Adaptive Signal Processing
Final Project

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The project formulates a moving source and interference propagating to a uniform linear array with N elements and the inter-element spacing $d = \frac{\lambda}{2}$. The input $\mathbf{x}(t)$ of the array is

$$\mathbf{x}(t) = \mathbf{a}(\theta_s(t))\mathfrak{s}(t) + \mathbf{a}(\theta_i(t))\mathfrak{i}(t) + \mathbf{n}(t)$$

with time index $t = 1, 2, \ldots, L$. The steering vector is defined as

$$\mathbf{a}(\theta) = [1 \ e^{j\pi \sin\theta} \ e^{j2\pi \sin\theta} \ \ldots \ e^{j(N-1)\pi \sin\theta}]^T$$

The source signal is $\mathfrak{s}(t)$, the interference signal is $\mathfrak{i}(t)$, the noise vector is $\mathbf{n}(t)$, the source DOA is $\theta_s(t)$, and the interference DOA is $\theta_i(t)$. The goal of the project is to estimate $\theta_s(t), \theta_i(t)$ and design a beamformer to estimate the source signal.

1 Details of the following beamformers

Consider a uniform linear array with N isotropic antennas and the inter-element spacing $d = \frac{\lambda}{2}$. The receiving signal is

$$\mathbf{x}(t) = \mathbf{a}(\theta)s(t) + \mathbf{n}(t)$$

with the steering vector

$$\mathbf{a}(\theta) = [1 \ e^{j\pi \sin\theta} \ e^{j2\pi \sin\theta} \ \ldots \ e^{j(N-1)\pi \sin\theta}]^T$$

1.1 Beamformer with uniform weights

The weight vector of uniformly weighted beamformer is

$$\mathbf{w} = \frac{1}{N} \underbrace{\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}}_{\mathrm{N}}$$

with the output

$$y(t) = \mathbf{w}^H\mathbf{x} = \mathbf{w}^H[\mathbf{a}(\theta)s(t) + \mathbf{n}(t)] = \mathbf{w}^H\mathbf{a}(\theta)s(t) + \mathbf{w}^H\mathbf{n}(t) = \mathbf{B}_\theta(\theta) + \mathbf{w}^H\mathbf{n}(t)$$

where $\mathbf{B}_\theta(\theta) = \mathbf{w}^H\mathbf{a}(\theta)$ is the beampattern of the beamformer.

The beampattern of uniformly weighted beamformer is

$$
\begin{aligned}
\mathbf{B}_\theta(\theta) =& \frac{1}{N}\begin{bmatrix}1\\1\\\vdots\\1\end{bmatrix}^H \begin{bmatrix}1\\e^{j\pi\sin\theta}\\\vdots\\e^{j(N-1)\pi\sin\theta}\end{bmatrix}\\
=& \frac{1}{N}\sum_{n=0}^{N-1} e^{j\pi\sin\theta n}\\
=& \begin{cases}\frac{1}{N}\times\frac{1-e^{jN\pi\sin\theta n}}{1-e^{j\pi\sin\theta n}} & \text{if } e^{j\pi\sin\theta n}\neq 1\\ 1 & \text{if } e^{j\pi\sin\theta n}=1\end{cases}\\
=& e^{j\frac{N-1}{2}\pi\sin\theta}\times\frac{1}{N}\times\frac{\sin(\frac{N}{2}\pi\sin\theta)}{\sin(\frac{1}{2}\pi\sin\theta)}, \text{ where } -90^\circ\leq\theta\leq 90^\circ
\end{aligned}
$$

It is obvious that $\mathbf{B}_\theta(\theta)$ has maximum 1 at $\theta = 0^\circ$.

1.2 Beamformer with array steering

If the direction of arrival (DOA) information is given, then we can steer the beamformer to the signal direction.

$$
\mathbf{w} = \frac{1}{N}\mathbf{a}(\theta_s) = \frac{1}{N}\begin{bmatrix}1\\e^{j\pi\sin\theta_s}\\\vdots\\e^{j(N-1)\pi\sin\theta_s}\end{bmatrix}
$$

Then the beampattern will be

$$
\begin{aligned}
\mathbf{B}_\theta(\theta) =& \frac{1}{N}\mathbf{a}^H(\theta_s)\mathbf{a}(\theta)\\
=& \frac{1}{N}\begin{bmatrix}1\\e^{j\pi\sin\theta_s}\\\vdots\\e^{j(N-1)\pi\sin\theta_s}\end{bmatrix}^H \begin{bmatrix}1\\e^{j\pi\sin\theta}\\\vdots\\e^{j(N-1)\pi\sin\theta}\end{bmatrix}\\
=& \frac{1}{N}\sum_{n=0}^{N-1} e^{-j\pi\sin\theta_s n}e^{j\pi\sin\theta n}\\
=& \frac{1}{N}\sum_{n=0}^{N-1} e^{j(\pi\sin\theta-\pi\sin\theta_s)n}\\
=& e^{j\frac{N-1}{2}(\pi\sin\theta-\pi\sin\theta_s)}\times\frac{1}{N}\times\frac{\sin[\frac{N}{2}(\pi\sin\theta-\pi\sin\theta_s)]}{\sin[\frac{1}{2}(\pi\sin\theta-\pi\sin\theta_s)]}, \text{ where } -90^\circ\leq\theta\leq 90^\circ
\end{aligned}
$$

Same as previous, $\mathbf{B}_\theta(\theta)$ has maximum 1 at $\theta = \theta_s$.

1.3 MVDR Beamformer

The minimum variance distortionless response (MVDR) beamformer is the result of the following optimization problem

$$
\begin{aligned}
\mathbf{w}_{MVDR} = \ & \underset{\mathbf{w}}{\arg\min} && \mathbb{E}[|\mathbf{w}^H\mathbf{x}|^2] = \mathbf{w}^H\mathbf{R}\mathbf{w}\\
& \text{subject to} && \mathbf{w}^H\mathbf{a}(\theta_s) = 1
\end{aligned}
$$

where $\mathbf{R} = \mathbb{E}[\mathbf{x}\mathbf{x}^H]$.
Solving the problem by Lagrange multiplier,

$$\begin{gathered}
\mathcal{L} = \mathbf{w}^H\mathbf{R}\mathbf{w} - \frac{1}{2}\lambda[\mathbf{w}^H\mathbf{a}(\theta_s) - 1] - \frac{1}{2}\lambda^*[\mathbf{w}^T\mathbf{a}^*(\theta_s) - 1] \\
\frac{\partial \mathcal{L}}{\partial \mathbf{w}^*} = \mathbf{R}\mathbf{w} - \frac{1}{2}\lambda\mathbf{a}(\theta_s) = 0, \quad \mathbf{w} = \frac{1}{2}\lambda\mathbf{R}^{-1}\mathbf{a}(\theta_s) \\
\mathbf{w}^H\mathbf{a}(\theta_s) = \frac{1}{2}\lambda\mathbf{a}^H(\theta_s)\mathbf{R}^{-1}\mathbf{a}(\theta_s) = 1, \quad \lambda = \frac{2}{\mathbf{a}^H(\theta_s)\mathbf{R}^{-1}\mathbf{a}}
\end{gathered}$$

We obtain the solution of the optimization problem,

$$\mathbf{w}_{MVDR} = \frac{\mathbf{R}^{-1}\mathbf{a}(\theta_s)}{\mathbf{a}^H(\theta_s)\mathbf{R}^{-1}\mathbf{a}(\theta_s)}$$

1.4 LCMV Beamformer

Although the MVDR beamformer can suppress noise optimally, the performance decreases in interfered environment. Thus, we add more constraint to the optimization problem.

$$\begin{aligned}
\mathbf{w}_{LCMV} = \ & \underset{\mathbf{w}}{\arg\min} \quad \mathbb{E}[|\mathbf{w}^H\mathbf{x}|^2] = \mathbf{w}^H\mathbf{R}\mathbf{w} \\
& \text{subject to} \quad \mathbf{C}^H\mathbf{w} = \mathbf{g}
\end{aligned}$$

where $\mathbf{C}^H \in \mathbb{C}^{N\times M}$ and $\mathbf{g} \in \mathbb{C}^{M\times 1}$.
Solving the problem again by Lagrange multiplier,

$$\begin{gathered}
\mathcal{L} = \mathbf{w}^H\mathbf{R}\mathbf{w} - \frac{1}{2}\boldsymbol{\lambda}^H[\mathbf{C}^H\mathbf{w} - \mathbf{g}] - \frac{1}{2}\boldsymbol{\lambda}^T[\mathbf{C}^T\mathbf{w}^* - \mathbf{g}^*] \\
\frac{\partial \mathcal{L}}{\partial \mathbf{w}^*} = \mathbf{R}\mathbf{w} - \frac{1}{2}\mathbf{C}\boldsymbol{\lambda} = \mathbf{0}, \quad \mathbf{w} = \frac{1}{2}\mathbf{R}^{-1}\mathbf{C}\boldsymbol{\lambda} \\
\mathbf{C}^H\mathbf{w} = \mathbf{g} = \frac{1}{2}\mathbf{C}^H\mathbf{R}^{-1}\mathbf{C}\boldsymbol{\lambda} = \mathbf{g}, \quad \lambda = 2(\mathbf{C}^H\mathbf{R}^{-1}\mathbf{C})^{-1}\mathbf{g}
\end{gathered}$$

We obtain the solution of the optimization problem,

$$\mathbf{w}_{LCMV} = \mathbf{R}^{-1}\mathbf{C}(\mathbf{C}^H\mathbf{R}^{-1}\mathbf{C})^{-1}\mathbf{g}$$

The result of the optimization problem is called the linearly constrained minimum variance (LCMV) beamformer.

2 DOA Denosing

2.0.1 Hilbert-Huang Transform and Empirical Mode Decomposition

For the denoising part, I decide to deploy the Hilbert-Huang Transform (HHT) from the Time-Frequency Analysis and Wavelet Transforms course this semester. The reason I choose HHT is the DOA of the signal and interference has a trend, which is the perfect scenario for HHT compared with other time-frequency analysis techniques.

The Hilbert-Huang Transform decomposes the signal into the sum of intrinsic mode functions (IMF) and the residual signal, which can be expressed as

$$x(t) = x_0(t) + \sum_{s=1}^{n} c_s(t)$$

where $x_0(t)$ is the residual signal, $c_s(t),\ s = 1, 2, \ldots, n$ are the intrinsic mode functions with different frequencies. As s getting larger, the frequency of $c_s(t)$ will be higher.

Intrinsic mode function is the relaxation of sinusoidal function, which its amplitude and frequency can vary with time. But two requirements are demanded :

- The total number of local maxima and local minima and the number of zero-crossings must either equal or differ at most by one.

- The mean function, defined as the mean of the upper envelope and the lower envelope of the function, must near to zero at defined interval.

The instantaneous frequency of the IMF can be calculate by counting the number of zero-crossings. Specifically speaking,

$$F_s(t) = \frac{Number\ of\ zero\ crossings\ of\ c_s(t)\ between\ t-B\ and\ t+B}{4B}$$

To simplify the discussion, we assume the instantaneous frequency of every IMF are all constants in this project.

The residual signal the the component of the original signal that has no frequency, which implies the trend of the signal.

The procedure of obtaining the residual signal and IMFs is called the empirical mode decomposition (EMD). The procedure is described as step 1 to step 25 at Algorithm 1.

Because noise is a high-frequency signal, so we can denoise the DOA signal by getting rid of the IMFs with high frequency, which is more likely to be the noise component. As for how much IMFs should be discarded, my approach is to roughly estimate the frequency by finding the number of the prominence extremes of the signal. Then compare the frequency between the signal and each IMF by comparing the signal's prominence extremes and each IMF's zero-crossing amount. When the ith IMF's frequency is higher than the signal's frequency, I will discard the $(i+1)$th IMF to the last IMF and sum up the rest component. The result is the denoised signal, which is shown in figure 1. The steps of DOA denoising are described in detail at Algorithm 1.

2.0.2 Algorithm Summarization

Algorithm 1 DOA Denoising

$x(t) \leftarrow \tilde{\theta}(t), n \leftarrow 1, k \leftarrow 1$
while 1 **do**
  while 1 **do**
    Find the local peaks
    Connect local peaks $e_{max}(t)$
    Find the local dips
    Connect local dips $e_{min}(t)$
    Compute the mean $z(t) = \frac{1}{2}[e_{max}(t) + e_{min}(t)]$
    Compute the residue $h_k(t) = x(t) - z(t)$
    if *All local maxima* > 0 *and local minima* < 0 *and* $|mean\ of\ h_k(t)| < \epsilon\ \forall t$ **then**
      $c_n(t) \leftarrow h_k(t)$ ▷ Determine whether $h_k(t)$ is an IMF
      break
    else
      $x(t) \leftarrow h_k(t)$
      $k \leftarrow k + 1$
    end if
  end while
  $\tilde{\theta}_0(t) \leftarrow \tilde{\theta}(t) - \sum_{s=1}^{n} c_s(t)$
  if $x_0(t)$ *has no more than one extreme point* **then**
    break ▷ Determine whether $x_0(t)$ is the residual signal
  else
    $x(t) \leftarrow \tilde{\theta}_0(t)$
    $n \leftarrow n + 1$
  end if
end while
Find the number of zero-crossings of each IMF $r_s,\ s = 1, 2, \ldots, n$
Find the number of prominence extremes of residual signal Q
if $r_i > Q$ **then**
  $\hat{\theta}(t) \leftarrow \sum_{s=1}^{i+1} c_s(t)$
end if

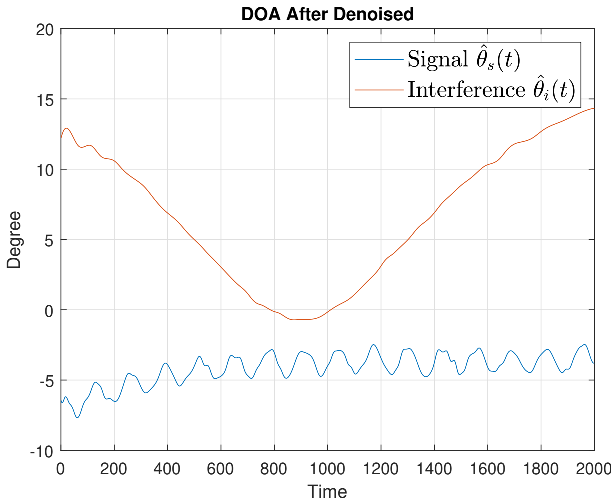


Figure 1: Denoised DOA

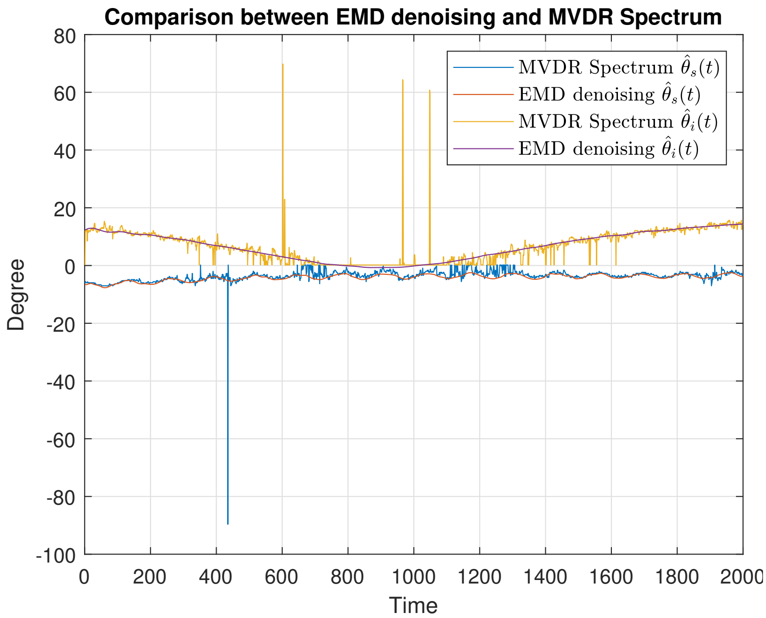


Figure 2: The comparision between EMD-based denoising and MVDR spectrum

2.0.3 Comparison and Advantages

Another commonly adopted method for DOA estimation is the MVDR spectrum. The MVDR spectrum is defined as

$$P_{MVDR} = \frac{1}{\mathbf{a}^H(\theta)\mathbf{R}^{-1}\mathbf{a}(\theta)}$$

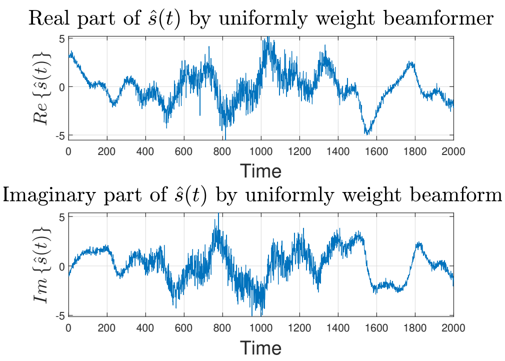
Figure 2 is a comparison between EMD-based denoising and MVDR spectrum. We can find that EMD-based denoising performs better than MVDR spectrum. The main reason is because the MVDR spectrum depends on correlation matrix $\mathbf{R}$, which depends on the received signal $x(t)$ containing $\tilde{\theta}_s(t)$, $\tilde{\theta}_i(t)$ and $\mathbf{n}(t)$. So the MVDR spectrum of source signal, for example, will be affected by not only the noise of $\tilde{\theta}_s(t)$, but also the noise of $\tilde{\theta}_i(t)$ and vector noise $\mathbf{n}(t)$, causing the result more worse. On the other side, EMD-based denoising will only be affected by the noise of $\tilde{\theta}_s(t)$, providing better result than MVDR spectrum.

3 Beamformer Design

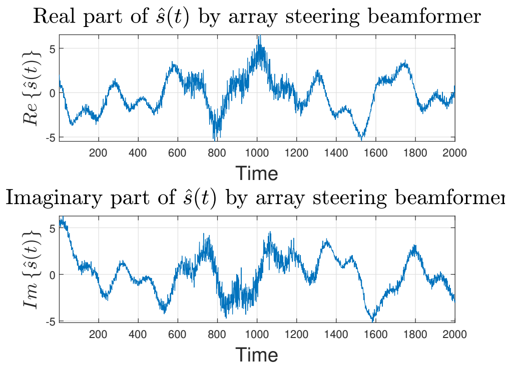
3.1 Results of the previous beamformers

Before discussing the proposed beamformer, I have simulated the results of all beamformers in section 1, which are shown in figure 3.

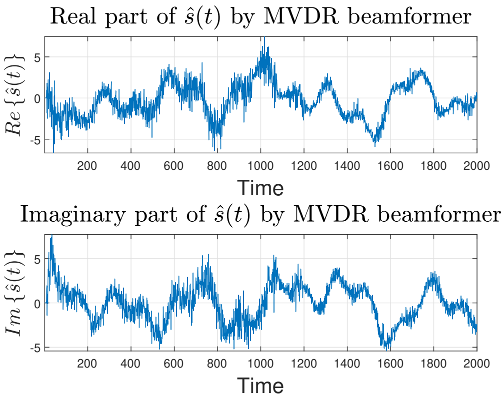
Among these results, I realized that the array steering beamformer has the best performance. In my consideration, there are two possible reasons : The source DOA I estimated is very accurate, and the interference power is much bigger than noise power, causing the MVDR beamformer has difficulty dealing with this scenario. Therefore, my main idea is to design a array-steering-based beamformer, while preserving the original array steering beampattern as much as I can.



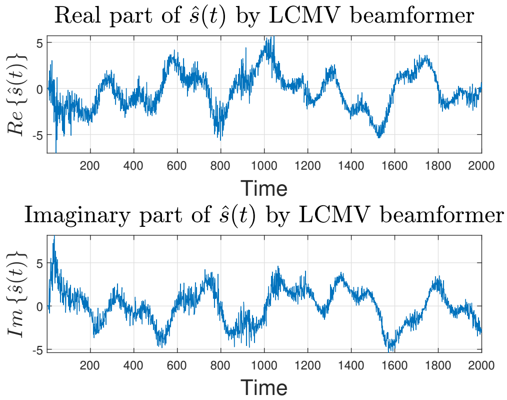
(a) Uniformly weighted beamformer



(b) Array steering beamformer



(c) MVDR beamformer



(d) LCMV beamformer

Figure 3: The results of the beamformers in section 1

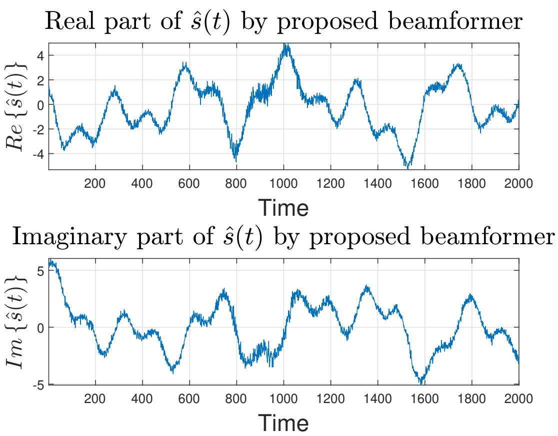


Figure 4: The result of the proposed beamformer

3.2 Proposed Beamformer

3.2.1 Null Steering

In order to design the beamformer I want, I employed a design method called null steering. The main concept of null steering can be formulated as an optimization problem :

Given a desired beamformer with beampattern $\mathbf{B}_{\theta_d}(\theta) = \mathbf{B}_d(\psi)$, where $\psi = \pi \sin\theta$,

$$\begin{aligned} \mathbf{w} = \ & \underset{\mathbf{w}}{\arg\min} \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} |\mathbf{B}(\psi) - \mathbf{B}_d(\psi)|^2 \mathrm{d}\psi \\ & \text{subject to} \quad \mathbf{w}^H \mathbf{a}(\theta_i) = 0 \end{aligned}$$

Where $\theta_i = \theta_i(t), \ \forall t$ in the given time interval.

The interpretation of the optimization problem is to create a notch at θ_i direction, and minimizes the difference, i.e. the mean-square-error, between the desired beampattern and the new beampattern simultaneously.

We have known that beamparttern is the spaital version of the discrete-time Fourier transform, where the ω in the discrete-time Fourier transform is substituted by $-\psi = -\pi \sin\theta$. So every property of the discrete-time Fourier transform holds good with beamparttern. By Parseval's identity,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |\mathbf{B}(\psi) - \mathbf{B}_d(\psi)|^2 \mathrm{d}\psi = ||\mathbf{w} - \mathbf{w}_d||^2$$

Then the optimization problem becomes

$$\begin{aligned} \mathbf{w} = \ & \underset{\mathbf{w}}{\arg\min} \quad ||\mathbf{w} - \mathbf{w}_d||^2 \\ & \text{subject to} \quad \mathbf{w}^H \mathbf{a}(\theta_i) = 0, \ \theta_i = \theta_i(t) \end{aligned}$$

Although we can solve the problem by Lagrange multiplier again, there is a more elegant and faster way to solve it by the concept of orthogonal projection. We can find that the constraint function is actually an inner product, which implies the optimal weight vector must be orthogonal to the steering vector towards θ_i, while minimizing the 2-norm distance between $\mathbf{w}$ and $\mathbf{w}_d$. Obviously, the solution of the optimization problem is the projection of $\mathbf{w}_d$ onto $\mathscr{B}$, the orthogonal complement of the space spanned by $\mathbf{a}(\theta_i)$, which is

$$\mathbf{w}_{opt} = Proj_{\mathscr{B}}(\mathbf{w}_d) = \left[\mathbf{I} - \frac{\mathbf{a}(\theta_i)\mathbf{a}(\theta_i)^H}{\mathbf{a}(\theta_i)^H \mathbf{a}(\theta_i)}\right] \mathbf{w}_d$$

Where $\mathbf{I}$ is the identity matrix.

But the biggest problem of null steering is there is no constraint of distortionless response towards the signal direction. That means the source signal may be suppressed at some time. As to solve this problem, we have to give up the optimal solution.

3.2.2 Suboptimal Null Steering

In order to preserve the distortionless response at all time, and to guarantee the suppression of the interference, we can find a suboptimal solution which can satisfy both constraints simultaneously. Even though we still hope that the new beampattern is as close as possible to the array steering beamparrtern, the problem is no longer an optimization problem.

$$\begin{cases} \mathbf{w}^H \mathbf{a}(\theta_s) = 1 \\ \mathbf{w}^H \mathbf{a}(\theta_i) = 0 \end{cases}$$

It is clear that the system of equations has infinitely many solutions if the number of rows of $\mathbf{w}$ is greater than two.

Because the desired beamformer is the array steering beamformer, so we have

$$\mathbf{w}_d^H \mathbf{a}(\theta_s) = 1$$

Now, define the suboptimal solution

$$\mathbf{w}_{subopt} = \mathbf{w}_d + \mathbf{e}$$

where $\mathbf{e}$ is the difference between $\mathbf{w}_{subopt}$ and $\mathbf{w}_d$.

In order to satisfy the first constraint,

$$\begin{aligned}
\mathbf{w}_{subopt}^H \mathbf{a}(\theta_s) =& (\mathbf{w}_d + \mathbf{e})^H \mathbf{a}(\theta_s) = \mathbf{w}_d^H \mathbf{a}(\theta_s) + \mathbf{e}^H \mathbf{a}(\theta_s) \\
=& \; 1 + \mathbf{e}^H \mathbf{a}(\theta_s) = 1 \\
\mathbf{e}^H \mathbf{a}(\theta_s) = 0, & \quad \mathbf{e} \perp \mathbf{a}(\theta_s)
\end{aligned}$$

This implies that no matter what value of $\mathbf{e}$ is, it will not violate the first constraint. So we can find the suboptimal solution by adjusting $\mathbf{e}$. In fact, the suboptimal solution set is the intersection between orthogonal complement of the space spanned by $\mathbf{a}(\theta_i)$, $\mathscr{B}$, and the space spanned by $\mathbf{e}$, $\mathscr{E}$. Because $\mathbf{w}_{subopt} = \mathbf{w}_d + \mathbf{e}$, and the second constraint, that $\mathbf{w}_{subopt}$ must be in $\mathscr{B}$.

Because $\mathbf{e} \perp \mathbf{a}(\theta_s)$, then $\mathbf{e} \perp \mathbf{w}_d$. And because that $\mathbf{w}_{opt}$ is the orthogonal projection of $\mathbf{w}_d$ onto $\mathscr{B}$, so $\mathbf{w}_{opt}$ is orthogonal to $\mathscr{B} \cap \mathscr{E}$. To ensure the least perturbation of the suboptimal solution, we let

$$\mathbf{w}_{subopt} = \alpha \mathbf{w}_{opt}$$

such that

$$\mathbf{w}_{subopt}^H \mathbf{a}(\theta_s) = \alpha \mathbf{w}_{opt} \mathbf{a}(\theta_s) = 1, \;\; \alpha = \frac{1}{\mathbf{w}_{opt}^H \mathbf{a}(\theta_s)}$$

The suboptimal solution, which is the weight vector of the proposed beamformer, is obtained.

$$\mathbf{w}_{subopt} = \alpha \mathbf{w}_{opt} = \frac{\mathbf{w}_{opt}}{\mathbf{w}_{opt}^H \mathbf{a}(\theta_s)}$$

$$\text{where} \qquad \mathbf{w}_{opt} = \left[\mathbf{I} - \frac{\mathbf{a}(\theta_i)\mathbf{a}(\theta_i)^H}{\mathbf{a}(\theta_i)^H \mathbf{a}(\theta_i)} \right] \mathbf{w}_d.$$

3.2.3 Algorithm Summarization

Algorithm 2 Proposed beamformer

Require: The $\mathbf{a}[\theta_s(t)], \mathbf{a}[\theta_i(t)]$ from Algorithm 1

1: **for** $t = 1, 2, \ldots, L$ **do**

2: Get the $\mathbf{w}_d(t)$ from the array steering beamformer

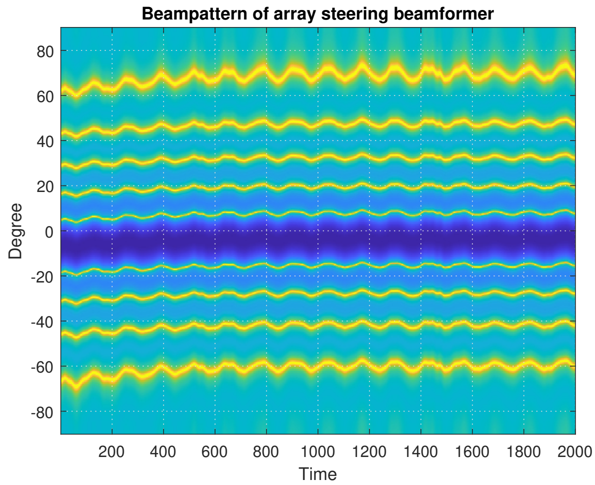
3: $\mathbf{v}(t) \leftarrow \left[\mathbf{I} - \frac{\mathbf{a}[\theta_i(t)]\mathbf{a}[\theta_i(t)]^H}{\mathbf{a}[\theta_i(t)]^H \mathbf{a}[\theta_i(t)]} \right] \mathbf{w}_d(t)$

4: $\mathbf{w}(t) \leftarrow \frac{\mathbf{v}(t)}{\mathbf{v}^H(t)\mathbf{a}[\theta_s(t)]}$

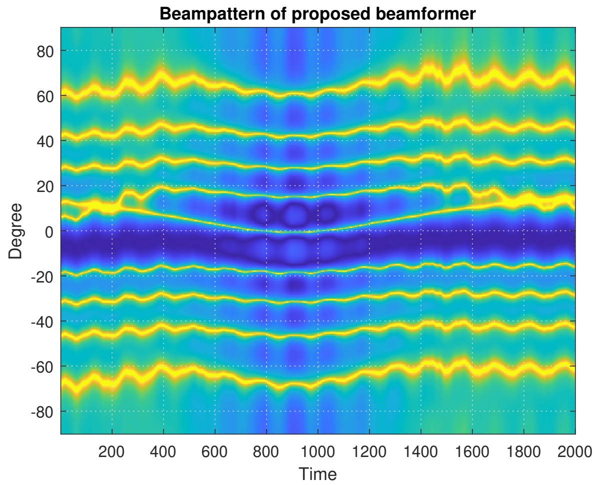
5: **end for**

3.2.4 Comparison and Advantages

Even though the weight vector derived from the proposed method is suboptimal, the result still outperforms all the beamformers discussed in section 1 massively (figure 4). Because the proposed beamformer does not rely on the correlation matrix $\mathbf{R}$, which has to be estimated by severely contaminated $\mathbf{x}(t)$. In additional, compared with the array steering beamformer, the proposed beamformer has fully suppressed the interference which array steering beamformer has failed to acheive, and the proposed beampattern is also very similar to the array steering beampattern, which is shown in figure 5. Last but not least, because the proposed beamformer does not to estimate $\mathbf{R}$, recursive method is not required, which means that the proposed beamformer has less complexity than the methods that require $\mathbf{R}$, such like MVDR and LCMV beamformers.



(a) Array steering beamformer



(b) Proposed beamformer

Figure 5: Beampattern of Array steering beamformer and Proposed beamformer. As the color getting more closer to yellow, the attenuation becomes larger.