

## ON A PROBLEM OF FORMAL LOGIC

By F. P. RAMSEY.

[Received 28 November, 1928.—Read 13 December, 1928.]

This paper is primarily concerned with a special case of one of the leading problems of mathematical logic, the problem of finding a regular procedure to determine the truth or falsity of any given logical formula\*. But in the course of this investigation it is necessary to use certain theorems on combinations which have an independent interest and are most conveniently set out by themselves beforehand.

## I.

The theorems which we actually require concern finite classes only, but we shall begin with a similar theorem about infinite classes which is easier to prove and gives a simple example of the method of argument.

**THEOREM A.** *Let  $\Gamma$  be an infinite class, and  $\mu$  and  $r$  positive integers; and let all those sub-classes of  $\Gamma$  which have exactly  $r$  members, or, as we may say, let all  $r$ -combinations of the members of  $\Gamma$  be divided in any manner into  $\mu$  mutually exclusive classes  $C_i$  ( $i = 1, 2, \dots, \mu$ ), so that every  $r$ -combination is a member of one and only one  $C_i$ ; then, assuming the axiom of selections,  $\Gamma$  must contain an infinite sub-class  $\Delta$  such that all the  $r$ -combinations of the members of  $\Delta$  belong to the same  $C_i$ .*

Consider first the case  $\mu = 2$ . (If  $\mu = 1$  there is nothing to prove.) The theorem is trivial when  $r$  is 1, and we prove it for all values of  $r$  by induction. Let us assume it, therefore, when  $r = \rho - 1$  and deduce it for  $r = \rho$ , there being, since  $\mu = 2$ , only two classes  $C_i$ , namely  $C_1$  and  $C_2$ .

---

\* Called in German the *Entscheidungsproblem*; see Hilbert und Ackermann, *Grundzüge der Theoretischen Logik*, 72–81.

It may happen that  $\Gamma$  contains a member  $x_1$  and an infinite sub-class  $\Gamma_1$ , not including  $x_1$ , such that the  $\rho$ -combinations consisting of  $x_1$  together with any  $\rho-1$  members of  $\Gamma_1$ , all belong to  $C_1$ . If so,  $\Gamma_1$  may similarly contain a member  $x_2$  and an infinite sub-class  $\Gamma_2$ , not including  $x_2$ , such that all the  $\rho$ -combinations consisting of  $x_2$  together with  $\rho-1$  members of  $\Gamma_2$ , belong to  $C_1$ . And, again,  $\Gamma_2$  may contain an  $x_3$  and a  $\Gamma_3$  with similar properties, and so on indefinitely. We thus have two possibilities: either we can select in this way two infinite sequences of members of  $\Gamma$  ( $x_1, x_2, \dots, x_n, \dots$ ), and of infinite sub-classes of  $\Gamma$  ( $\Gamma_1, \Gamma_2, \dots, \Gamma_n, \dots$ ), in which  $x_n$  is always a member of  $\Gamma_{n-1}$ , and  $\Gamma_n$  a sub-class of  $\Gamma_{n-1}$  not including  $x_n$ , such that all the  $\rho$ -combinations consisting of  $x_n$  together with  $\rho-1$  members of  $\Gamma_n$ , belong to  $C_1$ ; or else the process of selection will fail at a certain stage, say the  $n$ -th, because  $\Gamma_{n-1}$  (or if  $n=1$ ,  $\Gamma$  itself) will contain no member  $x_n$  and infinite sub-class  $\Gamma_n$  not including  $x_n$  such that all the  $\rho$ -combinations consisting of  $x_n$  together with  $\rho-1$  members of  $\Gamma_n$  belong to  $C_1$ . Let us take these possibilities in turn.

If the process goes on for ever let  $\Delta$  be the class ( $x_1, x_2, \dots, x_n, \dots$ ). Then all these  $x$ 's are distinct, since if  $r > s$ ,  $x_r$  is a member of  $\Gamma_{r-1}$  and so of  $\Gamma_{r-2}, \Gamma_{r-3}, \dots$ , and ultimately of  $\Gamma_s$  which does not contain  $x_s$ . Hence  $\Delta$  is infinite. Also all  $\rho$ -combinations of members of  $\Delta$  belong to  $C_1$ ; for if  $x_s$  is the term of such a combination with least suffix  $s$ , the other  $\rho-1$  terms of the combination belong to  $\Gamma_s$ , and so form with  $x_s$  a  $\rho$ -combination belonging to  $C_1$ .  $\Gamma$  therefore contains an infinite sub-class  $\Delta$  of the required kind.

Suppose, on the other hand, that the process of selecting the  $x$ 's and  $\Gamma$ 's fails at the  $n$ -th stage, and let  $y_1$  be any member of  $\Gamma_{n-1}$ . Then the  $(\rho-1)$ -combinations of members of  $\Gamma_{n-1}-(y_1)$  can be divided into two mutually exclusive classes  $C'_1$  and  $C'_2$  according as the  $\rho$ -combinations formed by adding to them  $y_1$  belong to  $C_1$  or  $C_2$ , and by our theorem (A), which we are assuming true when  $r = \rho-1$  (and  $\mu = 2$ ),  $\Gamma_{n-1}-(y_1)$  must contain an infinite sub-class  $\Delta_1$  such that all  $(\rho-1)$ -combinations of the members of  $\Delta_1$  belong to the same  $C'_i$ ; i.e. such that the  $\rho$ -combinations formed by joining  $y_1$  to  $\rho-1$  members of  $\Delta_1$  all belong to the same  $C_i$ . Moreover, this  $C_i$  cannot be  $C_1$ , or  $y_1$  and  $\Delta_1$  could be taken to be  $x_n$  and  $\Gamma_n$  and our previous process of selection would not have failed at the  $n$ -th stage. Consequently the  $\rho$ -combinations formed by joining  $y_1$  to  $\rho-1$  members of  $\Delta_1$  all belong to  $C_2$ . Consider now  $\Delta_1$  and let  $y_2$  be any of its members. By repeating the preceding argument  $\Delta_1-(y_2)$  must contain an infinite sub-class  $\Delta_2$  such that all the  $\rho$ -combinations got by joining  $y_2$  to  $\rho-1$  members of  $\Delta_2$  belong to the same  $C_i$ .

And, again, this  $C_i$  cannot be  $C_1$ , or, since  $y_2$  is a member and  $\Delta_2$  a sub-class of  $\Delta_1$  and so of  $\Gamma_{n-1}$  which includes  $\Delta_1$ ,  $y_2$  and  $\Delta_2$  could have been chosen as  $x_n$  and  $\Gamma_n$  and the process of selecting these would not have failed at the  $n$ -th stage. Now let  $y_3$  be any member of  $\Delta_2$ ; then  $\Delta_2 - (y_3)$  must contain an infinite sub-class  $\Delta_3$  such that all  $\rho$ -combinations consisting of  $y_3$  together with  $\rho-1$  members of  $\Delta_3$ , belong to the same  $C_i$ , which, as before, cannot be  $C_1$  and must be  $C_2$ . And by continuing in this way we shall evidently find two infinite sequences  $y_1, y_2, \dots, y_n, \dots$  and  $\Delta_1, \Delta_2, \dots, \Delta_n, \dots$  consisting respectively of members and sub-classes of  $\Gamma$ , and such that  $y_n$  is always a member of  $\Delta_{n-1}$ ,  $\Delta_n$  a sub-class of  $\Delta_{n-1}$  not including  $y_n$ , and all the  $\rho$ -combinations formed by joining  $y_n$  to  $\rho-1$  members of  $\Delta_n$  belong to  $C_2$ ; and if we denote by  $\Delta$  the class  $(y_1, y_2, \dots, y_n, \dots)$  we have, by a previous argument, that all  $\rho$ -combinations of members of  $\Delta$  belong to  $C_2$ .

Hence, in either case,  $\Gamma$  contains an infinite sub-class  $\Delta$  of the required kind, and Theorem A is proved for all values of  $r$ , provided that  $\mu = 2$ . For higher values of  $\mu$  we prove it by induction; supposing it already established for  $\mu = 2$  and  $\mu = \nu-1$ , we deduce it for  $\mu = \nu$ .

The  $r$ -combinations of members of  $\Gamma$  are then divided into  $\nu$  classes  $C_i$  ( $i = 1, 2, \dots, \nu$ ). We define new classes  $C'_i$  for  $i = 1, 2, \dots, \nu-1$  by

$$C'_i = C_i \quad (i = 1, 2, \dots, \nu-2),$$

$$C'_{\nu-1} = C_{\nu-1} + C_\nu.$$

Then by the theorem for  $\mu = \nu-1$ ,  $\Gamma$  must contain an infinite sub-class  $\Delta$  such that all  $r$ -combinations of the members of  $\Delta$  belong to the same  $C'_i$ . If, in this  $C'_i$ ,  $i \leq \nu-2$ , they all belong to the same  $C_i$ , which is the result to be proved; otherwise they all belong to  $C'_{\nu-1}$ , *i.e.* either to  $C_{\nu-1}$  or to  $C_\nu$ . In this case, by the theorem for  $\mu = 2$ ,  $\Delta$  must contain an infinite sub-class  $\Delta'$  such that the  $r$ -combinations of members of  $\Delta'$  either all belong to  $C_{\nu-1}$  or all belong to  $C_\nu$ ; and our theorem is thus established.

Coming now to finite classes it will save trouble to make some conventions as to notation. Small letters other than  $x$  and  $y$ , whether Italic or Greek (*e.g.*  $n, r, \mu, m$ ) will always denote finite cardinals, positive unless otherwise stated. Large Greek letters (*e.g.*  $\Gamma, \Delta$ ) will denote classes, and their suffixes will indicate the number of their members (*e.g.*  $\Gamma_m$  is a class with  $m$  members). The letters  $x$  and  $y$  will represent members of the classes  $\Gamma, \Delta$ , etc., and their suffixes will be used merely to distinguish them. Lastly, the letter  $C$  will stand, as before, for classes of combinations, and its suffixes will not refer to the

number of members, but serve merely to distinguish the different classes of combinations considered.

Corresponding to Theorem A we then have

**THEOREM B.** *Given any  $r$ ,  $n$ , and  $\mu$  we can find an  $m_0$  such that, if  $m \geq m_0$  and the  $r$ -combinations of any  $\Gamma_m$  are divided in any manner into  $\mu$  mutually exclusive classes  $C_i$  ( $i = 1, 2, \dots, \mu$ ), then  $\Gamma_m$  must contain a sub-class  $\Delta_n$  such that all the  $r$ -combinations of members of  $\Delta_n$  belong to the same  $C_i$ .*

This is the theorem which we require in our logical investigations, and we should at the same time like to have information as to how large  $m_0$  must be taken for any given  $r$ ,  $n$ , and  $\mu$ . This problem I do not know how to solve, and I have little doubt that the values for  $m_0$  obtained below are far larger than is necessary.

To prove the theorem we begin, as in Theorem A, by supposing that  $\mu = 2$ . We then take, not Theorem B itself, but the equivalent

**THEOREM C.** *Given any  $r$ ,  $n$ , and  $k$  such that  $n+k \geq r$ , there is an  $m_0$  such that, if  $m \geq m_0$  and the  $r$ -combinations of any  $\Gamma_m$  are divided into two mutually exclusive classes  $C_1$  and  $C_2$ , then  $\Gamma_m$  must contain two mutually exclusive sub-classes  $\Delta_n$  and  $\Delta_k$  such that all the combinations formed by  $r$  members of  $\Delta_n + \Delta_k$  which include at least one member from  $\Delta_n$  belong to the same  $C_i$ .*

That this is equivalent to Theorem B with  $\mu = 2$  is evident from the fact that, for any given  $r$ , Theorem C, for  $n$  and  $k$ , asserts more than Theorem B for  $n$ , but less than Theorem B for  $n+k$ .

The proof of Theorem C must be performed by mathematical induction, and can conveniently be set out as a demonstration that it is possible to define by recursion a function  $f(r, n, k)$  which will serve as  $m_0$  in the theorem.

If  $r = 1$ , the theorem is evidently true with  $m_0$  equal to the greater of  $2n-1$  and  $n+k$ , so that we may define

$$f(1, n, k) = \max(2n-1, n+k) \quad (n \geq 1, k \geq 0).$$

For other values of  $r$  we define  $f(r, n, k)$  by recursion formulae involving an auxiliary function  $g(r, n, k)$ . Suppose that  $f(r-1, n, k)$  has been defined for a certain  $r-1$ , and all  $n, k$  such that  $n+k \geq r-1$ , then we define it for  $r$  by putting

$$f(r, 1, k) = f(r-1, k-r+2, r-2) + 1 \quad (k+1 \geq r),$$

$$g(r, 0, k) = \max(r-1, k),$$

$$g(r, n, k) = f\{r, 1, g(r, n-1, k)\} \quad (n \geq 1),$$

$$f(r, n, k) = f\{r, n-1, g(r, n, k)\} \quad (n > 1).$$

These formulae can be easily seen to define  $f(r, n, k)$  for all positive values of  $r, n$  and  $k$  satisfying  $n+k \geq r$ , and  $g(r, n, k)$  for all values of  $r$  greater than 1, and all positive values of  $n$  and  $k$ ; and we shall prove that Theorem C is true when we take  $m_0$  to be this  $f(r, n, k)$ . We know that this is so when  $r=1$ , and we shall therefore assume it for all values up to  $r-1$  and deduce it for  $r$ .

When  $n=1$ , and  $m \geq m_0 = f(r-1, k-r+2, r-2)+1$ , we may take any member  $x$  of  $\Gamma_m$  to be sole member of  $\Delta_1$  and there remain at least  $f(r-1, k-r+2, r-2)$  members of  $\Gamma_m - (x)$ ; the  $(r-1)$ -combinations of these members of  $\Gamma_m - (x)$  can be divided into classes  $C'_1$  and  $C'_2$  according as they belong to  $C_1$  or  $C_2$  when  $x$  is added to them, and, by our theorem for  $r-1$ ,  $\Gamma_m - (x)$  must contain two mutually exclusive classes  $\Delta_{k-r+2}, \Lambda_{r-2}$  such that every combination of  $r-1$  terms from  $\Delta_{k-r+2} + \Lambda_{r-2}$  (since one of its terms must come from  $\Delta_{k-r+2}, \Lambda_{r-2}$  having only  $r-2$  members) belongs to the same  $C'_i$ . Taking  $\Lambda_k$  to be this  $\Delta_{k-r+2} + \Lambda_{r-2}$  all combinations consisting of  $x$ , together with  $r-1$  members of  $\Lambda_k$ , belong to the same  $C_i$ . The theorem is therefore true for  $r$  when  $n=1$ .

For other values of  $n$  we prove it by induction, assuming it for  $n-1$  and deducing it for  $n$ . Taking

$$m \geq m_0 = f(r, n, k) = f\{r, n-1, g(r, n, k)\},$$

$\Gamma_m$  must, by the theorem for  $n-1$ , contain a  $\Delta_{n-1}$  and a  $\Lambda_{g(r, n, k)}$  such that every combination of  $r$  members of  $\Delta_{n-1} + \Lambda_{g(r, n, k)}$ , at least one term of which comes from  $\Delta_{n-1}$ , belongs to the same  $C_i$ , say to  $C_1$ . If, now,  $\Lambda_{g(r, n, k)}$  contains a member  $x$  and a sub-class  $\Lambda_k$  not including  $x$ , such that every combination of  $x$  and  $r-1$  members of  $\Lambda_k$  belongs to  $C_1$ , then, taking  $\Delta_n$  to be  $\Delta_{n-1} + (x)$  and  $\Lambda_k$  to be this  $\Lambda_k$ , our theorem is true. If not, there can be no member of  $\Lambda_{g(r, n, k)}$  which has a sub-class of  $k$  members of  $\Lambda_{g(r, n, k)}$  connected with it in this way. But since

$$g(r, n, k) = f\{r, 1, g(r, n-1, k)\},$$

$\Lambda_{g(r, n, k)}$  must contain a member  $x_1$  and a sub-class  $\Lambda_{g(r, n-1, k)}$ , not including  $x_1$ , such that  $x_1$  combined with any  $r-1$  members of  $\Lambda_{g(r, n-1, k)}$  gives a combination belonging to the same  $C_i$ , which cannot be  $C_1$ , or  $x_1$  and any  $k$  members of  $\Lambda_{g(r, n-1, k)}$  could have been taken as the  $x$  and  $\Lambda_k$  above. Hence the combinations formed by  $x_1$  together with any  $r-1$  members of  $\Lambda_{g(r, n-1, k)}$  all belong to  $C_2$ . But now

$$g(r, n-1, k) = f\{r, 1, g(r, n-2, k)\},$$

and  $\Lambda_{g(r, n-1, k)}$  must contain an  $x_2$  and a  $\Lambda_{g(r, n-2, k)}$ , not including  $x_2$ , such that the combinations formed by  $x_2$  and  $r-1$  members of  $\Lambda_{g(r, n-2, k)}$  all

belong to the same  $C_i$ , which must, as before, be  $C_2$ , since  $x_2$  and  $\Lambda_{g(r, n-2, k)}$  are both contained in  $\Lambda_{g(r, n, k)}$  and  $g(r, n-2, k) \geq k$ . Continuing in this way we can find  $n$  distinct terms  $x_1, x_2, \dots, x_n$  and a  $\Lambda_{g(r, 0, k)}$  such that every combination of  $r$  terms from  $(x_1, x_2, \dots, x_n) + \Lambda_{g(r, 0, k)}$  belongs to  $C_2$ , provided that at least one term of the combination comes from  $(x_1, x_2, \dots, x_n)$ . Since  $g(r, 0, k) \geq k$  this proves our theorem, taking  $\Delta_n$  to be  $(x_1, x_2, \dots, x_n)$  and  $\Lambda_k$  to be any  $k$  terms of  $\Lambda_{g(r, 0, k)}$ .

Theorem C is therefore established for all values of  $r, n$ , and  $k$ , with  $m_0$  equal to  $f(r, n, k)$ . It follows that, if  $\mu = 2$ , Theorem B is true for all values of  $r$  and  $n$  with  $m_0$  equal to  $f(r, n-r+1, r-1)$ , which we shall also call  $h(r, n, 2)$ .

For other values of  $\mu$  we prove Theorem B by induction, taking  $m_0$  to be  $h(r, n, \mu)$ , where

$$h(r, n, 2) = f(r, n-r+1, r-1)$$

$$h(r, n, \mu) = h\{r, h(r, n, \mu-1), 2\} \quad (\mu > 2).$$

For, assuming the theorem for  $\mu-1$ , we prove it for  $\mu$  by defining new classes of combinations

$$C'_1 = C_1,$$

$$C'_2 = \sum_{i=2}^{\mu} C_i.$$

If then  $m \geq h(r, n, \mu) = h\{r, h(r, n, \mu-1), 2\}$ , by the theorem for  $\mu = 2$ ,  $\Gamma_m$  must contain a  $\Gamma_{h(r, n, \mu-1)}$  the  $r$ -combinations of whose members belong either all to  $C'_1$  or all to  $C'_2$ . In the first case there is no more to prove; in the second we have only to apply the theorem for  $\mu-1$  to  $\Gamma_{h(r, n, \mu-1)}$ .

In the simplest case in which  $r = \mu = 2$  the above reasoning gives  $m_0$  equal to  $h(2, n, 2)$ , which is easily shown to be  $2^{n(n-1)/2}$ . But for this case there is a simple argument which gives the much lower value  $m_0 = n!$ , and shows that our value  $h(r, n, \mu)$  is altogether excessive.

For, taking Theorem C first, we can prove by induction with regard to  $n$  that, for  $r = 2$ , we may take  $m_0$  to be  $k \cdot (n+1)!$ . ( $k$  is here supposed greater than or equal to 1.) For this is true when  $n = 1$ , since, if  $m \geq 2k$ , of the  $m-1$  pairs obtained by combining any given member of  $\Gamma_m$  with the others, at least  $k$  must belong to the same  $C_i$ . Assuming it, then, for  $n-1$ , let us prove it for  $n$ .

If  $m \geq k \cdot (n+1)! = k(n+1) \cdot n!$ ,  $\Gamma_m$  must, by the theorem for  $n-1$ , contain two mutually exclusive sub-classes  $\Delta_{n-1}$  and  $\Lambda_{k(n+1)}$  such that all pairs from  $\Delta_{n-1} + \Lambda_{k(n+1)}$ , at least one term of which comes from  $\Delta_{n-1}$ , belong to the same  $C_i$ , say  $C_1$ . Now consider the members of  $\Lambda_{k(n+1)}$ ; in

the first place, there may be one of these,  $x$  say, which is such that there are  $k$  other members of  $\Lambda_{k(n+1)}$  which combined with  $x$  give pairs belonging to  $C_1$ . If so, the theorem is true, taking  $\Delta_n$  to be  $\Delta_{n-1} + (x)$ ; if not, let  $x_1$  be any member of  $\Lambda_{k(n+1)}$ . Then there are at most  $k-1$  other members of  $\Lambda_{k(n+1)}$  which combined with  $x_1$  give pairs belonging to  $C_1$ , and  $\Lambda_{k(n+1)} - (x_1)$  must contain a  $\Lambda_{kn}$  any member of which gives when combined with  $x_1$  a pair belonging to  $C_2$ . Let  $x_2$  be any member of  $\Lambda_{kn}$ , then, since  $x_2$  and  $\Lambda_{kn}$  are both contained in  $\Lambda_{k(n+1)}$ , there are at most  $k-1$  other members of  $\Lambda_{kn}$  which when combined with  $x_2$  give pairs belonging to  $C_1$ . Hence  $\Lambda_{kn} - (x_2)$  contains a  $\Lambda_{k(n-1)}$  any member of which combined with  $x_2$  gives a pair belonging to  $C_2$ . Continuing in this way we obtain  $x_1, x_2, \dots, x_n$  and  $\Lambda_k$ , such that every pair  $x_i, x_j$  and every pair consisting of an  $x_i$  and a member of  $\Lambda_k$  belongs to  $C_2$ . Theorem C is therefore proved.

Theorem B for  $n$  then follows, with the  $m_0$  of Theorem C for  $n-1$  and 1, *i.e.* with  $m_0$  equal to  $n!^*$ ; and it is an easy extension to show that, if in Theorem B  $r=2$  but  $\mu \neq 2$ , we can take  $m_0$  to be  $n!!! \dots$ , where the process of taking the factorial is performed  $\mu-1$  times.

## II.

### Predicate logic

We shall be concerned with logical formulae containing variable propositional functions, *i.e.* predicates or relations, which we shall denote by Greek letters  $\phi, \chi, \psi$ , etc. These functions have as arguments individuals denoted by  $x, y, z$ , etc., and we shall deal with functions with any finite number of arguments, *i.e.* of any of the forms

$$\phi(x), \chi(x, y), \psi(x, y, z), \dots$$

In addition to these variable functions we shall have the one constant function of identity  $x = y$  or  $=(x, y)$ .

By operating on the values of  $\phi, \chi, \psi, \dots$ , and  $=$  with the logical operations

$\sim$	meaning	<i>not</i> ,
$\vee$	„	<i>or</i> ,
$\cdot$	„	<i>and</i> ,
$(x)$	„	<i>for all</i> $x$ .
$(Ex)$	„	<i>there is an</i> $x$ <i>for which</i> ,

---

\* But this value is, I think, still much too high. It can easily be lowered slightly even when following the line of argument above, by using the fact that if  $k$  is even it is impossible for every member of an odd class to have exactly  $k-1$  others with which it forms a pair of  $C_1$ , for then twice the number of these pairs would be odd; we can thus start when  $k$  is even with a  $\Lambda_{k(n+1)-1}$  instead of a  $\Lambda_{k(n+1)}$

we can construct expressions such as

$$[(x, y) \{ \phi(x, y) \vee x = y \} ] \vee \{ (Ez) \chi(z) \}$$

in which all the individual variables are made "apparent" by prefixes  $(x)$  or  $(Ex)$ , and the only real variables left are the functions  $\phi, \chi, \dots$ . Such an expression we shall call a *first order formula*.

If such a formula is true for all interpretations\* of the functional variables  $\phi, \chi, \psi$ , etc., we shall call it *valid*, and if it is true for no interpretations of these variables we shall call it *unsatisfiable*. If it is true for some interpretations (whether or not for all) we shall call it *satisfiable* †. *consistent* ‡.

The *Entscheidungsproblem* is to find a procedure for determining whether any given formula is valid, or, alternatively, whether any given formula is consistent; for these two problems are equivalent, since the necessary and sufficient condition for a formula to be consistent is that its contradictory should not be valid. We shall find it more convenient to take the problem in this second form as an investigation of *consistency*. The consistency of a formula may, of course, depend on the number of individuals in the universe considered, and we shall have to distinguish between formulae which are consistent in every universe and those which are only consistent in universes with some particular numbers of members. Whenever the universe is infinite we shall have to assume the axiom of selections.

The problem has been solved by Behmann‡ for formulae involving only functions of one variable, and by Bernays and Schönfinkel§ for formulae involving only two individual apparent variables. It is solved below for the further case in which, when the formula is written in "normal form"; there are any number of prefixes of generality  $(x)$  but none of existence  $(Ex)$ ||. By "normal form"¶ is here meant that all the prefixes stand at the beginning, with no negatives between or in front of them, and have scopes extending to the end of the formula.

\* To avoid confusion we call a constant function substituted for a variable  $\phi$ , not a value but an *interpretation* of  $\phi$ ; the *values* of  $\phi(x, y, z)$  are got by substituting constant individuals for  $x, y$ , and  $z$ .

† German *erfüllbar*.

‡ H. Behmann, "Beiträge zur Algebra der Logik und zum Entscheidungsproblem", *Math. Annalen*, 86 (1922), 163-229.

§ P. Bernays und M. Schönfinkel, "Zum Entscheidungsproblem der mathematischen Logik", *Math. Annalen*, 99 (1928), 342-372. These authors do not, however, include identity in the formulae they consider.

|| Later we extend our solution to the case in which there are also prefixes of existence provided that these all precede all the prefixes of generality.

¶ Hilbert und Ackermann, *op. cit.*, 63-4.



The formulae to be considered are thus of the form

$$(x_1, x_2, \dots, x_n)F(\phi, \chi, \psi; \dots, =, x_1, x_2, \dots, x_n),$$

where the matrix  $F$  is a truth-function of values of the functions  $\phi, \chi, \psi$ , etc., and  $=$  for arguments drawn from  $x_1, x_2, \dots, x_n$ .

This type of formula is interesting as being the general type of an axiom system consisting entirely of "general laws"\*. The axioms for order, betweenness, and cyclic order are all of this nature, and we are thus attempting a general theory of the consistency of axiom systems of a common, if very simple, type.

If identity does not occur in  $F$  the problem is trivial, since in this case whether the formula is consistent or not can be shown to be independent of the number of individuals in the universe, and we have only the easy task of testing it for a universe with one member only†.

But when we introduce identity the question becomes much more difficult, for although it is still obvious that if the formula is consistent in a universe  $U$  it must be consistent in any universe with fewer members than  $U$ , yet it may easily be consistent in the smaller universe but not in the larger. For instance,

$$(x_1, x_2)[x_1 = x_2 \vee \{\phi(x_1) \cdot \sim \phi(x_2)\}]$$

is consistent in a universe with only one member but not in any other.

We begin our investigation by expressing  $F$  in a special form.  $F$  is a truth-function of the values of  $\phi, \chi, \psi, \dots$ , and  $=$  for arguments drawn from  $x_1, x_2, \dots, x_n$ . If  $\phi$  is a function of  $r$  variables there will be  $n^r$ ‡ values of  $\phi$  which can occur in  $F$ , and  $F$  will be a truth-function of  $\Sigma n^r$  values of  $\phi, \chi, \psi, \dots$ , and  $=$ , which we shall call *atomic propositions*. With regard to these  $\Sigma n^r$  atomic propositions there are  $2^{\Sigma n^r}$  possibilities of truth and falsity which we shall call *alternatives*, each alternative being a conjunction of  $\Sigma n^r$  propositions which are either atomic propositions or their contradictions. In constructing the alternatives all the  $\Sigma n^r$  atomic propositions are to be used whether or not they occur in  $F$ .  $F$  can then be expressed as a disjunction of some of these alternatives, namely those with which it is compatible. It is well known that such an

\* C. H. Langford, "Analytic completeness of postulate sets", *Proc. London Math. Soc.* (2), 25 (1926), 115-6.

† Bernays und Schönfinkel, *op. cit.*, 359. We disregard altogether universes with no members.

‡ Here and elsewhere numbers are given not because they are relevant to the argument, but to enable the reader to check that he has in mind the same class of entities as the author.

expression is possible; indeed, it is the dual of what Hilbert and Ackermann call the "ausgezeichnete konjunktive Normalform"\*, and is fundamental also in Wittgenstein's logic. The only exception is when  $F$  is a self-contradictory truth-function, in which case our formula is certainly not consistent.

$F$  having been thus expressed as a disjunction of alternatives (in our special sense of the word), our next task is to show that some of these alternatives may be able to be removed without affecting the consistency or inconsistency of the formula. If all the alternatives can be removed in this way the formula will be inconsistent; otherwise we shall have still to consider the alternatives that remain.

In the first place an alternative may violate the laws of identity by containing parts of any of the following forms:—

$$x_i \neq x_i^\dagger,$$

$$x_i = x_j \cdot x_j \neq x_i \quad (i \neq j),$$

$$x_i = x_j \cdot x_j = x_k \cdot x_i \neq x_k \quad (i \neq j, j \neq k, k \neq i),$$

or by containing  $x_i = x_j$  ( $i \neq j$ ) and values of a function  $\phi$  and its contradictory  $\sim \phi$  for sets of arguments which become the same when  $x_i$  is substituted for  $x_j$  [e.g.  $x_1 = x_2 \cdot \phi(x_1, x_2, x_3) \cdot \sim \phi(x_2, x_1, x_3)$ ].

Any alternative which violates these laws must always be false and can evidently be discarded without affecting the consistency of the formula. The remaining alternatives can then be classified according to the number of  $x$ 's they make to be different, which may be anything from 1 up to  $n$ .

Suppose that for a given alternative this number is  $v$ , then we can derive from it what we will call the corresponding  $y$  alternative by the following process:—

For  $x_1$ , wherever it occurs in the given alternative, write  $y_1$ ; next, if in the alternative  $x_2 = x_1$ , for  $x_2$  write  $y_1$  again, if not for  $x_2$  write  $y_2$ . In general, if  $x_i$  is in the given alternative identical with any  $x_j$  with  $j$  less than  $i$ , write for  $x_i$  the  $y$  previously written for  $x_j$ ; otherwise write for  $x_i$ ,  $y_{k+1}$ , where  $k$  is the number of  $y$ 's already introduced. The expression which results contains  $v$   $y$ 's all different instead of  $n$   $x$ 's, some of which are identical, and we shall call it the  $y$  alternative corresponding to the given  $x$  alternative.

\* *Op. cit.*, 16.

† We write  $x \neq y$  for  $\sim (x = y)$ .

Thus to the alternative

$$\phi(x_1) \cdot \sim \phi(x_2) \cdot \phi(x_3) \cdot \sim \phi(x_4) \cdot x_1 = x_3 \cdot x_2 = x_4 \cdot x_1 \neq x_2^*$$

corresponds the  $y$  alternative

$$\phi(y_1) \cdot \sim \phi(y_2) \cdot y_1 \neq y_2.$$

We call two  $y$  alternatives *similar* if they contain the same number of  $y$ 's and can be derived from one another by permuting those  $y$ 's, and we call two  $x$  alternatives *equivalent* if they correspond to similar (or identical)  $y$  alternatives.

Thus

$$\phi(x_1) \cdot \sim \phi(x_2) \cdot \phi(x_3) \cdot \sim \phi(x_4) \cdot x_1 = x_3 \cdot x_2 = x_4 \cdot x_1 \neq x_2, \quad (\alpha)$$

is equivalent to

$$\sim \phi(x_1) \cdot \phi(x_2) \cdot \phi(x_3) \cdot \phi(x_4) \cdot x_1 \neq x_2 \cdot x_2 = x_3 = x_4, \quad (\beta)$$

since they correspond to the similar  $y$  alternatives

$$\phi(y_1) \cdot \sim \phi(y_2) \cdot y_1 \neq y_2$$

$$\sim \phi(y_1) \cdot \phi(y_2) \cdot y_1 \neq y_2$$

derivable from one another by interchanging  $y_1$  and  $y_2$ , although  $(\alpha)$  and  $(\beta)$  are not so derivable by permuting the  $x$ 's.

We now see that we can discard any alternative contained in  $F$  unless  $F$  also contains all the alternatives equivalent to it; *e.g.* if  $F$  contains  $(\alpha)$  but not  $(\beta)$ ,  $(\alpha)$  may be discarded from it. For omitting alternatives clearly cannot make the formula consistent if it was not so before; and we can easily prove that, if it was consistent before, omitting *these* alternatives cannot make it inconsistent.

For suppose that the formula is consistent, *i.e.* that for some particular interpretation of  $\phi, \chi, \psi, \dots$ ,  $F$  is true for every set of  $x$ 's, and let  $p$  be an alternative contained in  $F$ ,  $q$  an alternative equivalent to  $p$  but not contained in  $F$ . Then for every set of  $x$ 's one and only one alternative in  $F$  will (on this interpretation of  $\phi, \chi, \psi, \dots$ ) be the true one, and this alternative can never be  $p$ . For if it were  $p$ , the corresponding  $y$  alternative would be true for some set of  $y$ 's, and the similar  $y$  alternative corresponding to  $q$  would be true for a set of  $y$ 's got by permuting this last set. Giving the  $x$ 's suitable values in terms of the  $y$ 's,  $q$  would then

---

\* We take one function of one variable only for simplicity; also to save space we omit expressions which may be taken for granted, such as  $x_1 = x_1, x_1 \neq x_1$ .

be true for a certain set of  $x$ 's and  $F$  would be false for these  $x$ 's contrary to hypothesis. Hence  $p$  is never the true alternative and may be omitted without affecting the consistency of the formula.

When we have discarded all these alternatives from  $F$ , the remainder will fall into sets each of which is the complete set of all alternatives equivalent to a given alternative. To such a set of  $x$  alternatives will correspond a complete set of similar  $y$  alternatives, and the disjunction of such a complete set of similar  $y$  alternatives (*i.e.* of all permutations of a given  $y$  alternative) we shall call a *form\**. A form containing  $v$   $y$ 's we shall denote by an *Italic capital* with suffix  $v$ , *e.g.*  $A_v$ ,  $B_v$ .

The force of our formula can now be represented by the following conjunction, which we shall call  $P$ .

$$\left. \begin{array}{l} \text{For every } y_1, \quad A_1 \text{ or } B_1 \text{ or } \dots \\ \text{For every distinct } y_1, y_2, \quad A_2 \text{ or } B_2 \text{ or } \dots \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ \text{For every distinct } y_1, y_2, \dots, y_v, \quad A_v \text{ or } B_v \text{ or } \dots \\ \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \quad \dots \\ \text{For every distinct } y_1, y_2, \dots, y_n, \quad A_n \text{ or } B_n \text{ or } \dots \end{array} \right\} (P)$$

where  $A_v$ ,  $B_v$ , etc., are the forms corresponding to the  $x$  alternatives still remaining in  $F$ . If for any  $v$  there are no such forms, *i.e.* if no alternatives with  $v$  different  $x$ 's remain in  $F$ , our formula implies that there are no such things as  $v$  distinct individuals, and so cannot be consistent in a world of  $v$  or more members.

We have now to define what is meant by saying that one form is *involved* in another. Consider a form  $A_v$  and take one of the  $y$  alternatives contained in it. This  $y$  alternative is a conjunction of the values of  $\phi$ ,  $\chi$ ,  $\psi$ , ..., and their negatives for arguments drawn from  $y_1, y_2, \dots, y_v$ . (We may leave out the values of identity and difference, since it is taken for granted that  $y$ 's are always different.) If  $\mu < v$  we can select  $\mu$  of these  $y$ 's in any way and leave out from the alternative all the terms in it which contain any of the  $v - \mu$   $y$ 's not selected. We have left an alternative in  $\mu$   $y$ 's which we can renumber  $y_1, y_2, \dots, y_\mu$ , and the form  $E_\mu$  to which this new alternative belongs we shall describe as being *involved* in the  $A_v$  with which we started. Starting with one particular  $y$  alternative in  $A_v$  we shall get a large number of different  $E_\mu$ 's by

\* Cf. Langford, *op. cit.*, 116-120.

† The notation is partially misleading, since  $A_v$  has no closer relation to  $A_\mu$  than to  $B_\mu$ .

choosing differently the  $\mu$   $y$ 's which we select to preserve; and from whichever  $y$  alternative in  $A_\nu$  we start, the  $E_\mu$ 's which we find to be involved in  $A_\nu$  will be the same.

For example,

$$\{\phi(y_1, y_1) \cdot \phi(y_1, y_2) \cdot \phi(y_2, y_1) \cdot \sim \phi(y_2, y_2)\} \\ \vee \{\sim \phi(y_1, y_1) \cdot \phi(y_1, y_2) \cdot \phi(y_2, y_1) \cdot \phi(y_2, y_2)\}$$

is a form  $A_2$  which involves the two  $E_1$ 's

$$\phi(y_1, y_1) \\ \sim \phi(y_1, y_1).$$

It is clear that if for some distinct set of  $\nu$   $y$ 's a form  $A_\nu$  is true, then every form  $E_\mu$  involved in  $A_\nu$  will be true for some distinct set of  $\mu$   $y$ 's contained in the  $\nu$ .

We are now in a position to settle the consistency or inconsistency of our formula when  $N$ , the number of individuals in the universe, is less than or equal to  $n$ , the number of  $x$ 's in our formula. In fact, if  $N \leq n$ , it is necessary and sufficient for the consistency of the formula that  $P$  should contain a form  $A_N$  together with all the forms  $E_\mu$  involved in it for every  $\mu$  less than  $N$ .

This condition is evidently necessary, since the  $N$  individuals in the universe must, taken as  $y_1, y_2, \dots, y_N$ , have some form  $A_N$  in regard to any  $\phi, \chi, \psi, \dots$ ; and all forms involved in this  $A_N$  must be true for different selections of  $y$ 's, and so contained in  $P$  if  $P$  is to be true for this  $\phi, \chi, \psi, \dots$ .

Conversely, suppose that  $P$  contains a form  $A_N$  together with all forms involved in  $A_N$ ; then, calling the  $N$  individuals in the universe  $y_1, y_2, \dots, y_N$ , we can define functions  $\phi, \chi, \psi, \dots$  to make any assigned  $y$  alternative in  $A_N$  true; for any permutation of these  $N$   $y$ 's another alternative in  $A_N$  will be the true one, and for any subset of  $y$ 's some  $y$  alternative in a form involved in  $A_N$ . Since all these  $y$  alternatives are by hypothesis contained in  $P$ ,  $P$  will be true for these  $\phi, \chi, \psi, \dots$ , and our formula consistent.

When, however,  $N > n$  the problem is not so simple, although it clearly depends on the  $A_n$ 's in  $P$  such that all forms involved in them are also contained in  $P$ . These  $A_n$ 's we may call *completely contained* in  $P$ , and if there are no such  $A_n$ 's a similar argument to that used when  $N \leq n$  will show that the formula is inconsistent. But the converse argument, that if there is an  $A_n$  completely contained in  $P$  the formula must be consistent, no longer holds good; and to proceed further we have to introduce a new conception, the conception of a form being *serial*.

But before proceeding to explain this idea it is best to simplify matters by the introduction of new functions. Let  $\phi$  be one of the variable functions in our formula, with, say,  $r$  arguments. Then, if  $r < n$ ,  $\phi$  will occur in  $P$  with all its arguments different [e.g.  $\phi(y_1, y_2, \dots, y_r)$ ] and also with some of them the same [e.g.  $\phi(y_1, y_2, \dots, y_{r-1}, y_1)$ ]; but we can conveniently eliminate values of the second kind by introducing new functions of fewer arguments than  $r$ , which, when all their arguments are different, take values equivalent to those of  $\phi$  with some of its arguments identical.

E.g. we may put

$$\phi_1(y_1, y_2, \dots, y_{r-1}) = \phi(y_1, y_2, \dots, y_{r-1}, y_1).$$

In this way  $\phi$  gives rise to a large number of functions with fewer arguments; each of these functions we define only for the case in which all its arguments are different, as is secured by these arguments being  $y$ 's with different suffixes. If  $r > n$ , there is no difference except that  $\phi$  can never occur with all its arguments different, and so is entirely replaced by the new functions.

If we do this for all the functions  $\phi, \chi, \psi, \dots$ , and replace them by new functions wherever they occur in  $P$  with some of their arguments the same,  $P$  will contain a new set of variable functions (including all the old ones which have no more than  $n$  arguments), and these will never occur in  $P$  with the same argument repeated.

It is easy to see that this transformation does not affect the consistency of the formula, for, if it were consistent before, it must be consistent afterwards, since the new functions have simply to be replaced by their definitions. And if it is consistent afterwards it must have been so before, since any function of the old set has only to be given for any set of arguments the value of the appropriate function of the new set\*.

---

\* For instance, if  $\phi(y_1, y_2, y_3)$  is a function of the old set, we have five new functions

$$\begin{aligned}\phi_0(y_1, y_2, y_3) &= \phi(y_1, y_2, y_3), \\ \chi_0(y_1, y_2) &= \phi(y_1, y_1, y_2), \\ \psi_0(y_1, y_2) &= \phi(y_1, y_2, y_1), \\ \pi_0(y_1, y_2) &= \phi(y_1, y_1, y_2), \\ \rho_0(y_1) &= \phi(y_1, y_1, y_1),\end{aligned}$$

and any value of  $\phi$  is equivalent to a value of one and only one of the new functions. It must be remembered that the new functions are used only with all their arguments different; for otherwise they would not be independent, since we should have, for instance,  $\chi_0(y_1, y_1)$  equivalent to  $\rho_0(y_1)$ . But  $\chi_0(y_1, y_1)$  never occurs, and  $\phi(y_1, y_1, y_1)$  is equivalent not to any value of  $\chi_0$  but only to  $\rho_0(y_1)$ .

In view of this fact we shall find it more convenient to take  $P$  in its new form, and denote the new set of functions by  $\phi_0, \chi_0, \psi_0, \dots$

Suppose, then, that  $\phi_0$  is a function of  $r$  variables; there are

$$n(n-1) \dots (n-r+1)$$

values of  $\phi_0$  with  $r$  different arguments drawn from  $y_1, y_2, \dots, y_n$  and every  $y$  alternative must contain each of these values or its contradictory.  $r!$  of these values will have as arguments permutations of  $y_1, y_2, \dots, y_r$ . Any other set of  $r$   $y$ 's can be arranged in the order of their suffixes as  $y_{s_1}, y_{s_2}, \dots, y_{s_r}$ ,  $s_1 < s_2 < s_3 \dots < s_r$ , and it may happen that a given alternative contains the values of  $\phi_0$  for those and only those permutations of  $y_{s_1}, y_{s_2}, \dots, y_{s_r}$  which correspond (in the obvious way) to the permutations of  $y_1, y_2, \dots, y_r$  for which it (the alternative) contains the values of  $\phi_0$ ; e.g. if the alternative contains  $\phi_0(y_1, y_2, \dots, y_r)$  and  $\phi_0(y_r, y_{r-1}, \dots, y_1)$ , but for every other permutation of  $y_1, y_2, \dots, y_r$  contains the corresponding value of  $\sim \phi_0$ , then it may happen that the alternative contains  $\phi_0(y_{s_1}, y_{s_2}, \dots, y_{s_r})$  and  $\phi_0(y_{s_r}, y_{s_{r-1}}, \dots, y_{s_1})$ , but for every other permutation of  $y_{s_1}, y_{s_2}, \dots, y_{s_r}$  contains the corresponding value of  $\sim \phi_0$ .

If this happens, no matter how the set of  $r$   $y$ 's,  $y_{s_1}, y_{s_2}, \dots, y_{s_r}$  is chosen from  $y_1, y_2, \dots, y_n$ , then we say that the *alternative* is *serial* in  $\phi_0^*$ , and if an alternative is serial in every function of the new set we shall call it *serial* simply.

Consider, for example, the following alternative, in which we may imagine  $\phi_0$  and  $\psi_0$  to be derived from one "old" function  $\phi$  by the definitions

$$\phi_0(y_i, y_k) = \phi(y_i, y_k),$$

$$\psi_0(y_i) = \phi(y_i, y_i),$$

$$\phi_0(y_1, y_2) \cdot \sim \phi_0(y_2, y_1) \cdot \phi_0(y_1, y_3) \cdot \sim \phi_0(y_3, y_1) \cdot \phi_0(y_2, y_3) \cdot \sim \phi_0(y_3, y_2),$$

$$\psi_0(y_1) \cdot \sim \psi_0(y_2) \cdot \psi_0(y_3).$$

This is serial in  $\phi_0$ , since we always have  $\phi_0(y_{s_1}, y_{s_2}) \cdot \sim \phi_0(y_{s_2}, y_{s_1})$ ; but not in  $\psi_0$ , since we sometimes have  $\psi_0(y_{s_1})$ , but sometimes  $\sim \psi_0(y_{s_1})$ . Hence it is not a serial alternative.

We call a *form* *serial* when it contains at least one serial alternative, and can now state our chief result as follows.

\* Thus, if  $\phi_0$  is a function of  $n$  variables, all alternatives are serial in  $\phi_0$ .

**THEOREM.**—*There is a finite number  $m$ , depending on  $n$ , the number of functions  $\phi, \chi, \psi, \dots$ , and the numbers of their arguments, such that the necessary and sufficient condition for our formula to be consistent in a universe with  $m$  or more members is that there should be a serial form  $A_n$  completely contained in  $P$ . For consistency in a universe of fewer than  $m$  members this condition is sufficient but not necessary.*

We shall first prove that, whatever be the number  $N$  of individuals in the universe, the condition is sufficient for the consistency of the formula. If  $N \leq n$ , this is a consequence of a previous result, since, if  $A_n$  is completely contained in  $P$ , so is any  $A_N$  involved in  $A_n$ .

If  $N > n$ , we suppose the universe ordered in a series by a relation  $R$ . (If  $N$  is infinite this requires the Axiom of Selections.) Let  $q$  be any serial alternative contained in  $A_n$ . If  $\phi_0$  is a function of  $r$  arguments,  $q$  will contain the values of either  $\phi_0$  or  $\sim \phi_0$  (but not both) for every permutation of  $y_1, y_2, \dots, y_r$ . Any such permutation can be written  $y_{\rho_1}, y_{\rho_2}, \dots, y_{\rho_r}$  where  $\rho_1, \rho_2, \dots, \rho_r$  are  $1, 2, \dots, r$  rearranged. We make a list of all those permutations  $(\rho_1, \rho_2, \dots, \rho_r)$  for which  $q$  contains the values of  $\phi_0$ , and call this list  $\Sigma$ . We now give  $\phi_0$  the constant interpretation that  $\phi_0(z_1, z_2, \dots, z_r)$  is to be true if and only if the order of the terms  $z_1, z_2, \dots, z_r$  in the series  $R$  is given by one of the permutations  $(\rho_1, \rho_2, \dots, \rho_r)$  contained in  $\Sigma$ , in the sense that, for each  $i$ ,  $z_i$  is the  $\rho_i$ -th of  $z_1, z_2, \dots, z_r$  as they are ordered by  $R$ .

Let us suppose now that  $y_1, y_2, \dots, y_n$  are numbered in the order in which they occur in  $R$ , i.e. that in the  $R$  series  $y_1$  is the first of them,  $y_2$  the second, and so on. Then we shall see that, if  $\phi_0$  is given the constant interpretation defined above, all the values of  $\phi_0$  and  $\sim \phi_0$  in  $q$  will be true. Indeed, for values whose arguments are obtained by permuting  $y_1, y_2, \dots, y_r$  this follows at once from the way in which  $\phi_0$  has been defined. For  $\phi_0(y_{\sigma_1}, y_{\sigma_2}, \dots, y_{\sigma_r})$  is true if and only if the order of  $y_{\sigma_1}, y_{\sigma_2}, \dots, y_{\sigma_r}$  in the  $R$  series is given by a permutation  $(\rho_1, \rho_2, \dots, \rho_r)$  contained in  $\Sigma$ . But the order in the series of  $y_{\sigma_1}, y_{\sigma_2}, \dots, y_{\sigma_r}$  is in fact given (on our present hypothesis that the order of the  $y$ 's is  $y_1, y_2, \dots, y_n$ ) by  $(\sigma_1, \sigma_2, \dots, \sigma_r)$ , which is contained in  $\Sigma$  if and only if  $\phi_0(y_{\sigma_1}, y_{\sigma_2}, \dots, y_{\sigma_r})$  is contained in  $q$ . Hence values of  $\phi_0$  for arguments consisting of the first  $r$   $y$ 's are true when they are contained in  $q$  and false otherwise, i.e. when the corresponding values of  $\sim \phi_0$  are contained in  $q$ .

For sets of arguments not confined to the first  $r$   $y$ 's our result follows from the fact that  $q$  is serial, i.e. that if  $s_1 < s_2 < \dots < s_r$ , so that  $y_{s_1}, y_{s_2}, \dots, y_{s_r}$  are in the order given by the  $R$  series,  $q$  contains the



values of  $\phi_0$  for just those permutations of  $y_{s_1}, y_{s_2}, \dots, y_{s_r}$  which correspond to the permutations of  $y_1, y_2, \dots, y_r$  for which it contains the values of  $\phi_0$ , *i.e.* by the definition of  $\phi_0$  and the preceding argument, for just those permutations of  $y_{s_1}, y_{s_2}, \dots, y_{s_r}$  which make  $\phi_0$  true.

Hence all the values of  $\phi_0$  and  $\sim \phi_0$  in  $q$  are true when  $y_1, y_2, \dots, y_n$  are in the order given by the  $R$  series.

If, then, we define analogous constant interpretations for  $\chi_0, \psi_0$ , etc., and combine these with our interpretation of  $\phi_0$ , the whole of  $q$  will be true provided that  $y_1, y_2, \dots, y_n$  are in the order given by the  $R$  series, and if  $y_1, y_2, \dots, y_n$  are in any other order the true alternative will be obtained from  $q$  by suitably permuting the  $y$ 's, *i.e.* will be an alternative similar to  $q$  and contained in the same form  $A_n$ . Hence  $A_n$  is true for any set of distinct  $y_1, y_2, \dots, y_n$ . Moreover, for any set of distinct  $y_1, y_2, \dots, y_v$  ( $v < n$ ) the true form will be one involved in  $A_n$ , and since  $A_n$  and all forms involved in it are contained in  $P$ ,  $P$  will be true for these interpretations of  $\phi_0, \chi_0, \psi_0, \dots$ , and our formula must be consistent.

Having thus proved our condition for consistency sufficient in any universe, we have now to prove it necessary in any infinite or sufficiently large finite universe, and for this we have to use the Theorem B proved in the first part of the paper.

Our line of argument is as follows: we have to show that, whatever  $\phi_0, \chi_0, \psi_0, \dots$  we take  $P$  will be false unless it completely contains a serial  $A_n$ . For this it is enough to show that, given any  $\phi_0, \chi_0, \psi_0, \dots$ , there must be a set of  $n$   $y$ 's for which the true form is serial\*, or, since a serial form is one which contains a serial alternative, that there must be a set of values of  $y_1, y_2, \dots, y_n$  for which the true alternative is serial.

Let us suppose that among our functions  $\phi_0, \chi_0, \psi_0, \dots$  there are  $a_1$  functions of one variable,  $a_2$  of two variables,  $\dots$ , and  $a_n$  of  $n$  variables, and let us order the universe by a serial relation  $R$ .

The  $N$  individuals in the universe are divided by the  $a_1$  functions of one variable into  $2^{a_1}$  classes according to which of these functions they make true or false, and if  $N \geq 2^{a_1} k_1$  we can find  $k_1$  individuals which all belong to the same class, *i.e.* agree as to which of the  $a_1$  functions they make true and which false, where  $k_1$  is a positive integer to be assigned later. Let us call this set of  $k_1$  individuals  $\Gamma_{k_1}$ .

Now consider any two distinct members of  $\Gamma_{k_1}$ ,  $z_1$  and  $z_2$  say, and let  $z_1$  precede  $z_2$  in the  $R$  series. Then in regard to any of the  $a_2$  functions of two variables,  $\phi_0$  say, there are four possibilities. We may either

---

\* For then  $P$  can only be true for  $\phi_0, \chi_0, \psi_0, \dots$  by completely containing this true serial form.

have

$$(1) \quad \phi_0(z_1, z_2) \cdot \phi_0(z_2, z_1),$$

or

$$(2) \quad \phi_0(z_1, z_2) \cdot \sim \phi_0(z_2, z_1),$$

or

$$(3) \quad \sim \phi_0(z_1, z_2) \cdot \phi_0(z_2, z_1),$$

or

$$(4) \quad \sim \phi_0(z_1, z_2) \cdot \sim \phi_0(z_2, z_1).$$

$\phi_0$  thus divides the combinations two at a time of the members of  $\Gamma_{k_1}$  into four distinct classes according to which of these four possibilities is realised when the combination is taken as  $z_1, z_2$  in the order in which its terms occur in the  $R$  series; and the whole set of  $a_2$  functions of two variables divide the combinations two at a time of the members of  $\Gamma_{k_1}$  into  $4^{a_2}$  classes, the combinations in each class agreeing in the possibility they realise with respect to each of the  $a_2$  functions. Hence, by Theorem B, if  $k_1 = h(2, k_2, 4^{a_2})$ ,  $\Gamma_{k_1}$  must contain a sub-class  $\Gamma_{k_2}$  of  $k_2$  members such that all the pairs out of  $\Gamma_{k_2}$  agree in the possibilities they realise with respect to each of the  $a_2$  functions of two variables.

We continue to reason in the same way according to the following general form:—

Consider any  $r$  distinct members of  $\Gamma_{k_{r-1}}$ ; suppose that in the  $R$  series they have the order  $z_1, z_2, \dots, z_r$ . Then with respect to any function of  $r$  variables there are  $2^r$  possibilities in regard to  $z_1, z_2, \dots, z_r$ , and the  $a_r$  functions of  $r$  variables divide the combinations  $r$  at a time of the members of  $\Gamma_{k_{r-1}}$  into  $2^{r \cdot a_r}$  classes. By Theorem B, if  $k_{r-1} = h(r, k_r, 2^{r \cdot a_r})^*$ ,  $\Gamma_{k_{r-1}}$  must contain a sub-class  $\Gamma_{k_r}$  of  $k_r$  members such that all the combinations  $r$  at a time of the members of  $\Gamma_{k_r}$  agree in the possibilities they realise with respect to each of the  $a_r$  functions of  $r$  variables.

We proceed in this way until we reach  $\Gamma_{k_{n-1}}$ , all combinations  $n-1$  at a time of whose members agree in the possibilities they realise with respect to each of the  $a_{n-1}$  functions of  $n-1$  variables. We then determine that  $k_{n-1}$  shall equal  $n$ , which fixes  $k_{n-2}$  as  $h(n-1, n, 2^{(n-1) \cdot a_{n-1}})$  and so on back to  $k_1$ , every  $k_{r-1}$  being determined from  $k_r$ .

If, then,  $N \geq 2^{a_1} k_1$ , the universe must contain a class  $\Gamma_{k_{n-1}}$  or  $\Gamma_n$  (since  $k_{n-1} = n$ ) of  $n$  members which is contained in  $\Gamma_{k_r}$  for every  $r$ ,  $r = 1, 2, \dots, n-1$ . Let its  $n$  members be, in the order given them by  $R$ ,  $y_1, y_2, \dots, y_n$ . Then for every  $r$  less than  $n$ ,  $y_1, y_2, \dots, y_n$  are contained in  $\Gamma_{k_r}$  and all  $r$  combinations of them agree in the possibilities they realise

---

\* If  $a_r = 0$  we interpret  $h(r, k_r, 1)$  as  $k_r$  and identify  $\Gamma_{k_{r-1}}$  and  $\Gamma_{k_r}$ .

with respect to each function of  $r$  variables. Let  $y_{s_1}, y_{s_2}, \dots, y_{s_r}$  ( $s_1 < s_2 < \dots < s_r$ ) be such a combination, and  $\chi_0$  a function of  $r$  variables. Then  $y_{s_1}, y_{s_2}, \dots, y_{s_r}$  are in the order given them by  $R$ , and so are  $y_1, y_2, \dots, y_r$ ; consequently the fact that these two combinations agree in the possibilities which they realise with respect to  $\chi_0$  means that  $\chi_0$  is true for the same permutations of  $y_{s_1}, y_{s_2}, \dots, y_{s_r}$  as it is of  $y_1, y_2, \dots, y_r$ . The true alternative for  $y_1, y_2, \dots, y_n$  is therefore serial in  $\chi_0$ , and similarly it is serial in every other function of any number  $r$  of variables\*; it is therefore a serial alternative.

Our condition is, therefore, shown to be necessary in any universe of at least  $2^{a_1} k_1$  members where  $k_1$  is given by

$$\begin{aligned} k_{n-1} &= n, \\ k_{r-1} &= \left. \begin{aligned} h(r, k_r, 2^{r! a_r}) & \text{ if } a_r \neq 0 \\ = k_r & \text{ if } a_r = 0 \end{aligned} \right\} (r = n-1, n-2, \dots, 2). \end{aligned}$$

For universes lying between  $n$  and  $2^{a_1} k_1$  we have not found a necessary and sufficient condition for the consistency of the formula, but it is evidently possible to determine by trial whether any given formula is consistent in any such universe.

### III.

We will now consider what our result becomes when our formula

$$(x_1, x_2, \dots, x_n) F(\phi, \chi, \psi, \dots, =, x_1, x_2, \dots, x_n)$$

contains in addition to identity only one function  $\phi$  of two variables.

In this case we have two functions  $\phi_0, \psi_0$  given by

$$\begin{aligned} \phi_0(y_i, y_k) &= \phi(y_i, y_k) \quad (i \neq k), \\ \chi_0(y_i) &= \phi(y_i, y_i), \end{aligned}$$

so that  $a_1 = 1, a_2 = 1, a_r = 0$  when  $r > 2$ . Consequently

$$k_2 = k_3 = \dots = k_{n-1} = n \quad \text{and} \quad k_1 = h(2, n, 4);$$

but the argument at the end of I shows that we may take instead  $k_1 = n!!!$ , and our necessary and sufficient condition for consistency applies to any universe with at least  $2 \cdot n!!!$  individuals.

In this simple case we can present our condition in a more striking form as follows.

---

\* We have shown this when  $r < n$ ; we may also have  $r = n$ , but then there is nothing to prove since in a function of  $n$  variables every alternative is serial.

It is necessary and sufficient for the consistency of the formula that it should be true when  $\phi$  is replaced by at least one of the following types of function :—

- (1) The universal function  $x = x . y = y$ .
- (2) The null function  $x \neq x . y \neq y$ .
- (3) Identity  $x = y$ .
- (4) Difference  $x \neq y$ .
- (5) A serial function ordering the whole universe in a series, *i.e.* satisfying
  - (a)  $(x) \sim \phi(x, x)$ ,
  - (b)  $(x, y)[x = y \vee \{ \phi(x, y) . \sim \phi(y, x) \} \vee \{ \phi(y, x) . \sim \phi(x, y) \} ]$ ,
  - (c)  $(x, y, z) \{ \sim \phi(x, y) \vee \sim \phi(y, z) \vee \phi(x, z) \}$ .
- (6) A function ordering the whole universe in a series, but also holding between every term and itself, *i.e.* satisfying
  - (a')  $(x) \phi(x, x)$

and (b) and (c) as in (5).

Types (1)–(4) include only one function each; in regard to types (5) and (6) it is immaterial what function of the type we take, since if one satisfies the formula so, we shall see, do all the others\*.

We have to prove this new form of our condition by showing that  $P$  will completely contain a serial  $A_n$  if and only if it is satisfied by functions of at least one of our six types. Now an alternative in  $n$   $y$ 's is serial in  $\chi_0$  if it contains

either (i)  $\chi_0(y_1) . \chi_0(y_2) . \dots \chi_0(y_n)$  or, for short,  $\prod_r \chi_0(y_r)$ ,

or (ii)  $\sim \chi_0(y_1) . \dots \sim \chi_0(y_n)$  „ „  $\prod_r \sim \chi_0(y_r)$ ,

but not otherwise, and it will be serial in  $\phi_0$  if it contains

either (a)  $\prod_{r < s} \phi_0(y_r, y_s) . \phi_0(y_s, y_r)$ ,

or (b)  $\prod_{r < s} \phi_0(y_r, y_s) . \sim \phi_0(y_s, y_r)$ ,

or (c)  $\prod_{r < s} \sim \phi_0(y_r, y_s) . \phi_0(y_s, y_r)$ ,

or (d)  $\prod_{r < s} \sim \phi_0(y_r, y_s) . \sim \phi_0(y_s, y_r)$ .

\* A result previously obtained for type (5) by Langford, *op. cit.*

There are thus altogether eight alternatives serial in both  $\phi_0$  and  $\chi_0$  got by combining either of (i), (ii) with any of (a), (b), (c), (d); but these eight serial alternatives only give rise to six serial forms, since the alternatives (i) (b) and (i) (c) can be obtained from one another by reversing the order of the  $y$ 's and so belong to the same form, and so do the alternatives (ii) (b) and (ii) (c).

It is also easy to see that any formula completely containing one of these six serial forms will be satisfied by all functions of one of the six types according to the scheme

<i>Form</i>	(i) (a)	(i) (b and c)	(i) (d)	(ii) (a)	(ii) (b and c)	(ii) (d)
<i>Type of function</i>	1	6	3	4	5	2

and that conversely a formula satisfied by a function of one of the six types must completely contain the corresponding form. For instance, a function of type 6 will satisfy the alternative (i) (b) when  $y_1, y_2, \dots, y_n$  are in their order in the series determined by the function, and when  $y_1, y_2, \dots, y_n$  are in any other order the function will satisfy an alternative of the same form.

In the language of the theory of postulate systems we can interpret our universe as a class  $K$ , and conclude that a postulate system on a base  $(K, R)$  consisting only of general laws involving at most  $n$  elements will be compatible with  $K$  having as many as  $2 \cdot n!!!$  members if and only if it can be satisfied by an  $R$  of one of our six types.

#### IV.

Let us, in conclusion, briefly indicate how to extend our method in order to determine the consistency or inconsistency of formulae of the more general type

$$(Ez_1, z_2, \dots, z_m)(x_1, x_2, \dots, x_n)F(\phi, \chi, \psi, \dots, =, z_1, z_2, \dots, z_m, x_1, x_2, \dots, x_n)$$

which have in normal form both kinds of prefix, but satisfy the condition that all the prefixes of existence precede all those of generality.

As before, we can suppose  $F$  represented as a disjunction of alternatives and discard those which violate the laws of identity. Those left we can group according to the values of identity and difference for arguments drawn entirely from the  $z$ 's. Such a set of values of identity and difference we can denote by  $H_i (=, z_1, z_2, \dots, z_m)$ , and  $F$  can be put in the form

$$(H_1 \cdot F_1) \vee (H_2 \cdot F_2) \vee (H_3 \cdot F_3) \vee \dots,$$

and the whole formula is equivalent to a disjunction of formulae.

$$\begin{aligned} & (Ez_1, z_2, \dots, z_m) \{ H_1(=, z_1, z_2, \dots, z_m) \\ & \quad \cdot (x_1, x_2, \dots, x_n) F_1(\phi, \dots, =, z_1, \dots, z_m, x_1, \dots, x_n) \} \\ & \vee (Ez_1, z_2, \dots, z_m) \{ H_2(=, z_1, \dots, z_m) \\ & \quad \cdot (x_1, x_2, \dots, x_n) F_2(\phi, \dots, =, z_1, \dots, z_m, x_1, \dots, x_n) \} \\ & \vee \text{etc.} \end{aligned}$$

Since if any one of these formulae is consistent so is their disjunction, and if their disjunction is consistent one at least of its terms must be consistent, it is enough for us to show how to determine the consistency of any one of them, say the first. In this  $H_1(=, z_1, z_2, \dots, z_m)$  is a consistent set of values of identity and difference for every pair of  $z$ 's. We renumber the  $z$ 's  $z_1, z_2, \dots, z_\mu$  using the same suffix for every set of  $z$ 's that are identical in  $H_1$ , and our formula becomes

$$(Ez_1, z_2, \dots, z_\mu)(x_1, x_2, \dots, x_n) F_1(\phi, \chi, \dots, =, z_1, z_2, \dots, z_\mu, x_1, x_2, \dots, x_n), \quad (i)$$

in which it is understood that two  $z$ 's with different suffixes are always different.

Now supposing the universe to have at least  $\mu + n$  members, we consider the different possibilities in regard to the  $x$ 's being identical with the  $z$ 's, and rewrite our formula

$$(Ez_1, z_2, \dots, z_\mu)(x_1, x_2, \dots, x_n) \left\{ \prod_{\substack{i=1, \dots, n \\ j=1, \dots, \mu}} x_i \neq z_j \rightarrow G(\phi, \chi, \dots, =, z_1, \dots, z_\mu, x_1, \dots, x_n) \right\},$$

in which  $\rightarrow$  means "if, then" and

$$G(\phi, \dots, x_n) = \Pi F_1(\phi, \chi, \dots, =, z_1, \dots, z_\mu, \theta_1, \theta_2, \dots, \theta_n),$$

the product being taken for

$$\begin{aligned} \theta_1 &= x_1, z_1, z_2, \dots, z_\mu, \\ \theta_2 &= x_2, z_1, z_2, \dots, z_\mu, \\ &\dots \quad \dots \quad \dots \\ \theta_n &= x_n, z_1, z_2, \dots, z_\mu, \end{aligned}$$

and in  $G$  any term  $x_i = z_j$  is replaced by a falsehood (*e.g.*  $x_i \neq x_i$ ) not involving any  $z$ .

Next we modify  $G$  by introducing new functions. In  $G$  occur values of, *e.g.*  $\phi$ , with arguments some of which are  $z$ 's and some  $x$ 's; from

these we define functions of the  $x$ 's only by simply regarding the  $z$ 's as constants, and call these new functions  $\phi_0, \chi_0, \dots$ . Values of  $\phi, \chi, \psi, \dots$ , which include no  $x$ 's among their arguments, we replace by constant propositions  $p, q, \dots$ . The only values of identity in  $G$  are of the form  $x_i = x_j$  and these we leave alone. Suppose that by this process  $G$  turns into

$$L(\phi, \chi, \psi, \dots, \phi_0, \chi_0, \dots, p, q, \dots, =, x_1, x_2, \dots, x_n).$$

Then the consistency of formula (i) in a universe of  $N$  individuals is evidently equivalent to the consistency in a universe of  $N-\mu$  individuals of the formula

$$(x_1, x_2, \dots, x_n) L(\phi, \chi, \psi, \dots, \phi_0, \chi_0, \dots, p, q, \dots, =, x_1, x_2, \dots, x_n).$$

But this is a formula of the type previously dealt with, except for the variable propositions  $p, q, \dots$ , which are easily eliminated by considering the different cases of their truth and falsity, the formula being consistent if it is consistent in one such case.