International Journal of Control
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/tcon20

A nonlinear H_∞ control design in robotic systems under parameter perturbation and external disturbance

BOR-SEN CHEN [a] , TZANN-SHIN LEE [a] & JUI-HSUAN FENG [a]
[a] Department of Electrical Engineering, National Tsing-Hua University, Hsin Chu 300, Taiwan, Republic of China
Published online: 11 Aug 2011.

INT. J. CONTROL, 1994, VOL. 59, NO. 2, 439–461

A nonlinear H_∞ control design in robotic systems under parameter perturbation and external disturbance

BOR-SEN CHEN†, TZANN-SHIN LEE† and JUI-HSUAN FENG†

A state feedback H_∞ optimal disturbance attenuation for the model reference control of rigid robotic systems is studied. The disturbances affecting the system dynamics come from the residue of the applied torques due to perturbations of the system parameters and external noise. The design objective is that the disturbance attentuation level must be less than or equal to a desired positive value γ. By combining the nonlinear minimax control technique with the work of Johansson (1990) we are able to present an explicit global solution to this nonlinear time-varying H_∞-control problem. In particular, it turns out that if γ, the desired attenuation level, is greater than the magnitude of the weighting matrix on control inputs, then a state feedback law achieving the desired performance can be explicitly constructed. Hence, it is advantageous to consider our approach for H_∞ tracking control of other physical systems, Finally, extensive simulations are made for tracking control of a two-link robotic manipulator with the proposed H_∞ designs.

1. Introduction

The motion control of an unconstrained robotic manipulator has received a great deal of attention in the past decade. Many approaches have been introduced to treat this control problem. The linear optimal control approach is standard and relies on a linearized equation with respect to an operating point. There is some research which relates to this topic; for example, Saridis and Lee (1979) performed some early work on linear self-tuning optimizing control in robotics, and Luo and Saridis (1985) studied linear quadratic design of PID controllers. Some studies, concerning suboptimal control based on nonlinear robotic dynamic equations from the viewpoint of an approximate solution, have also been presented: for example, Lee and Chen (1983) proposed a suboptimal nonlinear control design based on quasi-linearization and linear optimal control.

Aside from local linearization, 'feedback linearization' of nonlinear systems gives an alternative strategy to the control of robotics (it is a kind of 'computed torque') (Freund 1975, Lee 1982). This method provides a control law separated into two parts: one is a nonlinear state feedforward to exactly cancel the nonlinear terms and factors, and the other is based on an optimal control design for the simplified system. This optimization is based on linear system dynamics with double integrator action and does not actually include the nonlinear dynamics of the robotic system. Thus, control is not really optimized with respect to the applied torques. Recently, Johansson has studied the optimal motion control with minimization of the applied torques (forces) by finding an explicit solution to the corresponding Hamilton–Jacobi equation (Johansson 1990).

Received 13 October 1992. Revised 29 January 1993.

† Department of Electrical Engineering, National Tsing-Hua University, Hsin Chu 300, Taiwan, Republic of China.

0020-7179/94 $10.00

Since, in practical robotic systems, the system parameters cannot be *a priori* known exactly (for example, the load may vary while performing different tasks, the friction coefficients may change in different configurations and some neglected nonlinearity, such as backlash, may appear as a disturbance at the control input) the robot arm may receive unpredictable interference from the environment where it resides. Therefore, it is necessary to consider these effects due to parameter perturbations and external disturbances. Various adaptive control algorithms (Tomizuka *et al*. 1985, Dubowsky and Desforges 1979, Koivo and Guo 1983) have been proposed to cope with uncertain or unknown parameters from the viewpoint of linearization. On the other hand, several authors have proposed adaptive control solutions which take the nonlinear actions into consideration (Johansson 1990, Craig *et al*. 1987, Slotine and Li 1987). However, the use of an adaptive control algorithm causes more complexity, especially for higher order nonlinear time-varying systems.

The approach of H_∞ optimal control has been widely discussed for robustness and its capability of disturbance attenuation in linear control systems (Doyle *et al*. 1989, Francis 1988) under a state space formulation. Solutions for nonlinear time-invariant systems have also been discovered in recent literature (van de Schaft 1991, 1992, Isidori and Astolfi 1992). The results require solving nonlinear partial differential equations whose global solutions are usually difficult to obtain.

In this paper, we consider the motion control of robotic systems under parametric uncertainties and external disturbances in an H_∞ setting. By so doing, our design can counteract the effects due to both parametric uncertainties and external disturbances. This is because the parameter uncertainties may act as internally generated disturbances. In this spirit, a model reference robotic control problem with desired disturbance attenuation (or robotic H_∞ control problem, for short) is proposed. The solution will not only achieve optimal tracking while perturbations are absent, but also make the worst-case effect on the tracking error due to the combined disturbance (including internally generated and external disturbances) to be less than or equal to a desired level γ. This is an application of nonlinear H_∞ optimal theory to the robotic tracking control case and extends the work of Johansson from H_2-optimal control to H_∞-optimal control.

In our approach, to achieve desired attenuation on the combined disturbance, the robotic H_∞ disturbance attenuation problem will be transformed into a nonlinear constrained minimax cost control problem. Dynamic game theory is then used to treat this leader–follower game (Basar and Bernhard 1990) which involves solving a so-called Bellman–Isaacs equation. This leads to a sufficient condition for the global solvability of the model reference control with desired disturbance attenuation problem. Thus, our work presents a nice paradigm of a nonlinear system in which the H_∞ control problem admits an explicit global solution by combining current nonlinear H_∞ state feedback control results with the work of Johansson (1990). More appealingly, we also discover that the minimum achievable level of attenuation, γ, is bounded from below by the 'size' of the weighting matrix on the control inputs. This is reasonable since it can be checked that the combined disturbance lies within the span of the input matrix and it is then possible to eliminate it (i.e. $\gamma \to 0$) by using a high gain control. However, non-smooth control action may arise.

The rest of this paper is organized as follows. Section 2 gives the motivation and formulation of the robotic H_∞-control problem which takes into account the effects resulting from both the parameter uncertainties and exogenous disturbances. In § 3, a related minimax cost control problem is proposed to provide a possible solution to our problem. The Bellman–Isaacs equation is then employed to solve globally the minimax cost control problem whose solution yields the desired state feedback law. In § 4, we test our design on the tracking control of a two-link robot by using a computer. Finally, the conclusions are given in § 5.

2. H_2 optimal model reference control and problem formulation

In this section, the H_2 optimal model reference control of robotic systems via minimizing tracking errors and the applied torques will be discussed. Based on this quadratic optimal control problem, the robotic H_∞-control problem, i.e. the model reference robotic control problem with a desired level of disturbance attenuation is formulated. The disturbance comes from two different perturbation sources: parametric uncertainty and exogenous input disturbance.

2.1. *H_2 optimal model reference control*

According to the Denavit–Hartenberg representation (Yoshikawa 1990), the geometric configuration of a robotic manipulator with n links can be determined by n variables. Set these variables as the entries of a column vector q, and from the context of analytical dynamics, a systematic way to derive the equation of motion is to apply the methods of the Lagrange theory (Goldstein 1950). More precisely, for a given mechanical system having n rigid bodies with kinetic energy $\mathcal{T}(q, \dot{q})$ and potential energy $\mathcal{U}(q)$ we first recall that the lagrangian $\mathcal{L}(q, \dot{q})$ of this system is defined to be

$$\mathcal{L}(q, \dot{q}) =: \mathcal{T}(q, \dot{q}) - \mathcal{U}(q) \tag{1}$$

and the Lagrange equation has the form

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{q}}\right) - \left(\frac{\partial \mathcal{L}}{\partial q}\right) = \mathcal{Q} \tag{2}$$

in which $\mathcal{Q} \in \mathbb{R}^n$ is the non-conservative generalized forces along the directions of their corresponding generalized coordinates q. For rigid robotic systems, the kinetic energy $\mathcal{T}(q, \dot{q})$ is a quadratic function of the vector $\dot{q}$ of the form

$$\mathcal{T}(q, \dot{q}) = \tfrac{1}{2}\dot{q}^{\mathrm{T}} M(q)\dot{q}$$

Hence, the equations of motion for such systems can be derived from (2) as

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + G(q) = \tau \tag{3}$$

where the matrix $M(q) \in \mathbb{R}^{n \times n}$ denotes the moment of inertia which is symmetric and positive definite for every $q \in \mathbb{R}^n$. Other terms of (3) include coriolis, centripetal forces

$$C(q, \dot{q})\dot{q} =: \dot{M}(q)\dot{q} - \frac{1}{2}\frac{\partial}{\partial q}(\dot{q}^{\mathrm{T}} M(q)\dot{q}) \tag{4}$$

the gravitational forces

$$G(q) =: \frac{\partial \mathcal{U}(q)}{\partial q} \tag{5}$$

and the column vector of applied torques, $\tau \in \mathbb{R}^n$.

The desired reference trajectory to follow is assumed to be available as bounded functions of time in terms of generalized positions $q_r \in \mathscr{C}^2$ (the class of twice continuously differentiable functions) and their corresponding accelerations $\ddot{q}_r$ and velocities $\dot{q}_r$.

All variables $\ddot{q}_r$, $\dot{q}_r$, q_r are assumed to be within the physical and kinematic limits of the control object. The variables $\ddot{q}_r$, $\dot{q}_r$, q_r may be conveniently generated from a reference model of the type

$$\ddot{q}_r + K_v\dot{q}_r + K_p q_r = K_r r \quad r, q_r \in \mathbb{R}^n \tag{6}$$

with some bounded driving signal r. Of course, the dynamic system (6) with $n \times n$-matrices K_v, K_p and K_r needs to be stable. The state tracking error is defined as

$$\tilde{x} =: \begin{bmatrix} \dot{\tilde{q}} \\ \tilde{q} \end{bmatrix} =: \begin{bmatrix} \dot{q} - \dot{q}_r \\ q - q_r \end{bmatrix} \tag{7}$$

Then, the tracking problem of generalized position q is reduced to the regulation problem of error state $\tilde{x}$. Using (6) and (3), the dynamic equation for the state tracking error $\tilde{x}$ is obtained as

$$\dot{\tilde{x}} = \begin{bmatrix} \ddot{\tilde{q}} \\ \dot{\tilde{q}} \end{bmatrix} = A(q, \dot{q})\tilde{x} + B_0(\ddot{q}_r, \dot{q}_r, \dot{q}, q) + BM^{-1}(q)\tau \tag{8}$$

where

$$A(q, \dot{q}) =: \begin{bmatrix} -M^{-1}(q)C(q, \dot{q}) & 0_{n\times n} \\ I_{n\times n} & 0_{n\times n} \end{bmatrix}$$

$$B_0(\ddot{q}_r, \dot{q}_r, \dot{q}, q) =: \begin{bmatrix} -\ddot{q}_r - M^{-1}(q)(G(q) + C(q, \dot{q})\dot{q}_r) \\ 0_{n\times n} \end{bmatrix}$$

$$B =: \begin{bmatrix} I_{n\times n} \\ O_{n\times n} \end{bmatrix}$$

One of the nice contributions in the work of Johansson (1990) is to propose a selective applied torque

$$\begin{aligned} u &=: [M(q) \quad C(q, \dot{q})] \begin{bmatrix} \dot{\tilde{z}}_1 \\ \tilde{z}_1 \end{bmatrix} \\ &=: M(q)T_1\dot{\tilde{x}} + C(q, \dot{q})T_1\tilde{x} \end{aligned} \tag{9}$$

which affects the kinetic energy only, since during the process of motion, it may not be necessary to consider the consumption due to the potential energy and to optimize gravitation-dependent torques or forces. The matrix T_1 in (9) is introduced via the following state-space transformation of $\tilde{x}$ (Johansson 1990)

$$\tilde{z} =: \begin{bmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{bmatrix} =: T_0\tilde{x} =: \begin{bmatrix} T_{11} & T_{12} \\ 0 & I_{n\times n} \end{bmatrix} \begin{bmatrix} \dot{\tilde{q}} \\ \tilde{q} \end{bmatrix} \tag{10}$$

where $T_{11}, T_{12} \in \mathbb{R}^{n\times n}$ are constant matrices to be adequately determined later. Apparently, (9) must conform to (8). We get the relation between the control varaible u and the applied torque τ as follows

$$\tau = M(q)(\ddot{q}_{\mathrm{r}} - T_{11}^{-1}T_{12}\dot{\tilde{q}} - T_{11}^{-1}M^{-1}(q)(C(q,\dot{q})B^{\mathrm{T}}T_0\tilde{x} - u)) + C(q,\dot{q})\dot{q} + G(q) \tag{11}$$

Then, after a suitable operation (Johansson 1990), the state tracking error equation, driven by the selective applied torque u, can be written as

$$\begin{aligned}\dot{\tilde{x}} &= T_0^{-1}\begin{bmatrix}\dot{\tilde{z}}_1\\ \dot{\tilde{z}}_2\end{bmatrix}\\ &= T_0^{-1}\begin{bmatrix}-M^1(q)C(q,\dot{q}) & 0_{n\times n}\\ T_{11}^{-1} & -T_{11}^{-1}T_{12}\end{bmatrix}T_0\tilde{x} + T_0^{-1}\begin{bmatrix}M^{-1}(q)\\ 0\end{bmatrix}u\end{aligned} \tag{12}$$

Subject to the above tracking error dynamics, one is naturally led to find an optimal control u_{o} so that the following cost functional $J_2(u)$

$$\min_{u(\cdot)} J_2(u) = \min_{u(\cdot)} \frac{1}{2}\int_0^\infty (\tilde{x}(t)^{\mathrm{T}}Q\tilde{x}(t) + u(t)^{\mathrm{T}}Ru(t))\,\mathrm{d}t \tag{13}$$

is minimized (Johansson 1990) for some given weighting matrices $R = R^{\mathrm{T}} > 0$, $Q = Q^{\mathrm{T}} > 0$. This is the so-called H_2 (quadratic) optimal control problem. Once the optimal control u_{o} is obtained, the corresponding optimal applied torques can be given from (11).

2.2. *Problem formulation with H_∞ performance*

The H_2 optimal model reference control of the robotic system described in the previous subsection has been solved by Johansson (1990). However, the design that makes the H_2 performance index small does not, in general, guarantee any robustness with respect to parametric uncertainty and any performance in the face of exogenous disturbance (Flamm *et al.* 1987). In practical robotic systems however, perturbations in system parameters are inevitable. Hence, it is significant to consider the effect due to these parameter perturbations:

- ΔM the uncertainty of the $M(q)$ due to the changes of load;
- ΔC the perturbation of the term $C(q,\dot{q})\dot{q}$ due to the change in moment of inertia $\Delta M(q)$ and friction;
- ΔG the perturbation of the gravitational force due to the configuration and changes of the total mass of the robotic manipulator.

That is, the parameter matrices in the robot model (3) can be divided into a nominal part and a perturbed part

$$\begin{aligned}M(q) &=: M_0(q) + \Delta M(q)\\ C(q,\dot{q}) &=: C_0(q,\dot{q}) + \Delta C(q,\dot{q})\\ G(q) &=: G_0(q) + \Delta G(q)\end{aligned}$$

where $M_0(q)$, $C_0(q,\dot{q})$ and $G_0(q)$ are the nominal parameter matrices and are assumed to be known exactly, whereas the parameter perturbations $\Delta M(q)$,

$\Delta C(q,\dot{q})$ and $\Delta G(q)$ are unknown. On the other hand, a certain neglected unmodelled nonlinearity may appear as a disturbance at the input terminals. Taking into account these effects, the dynamical equations for real robotic systems should take the following form

$$[M_0(q) + \Delta M(q)]\ddot{q} + [C_0(q,\dot{q}) + \Delta C(q,\dot{q})]\dot{q} + [G_0(q) + \Delta G(q)] = \tau + w \tag{14}$$

where w denotes the finite energy (square integrable) exogenous disturbance. Now, since the perturbations $\Delta M(q)$, $\Delta C(q,\dot{q})$ and $\Delta G(q)$ are uncertain, the torque (11) can only be applied with nominal parameter matrices $M_0(q)$, $C_0(q,\dot{q})$ and $G_0(q)$. That is

$$\begin{aligned}\tau = {} & M_0(q)(\ddot{q}_r - T_{11}^{-1}T_{12}\dot{\tilde{q}} - T_{11}^{-1}M_0^{-1}(q)(C_0(q,\dot{q})B^{\mathrm{T}}T_0\tilde{x} - u)) \\ & + C_0(q,\dot{q})\dot{q} + G_0(q)\end{aligned} \tag{15}$$

The effect of perturbations has been described in the perturbed model (14). Obviously, this model gives a more complete characterization for a robotic system in the real world. By letting $\delta = -(\Delta M(q)\ddot{q} + \Delta C(q,\dot{q})\dot{q} + \Delta G(q) - w)$ denote the combined disturbance due to the parameter uncertainties and the external disturbance w, equation (14) can be rewritten as

$$M_0(q)\ddot{q} + C_0(q,\dot{q})\dot{q} + G_0(q) = \tau + \delta \tag{16}$$

With the applied torque (15) in terms of nominal parameters, one is able to form perturbed state tracking error dynamics (from u to state tracking error $\tilde{x}$) by following the derivation for the unperturbed case, which has the form

$$\begin{aligned}\dot{\tilde{x}} = {} & T_0^{-1}\begin{bmatrix} -M_0^{-1}(q)C_0(q,\dot{q}) & 0_{n\times n} \\ T_{11}^{-1} & -T_{11}^{-1}T_{12}\end{bmatrix} T_0\tilde{x} + T_0^{-1}\begin{bmatrix} I_{n\times n} \\ 0_{n\times n}\end{bmatrix} M_0^{-1}(q)u \\ & + \begin{bmatrix} I_{n\times n} \\ 0_{n\times n}\end{bmatrix} M_0^{-1}(q)\delta\end{aligned} \tag{17}$$

Without loss of generality, we define a new disturbance vector

$$d =: M_0(q)T_{11}M_0^{-1}(q)\delta$$

Since

$$\begin{bmatrix} M_0^{-1}(q)\delta \\ 0_{n\times n}\end{bmatrix} = \begin{bmatrix} T_{11}^{-1}M_0^{-1}(q)\{M_0(q)T_{11}M_0^{-1}(q)\delta\} \\ 0_{n\times n}\end{bmatrix} = T_0^{-1}\begin{bmatrix} I_{n\times n} \\ 0_{n\times n}\end{bmatrix} M_0^{-1}(q)d$$

the perturbed state tracking error equation (17) becomes

$$\dot{\tilde{x}} = A_T(\tilde{x},t)\tilde{x} + B_T(\tilde{x},t)u + B_T(\tilde{x},t)d \tag{18}$$

where

$$A_T(\tilde{x},t) = T_0^{-1}\begin{bmatrix} -M_0^{-1}(q)C_0(q,\dot{q}) & 0_{n\times n} \\ T_{11}^{-1} & -T_{11}^{-1}T_{12}\end{bmatrix} T_0$$

$$B_T(\tilde{x},t) = T_0^{-1}\begin{bmatrix} I_{n\times n} \\ 0_{n\times n}\end{bmatrix} M_0^{-1}(q)$$

In general, the parameter perturbations ΔM, ΔC, ΔG and external disturbance w cannot *a priori* be estimated precisely. Hence, the combined disturbance

d is uncertain. It is not easy to find a good control strategy to cancel out this effect completely. In comparison with the unperturbed error dynamics (12), we found an extra disturbance term d appears in the perturbed error equation (18) and so a modified model reference control, with consideration of the disturbance attenuation, is appealing. We propose a new performance criterion including a desired disturbance attenuation level γ for the perturbed tracking error dynamics (18) as follows (Rhee and Speyer 1991, Limebeer *et al.* 1992)

$$\min_{u(\cdot)\in L_2} \max_{0\neq d(\cdot)\in L_2} \frac{\int_0^\infty (\frac{1}{2}\tilde{x}^{\mathrm{T}}(t)Q\tilde{x}(t) + \frac{1}{2}u^{\mathrm{T}}(t)Ru(t))\,\mathrm{d}t}{\int_0^\infty \frac{1}{2}d^{\mathrm{T}}(t)d(t)\,\mathrm{d}t} \leqslant \gamma^2 \tag{19}$$

where the state trajectory $\tilde{x}(\cdot)$ starts from initial condition $\tilde{x}(0) = 0$.

As γ approaches ∞, the performance criterion (19) degenerates to the H_2 (quadratic) performance index (13). So, supposing that the performance criterion (19) is achieved by the minimizer u^* and the maximizer d^*, we shall call u^* the optimal control and d^* the worst case disturbance. In this case, take the state feedback law $u = u^*$ then, for the closed loop system, the worst case effect of parameter perturbations and exogenous disturbances, to the tracking error and control signal, is guaranteed to be less than or equal to γ, i.e.

$$\max_{0\neq d(\cdot)\in L_2} \frac{\|z^*(t)\|_{L_2}}{\|d(t)\|_{L_2}} \leqslant \gamma^2 \tag{20}$$

where

$$z^*(t) =: \begin{bmatrix} Q^{1/2}\tilde{x}(t) \\ R^{1/2}u^*(t) \end{bmatrix}.$$

The above expression means that the H_∞ (induced L_2) norm from d to z^* is less than or equal to γ. From this reasoning, we are actually dealing with the H_∞ optimal disturbance attenuation problem for the model reference robotic control or *robotic H_∞-control problem*, for short.

Once the solution u^* is found for a given γ, we can apply successively smaller values of γ to obtain a minimum (optimal) level of disturbance attenuation. Hence, we have solved the minimum H_∞ norm problem. One can also expect that the smaller the value of γ for which the performance index (19) holds, the better the response (or the more robustness) which will result.

In general, when ΔM, ΔC and ΔG are all equal to zero, $\gamma < 1$ is necessary for pure attenuation of the external disturbance w.

A few remarks concerning the proposed performance criterion (19) are in order.

Remark 1—Relation to H_∞-control: Formally, subject to the perturbed tracking error dynamics (18) a (full information) H_∞-control problem is to find a state feedback law u such that

$$\max_{0\neq d(\cdot)\in L_2} \frac{\|z(t)\|_{L_2}}{\|d(t)\|_{L_2}} \leqslant \gamma^2 \tag{21}$$

in which $z(t)$ is defined as z^* with only u^* substituted by u. From the above discussion, we already know that the fulfillment of the performance criterion (19) implies the existence of a solution to the H_∞-control problem. Conversely, it is clear that the solvability of the H_∞-control problem makes the performance criterion (19) hold. Hence, considering the performance criterion, (19) is equivalent to solving the H_∞-control problem. □

Remark 2—Finite L_2-gain characterization: Due to the causality of robotic systems, the optimal control law u^* makes the closed loop system satisfy (Desoer and Vidyasagar 1975)

$$\max_{0\neq d(\cdot)L_{2T}} \frac{\|z^*(t)\|_{L_{2T}}}{\|d(t)\|_{L_{2T}}} \leqslant \gamma^2, \quad \forall T \geqslant 0 \tag{22}$$

where $\|\cdot\|_{L_{2T}}$ denotes the truncated L_2 norm at time T. In other words, the L_2-gain from the disturbance d to z^* is less than or equal to γ. □

Remark 3—Estimation of tolerable parametric uncertainty: Suppose that (19) holds. When $w = 0$, by the above remark and small gain theorem we can give, as a rule of thumb, a rough quantification on the tolerable parametric perturbation. Indeed, in this case, with $u = u^*$, (22) implies that the gain from d to $\sqrt{Q}\tilde{x}$ is less than or equal to γ. By definition, $d = -M_0(q)T_{11}M_0^{-1}(q)(\Delta M(q)\ddot{q} + \Delta C(q, \dot{q})\dot{q} + \Delta G(q))$. The disturbance d now can be viewed as an output of a (time-varying) operator H, induced by the parametric uncertainties $\Delta M(q)$, $\Delta C(q, \dot{q})$ and $\Delta G(q)$, with input $\tilde{x}$ that is essentially a function of $\dot{q}$, q and t. Then, from the small gain theorem, the closed loop system with H in the feedback path will remain stable for those parametric uncertainties $\Delta M(q)$, $\Delta C(q, \dot{q})$ and $\Delta G(q)$ such that the loop gain (in the sense of L_2) is strictly less than 1. □

3. Main results

From the analysis in the above section, the model reference robotic control problem with desired attenuation on parameter perturbation and exogenous disturbance is formulated as an H_∞-control problem. In this section, we are going to show that, with the help of the technique of completing the squares and an adequate choice of the value function (energy storage function), the robotic H_∞-control problem (19) can be globally solved in terms of an explicit formula via the Bellman–Isaacs equation from the theory of differential games (Basar and Olsder 1982, Basar and Bernhard, 1990).

After some rearrangement, the performance criterion (19) is equivalent to the following minimax problem (see also Rhee and Speyer 1991, Limebeer *et al.* 1992)

$$\min_{u(\cdot)} \max_{0\neq d(\cdot)\in L_2} \int_0^\infty (\tfrac{1}{2}\tilde{x}^{\mathrm{T}}(t)Q\tilde{x}(t) + \tfrac{1}{2}u^{\mathrm{T}}(t)Ru(t) - \tfrac{1}{2}\gamma^2 d^{\mathrm{T}}(t)d(t))\,\mathrm{d}t \leqslant 0, \quad \tilde{x}(0) = 0 \tag{23}$$

Now, in view of (23) we define the cost functional

$$J(\tilde{x}(t), u, d, t) =: \int_t^\infty L(\tilde{x}(s), u(s), d(s))\,\mathrm{d}s \tag{24}$$

with the lagrangian

$$L(\tilde{x}, u, d) =: \tfrac{1}{2}\tilde{x}^{\mathrm{T}}Q\tilde{x} + \tfrac{1}{2}u^{\mathrm{T}}Ru - \tfrac{1}{2}\gamma^2 d^{\mathrm{T}}d \tag{25}$$

By introducing the value function $V(\tilde{x}(t), t) = \min_{u(\cdot)} \max_{d(\cdot)} J(\tilde{x}(t), u, d, t)$, the performance criterion of (19) is equivalent to

$$V(\tilde{x}(0), 0) = \min_{u(\cdot)} \max_{d(\cdot)} J(\tilde{x}(0), u, d, 0) \leqslant 0, \quad \text{when } \tilde{x}(0) = 0 \tag{26}$$

Therefore, from the above analysis, it is seen that the solution for the robotic H_∞-control problem can be divided into the following two steps.

Step 1. Solve the minimax cost control problem (or leader-follower game)

$$V(\tilde{x}(0), 0) = \min_{u(\cdot)} \max_{d(\cdot)} J(\tilde{x}(0), u, d, 0) \tag{27}$$

subject to the perturbed tracking error dynamical equation (18).

Step 2. Find the condition so that the following inequality

$$V(\tilde{x}(0), 0) \leqslant 0 \tag{28}$$

will hold when $\tilde{x}(0) = 0$.

For clarity of presentation, the solution to the above two steps will be discussed in the following two subsections. In § 3.1, we will first proceed to Step 1 and find out the optimal control (minimizer) u^* and worst case disturbance (maximizer) d^* that achieve the minimax problem (27) and the associated minimax value function V through the Bellman–Isaacs equation. Secondly, with the value function V, we seek to solve Step 2, and a sufficient condition for the solvability of the robotic H_∞-control problem is thus obtained. In § 3.2, the optimal control input u^* is explicitly constructed via the solution of an algebraic matrix equation which is a reduction of the Bellman–Isaacs equation.

3.1. *The Bellman–Isaacs equation*

Perhaps, the most powerful tool to treat the minimax problem (27) is the theory of differential games (Basar and Olsder 1982, Basar and Bernhard 1990). Accordingly, we start by applying the minimax Bellman–Isaacs equation to solve our leader–follower game. Under the assumption that the value function V exists and is continuously differentiable, the minimax control problem (27) subject to (18) is achieved by an equilibrium pair u^* and d^*, if and only if, $V(\tilde{x}, t)$ satisfies the following minimax Bellman–Isaacs equation

$$\begin{aligned} -\frac{\partial V(\tilde{x}, t)}{\partial t} &= \min_{u(\cdot)} \max_{d(\cdot)} \left\{ L(\tilde{x}, u, d) + \left(\frac{\partial V(\tilde{x}, t)}{\partial \tilde{x}}\right)^{\mathrm{T}} \dot{\tilde{x}} \right\} \\ &=: \min_{u(\cdot)} \max_{d(\cdot)} H(\tilde{x}, u, d, t) \qquad (29) \\ &=: H^*(\tilde{x}, u^*, d^*, t) \qquad (30) \end{aligned}$$

where H is the hamiltonian. With reference to the cost functional J, the value function V must agree with the following terminal constraint

$$V(\tilde{x}(\infty), \infty) = 0 \tag{31}$$

As in the H_2 optimal control problem considered in Johansson (1990), the special structure of the perturbed state tracking error equation (18) will help in deriving the solution to the Bellman–Isaacs equation by chosing a suitable value function V of the form

$$V(\tilde{x}, t) =: \tfrac{1}{2}\tilde{x}^{\mathrm{T}} P(\tilde{x}, t)\tilde{x} \tag{32}$$

where $P(\tilde{x}, t)$ is a positive definite symmetric matrix for all $\tilde{x}$ and t. Using the technique of completing the squares and the identity

$$\left(\frac{\partial V(\tilde{x}, t)}{\partial \tilde{x}}\right)^{\mathrm{T}} \dot{\tilde{x}} = \tilde{x}^{\mathrm{T}} P(\tilde{x}, t)\dot{\tilde{x}} + \frac{1}{2}\sum_{i=1}^{2n} \tilde{x}^{\mathrm{T}}\left(\frac{\partial P(\tilde{x}, t)}{\partial \tilde{x}_i}\dot{\tilde{x}}_i\right)\tilde{x} \tag{33}$$

the Bellman–Isaacs equation (30) becomes

$$\begin{aligned} -\frac{1}{2}\tilde{x}^{\mathrm{T}} \frac{\partial P(\tilde{x}, t)}{\partial t}\tilde{x} &= \frac{1}{2}\tilde{x}^{\mathrm{T}} Q\tilde{x} + \tilde{x}^{\mathrm{T}} P(\tilde{x}, t)A_T(\tilde{x}, t)\tilde{x} - \frac{1}{2}G_1^{\mathrm{T}}(\tilde{x}, t)R^{-1}G_1(\tilde{x}, t) \\ &\quad + \frac{1}{2\gamma^2}G_1^{\mathrm{T}}(\tilde{x}, t)G_1(\tilde{x}, t) + \frac{1}{2}\sum_{i=1}^{2n} \tilde{x}^{\mathrm{T}}\left(\frac{\partial P(\tilde{x}, t)}{\partial \tilde{x}_i}\dot{\tilde{x}}_i\right)\tilde{x} \end{aligned} \tag{34}$$

where

$$G_1(\tilde{x}, t) =: B_{\mathrm{T}}^{\mathrm{T}}(\tilde{x}, t)P(\tilde{x}, t)\tilde{x}$$

That is

$$\begin{aligned} \tilde{x}^{\mathrm{T}}\Big\{ &\dot{P}(\tilde{x}, t) + P(\tilde{x}\ t)A_T(\tilde{x}, t) + A_{\mathrm{T}}^{\mathrm{T}}(\tilde{x}, t)P(\tilde{x}, t) \\ &- P(\tilde{x}, t)B_T(\tilde{x}, t)\left(R^{-1} - \frac{1}{\gamma^2}I\right)B_T^{\mathrm{T}}(\tilde{x}, t)P(\tilde{x}, t) + Q\Big\}\tilde{x} = 0, \\ &\qquad \forall \tilde{x} \in \mathbb{R}^n \end{aligned}$$

The corresponding optimal control u^* and the worst case disturbance w^* are

$$u^*(t) = -R^{-1}G_1(\tilde{x}, t) = -R^{-1}B_T^{\mathrm{T}}(\tilde{x}, t)P(\tilde{x}, t)\tilde{x} \tag{35}$$

and

$$d^*(t) = \frac{1}{\gamma^2}G_1(\tilde{x}, t) = \frac{1}{\gamma^2}B_T^{\mathrm{T}}(\tilde{x}, t)P(\tilde{x}, t)\tilde{x} \tag{36}$$

respectively. Therefore, a nonlinear Riccati equation is obtained as follows

$$\begin{aligned} &\dot{P}(\tilde{x}, t) + P(\tilde{x}, t)A_T(\tilde{x}, t) + A_T^{\mathrm{T}}(\tilde{x}, t)P(\tilde{x}, t) \\ &\quad - P(\tilde{x}, t)B_T(\tilde{x}, t)\left(R^{-1} - \frac{1}{\gamma^2}I\right)B_T^{\mathrm{T}}(\tilde{x}, t)P(\tilde{x}, t) + Q = 0 \end{aligned} \tag{37}$$

In general, it is not easy to solve the above nonlinear Riccati equation. However, in the robotic system, the nonlinear Riccati equation (37) can be further simplified to an algebraic matrix equation with an adequate choice of the matrix function P and by use of the skew symmetric matrix $N(q, \dot{q})$ (Johansson 1990, You and Chen 1992)

$$N(q, \dot{q}) = C_0(q, \dot{q}) - \tfrac{1}{2}\dot{M}_0(q, \dot{q}) \tag{38}$$

Because the state transformation (10) has been involved in the process of optimization, without loss of generality, we suggest P to be in a more explicit form (Johansson 1990, You and Chen 1992)

$$P(\tilde{x}, t) = T_0^{\mathrm{T}} \begin{bmatrix} M_0(\tilde{x}, t) & 0 \\ 0 & K \end{bmatrix} T_0$$

where K is a positive definite symmetric constant matrix representing the stiffness of a spring action around the given reference position, whereas the term containing $M_0(q)$ represents kinetic energy. We can view the above value function $V(\tilde{x}, t)$ as a total energy storage at error state $\tilde{x}$ and time t.

In the following paragraphs, we will show that under some conditions this suggested matrix function $P(\tilde{x}, t)$ is the solution of the nonlinear Riccati equation (37) and the constant matrices T_0 and K can be solved from an algebraic Riccati-like equation.

Consider the second and third terms of the nonlinear Riccati equation (37). Using (38), and after some algebraic manipulations, we get

$$P(\tilde{x}, t)A_T(\tilde{x}, t) + A_T^{\mathrm{T}}(\tilde{x}, t)P(\tilde{x}, t) = \begin{bmatrix} 0_{n\times n} & K \\ K & 0_{n\times n} \end{bmatrix} + T_0^{\mathrm{T}} \begin{bmatrix} -\dot{M}_0(q, \dot{q}) & 0_{n\times n} \\ 0_{n\times n} & 0_{n\times n} \end{bmatrix} T_0 \tag{39}$$

It can also be easily checked that

$$G_1(\tilde{x}, t) = B_T^{\mathrm{T}}(\tilde{x}, t)P(\tilde{x}, t) = B^{\mathrm{T}}T_0 \tag{40}$$

Hence, from (35) and (36) the optimal control u^* and the worst case disturbance w^* can be rewritten as

$$u^* = -R^{-1}B^{\mathrm{T}}T_0\tilde{x} \tag{41}$$

$$d^* = \frac{1}{\gamma^2}B^{\mathrm{T}}T_0\tilde{x} \tag{42}$$

With the substitution of (39) and (40) into (37), the nonlinear Riccati equation (37) becomes an algebraic equation

$$\begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix} + Q - T_0^{\mathrm{T}}B\left(R^{-1} - \frac{1}{\gamma^2}\right)B^{\mathrm{T}}T_0 = 0 \tag{43}$$

Remark 4 (van der Schaft 1992): As $\gamma \to \infty$ (i.e. no requirement is imposed on the attenuation of the disturbance d), the algebraic matrix equation (43) degenerates to

$$\begin{bmatrix} 0 & K \\ K & 0 \end{bmatrix} + Q - T_0^{\mathrm{T}}BR^{-1}B^{\mathrm{T}}T_0 = 0 \tag{44}$$

which is the result of the H_2 optimal control considered by Johansson (1990). □

From the above analysis, it now follows that if the value function V of (32) will be a solution to the Bellman–Issacs equation with $u^* = -R^{-1}B^{\mathrm{T}}T_0\tilde{x}$ and $d^* = (1/\gamma^2)B^{\mathrm{T}}T_0\tilde{x}$, the matrices K and T_0 must solve the algebraic matrix equation (43). However, except for K and T_0, to be the solution of (43) it still needs to check the terminal condition (31) to guarantee that V of (32) will be the solution of the minimax cost control problem (27).

Lemma 1: If $\gamma^2 I > R$, *then the scalar function*

$$V(\tilde{x}, t) = \frac{1}{2}\tilde{x}^{\mathrm{T}} T_0^{\mathrm{T}} \begin{bmatrix} M_0(q) & 0 \\ 0 & K \end{bmatrix} T_0 \tilde{x} \tag{38}$$

with $K > 0$ *and non-singular* T_0 *solving the algebraic matrix equation (43), is the solution of the Bellman–Isaacs equation with the terminal condition (31). The optimal control* u^* *and the worst case disturbance* w^* *are given in (41) and (42) respectively.*

Proof: We need only to prove that V satisfies the terminal condition $V(\tilde{x}(\infty), \infty) = 0$. Since the matrix $K > 0$ and T_0 is non-singular, we have

$$V(\tilde{x}, t) > 0, \quad \text{whenever} \quad \tilde{x} \neq 0$$

Taking the time derivative of this value function with $u = u^*$ and $d = d^*$, we have

$$\frac{\mathrm{d}V(\tilde{x}, t)}{\mathrm{d}t} = \left(\frac{\partial V(\tilde{x}, t)}{\partial \tilde{x}}\right)^{\mathrm{T}} \dot{\tilde{x}}\Bigg|_{u^*, d^*} + \frac{\partial V(\tilde{x}, t)}{\partial t}$$

From (29), we get

$$\frac{\partial V(\tilde{x}, t)}{\partial t} = -L(\tilde{x}, u^*, d^*) - \left(\frac{\partial V(\tilde{x}, t)}{\partial \tilde{x}}\right)^{\mathrm{T}} \dot{\tilde{x}}\Bigg|_{u^*, d^*}$$

then

$$\begin{aligned} \frac{\mathrm{d}V(\tilde{x}, t)}{\mathrm{d}t} &= -\tfrac{1}{2}\tilde{x}^{\mathrm{T}} Q \tilde{x} - \tfrac{1}{2}u^{*\mathrm{T}} R u^* + \tfrac{1}{2}\gamma^2 d^{*\mathrm{T}} d^* \\ &= -\tfrac{1}{2}\tilde{x}^{\mathrm{T}}\left\{Q + T_0^{\mathrm{T}} B\left(R^{-1} - \frac{1}{\gamma^2} I\right) B^{\mathrm{T}} T_0\right\}\tilde{x} \\ &< 0, \quad \text{whenever } \tilde{x} \neq 0 \end{aligned}$$

The last inequality is due to $\gamma^2 I > R$ and $Q > 0$. Following directly from the standard Lyapunov argument, we know that $\lim_{t\to\infty} \tilde{x}(t) = 0$. Since V is continuous

$$\lim_{t\to\infty} V(\tilde{x}(t), t) = V(\lim_{t\to\infty} \tilde{x}(t), \infty) = 0$$

Hence the result follows. □

With the above lemma, we can now proceed to show that the minimax cost control problem (27) in Step 1 is solved by the pair $u^* = -R^{-1}B^{\mathrm{T}}T_0\tilde{x}$ and $d^* = (1/\gamma^2)B^{\mathrm{T}}T_0\tilde{x}$. On the other hand, Step 2 can be implied by substituting $\tilde{x}(0) = 0$ into (38) and a sufficient condition for the solvability of the robotic H_∞-control problem (19) is thus obtained. Detailed derivations are presented in the theorem that follows.

Theorem 1: *Subject to the perturbed error dynamics (18). The robotic* H_∞-*control problem (19) is solvable if* $\gamma^2 I > R$ *and the algebraic matrix equation (43) has a pair of solutions* $K > 0$ *and non-singular* T_0. *The optimal control* $u^*(t)$ *and the worst case disturbance* $d^*(t)$ *are given in (41) and (42), respectively, i.e.*

$$u^* = -R^{-1}B^{\mathrm{T}}T_0\tilde{x}$$

$$d^* = \frac{1}{\gamma^2} B^{\mathrm{T}} T_0 \tilde{x}$$

Also, from (15), the corresponding applied torque which guarantees the desired H_∞ performance (19) is given by

$$\begin{aligned}\tau^* = {} & M_0(q)(\ddot{q}_r - T_{11}^{-1} T_{12} \dot{\tilde{q}} - T_{11}^{-1} M_0^{-1}(q)(C_0(q, \dot{q}) B^{\mathrm{T}} T_0 \tilde{x} - u^*)) \\ & + C_0(q, \dot{q})\dot{q} + G_0(q) \end{aligned} \tag{45}$$

Proof: To initiate Step 1, noting as shown in Lemma 1 that $V(\tilde{x}(\infty), \infty) = 0$, we only need to verify that V satisfies (27). Consider the tracking error dynamics (18) with the optimal control $u^*(t)$ and the worst case disturbance $d^*(t)$. Then, by (30)

$$\begin{aligned}\frac{\mathrm{d}V(\tilde{x}, t)}{\mathrm{d}t} &= \left(\frac{\partial V(\tilde{x}, t)}{\partial \tilde{x}}\right)^{\mathrm{T}} \dot{\tilde{x}} \bigg|_{u^*, d^*} + \frac{\partial V(\tilde{x}, t)}{\partial t} \\ &= -L(\tilde{x}, u^*, d^*) \\ &= -(\tfrac{1}{2}\tilde{x}^{\mathrm{T}} Q \tilde{x} + \tfrac{1}{2} u^{*\mathrm{T}} R u^* - \tfrac{1}{2} d^{*\mathrm{T}} d^*)\end{aligned} \tag{46}$$

Integrating the above equation, we have

$$\begin{aligned}\int_0^\infty \mathrm{d}V(\tilde{x}, t) &= V(\tilde{x}(\infty), \infty) - V(\tilde{x}(0), 0) \\ &= -\min_{u(\cdot)} \max_{d(\cdot)} \int_0^\infty (\tfrac{1}{2}\tilde{x}^{\mathrm{T}}(t) Q \tilde{x}(t) + \tfrac{1}{2} u^{\mathrm{T}}(t) R u(t) - \tfrac{1}{2}\gamma^2 d^{\mathrm{T}}(t) d(t))\, \mathrm{d}t\end{aligned}$$

According to Lemma 1 we have

$$\begin{aligned}V(\tilde{x}(0), 0) &= \min_{u(\cdot)} \max_{d(\cdot)} \int_0^\infty (\tfrac{1}{2}\tilde{x}^{\mathrm{T}}(t) Q \tilde{x}(t) + \tfrac{1}{2} u^{\mathrm{T}}(t) R u(t) - \tfrac{1}{2}\gamma^2 d^{\mathrm{T}}(t) d(t))\, \mathrm{d}t \\ &= \min_{u(\cdot)} \max_{d(\cdot)} J(\tilde{x}(0), u, d, 0)\end{aligned}$$

which shows that V is the desired value function solving the minimax cost control problem (27) in Step 1.

For Step 2, since

$$V(\tilde{x}(t), t) = \tfrac{1}{2}\tilde{x}^{\mathrm{T}}(t) T_0^{\mathrm{T}} \begin{bmatrix} M_0(q(t)) & 0 \\ 0 & K \end{bmatrix} T_0 \tilde{x}(t)$$

upon setting $\tilde{x}(0) = 0$, we obtain that $V(0, 0) = 0$ which implies (28). From the equivalence between (19) and (26), the proof is completed. □

3.2. *Solution to the algebraic matrix equation (43)*

In order to implement the applied torque (45), we must find matrices T_0 and K which solve the Riccati-like algebraic equation (43), with some desired properties. Comparing (43) with its H_2 case (44) we find that one more extra term is involved. This will impose further constraints on the weighting matrices Q and R. Hence, the results of Johansson (1990) cannot be applied directly.

However, its solution can still be characterized as follows. Let the positive definite symmetric matrix Q be factorized as

$$Q = \begin{bmatrix} Q_1^T Q_1 & Q_{12} \\ Q_{12}^T & Q_2^T Q_2 \end{bmatrix} \tag{47}$$

With this factorization, (43) splits into the following four equations

$$Q_1^T Q_1 - T_{11}^T \left(R^{-1} - \frac{1}{\gamma^2} I \right) T_{11} = 0 \tag{48}$$

$$K + Q_{12}^T - T_{12}^T \left(R^{-1} - \frac{1}{\gamma^2} I \right) T_{11} = 0 \tag{49}$$

$$K + Q_{12} - T_{11}^T \left(R^{-1} - \frac{1}{\gamma^2} I \right) T_{12} = 0 \tag{50}$$

$$Q_2^T Q_2 - T_{12}^T \left(R^{-1} - \frac{1}{\gamma^2} I \right) T_{12} = 0 \tag{51}$$

From (48) and (51), the matrix $(R^{-1} - (1/\gamma^2)I)$ must be positive definite since the matrices $Q_1^T Q_1$ and $Q_2^T Q_2$ are positive definite. Let

$$R_1^T R_1 = \left(R^{-1} - \frac{1}{\gamma^2} I \right)^{-1} \tag{52}$$

Substituting this into (48) and (51), the sub-matrices T_{11}, T_{12} and hence matrix T_0 can be obtained as

$$T_0 = \begin{bmatrix} R_1^T Q_1 & R_1^T Q_2 \\ 0 & I \end{bmatrix} \tag{53}$$

Again, substituting these values of T_{11} and T_{12} into (49) and (50), respectively, we get the symmetric solution matrix K as

$$K = \tfrac{1}{2}(Q_1^T Q_2 + Q_2^T Q_1) - \tfrac{1}{2}(Q_{12}^T + Q_{12}) \tag{54}$$

The above derivation has provided us with a solution to (43) and hence the robotic H_∞-control problem.

Corollary 1: *Given a desired level of disturbance attenuation γ, and weighting matrices $Q > 0$ and $R > 0$, let the weighting matrix Q be further taken as*

$$Q = \begin{bmatrix} Q_1^T Q_1 & Q_{12} \\ Q_{12}^T & Q_2^T Q_2 \end{bmatrix}$$

with Q_1, Q_2 and Q_{12} satisfying

$$(Q_1^T Q_2 + Q_2^T Q_1) - (Q_{12}^T + Q_{12}) > 0 \tag{55}$$

Then if

$$R < \gamma^2 I \tag{56}$$

the robotic H_∞-control problem (19) is solved by the pair

$$u^* = -R^{-1} B^T T_0 \tilde{x}$$

$$d^* = \frac{1}{\gamma^2} B^T T_0 \tilde{x}$$

where T_0 was given in (53).

Note that from (55) and (53), the matrix K of (54) is positive definite and symmetric and T_0 is non-singular as required for the value function V of (32).

Remark 5: The inequality (56) imposes a constraint on the achievable level of disturbance attenuation, γ, due to the penalty on the control effort u in the performance index (19). However, as $R \rightarrow 0$, i.e. there is no penalty on the control input u, γ may be as small as possible but may require large control energy and yield non-smooth responses. □

Based on this corollary, the robotic H_∞-control design can be outlined as the following design steps.

Design algorithm

Step 1. Choose a desired level of disturbance attenuation, $\gamma > 0$.

Step 2. Select the weighting matrix $R > 0$ such that $\lambda_{\max}(R) < \gamma^2$ and the weighting matrix

$$Q = \begin{bmatrix} Q_1^{\mathrm{T}} Q_1 & Q_{12} \\ Q_{12}^{\mathrm{T}} & Q_2^{\mathrm{T}} Q_2 \end{bmatrix} \quad \text{with } Q_1^{\mathrm{T}} Q_2 + Q_2^{\mathrm{T}} Q_1 - (Q_{12}^{\mathrm{T}} Q_{12}) > 0$$

Step 3. Calculate the Cholesky factorization

$$R_1^{\mathrm{T}} R_1 = \left(R^{-1} - \frac{1}{\gamma^2} I \right)^{-1}$$

and

$$T_0 = \begin{bmatrix} R_1^{\mathrm{T}} Q_1 & R_1^{\mathrm{T}} Q_2 \\ 0 & I \end{bmatrix}$$

Step 4. Obtain the corresponding optimal applied torque

$$\begin{aligned} \tau^* = {} & M_0(q)(\ddot{q}_r - T_{11}^{-1} T_{12} \dot{\tilde{q}} - T_{11}^{-1} M_0^{-1}(q)(C_0(q, \dot{q}) B^{\mathrm{T}} T_0 \tilde{x} - u^*)) \\ & + C_0(q, \dot{q})\dot{q} + G_0(q) \end{aligned} \tag{57}$$

where

$$u^* = -R^{-1} B^{\mathrm{T}} T_0 \tilde{x} \tag{58}$$

Remark 6—Irrelevance of Q to system response: It is quite surprising to note that the H_∞ design does not concern the magnitude of Q. However, this can be observed from the applied torque (57). Let $Q = \alpha I$ and $R = \beta I$ for some positive real α and $\beta < \gamma^2$. With these choices, K is always positive definite. Also, $T_{11} = T_{12} = \rho I$, for some real ρ. Hence, after using (7) and (58), the applied torque (57) becomes

$$\begin{aligned} \tau^* = {} & M_0(q)[\ddot{q}_r - \dot{\tilde{q}} - M_0^{-1}(q)(C_0(q, \dot{q}) - R^{-1})(\dot{\tilde{q}} + \tilde{q})] \\ & + C_0(q, \dot{q})\dot{q} + G_0(q) \end{aligned}$$

which is irrelevant to Q. □

4. A simulation example

Consider a two-link mainpulator shown in Fig. 1, with system parameters: link mass m_1, m_2 (kg), lengths l_1, l_2 (m), angular positions q_1, q_2 (rad), applied

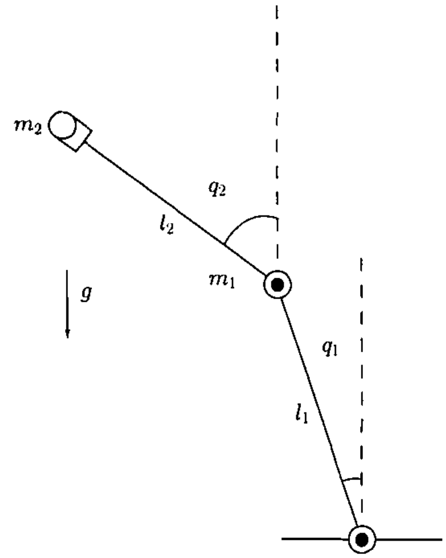

Figure 1. The two-link robotic manipulator.

torques τ_1, τ_2 (Nm). The parameters for the equation of motion (3) are

$$M(q) = \begin{bmatrix} (m_1 + m_2)l_1^2 & m_2 l_1 l_2(s_1 s_2 + c_1 c_2) \\ m_2 l_1 l_2(s_1 s_2 + c_1 c_2) & m_2 l_2^2 \end{bmatrix}$$

$$C(q, \dot{q}) = m_2 l_1 l_2(c_1 s_2 - s_1 c_2)\begin{bmatrix} 0 & -\dot{q}_2 \\ -\dot{q}_1 & 0 \end{bmatrix}$$

$$G(q) = \begin{bmatrix} -(m_1 + m_2)l_1 g s_1 \\ -m_2 l_2 g s_2 \end{bmatrix}$$

where $q \in \mathbb{R}^2$, $\tau \in \mathbb{R}^2$ and the short-hand notations $c_1 = \cos(q_1)$, $s_1 = \sin(q_1)$, etc. are used. Suppose that the trajectory planning problem for a weight-lifting operation is considered and the two-link manipulator suffers from the parameter uncertainties $\Delta M(q)$, ΔC and ΔG due to the changes of the load and an exogenous disturbance w.

The model reference control with the desired H_∞ performance (19) is employed to treat this robotic trajectory planning problem. For the convenience of simulation, the nominal parameters of the robotic system are given as: $m_1 = 1$ (kg), $m_2 = 10$ (kg), $l_1 = 1$ (m), $l_1 = 1$ (m), and the initial conditions $q_1(0) = -2$, $q_2(0) = -2$, $\dot{q}_1(0) = 0$, $\dot{q}_2(0) = 0$. The desired reference trajectory vector q_r starting from $q_r(0) = [2\ 2]^T$ is characterized by (6) with $K_p = 0_{2\times 2}$, $K_d = I_{2\times 2}$ and $r = [\pi/3\ \pi/6]^T$, which generates sinusoidal motions.

The parameter perturbations are assumed of the following forms

$$\Delta M(q) = \begin{bmatrix} -5 & -5(s_1 s_2 + c_1 c_2) \\ -5(s_1 s_2 + c_1 c_2) & -5 \end{bmatrix}$$

$$\Delta C(q,\dot q) = (c_1 s_2 - s_1 c_2)\begin{bmatrix} 0 & 5\dot q_1^2 \\ 5\dot q_2^2 & 0 \end{bmatrix}$$

$$\Delta G(q) = \begin{bmatrix} 5gs_1 \\ 5gs_2 \end{bmatrix}$$

which correspond to a change of load from 10 (kg) to 5 (kg). Adopting the above parameters, the perturbed equation of motion (18) has the form

$$\begin{bmatrix} 11 & 10(s_1s_2 + c_1c_2) \\ 10(s_1s_2 + c_1c_2) & 10 \end{bmatrix}\begin{bmatrix} \ddot q_1 \\ \ddot q_2 \end{bmatrix} + 10(c_1s_2 - s_1c_2)\begin{bmatrix} -\dot q_2^2 \\ -\dot q_1^2 \end{bmatrix} + \begin{bmatrix} -11gs_1 \\ -10gs_2 \end{bmatrix} = \begin{bmatrix} \tau_1 + \delta_1 \\ \tau_2 + \delta_2 \end{bmatrix}$$

in which the combined disturbance δ due to plant-model mismatch and exogenous disturbance equals

$$\delta = \begin{bmatrix} \delta_1 \\ \delta_2 \end{bmatrix} = \begin{bmatrix} 5\ddot q_1 + 5(s_1s_2 + c_1c_2)\ddot q_2 - 5(c_1s_2 - s_1c_2)\dot q_1^2\dot q_2 - 5gs_1 + w_1 \\ 5(s_1s_2 + c_1c_2)5\ddot q_1 + 5\ddot q_2 - 5(c_1s_2 - s_1c_2)\dot q_2^2\dot q_1 - 5gs_2 + w_2 \end{bmatrix}$$

where the exogenous disturbances w_1 and w_2 are square waves with amplitude 2. In order to illustrate the capability of disturbance attenuations with different choices of γ, we deliberately design the control laws to achieve the following four different levels of disturbance attenuation. The simulation results are shown in Figs 2 to 5. The angular positions q_1 and q_2 are represented in Figs 2 and 3 respectively. The angular velocities $\dot q_1$ and $\dot q_2$ are depicted in Figs 4 and 5 respectively. The applied torques τ_1 and τ_2 are plotted in Figs 6–9. The dotted curves in these figures are designed by the optimal control law in Johansson (1990) without parameter perturbations (i.e. in the situation that $\Delta M(q) = 0$, $\Delta C(q,\dot q) = 0$ and $\Delta G(q) = 0$) and exogenous disturbances.

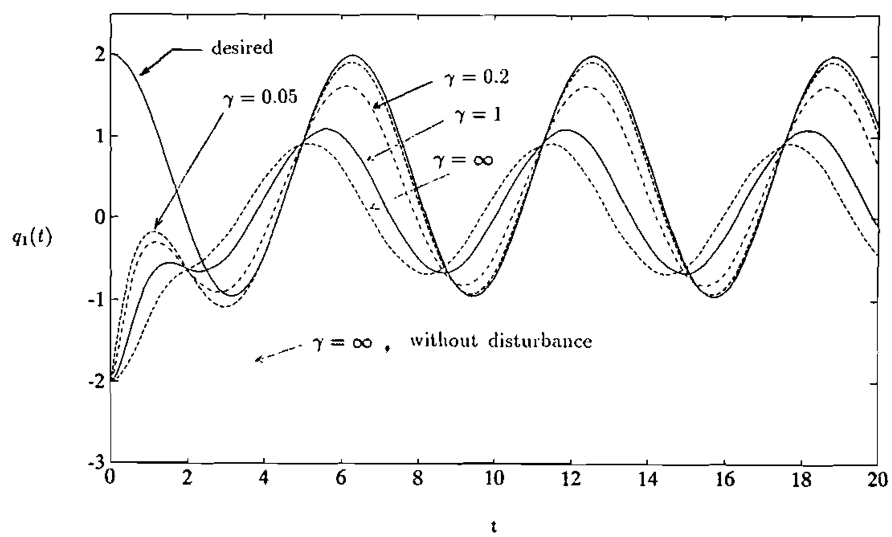

Figure 2. The angular position $q_1(t)$.

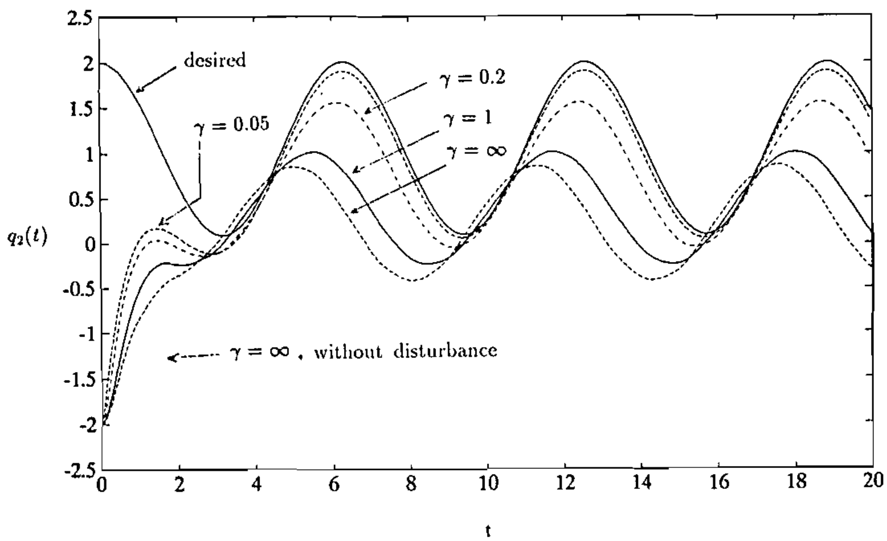

Figure 3. The angular position $q_2(t)$.

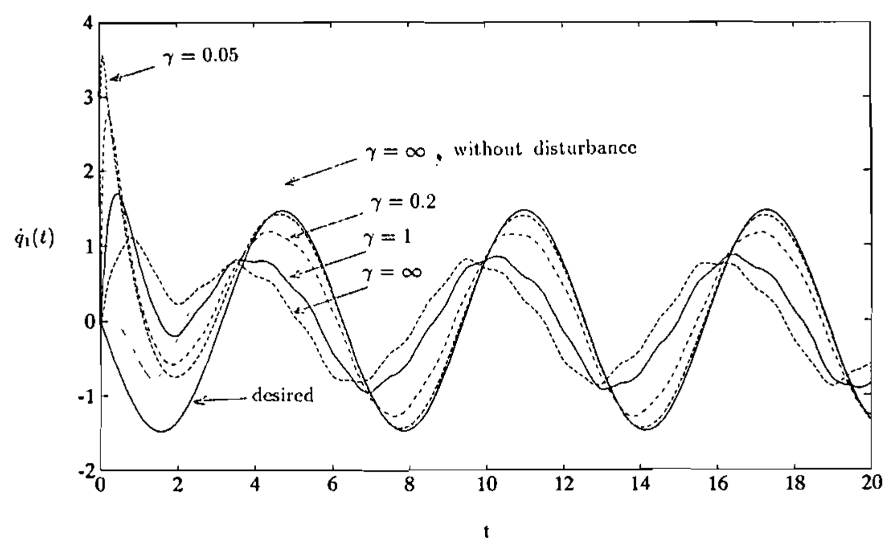

Figure 4. The angular velocity $\dot{q}_1(t)$.

Since the H_∞ design is not related to the choice of the 'size' of weighting matrix Q, we set in each case $Q = I_{4\times4}$ and, according to (54), we get $K = I_{2\times2}$ which is positive definite. So, following the design algorithm given in the previous section, we only need the solution matrix T_0 to the Riccati-like algebraic equation (43) in each case. The corresponding optimal control u^* and optimal applied torque τ^* are readily obtained from (58) and (57) in Step 4 of the Design algorithm.

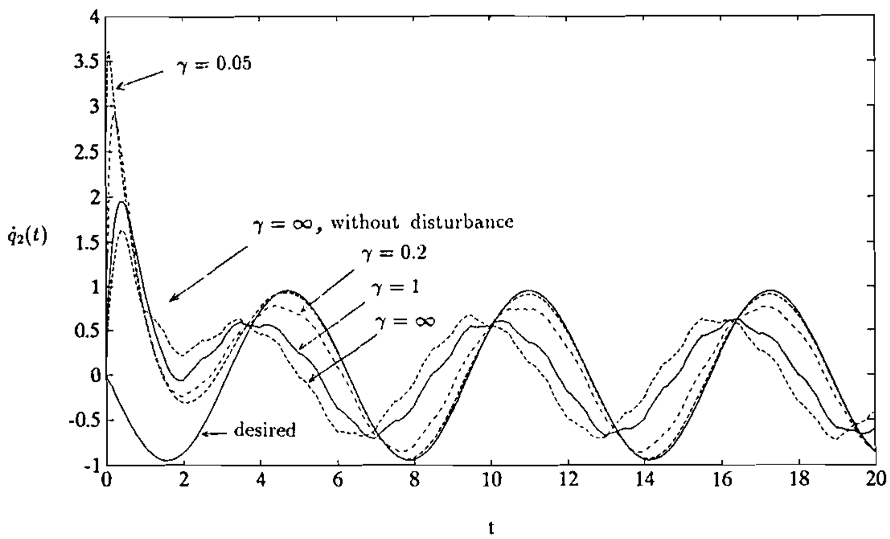

Figure 5. The angular velocity $\dot{q}_2(t)$.

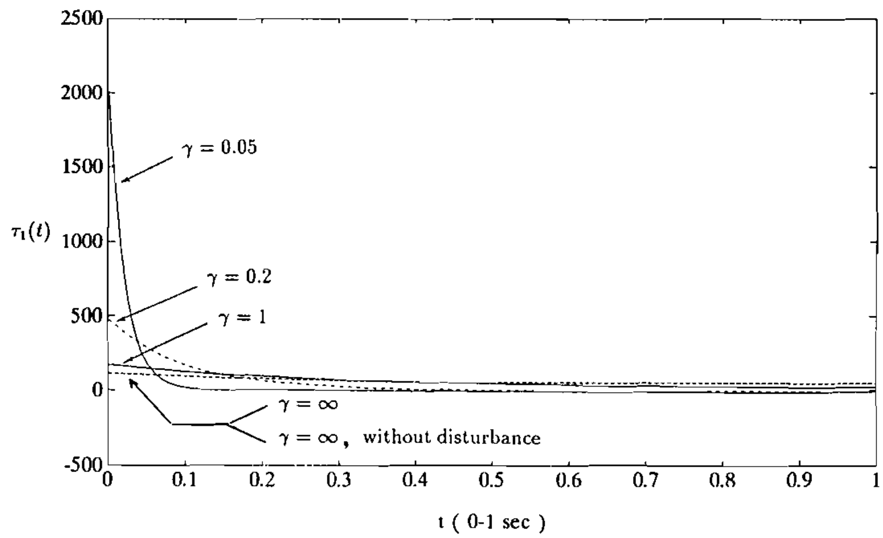

Figure 6. The applied torque τ_1 (transient response).

Case 1: $\gamma = \infty$, $Q = I_{4\times4}$, $R = 0{\cdot}09 I_{2\times2}$

This design is the H_2 optimal model reference control which had been considered in Johansson (1990) but the perturbations $\Delta M(q)$, $\Delta C(q, \dot{q})$, $\Delta G(q)$ and exogenous disturbances are included. According to (53)

$$T_0 = \begin{bmatrix} 0{\cdot}3 & 0 & 0{\cdot}3 & 0 \\ 0 & 0{\cdot}3 & 0 & 0{\cdot}3 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

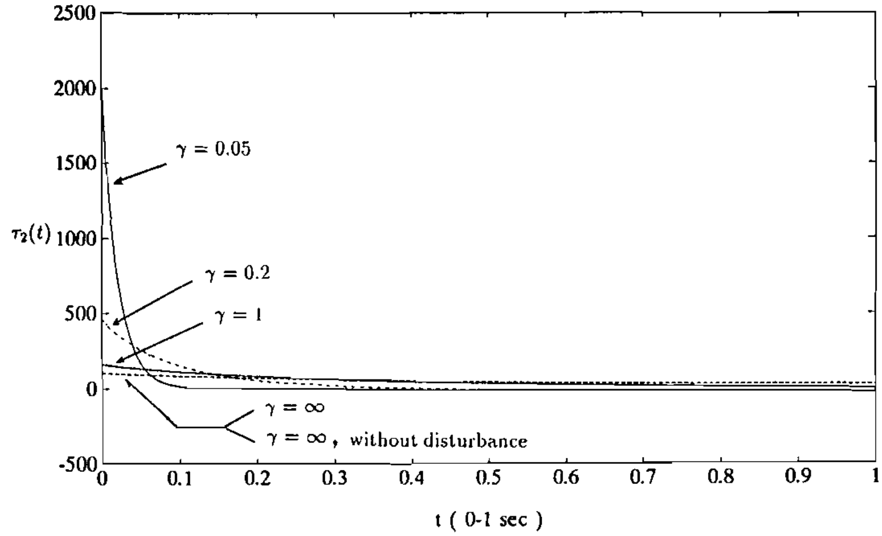

Figure 7. The applied torque τ_2 (transient response).

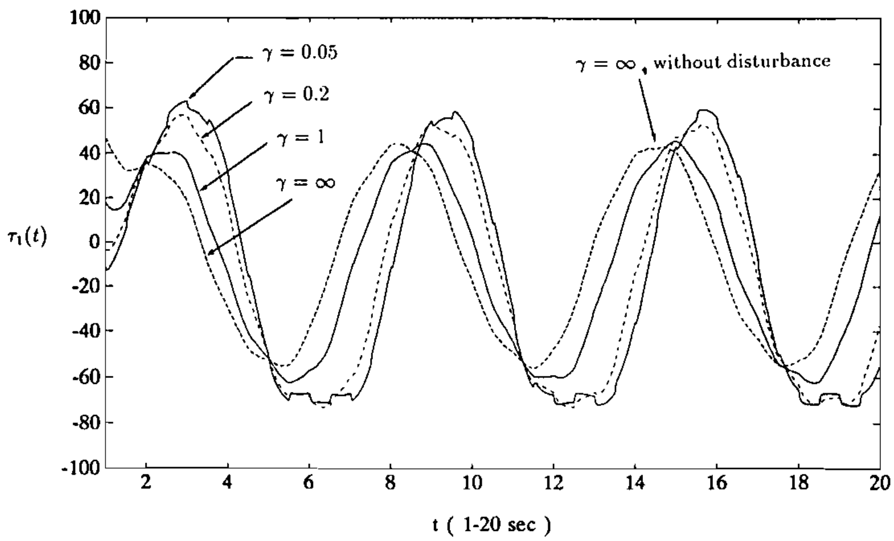

Figure 8. The applied torque τ_1 (steady-state response).

Case 2: $\gamma = 1,\ Q = I_{4\times4},\ R = 0{\cdot}04 I_{2\times2}$

$$T_0 = \begin{bmatrix} 0{\cdot}204124 & 0 & 0{\cdot}204124 & 0 \\ 0 & 0{\cdot}204124 & 0 & 0{\cdot}204124 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

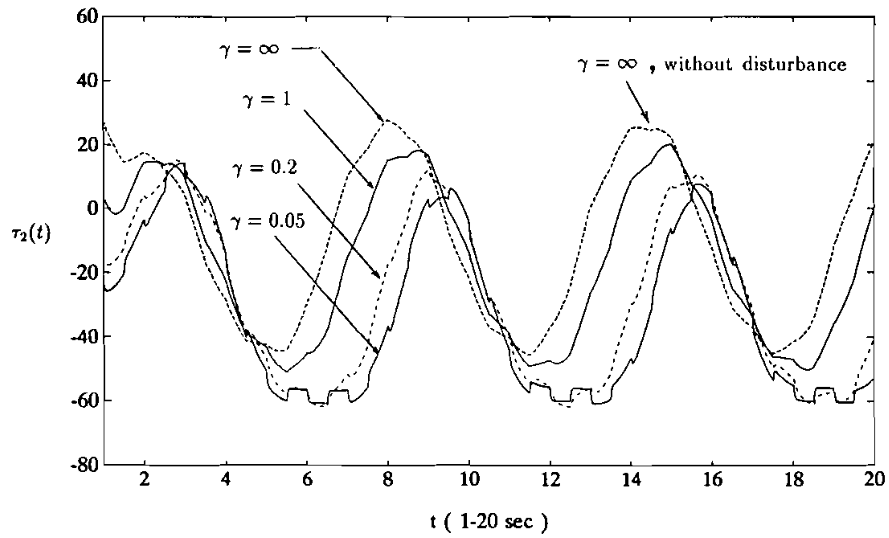

Figure 9. The applied torque τ_2 (steady-state response).

Case 3: $\gamma = 0{\cdot}2,\ Q = I_{4\times4},\ R = {\cdot}01 I_{2\times2}$

$$T_0 = \begin{bmatrix} 0{\cdot}1155 & 0 & 0{\cdot}1155 & 0 \\ 0 & 0{\cdot}1155 & 0 & 0{\cdot}1155 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Case 4: $\gamma = 0{\cdot}05,\ Q = I_{4\times4},\ R = 0{\cdot}002 I_{2\times2}$

$$T_0 = \begin{bmatrix} 0{\cdot}1 & 0 & 0{\cdot}1 & 0 \\ 0 & 0{\cdot}1 & 0 & 0{\cdot}1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

From the simulation results of the above four cases, we found that smaller γ may yield better performance in attenuating the effect of the combined disturbance. Hence, the desired attenuation properties of the proposed designs have been achieved. They can be used to diminish the effects due to the parameter perturbation and/or external disturbance in robotic systems.

5. Conclusions

In this paper, we have, explicitly and globally, solved the model reference control problem for a robotic system under parameter uncertainties and exogenous disturbances from the viewpoint of H_∞ disturbance attenuation. The parameter uncertainties of robotic dynamics are transformed into internally generated disturbances. The control design is to attenuate the (L_2)-gain, from the disturbances to the tracking errors and control energy, to a desired value. We have shown that this H_∞ problem can be globally solved via a constrained minimax cost control problem (or leader–follower game) and thus extends the H_2-optimal control results provided by Johansson (1990) to the H_∞-optimal case, which serves as an elegant example where the Bellman–Isaac equation admits an explicit global solution.

The solution relies only on an algebraic matrix equation rather than a nonlinear time-varying partial differential equation. The minimum achievable level of attenuation suffers a lower bound which merely relates to the magnitude of the weighting matrix R.

As the H_∞ control becomes a popular design tool for linear systems, our approach may be advantageous for the application of H_∞-control design to other physical tracking control systems.

Finally, from the simulations with various prespecified disturbance attenuation levels, it is seen that our design gives desired results. Further study may be necessary on the case when joint flexibility is presented or speed measurements are not available. Also, it would be interesting to apply the ideal of H_∞ design to other mechanical systems.

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