognise the ω-regular languages. This is illustrated in Figure 1.17

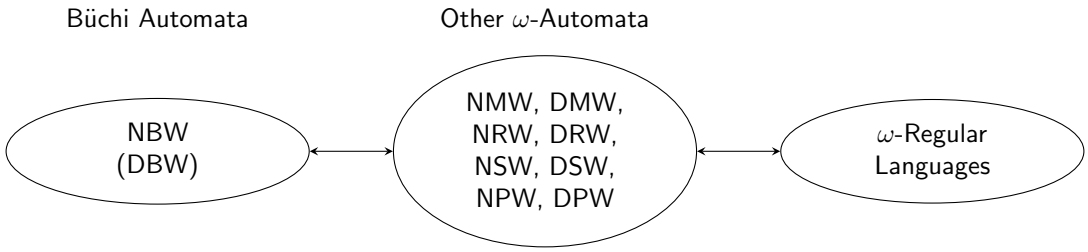


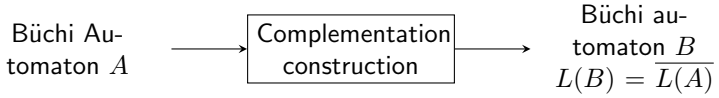
Figure 1.17: Non-deterministic Büchi automata (NBW) are expressively equivalent to Muller, Rabin, Streett, and parity automata (both deterministic and non-deterministic), and to the ω-regular languages. Deterministic Büchi automata (DBW) are less expressive than NBW.

1.7.3 Complementation of Büchi Automata

Büchi automata are closed under complementation. This result has been proved by Büchi himself when he introduced Büchi automata in [4]. Basically, this means that for every Büchi automata A, there exists another Büchi automaton B that recognises the complement language of A, that is, $L(B) = \overline{L(A)}$.

It is interesting to see that this closure does not hold for the specific case of DBW. That means that while for every DBW a complement Büchi automaton does indeed exist, following from the above closure property for Büchi automata in general, this automaton is not necessarily a DBW. The complement of a DBW may be, and often is, as we will see, a NBW. This result is proved in [39] (p. 15).

The problem of Büchi complementation consists now in finding a procedure (usually called a construction) that takes as input any Büchi automaton A and outputs another Büchi automaton B with $L(B) = \overline{L(A)}$, as shown below.



For complementation of automata in general, construction usually differ depending on whether the input automaton A is deterministic or non-deterministic. Complementation of deterministic automata is often simpler and may sometimes even provide a solution for the complementation of the non-deterministic ones.

To illustrate this, we can briefly look at the complementation of the ordinary finite state automata on finite words (FA). FA are also closed under complementation [10] (p. 133). A DFA can be complemented by simply switching its accepting and non-accepting states [10] (p. 133). Now, since NFA and DFA are equivalent [10] (p. 60), a NFA can be complemented by converting it to an equivalent DFA first, and then complement this DFA. Thus, the complementation construction for DFA provides a solution for the complementation of NFA.

Returning to Büchi automata, the case is more complicated due to the inequivalence of NBW and DBW. The complementation of DBW is indeed "easy", as was the complementation of DFA. There is a construction, introduced in 1987 by Kurshan [17], that can complement a DBW to a NBW in polynomial time. The size of the complement NBW is furthermore at most the double of the size of the input DBW.

If now for every NBW there would exist an equivalent DBW, an obvious solution to the general Büchi complementation problem would be to transform the input automaton to a DBW (if it is not already a DBW) and then apply Kurshan's construction to the DBW. However, as we have seen, this is not the case. There are NBW that cannot be turned into equivalent DBW.

Hence, for NBW, other ways of complementing them have to be found. In the next section we will review the most important of these "other ways" that have been proposed in the last 50 years since the introduction of Büchi automata. The Fribourg construction, that we present in Chapter **??**, is another alternative way of achievin this same aim.

1.7.4 Complexity of Büchi Complementation

Constructions for complementing NBW turned out to be very complex. Especially the blow-up in number of states from the input automaton to the output automaton is significant. For example, the original complementation construction proposed by Büchi [4] involved a doubly exponential blow-up. That is, if the input automaton has n states, then for some constant c the output automaton has, in the worst case, c^{c^n} states [37]. If we set c to 2, then an input automaton with six states would result in a complement automaton with about 18 quintillion (18×10^{18}) states.

Generally, state blow-up functions, like the c^{c^n} above, mean the absolute worst cases. It is the maximum number of states a construction *can* produce. For by far most input automata of size n a construction will produce much fewer states. Nevertheless, worst case state blow-ups are an important (the most important?) performance measure for Büchi complementation constructions. A main goal in the development of new constructions is to bring this number down.

A question that arises is, how much this number can be brought down? Researchers have investigated this question by trying to establish so called lower bounds. A lower bound is a function for which it is proven that no state blow-up of any construction can be less than it. The first lower bound for Büchi complementation has been established by Michel in 1988 at $n!$ [20]. This means that the state blow-up of any Büchi complementation construction can never be less than $n!$.

There are other notations that are often used for state blow-ups. One has the form $(xn)^n$, where x is a constant. Michel's bound of $n!$ would be about $(0.36n)^n$ in this case [54]. We will often use this notation, as it is convenient for comparisons. Another form has 2 as the base and a big-O term in the exponent. In this case, Michel's $n!$ would be $2^{O(n \log n)}$ [54].

Michel's lower bound remained valid for almost two decades until in 2006 Yan showed a new lower bound of $(0.76n)^n$ [54]. This does not mean that Michel was wrong with his lower bound, but just too reserved. The best possible blow-up of a construction can now be only $(0.76n)^n$ and not $(0.36n)^n$ as believed before. In 2009, Schewe proposed a construction with a blow-up of exactly $(0.76n)^n$ (modulo a polynomial factor) [35]. He provided thus an upper bound that matches Yan's lower bound. The lower bound of $(0.76n)^n$ can thus not rise any further and seems to be definitve.

Maybe mention note on exponential complexity in [49] p. 8.

1.8 Review of Büchi Complementation Constructions

1.8.1 Ramsey-Based Approaches

The method is called Ramsey-based because its correctness relies on a combinatorial result by Ramsey to obtain a periodic decomposition of the possible behaviors of a Büchi automaton on an infinite word [3].

1.8.2 Determinisation-Based Approaches

1.8.3 Rank-Based Approaches

1.8.4 Slice-Based Approaches

1.9 Empirical Performance Investigations

Chapter 2

The Fribourg Construction

Contents

In this chapter we describe the Fribourg construction, the Büchi complementation construction developed at the University of Fribourg by Joel Allred and Ulrich Ultes-Nitsche. The construction has been published in 2014 as a technical report entitled "Complementing Büchi Automata with a Subset-tuple Construction" [1].

We do not give a formal description of the Fribourg construction in this chapter, because this has already been done in [1]. Rather, our aim is to give an intuitive and practically oriented description. That means, demonstrating the concrete steps one has to do when sitting with a pencil in front of an automaton to be complemented. Similarly, this chapter does not contain any proofs of the correctness or complexity of the constructions, because they can be found in [1].

This chapter is structured as follows. In Section 2.1 we present some basic properties of the Fribourg construction and put it in relation with other complementation constructions. In Section 2.2, we describe the actual construction which consists of two stages, the construction of the upper part and the construction of the lower part. We present these two stages in separate sections, together with an example. Finally, in Section **??**, we describe three optimisations for the construction, that have the abbreviations R2C, M1, and M2. These optimisations will also be subject to our empirical performance investigation that we describe in the subsequent chapter.

A note on terminology: the authors themselves call their construction "subset-tuple construction". This is because a state of the output automaton consists of a tuple of subsets of states of the input automaton. However, this is also the case for other constructions. To make our construction more distinguishable from the other constructions, we decided to use the more striking name "Fribourg construction".

2.1 Basics

The Fribourg construction is a *slice-based* complementation construction. That means that, like the other constructions of the slice-based approach (see Section 1.3.4), it is based on reduced split trees. Furthermore, it works in a subset-construction manner, that is, the output automaton is constructed state-by-state, by starting from an initial state.

The output-states are internally structured as tuples of subsets of input-states (each state consists of one tuple). The subsets of a tuple are pairwise disjoint, that is, no tuple contains two times the same input-state. The input-states of a tuple are the same as in the subset construction. The difference to the subset construction is that the input-states are not all in the same set, but distributed over multiple sets. Furthermore, the order of these sets in the tuple matters. As a convention, we will refer to the subsets contained by a tuple of an output-state as the *components* of this output-state.

The components output-states are determined by levels of reduced split trees. According to the terminology of Vardi and Wilke [52], we call these levels *slices*. Figure shows how slices of a reduced split tree determine output-states of the Fribourg construction. The relation works also in the other direction, that is, an output-state of the Fribourg construction determines a slice of a reduced split tree. The simple rule is that each vertex of a slice becomes a component in in the tuple of the output-state, and the order of the vertices is preserved.

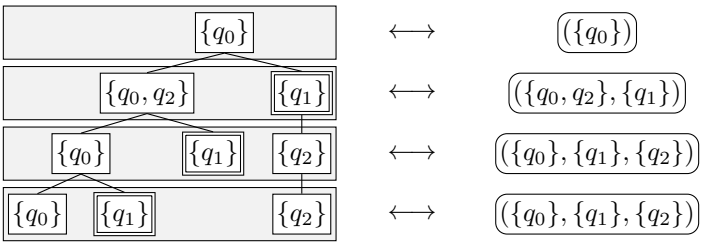


Figure 2.1: Translation from slices of a reduced split tree (shaded boxes on the left) to output-states of the Fribourg construction (right), and vice versa.

This relation between slices of reduced split trees and output-states determines the basic working of the Fribourg construction. It works as follows. Start with an initial output-state containing only the initial

input-state in its tuple. This state will be the current state. Translate the current state to a level of a reduced split tree, and, according to the input automaton, determine the next level of the reduced split tree for the first alphabet symbol, say a. Translate this new level back to an output-state, and set it as the a-successor of the current state. Repeat this procedure for all alphabet symbols, until the current state has a successor for each one. Finally, repeat the entire procedure for each output-state that has no successors yet. As in the subset construction, if a successor state happens to be identical to an already existing state, then just add a transition to this existing state.

What we just described is the very basic working of the Fribourg construction. However, there are additional details. First, there are two passes of this subset-construction-like procedure. The first one results in the so-called *upper part* of the final complement automaton. The second one is applied on the states of the upper part (and all the newly created states), and results in the so-called *lower part* attached to the upper part. These two parts together form the final complement automaton. The terminology "upper" and "lower" results from the fact that, when doing the construction by hand, the lower part is typically drawn below the upper part.

This distinction between upper and lower part is inspired by Kurshan's construction for complementing deterministic Büchi automata [17]. In fact, Kurshan's construction is a special case of the Fribourg construction [1].

The upper part of the Fribourg construction does not contain any accepting states. The lower part, in turn, may contain accepting states. A run of the final complement automaton starts in the upper part, and has in each upper-part state the non-deterministic choice to move to the lower part. Once in the lower part, a run cannot return to the upper part anymore. Semantically, the upper part represents anything that can happen in a finite prefix of an ω-word, and the lower part takes care of the infinite behaviour on ω-words.

The fact that the lower part determines the acceptance of a run, requires additional sophistication. This is achieved by the decoration of components. All components of the lower-part states are decorated with one of the three colours 0, 1 and 2. The colour of a component is determined by two things. First, whether the component is accepting or non-accepting. Second, the colour of its predecessor component. The predecessor component c_{pred} of a component c is the component of the predecessor state, that, in terms of reduced split trees, is the parent vertex of the vertex corresponding to component c.

Figure showing predecessor component relation

This colouration of components in the lower part requires that during the construction of the lower part, we keep track of two properties of each component. First, whether it is accepting or non-accepting, and second, its predecessor component (or just the colour of its predecessor component).

On a more general note, the Fribourg construction uses a similar idea as Vardi and Wilke's slice-based construction from 2007 [52]. The upper and lower part of the Fribourg construction correspond to the *initial phase* and *repetition phase* of Vardi and Wilke's construction. Furthermore, the colours 0, 1, and 2 of the Fribourg construction correspond to the decorations *inf*, *new*, and *die* of Vardi and Wilke's construction. However, the two constructions still differ in details, especially in the transition from the upper part to the lower part. In any case, the Fribourg construction has been developed independently and is not based on Vardi and Wilke's construction. Rather, the development of the Fribourg construction was based on Kurshan's construction for complementing DBW, which is to the best of our knowledge, not the case for Vardi and Wilke's construction.

Another difference is that the Fribourg construction uses right-to-left reduced split trees, whereas Vardi and Wilke's construction (as well as Kähler and Wilke's construction [11]) uses left-to-right reduced split trees. This is howevrer an arbitrary choice, and it has no effect on the final complement automaton. It would be possible to describe the Fribourg construction with left-to-right reduced split trees, and Vardi and Wilke's construction with right-to-left reduced split trees. In this thesis, we will stick with the right-to-left reduced split trees for the Fribourg construction.

By using reduced split trees, we consider only greedy runs on prefixes of words. That is, if two or more runs on the same words are after a certain number of steps in the same state, then only one of them is considered, the others are omitted.

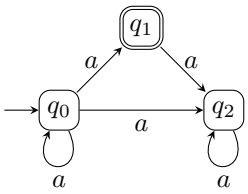


Figure 2.2: Example automaton.

2.2 The Construction

In this section, we describe the Fribourg construction in some more detail, and we illustrate its application with an example.

2.2.1 Upper Part

Description

The construction of the upper part is simple and works basically as described above. We start with an initial output-state containing only a single component with the initial state of the input automaton. Then, for each output-state q that has no successors so far, we creat a successor slice for each symbol of the alphabet α, translate it to a state, and set it as the α-successor of q. This is repeated until all states have a been processed.

In case that the resulting automaton is not complete (that is, one or more states do not have successors for certain alphabet symbols), then it is made complete by adding an accepting sink state. This sink state is not actually a part of the upper part, and it is not further processed during the rest of the construction. However, its presence is important in case the upper part is not complete.

The result of this first stage of the construction is a deterministic and complete automaton that does not contain any accepting states (except a possible sink state, but which, as mentioned, does not really belong to the upper part).

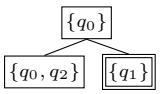
Example

We will illustrate the application of the Fribourg construction with the example automaton in Figure 2.2. This automaton has only one single alphabet symbol a. This choice was made to keep the example simple, because an alphabet size of, say 2, would double the number of steps and the number of transitions in the output-automaton. However, the procedure for automata with larger alphabets is exactly the same, just repeat the step of creating a successor state for the current state for each alphabet symbol.

The example automaton in Figure 2.2 does not accept any word, because it is impossible for any run to visit the only accepting state q_1 infinitely many times. The automaton A is thus empty. Consequently, the complement of A is universal, that is, it accepts every possible word[1].

Figure 2.3 shows the complete steps for creating the upper part from the example automaton in Figure 2.2. In Figure 2.3 (a), we start with a state containing only the component $\{q_0\}$, because q_0 is the initial state of A.

In Figure 2.3 (b), we determine the a-successor of the state $(\{q_0\})$. As explained, this works by looking at the state as a slice of a reduced split tree, and then creating the succeeding slice. For the case of $(\{q_0\})$, this gives the following two slices:



[1]The only possible ω-word with the alphabet $\Sigma = \{a\}$ is a^ω, that is an infinite sequence of a's.

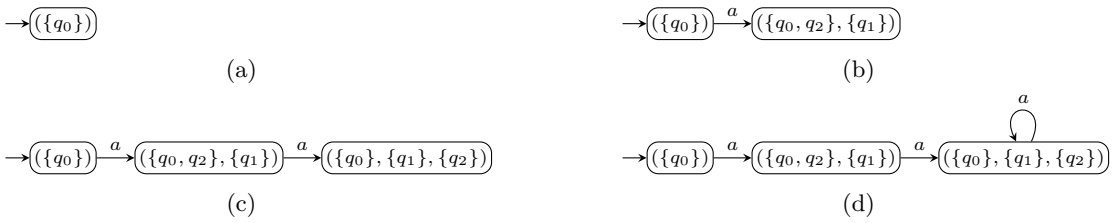
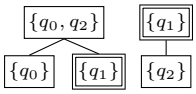


Figure 2.3: Steps for creating the upper part of the complement from the example input-automaton in Figure 2.2.

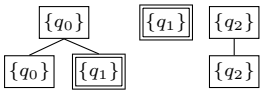
From the state q_0 in the example automaton A, we can reach states q_0, q_1, and q_2 with the symbol a. Since q_1 is accepting, it is separated from the other states and put as a separate child to the right of the other child in the new slice. Transforming this new slice back to a state yields $(\{q_0, q_2\}, \{q_1\})$, which is the a-successor of $(\{q_0\})$. The component $\{q_1\}$ is furthermore an accepting component, however, this does not matter in the construction of the upper part. Since there is only the single alphabet symbol a, we have already created all the successors of $(\{q_0\})$. If there would be more alphabet symbols, we would have to repeat the same procedure for each symbol.

In Figure 2.3 (c), we create the a-successor of the previously created $(\{q_0, q_2\}, \{q_1\})$. Applying the same procedure as before, we get the following two slices:



Since we use right-to-left reduced split trees, we have to create the new slice from right to left, that is, first determining the children of vertex q_1, and then of $\{q_0, q_2\}$. From q_1 we can only reach q_2 on symbol a, thus the only child of $\{q_1\}$ is $\{q_2\}$. From the states in $\{q_0, q_2\}$, we can reach q_0, q_1, and q_2 on symbol a. However, q_1 already appears in the new slice, so it drops out. From the remaining states q_0 and q_1, q_1 is accepting, so it becomes a separate right-child, whereas q_0 becomes the left-child. Translating this slice to a state yields $(\{q_0\}, \{q_1\}, \{q_2\})$, which is the a-successor of $(\{q_0, q_2\}, \{q_1\})$.

In Figure 2.3 (d), we determine in turn the a-successor of $(\{q_0\}, \{q_1\}, \{q_2\})$. Applying again the slice-procedure, we get the following:



Again, we process the vertices of the upper slice from right to left. The only a-successor of q_2 in automaton A is q_2, thus $\{q_2\}$ is the only child of the vertex $\{q_2\}$ in the upper slice. The state q_1 has q_2 as its a-successor. However, q_2 already appears in the new slice, and thus it drops out. Since q_1 has no remaining a-successors, the vertex $\{q_1\}$ in the upper slice remains childless[2]. Finally, the a-successors of q_0 are q_0, q_1, and q_2, however, as before, q_2 drops out because it already appears to the right, and q_0 and q_1 are separated because q_1 is accepting. Translating the new slice back to a state results in $(\{q_0\}, \{q_1\}, \{q_2\})$, whichi s identical to the current state. Thus, instead of adding a new state to the automaton, we just add an a-loop to $(\{q_0\}, \{q_1\}, \{q_2\})$.

At this stage, all the existing states have successors for all the alphabet symbols, and thus the construction of the upper part is completed. The next step is to attach the lower part to the upper part in order to form the final complement automaton.

[2]In the terminology that we will use starting from the next section, we say that the runs going through this q_1 "disappear".

2.2.2 Lower Part

The construction of the lower part takes the states of the previously constructed upper part as input. This means that these states are taken as the initial "states to be processed". Thus, the states of the upper part will get additional successors in the lower part, which makes the states of the upper part non-deterministic[3]. The states of the lower part, in turn, are deterministic, and do not have transitions back to the upper part. This means that once a run switches from the upper to the lower part, it stays there infinitely (or dies there).

The construction of the lower part proceeds principially in the same way as the construction of the upper part. However, it includes some additional "structure" in the form of a decoration of the components. In particular, every component of a lower-part state is assigned a colour. This colour is determined at creation time of the containing state, and is never changed[4]. These colours are distinguising features for the containing states. This means that if two states contain the same components in the same order, but the components have different colours, then the two states are different. Clearly, this makes the number of possible states of the lower part much larger than the number of possible states of the upper part.

We denote these three colours for the components of the lower part by 0, 1 and 2. In order to determine the accepting set at the end of the construction, it is necessary to ditinguish the states from the upper part from the states of the lower part. This problem is solved by previously assigning the special colour -1 to all components of upper part-states.

The assignment of a colour to a component of a lower-part state depends on three things:

1. Whether the component is accepting or non-accepting
2. The colour of the predecessor component
3. Whether the state containing the predecessor component contains any 2-coloured components

How much to describe the concept of predecessor component, as it is already described in Section 2.1?

The rules for assigning one of the colours 0, 1, and 2 to a component c are shown in Figure 2.4. There are two different sets of rules for the cases that the state containing the predecessor component *does* (Figure 2.4 (b)), or *does not* ((Figure 2.4 (a)) contain components with colour 2. In each of these cases, the two remaining criteria, whether c is accepting or non-accepting, and the colour of the predecessor component c_{pred}, determine a single colour that must be assigned to component c (bold in Figure 2.4).

In the first case, that the predecessor state contains no 2-coloured components, the possible colours of the predecessor components are -1, 0, and 1. Naturally, the predecessor component cannot be 2-coloured, but on the other hand, it might have colour -1, if the predecessor state is a state of the upper part. For the other case, that the predecessor state contains 2-coloured components, the possible colours for the predecessor component naturally include colour 2, but do not include colour -1, because a state containing 2-coloured components cannot be a state of the upper part.

The purpose of the colours is to signalise the presence or absence of certain runs of the input automaton on a specific word. Note that the complement of a non-deterministic automaton must accept a word if and only if *all* the runs of the input automaton on this words are rejecting. If there is a single run of the input automaton that accepts the word, then the complement automaton must not accept the word. Thus, we need a way to be sure that there are *no* accepting runs of the input automaton on a specific word, and then we can accept this word with the complement automaton.

Colour 2 Is used to signalise the presence of "dangerous" runs, that is, runs that have the potential to becom accepting. If a component has colour 2, it means that there are input-runs that made a "right-turn", in terms of slices of reduced split trees, that is, visited an accepting state (see rules in Figure 2.4 (a) line 1 and 2). Note that all the successor components of a 2-coloured component are also 2-coloured, no matter if they are accepting or non-accepting (Figure 2.4 (b) line 3). A "string" of 2-coloured components can only be cut if a 2-coloured component has no successor components

[3] Two points about this non-determinism are interesting. First, the states of the upper part are the only non-deterministic states, all the other states are deterministic. Second, the degree of non-determinism of these states is at most 2, and thus the degree of non-determinism of the entire complement automaton is at most 2.

[4] The colour of a component may be changed with the optimisations described in Section 2.3.

Colour of c_{pred}	c is non-accepting	c is accepting
-1	**0**	**2**
0	**0**	**2**
1	**2**	**2**

(a) Case A: the predecessor state has *no* 2-coloured components

Colour of c_{pred}	c is non-accepting	c is accepting
0	**0**	**1**
1	**1**	**1**
2	**2**	**2**

(b) Case B: the predecessor state *has* 2-coloured components

Figure 2.4: Rules for determining the colour of a component c, based on (1) the colour of the predecessor component c_{pred}, and (2) whether c is an accepting or non-accepting component. There are two set of rules that are shown in the two subfigures: (a) the predecessor state does not have any components with colour 2, and (b) the predecessor state does have one or more components with colour 2.

(no children in terms of reduced split trees). In this case we say that the 2-coloured component "disappears".

Colour 1 Means basically the same as colour 2, namely that there are "dangerous" runs. The reason that colour 1 exists is a caveat that could arise if we would assign colour 2 to *every* component that just made a right-turn. In this case, it could happen that we miss the disappearance of 2-coloured components, and thus keep the containing output-state non-accepting instead of accepting. For this reason, the trick with colour 1 works as follows. If the predecessor state already contains 2-coloured components, then every component of the current state that deserves to be 2-coloured gets colour 1 instead of colour 2. This can be seen in Figure 2.4 (b) line 1 and 2. These are actually "dangerous" components kept "on hold", becaus then trick goes on as follows. As soon as all the 2-coloured components of a state disappear, the successors of all the 1-coloured components get 2-coloured. This can be seen in Figure 2.4 (a) line 3. At this point, these "dangerous" components "on hold" get the real "dangerous" components.

Colour 0 Means the absence of "dangerous" runs. This is because the corresponding input-run did not make any right-turns, that is, they went only through non-accepting states. These runs are "safe" in the sense that so far they bear no risk of becoming accepting runs.

Wrapping up, for constructing the lower part of the output automaton, one just has to apply the same successor creation procedure as for the upper part, with the addition of assigning colours to the new states' components according to the rules in Figure 2.4. In the end, when every state has been processed and the procedure ends, the only thing that is missing is to determine the accepting states of the resulting automaton. The rule is that every state of the lower part that does *not* contain any 2-coloured components is an accepting states. That means that every state of the automaton that contains exclusively 0-coloured and/or 1-coloured components is an accepting state.

In the following, we demonstrate the application of the lower-part construction by an example.

Example

We continue the complementation of the example automaton in Figure 2.2 with the construction of the lower part. Therefore, we start where we left off the example in the last section, namely with the upper part. For this example, we will use the following notation for specifying the colour of a component $\{q\}$:

- $\widehat{\{q\}}$: colour -1
- $\{q\}$: colour 0

- $\overline{\{q\}}$: colour 1
- $\overline{\overline{\{q\}}}$: colour 2

Figure 2.5 shows some of the steps of the construction of the lower part. In Figure 2.5 (a), we start with the upper part that we previously constructed in the last section. The only difference is that we assigned colour -1 to all of the components of the upper part.

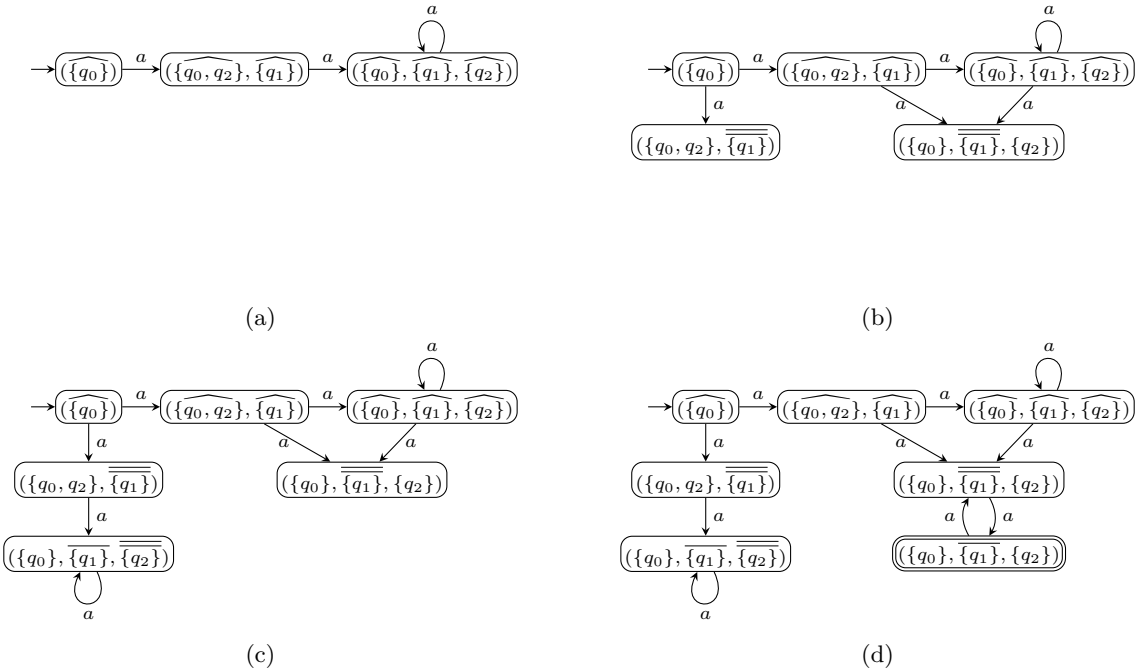
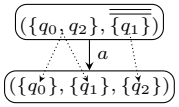


Figure 2.5: Selected steps of the construction of the lower part, starting with the upper part

In Figure 2.5 (b), we created the a-successors of the three states of the upper part. The structure of these new states, apart from the colours, is determined by the same method that we used for the upper part. The only difference in the construction of the lower part is that each component is assigned a colour. For both new states, the predecessor states do not contain any 2-coloured components, thus we only need to consider the colour rules in Figure 2.4 (a). Regarding the colour of the predecessor components, they all have colour -1, thus we have to use the rule in Figure 2.4 (a) line 1 for all the new components. In this way, the components $\{q_0, q_2\}$, $\{q_0\}$, and $\{q_2\}$ are assigned colour 0, because they are non-accepting, and component $\{q_1\}$ gets colour 2, because it is accepting.

Note how we have to keep track for each component of the lower part whether it is accepting or non-accepting, and which is its predecessor component in the predecessor state.

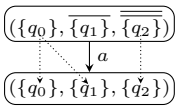
In Figure 2.5 (c), we added the a-successor to the state $(\{q_0, q_2\}, \overline{\overline{\{q_1\}}})$. Disregarding the colours, this state has the form $(\{q_0\}, \{q_1\}, \{q_2\})$. Its predecessor state $(\{q_0, q_2\}, \overline{\overline{\{q_1\}}})$ contains a 2-coloured component, thus we have to use the colour rules in Figure 2.4 (b). Now we need to know which are the predecessor components of the components in $(\{q_0\}, \{q_1\}, \{q_2\})$. This information is contained in the two slices of the reduced split tree that were used to determine the structure of the new state. For the case of our two states, the successor relation of their components is as follows:



The predecessor component of $\{q_0\}$ is $\{q_0, q_2\}$, which has colour 0. Thus, we have to use the rule in Figure 2.4 (b) line 1, and $\{q_0\}$ gets the colour 0, because it is non-accepting. The predecessor component of $\{q_1\}$ is also $\{q_0, q_2\}$, and we have to use the same rule. However, since $\{q_1\}$ is accepting, it gets colour 1. Note how the use of colour 1 here prevents the introduction of a further 2-coloured component before

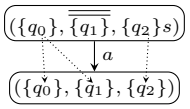
an already existing 2-coloured component has disappeared. The predecesor component of $\{q_2\}$ is the 2-coloured $\overline{\overline{\{q_1\}}}$, and thus, according to the rule in Figure 2.4 (b) line 3, $\{q_2\}$ also gets colour 2.

Next, still in Figure 2.5 (c), we create in turn the successor state of $(\{q_0\}, \overline{\{q_1\}}, \overline{\overline{\{q_2\}}})$. The structure of the components stays the same for the successor. The succesor relation of the two states is as follows:



The result is that $\{q_0\}$ gets colour 0, $\{q_1\}$ gets colour 1, and $\{q_2\}$ gets colour 2. Thus, the successor state is identical to the current state, and we add loop.

Figure 2.5 (d) includes the remaining for arriving at the final complement automaton. First, we created the a-successor of the state $(\{q_0\}, \overline{\overline{\{q_1\}}}, \{q_2\})$. The complement successor relation of this state with its successor is as follows:



According to the rules in Figure 2.4 (b), component $\{q_0\}$ gets colour 0, $\{q_1\}$ gets colour 1, and $\{q_2\}$ gets colour 0. This results in a new state, since the colours are different from the ones in the current state. Interesting here is that we have the case that a 2-coloured component "disappears". This is because the component $\overline{\overline{\{q_1\}}}$ has no successor component in the successor state. For the a-successor of the new state $(\{q_0\}, \overline{\{q_1\}}, \{q_2\})$, this means in turn that we have to use the rules in Figure 2.4 (a), what results in the already existing state $(\{q_0\}, \overline{\overline{\{q_1\}}}, \{q_2\})$.

At this point, all the states in the automaton have been processed, and the construction is therefore completed. The only thing that remains to be done is to determine the accepting states of the automaton. The rule is that each state of the lower part that does not contain any 2-coloured component is an accepting state. In our automaton this applies only to the state $(\{q_0\}, \overline{\{q_1\}}, \{q_2\})$, which is thus the only accepting state of the automaton.

It can be easily seen that the automaton in Figure 2.5 accepts the word a^ω, which is not accepted by the input automaton in Figure 2.2. Thus, the result of our construction is a correct complement of the input automaton.

A loose upper bound for the worst-case state complexity of the entire construction (including the construction of both, the upper and the lower part) has been calculated to be in $O((1.59n)^n)$, where n is the number of states of the input automaton [1]. This result is subject to refinement in ongoing research by the authors of the Fribourg construction. As mentioned, this publication ([1]) also contains a formal desciption and a proof of correctness of the construction.

2.3 Optimisations

There are several optimisations to the basic Fribourg construction, which all have the goal to reduce the size of the output automaton. These optimisations are like "add-ons" and can be added to the basic construction and combined in different ways.

In this section, we describe three of these optimisations. For easier reference, we define an abbreviation for each optimisation: R2C, M1, and M2. These optimisations will also be subject to the empirical performance investigation of the Fribourg construction, starting from the next chapter, and there we will use the same abbreviations.

2.3.1 R2C: Remove States with Rightmost 2-Coloured Components

The R2C optimisation can be summarised as follows:

> If the input automaton is complete, then states of the lower part whose rightmost component has colour 2 can be omitted.

This means that if during the construction of the lower part we determine a state that has a 2-coloured rightmost component, then we do not need to add this state to the automaton. This consequently also omits all the potential successors of this state from the automaton.

The reason for this is the following fact. If the input automaton is complete, then the rightmost component of an output-state *always* has at least one successor component (because of the completeness of the input automaton). However, this applies only to the rightmost component, as the successors of the other components might be omitted, due to the right-to-left precedence in right-to-left reduced split trees.

If the colour of the rightmost component is 2, then the successor component inherits this colour 2 (Figure 2.4 (b) line 3). This means inductively that all the successors of a state with a 2-coloured rightmost component will have a 2-coloured rightmost component. Or in other words, there is a 2-coloured component that will never disappear. Since states containing a 2-coloured component are non-accepting, these states form a cycle without a single accepting state, and this cycle can be removed without changing the language of the automaton.

Note, however, that the omission of these states can only be done if the input automaton is complete.

2.3.2 M1: Merge Components

The M1 optimisation allows the merging of adjacent components depending on their colour. With the merging of adjacent components, we mean the replacement of these components with their union (if regarding components as sets). The three merging rules are as follows.

1. Two adjacent 1-coloured components can be merged to a single 1-coloured component
2. Two adjacent 2-coloured components can be merged to a single 2-coloured component
3. A 2-coloured component adjacent to a 1-coloured component (in this order from left to right) can be merged to a single 2-coloured component

These mergings can be done recursively. This means that the result of a merging can again be subject to further merging, until nothing more can be merged. For example, the state $(\{q_0\}, \overline{\{q_1\}}, \overline{\{q_2\}}, \overline{\{q_3\}}, \overline{\{q_4\}}, \overline{\{q_5\}})$ can be transformed to $(\{q_0\}, \overline{\{q_1, q_2, q_3, q_4, q_5\}})$, by the recursively applying the above three merging rules.

Consequent applying of these mergings reduces the maximal number of states in the output automaton. An upper bound on the number of states in the lower part of the automaton has been calculated to be in $O((1.195n)^n)$, where n is the number of states of the input automaton [1]. This is considerably lower than the $O((1.59n)^n)$ worst-case state growth of the Fribourg construction without the M1 optimisation. A formal description and a proof of correctness of the M1 optimisation can be found in [1] (Section 4.E).

2.3.3 M2:

The M2 optimisation is the most involved of the three optimisations. Furthermore, it can only be applied together with the M1 optimisation. The main point of the M2 optimisation can be summarised as follows:

> Every state of the lower part contains *at most* one 2-coloured component.

The purpose of this restriction is to further reduce the maximal number of states the construction can generate (that is, to reduce the worst-case state complexity of the construction).

- When determining the colours of the components of a new state, the components must be processed from right to left
- The first component that according to the rules in Figure 2.4 deserves colour 2 gets colour 2 as usual
  - If this component has a sibling[5] to its left, then this sibling gets colour 2 as well

[5] With sibling we mean having the same predecessor component.

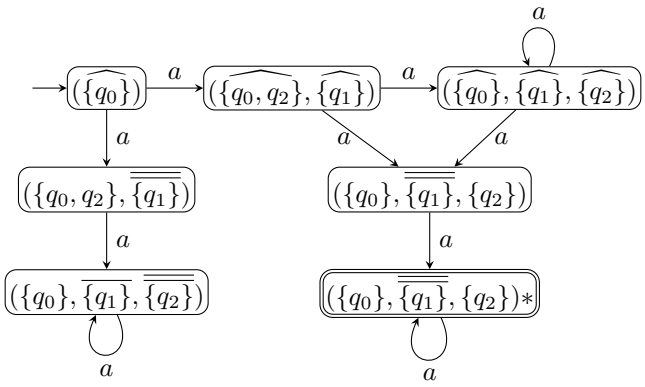


Figure 2.6: Result of applying the M2 optimisation to the complementation of the example automaton from Figure 2.2.

- From the point on where colour 2 has been assigned to a component (and possibly its sibling), all the further components that according to Figure 2.4 deserve colour 2 get colour 1 instead of colour 2

This leaves the new state in a situation where at most two components have colour 2, namely the first one that deserves it from the right, and its possible sibling. In the case that there was a sibling, the application of the M1 optimisation (which is necessary for the application of the M2 optimisation) merges these two components to a single 2-coloured component. This makes the state in any case having at most one 2-coloured component.

This modification of the construction requires further care to be taken when the only 2-coloured component of a state "disappears", that is, has no successor components in the state under construction. In this case, one of the 1-coloured components of this state is made accepting. In more detail, the points to consider are as follows.

- Disappearance of a 2-coloured component: uf the predecessor of a new state has a 2-coloured component, and if after the assignment of colours to the components of this new state, as described above, the new state has no 2-coloured component, then we say that the 2-coloured component of the predecesor state disappeared.
- When the disappearance of a 2-coloured component is detected, and if the state under construction contains at least one 1-coloured component, then the following is done:
  - The position where the successor component of the disappeared component *would* be is determined
  - Select the first 1-coloured component that is to the left of the first a 0-coloured component that is to the left of this position
    - If the search arrives at the leftmost component of the state, then continue it starting from the rightmost component
    - If there is exactly one 1-coloured component in the state, then directly select this component
  - Change the colour of the selected component from 1 to 2
  - Mark the state with a special mark (for example a $*$)
    - This mark distinguished states. That is, if two states have the same components with the same colours, but one has a mark and the other not, then they are two different states.

The result of this procedure is that the state has a single 2-coloured component again. However, as mentioned, this state must be accepting, even though it contains a 2-coloured component. It is the purpose of the mark to distinguish this state from other states with a 2-coloured component. The last thing we have to do is to change the acceptance condition to include all states of the lower part containing no 2-coloured components, plus all states having a mark.

In this case, the difference is the marked state $(\{q_0\}, \overline{\overline{\{q_1\}}}, \{q_2\})*$. It exists, because the 2-coloured component $\overline{\overline{\{q_1\}}}$ of the predecessor state disappears. According to the colour rules, the $\overline{\{q_1\}}$ in the new state then gets colour 1. This is where the construction without the M2 optimisation would left off. With the M2 optimisation, however, we have to select one of the 1-coloured components of the new state and change its colour to colour 2. There is only one 1-coloured component in the state, $\overline{\{q_1\}}$, and so we change its colour from 1 to 2. We also mark this state with a star. The successor of this marked state happens to be identical with itself (because the component $\overline{\overline{\{q_1\}}}$ again disappears), so we add a loop. In the end, the marked state is made accepting.

In [1] (Section 4.H) a very loose upper bound on the number of states of the lower part of $O((0.86n)^n)$ is proposed. This is a significant reduction from the upper bound of $O((1.195n)^n)$ that s achieved with the M1 optimisation. Furthermore, concrete calculations on the M2 optimisation suggest that the real upper bound might be as low as $O((0.76n)^n)$. This coincides with the established lower bound for Büchi complementation by Yan [55] and would make the construction optimal in the sense of worst-case state complexity. The actual complexity of the M2 optimisation is subject to further research by the authors of the Fribourg construction.

[1] Section 4.H

Appendix A

Plugin Installation and Usage

Since between the 2014–08–08 and 2014–11–17 releases of GOAL certain parts of the plugin interfaces have changed, and we adapted our plugin accordingly, the currently maintained version of the plugin works only with GOAL versions 2014-11-17 or newer. It is thus essential for any GOAL user to update to this version in order to use our plugin.

Appendix B

Median Complement Sizes of the GOAL Test Set

Bla bla bla

	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	269	308	254	236	238	297	266	156	207	68
1.2	960	1,407	1,479	2,150	1,152	1,090	942	1,206	718	104
1.4	3,426	2,915	2,752	3,393	2,693	3,265	2,263	2,425	1,844	154
1.6	3,799	3,698	4,901	3,926	3,960	3,655	2,580	1,905	2,124	155
1.8	3,375	3,169	3,420	3,967	3,943	3,132	2,246	1,144	971	114
2.0	1,906	2,261	2,383	2,884	2,354	2,096	1,169	932	568	98
2.2	1,467	1,633	1,795	1,942	1,611	1,640	569	499	330	78
2.4	924	1,232	1,319	1,317	1,056	886	514	314	182	59
2.6	625	763	880	945	828	684	316	175	132	44
2.8	483	584	836	690	575	395	240	151	103	41
3.0	319	450	557	523	367	313	155	116	84	32

(a) Fribourg

	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	269	308	254	236	238	297	266	156	207	68
1.2	960	1,407	1,479	2,150	1,152	1,090	942	1,206	718	104
1.4	3,426	2,915	2,752	3,393	2,693	3,265	2,263	2,425	1,844	154
1.6	3,799	3,698	4,901	3,926	3,960	3,655	2,580	1,905	2,124	155
1.8	3,375	3,169	3,420	3,967	3,943	3,093	2,246	1,144	971	114
2.0	1,906	2,184	2,383	2,818	2,354	1,989	1,127	885	568	97
2.2	1,410	1,561	1,639	1,884	1,609	1,588	496	464	284	78
2.4	884	1,200	1,234	1,184	939	806	373	256	165	55
2.6	575	731	815	860	751	575	246	162	114	43
2.8	431	530	672	466	371	274	174	120	85	36
3.0	232	325	344	360	269	169	91	85	53	27

(b) Fribourg+R2C

	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	390	438	434	324	328	459	337	204	227	40
1.2	1,576	2,394	2,505	2,996	1,613	1,551	1,166	1,542	1,002	58
1.4	5,007	4,336	4,652	4,877	3,458	3,956	3,169	3,380	1,868	86
1.6	5,067	5,032	6,444	4,868	4,575	3,864	3,211	1,731	1,892	85
1.8	4,016	3,701	3,647	4,523	3,548	3,009	1,808	451	336	62
2.0	1,663	2,276	2,676	3,035	1,925	1,932	464	307	150	54
2.2	989	1,514	1,621	1,826	1,121	846	155	127	93	45
2.4	560	821	919	771	529	267	133	87	55	32
2.6	388	519	524	441	259	219	84	50	41	26
2.8	311	317	396	242	165	95	64	44	33	22
3.0	173	224	211	169	102	72	41	34	27	18

(c) Fribourg+R2C+C

	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	225	223	195	181	187	199	189	124	161	68
1.2	731	971	946	1,071	629	562	488	568	388	104
1.4	2,228	1,701	1,543	1,732	1,241	1,287	945	944	727	154
1.6	2,489	2,263	2,331	2,133	1,777	1,443	964	757	889	155
1.8	2,381	2,027	2,009	2,075	1,618	1,243	1,005	592	515	114
2.0	1,390	1,569	1,416	1,573	1,093	1,008	594	464	330	98
2.2	1,118	1,197	1,150	1,151	879	809	317	330	241	78
2.4	712	885	836	809	580	535	316	231	145	59
2.6	498	569	601	627	497	412	217	137	113	44
2.8	391	455	578	456	374	263	173	119	90	41
3.0	258	350	392	354	253	208	119	97	74	32

(d) Fribourg+M1

	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	215	213	189	174	175	192	186	121	156	68
1.2	712	914	913	1,075	619	563	526	620	416	104
1.4	2,075	1,620	1,503	1,650	1,254	1,339	1,003	1,006	848	154
1.6	2,344	2,062	2,340	2,016	1,755	1,520	1,053	858	986	155
1.8	2,205	1,873	1,920	2,040	1,689	1,315	1,080	664	598	114
2.0	1,290	1,485	1,405	1,522	1,134	1,044	652	531	392	98
2.2	1,023	1,119	1,092	1,127	868	875	376	359	262	78
2.4	674	849	790	807	617	544	355	251	156	59
2.6	478	549	594	597	510	431	231	147	116	44
2.8	370	439	559	455	382	283	182	124	93	41
3.0	249	341	388	348	260	225	123	101	77	32

(e) Fribourg+M1+M2

	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	225	223	195	181	187	199	189	124	161	68
1.2	731	971	946	1,071	629	562	488	568	388	104
1.4	2,228	1,701	1,543	1,732	1,241	1,287	945	944	727	154
1.6	2,489	2,263	2,331	2,133	1,777	1,443	964	757	889	155
1.8	2,381	2,027	2,009	2,075	1,618	1,215	1,005	592	515	114
2.0	1,390	1,513	1,416	1,542	1,093	1,003	594	441	330	97
2.2	1,019	1,156	1,064	1,104	859	785	304	303	221	78
2.4	672	867	789	772	544	478	269	191	139	55
2.6	466	542	572	568	452	348	183	129	99	43
2.8	368	407	480	337	260	197	129	96	75	36
3.0	201	261	266	272	199	136	83	74	50	27

(f) Fribourg+M1+R2C

	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	329	303	279	240	229	288	230	157	160	40
1.2	988	1,392	1,356	1,352	751	741	608	704	516	58
1.4	2,939	2,581	2,066	2,190	1,351	1,622	1,132	1,261	932	86
1.6	3,150	2,900	2,842	2,218	1,885	1,563	1,177	821	896	85
1.8	2,782	2,485	2,047	2,180	1,625	1,269	855	395	309	62
2.0	1,338	1,638	1,544	1,566	979	957	349	261	147	54
2.2	838	1,125	993	1,027	667	521	153	125	93	45
2.4	494	700	624	524	296	214	126	87	55	32
2.6	327	434	383	334	212	163	82	50	41	26
2.8	283	273	305	202	144	95	60	44	33	22
3.0	164	200	173	142	92	72	41	34	27	18

(g) Fribourg+M1+R2C+C

	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	126	118	97	60	51	52	62	36	48	30
1.2	432	517	345	262	160	126	92	120	109	40
1.4	1,044	331	133	89	45	22	19	31	27	20
1.6	358	24	11	5	4	6	5	3	3	4
1.8	19	5	1	1	1	1	1	1	1	1
2.0	1	1	1	1	1	1	1	1	1	1
2.2	1	1	1	1	1	1	1	1	1	1
2.4	1	1	1	1	1	1	1	1	1	1
2.6	1	1	1	1	1	1	1	1	1	1
2.8	1	1	1	1	1	1	1	1	1	1
3.0	1	1	1	1	1	1	1	1	1	1

(h) Fribourg+R

Figure B.1: Median complement sizes of the 10,939 effective samples of the internal tests on the GOAL test set. The rows (1.0 to 3.0) are the transition densities, and the columns (0.1 to 1.0) are the acceptance densities.

	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	130	117	109	77	69	61	56	40	40	29
1.2	387	456	352	281	155	136	101	105	75	45
1.4	822	683	394	376	230	204	151	120	105	63
1.6	890	594	458	321	237	178	134	114	113	61
1.8	624	507	324	275	196	136	110	92	89	41
2.0	362	286	211	176	117	103	79	64	59	34
2.2	248	222	124	116	82	73	56	52	50	28
2.4	147	145	114	87	56	48	43	39	35	19
2.6	115	117	67	61	47	42	32	29	29	15
2.8	95	71	52	45	38	29	27	25	23	13
3.0	59	60	47	35	32	27	22	21	20	10

(a) Piterman+EQ+RO

	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	171	174	166	124	118	117	100	67	84	35
1.2	622	833	803	877	529	398	320	372	215	53
1.4	2,086	1,618	1,367	1,676	1,065	967	664	682	494	78
1.6	2,465	2,073	2,182	1,959	1,518	1,259	767	545	623	78
1.8	2,310	1,963	1,950	1,988	1,485	1,095	746	418	346	57
2.0	1,318	1,482	1,393	1,461	981	871	434	338	228	50
2.2	1,068	1,145	1,085	1,067	772	747	263	235	158	40
2.4	689	838	809	751	524	466	240	159	93	30
2.6	469	531	555	565	437	360	169	94	71	23
2.8	369	421	536	405	329	224	130	81	58	21
3.0	244	327	360	322	219	176	85	64	49	16

(b) Slice+P+RO+MADJ+EG

Figure B.2: Median complement sizes of the 10,998 effective samples of the external tests without the Rank construction. The rows (1.0 to 3.0) are the transition densities, and the columns (0.1 to 1.0) are the acceptance densities.

Appendix C

Execution Times

Construction	Mean	Min.	P25	Median	P75	Max.	Total	≈ hours
Fribourg	8.5	2.5	3.3	4.9	7.3	586.0	93,351.2	259
Fribourg+R2C	6.6	2.2	2.9	4.2	6.4	219.7	72,545.7	202
Fribourg+R2C+C	8.5	2.2	2.6	3.5	6.4	582.9	93,396.2	259
Fribourg+M1	4.9	2.5	3.2	4.1	5.9	55.1	54,061.3	150
Fribourg+M1+M2	4.6	2.2	2.9	3.8	5.1	38.4	49,848.0	138
Fribourg+M1+R2C	4.4	2.2	2.8	3.6	5.3	42.5	48,572.0	135
Fribourg+M1+R2C+C	5.6	2.5	3.2	4.0	6.5	147.4	60,918.9	169
Fribourg+R	7.5	2.2	3.0	3.9	6.3	470.5	82,387.3	229

Table C.1: Execution times in CPU time seconds for the 10,939 effectvie samples of the GOAL test set.

Construction	Mean	Min.	P25	Median	P75	Max.	Total	≈ hours
Piterman+EQ+RO	3.0	2.2	2.6	2.8	3.0	42.9	21,410.6	59
Slice+P+RO+MADJ+EG	3.7	2.2	2.7	3.2	4.1	36.7	26,398.9	73
Rank+TR+RO	16.0	2.3	2.8	3.7	9.3	443.3	115,563.9	321
Fribourg+M1+R2C	4.0	2.2	2.7	3.1	4.4	410.4	28,970.8	80

Table C.2: Execution times in CPU time seconds for the 7,204 effectvie samples of the GOAL test set.

Construction	Mean	Min.	P25	Median	P75	Max.	Total	≈ hours
Piterman+EQ+RO	3.6	2.2	2.7	2.9	3.4	365.7	39,663.4	110
Slice+P+RO+MADJ+EG	4.3	2.2	2.9	3.7	5.0	42.4	47,418.2	132
Fribourg+M1+R2C	4.7	2.2	2.8	3.6	5.3	410.4	52,149.0	145

Table C.3: Execution times in CPU time seconds for the 10,998 effectvie samples of the GOAL test set without the Rank construction.

Construction	Michel 1	Michel 2	Michel 3	Michel 4	Fitted curve	Std. error
Fribourg	2.3	4.0	88.8	100,976.0	$(1.14n)^n$	0.64%
Fribourg+R2C	2.3	3.4	27.4	27,938.3	$(0.92n)^n$	0.64%
Fribourg+M1	2.2	3.6	17.9	6,508.4	$(0.72n)^n$	0.63%
Fribourg+M1+M2	2.3	3.5	13.8	2,707.4	$(0.62n)^n$	0.62%
Fribourg+M1+M2+R2C	2.5	3.5	10.8	2,332.6	$(0.61n)^n$	0.62%
Fribourg+R	2.4	3.7	86.0	101,809.6	$(1.14n)^n$	0.64%

Table C.4: Execution times in CPU time seconds for the four Michel automata.

Construction	Michel 1	Michel 2	Michel 3	Michel 4	Fitted curve	Std. error
Piterman+EQ+RO	2.5	3.8	42.6	75,917.4	$(1.08n)^n$	0.64%
Slice+P+RO+MADJ+EG	2.3	3.6	11.4	159.5	$(0.39n)^n$	0.38%
Rank+TR+RO	2.2	3.0	6.4	30.0	$(0.29n)^n$	0.18%
Fribourg+M1+M2+R2C	2.5	3.5	10.8	2,332.6	$(0.61n)^n$	0.62%

Table C.5: Execution times in CPU time seconds for the four Michel automata.

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