Performance Investigation of a Subset-Tuple Büchi Complementation Construction

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Contents

Chapter 1

Introduction

A Büchi complementation construction takes as input a Büchi automaton A and produces as output another Büchi automaton B which accepts the complement language of the input automaton A. Complement language denotes the "contrary" language, that is, B must *accept* (over a given alphabet) every word that A *does not* accept, and must in turn *not accept* every word that A *accepts*.

Büchi automata are finite automata (that is, having a finite number of states) which operate on infinite words (that is, words that "never end"). Operating on infinite words, they belong thus to the category ω-automata. An important application of Büchi automata is in model checking which is a formal system verification technique. There, they are used to represent both, the description of the system to be checked for the presence of a correctness property, and (the negation of) this correctness property itself.

In one approach to model checking, the correctness property is directly specified as a Büchi automaton One approach to model checking requires that the Büchi automaton representing the correctness property is complemented. It is here that the problem of Büchi complementation has one of its practical applications.

The complementation of non-deterministic Büchi automata is hard. It has been proven to have an exponential lower bound in the number of generated states [cite]. That is, the number of states of the output automaton is, in the worst case, an exponential function of the number of states of the input automaton. However, since the introduction of Büchi automata in the 1960's, significant process in reducing the complexity (in other words, the degree of exponentiality) of the Büchi complementation problem has been made. Some numbers [list complexities of the different constructions].

Chapter 2

The Büchi Complementation Problem

2.1 Preliminaries

2.1.1 Büchi Automata

Büchi automata have been introduced in 1962 by Büchi [1] in order to show the decidability of monadic second order logic; over the successor structure of the natural numbers [**?**].s

Definitions

Informally speaking, a Büchi automaton is a finite state automaton running on input words of infinite length. That is, once started reading a word, a Büchi automaton never stops. A word is accepted if it results in a run (sequence of states) of the Büchi automaton that includes infinitely many occurrences of at least one accepting state.

More formally, a Büchi automaton A is defined by the 5-tuple $A = (Q, \Sigma, q_0, \delta, F)$ with the following components.

- Q: a finite set of states
- Σ: a finite alphabet
- q_0: an initial state, $q_0 \in Q$
- δ: a transition function, $\delta : Q \times \Sigma \to 2^Q$
- F: a set of accepting states, $F \in 2^Q$

We denote by Σ^ω the set of all words of infinite length over the alphabet Σ. A Büchi automaton runs on the elements of Σ^ω. In the following, we define the acceptance behaviour of a Büchi automaton A on a word $\alpha \in \Sigma^\omega$.

- A *run* of Büchi automaton A on a word $\alpha \in \Sigma^\omega$ is a sequence of states $q_0 q_1 q_2 \ldots$ such that q_0 is A's initial state and $\forall i \geq 0 : q_{i+1} \in \delta(q_i, \alpha_i)$
- $\inf(\rho) \in 2^Q$ is the set of states that occur infinitely often in a run ρ
- A run ρ is accepting if an only if $\inf(\rho) \cap F \neq \varnothing$
- A Büchi automaton A accepts a word $\alpha \in \Sigma^\omega$ if and only if there is an accepting run of A on α

The set of all the words that are accepted by a Büchi automaton A is called the *language* $L(A)$ of A. Thus, $L(A) \subseteq \Sigma^\omega$. On the other hand, the set of all words of Σ^ω that are rejected by A is called the *complement language* $\overline{L(A)}$ of A. The complement language can be defined as $\overline{L(A)} = \Sigma^\omega \setminus L(A)$.

Büchi automata are closed under union, intersection, concatenation, and complementation [19].

A deterministic Büchi automaton (DBW) is a special case of a non-determnistic Büchi automaton (NBW). A Büchi automaton is a DBW if $|\delta(q, \alpha)| = 1$, $\forall q \in Q, \forall \alpha \in \Sigma$. That is, every state has for every alphabet symbol exactly one successor state. A DBW can also be defined directly by replacing the transition function $\delta : Q \times \Sigma \to 2^Q$ with $\delta : Q \times \Sigma \to Q$ in the above definition.

Expressiveness

It has been showed by Büchi that NBW are expressively equivalent the ω-regular languages [1]. That means that every language that is recognised by a NBW is a ω-regular language, and on the other hand, for every ω-regular language there exists a NBW recognising it.

However, this equivalence does not hold for DBW. There are ω-regular languages that cannot be recognised by any DBW. A typical example is the language $(0+1)^*1^\omega$. This is the language of all infinite words of 0 and 1 with only finitely many 0. It can be shown that this language can be recognised by a NBW (it is thus a ω-regular language) but not by a DBW [19][14]. The class of languages recognised by DBW is thus a strict subset of ω-regular languages recognised by NBW. We say that DBW are less expressive than NBW.

An implication of this is that there are NBW for which no DBW recognising the same language exists. Or in other words, there are NBW that cannot be converted to DBW. Such an inequivalence is not the case, for example, for finite state automata on finite words, where every NFA can be converted to a DFA with the subset construction [2][12]. In the case of Büchi automata, this inequivalence is the main cause that Büchi complementation problem is such a hard problem [11] and until today regarded as unsolved.

2.1.2 Other ω-Automata

After the introduction of Büchi automata in 1962, several other types of ω-automata have been proposed. The best-known ones are by Muller (Muller automata, 1963) [10], Rabin (Rabin automata, 1969) [13], Streett (Streett automata, 1982) [17], and Mostowski (parity automata, 1985) [9].

All these automata differ from Büchi automata, and among each other, only in their acceptance condition, that is, the condition for accepting or rejecting a run ρ. We can write a general definition of ω-automata that covers all of these types as $(Q, \Sigma, q_0, \delta, Acc)$. The only difference to the 5-tuple defining Büchi automata is the last element, Acc, which is a general acceptance condition. We list the acceptance condition of all the different ω-automata types below [6]. Note that again a run ρ is a sequence of states, and $\inf(\rho)$ is the set of states that occur infinitely often in run ρ.

Type	**Definitions**	**Run ρ accepted if and only if...**
Büchi	$F \subseteq Q$	$\inf(\rho) \cap F \neq \varnothing$
Muller	$F \subseteq 2^Q$	$\inf(\rho) \in F$
Rabin	$\{(E_1, F_1), \ldots, (E_r, F_r)\}$, $E_i, F_i \subseteq Q$	$\exists i : \inf(\rho) \cap E_i = \varnothing \wedge \inf(\rho) \cap F_i \neq \varnothing$
Streett	$\{(E_1, F_1), \ldots, (E_r, F_r)\}$, $E_i, F_i \subseteq Q$	$\forall i : \inf(\rho) \cap E_i \neq \varnothing \vee \inf(\rho) \cap F_i = \varnothing$
Parity	$c : Q \to \{1, \ldots, k\}$, $k \in \mathbb{N}$	$\min\{c(q) \mid q \in \inf(\rho)\} \bmod 2 = 0$

In the Muller acceptance condition, the set of infinitely occuring states of a run ($\inf(\rho)$) must match a predefined set of states. The Rabin and Streett conditions use pairs of state sets, so-called accepting pairs. The Rabin and Streett conditions are the negations of each other. This allows for easy complementation of deterministic Rabin and Streett automata [6], which will be used for certain Büchi complementation construction, as we will see in Section2.2. The parity condition assigns a number (color) to each state and accepts a run if the smallest-numbered of the infinitely often occuring states has an even number. For all of these automata there exist non-deterministic

and deterministic versions, and we will refer to them as NMW, DMW (for non-deterministic and deterministic Muller automata), and so on.

In 1966, McNaughton made an important proposition, known as *McNaughton's Theorem* [7]. Another proof given in [18]. It states that the class of languages recognised by deterministic Muller automata are the ω-regular languages. This means that non-deterministic Büchi automata and deterministic Muller automata are equivalent, and consequently every NBW can be turned into a DMW. This result is the base for the determinisation-based Büchi complementation constructions, as we will see in Section 2.2.2.

It turned out that also all the other types of the just introduced ω-automata, non-deterministic and determinstic, are equivalent among each other [14][4][3][6][18]. This means that all the ω-automata mentioned in this thesis, with the exception of DBW, are equivalent and recognise the ω-regular languages. This is illustrated in Figure 2.1

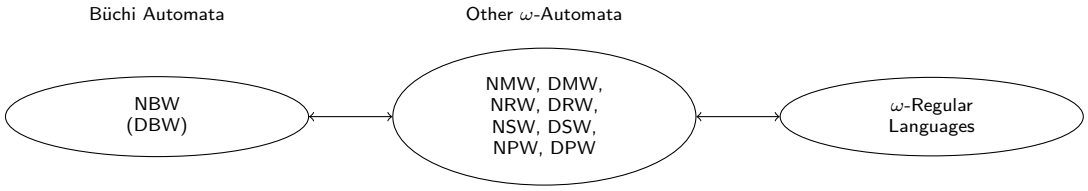


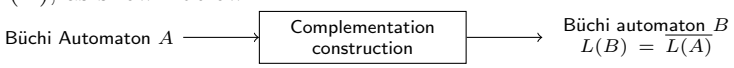
Figure 2.1: Non-deterministic Büchi automata (NBW) are expressively equivalent to Muller, Rabin, Streett, and parity automata (both deterministic and non-deterministic), and to the ω-regular languages. Deterministic Büchi automata (DBW) are less expressive than NBW.

2.1.3 Complementation of Büchi Automata

Büchi automata are closed under complementation. This result has been proved by Büchi himself when he introduced Büchi automata in [1]. Basically, this means that for every Büchi automata A, there exists another Büchi automaton B that recognises the complement language of A, that is, $L(B) = \overline{L(A)}$.

It is interesting to see that this closure does not hold for the specific case of DBW. That means that while for every DBW a complement Büchi automaton does indeed exist, following from the above closure property for Büchi automata in general, this automaton is not necessarily a DBW. The complement of a DBW may be, and often is, as we will see, a NBW. This result is proved in [18] (p. 15).

The problem of Büchi complementation consists now in finding a procedure (usually called a construction) that takes as input any Büchi automaton A and outputs another Büchi automaton B with $L(B) = \overline{L(A)}$, as shown below.



For complementation of automata in general, construction usually differ depending on whether the input automaton A is deterministic or non-deterministic. Complementation of deterministic automata is often simpler and may sometimes even provide a solution for the complementation of the non-deterministic ones.

To illustrate this, we can briefly look at the complementation of the ordinary finite state automata on finite words (FA). FA are also closed under complementation [2] (p. 133). A DFA can be complemented by simply switching its accepting and non-accepting states [2] (p. 133). Now, since NFA and DFA are equivalent [2] (p. 60), a NFA can be complemented by converting it to an equivalent DFA first, and then complement this DFA. Thus, the complementation construction for DFA provides a solution for the complementation of NFA.

Returning to Büchi automata, the case is more complicated due to the inequivalence of NBW and DBW. The complementation of DBW is indeed "easy", as was the complementation of DFA.

There is a construction, introduced in 1987 by Kurshan [5], that can complement a DBW to a NBW in polynomial time. The size of the complement NBW is furthermore at most the double of the size of the input DBW.

If now for every NBW there would exist an equivalent DBW, an obvious solution to the general Büchi complementation problem would be to transform the input automaton to a DBW (if it is not already a DBW) and then apply Kurshan's construction to the DBW. However, as we have seen, this is not the case. There are NBW that cannot be turned into equivalent DBW.

Hence, for NBW, other ways of complementing them have to be found. In the next section we will review the most important of these "other ways" that have been proposed in the last 50 years since the introduction of Büchi automata. The Fribourg construction, that we present in Chapter 3, is another alternative way of achievin this same aim.

2.1.4 Complexity of Büchi Complementation

Constructions for complementing NBW turned out to be very complex. Especially the blow-up in number of states from the input automaton to the output automaton is significant. For example, the original complementation construction proposed by Büchi [1] involved a doubly exponential blow-up. That is, if the input automaton has n states, then for some constant c the output automaton has, in the worst case, c^{c^n} states [16]. If we set c to 2, then an input automaton with six states would result in a complement automaton with about 18 quintillion (18×10^{18}) states.

Generally, state blow-up functions, like the c^{c^n} above, mean the absolute worst cases. It is the maximum number of states a construction *can* produce. For by far most input automata of size n a construction will produce much fewer states. Nevertheless, worst case state blow-ups are an important (the most important?) performance measure for Büchi complementation constructions. A main goal in the development of new constructions is to bring this number down.

A question that arises is, how much this number can be brought down? Researchers have investigated this question by trying to establish so called lower bounds. A lower bound is a function for which it is proven that no state blow-up of any construction can be less than it. The first lower bound for Büchi complementation has been established by Michel in 1988 at $n!$ [8]. This means that the state blow-up of any Büchi complementation construction can never be less than $n!$.

There are other notations that are often used for state blow-ups. One has the form $(xn)^n$, where x is a constant. Michel's bound of $n!$ would be about $(0.36n)^n$ in this case [20]. We will often use this notation, as it is convenient for comparisons. Another form has 2 as the base and a big-O term in the exponent. In this case, Michel's $n!$ would be $2^{O(n \log n)}$ [20].

Michel's lower bound remained valid for almost two decades until in 2006 Yan showed a new lower bound of $(0.76n)^n$ [20]. This does not mean that Michel was wrong with his lower bound, but just too reserved. The best possible blow-up of a construction can now be only $(0.76n)^n$ and not $(0.36n)^n$ as believed before. In 2009, Schewe proposed a construction with a blow-up of exactly $(0.76n)^n$ (modulo a polynomial factor) [15]. He provided thus an upper bound that matches Yan's lower bound. The lower bound of $(0.76n)^n$ can thus not rise any further and seems to be definitve.

Maybe mention note on exponential complexity in [19] p. 8.

2.2 Review of Büchi Complementation Constructions

2.2.1 Ramsey-Based Approaches

The method is called Ramsey-based because its correctness relies on a combinatorial result by Ramsey to obtain a periodic decomposition of the possible behaviors of a Büchi automaton on an infinite word [**?**].

Chapter 3

The Fribourg Construction

Chapter 4

Performance Investigation of the Fribourg Construction

Chapter 5

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Chapter 6

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Conclusions

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