Rank-based construction

NBW --> WAPA (weak alternating parity automaton)
WAPA --> complement WAPA
WAPA --> NBW

Implemented in GOAL as "wapa"

Refers to Kupferman, Vardi, 1997
(Weak alternating automata are not that weak)
which introduced the rank-based approch.

Complementation of Büchi Automata Revisited

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Abstract. As an alternative to the two classical proofs for complementation of Büchi automata, due to Büchi himself and to McNaughton, we outline a third approach, based on stratified alternating automata with a "weak" acceptance condition. Building on work by Muller, Saoudi, Schupp (1986) and Kupferman and Vardi (1997), we present a streamlined version of this complementation proof. An essential point is a determinacy result on infinite games with a weak winning condition. In a unifying logical setting, the three approaches are shown to correspond to three different types of second-order definitions of ω-languages.

1 Introduction

In his seminal paper [Bü62], Büchi introduced a framework for defining sets of sequences, today called the framework of Büchi automata, and showed that the class of properties definable therein is closed under complementation. This result was the key to establish a bridge between ω-automata theory and monadic second-order logic over infinite strings, which in turn opened a new chapter of automata theory with interesting applications in logic and computer science. Even after forty years this chapter is far from being closed, and it is worthwhile to reconsider the beginnings.

There are two "classical" approaches to the complementation of Büchi automata. The first, as found by Büchi himself, stays in the framework of Büchi automata and provides a transformation of a nondeterministic Büchi automaton into such an automaton for the complement language. The second proof, due to McNaughton [McN66] and later sharpened by Safra [Sa88], involves a transformation to deterministic automata with a more general acceptance condition, the Muller condition [Mu63], and uses the fact that for deterministic Muller automata the complementation step is obvious. Both approaches involve nontrivial arguments: In the first case, a combinatorial result is applied (Ramsey's Theorem B [Ra29]), and in the second a very intricate automaton construction is required.

In the present paper, we expose a third proof strategy which so far did not attract much attention in the literature but has some advantages. Instead of reducing the nondeterminism of Büchi automata to determinism with a more complex acceptance condition, this third approach is based on a more general transition mode than nondeterminism, namely alternation, but at the same time the acceptance condition is made simpler: Instead of Büchi acceptance the so-called "weak acceptance" condition is used. As in the deterministic case, the weak acceptance condition is closed under negation, whence complementation is relatively easy. Moreover, the task of connecting Büchi automata to weak alternating automata is not as complex as to show that Büchi automata and deterministic automata are equivalent.

The idea of weak alternating automata is due to Muller, Saoudi, and Schupp ([MSS86], [MS87], [MSS88]). However, in their work they emphasize complexity issues (especially regarding program logics and temporal logics) and not so much a reconsideration of the complementation problem. Recently, Vardi and Kupferman [KV97] have taken up the approach and supplied a self-contained complementation proof for Büchi automata. Their proof strategy does not make use, however, of the duality phenomenon which is characteristic for alternating automata. In the subsequent sections we outline such a more symmetric proof. The key ingredient is a result on determinacy of infinite games with a weak winning condition. An advantage of this proof architecture is its composition from rather elementary, easily verified "modules". Moreover, some conceptual points are clarified. For example, one sees that for defining regular ω-languages by automata, the use of liveness conditions (such as the Büchi acceptance condition) can be avoided, if one works with alternating automata. The complementation proof via infinite games also sheds some light on the relation between automata on infinite words and automata on infinite trees. In the game-theoretic framework, the proofs of complementation for ω-automata and for tree automata can be compared via the respective determinacy results. For Büchi automata complementation we shall need only a very simple determinacy proof based on a reachability analysis, whereas tree automata complementation requires the more complicated determinacy proof for parity games (cf. [Th97]). This pinpoints a characteristic difference between ω-automata theory and tree automata theory.

Kupferman and Vardi
NBW —> UCW (complement)
UCW —> WAA
WAA —> NBW

Thomas
NBW —> WAPA
WAPA —> WAPA (complement)
WAPA —> NBW

Kupferman and Vardi do the complementation not in the framework of alternating automata.

first rank-based construction

This paper provides an introduction to results obtained in collaboration with C. Löding (see his diploma thesis [Lö98]) on alternating ω-automata. We confine ourselves to the use of weak alternating automata in the complementation proof, mentioning only briefly how the transformation of Büchi automata into this model and conversely works, and leaving aside complexity issues and further applications of alternating automata. A joint paper with a more detailed exposition and further results is in preparation.

2 Review of the classical proofs

A Büchi automaton over the alphabet A is a finite automaton of the form $\mathcal{A} = (Q, A, q_0, \Delta, F)$ with finite set Q of states, initial state q_0, transition relation $\Delta \subseteq Q \times A \times Q$, and a set $F \subseteq Q$ of final states. It accepts an ω-word $\alpha = \alpha(0)\alpha(1)\ldots$ from A^ω if there is a run $\rho = \rho(0)\rho(1)\ldots$ from Q^ω with $\rho(0) = q_0$ and $(\rho(i), \alpha(i), \rho(i+1)) \in \Delta$ for $i \geq 0$, which is *Büchi accepting*, i.e. such that $\rho(i) \in F$ for infinitely many i. Formally, we write

$$(*) \qquad \exists \rho\ (\rho(0) = q_0\ \wedge\ \forall i((\rho(i), \alpha(i), \alpha(i+1)) \in \Delta\ \wedge\ \forall i \exists j > i\ \rho(j) \in F)$$

The ω-language $L(\mathcal{A})$ recognized by $\mathcal{A}$ consists of all ω-words α for which $(*)$ holds.

In the logical classification of quantifier alternation hierarchies, one calls $(*)$ a Σ_1^1-formula, referring to the existential sequence quantifier in front (which in an arithmetical setting is captured by a tuple of existential second-order quantifiers ranging over sets of natural numbers). We shall also take into account the logical status of the acceptance condition, which in the case above is called a Π_2^0-condition, referring to its two first-order quantifiers, with a universal quantifier coming first. Taking both aspects together, we speak of a $\Sigma_1^1[\Pi_2^0]$-definition.

Büchi's complementation theorem says that the negation of the formula $(*)$ may again be written in this form, with different Q, Δ, F. Büchi stated the result in this "logical" form (see [Bü62, Lemma 9]). Let us sketch his proof. Given a Büchi automaton $\mathcal{A}$ as above, define a congruence $\sim_{\mathcal{A}}$ over A^+ by declaring two finite words u, v as equivalent iff the following holds: for any $p, q \in Q$, $\mathcal{A}$ can reach q from p via the input u iff this is possible via the input v, and furthermore $\mathcal{A}$ can reach q from p via the input u by passing through a state of F iff this is possible via the input v. It is easy to verify that $\sim_{\mathcal{A}}$ is a congruence with finitely many (regular) equivalence classes. Denoting the $\sim_{\mathcal{A}}$-class of the word u by $[u]$, one observes that an ω-language $[u] \cdot [v]^\omega$ is either contained in $L(\mathcal{A})$ or disjoint from $L(\mathcal{A})$. Invoking Ramsey's Theorem B ([Ra29]), one shows that any ω-word can be cut into a sequence $u_0 u_1 \ldots$ of finite words where all u_i for $i > 1$ belong to a fixed $\sim_{\mathcal{A}}$-class. Applying this decomposition to the ω-words outside $L(\mathcal{A})$, one sees that $A^\omega \setminus L(\mathcal{A})$ is representable as a union of sets $[u] \cdot [v]^\omega$, taking those pairs u, v where $u \cdot v^\omega$ is not in $L(\mathcal{A})$. The union is of course a finite one since there are only finitely many equivalence classes. This representation of $A^\omega \setminus L(\mathcal{A})$ is easily converted into a Büchi automaton.

The complementation theorem can also be shown via a transformation of Büchi automata into deterministic Muller automata. This is the content of McNaughton's Theorem ([McN66]). A Muller automaton is specified in the form $\mathcal{A} = (Q, A, q_0, \delta, \mathcal{F})$ where $\delta : Q \times A \to Q$ is the transition function and $\mathcal{F}$ is a subset of the powerset of Q. Acceptance of an ω-word α means that in the unique run ρ of $\mathcal{A}$ on α, the set $\mathrm{In}(\rho)$ of states visited infinitely often belongs to $\mathcal{F}$. Formally, we express this as follows:

$$(**) \qquad \exists \rho\ ((\rho(0) = q_0\ \wedge\ \forall i\ \rho(i+1) = \delta(\rho(i), \alpha(i)))\ \wedge\ \mathrm{In}(\rho) \in \mathcal{F}).$$

We have $\mathrm{In}(\rho) \in \mathcal{F}$ iff for some $F \in \mathcal{F}$, precisely the states in F are infinitely often visited in ρ. A formalization of this condition reads as follows:

$$\bigvee_{F \in \mathcal{F}} \Big(\bigwedge_{q \in F} \forall i\ \exists j > i\ \rho(j) = q\ \wedge \bigwedge_{q \in Q \setminus F} \neg\, \forall i\ \exists j > i\ \rho(j) = q \Big)$$

Taking the set $2^Q \setminus \mathcal{F}$ instead of $\mathcal{F}$, one obtains an automaton recognizing the complement language.

Let us analyze the logical status of deterministic Muller automata. The acceptance condition is a Boolean combination of Π_2^0-formulas, and thus we shall call the formula $(**)$ a $\Sigma_1^1[\mathrm{Bool}(\Pi_2^0)]$-formula. Since the run required in $(**)$ is unique, one can also use a universal condition instead:

$$(**') \qquad \forall \rho\ (\rho(0) = q_0\ \wedge\ \forall i\ \rho(i+1) = \delta(\rho(i), \alpha(i))\ \ \to\ \ \mathrm{In}(\rho) \in \mathcal{F})$$

This condition $(**')$ is a $\Pi^1_1[\mathrm{Bool}(\Pi^0_2)]$-formula, equivalent to the Σ^1_1-definition $(**)$ above. Properties which are definable in Σ^1_k-form and also in Π^1_k-form are said to have a Δ^1_k-representation. (Note that this involves two separate definitions.) So a deterministic Muller automaton provides a $\Delta^1_1[\mathrm{Bool}(\Pi^0_2)]$-representation of the recognized ω-language.

In this logical setting, McNaughton's Theorem says that the $\Sigma^1_1[\Pi^0_2]$-definitions as provided by Büchi automata can be brought into $\Delta^1_1[\mathrm{Bool}(\Pi^0_2)]$-form, which means a "decrease" of the second-order quantifier complexity at the cost of a more complicated first-order kernel.

3 Alternating automata

The concept of alternating automaton combines the idea of existential branching, as found in nondeterministic automata, with its dual, universal branching. The two branching modes are specified by Boolean expressions over the state set Q. For example, $q_1 \vee (q_2 \wedge q_3)$ denotes the nondeterministic choice of going either to q_1 or to q_2, q_3 simultaneously. The set of such positive (i.e., negation-free) Boolean expressions is denoted by $\mathrm{B}_+(Q)$. We introduce alternating automata here with the so-called weak acceptance condition. It refers to a ranking r of the states by natural numbers.

So an alternating automaton is presented in the form $\mathcal{A} = (Q, A, q_0, \delta, r)$ with entries Q, A, q_0 as before for Büchi automata and functions $\delta : Q \times A \to \mathrm{B}_+(Q)$ and $r : Q \to \{0, \ldots, m\}$ for some m. Moreover, the ranking function r defines a stratification of Q in the sense that for any q occurring in an expression $\delta(p, a)$ we have $r(p) \geq r(q)$. This means that by applying transitions we can only keep or decrease the ranks of states. every successor state gets a non-increasing rank

The definition of acceptance by alternating automata is somewhat involved. Usually, it refers to the notion of run tree (or computation tree). We use here a different terminology which allows a more elegant logical comparison with the previous modes of acceptance (nondeterministic and deterministic).

A run of an alternating automaton is a dag (directed acyclic graph) whose elements are labelled with states from Q. The dag can be presented as a sequence of "slices" $S_0, S_1, \ldots$, where the states occurring in S_i are the simultaneously "active" ones at the i-th letter of the input word. Acceptance will mean that a run dag exists (which represents the existential branching in the automaton) such that for all paths through the dag (representing its universal branching) a condition regarding the ranks of the states occurring on this path is satisfied.

In the definition of run dags we refer to the models of Boolean expressions in $\mathrm{B}_+(Q)$. We shall identify such a Boolean model with a subset S of Q, given by the assignment of states to truth values which sends the states in S to value 1 and the states in $Q \setminus S$ to value 0. Since our expressions are positive, a superset S' of a model S of an expression β is again a model of β. By a minimal model of β we mean a model S of which no proper subset is again a model. If the expression β from $\mathrm{B}_+(Q)$ is presented in disjunctive normal form, the minimal models of β are given by the sets S which constitute the individual conjuncts of the disjunctive form.

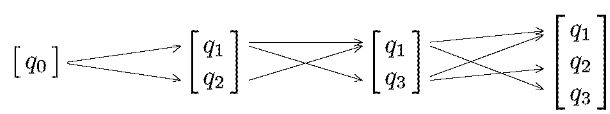
Let us make precise how a run dag is built up. It is started with the slice S_0 consisting solely of the initial state q_0 (more precisely: of a node labelled with q_0). From a given S_i one obtains a slice S_{i+1} as follows, assuming that the input letter $\alpha(i)$ is the letter a: For each p from S_i one chooses a minimal model of the expression $\delta(p, a)$; the union of all these minimal models for $p \in S_i$ is taken to be S_{i+1}. One inserts an edge from $p \in S_i$ to $q \in S_{i+1}$ (more precisely: from the node labelled p in S_i to the node labelled q in S_{i+1}) if q belongs to the minimal model chosen for $\delta(p, a)$.

If the expressions $\delta(p, a)$ are given in disjunctive normal form, the formation of a run dag can be described more easily. Again S_0 consists of q_0 only. Starting from a slice S_i and assuming again $\alpha(i) = a$, pick one disjunction member from $\delta(p, a)$ for each p in S_i. Collect all states arising by these choices to form the slice S_{i+1}, and introduce an edge from p in S_i to q in S_{i+1} if q occurs in the chosen disjunction member of $\delta(p, a)$.

Example 1. Given $Q = \{q_0, q_1, q_2, q_3\}$, we may have, for a certain letter $a \in A$:

$$\begin{aligned}
&\delta(q_0, a) = q_1 \wedge q_2 \\
&\delta(q_1, a) = (q_1 \wedge q_3) \vee (q_2 \wedge q_3) \\
&\delta(q_2, a) = q_1 \\
&\delta(q_3, a) = (q_1 \wedge q_2) \vee q_3
\end{aligned}$$

With this function δ, a run dag on the input word $aaa\ldots$ may start with the following slices:



Now we introduce the weak acceptance condition. This condition refers to the infinite paths through a run dag. By the stratification property of the rank function on states, the ranks on such a path will stay constant from some point onwards. The path is accepting if this ultimately assumed rank is even. Equivalently, the set $\mathrm{Occ}(r(\pi))$ of r-ranks occurring in the path π under consideration has an even minimum. So the acceptance condition reads as follows:

$$\exists \text{ run dag } \rho \ \ \forall \text{ paths } \pi \text{ through } \rho : \min(\mathrm{Occ}(r(\pi)) \text{ is even}$$

A *weak alternating automaton* is an alternating automaton used with this "weak" acceptance condition.

In order to fix the quantifier complexity of weak acceptance, we rewrite the minimum condition in a more formal way. Denoting by $\pi(i)$ the i-th state of the path π, we can express it as follows:

$$\bigvee_{k \text{ even}} (\exists i \ r(\pi(i)) = k \ \wedge \ \bigwedge_{l<k} \neg\exists i \ r(\pi(i)) = l)$$

So weak acceptance for a given path π is a Bool(Σ_1^0)-condition. In the quantifier prefix "$\exists$ run dag ρ $\forall$ paths π" of the acceptance clause, the existential quantifier on run dags and the universal quantifier on paths can both be coded as ranging over infinite sequences over appropriate finite alphabets: A run dag can be represented by a sequence of slices and pointers between adjacent slices, so the possible entries of the sequence are representable by the letters of a finite alphabet. Similarly, a path through a run dag is codable in this way. (At this point one sees an advantage of the notion of run dag over the standard approach involving computation trees. Their existence is not directly formalizable by sequence quantifiers.)

In summary, the definition of an ω-language by a weak alternating automaton is a Σ_2^1[Bool(Σ_1^0)]-representation. In comparison to the $\Sigma_1^1[\Pi_2^0]$-definition given by Büchi automata, the second-order quantifier complexity is increased, while the acceptance condition is simpler: Instead of a requirement that certain states are visited infinitely often, we only ask for the visit of certain states and the avoidance of others.

In the next section we show that the class of ω-languages accepted by weak alternating automata is closed under complementation. As a consequence we can improve the Σ_2^1[Bool(Σ_1^0)]-form of the definition to obtain even a Δ_2^1[Bool(Σ_1^0)]-representation.

4 Duality and weak determinacy

The purpose of this section is to show closure under complement for weak alternating automata. In this analysis, game-theoretic notions are useful. With each (weak) alternating automaton $\mathcal{A} = (Q, A, q_0, \delta, r)$ and each ω-word $\alpha \in A^\omega$ we associate an infinite game $G(\mathcal{A}, \alpha)$, played by two persons called Automaton and Pathfinder. The idea is that in the process of scanning the input word α, Automaton picks sets of simultaneously active states according to the transition function of $\mathcal{A}$, whereas Pathfinder picks, at each point, one of these momentary active states. Making such choices in alternation they build up a path through a run dag of $\mathcal{A}$ on α, and Automaton is declared the winner of the play if the acceptance condition of $\mathcal{A}$ is satisfied on this path.

Formally, a game position refers to the number i supplying the momentary input letter $\alpha(i)$. If Automaton has the next move, the game position is of the form (i, p) with $p \in Q$, and if Pathfinder has to make the next move, the game position has the form (i, S) with $S \subseteq Q$. The initial position is $(0, q_0)$; so Automaton starts. A move of Automaton from position (i, p) consists in the choice of a minimal model of $\delta(p, \alpha(i))$, i.e. a set $S \subseteq Q$, yielding the game position $(i + 1, S)$. Pathfinder reacts by picking a state s from S, producing the game position $(i + 1, s)$. The play determines a sequence of states (extracted from the positions of Automaton), and Automaton wins the play if the minimal rank occurring in this sequence of states is even.

A *local strategy* (also called memoryless strategy) for a player P is a function which associates with any game position of P a move which can be performed in this position. Such a function is called a winning strategy for player P from game position *pos* if its application will produce, when starting from *pos*, for any moves of the opponent, a play won by P.

Proposition 2. *The weak alternating automaton $\mathcal{A}$ accepts α iff in the game $G(\mathcal{A}, \alpha)$, Automaton has a local winning strategy from the initial position.*

Proof. First assume that there is an accepting run dag of $\mathcal{A}$ on α, say with slices $S_0, S_1, \ldots$. Define a strategy for Automaton by choosing, given game position (i, p), the set S of states from S_{i+1} which are reachable from p by an edge of the run dag. In this way, starting from position $(0, q_0)$, Automaton ensures that the play proceeds along a path through the run dag. Since the run dag is accepting, Automaton wins by this local strategy.

Conversely, a local strategy for Automaton defines an accepting run dag: For $i = 0, 1, \ldots$ the slices S_i are built up inductively, beginning with the singleton $S_0 = \{q_0\}$: For any game position (i, p) as picked by Pathfinder, Automaton's local strategy prescribes a set of states as next move; the union of these is taken to form S_{i+1}, and edges are inserted which allow to trace the connections to the different choices of p. It is clear that the constructed run dag is accepting. $\square$

The complementation proof for weak alternating automata will be given in this game-theoretic setting. It has two steps: the dualization of alternating automata and a determinacy result for infinite games.

For the first step, we introduce the *dual automaton* $\widetilde{\mathcal{A}}$ of a given weak alternating automaton $\mathcal{A} = (Q, A, q_0, \delta, r)$. The definition uses the dualization of Boolean expressions: Given an expression $\beta \in \mathrm{B}_+(Q)$, let its dual $\widetilde{\beta}$ arise from β by exchanging $\vee$ and $\wedge$. Now the dualized transition function $\widetilde{\delta}$ is defined by $\widetilde{\delta}(p, a) = \widetilde{\delta(p, a)}$. The dual automaton $\widetilde{\mathcal{A}}$ is obtained as $(Q, A, q_0, \widetilde{\delta}, r)$ with the convention that in an accepting run dag, on each path the minimal rank of visited states should be *odd*.

We need a remark on models of the dual of an expression in $\mathrm{B}_+(Q)$. Recall that a model of an expression β is considered as a subset S of Q.

Remark 3. A set S is a model of $\widetilde{\beta}$ iff every minimal model R of β contains a state from S.

Proof. Let $MM(\beta)$ be the set of minimal models of β. We have the logical equivalence

$$\beta \equiv \bigvee_{R \in MM(\beta)} \bigwedge_{q \in R} q$$

By duality, we have

$$\widetilde{\beta} \equiv \bigwedge_{R \in MM(\beta)} \bigvee_{q \in R} q$$

This shows the claim. $\square$

Now we are able to connect local winning strategies in the two games $G(\mathcal{A}, \alpha)$ and $G(\widetilde{\mathcal{A}}, \alpha)$:

Proposition 4. *Automaton has a local winning strategy in $G(\mathcal{A}, \alpha)$ from the initial position iff Pathfinder has a local winning strategy in $G(\widetilde{\mathcal{A}}, \alpha)$ from the initial position.*

Proof. We show how to transform a local winning strategy of Automaton in $G(\mathcal{A}, \alpha)$ into a local Pathfinder strategy for the dual game. The desired strategy in $G(\widetilde{\mathcal{A}}, \alpha)$ has to tell Pathfinder which state to take for any game position $(i+1, S)$ (where $i \geq 0$). Note that in fixing the strategy it suffices to consider only game positions $(i+1, S)$ which are reachable, i.e. for which a sequence of moves in $G(\widetilde{\mathcal{A}}, \alpha)$ exists starting in the initial position $(0, q_0)$ and ending in $(i+1, S)$. The set S of the game position $(i+1, S)$ is produced by Automaton from a game position (i, s), such that S is a minimal model of $\widetilde{\delta(s, \alpha(i))}$. Pathfinder chooses such a state s which could produce S via $\alpha(i)$. Now in the game $G(\mathcal{A}, \alpha)$ at position (i, s), the given local winning strategy of Automaton picks a minimal model R of $\delta(s, \alpha(i))$. By the remark above, there is a state in $R \cap S$. For his move from the game position $(i+1, S)$, Pathfinder chooses such a state. Then in $G(\widetilde{\mathcal{A}}, \alpha)$ a state sequence is built up which is compatible with Automaton's winning strategy in $G(\mathcal{A}, \alpha)$ and hence is won in that game by Automaton. In the game $G(\widetilde{\mathcal{A}}, \alpha)$, where the roles of even and odd ranks are exchanged, this state sequence gives a play won by Pathfinder. So the described strategy is a winning strategy for Pathfinder in $G(\widetilde{\mathcal{A}}, \alpha)$, and its specification shows that it is local.

The other direction is shown analogously, by exchanging the roles of $\mathcal{A}$ and $\widetilde{\mathcal{A}}$. $\square$

Proposition 5. *Let $\mathcal{A}$ be a weak alternating automaton. From any game position in $G(\mathcal{A}, \alpha)$, either Automaton or Pathfinder has a local winning strategy.*

Proof. Let $\mathcal{A} = (Q, A, q_0, \delta, r)$ be a weak alternating automaton, where $r : Q \to \{0, \dots, m\}$ and $Q_i := \{q \in Q \mid r(q) = i\}$. Let Pos_0 be the set of game positions of the game $G(\mathcal{A}, \alpha)$. It is divided into the sets Pos_A and Pos_P where Automaton, respectively Pathfinder has the next move.

As a preparation we need the definition of "attractor set of a set T of game positions". This attractor set (for player Automaton, say) is denoted $\mathrm{Attr}_A(T)$; it contains all game positions from which Automaton can force in finitely many moves a visit in the target set T. The set is constructed inductively by collecting, for $i \geq 0$, the positions from where a visit to T can be forced within i moves: Let $\mathrm{Attr}_A^0 := T$ and set

$$\begin{aligned}\mathrm{Attr}_A^{i+1}(T) := \mathrm{Attr}_A^i(T) &\\ \cup \{p \in Pos_A \mid &\text{ there is a move from } p \text{ to } \mathrm{Attr}_A^i(T)\}\\ \cup \{p \in Pos_P \mid &\text{ all moves from } p \text{ lead to } \mathrm{Attr}_A^i(T)\}\end{aligned}$$

Now let $\mathrm{Attr}_A(T) = \bigcup_{i \geq 0} \mathrm{Attr}_A^i(T)$. From the positions in this set player Automaton can force a decrease of distance to T in each step (which defines a local strategy). Also note that for the game positions *pos* outside $\mathrm{Attr}_A(T)$, Pathfinder will be able to avoid entering this set. (If at *pos* it is Pathfinder's turn, one move to the complement of $\mathrm{Attr}_A(T)$ is possible, and if it is Automaton's turn, all moves lead to the complement of $\mathrm{Attr}_A(T)$; otherwise *pos* would be already in $\mathrm{Attr}_A(T)$ itself.) So from outside $\mathrm{Attr}_A(T)$, Pathfinder can avoid, by a local strategy, to enter this set and hence can avoid the visit of T.

The set $\mathrm{Attr}_P(T)$ for player Pathfinder is constructed analogously.

Using the notion of attractor set and corresponding local strategies we determine inductively the game positions in $G(\mathcal{A}, \alpha)$ from where player Automaton, respectively Pathfinder, wins.

Clearly, from the positions in $A_0 := \mathrm{Attr}_A(Q_0)$, Automaton can force, by a local strategy, to reach states of rank 0 and thus win. Consider the subgame whose set of positions is $Pos_1 := Pos_0 \setminus A_0$ (all of which have rank ≥ 1). From the positions in $A_1 := \mathrm{Attr}_P(Pos_1 \cap Q_1)$, Pathfinder can force, again by a local strategy, to reach (and stay in) states of rank 1 and hence win. (Note that Pathfinder can avoid to enter A_0, as explained above.) In this way we continue: In the game with position set $Pos_2 := Pos_1 \setminus A_1$ (containing only states of rank ≥ 2) we form the attractor set $A_2 := \mathrm{Attr}_A(Pos_2 \cap Q_2)$, etc. Then the positions from which Automaton wins (by the local attractor strategies) are those in the sets A_i with even $i \leq m$. Similarly, Pathfinder wins from the positions in the sets A_i with odd $i \leq m$ (again by his local attractor strategies). □

Now we have all prerequisites for the complementation of weak alternating automata:

Theorem 6. *For any weak alternating automaton $\mathcal{A}$ over the alphabet A, we have $A^\omega \setminus L(\mathcal{A}) = L(\widetilde{\mathcal{A}})$.*

Proof. By Proposition 2, the automaton $\mathcal{A}$ does not accept the input word α iff Automaton does not have a local winning strategy in $G(\mathcal{A}, \alpha)$ from the initial position. By Proposition 4, this means that Pathfinder does not have a local winning strategy in $G(\widetilde{\mathcal{A}}, \alpha)$ from the initial position. By the determinacy result (Proposition 5), this holds iff Automaton has a local winning strategy in $G(\widetilde{\mathcal{A}}, \alpha)$ from the initial position, which in turn means that $\widetilde{\mathcal{A}}$ accepts α. □

The present game-theoretic complementation proof for Büchi automata has the same general structure as the complementation of nondeterministic tree automata in the framework of parity games (or "Rabin chain games"; see for example [Th97]). So it is possible to compare the two proofs. (Note that the two classical proofs of Büchi automata complementation, via deterministic automata or via a finite congruence saturating the given ω-language, do not extend – as far as we know – to tree automata, which makes a direct comparison difficult.) The parity games associated with tree automata have a winning condition defined by a Boolean combination of Σ_2^0-formulas, and the corresponding determinacy proof, also by induction on the ranks of game positions, involves a nontrivial combination of attractor strategies with strategies given by the inductive hypothesis. In the "weak" games considered in the present paper, where Boolean combinations of Σ_1^0-conditions serve as winning conditions, a straightforward reachability analysis, yielding the attractor sets $A_0, \ldots, A_m$, suffices for the determinacy proof. This direct comparison in the game-theoretical setting shows in which sense complementation is easier for ω-automata than for tree automata and thus reveals a characteristic difference between ω-automata theory and tree automata theory.

5 Equivalence of Büchi automata and weak alternating automata

In order to complete the complementation proof for Büchi automata, we have to supply transformations from Büchi automata to weak alternating automata and conversely. These transformations, as developed by C. Löding in [Lö98], are described here very briefly.

Proposition 7. *For any Büchi automaton $\mathcal{A}$ there is a weak alternating automaton $\mathcal{A}'$ with $L(\mathcal{A}) = L(\mathcal{A}')$.*

Proof. Let $\mathcal{A} = (Q, A, q_0, \Delta, F)$ be a Büchi automaton with n states. The desired weak alternating automaton is constructed over the state set $Q \times \{0, \ldots, 2n\}$, taking $(q_0, 2n)$ as initial state, and defining the rank function r by $r((q,i)) = i$. The transition function δ associates to a state $(p, 0)$ of rank 0 and letter a simply the disjunction over all $(q, 0)$ with $(p, a, q) \in \Delta$. For even ranks $i > 0$, however, we take the disjunction over all expressions $(q, i) \wedge (q, i-1)$ with $(p, a, q) \in \Delta$. This will open a track of states of odd rank until some final state of the Büchi automaton is reached: Namely, as long as $p \notin F$ we set, for odd i, $\delta((p, i), a)$ to be the disjunction over all (q, i), again of rank i, with $(p, a, q) \in \Delta$, while for $p \in F$ we take the disjunction over all $(q, i-1)$ with $(p, a, q) \in \Delta$.

From an accepting run ρ of $\mathcal{A}$ one obtains an accepting run dag ρ' of $\mathcal{A}'$ and conversely. The construction of ρ' from ρ is straightforward: With the states of even rank, one simulates the given run ρ, and by the definition of the transition function δ of $\mathcal{A}$ no path eventually stays on an odd level. The converse direction requires to compose an accepting Büchi run ρ from an accepting run dag ρ'. This composition is achieved by concatenating run segments leading from a state $(p, 2i)$ to a state $(q, 2j)$ with $i > j$, for in this case an intermediate visit in F is ensured by the construction of δ. In order to be able to do this infinitely often, one has to reset the (even) rank to a higher level infinitely often. This in turn is made possible by the presence of $n+1$ even ranks (from 0 to $2n$), which means that once rank 0 is present, some state occurs on two ranks. □

Proposition 8. *For any weak alternating automaton $\mathcal{A}$, there is a Büchi automaton $\mathcal{A}'$ with $L(\mathcal{A}) = L(\mathcal{A}')$.*

Proof. Apply a subset construction as given by Miyano and Hayashi [MH84]: The desired Büchi automaton $\mathcal{A}'$ has states (S, R) where S, R are subsets of the state set Q of the given weak alternating automaton $\mathcal{A}$. In the first component S, $\mathcal{A}'$ guesses the slices of a run dag of $\mathcal{A}$ and thus a run dag, while in the second component R keeps track of those states from S which are of odd rank. If they eventually vanish, R is reset to the whole set S again. Then infinitely many such reset operations (captured by the Büchi acceptance condition) signal that no path in the run dag finally stays in states of odd rank. □

6 Discussion

We have outlined a complementation proof for Büchi automata by invoking a determinacy theorem on weak infinite games. Referring to the logical classification of automata theoretic definability explained in Section 2, we passed from $\Sigma_1^1[\Pi_2^0]$-representations of ω-languages, as given by Büchi automata, to the level of $\Delta_2^1[\mathrm{Bool}(\Sigma_1^0)]$-representations, as given by weak alternating automata. This is an alternative to the option to pass to $\Delta_1^1[\mathrm{Bool}(\Sigma_2^0)]$-representations, as provided by deterministic automata.

None of the steps as described in the propositions above is very difficult; so the complementation proof via weak alternation is composed of simple "modules", in some contrast to the two classical proofs (which rely on a nontrivial combinatorial result or a complicated automaton construction). Of course, there is a price to be paid in using the more involved definition of acceptance of alternating automata.

Another conceptual advantage of weak alternating automata may be the fact that acceptance is defined without resorting to liveness conditions; in the kernel of the second-order definition of acceptance one finds here only conditions on mere reachability or non-reachability of states. This phenomenon may be helpful in the investigation of still unsolved problems of ω-automata theory, for instance in the question of finding a good framework for the minimization of ω-automata.

References

[Bü62] J.R. Büchi, on a decision method in restricted second order arithmetic, in: *Proc. 1960 Int. Congr. on Logic, Methodology and Philosophy of Science*, Stanford Univ. Press, Stanford, 1962, pp. 1-11.

[KV97] O. Kupferman, M.Y. Vardi, Weak alternating automata are not so weak, in: *Proc. 5th Israeli Symp. on Theory of Computing and Systems*, IEEE Computer Soc. Press, 1997, pp. 147-158.

[Lö98] C. Löding, *Methods for the Transformation of ω-Automata: Complexity and Connection to Second Order Logic* Diploma Thesis, Universität Kiel, Inst. f. Informatik u. Prakt. Math., 1998.

[McN66] R. McNaughton, Testing and generating infinite sequences by a finite automaton, *Inf. Contr.* **9** (1966), 521-530.

[MH84] S. Miyano, T. Hayashi, Alternating finite automata on ω-words, *Theor. Comput. Sci.* **32** (1984), 321-330.

[MS87] D.E. Muller, P.E. Schupp, Alternating automata on infinite trees, *Theor. Comput. Sci.* **54** (1987), 267-276.

[MSS86] D.E. Muller, A. Saoudi, P.E. Schupp, Alternating automata, the weak monadic theory of the tree and its complexity, in: *Proc. 13th ICALP*, Lecture Notes in Computer Science **226**, Springer-Verlag, Berlin 1986, pp. 275-283.

[MSS88] D.E. Muller, A. Saoudi, P.E. Schupp, Weak alternating automata give a simple explanation of why most temporal and dynamic logics are decidable in exponential time, in: *Proc. 3rd IEEE Symp. on Logic in Computer Science*, 1988, pp. 422-427.

[Mu63] D.E. Muller, Infinite sequences and finite machines, in: *Proc. 4th Ann. Symp. on Switching Circuit Theory and Logical Design*, IEEE, 1963, pp. 3-16.

[Ra29] F.P. Ramsey, On a problem of formal logic, *Proc. London Math. Soc.* (2) **30** (1929), 264-286.

[Sa88] S. Safra, On the complexity of omega-automata, in: *Proc. 29th Ann. Symp. on Foundations of Computer Science*, IEEE Computer Society, 1988, pp. 319-327.

[Th97] W. Thomas, Languages, automata, and logic, in *Handbook of Formal Language Theory* (G. Rozenberg, A. Salomaa, Eds.), Vol 3, Springer-Verlag, Berlin 1997, pp. 389-455.