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ALMOST DETERMINISTIC ω-AUTOMATA WITH EXISTENTIAL OUTPUT CONDITION

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ABSTRACT. The theorem on reduction in the nondeterminateness degree of ω-automata has been formulated.

1. **Notation.** ω denotes the set of natural numbers. An ordinal will be identified with the set of all its predecessors. $|A|$ and $P(A)$ denote the cardinality and the power set of A. For a set A and an ordinal α, denote A^α to be the set of all α-sequences over A, $A^\alpha = \{v|v: \alpha \to A\}$. $A^* = \bigcup_{n \in \omega} A^n$. We shall write $l(v) = n$ if $v \in A^n$. If $v_1 \in A^*$, $v_2 \in A^\alpha$ ($B \subseteq A^*$, $C \subseteq A^\alpha$) for an ordinal, then v_1v_2 (BC) will denote the result of concatenating v_1 with v_2 (B with C).

For a function $f: A \to B$, define

$$In(f) = \{b|b \in B, |f^{-1}(b)| \geq \omega\}.$$

2. **ω-definability.** We shall keep the terminology of [6].

A (nondeterministic) automaton over an alphabet Σ is a quadruple $\mathfrak{A} = \langle S, M, S_0, F\rangle$ where S is a finite set, the *set of states*, M is a function $M: S \times \Sigma \to P(S)$, the *transition function*, $S_0 \subseteq S$ is the set of *initial states*, and $F \subseteq S$ is the set of *final states*.

The *rank* of $\mathfrak{A}$ is the least number n such that $|S_0| \leq n$ and $|M(s, \sigma)| \leq n$ for every $s \in S$, $\sigma \in \Sigma$. An automaton of rank 1 is called *deterministic* (d.). An automaton $\mathfrak{A} = \langle S, M, S_0, F\rangle$ is called *limitary deterministic* (l. d.) if there is a d. automaton $\mathfrak{B} = \langle T, N, T_0, G\rangle$ over Σ with $G = F$ and $N \subseteq M$.

Given $n < \omega$. An $\mathfrak{A}$*-run* on $v \in \Sigma^n$ is a function $r: n + 1 \to S$ such that $r(0) \in S_0$ and $r(i + 1) \in M(r(i), v(i))$, $i < n$.

An $\mathfrak{A}$*-run* on $v \in \Sigma^\omega$ is a function $r: \omega \to S$ satisfying the above for any $i < \omega$.

A word $v \in \Sigma^n$, $n < \omega$, is *accepted* by $\mathfrak{A}$ if there is an $\mathfrak{A}$-run on v such that $r(n) \in F$. The set of all words $v \in \Sigma^*$ accepted by $\mathfrak{A}$ will be denoted by $L(\mathfrak{A})$. A set $A \subseteq \Sigma^*$ is called *regular* if for some automaton $\mathfrak{A}$, $L(\mathfrak{A}) = A$.

Proceeding to ω-sequences, we introduce two different output conditions attached to two different notions of finite automata. And so with the automaton

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$\mathfrak{A} = \langle S, M, S_0, F\rangle$ we associate, following J. R. Büchi [1], the *existential* rule of ω-definability:

$\mathfrak{A}$ *defines* $v \in \Sigma^\omega$ iff there is an $\mathfrak{A}$-run on v such that $In(r) \cap F \neq \emptyset$.

The set of all $v \in \Sigma^\omega$ defined by $\mathfrak{A}$ will be denoted by $E(\mathfrak{A})$. A set $A \subseteq \Sigma^\omega$ is *existentially definable* (ED) if for some automaton $\mathfrak{A}$, $E(\mathfrak{A}) = A$. $A \subseteq \Sigma^\omega$ is ED of *rank* n if there is an automaton $\mathfrak{A}$ of rank n such that $E(\mathfrak{A}) = A$. An ED set of rank 1 will be called *deterministic* (d.).

The second notion of the finite acceptor is due to D. Muller [4].

A *Muller automaton* over Σ is a system $\mathfrak{A} = \langle S, M, s_0, F\rangle$ where S is a finite set, $M: S \times \Sigma \to S$, $s_0 \in S$, and $F \subseteq P(S)$ is the set of *designated subsets of* S.

The *Muller* (*universal*) rule of ω-definability states:

$\mathfrak{A}$ *defines* $v \in \Sigma^\omega$ iff there is a function $r: \omega \to S$ satisfying $r(0) = s_0$, $r(i+1) = M(r(i), v(i))$, $i < \omega$, and such that $In(r) \in F$.

The set of all ω-sequences defined in such a manner will be denoted by $U(\mathfrak{A})$. A set $A \subseteq \Sigma^\omega$ is called *universally definable* (UD) if for some Muller automaton $\mathfrak{A}$, $U(\mathfrak{A}) = A$.

By the fundamental result of McNaughton [3] we have

Theorem 1. *Given a set* $A \subseteq \Sigma^\omega$. *$A$ is ED if and only if A is UD.*

3. **Rank and limitary determinism of ED sets.** It is trivially verifiable that there are ED sets which are not of rank 1. This fact naturally raises the question of the possible reductions in the degree of nondeterminateness of such sets. In answer to this we have

Theorem 2. *For every automaton $\mathfrak{A}$ there exists a l.d. automaton $\mathfrak{B}$ of rank 2 such that* $E(\mathfrak{B}) = E(\mathfrak{A})$.

Proof. Let $\mathfrak{A} = \langle S, M, s_0, F\rangle$ be a Muller automaton with $F = \{A_j\}_{j<n}$. Construct the set $T = \{(a_0, \ldots, a_{n-1})s \mid a_j \in P(A_j)$ or $a_j = \rho$, $s \in S\}$, $\rho \notin P(S)$, and the function $N: T \times \Sigma \to T$ by

$$N((a_0, \ldots, a_{n-1})s, \sigma) = \{(b_0, \ldots, b_{n-1})M(s, \sigma)\}$$

and

$$\begin{aligned} b_j &= a_j \cup \{s\} && \text{if } s \in A_j \text{ and } a_j \neq A_j,\ a_j \neq \rho, \\ &= \emptyset && \text{if } s \in A_j \text{ and } a_j = A_j, (a_j \neq \rho), \\ &= \rho && \text{otherwise.} \end{aligned}$$

Let us define the automaton $\mathfrak{B} = \langle S \cup T, H, \{s_0\}, G\rangle$ where $H(s, \sigma) = \{(\emptyset, \ldots, \emptyset)s', s'\}$ for $s' = M(s, \sigma)$, $s \in S$, $H|T \times \Sigma = N$ and $G = \{(a_0, \ldots, a_{n-1})s \mid a_j = A_j$ for some $j \in n\}$. $\mathfrak{B}$ is l.d. and of rank 2.

With the above, for any function $v \in \Sigma^\omega$ the following occurences are equivalent:

(1) There is a function $r \in (S \cup T)^{\omega}$ such that $r(i+1) \in H(r(i), v(i))$, $i < \omega$, and $In(r) \cap G \neq \varnothing$.

(2) There is a function $r \in S^{\omega}$ such that $r(i+1) = M(r(i), v(i))$, $i < \omega$, and $In(r) \in F$.

To display this, suppose that (1) is fulfilled. From the construction of the function N it follows that $r_s(i+1) = M(r_s(i), v(i))$, where $r_s(i) = s$ if $r(i) = s$ or $r(i) = (a_0, \ldots, a_{n-1})s$. If $(a_0, \ldots, a_{n-1}) \in \pi_1(In(r) \cap G)$, with π_1 the 1st projection, then there is an index j such that $a_j = A_j \in F$ and $a_{j'} \subset A_j$ or $a_{j'} = \rho$ for $j' \neq j$. Suppose now that there is a second $(b_0, \ldots, b_{n-1}) \in \pi_1(In(r) \cap G)$ with $a_j \neq b_j$ for some $j \in n$. This implies immediately that $(a_0, \ldots, a_{n-1}) \notin \pi_1(In(r) \cap G)$, a contradiction. So it must be exactly $In(r_s) = A_j$. On the other hand, let (2) be satisfied with $In(r) = A_j \in F$. There is an integer k such that $r(i) \in A_j$ for $i \geq k$. Construct the function $r' \in (S \cup T)^{\omega}$ by $r'|k = r|k$,

$$r'(k) = (\varnothing, \ldots, \varnothing)M(r(k-1), v(k-1))$$

and

$$r'(k+1+i) = \bar{H}((\varnothing, \ldots, \varnothing)M(r(k-1), v(k-1)), v_k(i))$$

for $v_k(i)\colon i+1 \to \Sigma$ defined by $v_k(i') = v(k+i')$ and $\bar{H}$ being the sequential extension of H. For $i \geq k$ we have $r'(i) = (a_0, \ldots, a_{n-1})s$ provided $a_j \subseteq A_j$. Here again by the second part of (2) we have $(a_0, \ldots, a_{n-1})$ with $a_j = A_j$ belonging to $\pi_1(In(r'))$.

The above entails the identity $E(\mathfrak{B}) = U(\mathfrak{A})$, and thus, by Theorem 1, our thesis follows.

Now let $\mathfrak{A} = \langle S, M, S_0, F\rangle$ be a l.d. automaton over Σ. Define the automata $\mathfrak{A}_1(s) = \langle S, M, S_0, \{s\}\rangle$, $s \in F$. There exist d. automata $\mathfrak{A}_2(s) = \langle T, N, \{s\}, F\rangle$, $s \in F$, over Σ with $N \subseteq M$. We have $E(\mathfrak{A}) = \bigcup_{s \in F} L(\mathfrak{A}_1(s))E(\mathfrak{A}_2(s))$.

Since the regular sets concatenated with ED sets are again ED sets and ED sets are closed under the union, Theorem 2 will yield the following expansion result.

Theorem 3. *Given a set $A \subseteq \Sigma^{\omega}$. A is ED if and only if there are regular sets $B_i \subseteq \Sigma^*$, and d. ED sets $C_i \subseteq \Sigma^{\omega}$, $i < n < \omega$, satisfying the identity $A = \bigcup_{i<n} B_i C_i$.*

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