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# A Unified Approach for Showing Language Containment And Equivalence between Various Types of Omega-Automata

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# A Unified Approach For Showing Language Containment And Equivalence Between Various Types Of $\omega$-Automata

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**Abstract** We consider the language containment and equivalence problems for six different types of $\omega$–automata: Büchi, Muller, Rabin, Streett, the L-automata of Kurshan, and the $\forall$-automata of Manna and Pnueli. We give a six by six matrix in which each row and column is associated with one of these types of automata. The entry in the $i^{th}$ row and $j^{th}$ column is the complexity of showing containment between the $i^{th}$ type of automaton and the $j^{th}$. Thus, for example, we give the complexity of showing language containment and equivalence between a Büchi automaton and a Muller or Streett automaton. Our results are obtained by a uniform method that associates a formula of the logic CTL* with each type of automaton. Our algorithms use a *model checking* procedure for the logic with the formulas obtained from the automata. The results of our paper are important for verification of finite state concurrent systems with fairness constraints. A natural way of reasoning about such systems is to model the finite state program by one $\omega$-automaton and its specification by another.

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# 1. Introduction

## 1.1. Background

*$\omega$–Automata* were first used by Büchi in a paper on the decision problem for the logic S1S [7]. A short time later Muller showed that such automata were also useful for modeling the behavior of asynchronous circuits [3]. Like a conventional automaton on finite words, an $\omega$–automaton consists of a set of states, an input alphabet, a transition relation and a start state. The difference between the two occurs in the definition of what it means for a word to be *accepted* by an automaton. Since the notion of a final state is not appropriate for a machine that accepts infinite words, another method must be used for defining acceptance. In Büchi's definition, some states were specified as *accepting states*. In order for a word to be accepted, these states must occur infinitely often during a run of the machine on the word. The definition that Muller used was somewhat more complicated. His acceptance condition consisted of set in which each element was a set of states. In order for a word to be accepted by an automaton, the set of states that occurred infinitely often during a run of the machine on the word must be one of the elements of the acceptance set. Other acceptance conditions have been given by Rabin [10], Streett [11], Kurshan [9], and Manna and Pnueli [16]. It can be shown that each type of automaton accepts the same class of languages (i.e. the $\omega$–regular langauges); however, the translation from one type of automaton to another may be quite complex [12].

The language containment and equivalence problems for $\omega$–automata are defined in exactly the same way as for automata on finite words. Let $M_1$ and $M_2$ be two automata on infinite words with the same alphabet $\Sigma$. $\mathcal{L}(M_i)$ will be the language accepted by $M_i$. The *language containment problem* ( or simply the *containment* problem when this is unambiguous) is the problem of determining whether $\mathcal{L}(M_1) \subseteq \mathcal{L}(M_2)$. The *equivalence problem*, on the other hand, is the problem of deciding whether $\mathcal{L}(M_1) = \mathcal{L}(M_2)$. Given an algorithm for the containment problem, we can easily obtain an algorithm for the equivalence problem, since $\mathcal{L}(M_1) = \mathcal{L}(M_2)$ iff $\mathcal{L}(M_1) \subseteq \mathcal{L}(M_2)$ and $\mathcal{L}(M_2) \subseteq \mathcal{L}(M_1)$. If $M_2$ is nondeterministic, then determining whether $\mathcal{L}(M_1) \subseteq \mathcal{L}(M_2)$ will in general be PSPACE hard, since the corresponding problem for ordinary automata on finite words has this complexity [1]. Consequently, in this paper we will only consider the case in which $M_2$ is deterministic. $M_1$, however, can be either deterministic or nondeterministic.

In recent years the study of $\omega$–automata has experienced a somewhat surprising rebirth. The renewed interest is apparently due to several factors. First of all, there has been a significant amount of research during this period on abstract models for concurrent programs. An important part of this research has been the study of various notions of *fairness*. Several of these notions involve some event holding infinitely often. Because of the similarity to the way that acceptance is defined for Büchi automata, it is natural to use such automata in modeling programs with this type of fairness constraint. Some automatic verification techniques for finite state concurrent programs have exploited this similarity with considerable success. The approach used by Kurshan

[1] This complexity is reversed in the case of $\forall$–automata. See Section 5.

[9], models both the program and its specification by $\omega$–automata. To show that a program is correct, he uses an algorithm for testing containment between the two such automata [8]. A second reason for interest in $\omega$-automata comes from research on temporal logic. There is a close relationship between $\omega$–automata and the models for a formula of linear temporal logic. Specifically, given a formula $f$ of linear temporal logic, it is possible to construct a Büchi automaton that accepts those infinite sequences that are models for $f$. This relationship has also been exploited in an approach to automatic verification called temporal logic *model checking* ([1], [2]). In this case the specification of a finite state program is given by a temporal logic formula. By the property mentioned above it is possible to extract a Büchi automaton from the temporal logic formula and show containment in the same way that Kurshan does. Finally, research in VLSI on problems like clock skew has led to increased interest in asynchronous circuits. Models for such circuits like the one originally proposed by Muller have been resurrected in hopes of obtaining a better understanding for this class of circuits [4].

### 1.2. New Results of this paper

We consider the problem of deciding containment between *all* of the various types of $\omega$–automata mentioned in the first paragraph. We give a $6 \times 6$ matrix where each row and column corresponds to one of the types of automata (See the figure at the end of Section 5.). The entry in the $i^{th}$ row and $j^{th}$ column is the complexity of showing containment between the $i^{th}$ type of automata and $j^{th}$. The entries on the diagonal of the matrix give the complexity of deciding containment of two automata of the same type. We give a single uniform framework for establishing all of these results. We show how each entry in the matrix can be reduced to the problem of determining whether a certain temporal logic formula is true of a Kripke structure obtained from the two automata. We can efficiently determine whether the formula is true of the structure by using a model checking algorithm for the logic.

The particular logic that we use is called CTL* ([1], [2], [5] ). It combines both branching-time and linear-time operators and is quite expressive. The syntax includes *path quantifiers*, A ("for all paths") and E ("for some path"), that are used as prefixes for formulas containing arbitrary combinations of the usual linear time operators G ("always"), F ("sometimes"), X ("nexttime"), and U ("until"). Although the model checking problem for full CTL* is PSPACE-complete [13], Emerson and Lei [6] give a restricted class of CTL* formulas (called *fair–CTL*) for which there is a model checking algorithm with polynomial complexity in the size of the CTL* formula and also in the size of the Kripke structure. We use a modification of this algorithm to obtain our results.

Our strategy for all of the cases in the matrix is essentially the same. We first express the acceptance conditions for the two automata by a formula in CTL*. Then we manipulate the formula to obtain one that can be handled by the model checking algorithm of Emerson and Lei. Since we are able to solve the containment problem by using an an efficient algorithm with practical complexity, the algorithms that we obtain for the entries in the matrix have practical complexity as well and are reasonably easy to implement. Moreover, since we use a uniform approach for obtaining our results, it is relatively simple to understand how the differences in

the complexity among the various entries arise.

Although some of our results were previously known (See [8] for instance.), most of our results are new, because no one else has considered the hybrid cases (Büchi contained in Streett, etc.) that we consider. Even some of the cases on the diagonal are new. For example, we give a low order polynomial algorithm for deciding containment between deterministic Muller automata. As far as we know, no polynomial algorithm has been given for this case before. A naive algorithm to solve this problem would probably have exponential complexity.

### 1.3. Outline of paper

Our paper is organized as follows: In Section 2 we give formal definitions for the various types of $\omega$–automata that we consider in this paper. In Section 3 we give the syntax and semantics for the branching-time temporal logic CTL*, and briefly discuss the model checking algorithm of Emerson and Lei. We precisely state the problem that the algorithm solves and give its complexity in the size of the CTL* formula and the size of the Kripke structure. Section 4 is the heart of the paper. In this section we show how to describe the various types of automata in CTL* and tell how to use the fair–CTL model checking algorithm for deciding containment between different types of machines. The paper concludes in Section 5 with a discussion of our results and some directions for future research.

## 2. $\omega$-Automata

A (*nondeterministic*) *$\omega$-automaton* over an *alphabet* $\Sigma$ is a tuple $(S, s_0, \delta, F)$ where $S$ is a finite set of *states*, $s_0$ is an *initial state*, $\delta : S \times \Sigma \rightarrow \mathcal{P}(S)$ is a *transition relation* and $F$ is an *acceptance condition*. The automaton is *deterministic* if $\forall s \in S, \forall a \in \Sigma \ : \mid \delta(s,a) \mid \leq 1$. The automaton is *complete* if $\forall s \in S, \forall a \in \Sigma \ : \mid \delta(s,a) \mid \geq 1$. In this paper we will always assume that the automata are complete. It is easy to see that this does not affect the complexity of containment.

A *path* in $M$ is an infinite sequence of states $s_0 s_1 s_2 \ldots \in S$ that starts in the initial state and has the property that $\forall i \geq 1, \exists a_i \in \Sigma \ : \ \delta(s_i, a_i) \ni s_{i+1}$. A path $s_0 s_1 s_2 \ldots \in S^\omega$ in $M$ is a run of an infinite word $a_1 a_2 \ldots \in \Sigma^\omega$ if $\forall i \geq 1 \ : \ \delta(s_i, a_i) \ni s_{i+1}$.

An infinite word is accepted by a Büchi, Muller, Rabin, Streett or L automaton if it has an accepting run in the automaton. An infinite word is accepted by a ∀-automaton if all its possible runs in the automaton are accepted.

$\mathcal{L}(M) = \{a_1 a_2 \ldots \in \Sigma^\omega \mid a_1 a_2 \ldots \text{ is accepted by } M\}$.

The *infinitary set* of a sequence $s_0 s_1 s_2 \ldots \in S^\omega$, $inf(s_0 s_1 \ldots)$, is the set of all the states that appear infinitely many times in the sequence.

If $M$ is a Büchi automaton then $F \subseteq S$ is a set of states (as in the case of automata on finite words) and a path $p$ is accepted by $M$ if $inf(p) \cap F \neq \emptyset$.

The acceptance condition of a Muller automaton is a set $F \subseteq \mathcal{P}(S)$ of sets of states. A path is accepted by the Muller automaton if $inf(p) \in F$.

In the case of Rabin automata, the acceptance condition has the form $F = \{(U_1, V_1), \ldots, (U_n, V_n)\}$, where $U_i, V_i \subseteq S$. In this case, a path is accepted by $M$ if there exists $i \in \{1, \ldots, n\}$ such that $inf(r) \subseteq U_i$ and $inf(r) \cap V_i \neq \emptyset$.

The Streett acceptance condition has the same form as that of Rabin, but the semantics is different. A path is accepted by a Streett automaton with $F = \{(U_i, V_i), \ldots, (U_n, V_n)\}$ if for every $i \in \{1, \ldots, n\}$, either $inf(r) \subseteq U_i$ or $inf(r) \cap V_i \neq \emptyset$.

If $M$ is an L automaton, the acceptance condition is a pair $F = (Z, V)$, where $Z \subseteq \mathcal{P}(S)$ and $V \subseteq S$. A path is accepted by the automaton if either $inf(r) \subseteq U$ for some $U \in Z$ or $inf(r) \cap V \neq \emptyset$.

The acceptance condition of a $\forall$-automaton is $F = (U, V) \subseteq S \times S$. A path is accepted by the automaton if either $inf(r) \subseteq U$ or $inf(r) \cap V \neq \emptyset$.

## 3. The Computation Tree Logic CTL*

There are two types of formulas in CTL*: *state formulas* (which are true in a specific state) and *path formulas* (which are true along a specific path). Let $AP$ be the set of atomic proposition names. A state formula is either:

- $A$, if $A \in AP$.
- If $f$ and $g$ are state formulas, then $\neg f$ and $f \vee g$ are state formulas.
- If $f$ is a path formula, then $\mathbf{E}f$ is a state formula.

A path formula is either:

- A state formula.
- If $f$ and $g$ are path formulas, then $\neg f$, $f \vee g$, $\mathbf{X}f$, and $f\mathbf{U}g$ are path formulas.

CTL* is the set of state formulas generated by the above rules.

We define the semantics of CTL* with respect to a structure $M = (S, \mathcal{R}, \mathcal{L})$, where

- $S$ is a set of states.
- $\mathcal{R} \subseteq S \times S$ is the transition relation, which must be total. We write $s_1 \rightarrow s_2$ to indicate that $(s_1, s_2) \in \mathcal{R}$.

- $\mathcal{L} : S \rightarrow \mathcal{P}(AP)$ is a function that labels each state with a set of atomic propositions true in that state.

Unless otherwise stated, all of our results apply only to *finite* Kripke structures.

We define a *path in M* to be a sequence of states, $\pi = s_0 s_1 \ldots$ such that for every $i \geq 0$, $s_i \rightarrow s_{i+1}$. $\pi^i$ will denote the *suffix* of $\pi$ starting at $s_i$.

We use the standard notation to indicate that a state formula $f$ holds in a structure: $M, s \models f$ means that $f$ holds at state $s$ in structure $M$. Similarly, if $f$ is a path formula, $M, \pi \models f$ means that $f$ holds along path $\pi$ in structure $M$. The relation $\models$ is defined inductively as follows (assuming that $f_1$ and $f_2$ are state formulas and $g_1$ and $g_2$ are path formulas):

1. $s \models A$ *iff* $A \in L(s)$.
2. $s \models \neg f_1$ *iff* $s \not\models f_1$.
3. $s \models f_1 \vee f_2$ *iff* $s \models f_1$ or $s \models f_2$.
4. $s \models \mathbf{E}(g_1)$ *iff* there exists a path $\pi$ starting with $s$ such that $\pi \models g_1$.
5. $\pi \models f_1$ *iff* $s$ is the first state of $\pi$ and $s \models f_1$.
6. $\pi \models \neg g_1$ *iff* $\pi \not\models g_1$.
7. $\pi \models g_1 \vee g_2$ *iff* $\pi \models g_1$ or $\pi \models g_2$.
8. $\pi \models \mathbf{X} g_1$ *iff* $\pi^1 \models g_1$.
9. $\pi \models g_1 \mathbf{U} g_2$ *iff* there exists a $k \geq 0$ such that $\pi^k \models g_2$ and for all $0 \leq j < k$, $\pi^j \models g_1$.

We will also use the following abbreviations in writing CTL* formulas:

$$\bullet f \wedge g \equiv \neg(\neg f \vee \neg g) \qquad \bullet \mathbf{A}(f) \equiv \neg \mathbf{E}(\neg f)$$
$$\bullet \mathbf{F} f \equiv true \mathbf{U} f \qquad \bullet \mathbf{G} f \equiv \neg \mathbf{F} \neg f.$$

Let $K = (S, \mathcal{R}, \mathcal{L})$ be a finite Kripke structure. The *model checking problem* for a logic L is to determine which states in $S$ satisfy a given formula $f$ of L. This problem is PSPACE–complete for CTL* [13]. However, for restricted CTL* formulas of the form

$$\mathbf{E}[\bigvee_{i=1}^{n} (\bigwedge_{j=1}^{n_i} (\mathbf{FG} p_{ij} \vee \mathbf{GF} q_{ij}))]$$

where $p_{ij}$ and $q_{ij}$ are propositional formulas, Emerson and Lei [6] give a polynomial model checking algorithm.

**Theorem 1** *Let $K = (S, \mathcal{R}, \mathcal{L})$ be a Kripke structure and $f$ be a CTL* formula of the above form. There is an algorithm for finding the states of $S$ where $f$ is true that runs in time*

$$\mathcal{O}(\sum_{i=1}^{n} n_i \mid \mathcal{R} \mid + \sum_{i=1}^{n} n_i^2 \mid S \mid + T)$$

*where $T$ is the time necessary to label the states satisfying $p_{ij}$ and $q_{ij}$ for $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, n_i\}$.*

In this paper we will need somewhat more complicated (although equivalent) CTL* formulas in order to obtain better time bounds for our algorithms. The new formulas have the form

$$\mathbf{E}[\bigvee_{i=1}^{n}(\bigwedge_{j=1}^{n_i}(\mathbf{FG}p_{ij} \vee \mathbf{GF}q_{ij}) \wedge (\bigwedge_{j=1}^{m_i}\mathbf{GF}r_{ij}) \wedge \mathbf{FG}p_i)].$$

In this case the complexity is

$$\mathcal{O}((\sum_{i=1}^{n}(n_i+1)+1) \mid \mathcal{R} \mid + \sum_{i=1}^{n}(n_i+1)(n_i+m_i+1) \mid \mathcal{S} \mid + T)$$

where $T$ is the time required to find the sets of states satisfying $p_{ij}$, $q_{ij}$, $r_{ij}$ and $p_i$ for all $i \in \{1,\ldots,n\}$ and $j \in \{1,\ldots,n_i\}$. Essentially the same algorithm and proof can be used in this case as well.

## 4. Complexity Results for Varous Types of $\omega$-automata

### 4.1. Büchi, Muller, Rabin, Streett and L automata

Let $M = (S, s_0, \delta, F)$ be a Büchi, Muller, Rabin, Streett or L automaton.

Let $\phi_F$ be a linear formula over $S$ that expresses the acceptance condition of $M$, more precisely, $\phi_F$ is a linear formula over $S$ that has the property : an infinite path in $M$ is accepted by $F$ if and only if it satisfies $\phi_F$.

We will show below that it is possible to express both the acceptance condition of $M$ and its negation in the form

$$\bigvee_{i=1}^{n}(\bigwedge_{j=1}^{n_i}(\mathbf{FG}p_{ij} \vee \mathbf{GF}q_{ij}) \wedge (\bigwedge_{j=1}^{m_i}\mathbf{GF}r_{ij}) \wedge \mathbf{FG}p_i)$$

where $p_{ij}, q_{ij}, r_{ij}$ and $p_i$ are propositional formulas. With each of these five different types of $\omega$–automata, we first give the acceptance condition and its negation as CTL* path formulas. Then we state the values of $n$, $n_i$, and $m_i$ that show why the formulas have the general form above.

- Büchi

$$\phi_F = \mathbf{GF}(\bigvee_{s\in F} s)$$
$$n = 1,\ n_1 = 0,\ m_1 = 1$$
$$\neg\phi_F = \mathbf{FG}(\bigvee_{s\in F} s)$$
$$n = 1,\ n_1 = 0,\ m_1 = 0$$

- Muller

$$\phi_F = \bigvee_{A \in F} (\mathbf{FG}(\bigvee_{s \in A} s) \wedge \bigwedge_{s \in A} \mathbf{GF}s) \text{ or}$$

$n = \mid F \mid$ and for every $A \in F$ : $n_A = 0$, $m_A = \mid A \mid$

We will show that $\phi_F$ is equivalent to the following formula

$$(\bigvee_{A \in F} \mathbf{FG}(\bigvee_{s \in A} s)) \wedge \bigwedge_{A \in F} \bigwedge_{t \in A} (\bigvee_{\substack{B \subset A \\ B \in F}} \mathbf{FG}(\bigvee_{s \in B} s) \vee \mathbf{GF}(\bigvee_{s \in \overline{A}} s) \vee \mathbf{GF}t)$$

To establish this result it is sufficient to prove that for a given $C \in S$, $C$ will be in $F$ if and only if the following condition holds :

1. $\exists A \in F$ : $C \subseteq A$ and
2. $\forall A \in F$, $\forall t \in A$ at least one of the following holds:
   (a) $\exists B \subset A$: $B \in F$, $C \subseteq B$ or
   (b) $C \cap \overline{A} \neq \emptyset$ or
   (c) $t \in C$

The "$\Rightarrow$" direction is proved by the following argument:

1. holds for $A = C$,
2. for any $A \in F$ either $A = C$ and therefore c) holds, or $C \cap \overline{A} \neq \emptyset$ then b) holds, or $C \subset A$ in which case a) holds.

The "$\Leftarrow$" direction is even simpler. By 1) there exists a minimal $A \in F$ such that $C \subseteq A$. Then by 2) for $A$ as a) and b) are false, it must be the case that $A \subseteq C$.

Using this result, it is easy to show that $\neg\phi$ can be expressed as follows:

$$\neg\phi_F = (\bigwedge_{A \in F} \mathbf{GF}(\bigvee_{s \in \overline{A}} s)) \vee \bigvee_{A \in F} \bigvee_{t \in A} (\bigwedge_{\substack{B \subset A \\ B \in F}} \mathbf{GF}(\bigvee_{s \in \overline{B}} s) \wedge \mathbf{FG}(\bigvee_{\substack{s \in A \\ s \neq t}} s))$$

$n = 1 + \Sigma_{A \in F} \mid A \mid$, $n_1 = 0$, $m_1 = \mid F \mid$

and for every $A \in F$, $t \in A$ : $n_{A,t} = 0$, $m_{A,t} = \mid \{B \subset A \mid B \in F\} \mid$

- Rabin

$$\phi_F = \bigvee_{(U,V) \in F} (\mathbf{FG}(\bigvee_{s \in U} s) \wedge \mathbf{GF}(\bigvee_{s \in V} s))$$

$n = \mid F \mid$, $n_{(U,V)} = 0$, $m_{(U,V)} = 1$

$$\neg\phi_F = \bigwedge_{(U,V) \in F} (\mathbf{GF}(\bigvee_{s \in \overline{V}} s) \vee \mathbf{FG}(\bigvee_{s \in \overline{U}} s))$$

$n = 1$, $n_1 = \mid F \mid$, $m_1 = 0$

- Streett

$$\phi_F = \bigwedge_{(U,V)\in F} (\mathbf{FG}(\bigvee_{s\in U} s) \vee \mathbf{GF}(\bigvee_{s\in V} s))$$

$$n = 1,\ n_1 = \mid F \mid,\ m_1 = 0$$

$$\neg\phi_F = \bigvee_{(U,V)\in F} (\mathbf{GF}(\bigvee_{s\in \overline{U}} s) \wedge \mathbf{FG}(\bigvee_{s\in \overline{V}} s))$$

$$n = \mid F \mid,\ n_{(U,V)} = 0,\ m_{(U,V)} = 1$$

- L

$$\phi_F = \bigvee_{U\in Z} \mathbf{FG}(\bigvee_{s\in U} s) \vee \mathbf{GF}(\bigvee_{s\in V} s)$$

$$n = 1 + \mid Z \mid,\ n_1 = 0,\ m_1 = 1 \text{ and for all } U \in Z :\ n_U = 0,\ m_U = 0$$

$$\neg\phi = \mathbf{FG}(\bigvee_{s\in \overline{V}} s) \wedge \bigwedge_{U\in Z} \mathbf{GF}(\bigvee_{s\in \overline{U}} s)$$

$$n = 1,\ n_1 = 0,\ m_1 = \mid Z \mid$$

We now show how to compute the complexity of the containment problem by using the formulas for $\phi_F$ and $\neg\phi_F$ for the five different types of $\omega$–automata. Let $M = (S, s_0, \delta, F)$ and $M' = (S', s'_0, \delta', F')$ be two complete Büchi, Muller, Rabin, Streett or L automata over $\Sigma$ such that $S \cap S' = \emptyset$.

Let $K(M, M') = (S \times S', (s_0, s'_0), \mathcal{L}, \mathcal{R})$ be the Kripke structure over $S \cup S'$ for which $\mathcal{L}(s, s') = \{s, s'\}$ and $(s, s')\mathcal{R}(t, t') \Leftrightarrow (\exists a \in \Sigma : \delta(s, a) \ni t$ and $\delta'(s', a) \ni t')$.

If $M'$ is deterministic, then

$$\mathcal{L}(M) \subseteq \mathcal{L}(M') \Leftrightarrow K(M, M') \models \neg\mathbf{E}(\phi_F \wedge \neg\phi_{F'})$$

where $\phi_F$ and $\phi_{F'}$ express the acceptance conditions of $M$ and $M'$ respectively. Suppose that $\phi_F$ and $\neg\phi'_F$ have the form : deterministic $\forall$-automaton with acceptance condition expressed by

$$\phi_F = \bigvee_{i=1}^{n} (\bigwedge_{j=1}^{n_i} (\mathbf{FG}p_{ij} \vee \mathbf{GF}q_{ij}) \wedge \bigwedge_{j=1}^{m_i} \mathbf{GF}r_{ij} \wedge \mathbf{FG}p_i)$$

$$\neg\phi_{F'} = \bigvee_{i=1}^{n'} (\bigwedge_{j=1}^{n'_i} (\mathbf{FG}p'_{ij} \vee \mathbf{GF}q'_{ij}) \wedge \bigwedge_{j=1}^{m'_i} \mathbf{GF}r'_{ij} \wedge \mathbf{FG}p'_i).$$

Then $K(M, M') \models \mathbf{E}(\phi_F \wedge \neg\phi_{F'})$ if and only if

$$K(M, M') \models \mathbf{E}[\bigvee_{i=1}^{n} \bigvee_{k=1}^{n'} (\bigwedge_{j=1}^{n_i} (\mathbf{FG}p_{ij} \vee \mathbf{GF}q_{ij}) \wedge \bigwedge_{l=1}^{n'_k} (\mathbf{FG}p'_{kl} \vee \mathbf{GF}q'_{kl}) \wedge \bigwedge_{j=1}^{m_i} \mathbf{GF}r_{ij} \wedge \bigwedge_{l=1}^{m'_k} \mathbf{GF}r'_{kl} \wedge \mathbf{FG}(p_i \wedge p'_k))]$$

and therefore, as shown in Section 3, the inclusion $\mathcal{L}(M) \subseteq \mathcal{L}(M')$ can be checked in time

$$\mathcal{O}(\sum_{i=1}^{n} \sum_{k=1}^{n'} (n_i + n'_k + 1)(\mid \delta \mid\mid \delta' \mid + \mid S \mid\mid S' \mid (n_i + n'_k + m_i + m'_k + 1)).$$

| $M$ \ $M'$ | Büchi det | Muller det | Rabin det | Streett det | L det | ∀ nondet |
|---|---|---|---|---|---|---|
| Büchi nondet | $ee' + vv'$ | $ee'g' + vv'f'g'$ | $ee'f' + vv'f'^2$ | $ee'f' + vv'f'$ | $ee' + vv'f'$ | $ee' + vv'$ |
| Muller nondet | $ee' + vv'g$ | $ee'fg' + vv'(ff'g' + gg')$ | $ee'ff' + vv'(ff'^2 + gf')$ | $ee'ff' + vv'gf'$ | $ee' + vv'(ff' + g)$ | $ee'f + vv'g$ |
| Rabin nondet | $ee'f + vv'f$ | $ee'fg' + vv'ff'g'$ | $ee'ff' + vv'ff'^2$ | $ee'ff' + vv'ff'^2$ | $ee'f + vv'ff'$ | $ee'f + vv'f$ |
| Streett nondet | $ee'f + vv'f^2$ | $ee'fg' + vv'f(f + f')g'$ | $ee'(f + f') + vv'(f + f')^2$ | $ee'ff' + vv'f^2f'$ | $ee'f + vv'f(f + f')$ | $ee' + vv'f$ |
| L nondet | $ee'f + vv'f$ | $ee'fg' + vv'ff'g'$ | $ee'ff' + vv'ff'^2$ | $ee'ff' + vv'ff'$ | $ee'f + vv'ff'$ | $ee'f + vv'f$ |
| ∀ det | $ee' + vv'$ | $ee'f' + vv'g'$ | $ee'f' + vv'f'$ | $ee'f' + vv'f'$ | $ee' + vv'f'$ | $ee' + vv'$ |

**Time complexity of the containment $\mathcal{L}(M) \subseteq \mathcal{L}(M')$**

where $e = \mid \delta \mid$, $e' = \mid \delta' \mid$, $v = \mid S \mid$, $v' = \mid S' \mid$, $f = \mid F \mid$, $f' = \mid F' \mid$, and $g = \sum_{A \in F} \mid A \mid$, $g' = \sum_{A \in F'} \mid A \mid$ if the automaton is a Muller automaton.

It is easy to see that the time required to label the states with the propositional subformulas $p_{ij}$, $q_{ij}$, etc. is dominated by the other terms of the complexity formula. Finally, to obtain each entry in the submatrix determined by the first five $\omega$-automata, we substitute the values of $n$, $n_i$, and $m_i$ for the acceptance condition of the first automaton and $n'$, $n'_k$, and $m'_k$ for the negation of the acceptance condition for the second automaton.

### 4.2. ∀-automata

The deterministic ∀-automata are a subclass of the deterministic Streett automata. Therefore we can complete the list in the previous section with

- deterministic ∀

$$\phi_F = \mathbf{FG}(\bigvee_{s\in U} s) \vee \mathbf{GF}(\bigvee_{s\in V} s)$$
$$n = 1,\ n_1 = 1,\ m_1 = 0$$
$$\neg\phi_F = \mathbf{GF}(\bigvee_{s\in U} s) \wedge \mathbf{FG}(\bigvee_{s\in V} s)$$
$$n = 2,\ n_1 = 0,\ m_1 = 1,\ n_2 = 0,\ m_2 = 0$$

Once we have the values for $n$, $n_i$ and $m_i$ we can use the same technique as in Subsection 4.1 for computing the complexity of each entry in the last row of the matrix.

In order to check $\mathcal{L}(M) \subseteq \mathcal{L}(M')$ where $M$ is a Büchi, Muller, Rabin, Streett, L or deterministic ∀-automaton and $M'$ is a nondeterministic ∀-automaton we must use a different approach.

Let $M = (S, s_0, \delta, F)$ be a Büchi, Muller, Rabin, Streett, L or deterministic ∀-automaton with the acceptance condition expressed by

$$\phi_F = \bigvee_{i=1}^{n}(\bigwedge_{j=1}^{n_i}(\mathbf{FG}p_{ij} \vee \mathbf{GF}q_{ij}) \wedge \bigwedge_{j=1}^{m_i}\mathbf{GF}r_{ij} \wedge \mathbf{FG}p_i)$$

and let $M' = (S', s'_0, \delta', (R', S'))$ be a complete ∀–automaton. Notice that the Rabin automaton $M'_{dual} = (S', s'_0, \delta', F'_{dual} = (R'_{dual}, S'_{dual}))$ with $R'_{dual} = \overline{S'}$ and $S'_{dual} = \overline{R'}$ is such that $\mathcal{L}(M'_{dual}) = \overline{\mathcal{L}(M')}$.

Then we have :

$$\mathcal{L}(M) \subseteq \mathcal{L}(M'_{dual}) \Leftrightarrow \mathcal{L}(M) \cap \mathcal{L}(M'_{dual}) \neq \emptyset \Leftrightarrow K(M, M'_{dual}) \models \mathbf{E}(\phi_F \wedge \phi_{F'_{dual}})$$

and therefore the time complexity to check $\mathcal{L}(M) \subseteq \mathcal{L}(M')$ in this case is

$$\mathcal{O}(\sum_{i=1}^{n}(n_i + 1)(\mid \delta \mid\mid \delta' \mid + \mid S \mid\mid S' \mid (n_i + m_i + 1))$$

Each entry in the column of the matrix for nondeterministic ∀-automata can be obtained by substituting the values of $n$, $n_i$, and $m_i$ for the acceptance condition of the automaton in the corresponding row.

## 5. Directions for Future Research

An obvious question is whether there are any reasonable acceptance conditions for $\omega$–automata that we have not considered. Certainly we have included all of the models that are commonly discussed in the literature, but are there any that don't fit in our framework? One possibility is to use a formula of linear-temporal logic as the acceptance condition for an automaton. For example, given an arbitrary formula $f$ of linear-temporal logic one can define an $f$-automaton that accepts an infinite word iff the word satisfies the formula $f$. The same question can be posed for Wolper's logic *ETL* [15] and, in fact, for any temporal logic with models that are sequences of states. This notion of acceptance at first appears more general than the previous ones that we have discussed, but it is really not. Assume that the alphabet for $f$ is $\Sigma$. By using a construction of [14] it is possible to obtain a nondeterministic Büchi automaton with alphabet $\mathcal{P}(\Sigma)$ that will accept an infinite word iff the word satisfies the formula $f$. Consequently, this problem is essentially the same as the problem of showing containment between some (possibly nondeterministic) $\omega$–automaton and a nondeterministic Büchi automaton.

The question of how to handle the containment problem when both automata are nondeterministic, is another important problem for research. In this case, the problem is at least PSPACE hard, since the containment problem for conventional automata on finite wordss has this complexity. In some cases it is possible to show that the containment problem for certain types of $\omega$–automata is also in PSPACE. We conjecture that this is true for all of the cases in the complexity matrix, but we have not been able to prove this result yet. Of course, to say that a particular problem is in PSPACE is not really very useful in practice. Concrete time bounds like $2^n$ or $2^{n\log n}$ are much more useful. We have already obtained some results of this type and we hope to complete the matrix for the nondeterministic case with bounds of this sort in the near future.

Finally, although our algorithms for testing containment are the best that we know, we are currently unable to show that many of them are optimal. In fact, we suspect that some are not optimal and may be improved in the future. We believe that additonal research would be valuable in this direction. To aid in the search for better algorithms, it would be quite helpful to have lower bounds for the non-optimal cases in the matrix. A related question that we have not fully considered is the usefulness of our present complexity measure. Currently, the entries in our matrix are worst case execution times. While this measure is certainly an important factor in evaluating the efficiency of our algorithms, it is not the only factor of interest. Since the automata may have many thousands of states, the amount of memory available is frequently the limiting factor rather than the execution time. Complexity measures that take this into account, are probably more useful than time complexity alone. Devising such measures and finding algorithms that are efficient with respect to the new measures are also important directions for research.

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