Empirical Performance Investigation of a Büchi Complementation Construction

Daniel Weibel

July 30, 2015

Abstract

This will be the abstract.

Acknowledgements

Contents

Chapter 1

Introduction

At the beginning of the 1960's, a Swiss logician named Julius Richard Büchi was looking for a way to decide the satisfiability of formulas of the monadic second order logic with one successor (S1S). In his quest, Büchi observed that an S1S formula can be represented by a certain type of finite state automaton that runs on infinite words, such that this automaton accepts a word if and only if the corresponding interpretation satisfies the formula. The proof of this equivalence between S1S formulas and this type of autoamtaon, which is known as *Büchi's Theorem*, led Büchi to his desired decision procedure: to test whether an S1S formula φ is satisfiable, translate it to an equivalent automaton A, and test whether A is empty (that is, accepts no words at all). If A is empty, then φ is unsatisfiable, if A is non-empty, then φ is satsifiable. [4]

The type of automaton that Büchi used for solving this logical problem is called *Büchi automaton*. The application of Büchi automata to logic, that was established by Büchi, had a large impact on other fields, especially model checking, which is a technique of formal verification. In particular, Büchi automata allow to solve the model checking question automata-theoretically, which has many advantages [49].

However, there is one operation on Büchi automata that is giving a "headache" to the research community since the introduction of Büchi automata more than 50 years ago, namely the problem of *complementation*. Algorithms for carrying out this operation, although possible[1], turn out to be very complex, in many cases too complex for practical application. Yet, Büchi complementation has a practical application in the automata-theoretic approach to model checking. This discrepancy led to an ongoing quest for finding more efficient *Büchi complementation constructions*, and generally better understanding the complexity of Büchi complementation. The work in this thesis is situated in this area of research.

In this introductory chapter, we will first

1.1 Context

1.1.1 Büchi Automata and Büchi Complementation

Büchi automata are finite state automata that process words of infinite length, so called ω-words. If Σ is the alphabet of a Büchi automaton, then the set of all the possible ω-words that can be generated from this alphabet is denoted by Σ^ω. A word $\alpha \in \Sigma^\omega$ is accepted by a Büchi automaton if it results in at least one run that contain infinitely many occurrences of at least one accepting state. A run of a Büchi automaton on a word is an infinite sequence of states. Deterministic Büchi automata have at most one run for each word in Σ^ω, whereas non-deterministic Büchi automata may have multiple runs for a word.

The complement of a Büchi automaton A is another Büchi automaton[2] denoted by $\overline{A}$. Both, A and $\overline{A}$, share the same alphabet Σ. Regarding a word $\alpha \in \Sigma^\omega$, the relation between an automaton and its complement is as follows:

$$\alpha \text{ accepted by } A \iff \alpha \text{ not accepted by } \overline{A}$$

That is, all the words of Σ^ω that are *accepted* by an automaton are *rejected* by its complement, and all the words of Σ^ω that are *rejected* by an automaton are *accepted* by its complement. In other words, there is no single word of Σ^ω that is accepted or rejected by *both* of an automaton and its complement.

A Büchi complementation construction is an algorithm that, given a Büchi automaton, creates the complement of this Büchi automaton. The difficulty of this operation depends on whether the input automaton is determinstic or non-deterministic. The complementation of deterministic Büchi automata is "easy" and can be done in polynomial time and linear space [17]. The complementation of non-deterministic Büchi automata, however, is very complex. The understanding and reduction of its complexity is a domain of active research and lies at the centre of this thesis.

Consequently, when in the following we talk about "Büchi complementation", we specifically mean the complementation of *non-determinstic* Büchi automata. The main problem with the complexity of Büchi complementation is the so-called state growth or state complexity (sometimes also called state blow-up or

[1] Büchi himself has proved that Büchi automata are closed under complementation [4].

[2] Büchi automata are closed under complementation. This has been proved by Büchi [4], who, to this end, described the first Büchi complementation construction in history.

state explosion). This is the number of states of the output automaton in relation to the number of states of the input automaton. In simple words, Büchi complementation constructions produce complements that may be very, very large.

This inhibits the practial applicaion of Büchi complementation, because in this case the limited computing and time resources may not be high enough to accommodate for this high complexity. In the following subsections we highlight an important application that Büchi complementation has in practice, and thereby motivate the research on Büchi complementation and of this thesis.

1.1.2 Applications of Büchi Complementation

Language Containment of ω-Regular Languages

Büchi complementation is used for testing language containment of ω-regular languages. The ω-regular languages are the class of formal languages that is equivalent to non-deterministic Büchi automata[3]. At this point, we briefly describe the language containment in general, before in turn describing an application of the language containment problem in the next subsection.

Given two ω-regular languages L_1 and L_2 over alphabet Σ^ω the language containment problem consists in testing whether $L_1 \subseteq L_2$. This is true if all the words of L_1 are also in L_2, and false if L_1 contains at least one word that is not in L_2. The way this problem is algorithmically solved is by testing $L_1 \cap \overline{L_2} = \varnothing$. Here, $\overline{L_2}$ denotes the complement language of L_2, which means $\overline{L_2}$ contains all the words of Σ^ω that are *not* in L_2. The steps for testing $L_1 \cap \overline{L_2} = \varnothing$ are the following:

- Create the complement language $\overline{L_2}$ of L_2
- Create the intersection language $L_{1,\overline{2}}$ of L_1 and $\overline{L_2}$
- Test whether $L_{1,\overline{2}}$ is empty (that is, contains no words at all)

Thus, the language containment problem is reduced to three operations on languages, *complementation*, *intersection*, and *emptiness testing*. The common way to work with formal languages is not to handle the languages themselves, but more compact structures that represent them, such as automata. In the case of ω-regular languages, these are non-deterministic Büchi automata.

For solving $L_1 \subseteq L_2$, one thus works with two Büchi automata A_1 and A_2 that represent L_1 and L_2, respectively. The problem then becomes $L(A_1) \subseteq L(A_2)$, and equivalently, $L(A_1) \cap \overline{L(A_2)} = \varnothing$. This is automata-theoretically solved as $\mathsf{empty}(A_1 \cap \overline{A_2})$, which includes the three following steps:

- Construct the complement automaton $\overline{A_2}$ of A_2
- Construct the intersection automaton $A_{1,\overline{2}}$ of A_1 and A_2
- Test whether $A_{1,\overline{2}}$ is empty (that is, accepts no words at all)

If the final emptiness test on automaton $A_{1,\overline{2}}$ is true, then $L_1 \subseteq L_2$ is true, and if the emptiness test is false, then $L_1 \subseteq L_2$ is false. In this way, the language containment problem of ω-regular languages is reduced to three operations of *complementation*, *intersection*, and *emptiness testing* of non-deterministic Büchi automata. Thus, Büchi complementation is an intergral part of language contaiment of ω-regular languages.

Automata-Theoretic Model Checking via Language Containment

In the last subsection, we have seen that Büchi complementation is used for testing language containment of ω-regular languages. In this subsection, we will see what in turn language containment of ω-regular languages is used for. To this end, we describe one important application of it, namely the language containment approach to automata-theoretic model checking. In the following, we first describe basic working of this technique in general, and then point out the significance that Büchi complementation bears for it.

[3] Note that deterministic Büchi automata have a lower expressivity than non-deterministic Büchi automata, and are equivalent to only a subset of the ω-regular languages.

Basics

The language containment approach to automata-theoretic model checking is an approach to automata-theoretic model checking, which is an approach to model checking, which in turn is an approch to formal verification [?]. Figure 1.1 shows the branch of the family of formal verification techniques that contains the language containment approach to automata-theoretic model checking.

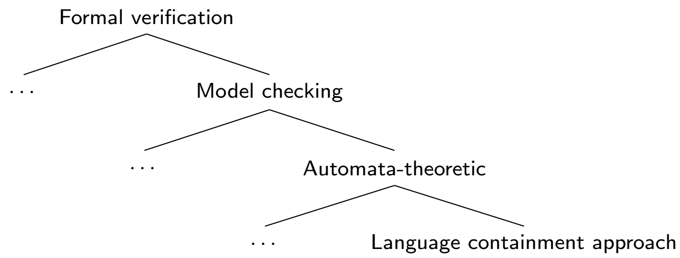

Figure 1.1: Branch of the family of formal verification techniques that contains the language containment approach to automata-theoretic model checking.

Formal verification is the use of mathematical techniques for proving the correctness of a system (software of hardware) with respect to a specified property [?]. A typical example is to verify that a program is deadlock-free (in which case the property would be "deadlock-freeness"). In general, formal verification techniques consist of the following three parts [?]:

1. A framework for modelling the system to verify
2. A framework for specifying the property to be verified
3. A verification method for testing whether the system satisfies the property

For the language containment approach to automata-theoretic model checking, the frameworks for points 1 and 2 are Büchi automata representing ω-regular languages. The verification method (point 3) is testing language containment of the system automaton's language in the property automaton's language. In some more detail, this works as follows. [49][?]

The system s to verify is modelled as a Büchi automaton, say S. This Büchi automaton represents an ω-regular language $L(S)$, and each word of $L(S)$ represents in turn a possible *computation trace* of the system. A computation trace is an infinite[4] sequence of "situations" that the system is in at any point in time. Such a situation consists of a finite amount of information of, for example, the values of variables, registers, or buffers. The observation that such a trace can be represented as a word of an ω-regular langauges comes from the fact that it can be represented intuitively as a linear Kripke structure (which in turn is an interpretation for a temporal logic formula that can also be used to represent computations), which in turn can be represented by a word of a language whose alphabet is ranges over the powerset of the atomic propositions of the Kripke strucutre. A work that explains these intimate relations between computation, temporal logic, formal languages, and automata in more detail is [49]. In simple words, the language $L(S)$, represented by the system automaton S, represents everything that the system *can* do.

Similarly, a property p to be verified is represented as a Büchi automaton, say P, which represents the ω-regular language $L(P)$, whose words represent computation traces. These computation traces are all the computations of a system like s that satisfy the property p. If for example p is "deadlock-freeness", then the words of $L(P)$ represent all the possible computation traces that do *not* contain a deadlock. In this way, the language $L(P)$ represents everything that the system is *allowed* to do, with respect to a certain property.

The verification step is finally done by testing $L(S) \subseteq L(P)$. If this is true, then everything that the system *can* do is *allowed* to do, and the system satisfies the property p. If the language containment test

[4]The infinity of computation traces suggests that this type of formal verification (and model checking in general) is used for systems that are not expected to terminate and may run indefinitely. This type of systems is called *reactive* systems. They contrast with systems that are expected to terminate and produce a result. For this latter type of systems other formal verification techniques than model checking are used. See for example [?] and [?] for works that cover the formal verification of both types of systems.

is false, then the system *can* do a computation that is *not allowed* to do, and the system does not satisify the property p.

Summarising, the language containment approach to automata-theoretic model checking requires language containment of ω-regular languages, which, as we have seen, requires Büchi complementation. In the following, we will highlight the particular importance of Büchi complementation for this type of formal verification.

1.1.3 Significance of Büchi Complementation

The complexity of Büchi complementation makes the just described model checking approach nearly inapplicable in practice [**?**]. According to [**?**], there are so far no tools that include this approach. This is unfortunate, because the other Büchi operations for language containment, intersection and emptiness testing, have highly efficient solutions [**?**] (cf. [49]), and thus Büchi complementation is the only bottleneck. Existing practical applications are thus forced to circumvent the need for Büchi complementation. This is possible, has however certain disadvantages as we will see in the following.

One way to circumvent the complementation of non-deterministic Büchi automata is to specify the property as a deterministic Büchi automaton [**?**][**?**]. As we have mentioned, the complementation of deterministic Büchi automata has an efficient solution. The disadvantage of this approach is, however, that the property automaton may become exponentially larger, and that it is generally more complicated and less intuitive to represent a language as a deterministic automaton [**?**].

Another way is to use a different model checking approach altogether, which leads us back to the essence of model checking. In basic model checking, the property to be verified is represented as a formula φ of a temporal logic (typically LTL). The system to verify is represnted as a Kripke structure K, which serves as an interpretation of the formula φ. The verification step consists in checking whether K is a *model* of φ. An interpretation K is a model of a formula φ, if every state of the interpretation *satisifes* the formula, written as $K \models \varphi$. This test of modelhood is the reason that this verification approach is called *model checking* [49].

The modelhood test can be done automata-theoretically without the need for Büchi complementation [**?**] (in Figure 1.1, this would be a sibling to the language containment approach). The Kripke structure K is translated to a Büchi automaton A_K. The formula φ is negated and translated to the Büchi automaton $A_{\neg\varphi}$. Finally, one tests $\mathsf{empty}(A_K \cap A_{\neg\varphi})$. This correponds to teh language containment test $L(K) \subseteq L(\varphi)$, which is equivalent to the modelhood test $K \models \varphi$. The trick is that the complementation of the property, that is required for the language containment test, is pushed off from the complementation of a Büchi automaton to the negation of a temporal logic formula, which is trivial. This approach is used, for example, by the SPIN model checker [**?**]. The disadvantage is that the typically used temporal logic LTL is less expressive than Büchi automata, and hence the breadth of properties that can be expressed is limited. It has been stated that the expressivity of LTL is unsufficient for industrial verification applications [**?**].

For more information on model checking, as well as other formal verification techniques, we refer to the following works: [**?**][**?**][**?**].

As can be seen from these elaborations, having efficient procedures for Büchi complementation would be of great practical value. Even though handling the "worst-cases" will forever be unefficient,

1.2 Motivation

In the previous section we have seen that Büchi complementation is complex, and that it would be of practical value to better understand this complexity. In this section, we highlight the need for looking at this complexity in a way that has not received much attention in the past, namely empirically rather than theoretically.

In the following, we first present the traditional way of analysing the worst-case performance of complementation constructions, and then describe the empirical way for investigating their actual peformance.

This includes a review of the work that has been done so far. Note that we are using the terms complexity and performance interchangeably, and they both mean basically state growth.

1.2.1 Theoretical Investigation of Worst-Case Performance

The traditional performance measure for Büchi complementation constructions is their *worst-case state growth*[5]. This is the maximum number of states the construction *can* generate, in relation to the number of states of the input automaton.

For example, the initial complementation construction by Büchi (1962) [4] has a worst-case state growth of $2^{2^{O(n)}}$ does not mean that it produces a larger complement than Schewe's construction, for this concrete example. It might well be smaller. In fact, worst-case state complexities only allow to adequately deduce something about the specific worst-cases, and not about all the other automata. From a practical point of view, these worst cases are however not interesting, as their application is impracticable anyway (at least starting from a certain input automaton size). , where n is the number of states of the input automaton. At this point, two short comments. First, the state growth is often not given as an exact function, but uses the big-O notation. Second, for notating state growths, we will consistenly use the variable n, whose meaning is the number of states of the input automaton. This means, for example, that for an input automaton with 8 states, the maximum number of states that the output automaton of Büchi's construction can have is 1.16×10^{77} (if assuming the concrete function 2^{2^n}).

Different constructions exhibit different worst-case state growths, and one of the main objectives of construction creators is to reduce this number. For example, the much more recent construction by Schewe (2009) [35] has a worst-case state growth as low as $(0.76n)^n + n^2$. Given an input automaton with 8 states, the maximum number of states of the output automaton is approximately 119.5 million.

A related objective of research is the quest for the theoretical worst-case state growth of Büchi complementation *per se*. A first result of $n!$ has been proposed in 1988 by Michel [20]. He proved that there exists a family of automata whose complement *cannot* have less than $n!$ states (these automata are known as Michel automata, and we will use them as part of the test data for our experiments). This proves a *lower bound* for the fundamental worst-case complexity of Büchi complementation, as it is not known whether the Michel automata are the real worst cases, or if there are even worse cases. Indeed, in 2007, Yan [55] proved a new higher lower bound of $(0.76n)^n$ (Michel's $n!$ corresponds to approximately $(0.36n)^n$ [55]). The worst-case state growth of a concrete construction naturally serves as an *upper bound* to a known lower bound. Given Schewe's number $(0.76n)^n n^2$, the lower bound of $(0.76n)^n$ by Yan is regarded as "sharp", as the gap between the lower and upper bound is very narrow, and consequently, the lower bound canot rise much anymore.

Many construction developers aim at bringing the worst-case state growth of their construction close to the currently known lower bound. It goes so far that a construction matching this lower bound is regarded as "optimal".

1.2.2 Need for Empirical Investigation of Actual Performance

Worst-case state growths are interesting from a theoretical point of view, but they are poor guides to the actual performance of a construction [42]. For example, if we have a concrete automaton of, say, 15 states, and we complement it with Schewe's construction, the fact that the worst-case state complexity is $(0.76n)^n n^2$ does not reveal anything about how the construction will perform on this concrete automaton. In any case, we are not expecting the complement to have 1.6 quintillion (1.6×10^{18}) states (which would be the worst case), because this would most likely be practically infeasible.

Furthermore, if a construction has a higher worst-case state growth than another, it does not mean that it performs worse on a concrete case. In fact, worst-case state complexities only allow to adequately deduce the performance on the worst-case automata, but not on all the other automata. However, from a practical point of view, these worst cases are not interesting, as their application in practice is anyway infeasible [**?**] (at least starting from a certain input automaton size).

[5] As mentioned previously, also known as state complexity, state blow-up, or state explosion.

From a practical perspective we are interested how constructions perform on automata as they could occur in a concrete application of Büchi complementation, such as automata-theoretic model checking. This may include questions like the following. What is a reasonable complement size to expect for the given automaton with n states? Are there generally easier and harder automata? What are the factors that make an automaton especially easy or hard? How does the performance of different constructions on the same automata vary? Are there constructions which are better suited for a certain type of automata than other constructions?

Questions like this can be attempted to answer by empirical peformance investigations. As its two most important elements this includes an *implementation* of the investigated constructions and *test data*. With test data, we mean a set of concrete automata on which the implementations of the constructions are run. The analysis is then done on the generated complement automata.

There have been relatively few empirical attempts in the history of Büchi complementation [42], compared to the number of theoretical works. In the following, we give (non-exhaustive) list of empirical works in the past that illustrate the approach, and also show the line of research in which the work of this thesis is situated.

1995 Tasiran et al. [?] create an efficient implementation of Safra' construction[33] (determinisation-based) and used it for for automata-theoretic model checking tasks with the HSIS verification tool [?]. They state that they could easily complement property automata with some hundreds of states, however, they do not provide a statistical evaluation of the results.

2003 Gurumurthy et al. [?] implement Friedgut, Kupferman, and Vardi's construction [16] (rank-based) along with various optimisations that they propose as a part of the tool Wring [?]. They complement 1000 small automata, generated by translation from LTL formulas, and evaluate exeuction time, and number of states and transitions of the complement for the different versions of the construction.

2006 Althoff et al. [2] implement Safra's [33] and Muller and Schupp's [24] determinisation constructions[6] in a tool called OmegaDet, applied them on the Michel automata with 2 to 6 states, and compared the number of states of the determinised output automata.

2008 Tsay et al. [45] carry out a first comparative experiment with the publicly available[7] GOAL tool [44][45][46][43]. They include the constructions by Safra [33] (determinisation-based), Piterman [28] (determinisation-based), Thomas [41] (WAPA[8]), and Kupferman and Vardi[16] (rank-based or WAA[9]). These constructions are pre-implemented in GOAL. As the test data, they use 300 Büchi automata, translated from LTL formulas, with an average size of 5.4 states. They evaluate and compare execution times, as well as number of states and transitions of the complements.

2009 Kamarkar and Chakraborty [?] propose an improvement of Schewe's construction [35] (rank-based) and implement it, as well as Schewe's original construction, on top of the model checker NuSMV [?][?]. They run the constructions on 12 test automata and compare the sizes of the complements. Furthermore, they run the same tests with the constructions by Kupferman, and Vardi [16] (rank-based or WAA) and Piterman [28] (determinisation-based) within GOAL, and compare the results to the ones of their implementation of Schewe's construction.

2010 Tsai et al. [42] (paper entitled "State of Büchi Compelentation") carry out another experiment with GOAL. They compare the constructions by Piterman [28] (determinisation-based), Schewe [35] (rank-based), and Vardi and Wilke [52] (slice-based), with various optimsiations that they propose in the same paper. As the test data, they use 11,000 randomly generated automata with 15 states and an alphabet size of 2. The test set is organised into 110 automata classes that consist of the combinations of 11 transition densities and 10 accceptance densities. This test set is repeatedly used in subsequent work (including in this thesis), and we will refer to it as the GOAL test set (because it has been generated with the GOAL tool). Tsai et al. provide sophisticated evaluation of the states of the complements for all the tested constructions and construction versions.

[6] These determinisation constructions transform a non-deterministic Büchi automaton to a deterministic Rabin automataon (see Section ??), however, the are used as the base for determinisation-based complementation constructions.

[7] http://goal.im.ntu.edu.tw/wiki/doku.php

[8] Via **W**eak **A**lternating **P**arity **A**utomaton

[9] Via **W**eak **A**lternating **A**utomaton

2010 Breuers [?] proposes an improvement for the construction by Sistla, Vardi, and Wolper [37] (Ramsey-based), and creates an implementation of it. He generates his own test data (inspired by the work of Tsai et al. [42]) consisting of *easy*, *medium*, and *hard* automata, based on different transition density and acceptance density values. He evaluates the complement sizes produced by the construction for autoamta of sizes 5, 10, and 15 of all these difficulty categories.

2012 Breuers et al. [3] wrap the implementation of their improvement of Sistla, Vardi, and Wolper's construction [37] in the publicly available tool Alekto[10], and and run it on the GOAL test set. They compare the generated complement sizes, as well as the number of aborted complementation tasks (due to exceeding resource requirements) to the corresponding result for different constructions on the same test set by Tsai et al. [42].

2013 Göttel [8] creates a C implementation of the Fribourg construction [1], including the R2C optimisation (see Chapter **??**), and executes it on the GOAL test set, as well as on the Michel automata with 3 to 6 states. He analyses the resulting complement sizes and execution times separately for each of the 110 classes that the GOAL test set consists of. The Fribourg construction[11] is a slice-based complementation construction that is being developed at the university of Fribourg, and which lies at the heart of this thesis. The entire Chapter **??** of this thesis is dedicated to explaining the Fribourg construction.

1.3 Aim and Scope

The aim of this thesis is an in-depth empirical performance investigation of the Fribourg construction. As mentioned, the Fribourg construction is a Büchi complementation construction that is being developed at the University of Fribourg [1]. By empirically investigating the behaviour of this specific construction, we want to follow up the track of empirical research that we have outlined in the last section.

This thesis is certainly not sufficient to describe the performance of the Fribourg construction in its entiretey, or in a way that is adequate to be relied on in industiral applications. Neither this thesis can answer general questions about the observed behaviour of Büchi complementation. Rather, we see this piece of work as a mosaic stone that we add to the very complex and multi-faceted picture of the complexity of Büchi complementation.

The empirical performance investigation will include testing of different versions of the construction, and comparison with other complementation constructions...

Aim: empirical performance investigation of a specific Büchi complementaiton construction, comparison with other constructions

Scope: two test sets, relatively small automata, no real world or "typical" examples,

1.4 Overview

[10] http://www.automata.rwth-aachen.de/research/Alekto/

[11] The authors of the constructions use the name *subset-tuple construction* (see [1]), however, in this thesis, we will use the name *Fribourg construction*.

Chapter 2

Background

Contents

In this chapter we treat several topics that serve as a background for the rest of the thesis. In particular, the goal of this chapter is to set the stage for our description of the Fribourg construction in Chapter **??**, and the setup of our empirical performance investigation of the Fribourg construction in Chapter **??**, as well as its results in Chapter **??**.

In Section 2.1 we summarise some aspects about Büchi automata, as well as about other types of automata on infinite words, that are relevant to the purpose of Büchi complementation. In Section 2.2, we describe the principal techniques for run analysis of non-deterministic automata. This topic has a particular relation to Büchi complementation, and one of the run analysis techniques, reduced split trees, is the core of the Fribourg construction. In Section 2.3, we provide a review of proposed Büchi complementation constructions from the introduction of Büchi automata in 1962 until today. We organise the presentation of these constructions along the four main Büchi complementation approaches Ramsey-based, determinisation-based, rank-based, and slice-based. Some of the constructions that we describe in this section will be used in the performance investigation of the Fribourg construction in Chapter **??**.

2.1 Büchi Automata and Other Omega-Automata

Büchi automata are a type of ω-automata. These are fninite state automata that run on infinite words (so-called ω-words). Externally, ω-automata look the same as the traditional finite state automata on finite words. It is possible to interpret any such automaton on finite words as an ω-automaton, and vice versa.

The difference between ω-automata and automata on finite words is their acceptance condition. An automaton on finite words accepts a word, if after finishing reading it, the automaton is in an accepting state. For ω-automata, this acceptance condition is not possible, because an ω-automaton never finishes reading a word (because the word "never ends"). Instead, the acceptance condition of ω-automata is defined on the set of the so-called *infinitely recurring states*. We are going to describe this concept in Subsection 2.1.1 below.

In Subsection 2.1.1 of this section, wer first treat Büchi automata, and in Subsection 2.1.2 the principal other types of ω-automata, in particular, Muller, Rabin, Streett, and parity automata. In the latter subsection, we also introduce a shorthand notation for different types of ω-automata that we will use throughout the thesis.

Note that in the entire section we omit overly formal notation and proofs of any kind, because it is not necessary for the aim of this thesis. More comprehensive and formally rigorous treatments of ω-automata can be found, for example, in the works by Thomas [39, 40], or Wilke [53].

2.1.1 Büchi Automata

Below we summarise the aspects of Büchi automata that are most significant for our purposes, including the acceptance condition of Büchi automata, the expressivity of non-deterministic and deterministic Büchi automata, and basics about the complementation of non-deterministic and deterministic Büchi automata. In the course of this, we always stress the difference between deterministic and non-deterministic Büchi automata, as this is one of the main sources for the intricacy of Büchi complementation [25].

Definition and Acceptance Condition

A non-deterministic Büchi automaton A is defined by the 5-tuple $A = (Q, \Sigma, \delta, q_0 F)$ with the following components:

- Q: a finite set of states
- Σ: a finite alphabet
- δ: a transition function, $\delta : Q \times \Sigma \to 2^Q$
- q_0: an initial state, $q_0 \in Q$

- F: a set of accepting states, $F \subseteq Q$

Note that this is the same definition as for non-deterministic finite state automata on finite words [10]. The difference between Büchi automata and automata on finite words is only the acceptance condition of Büchi automata that we describe below. For deterministic Büchi automata, the definition is similar to the above one, but with a different transition function δ that returns none or a single state, instead of a set of states

An important concept in automata theory the notion of a *run* of an automaton on a given word. A run ρ of automaton A on word α is a sequence of states $qinQ$ that A visits in the process of reading α. For finite words, the length of a run is finite as well, however, for ω-words, the length of a run may be infinite. Note that deterministic automata have at most one run for a given word, whereas non-deterministic automata may have multiple possible runs for the same word.

The Büchi acceptance condition decides whether a run ρ is accepting or non-accepting. This in turn determines the acceptance or non-acceptance of a word: a word α is accepted by an automaton A, if and only if it has at least one accepting run in A. The decision whether a run ρ is accepting or non-accepting is based on the set of *infinitely recurring states* of ρ that we denote by $\mathsf{inf}(\rho)$. This set contains all the states that occur infinitely often in ρ. In particular, the Büchi acceptance condition is as follows:

$$\text{Run } \rho \text{ is accepting} \iff \mathsf{inf}(\rho) \text{ contains at least one accepting state}$$

That is, a run is accepting, if and only if the run contains at least one accepting state infintely often. Formally, this can be written as $\mathsf{inf}(\rho) \cap F \neq \varnothing$.

An intuitive way for describing the Büchi acceptance condition has been given by Vardi [49]. If we imagine the automaton having a green light that blinks whenever the automaton visits an accepting state, then the run is accepting if we observe the green light blinking infinitely many times. The fact that there are only finitely man accepting states, but the light blinks infinitely many times, implies that at least one accepting state is being visited infinitely often.

Expressivity

A particularity of Büchi automata is that deterministic and non-deterministic automata are *not* expressively equivalent. In particular, the class of languages corresponding to the deterministic Büchi automata is a strict subset of the class of languages corresponding to the non-deterministic automata. This result has been proved by Büchi himself in his 1962 paper [4].

This contrasts, for example, with finite state automata on finite words. In this case, non-deterministic and deterministic automata are expressively equivalent, and consequently every non-deterministic automaton can be turned into an equivalent deterministic automaton. With Büchi automata, however, this is not possible, because there exist languages that can be expressed by a non-deterministic Büchi automaton, but not by a deterministic one. An example of such a language is $(0+1)^*1^\omega$, that is, the language of all words of 0 and 1 ending with an infinite sequence of 1. A non-deterministic automaton representing this language, cannot be turned into an equivalent deterministic automaton [49, 32]. Because of this fact, we say that Büchi automata can *in general* not be determinised. This fact has implications on the complementation of non-deterministic Büchi automata, as we will see below.

The class of languages that is equivalent to the *non-deterministic* Büchi automata is the class of *ω-regular languages*. A formal description of the ω-regular languages can be found, for example, in [39, 40, 53]. Regarding deterministic Büchi automata, consequently the set of languages that is equivalent to them is a strict subset of the ω-regular languages.

Complementation

Non-deterministic Büchi automata are closed under complementation. This means that the complement of every non-deterministic Büchi automaton is another non-deterministic Büchi automaton. This result has been proved by Büchi in his 1962 paper [4][1]. Deterministic Büchi automata, on the other hand, are not closed under complementation [39]. In particular, this means that the complement of a deterministic Büchi automaton is still a Büchi automaton, however, possibly a non-deterministic one.

As we already mentioned, the algorithmic difficulty and complexity of complementation is very different for deterministic and non-deterministic automata. For deterministic Büchi automata, there exists a simple procedure, introduced in 1987 by Kurshan [17], that can complement a deterministic Büchi automaton to a non-deterministic Büchi automaton in polynomial time and linear space.

For non-deterministic Büchi automata, however, there exists no easy solution. The main reason is that Büchi automata can in general not be determinised. If they could be determinised, then a solution would be to transform a non-deterministic Büchi automaton to an equivalent deterministic one, and complement the deterministic Büchi automaton with Kurshan's construction. This is by the wy the approach that is used for the complementation of non-deterministic automata on finite words: determinise a non-deterministic automaton with the subset construction [29], and then trivially complement the deterministic automaton by making the accepting states non-accepting, and vice versa. Unfortunately, for Büchi automata such a simple procedure is not possible, and this can be seen as the main reason that Büchi complementation is such a hard problem [25].

2.1.2 Other Omega-Automata

After the introduction of Büchi automata in 1962, several other types of ω-automata have been introduced. These automata differ from Büchi automata only in their acceptance condition, that is, in the way they decide whether a run ρ is accepting or non-accepting. All these acceptance condition are however based on the set of infintely recurring states $\mathsf{inf}(\rho)$ of ρ. The following are the most important of these alternative ω-automata along with the year of their introduction.

- Muller automata (1963) [22]
- Rabin automata (1969) [30]
- Streett automata (1982) [38]
- Parity automata (1985) [21]

Some of these automata types are used in complementation constructions, especially in determinisation-based complementation constructions (see Section 2.3). Table 2.1 lists the Muller, Rabin, Streett, and parity acceptance conditions, along with the Büchi acceptance condition for comparison.

Type	Definitions	Condition
Muller	$U \subseteq 2^Q$	$\mathsf{inf}(\rho) \in U$
Rabin	$\{(U_1, V_1), \ldots, (U_r, V_r)\}$, $U_i, V_i \subseteq Q$	$\exists i : \mathsf{inf}(\rho) \cap U_i = \varnothing \land \mathsf{inf}(\rho) \cap V_i \neq \varnothing$
Streett	$\{(U_1, V_1), \ldots, (U_r, V_r)\}$, $U_i, V_i \subseteq Q$	$\forall i : \mathsf{inf}(\rho) \cap U_i \neq \varnothing \lor \mathsf{inf}(\rho) \cap V_i = \varnothing$
Parity	$\pi : Q \to \{1, \ldots, k\}$, $k \in \mathbb{N}$	$\mathsf{min}(\{\pi(q) \mid q \in \mathsf{inf}(\rho)\}) \bmod 2 = 0$
Büchi		$\mathsf{inf}(\rho) \cap F \neq \varnothing$

Table 2.1: Acceptance conditions of Muller, Rabin, Streett, parity, and Büchi automata. Note that Q denotes the set of states and F the set of accepting states of the corresponding automaton.

Below, we briefly explain the acceptance conditions of Muller, Rabin, Streett, and parity automata in words.

Muller acceptance condition Includes a set U of subsets of states. A run ρ is accepting if and only if $\mathsf{inf}(\rho)$ equals one of the pre-defined subsets in U. The Muller acceptance condition is the most general one, and the Rabin, Streett, parity, and Büchi acceptance conditions can be expressed as Muller acceptance conditions [53, 18].

[1] Actually, the proof of closure under complementation of non-deterministic Büchi automata was a necessary step for Büchi to prove the equivalence of Büchi automata and S1S formulas (Büchi's Theorem). In order to prove this closure under complementation, Büchi described the first Büchi complementation construction in history.

Rabin acceptance condition Inludes a list of pairs (U, V) where U and V are subsets of states. A run ρ is accepting if and only if *there is* at least one pair for which U does not contain any states of $\mathsf{inf}(\rho)$ *and* V contains at least one state of $\mathsf{inf}(\rho)$. A pair (U, V) is called a Rabin pair.

Streett acceptance condition Inludes a list of pairs (U, V) where U and V are subsets of states. A run ρ is accepting if and only if *for all* the pairs either U contains at least one state of $\mathsf{inf}(\rho)$ *or* V does not contain any states of $\mathsf{inf}(\rho)$. A pair (U, V) is called a Streett pair. Note that the Streett condition is the complement of the Rabin condition. This means that if we have two identical Rabin and Streett automata with an identical list of pairs, then the Streett automaton accepts a run if and only if the Rabin automaton does not accept the same run.

Parity acceptance condition Assigns a number to each of the states of the automaton. A run ρ is accepting if and only if the smallest-numbered element of $\mathsf{inf}(\rho)$ has an even number. The numbers that are assigned to the states are sometimes called colours [18].

Regarding the expressivity of these automata, it turns out that they are all equivalent to non-deterministic Büchi automata, and thus to the ω-regular languages [53]. This holds for non-determnistic *and* deterministic automata of these types. That means that, unlike Büchi automata, deterministic and non-deterministic Muller, Rabin, Streett, and parity automata are expressively equivalent.

At this point we introdue a notation that we will occasionally use for denoting different types of ω-automata. This notation has been used by Piterman [27] and later by Tsai et al. [42]. It consists of three-letter acronymes of the form

$$\{N, D\} \times \{B, M, R, S, P\} \times W$$

The first letter specifies whether the automaton is non-deterministic (N) or deterministic (D). The second letter stands for the acceptance condition: B for Büchi, M for Muller, R for Rabin, S for Streett, and P for parity. The last letter specifies on which structure the automaton runs. In our case these are always words, thus the last letter is always W. For example, NBW means non-deterministic Büchi automaton, DBW means deterministic Büchi automaton, NMW means non-deterministic Muller automaton, DMW means deterministic Muller automaton, and so on.

2.2 Run Analysis of Non-Deterministic Automata

As mentioned, in a deterministic automaton, every input word has at most one run. In a non-deterministic automaton, however, every input word may have multiple runs. The analysis of the different runs of a non-deterministic automaton on a given input word is called *run analysis* and is central to the Büchi complementation problem.

The reason that run analysis is central to Büchi complementation is as follows. In a Büchi complementation task, we are given a non-deterministic Büchi automaton A, and we attempt to construct its complement B. For constructing B we have in fact to decide for every word α whether B must accept it or not. This decision is intrinsically tied to the entirety of the runs of A on α in the following way:

$$B \text{ accepts } \alpha \iff \textit{All} \text{ the runs of } A \text{ on } \alpha \text{ are } \textit{non-accepting}$$

B must accept a word α, if and only if *all* the runs of A on α are non-accepting. If, for example, A has 10 runs on α, and 9 of them are non-accepting and one is accepting, then A still accepts the word α, and consequently, B must not accept it. Only if all of the 10 runs of A on α are non-accepting, A does not accept α, and consequently B must accept α. This means that for constructing the complement B, we need to consider *all* the runs of the input automaton A on specific words, and the way this can be done is by run analysis on A.

In this section, we present the principal run analysis techniques for non-deterministic Büchi automata [53]. We start with *run DAGs* (DAG stands for directed acyclic graphs) in Section 2.2.1. Then in Sections **??**, 2.2.3, and 2.2.4, we present three techniques based on trees with increasing sophistication, namely *run trees*, *split trees*, and *reduced split trees*. The last of the tree techniques, reduced split trees, lies at the heart of the Fribourg construction that we describe in Chapter **??**.

In the following subsections, we will give examples for the different run analysis techniques that are based on the non-deterministic Büchi automaton in Figure 2.1. Note that the alphabet of this automaton is $\Sigma = \{a\}$, and thus the only ω-word in Σ^ω is a^ω. However, the automaton has mutliple (infinitely many) runs for this word.

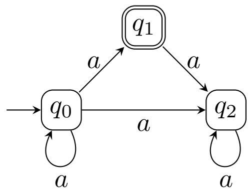

Figure 2.1: Non-deterministic Büchi automaton A used as example autoamton in this section.

2.2.1 Run DAGs

Run DAGs arrange all the runs of an automaton A on a word in a directed acyclic graph (DAG). This graph can be thought of as a matrix with rows and columns. The rows are called levels, and each column corresponds to a state of A. Consequently, the width (that is, number of columns) of a run DAG equals the number of states of A. Each level i (starting from 0) corresponds to the situation after reading the first i symbols of the word. Figure 2.2 shows the first five levels of the run DAG of the example automaton A in Figure 2.1 on the word a^ω.

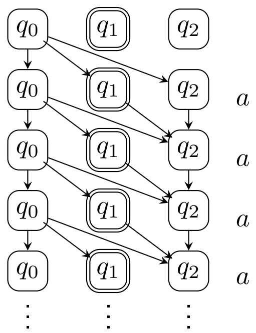

Figure 2.2: First few levels of the run DAG for the runs of automaton A (Figure 2.1) on the word a^ω.

Note that throughout this thesis, we are using rectangles with rounded corners for states of automata, and rectangles with sharp corners for vertices of graphs nodes of trees. In graphs and trees, we will however indicate vertices or nodes that correspond to accepting automaton states with double-lined rectangles.

As can be seen in Figure 2.2, every path of a run DAG corresponds to a run of the automaton on the given word. A run DAG is a structure that is able to represent an infinte number of runs by keeping a finite width. The number of levels of a run DAG is infinite for ω-words. A formal description of run DAGs can be found, for example, in [5].

Run DAGs are the basic structure for the rank-based complementation constructions, that we review in Section 2.3.3. Regarding this application, the fact that run DAGs have finite width is important, because the levels of run DAGs are mapped to states of the output automaton in these complementation constructions.

2.2.2 Run Trees

Trees in general are, after DAGs, the second structure that is used for run analysis. Run trees are the most basic of these tree structures. A run tree is basically a direct unwinding of all the runs of an automaton on a word as a tree. Figure 2.3 shows the first five levels of the run tree for the automaton A in Figure 2.1 on the word a^ω.

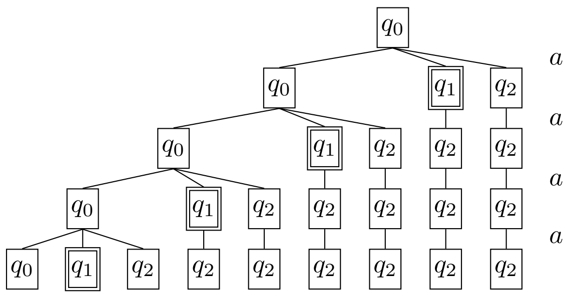

Figure 2.3: First five levels of the run tree for the runs of automaton A (Figure 2.1) on the word a^ω.

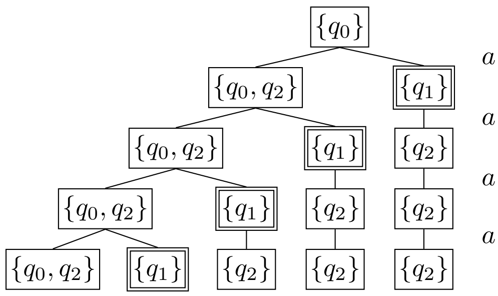

Figure 2.4: First five levels of the split tree for the runs of automaton A (Figure 2.1) on the word a^ω.

As can be see in Figure 2.3, there is a one-to-one mapping of paths in a run tree (from the root to the leaves) to runs of the corresponding automaton. This means that if the automaton has an infinite number of runs on a given word, then the corresponding run tree has an infinite maximum width. This makes run trees impractical to be used in Büchi complementation. The following tree techniques, split trees and reduced split trees, sacrifice a part of the information about individual runs, with the benefit of making the tree structure more compact.s

2.2.3 Split Trees

A split tree is basically a run tree where the accepting and non-accepting children of every node are merged. Consequently, a node of a split tree does not represent a single state of the automaton, but a set of states. The reduction of the number of children to at most two, makes the split tree furthermore a binary tree. Figure 2.4 shows the first five levels of the split tree of the automaton A (Figure 2.1) on the word a^ω.

Split trees sacrifice a part of the information about individual runs. For example, in Figure 2.4, we see that there must be a run that starts in q_0 (root node) and is in q_0 after reading four symbols (leftmost node on fifth level). However, the split tree does not provide the exact sequence of states. All that it reveals is that the sequence is $q_0 q_?^1 q_?^2 q_?^3 q_0$, where each $q_?^i$ is either q_0 or q_2. The run tree in Figure 2.3, on the other hand, shows the actual sequence unambiguously, which is $q_0 q_0 q_0 q_0 q_0$.

However, the split tree still provides an important piece of information, namely that the sequence $q_0 q_?^1 q_?^2 q_?^3 q_0$, whathever it actually looks like, does not contain any accepting states, because $q_?^i$ can only be q_0 or q_2, which are both non-accepting. It turns out that, with regard to the application of run analysis to Büchi complementation, this information is sufficient.

Split trees in fact embody a modified subset construction that does not mix accepting and non-accepting states. The result of applying such a construction to a non-deterministic automaton is an equivalent non-deterministic automaton whose degree of non-determinism is reduced to two (which means that each state has at most two successors for a given alphabet symbol). Such a construction has been described in [48].

A formal description of split trees can be found, for example, in [52] or [5]. Split trees are clearly more compact than run trees. However, their width may still grow infinitely. In the next section, we present a further reduction of the information contained in the tree, that bounds the maximum width of the tree to a finite number.

2.2.4 Reduced Split Trees

Reduced split trees are split trees that sacrifice even more information, but the information that is retained is still sufficient for Büchi complementation. The rule is that on each level, going from right to left or from left to right, only the first occurrence of each state is kept, and the others are omitted. This bounds the maximum width of the tree to the number n of states of the corresponding automaton, because each level of the tree may contain at most n nodes, one for each state of the automaton.

The direction in which the first occurrence of a state is determined depends on whether right-to-left or left-to-right split trees are used. In right-to-left split trees, the accepting child is put to the right of the non-accepting child. The split tree in Figure 2.4 is thus a right-to-left split tree. In left-to-right split trees, on the other hand, the accepting child is put to the left of the non-accepting child. In the literature both versions are used. For example, Vardi and Wilke [52] describe the left-to-right version, whereas Fogarty et al. [5] describe the right-to-left version. In this thesis, we will use exclusively the right-to-left version.

In a right-to-left reduced split tree, the direction in which the first occurrence of a state is determined is from right to left. This means that only the rightmost occurrence of each state on a level is kept and the others are omitted. Figure 2.5 shows the first five levels of the reduced split tree of the automaton A (Figure 2.1) on the word a^ω.

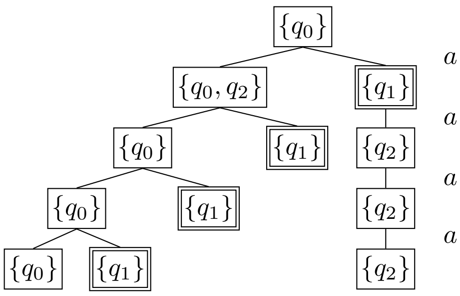

Figure 2.5: First five levels of the reduced split tree for the runs of automaton A (Figure 2.1) on the word a^ω.

As can be seen in Figure 2.5, each level contains at most one occurrence of each of the states q_0, q_1, and q_2. Comaring the reduced split tree in Figure 2.5 with the split tree in Figure 2.4 reveals which states are omitted in the reduced split tree. The root level and level 1 are similar in both trees. On level 2, the state q_2 in the leftmost node is omitted. This is because level 2 has already a q_2 to the right of it, namely in the rightmost node of the level. In level 3, there are two omissions of q_2 for the same reason. One of them causes an entire node to be omitted, because this node contained q_2 as its only state. Finally, on level 4, there are three omissions of q_2.

In the following, we illustrate what this omission of states entails and why the resulting "truncated" run analysis is still meaningful for Büchi complementaton. By omitting states, we obviously omit runs from the tree. This omission of runs is targeted at so-called *re-joining* runs. Re-joining runs are runs that afer a certain number of symbols end up in the same state again. For example, the automaton from Figure 2.1 has seven re-joining runs that after four steps end up in the state q_2. This can be easily seen in the corresponding run tree in Figure 2.3. A reduced split tree keeps of all the re-joining runs only exactly one, and the others are removed. This can be seen in Figure 2.5 which contains only one run from the root to q_2 on level 4.

The re-joining run that is kept is named *rightmost run*. It is called this way, because it is the first of a group of re-joining runs that makes a *right turn* in the corresponding path of the trees. Right turn means the transition from a parent to a right child, that is, an accepting child[2]. The rightmost run of a group of re-joining runs has the following property:

$$\text{Any re-joining run is accepting} \Longrightarrow \text{The rightmost re-joining run is accepting}$$

That is, if there is an accepting run in a group of re-joining runs, then the rightmost run is so too. On the other hand, if the rightmost run is non-accepting, then all the other re-joining runs are non-accepting as well. A proof of this relation can be found, for example, in [52][3].

Practically, this means that we can reduce the existential question about the presence of acceptance in a group of runs to a single run, namely the rightmost run. Remember that for Büchi complementation we are interested in exactly such an existential question: *is there* an accepting run of the automaton A on the word α? If yes (it does not matter how many accepting runs there are), then A accepts α and the complement must not accept it. If no, then A does not accept α and the complement must accept it. By considering only the rightmost runs, we can still reliably answer this question.

Thus, reduced split trees reduce the number of runs to include in the run analysis tremendously, but still retain all the needed information for Büchi complementation. Most importantly, they have a finite maximum width, what makes them usable in Büchi complementation constructions, such as the Fribourg construction (see Chapter **??**), and other slice-based constructions (see Section 2.3.4).

Note that an alternative name for rightmost run is *greedy run* [1]. This name refers to the fact that the rightmost run is the first of a group of re-joining runs that visits an accepting state, what makes it "greedy". It is interesting to note that reduced split trees are related to Muller-Schupp trees that are used in the Büchi determinisation construction by Muller and Schupp [24] (cf. [52, 5]). Finally, formal descriptions of reduced split trees can be found, for example, in [52] and [53] (left-to-right version), or [5] (right-to-left version).

2.3 Büchi Complementation Constructions

Since the introduction of Büchi automata in 1962, many constructions for complementing non-deterministic Büchi automata have been proposed.

2.3.1 Ramsey-Based Approach

The Ramsey-based approach has its name from a Ramsey-based combinatorial argument that is used in the complementation constructions. Ramsey was a British mathematician who lived at the beginning of the 20th century and founded a branch of combinatorics called the Ramsey theory [9].

Common to the Ramsey-based complementation constructions is that they stay completely within in the framework of Büchi automata. That is, they do not include intermediate automata of different types, as for example the determinization-based constructions. Rather, Ramsey-based constructions construct the complement automata directly by combinatorial operations on the states and transitions.

Büchi, 1962

The first Büchi complementation construction at all was described by Büchi himself, along with the introduction of Büchi automata in 1962 [4]. This complemenation construction is a Ramsey-based construction. It involves a combinatorial argument based on work by Ramsey [31]. The construction is complicated, and has a doubly-exponential worst-case state complexity of $2^{2^{O(n)}}$ [51]. This means that if we assume, for example, the concrete complexity to be 2^{2^n}, then an automaton with 6 states may result in a complement with at most 2^{2^6} states, which is more than 18 quintillions (18 billion billions).

[2]Note that everything we explain here is also valid for left-to-right split trees, but one has to exchange *right*, *rightmost*, and *right turn* with *left*, *leftmost*, and *left turn*, respectively.

[3]Lemma 2.6

The complexity of this worst-case is very high, and it would probably be impossible to complement such a worst-case automaton in practice. This is why all the subsequent complementation constructions, until today, have the goal to reduce this worst-case complexity. In this way, the worst-case state complexity became the main measure of performance for Büchi complementation constructions.

Sistla, Vardi, and Wolper, 1985

Another Ramsey-based construction has been introduced by Sistla, Vardi, and Wolper in 1987 [37] (first published in 1985 [36]). It is an improvement of Büchi's construction and the first one that involves only an exponential, instead of a doubly-exponential, worst-case state complexity. The complexity of this construction has been calculated to be $O\left(2^{4n^2}\right)$ (see [33][26]).

The Ramsey-based approach is the oldest of the four approaches and it was particularly

2.3.2 Determinization-Based Approach

The determinization-based complementation constructions proceed by converting an NBW to a deterministic automaton, complementing the deterministic automaton, and finally converting the complement automaton back to an NBW. The deterministic automaton cannot be a DBW (because NBW and DBW are not equivalent), however it can be a DMW, DRW, DSW, or DPW.

The idea behind this approach is that the complementation of deterministic ω-automata is easier than the complementation of non-deterministic ω-automata. The complementation problem is then in fact reduced to conversions between different types of automata. From these conversions, the conversion from the initial NBW to a deterministic ω-automaton is the most difficult and crucial one.

Safra, 1988

The first determinisation-based complementation construction has been described by Safra in 1988 [33]. Safra's main work was actually a determinisation construction for converting an NBW to a DRW. This is what today is known as *Safra's construction*. Safra then describes complementation as a possible application of his determinisation construction. He also presents the additional conversions that are needed for the entire complementation construction. The conversion steps of Safra's complementation procedure are as follows.

1. NBW $\longrightarrow$ DRW (Safra's construction)
2. DRW $\longrightarrow$ $\overline{\text{DSW}}$ (Complementation)
3. $\overline{\text{DSW}}$ $\longrightarrow$ $\overline{\text{DRW}}$
4. $\overline{\text{DRW}}$ $\longrightarrow$ $\overline{\text{NBW}}$

The complementation step from a DRW to a DSW that accepts the complement language can be trivially done by interpreting the Rabin acceptance condition as a Streett acceptance condition. This is possible, because these two acceptance conditions are the negations of each other (see Section **??**. The conversions from DSW to DRW, and from DRW to NBW are not of major difficulty or complexity, and are described by Safra in [33] (Lemma 3 and Lemma 5).

The core is the conversion from NBW to DRW (Safra's construction). This construction is basically a modified subset construction. That is, the output automaton is built up from an initial state step-by-step by adding new states and transitions. The main difference to the subset construction is that in Safra's construction, the output-states consist of trees of subsets of input-states, rather than just of subsets of input-states. These trees of subsets of states are called *Safra trees*. The details of the construction are rather intricate, but well described in [33]. The deterministic automaton that results from Safra's construction can then be interpreted as a Rabin automaton.

Description of Safra trees/construction: [2] [32]

The state growth of Safra's construction is $2^{O(n \log n)}$, where n is the number of states of the input automaton. The additional conversions (DSW to DRW, and DRW to NBW) have a lower state complexity than this, so that the overall complexity of the entire complementation procedure is still $2^{O(n \log n}$.

Muller and Schupp, 1995

Most other determinisation-based complementation constructions are based on improvements of Safra's construction. One of them is the construction for converting NBW to DRW proposed in 1995 by Muller an Schupp. This construction is said to be simpler and more intuitive than Safra's construction [32], however, often produces larger output automata in practice [2]. The theoretically caluclated state complexity of the Muller-Schupp construction is $2^{O(n \log n)}$, that is, similar to Safra's construction. A comparison of the Muller-Schupp construction and Safra's construction can be found in [2].

Descriptionof Muller-Schupp trees: [2]

Piterman, 2007

Another improvement of Safra's construction has been proposed in 2007 by Piterman from EPF Lausanne [28] (first presented at a conference in 2006 [27]). This construction converts a NBW to a DPW, rather than a DRW. Piterman's construction uses a more compact version of Safra trees, which allows it to produce smaller output automata. The concrete worst-case state growth of Piterman's construction is $2n^n n!$, opposed to $12^n n^{2n}$ of Safra's construction [28]. Complementation with Piterman's construction is done in the following steps.

1. NBW $\longrightarrow$ DPW (Piterman's construction)
2. DPW $\longrightarrow$ $\overline{\text{DPW}}$ (Complementation)
3. $\overline{\text{DPW}}$ $\longrightarrow$ $\overline{\text{NBW}}$

The complementation step from a DPW to a DPW accepting the complement language can be trivially done by, for example, increasing the number of each state by 1. The conversion from a DPW to an NBW can alse be done without major complexity [42].

2.3.3 Rank-Based Approach

The rank-based approach was the third of the four proposed main complementation approaches. It does neither include Ramsey theroy, nor determinisation. Rather, it is based on run analysis with run DAGs. The link of run analysis with run DAGs to complementation is as follows. A run DAG allows to summarise all the possible runs of an automaton on a specific word. If all these runs are rejecting, then we say that the entire run DAG is rejecting. In this case, the automaton does not accept the word, and consequently, the complement automaton must accept this word. Conversely, if one or more runs in the run DAG are not rejecting, then the entire run DAG is not rejecting. In this case, the automaton accepts the word, and consequently, the complement automaton must no accept this word.

The information of whether a run DAG is rejecting or not is expressed with so-called ranks. These are numbers that are assigned to the vertices of a run DAG, one rank per vertex. These ranks are assigned in a way that each path of a run DAG eventually gets trapped in a rank. From this information it is then possible to deduce whether the run DAG is rejecting or not. This in turn determines whether the complement automaton must accept the given word, or not.

This entire analysis of run DAGs with ranks is included in a subset construction. This means that the individual run DAGs are not constructed explicitly for each word, but rather implicitly "on-the-fly" within the complement automaton under construction. From a practical point of view, this means that rank-based constructions proceed in a subset construction based fashion. That is, the construction of the complement automaton is started with an initial state, and then step-by-step, successor states are added. Each output state consists of subsets of input-states.

Klarlund, 1991

The first rank-based construction has been proposed in 1991 by Klarlund [12]. However, Klarlund used the term *progress measure* instead of *rank*. This is because he looked at the ranks as a measure for the "progress" of a run towards the satisfaction of a certain property. The term *rank* has, to the best of our knowledge, been introduced by Thomas in 1999 [41]. Klarlund also did not mention run DAGs, but they are implicit in his description of the construction. The construction works as described above by performing a modified subset construction.

Kupferman and Vardi, 1997/2001

This construction by Kupferman and Vardi has been published as a preliminary conference version in 1997 [15], and as a journal version in 2001 [16]. Both publications are entitled "Weak Alternating Automata Are Not That Weak". The idea of the construction described by Kupferman and Vardi is the same as Klarlund's construction from 1991 [12]. However, Kupferman and Vardi provide two different descriptions for this idea.

The first description does not use run DAGs and ranks, but rather converst the input automaton to a weak alternating automaton, which is complemented, and then converted back to a non-deterministic Büchi automata. Weak alternating automata (WAA) have been introduced in 1986 by Muller, Saoudi, and Schupp [23]. Kupferman and Vardi state that this construction is conceptually simpler and easier implementable than Klarlund's construction [12]. This first version of Kupferman and Vardi's construction is described in both, the publications from 1997 [15] and 2001 [16].

Description of alternating automata: [49] (Section 2.5)

The second description in turn is rank-based, as described above, and works in the subset construction fashion without intermediate automata. Kupferman and Vardi point out that this version of the construction is identical to Klarlund's construction. What changes is just the terminology, for example "ranks" instead of "progress measure". This second version of Kupferman and Vardi's construction is to the best of our knowledge only described in the publication from 2001 [16], however, we are not sure, because we could not access the publication from 1997[15].

There is an *odd ranking* if and only if all the runs of the run DAG are rejecting. Odd ranking: all the paths get trapped in an odd rank. Only non-accepting states have odd ranks.

Description in [5] [50]

The automata produced by the two versions of Kupferman and Vardi's construction are identical. The worst-case state complexity has been calculated to be approximately $(6n)^n$ [35][50].

Thomas, 1999

This construction by Thomas [41] is based on the WAA construction by Kupferman and Vardi from 1997 [15]. It uses the concept of ranks, but does not proceed in the subset construction manner, as Klarlund's construction [12] and Kupferman and Vardi's second version [16]. Rather, it transforms the input NBW to an intermediate automaton, complements it, and converts the result back to an NBW. That is, it proceeds in a similar fashion as Kupferman and Vardi's first version [15]. The type of the intermediate automaton is a weak alternating parity automaton (WAPA), that is, a weak alternating automaton with the parity acceptance condition.

Friedgut, Kupferman, and Vardi, 2006

In 2006, Friedgut, Kupferman, and Vardi published a paper entitled "Büchi Complementation Made Tighter" [7] (a preliminary version of the paper has appeared in 2004 [6]). There, they describe an improvement to the second (rank-based) version of Kupferman and Vardi's construction from 2001 [16]. The improvement consists in the so-called *tight ranking*, a more sophisticated ranking function. It allows to massively reduce the worst-case state complexity of the construction to $(0.96n)^n$.

Tight rankings: description in [5] [50]

Schewe, 2009

In 2009, Schewe presented another improvement to the construction by Friedgut, Kupferman, and Vardi from 2006 [35]. His paper is entitled "Büchi Complementation Made Tight", which hints at the relation to the paper by Friedgut, Kupferman, and Vardi [7]. Schewe's improvement consists in a further refinement of the construction, in particular the use of turn-wise tests in the cut-point construction step. This improvement allows to further reduce the worst-case state complexity of the construction to $(0.76\,(n+1))^{n+1}$. This coincides, modulo a polynomial factor, with the lower bound for the state complexity of Büchi complementation of $(0.76n)^n$ that has been previously established by Yan in 2006 [54][55].

This result narrows down the possible range for the real worst-case state complexity of Büchi complementation considerably. It cannot be lower than the lower bound of $(0.76n)^n$ by Yan, and it cannot be higher than the complexity of Schwewe's construction of $(0.76\,(n+1))^{n+1}$. For this reason, we say that the proven worst-case complexity of a specific construction serves as an upper bound for the actual complexity of the problem.

2.3.4 Slice-Based Approach

The slice-based approach was the last approach that has been proposed. Its idea is very similar to the rank-based approach, but the main difference is the use of reduced split trees instead of run DAGs. The basic idea is to look at a state of the output automaton under construction as a horizontal level of a reduced split tree. Based on this, for each alphabet symbol, the succeeding level of the reduced split tree is determined, which results in a new state in the output automaton. These levels of reduced split trees are called *slices*, hence the name slice-based approach.

Like rank-based constructions, slice-based construction are essentially enhanced subset constructions. The slice-based constructions, however, include two runs of a subset construction, where the second one is typically more sophisticated than the first one.

Vardi and Wilke, 2007

The first slice-based Büchi complementation construction has been proposed in 2007 by Vardi and Wilke [52]. In this work, the authors review translations from various logics, including monadic second order logic of one successor (S1S), to ω-automata. They devise the slice-based complementation construction as a by-product of a determinisation construction for Büchi automata that they also introduce in this work.

Vardi and Wilke use left-to-right reduced split trees for their construction. That means, accepting states are put to the left of non-accepting states, and only the left most occurrence of each state is kept. The construction works by two passes of the enhanced subset construction. The first one (initial phase) is as described above. The second one (repetition phase), does additionally include decorations of the vertices of the reduced split trees (subsets) consisting of the three labels *inf*, *die*, and *new*. These decoration serves to keep track of the criterion that a word is rejected if and only if all of the branches of the corresponding reduced split tree contain only a finite number of left-turns. The worst-case state complexity of Vardi and Wilke's construction is $(3n)^n$ [52].

The slice-based construction by Vardi and Wilke is very similar to the Fribourg construction that we describe in Chapter **??**. An obvious difference is that the Fribourg construction uses right-to-left, rather than left-to-right, reduced split trees. However, this is an arbitrary choice, and has no influence on the result of the constructions. Another difference is that the transition from the initial phase to the repetition phase is handled quite differently by Vardi and Wilke, than for the corresponding automata parts in the Fribourg construction.

Kähler and Wilke, 2008

The slice-based construction by Kähler and Wilke from 2008 [11] is a generalisation of the construction by Vardi and Wilke from 2007 [52]. Kähler and Wilke proposed a construction idea that can be used for both, complementation and disambiguation. Consequently, this construction is less efficient than Vardi and Wilke's construction. It has a worst-case state complexity of $4(3n)^n$ [42].

A comparison of the rank-based and slice-based complementation approaches has been done by Fogarty, Kupferman, Wilke, and Vardi [5]. In this work, the authors also describe a translation of the slice-based construction by Kähler and Wilke [11] to a rank-based construction.

Appendix A

Plugin Installation and Usage

Since between the 2014–08–08 and 2014–11–17 releases of GOAL certain parts of the plugin interfaces have changed, and we adapted our plugin accordingly, the currently maintained version of the plugin works only with GOAL versions 2014-11-17 or newer. It is thus essential for any GOAL user to update to this version in order to use our plugin.

Appendix B

Median Complement Sizes of the GOAL Test Set

Bla bla bla

	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	269	308	254	236	238	297	266	156	207	68
1.2	960	1,407	1,479	2,150	1,152	1,090	942	1,206	718	104
1.4	3,426	2,915	2,752	3,393	2,693	3,265	2,263	2,425	1,844	154
1.6	3,799	3,698	4,901	3,926	3,960	3,655	2,580	1,905	2,124	155
1.8	3,375	3,169	3,420	3,967	3,943	3,132	2,246	1,144	971	114
2.0	1,906	2,261	2,383	2,884	2,354	2,096	1,169	932	568	98
2.2	1,467	1,633	1,795	1,942	1,611	1,640	569	499	330	78
2.4	924	1,232	1,319	1,317	1,056	886	514	314	182	59
2.6	625	763	880	945	828	684	316	175	132	44
2.8	483	584	836	690	575	395	240	151	103	41
3.0	319	450	557	523	367	313	155	116	84	32

(a) Fribourg

	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	269	308	254	236	238	297	266	156	207	68
1.2	960	1,407	1,479	2,150	1,152	1,090	942	1,206	718	104
1.4	3,426	2,915	2,752	3,393	2,693	3,265	2,263	2,425	1,844	154
1.6	3,799	3,698	4,901	3,926	3,960	3,655	2,580	1,905	2,124	155
1.8	3,375	3,169	3,420	3,967	3,943	3,093	2,246	1,144	971	114
2.0	1,906	2,184	2,383	2,818	2,354	1,989	1,127	885	568	97
2.2	1,410	1,561	1,639	1,884	1,609	1,588	496	464	284	78
2.4	884	1,200	1,234	1,184	939	806	373	256	165	55
2.6	575	731	815	860	751	575	246	162	114	43
2.8	431	530	672	466	371	274	174	120	85	36
3.0	232	325	344	360	269	169	91	85	53	27

(b) Fribourg+R2C

	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	390	438	434	324	328	459	337	204	227	40
1.2	1,576	2,394	2,505	2,996	1,613	1,551	1,166	1,542	1,002	58
1.4	5,007	4,336	4,652	4,877	3,458	3,956	3,169	3,380	1,868	86
1.6	5,067	5,032	6,444	4,868	4,575	3,864	3,211	1,731	1,892	85
1.8	4,016	3,701	3,647	4,523	3,548	3,009	1,808	451	336	62
2.0	1,663	2,276	2,676	3,035	1,925	1,932	464	307	150	54
2.2	989	1,514	1,621	1,826	1,121	846	155	127	93	45
2.4	560	821	919	771	529	267	133	87	55	32
2.6	388	519	524	441	259	219	84	50	41	26
2.8	311	317	396	242	165	95	64	44	33	22
3.0	173	224	211	169	102	72	41	34	27	18

(c) Fribourg+R2C+C

	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	225	223	195	181	187	199	189	124	161	68
1.2	731	971	946	1,071	629	562	488	568	388	104
1.4	2,228	1,701	1,543	1,732	1,241	1,287	945	944	727	154
1.6	2,489	2,263	2,331	2,133	1,777	1,443	964	757	889	155
1.8	2,381	2,027	2,009	2,075	1,618	1,243	1,005	592	515	114
2.0	1,390	1,569	1,416	1,573	1,093	1,008	594	464	330	98
2.2	1,118	1,197	1,150	1,151	879	809	317	330	241	78
2.4	712	885	836	809	580	535	316	231	145	59
2.6	498	569	601	627	497	412	217	137	113	44
2.8	391	455	578	456	374	263	173	119	90	41
3.0	258	350	392	354	253	208	119	97	74	32

(d) Fribourg+M1

	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	215	213	189	174	175	192	186	121	156	68
1.2	712	914	913	1,075	619	563	526	620	416	104
1.4	2,075	1,620	1,503	1,650	1,254	1,339	1,003	1,006	848	154
1.6	2,344	2,062	2,340	2,016	1,755	1,520	1,053	858	986	155
1.8	2,205	1,873	1,920	2,040	1,689	1,315	1,080	664	598	114
2.0	1,290	1,485	1,405	1,522	1,134	1,044	652	531	392	98
2.2	1,023	1,119	1,092	1,127	868	875	376	359	262	78
2.4	674	849	790	807	617	544	355	251	156	59
2.6	478	549	594	597	510	431	231	147	116	44
2.8	370	439	559	455	382	283	182	124	93	41
3.0	249	341	388	348	260	225	123	101	77	32

(e) Fribourg+M1+M2

	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	225	223	195	181	187	199	189	124	161	68
1.2	731	971	946	1,071	629	562	488	568	388	104
1.4	2,228	1,701	1,543	1,732	1,241	1,287	945	944	727	154
1.6	2,489	2,263	2,331	2,133	1,777	1,443	964	757	889	155
1.8	2,381	2,027	2,009	2,075	1,618	1,215	1,005	592	515	114
2.0	1,390	1,513	1,416	1,542	1,093	1,003	594	441	330	97
2.2	1,019	1,156	1,064	1,104	859	785	304	303	221	78
2.4	672	867	789	772	544	478	269	191	139	55
2.6	466	542	572	568	452	348	183	129	99	43
2.8	368	407	480	337	260	197	129	96	75	36
3.0	201	261	266	272	199	136	83	74	50	27

(f) Fribourg+M1+R2C

	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	329	303	279	240	229	288	230	157	160	40
1.2	988	1,392	1,356	1,352	751	741	608	704	516	58
1.4	2,939	2,581	2,066	2,190	1,351	1,622	1,132	1,261	932	86
1.6	3,150	2,900	2,842	2,218	1,885	1,563	1,177	821	896	85
1.8	2,782	2,485	2,047	2,180	1,625	1,269	855	395	309	62
2.0	1,338	1,638	1,544	1,566	979	957	349	261	147	54
2.2	838	1,125	993	1,027	667	521	153	125	93	45
2.4	494	700	624	524	296	214	126	87	55	32
2.6	327	434	383	334	212	163	82	50	41	26
2.8	283	273	305	202	144	95	60	44	33	22
3.0	164	200	173	142	92	72	41	34	27	18

(g) Fribourg+M1+R2C+C

	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	126	118	97	60	51	52	62	36	48	30
1.2	432	517	345	262	160	126	92	120	109	40
1.4	1,044	331	133	89	45	22	19	31	27	20
1.6	358	24	11	5	4	6	5	3	3	4
1.8	19	5	1	1	1	1	1	1	1	1
2.0	1	1	1	1	1	1	1	1	1	1
2.2	1	1	1	1	1	1	1	1	1	1
2.4	1	1	1	1	1	1	1	1	1	1
2.6	1	1	1	1	1	1	1	1	1	1
2.8	1	1	1	1	1	1	1	1	1	1
3.0	1	1	1	1	1	1	1	1	1	1

(h) Fribourg+R

Figure B.1: Median complement sizes of the 10,939 effective samples of the internal tests on the GOAL test set. The rows (1.0 to 3.0) are the transition densities, and the columns (0.1 to 1.0) are the acceptance densities.

	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	130	117	109	77	69	61	56	40	40	29
1.2	387	456	352	281	155	136	101	105	75	45
1.4	822	683	394	376	230	204	151	120	105	63
1.6	890	594	458	321	237	178	134	114	113	61
1.8	624	507	324	275	196	136	110	92	89	41
2.0	362	286	211	176	117	103	79	64	59	34
2.2	248	222	124	116	82	73	56	52	50	28
2.4	147	145	114	87	56	48	43	39	35	19
2.6	115	117	67	61	47	42	32	29	29	15
2.8	95	71	52	45	38	29	27	25	23	13
3.0	59	60	47	35	32	27	22	21	20	10

(a) Piterman+EQ+RO

	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
1.0	171	174	166	124	118	117	100	67	84	35
1.2	622	833	803	877	529	398	320	372	215	53
1.4	2,086	1,618	1,367	1,676	1,065	967	664	682	494	78
1.6	2,465	2,073	2,182	1,959	1,518	1,259	767	545	623	78
1.8	2,310	1,963	1,950	1,988	1,485	1,095	746	418	346	57
2.0	1,318	1,482	1,393	1,461	981	871	434	338	228	50
2.2	1,068	1,145	1,085	1,067	772	747	263	235	158	40
2.4	689	838	809	751	524	466	240	159	93	30
2.6	469	531	555	565	437	360	169	94	71	23
2.8	369	421	536	405	329	224	130	81	58	21
3.0	244	327	360	322	219	176	85	64	49	16

(b) Slice+P+RO+MADJ+EG

Figure B.2: Median complement sizes of the 10,998 effective samples of the external tests without the Rank construction. The rows (1.0 to 3.0) are the transition densities, and the columns (0.1 to 1.0) are the acceptance densities.

Appendix C

Execution Times

Construction	Mean	Min.	P25	Median	P75	Max.	Total	≈ hours
Fribourg	8.5	2.5	3.3	4.9	7.3	586.0	93,351.2	259
Fribourg+R2C	6.6	2.2	2.9	4.2	6.4	219.7	72,545.7	202
Fribourg+R2C+C	8.5	2.2	2.6	3.5	6.4	582.9	93,396.2	259
Fribourg+M1	4.9	2.5	3.2	4.1	5.9	55.1	54,061.3	150
Fribourg+M1+M2	4.6	2.2	2.9	3.8	5.1	38.4	49,848.0	138
Fribourg+M1+R2C	4.4	2.2	2.8	3.6	5.3	42.5	48,572.0	135
Fribourg+M1+R2C+C	5.6	2.5	3.2	4.0	6.5	147.4	60,918.9	169
Fribourg+R	7.5	2.2	3.0	3.9	6.3	470.5	82,387.3	229

Table C.1: Execution times in CPU time seconds for the 10,939 effectvie samples of the GOAL test set.

Construction	Mean	Min.	P25	Median	P75	Max.	Total	≈ hours
Piterman+EQ+RO	3.0	2.2	2.6	2.8	3.0	42.9	21,410.6	59
Slice+P+RO+MADJ+EG	3.7	2.2	2.7	3.2	4.1	36.7	26,398.9	73
Rank+TR+RO	16.0	2.3	2.8	3.7	9.3	443.3	115,563.9	321
Fribourg+M1+R2C	4.0	2.2	2.7	3.1	4.4	410.4	28,970.8	80

Table C.2: Execution times in CPU time seconds for the 7,204 effectvie samples of the GOAL test set.

Construction	Mean	Min.	P25	Median	P75	Max.	Total	≈ hours
Piterman+EQ+RO	3.6	2.2	2.7	2.9	3.4	365.7	39,663.4	110
Slice+P+RO+MADJ+EG	4.3	2.2	2.9	3.7	5.0	42.4	47,418.2	132
Fribourg+M1+R2C	4.7	2.2	2.8	3.6	5.3	410.4	52,149.0	145

Table C.3: Execution times in CPU time seconds for the 10,998 effectvie samples of the GOAL test set without the Rank construction.

Construction	Michel 1	Michel 2	Michel 3	Michel 4	Fitted curve	Std. error
Fribourg	2.3	4.0	88.8	100,976.0	$(1.14n)^n$	0.64%
Fribourg+R2C	2.3	3.4	27.4	27,938.3	$(0.92n)^n$	0.64%
Fribourg+M1	2.2	3.6	17.9	6,508.4	$(0.72n)^n$	0.63%
Fribourg+M1+M2	2.3	3.5	13.8	2,707.4	$(0.62n)^n$	0.62%
Fribourg+M1+M2+R2C	2.5	3.5	10.8	2,332.6	$(0.61n)^n$	0.62%
Fribourg+R	2.4	3.7	86.0	101,809.6	$(1.14n)^n$	0.64%

Table C.4: Execution times in CPU time seconds for the four Michel automata.

Construction	Michel 1	Michel 2	Michel 3	Michel 4	Fitted curve	Std. error
Piterman+EQ+RO	2.5	3.8	42.6	75,917.4	$(1.08n)^n$	0.64%
Slice+P+RO+MADJ+EG	2.3	3.6	11.4	159.5	$(0.39n)^n$	0.38%
Rank+TR+RO	2.2	3.0	6.4	30.0	$(0.29n)^n$	0.18%
Fribourg+M1+M2+R2C	2.5	3.5	10.8	2,332.6	$(0.61n)^n$	0.62%

Table C.5: Execution times in CPU time seconds for the four Michel automata.

Bibliography

[1] J. Allred, U. Ultes-Nitsche. Complementing Büchi Automata with a Subset-Tuple Construction. Tech. rep.. University of Fribourg, Switzerland. 2014.

[2] C. Althoff, W. Thomas, N. Wallmeier. Observations on Determinization of Büchi Automata. In J. Farré, I. Litovsky, S. Schmitz, eds., *Implementation and Application of Automata*. vol. 3845 of *Lecture Notes in Computer Science*. pp. 262–272. Springer Berlin Heidelberg. 2006.

[3] S. Breuers, C. Löding, J. Olschewski. Improved Ramsey-Based Büchi Complementation. In L. Birkedal, ed., *Foundations of Software Science and Computational Structures*. vol. 7213 of *Lecture Notes in Computer Science*. pp. 150–164. Springer Berlin Heidelberg. 2012.

[4] J. R. Büchi. On a Decision Method in Restricted Second Order Arithmetic. In *Proc. International Congress on Logic, Method, and Philosophy of Science, 1960*. Stanford University Press. 1962.

[5] S. J. Fogarty, O. Kupferman, T. Wilke, et al. Unifying Büchi Complementation Constructions. *Logical Methods in Computer Science*. 9(1). 2013.

[6] E. Friedgut, O. Kupferman, M. Vardi. Büchi Complementation Made Tighter. In F. Wang, ed., *Automated Technology for Verification and Analysis*. vol. 3299 of *Lecture Notes in Computer Science*. pp. 64–78. Springer Berlin Heidelberg. 2004.

[7] E. Friedgut, O. Kupferman, M. Y. Vardi. Büchi Complementation Made Tighter. *International Journal of Foundations of Computer Science*. 17(04):pp. 851–867. 2006.

[8] C. Göttel. Implementation of an Algorithm for Büchi Complementation. BSc Thesis, University of Fribourg, Switzerland. November 2013.

[9] R. L. Graham, B. L. Rothschild, J. H. Spencer. *Ramsey theory*. vol. 20. John Wiley & Sons. 1990.

[10] J. E. Hopcroft, R. Motwani, J. D. Ullman. *Introduction to Automata Theory, Languages, and Computation*. Addison-Wesley. 2nd edition ed.. 2001.

[11] D. Kähler, T. Wilke. Complementation, Disambiguation, and Determinization of Büchi Automata Unified. In L. Aceto, I. Damgård, L. Goldberg, et al, eds., *Automata, Languages and Programming*. vol. 5125 of *Lecture Notes in Computer Science*. pp. 724–735. Springer Berlin Heidelberg. 2008.

[12] N. Klarlund. Progress measures for complementation of omega-automata with applications to temporal logic. In *Foundations of Computer Science, 1991. Proceedings., 32nd Annual Symposium on*. pp. 358–367. Oct 1991.

[13] J. Klein. Linear Time Logic and Deterministic Omega-Automata. *Master's thesis, Universität Bonn*. 2005.

[14] J. Klein, C. Baier. Experiments with Deterministic ω-Automata for Formulas of Linear Temporal Logic. In J. Farré, I. Litovsky, S. Schmitz, eds., *Implementation and Application of Automata*. vol. 3845 of *Lecture Notes in Computer Science*. pp. 199–212. Springer Berlin Heidelberg. 2006.

[15] O. Kupferman, M. Y. Vardi. Weak Alternating Automata Are Not that Weak. In *Proceedings of the 5th Israeli Symposium on Theory of Computing and Systems*. pp. 147–158. IEEE Computer Society Press. 1997.

[16] O. Kupferman, M. Y. Vardi. Weak Alternating Automata Are Not that Weak. *ACM Trans. Comput. Logic*. 2(3):pp. 408–429. Jul. 2001.

[17] R. Kurshan. Complementing Deterministic Büchi Automata in Polynomial Time. *Journal of Computer and System Sciences*. 35(1):pp. 59 – 71. 1987.

[18] C. Löding. Optimal Bounds for Transformations of ω-Automata. In C. Rangan, V. Raman, R. Ramanujam, eds., *Foundations of Software Technology and Theoretical Computer Science*. vol. 1738 of *Lecture Notes in Computer Science*. pp. 97–109. Springer Berlin Heidelberg. 1999.

[19] R. McNaughton. Testing and generating infinite sequences by a finite automaton. *Information and Control*. 9(5):pp. 521 – 530. 1966.

[20] M. Michel. Complementation is more difficult with automata on infinite words. *CNET, Paris*. 15. 1988.

[21] A. Mostowski. Regular expressions for infinite trees and a standard form of automata. In A. Skowron, ed., *Computation Theory*. vol. 208 of *Lecture Notes in Computer Science*. pp. 157–168. Springer Berlin Heidelberg. 1985.

[22] D. E. Muller. Infinite Sequences and Finite Machines. In *Switching Circuit Theory and Logical Design, Proceedings of the Fourth Annual Symposium on*. pp. 3–16. Oct 1963.

[23] D. E. Muller, A. Saoudi, P. E. Schupp. Alternating automata, the weak monadic theory of the tree, and its complexity. In L. Kott, ed., *Automata, Languages and Programming*. vol. 226 of *Lecture Notes in Computer Science*. pp. 275–283. Springer Berlin Heidelberg. 1986.

[24] D. E. Muller, P. E. Schupp. Simulating Alternating Tree Automata by Nondeterministic Automata: New Results and New Proofs of the Theorems of Rabin, McNaughton and Safra. *Theoretical Computer Science*. 141(1–2):pp. 69 – 107. 1995.

[25] F. Nießner, U. Nitsche, P. Ochsenschläger. Deterministic Omega-Regular Liveness Properties. In S. Bozapalidis, ed., *Preproceedings of the 3rd International Conference on Developments in Language Theory, DLT'97*. pp. 237–247. Citeseer. 1997.

[26] J.-P. Pecuchet. On the complementation of Büchi automata. *Theoretical Computer Science*. 47(0):pp. 95 – 98. 1986.

[27] N. Piterman. From Nondeterministic Buchi and Streett Automata to Deterministic Parity Automata. In *Logic in Computer Science, 2006 21st Annual IEEE Symposium on*. pp. 255–264. 2006.

[28] N. Piterman. From Nondeterministic Buchi and Streett Automata to Deterministic Parity Automata. *Logical Methods in Computer Science*. 3(5):pp. 1–21. 2007.

[29] M. Rabin, D. Scott. Finite Automata and Their Decision Problems. *IBM Journal of Research and Development*. 3(2):pp. 114–125. April 1959.

[30] M. O. Rabin. Decidability of second-order theories and automata on infinite trees. *Transactions of the American Mathematical Society*. 141:pp. 1–35. July 1969.

[31] F. P. Ramsey. On a Problem of Formal Logic. *Proceedings of the London Mathematical Society*. s2-30(1):pp. 264–286. 1930.

[32] M. Roggenbach. Determinization of Büchi-Automata. In E. Grädel, W. Thomas, T. Wilke, eds., *Automata Logics, and Infinite Games*. vol. 2500 of *Lecture Notes in Computer Science*. pp. 43–60. Springer Berlin Heidelberg. 2002.

[33] S. Safra. On the Complexity of Omega-Automata. *Journal of Computer and System Science*. 1988.

[34] S. Safra. On the Complexity of Omega-Automata. In *Foundations of Computer Science, 1988., 29th Annual Symposium on*. pp. 319–327. Oct 1988.

[35] S. Schewe. Büchi Complementation Made Tight. In *26th International Symposium on Theoretical Aspects of Computer Science-STACS 2009*. pp. 661–672. 2009.

[36] A. Sistla, M. Vardi, P. Wolper. The complementation problem for Büchi automata with applications to temporal logic. In W. Brauer, ed., *Automata, Languages and Programming*. vol. 194 of *Lecture Notes in Computer Science*. pp. 465–474. Springer Berlin Heidelberg. 1985.

[37] A. P. Sistla, M. Y. Vardi, P. Wolper. The Complementation Problem for Büchi Automata with Applications to Temporal Logic. *Theoretical Computer Science*. 49(2–3):pp. 217 – 237. 1987.

[38] R. S. Streett. Propositional dynamic logic of looping and converse is elementarily decidable. *Information and Control*. 54(1–2):pp. 121 – 141. 1982.

[39] W. Thomas. Automata on Infinite Objects. In J. van Leeuwen, ed., *Handbook of Theoretical Computer Science (Vol. B)*. chap. Automata on Infinite Objects, pp. 133–191. MIT Press, Cambridge, MA, USA. 1990.

[40] W. Thomas. Languages, Automata, and Logic. In G. Rozenberg, A. Salomaa, eds., *Handbook of Formal Languages*. pp. 389–455. Springer Berlin Heidelberg. 1997.

[41] W. Thomas. Complementation of Büchi Automata Revisited. In J. Karhumäki, H. Maurer, G. Păun, et al, eds., *Jewels are Forever*. pp. 109–120. Springer Berlin Heidelberg. 1999.

[42] M.-H. Tsai, S. Fogarty, M. Vardi, et al. State of Büchi Complementation. In M. Domaratzki, K. Salomaa, eds., *Implementation and Application of Automata*. vol. 6482 of *Lecture Notes in Computer Science*. pp. 261–271. Springer Berlin Heidelberg. 2011.

[43] M.-H. Tsai, Y.-K. Tsay, Y.-S. Hwang. GOAL for Games, Omega-Automata, and Logics. In N. Sharygina, H. Veith, eds., *Computer Aided Verification*. vol. 8044 of *Lecture Notes in Computer Science*. pp. 883–889. Springer Berlin Heidelberg. 2013.

[44] Y.-K. Tsay, Y.-F. Chen, M.-H. Tsai, et al. Goal: A Graphical Tool for Manipulating Büchi Automata and Temporal Formulae. In O. Grumberg, M. Huth, eds., *Tools and Algorithms for the Construction and Analysis of Systems*. vol. 4424 of *Lecture Notes in Computer Science*. pp. 466–471. Springer Berlin Heidelberg. 2007.

[45] Y.-K. Tsay, Y.-F. Chen, M.-H. Tsai, et al. Goal Extended: Towards a Research Tool for Omega Automata and Temporal Logic. In C. Ramakrishnan, J. Rehof, eds., *Tools and Algorithms for the Construction and Analysis of Systems*. vol. 4963 of *Lecture Notes in Computer Science*. pp. 346–350. Springer Berlin Heidelberg. 2008.

[46] Y.-K. Tsay, Y.-F. Chen, M.-H. Tsai, et al. Tool support for learning Büchi automata and linear temporal logic. *Formal Aspects of Computing*. 21(3):pp. 259–275. 2009.

[47] Y.-K. Tsay, M.-H. Tsai, J.-S. Chang, et al. Büchi Store: An Open Repository of Büchi Automata. In P. Abdulla, K. Leino, eds., *Tools and Algorithms for the Construction and Analysis of Systems*. vol. 6605 of *Lecture Notes in Computer Science*. pp. 262–266. Springer Berlin Heidelberg. 2011.

[48] U. Ultes-Nitsche. A Power-Set Construction for Reducing Büchi Automata to Non-Determinism Degree Two. *Information Processing Letters*. 101(3):pp. 107 – 111. 2007.

[49] M. Vardi. An automata-theoretic approach to linear temporal logic. In F. Moller, G. Birtwistle, eds., *Logics for Concurrency*. vol. 1043 of *Lecture Notes in Computer Science*. pp. 238–266. Springer Berlin Heidelberg. 1996.

[50] M. Vardi. The Büchi Complementation Saga. In W. Thomas, P. Weil, eds., *STACS 2007*. vol. 4393 of *Lecture Notes in Computer Science*. pp. 12–22. Springer Berlin Heidelberg. 2007.

[51] M. Y. Vardi. Buchi Complementation: A Forty-Year Saga. In *5th symposium on Atomic Level Characterizations (ALC'05)*. 2005.

[52] M. Y. Vardi, T. Wilke. Automata: From Logics to Algorithms. In J. Flum, E. Grädel, T. Wilke, eds., *Logic and Automata: History and Perspectives*. vol. 2 of *Texts in Logic and Games*. pp. 629–736. Amsterdam University Press. 2007.

[53] T. Wilke. ω-Automata. In J.-E. Pin, ed., *Handbook of Automata Theory*. European Mathematical Society. To appear, 2015.

[54] Q. Yan. Lower Bounds for Complementation of ω-Automata Via the Full Automata Technique. In M. Bugliesi, B. Preneel, V. Sassone, et al, eds., *Automata, Languages and Programming*. vol. 4052 of *Lecture Notes in Computer Science*. pp. 589–600. Springer Berlin Heidelberg. 2006.

[55] Q. Yan. Lower Bounds for Complementation of omega-Automata Via the Full Automata Technique. *CoRR*. abs/0802.1226. 2008.