ECE 661 Homework 1

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Question 1

The general form of a point in the representational space $\mathbb{R}^3$ is $\begin{pmatrix} u \\ v \\ w \end{pmatrix}$, which is the homogeneous coordinate of the point $\begin{pmatrix} x \\ y \end{pmatrix}$ in the physical space $\mathbb{R}^2$.

We have $x = \frac{u}{w}$ and $y = \frac{v}{w}$ when $w \neq 0$.

For the origin in $\mathbb{R}^2$, $x = 0$ and $y = 0$, we get $u = 0$ and $v = 0$.

Conclusion: $\begin{pmatrix} 0 \\ 0 \\ w \end{pmatrix}$ where $w \in \mathbb{R}$ and $w \neq 0$ are points in the representational space $\mathbb{R}^3$ that are the homogeneous coordinates of the origin in the physical space $\mathbb{R}^2$.

Question 2

Since points at infinity in the physical space $\mathbb{R}^2$ are ideal points with the form of $\begin{pmatrix} u \\ v \\ 0 \end{pmatrix}$ in $\mathbb{R}^3$. For two points $\begin{pmatrix} u_1 \\ v_1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} u_2 \\ v_2 \\ 0 \end{pmatrix}$, they are both points at infinity in $\mathbb{R}^2$ since their coordinates in $\mathbb{R}^2$ are infinity. However, their coordinates in $\mathbb{R}^3$ are not necessarily the same, i.e. they are not the same point.

Conclusion: Not all points at infinity in the physical space $\mathbb{R}^2$ are the same.

Question 3

Since degenerate conic $\boldsymbol{C} = \boldsymbol{l}\boldsymbol{m}^{\boldsymbol{T}} + \boldsymbol{m}\boldsymbol{l}^{\boldsymbol{T}}$, $rank(\boldsymbol{C}) = rank(\boldsymbol{l}\boldsymbol{m}^{\boldsymbol{T}} + \boldsymbol{m}\boldsymbol{l}^{\boldsymbol{T}}) \leq rank(\boldsymbol{l}\boldsymbol{m}^{\boldsymbol{T}}) + rank(\boldsymbol{m}\boldsymbol{l}^{\boldsymbol{T}})$.

For $\boldsymbol{l}\boldsymbol{m}^{\boldsymbol{T}}$, since $\boldsymbol{l}\boldsymbol{m}^{\boldsymbol{T}}$'s columns are linearly dependent, $rank(\boldsymbol{l}\boldsymbol{m}^{\boldsymbol{T}}) = 1$. Similarly, $rank(\boldsymbol{m}\boldsymbol{l}^{\boldsymbol{T}}) = 1$.

$rank(\boldsymbol{C}) = rank(\boldsymbol{l}\boldsymbol{m}^{\boldsymbol{T}} + \boldsymbol{m}\boldsymbol{l}^{\boldsymbol{T}}) \leq rank(\boldsymbol{l}\boldsymbol{m}^{\boldsymbol{T}}) + rank(\boldsymbol{m}\boldsymbol{l}^{\boldsymbol{T}}) = 2$, so the rank of a degenerate conic cannot exceed 2.

Question 4

Step 1: $\boldsymbol{l_1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \\ 0 \end{pmatrix}$

Step 2: $\boldsymbol{l_2} = \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix} \times \begin{pmatrix} -7 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \\ 13 \end{pmatrix}$

Step 3: $\boldsymbol{x} = \begin{pmatrix} -5 \\ 3 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ -4 \\ 13 \end{pmatrix} = \begin{pmatrix} 39 \\ 65 \\ 23 \end{pmatrix}$, then its coordinate in $\mathbb{R}^2$ is $\begin{pmatrix} 39/23 \\ 65/23 \end{pmatrix}$

If the second line pass through $\begin{pmatrix} -7 \\ -5 \end{pmatrix}$ and $\begin{pmatrix} 7 \\ 5 \end{pmatrix}$, it takes two steps. Since the second line passes through the origin as well as the first line also passes through the origin, the intersection is the origin.

Question 5

$$\boldsymbol{l_1} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix}$$

$$\boldsymbol{l_2} = \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 15 \end{pmatrix}$$

$$\boldsymbol{x} = \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix} \times \begin{pmatrix} 3 \\ 5 \\ 15 \end{pmatrix} = \begin{pmatrix} 75 \\ -45 \\ 0 \end{pmatrix}$$

Since the third coordinate of $\boldsymbol{x}$ in $\mathbb{R}^3$ is 0, which means the intersection point is at infinity, i.e. the two lines are parallel.

Question 6

We can write the conic as $\frac{(x-3)^2}{1^2} + \frac{(y-2)^2}{(1/2)^2} = 1$, simplified as $x^2 + 4y^2 - 6x - 16y + 25 = 0$.

$$\boldsymbol{C} = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 4 & -8 \\ -3 & -8 & 25 \end{bmatrix}$$

Let $\boldsymbol{x} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$$\boldsymbol{l} = \boldsymbol{C}\boldsymbol{x} = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 4 & -8 \\ -3 & -8 & 25 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ -8 \\ 25 \end{bmatrix}$$

So the polar line is $-3x - 8y + 25 = 0$, it intersects with x-axis at $\begin{pmatrix} 25/3 \\ 0 \end{pmatrix}$, intersects with y-axis at $\begin{pmatrix} 0 \\ 25/8 \end{pmatrix}$

Question 7

$$\boldsymbol{l_1} = \begin{pmatrix} 1/2 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1/2 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \\ 0.5 \end{pmatrix}$$

$$\boldsymbol{l_2} = \begin{pmatrix} 0 \\ -1/3 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ -1/3 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 1/3 \end{pmatrix}$$

$$\boldsymbol{x} = \begin{pmatrix} -1 \\ 0 \\ 0.5 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 1/3 \end{pmatrix} = \begin{pmatrix} -1/2 \\ 1/3 \\ -1 \end{pmatrix}$$

The intersection of $x = 1/2$ and $y = -1/3$ is $\begin{pmatrix} 1/2 \\ -1/3 \end{pmatrix}$