Talbot Effect for Dispersive Partial Differential Equations

Presenter: Zhuozhen Cai, Spencer Rooke, Wei-Chen Huang, Hui Lyu

Department of Mathematics, College of LAS, University of Illinois at Urbana-Champaign

Mentor : Burak Erdoğan and Nikolaos Tzirakis

Graduate Leader: George Shakan

INTRODUCTION

The solution profiles of dispersive partial differential equations depends heavily on the algebraic properties of time. At rational times with respect to the period, the solution is a piece-wise step function. However, at irrational times with respect to the period, the solution is a continuous, but nowhere differentiable fractal curve. This phenomenon is known as the "Talbot Effect", which is when the solutions to dispersive P.D.E. exhibit two vastly different behaviours, depending on what time they are calculated.

Since the solution curves at irrational times are very fractal-like, they are quite chaotic and can oscillate very quickly. We can look at the box-dimension of these curves in order to get an idea of how chaotic each solution is.

AIM

1. Understand nonlinear dispersive P.D.E. on torus, Talbot Effect, Fast Fourier Transformation, and box dimensions.
2. Understand the solutions to the full nonlinear equation.
3. Numerically evaluate the full nonlinear Schrödinger Equation at rational and irrational times with respect to the period.
4. Prepare a software and experiment to calculate box dimensions.

METHOD 1

Box Dimension

Since the solution curves at irrational times are chaotic and oscillate quickly, we use box dimension of these curves in order to analyze the complexity of each solution.

Definition: $$dim = \lim_{\varepsilon \to 0} \frac{log(N_\varepsilon)}{log(\frac{1}{\varepsilon})}$$

where $N(\varepsilon)$ is the number of boxes that the solution touches, and ε is the width of each box.

As it is unreasonable to calculate this limit explicitly, we instead find N for varying values of ε by summing over small intervals as follows:

$$N(\varepsilon) = \sum_{i=0}^{N} \lceil \frac{\max(f(x))}{\varepsilon} \rceil - \lfloor \frac{\min(f(x))}{\varepsilon} \rfloor \quad (n\varepsilon < x < (n+1)\varepsilon)$$

We can then find the slope of the best fit line of $log(N)/log(\varepsilon^{-1})$, which is the box dimension.

METHOD 2

Split Step Method

The general equation we need to solve is:

$$iu_t + u_{xx} + |u|^p u = 0$$

In particular, for $p = 2$, we can derive the solutions corresponding to the linear and nonlinear parts of Schrödinger equation separately by:

Linear: $$\begin{cases} iu_t + u_{xx} = 0 \\ u(x,0) = g(x) \end{cases} \Rightarrow u(x,t) = \sum_{k=-\infty}^{\infty} e^{-itk^2} \hat{g}(k) e^{ikx}$$

Nonlinear: $$\begin{cases} iu_t + |u|^2 u = 0 \\ u(x,0) = g(x) \end{cases} \Rightarrow u(x,t) = e^{it|g(x)|^2} g(x)$$

To use the Split Step Method, we can apply the linear and nonlinear solutions at small times and compose them to obtain the total solution:

$$u(x, n\triangle t) = F^{-1}[e^{-iw(k)\triangle t} F(e^{i\triangle t|u(x,\triangle t(n-1))|^{p=2}} u(x, \triangle t(n-1))]$$

$$\text{for } n = 0, 1, 2, \ldots$$
$$\text{given } u(x,0) = u_0$$

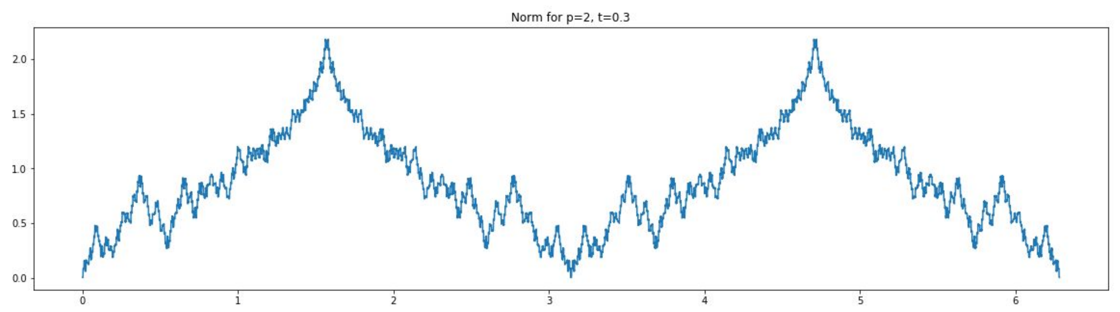
Where F is the Fourier transform, w is our dispersion relation, and we sum over n. The split step method works by applying the nonlinear and linear transformations for small times t, and iterating the initial conditions. For example, if we evaluate u at time Δt, we can then set that as our new $u_o(x,0)$, and so $u(x,2^*\Delta t)=u_o(x,\Delta t)$. By repeating this, we can approximate solutions $u(x,n^*\Delta t)$. We found that the most efficient way to evaluate $u(x,\Delta t)$ was to apply the nonlinear part in the time domain, and then apply the linear part in the frequency domain. This allows us to avoid calculating Fourier coefficients repeatedly for each new set of initial conditions, which saves us a great deal of time. This method can be shown to converge, [3].

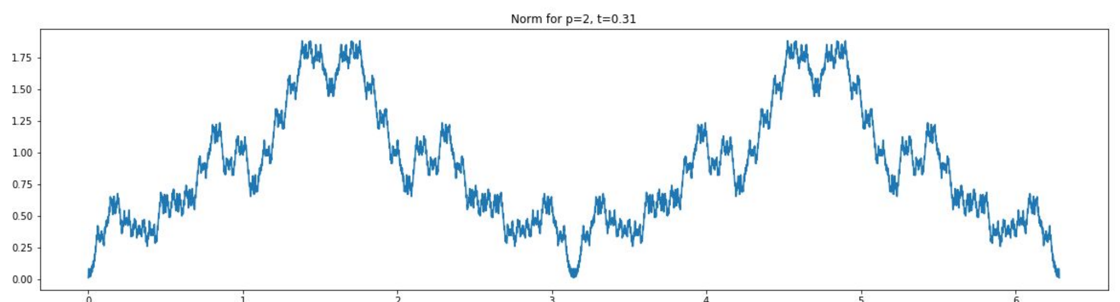
The initial function we are using here is:

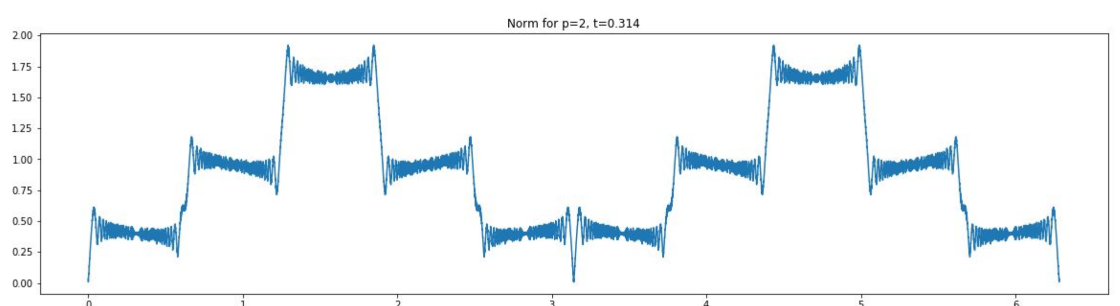
$$u(x,0) = g(x) = \begin{cases} 1 & : 0 \le x \le \pi \\ -1 & : \pi < x \le 2\pi \end{cases}$$

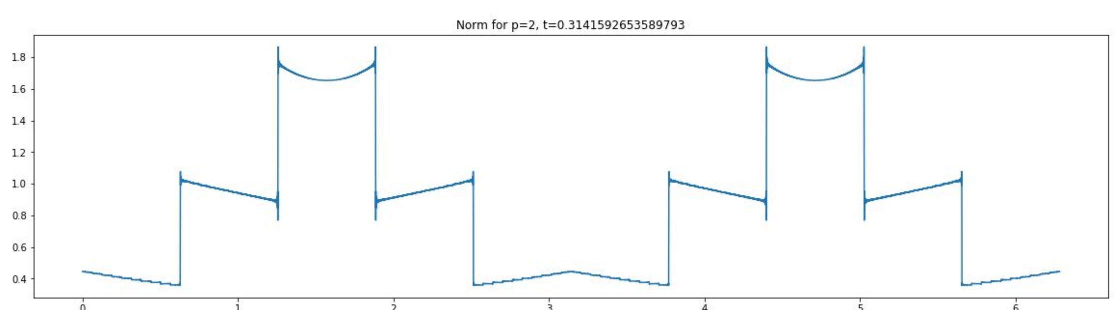
RESULTS

We are able to produce graphs for the real part, imaginary part, and the norm of the function we try to solve corresponding to different p values and evaluation times. The followings are graphs with fixed $p=2$ and $t=0.3$, $t=0.31$, $t=0.314$, and $t=0.1pi$ respectively.



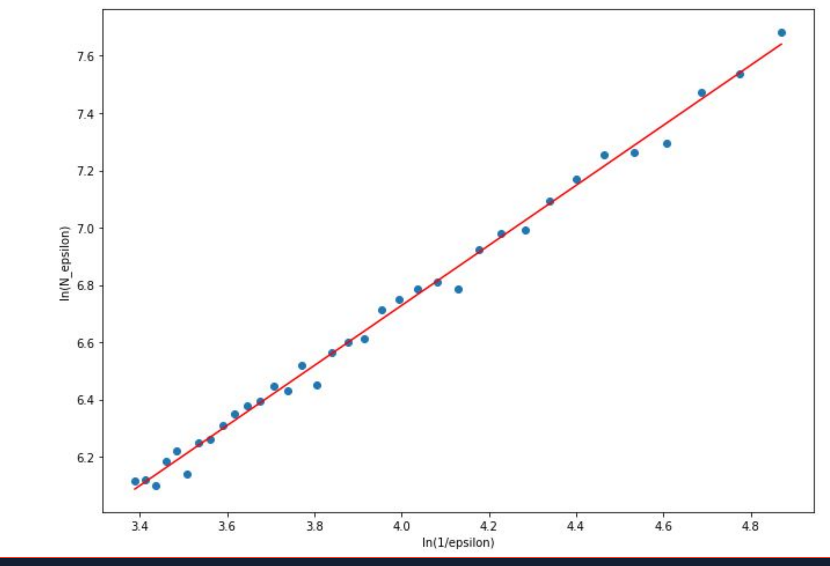






The time $t=0.3$, $t=0.31$ and $t=0.314$ will be irrational multiples of the space interval ($2pi$). The curves are chaotic and oscillates quickly, and the functions are continuous but nowhere differentiable fractal-like functions. For $t=0.1pi$, which is a rational multiple of the space interval, the solution is piece-wise step function, but discontinuous. Also, we notice that this behavior doesn't happen suddenly, instead, as we increase the accuracy of the decimal, which makes the t value closer to the actual $0.1pi$, the function behaves more similar to the step function.

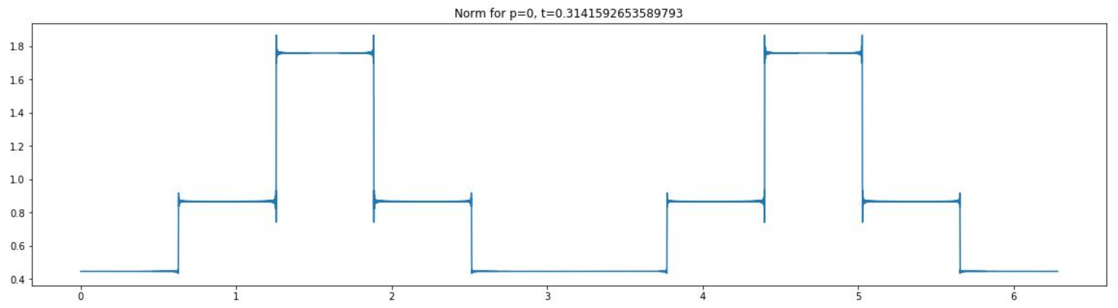
Here is the example graph of box dimension when $p=2$, $t=0.1$. The slope of the best fitting line equals to the box dimension which is in a range of $(1, 2)$. A steeper slope with a high value means that the solution curve is more "fractally", so it gains in complexity more quickly as the magnification decreases. The box dimension of the solution curve in this condition is 1.3520.

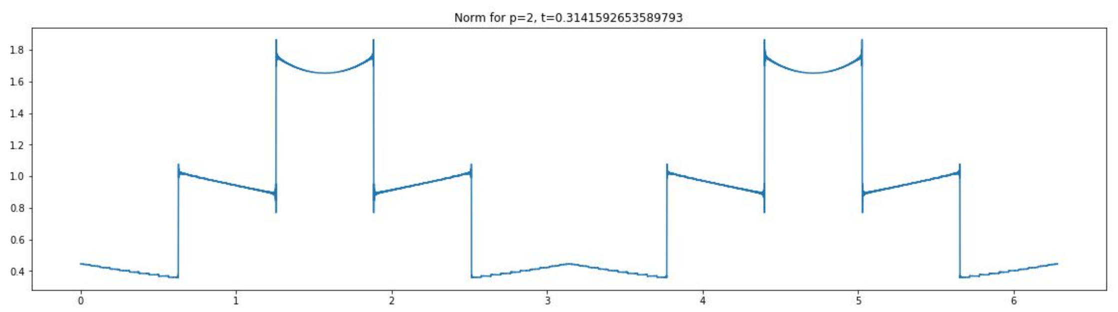


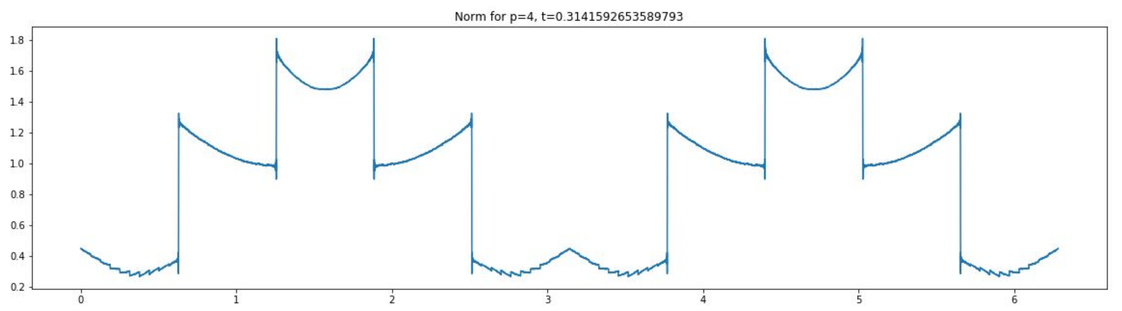
X-axis: $log(1/\epsilon)$
Y-axis: $log(N_\epsilon)$

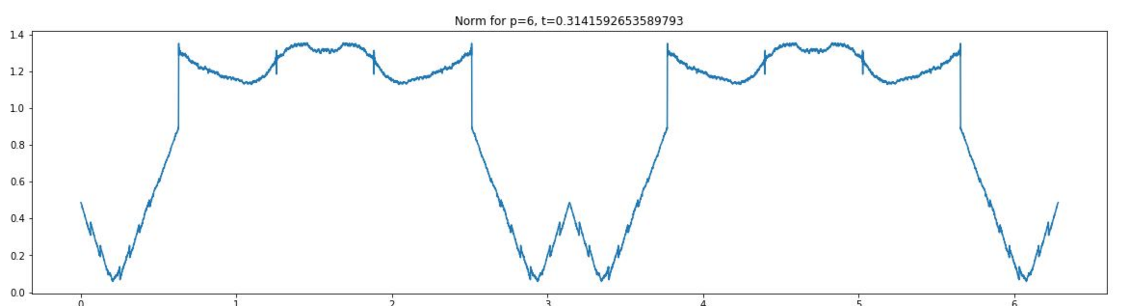
RESULTS

The followings are graphs with fixed $t=0.1pi$ and $p=0$, $p=2$, $p=4$, $p=6$ respectively.









When $p=2$, 4, and 6, the solutions are shifted vertically and they are no longer piecewise constant as what the function presents when $p=0$, but they do preserve the nature of step function. The curly shape exhibits the effect of nonlinearity.

FUTURE DIRECTIONS

Evaluate solutions to other nonlinear equations such as the Korteweg-deVries equation and calculate their box dimension for analysis.

ACKNOWLEDGEMENTS

[1] Gong Chen and Peter J. Olver, Numerical Simulation of Nonlinear Dispersive Quantization (2014)
[2] M. Berry, I. Marzoli. and F. Schleich. cuantum Carpets, carpets of Light Physics World, (2001)
[3] H. Holden, K. H. Karlsen, N. H. Risebro, and T. Tao, Operator Splitting for the KdV equation (2009)

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