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NOTE

ON THE COMPLEMENTATION OF BÜCHI AUTOMATA

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Abstract. Given a Büchi automaton with n states, we propose an elementary construction of a Büchi automaton with $O(16^{n^2})$ states which recognizes the complement of the ω-language recognized by the first one.

1. Introduction

Büchi [1] has shown that the class of ω-languages recognized by finite automata is closed under complementation. The first constructions of a Büchi automaton for the complement of the set recognized by a given n-state Büchi automaton [2, 3, 4, 7] involved a doubly exponential blow-up (at least 2^{2^n} states). More recently Sistla, Vardi and Wolper [8] have announced the construction of an automaton with only $O(16^{n^2})$ states. Here we give a new construction, which is simpler and more complete, of another Büchi automaton with $O(16^{n^2})$ states for the complement. The method is based on notions from [1] and a lemma from [6].

2. Preliminaries

A *Büchi automaton* on the finite alphabet A is a finite automaton $\mathfrak{A}=(Q, I, T, E)$ where Q is the finite set of states, I and T are the subsets of initial and terminal states respectively, and $E \subseteq Q\times A\times Q$ the set of edges. It *recognizes* the ω-regular language $\mathfrak{A}^\omega$ formed of all ω-words for which there is a run starting in an initial state and passing infinitely often through a terminal state.

With this automaton $\mathfrak{A}$ the *terminal transition semigroup* of state relations is associated which is the image of A^+ in the morphism $\theta : A^+ \to \mathrm{SS}(\mathfrak{A})$ defined by

$$\theta(a)=\begin{pmatrix}\sigma(a) & \tau(a)\\ \emptyset & \sigma(a)\end{pmatrix}. \tag{1}$$

Here, $\sigma(a)=\{(p, q)\in Q^2 \mid (p, a, q)\in E\}$ and $\tau(a)=\{(p, q)\in Q^2 \mid (p, a, q)\in E$ with $p\in T$ or $q\in T\}$.

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If $f: A^+ \to S$ is a semigroup morphism, we call every ω-language of the form $f^{-1}(m)f^{-1}(e)^\omega$ with $m = me$ and $e^2 = e$ an *elementary f-language.* We say that f *saturates* an ω-language $L \subseteq A^\omega$ if every elementary f-language intersecting L is included in L.

Let us recall some elementary facts.

Proposition ([1]). (1) *If f saturates L, it also saturates the complement* $\bar{L} = A^\omega / L$.

(2) *f saturates L iff L is the union of all elementary f-languages intersecting it.*

(3) *The terminal transition morphism* θ *saturates* $\mathfrak{A}^\omega$.

Proof. (1): Is an immediate consequence of the definition.

(2): The 'if'-part is evident. For the 'only-if'-part the union is trivially included in L; and the equality is obtained by showing, via Ramsey's Theorem, that every element of A^ω is contained in an elementary f-language.

(3): Let $\theta^{-1}(m)\theta^{-1}(e)^\omega$ be an elementary θ-language intersecting $\mathfrak{A}^\omega$, and let $\alpha = uv_1v_2\ldots$ be a factorization of an element α in the intersection, with $\theta(u) = m$ and $\theta(v_i) = e$ for every i. Because $me = e$ and $e^2 = e$, we can suppose, after possibly grouping some v_i's, that there is a run of $\mathfrak{A}$ over α of the form

$$i_0 \xrightarrow{u} q_0 \xrightarrow{v_1} q_0 \xrightarrow{v_2} q_0 \rightarrow \cdots,$$

where $i_0 \in I$ and every run $q_0 \rightarrow^{v_i} q_0$ goes through a terminal state. Let now β be another element of $\theta^{-1}(m)\theta^{-1}(e)^\omega$ and let $\beta = u'v_1'v_2'\ldots$ be a factorization with $\theta(u') = m$ and $\theta(v_i') = e$ for all i. We then have

$$(i_0, q_0) \in \sigma(u) = \{(p, q) \mid \text{there is a run } p \xrightarrow{u} q\} = \sigma(u')$$

and

$$(q_0, q_0) \in \tau(v_i) = \{(p, q) \mid \text{there is a run } p \xrightarrow{v_i} q \text{ going through a terminal state}\} = \tau(v_i').$$

So we have a run of $\mathfrak{A}$ over β of the form

$$i_0 \xrightarrow{u'} q_0 \xrightarrow{v_1'} q_0 \xrightarrow{v_2'} s_0 \rightarrow \cdots,$$

where every run $q_0 \rightarrow^{v_i'} q_0$ goes through a terminal state; this shows that $\beta \in \mathfrak{A}^\omega$. $\square$

Our construction relies on the two following lemmas. The first one is a technical result on elementary languages.

Lemma 1 ([5]). *Let u and v be two words with* $e = e^2 = \theta(v)$ *and* $m = me = \theta(u)$. *Then the elementary* θ*-language* $\theta^{-1}(m)\theta^{-1}(e)^\omega$ *intersects* $\mathfrak{A}^\omega$ *iff there exists* $i \in I$ *and* $q \in Q$ *such that* $(i, q) \in \sigma(u)$ *and* $(q, q) \in \tau(v)$.

Proof. First suppose that $\theta^{-1}(m)\theta^{-1}(e)^{\omega}$ intersects L. This elementary language is then included in L by the Proposition, and thus, the word uv^{ω} is accepted by $\mathfrak{A}$. Then if $p_0 \rightarrow^{u} p_1 \rightarrow^{v} p_2 \rightarrow^{v} \cdots$ is an accepting run, we can obtain a pair (i, q) with the desired property by taking p_0 for i and any state occurring infinitely often among the p_is for q.

The converse is evident. $\square$

We are now going to prove our key lemma. It relies on an idea of reverse determinism.

Lemma 2 ([6]). *Let $f: A^{+} \rightarrow S$ be a morphism, $e \in S$ an idempotent, and M a subset of S for which every element $m \in M$ satisfies $me = m$. Let S^1 be the monoid deduced from S by the adjunction of a new identity denoted by* 1. *Then the Büchi automaton*

$$\mathfrak{A} = \{S^1, M, \{1\}, \{(r, a, s) \mid f(a)s = r \text{ or } f(a)s = re\}\}$$

recognizes the language $L = \bigcup_{m \in M} f^{-1}(m) f^{-1}(e)^{\omega}$.

Proof. A simple verification shows that for this automaton the relation σ in (1) is defined by $\sigma(u) = \{(r, s) \mid f(u)s = r \text{ or } f(u)s = re\}$. Thus, for every $m \in M$, we have a path of the form $m \rightarrow^{u} 1$ iff $f(u) = m = me$, and a path of the form $1 \rightarrow^{u} 1$ iff $f(u) = e$. The conclusion follows. $\square$

Now we can give our construction.

Theorem. *Given a Büchi automaton $\mathfrak{A}$ with n states, one can effectively construct a Büchi automaton with $4^{n^2}(4^{n^2}+1)$ states which recognizes the complement of $\mathfrak{A}^{\omega}$.*

Proof. It is clear that we can effectively construct the terminal transition morphisms $\theta: A^{+} \rightarrow \mathrm{SS}(\mathfrak{A})$ and the semigroup $S = \mathrm{SS}(\mathfrak{A})$, whose cardinality is at most equal to $m = 2^{2n^2}$.

By the Proposition, θ saturates the language $L = A^{\omega}/\mathfrak{A}^{\omega}$, which can thus be expressed in the form $L = \bigcup_{e \in E} L_e$ with

$$E = \{e = e^2 \mid \exists m \colon \theta^{-1}(m)\theta^{-1}(e)^{\omega} \cap L \neq \emptyset\}, \qquad L_e = \bigcup_{m \in M_e} \theta^{-1}(m)\theta^{-1}(e)^{\omega}$$

and

$$M_e = \{m \mid me = m \text{ and } \theta^{-1}(m)\theta^{-1}(e)^{\omega} \cap L \neq \emptyset\}.$$

Further, E and the M_e can be effectively computed by Lemma 1. As in Lemma 2, we can then construct, for every $e \in E$, a Büchi automaton $\mathfrak{A}_e$ with $m+1$ states recognizing L_e. The disjoint union of these automata gives a Büchi automaton that recognizes L and whose number of states is at most equal to $m(m+1)$. $\square$

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