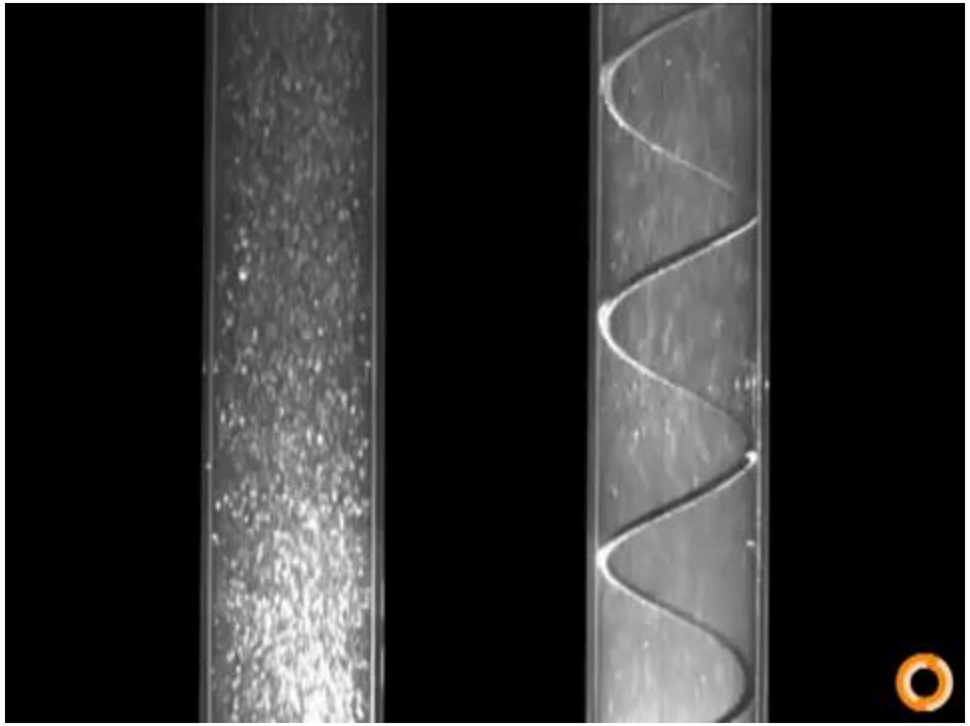
第4讲（第8-9章）

层流流动问题分析

1. 层流流体微元分析方法
2. 流体流动的微分方程式

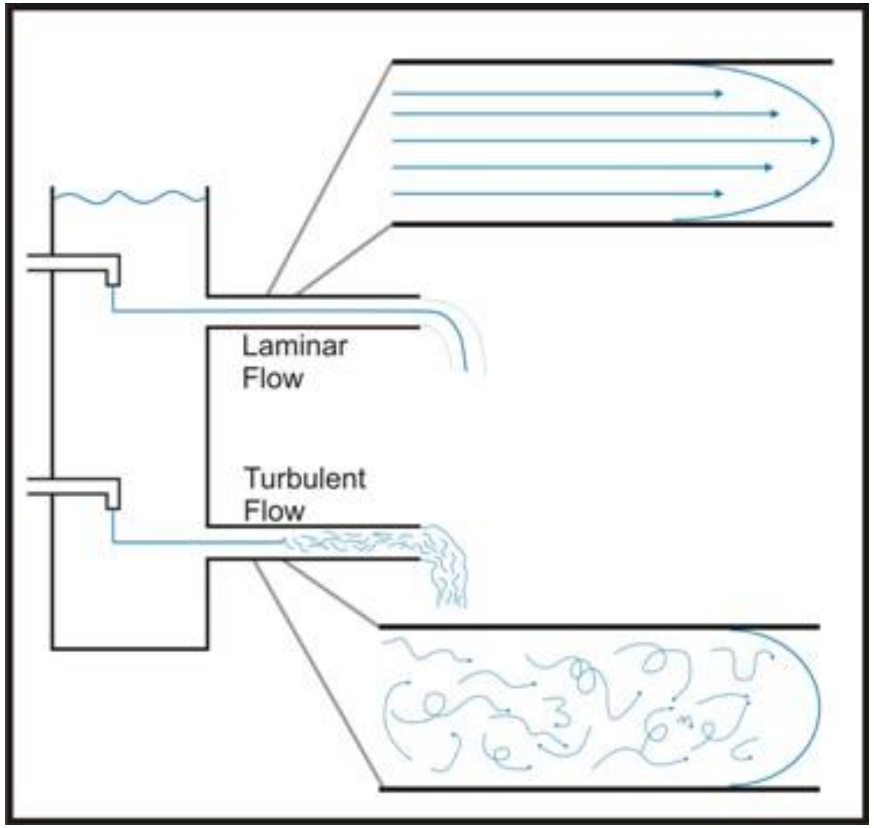
1. 层流流体微元分析方法

层流与湍流
(laminar flow & turbulent flow)



$$\mathrm{Re} \equiv L v_\infty \rho / \mu$$

雷诺数、Reynolds number



分层流动和强对流

从控制体 (control volume) 到流体微元 (differential element)

控制体	流体微元
不关心细节	从另一种角度
穿过表面的物理量	导出微分方程
物理量的总变化	理解各种物理量变化机制

积分 ⟷ 微分

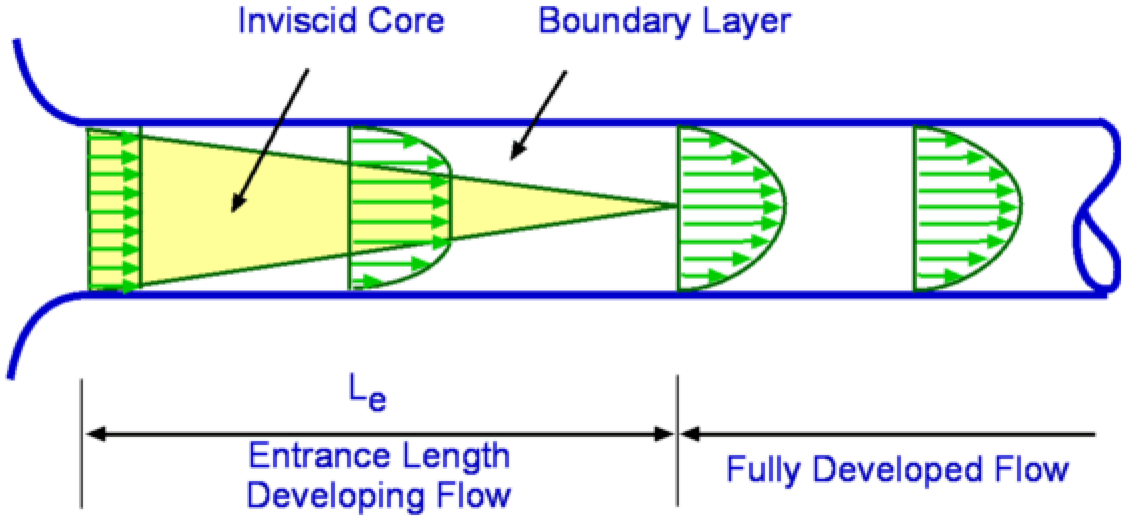
例1：等截面圆管内充分发展的层流流动

Fully developed：速度分布不随流向位置变化

充分开展 ？

充分发展 √

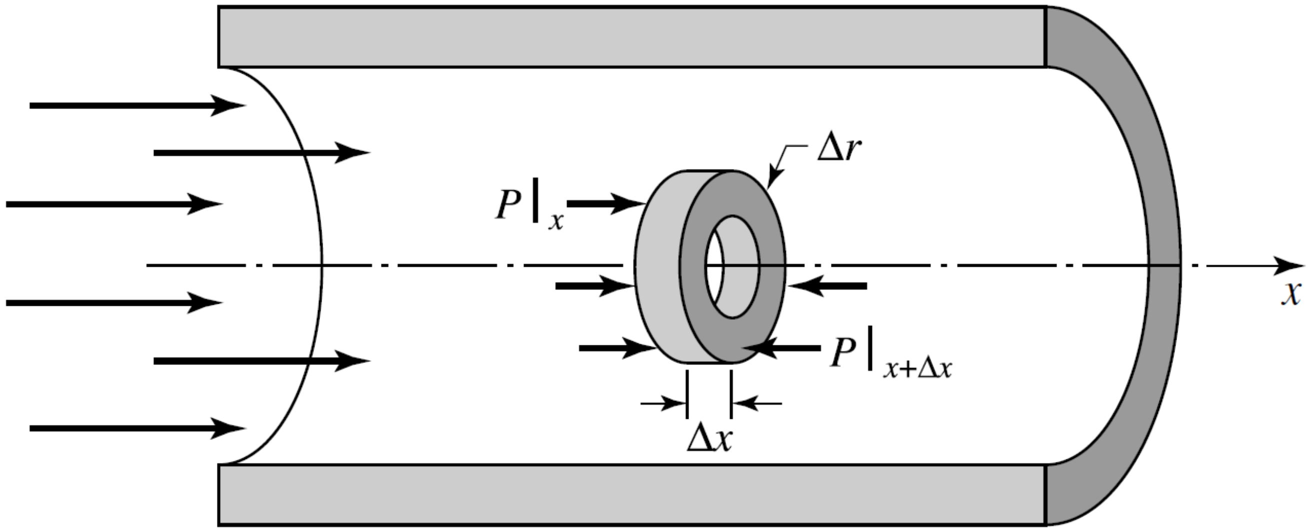
入口效应
(Entrance effect)

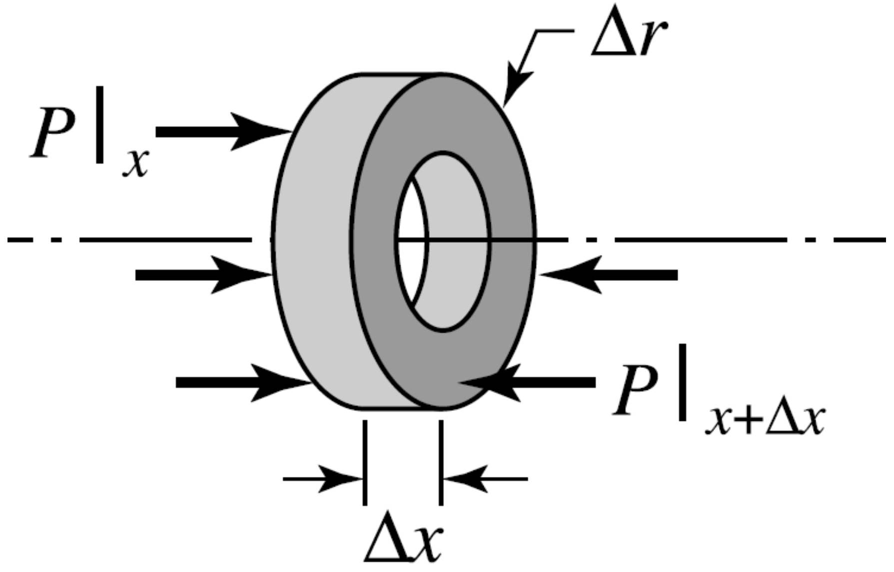


$$\frac{L_e}{d} \approx 1.6\,\mathrm{Re}_d^{1/4} \quad \text{for} \quad \mathrm{Re}_d \le 10^7$$

Some computed turbulent entrance-length estimates are thus

Re_d	4000	10^4	10^5	10^6	10^7
L_e/d	13	16	28	51	90


Δr
P|x
P|x+Δx
Δx
x


Δr
P|x
P|x+Δx
Δx

x方向动量守恒

$$\Sigma F_x = \iint_{\text{c.s.}} \rho v_x(\mathbf{v}\cdot\mathbf{n})dA + \frac{\partial}{\partial t}\iiint_{\text{c.v.}} \rho v_x dV$$

在微元上计算各项（略去高阶项）

$$\Sigma F_x = P(2\pi r\,\Delta r)|_x - P(2\pi r\,\Delta r)|_{x+\Delta x} + \tau_{rx}(2\pi r\,\Delta x)|_{r+\Delta r} - \tau_{rx}(2\pi r\,\Delta x)|_r$$

$$\iint_{\text{c.s.}} v_x\rho(\mathbf{v}\cdot\mathbf{n})dA = (\rho v_x)(2\pi r\,\Delta r v_x)|_{x+\Delta x} - (\rho v_x)(2\pi r\,\Delta r v_x)|_x$$

稳态流动，有

$$\frac{\partial}{\partial t}\iiint_{\text{c.v.}} v_x\rho\, dV = 0$$

充分发展流动，有

$$(\rho v_x)(2\pi r\,\Delta r v_x)|_{x+\Delta x} - (\rho v_x)(2\pi\,\Delta r v_x)|_x = 0$$

各项代入第一个方程，得

$$-[P(2\pi r\,\Delta r)|_{x+\Delta x} - P(2\pi r\,\Delta r)|_x] + \tau_{rx}(2\pi r\,\Delta x)|_{r+\Delta r} - \tau_{rx}(2\pi r\,\Delta x)|_r = 0$$

两边除以 $2\pi r\Delta x\Delta r$，得 $$-r\frac{P|_{x+\Delta x}-P|_x}{\Delta x}+\frac{(r\tau_{rx})|_{r+\Delta r}-(r\tau_{rx})|_r}{\Delta r}=0$$

求极限，有 $$-r\frac{dP}{dx}+\frac{d}{dr}(r\tau_{rx})=0$$

注意，这里P是x的函数，τ_{rx}是r的函数！

在充分发展流动中，流向压力梯度为常数，即 $$\frac{dP}{dx}=\text{const}$$

对r求积分，有 $$\tau_{rx}=\left(\frac{dP}{dx}\right)\frac{r}{2}+\frac{C_1}{r}$$

$r=0$处，剪应力不可能无穷大，所以C_1必为零，得 $$\tau_{rx}=\left(\frac{dP}{dx}\right)\frac{r}{2}$$

牛顿流体存在剪应力与剪应变关系式 $\tau_{rx}=\mu\dfrac{dv_x}{dr}$，所以有 $$\mu\frac{dv_x}{dr}=\left(\frac{dP}{dx}\right)\frac{r}{2}$$

对上式再一次积分，有 $v_x = \left(\frac{dP}{dx}\right)\frac{r^2}{4\mu} + C_2$

$$\mu\frac{dv_x}{dr} = \left(\frac{dP}{dx}\right)\frac{r}{2}$$

考虑到 ***r=R*** 壁面处的无滑移速度条件，得 $C_2 = -\left(\frac{dP}{dx}\right)\frac{R^2}{4\mu}$

得到抛物型速度分布 $v_x = -\left(\frac{dP}{dx}\right)\frac{1}{4\mu}(R^2 - r^2)$ 或者 $v_x = -\left(\frac{dP}{dx}\right)\frac{R^2}{4\mu}\left[1-\left(\frac{r}{R}\right)^2\right]$

r=0 通道中心，速度最大，有 $v_{\max} = -\left(\frac{dP}{dx}\right)\frac{R^2}{4\mu}$ 速度分布又写成 $v_x = v_{\max}\left[1-\left(\frac{r}{R}\right)^2\right]$

用上一讲的管道平均速度与最大速度的关系式，有 $v_{\text{avg}} = \frac{v_{\max}}{2} = -\left(\frac{dP}{dx}\right)\frac{R^2}{8\mu}$

压力梯度用平均速度表示，有 $-\frac{dP}{dx} = \frac{8\mu v_{\text{avg}}}{R^2} = \frac{32\mu v_{\text{avg}}}{D^2}$

如何用到管道流动？

Hagan-Poiseuille equation, 海根-泊肃叶方程

上面推导成立的条件

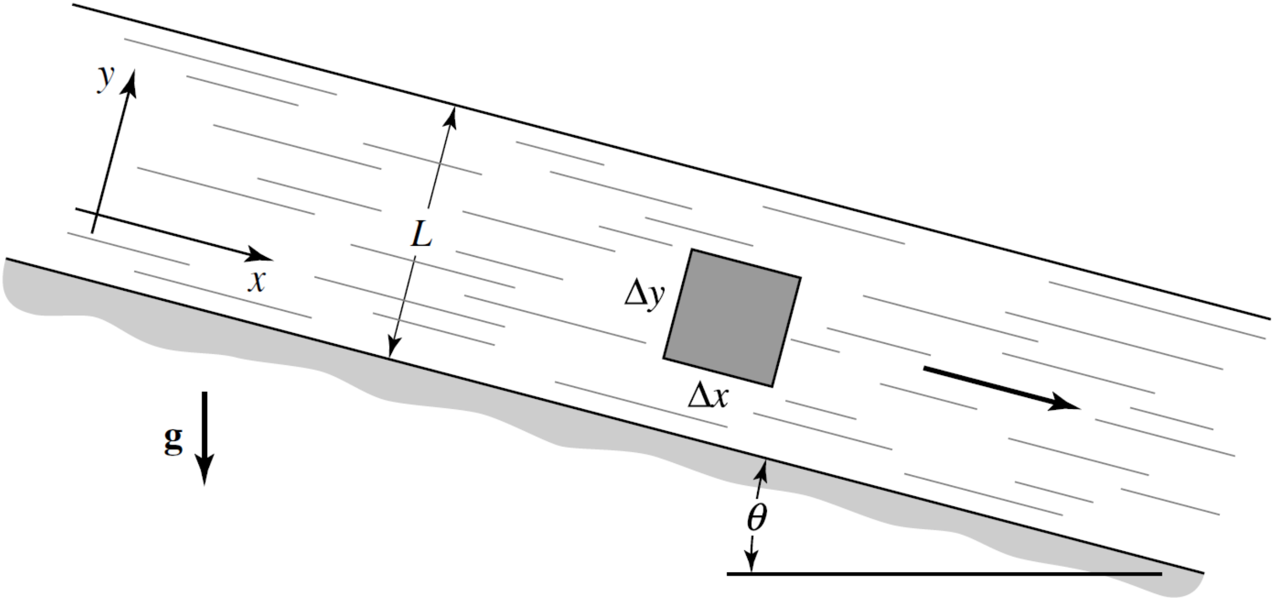
1. The fluid
- a. is Newtonian
- b. behaves as a continuum

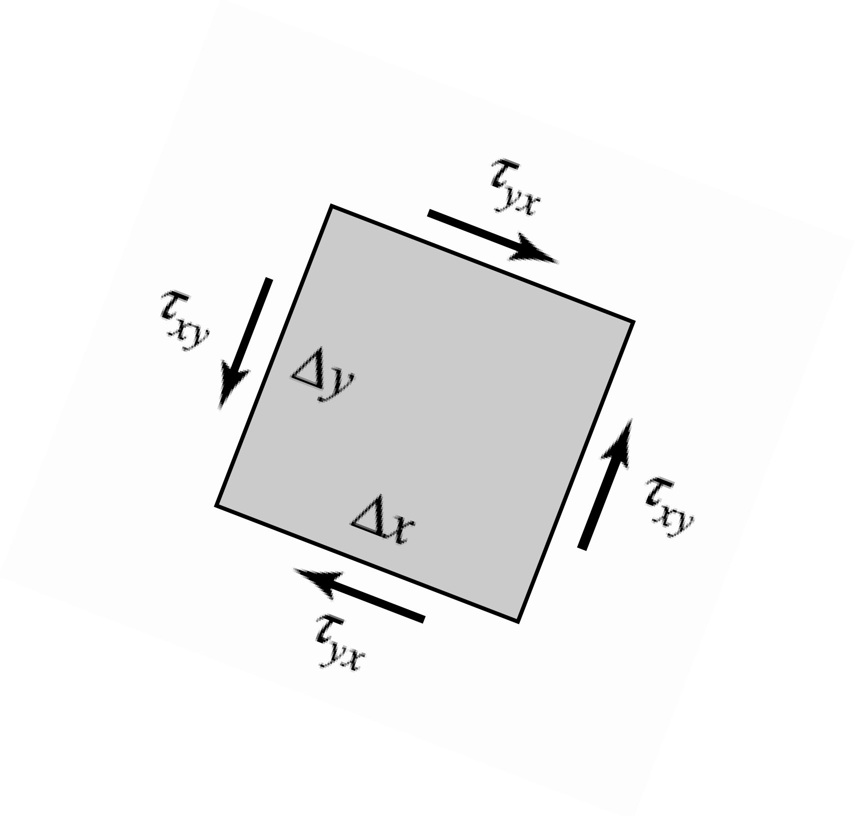
2. The flow is
- a. laminar
- b. steady
- c. fully developed
- d. incompressible

1. 流体
- **a.**牛顿流体
- **b.**连续介质

2. 流动
- **a.** 层流
- **b.** 稳态
- **c.** 完全发展
- **d.** 不可压缩

例2：牛顿流体沿倾斜平面向下的层流流动





注意坐标和微元的选取

回忆一下剪应力下标含义

x方向动量守恒

$$\Sigma F_x = \iint_{\text{c.s.}} \rho v_x(\mathbf{v}\cdot\mathbf{n})dA + \frac{\partial}{\partial t}\iiint_{\text{c.v.}} \rho v_x dV$$

在微元上计算各项

$$\Sigma F_x = P\Delta y|_x - P\Delta y|_{x+\Delta x} + \tau_{yx}\Delta x|_{y+\Delta y} - \tau_{yx}\Delta x|_y + \rho g\,\Delta x\,\Delta y\sin\theta$$

$$\iint_{\text{c.s.}} \rho v_x(\mathbf{v}\cdot\mathbf{n})dA = \rho v_x^2\Delta y|_{x+\Delta x} - \rho v_x^2\Delta y|_x$$

稳态流动，有

$$\frac{\partial}{\partial t}\iiint_{\text{c.v.}} v_x\rho\,dV = 0$$

充分发展流动，有

$$\rho v_x^2\Delta y|_{x+\Delta x} - \rho v_x^2\Delta y|_x = 0$$

各项代入第一个方程，得

$$\tau_{yx}\Delta x|_{y+\Delta y} - \tau_{yx}\Delta x|_y + \rho g\,\Delta x\,\Delta y\sin\theta = 0$$

$$\frac{\tau_{yx}\big|_{y+\Delta y} - \tau_{yx}\big|_{y}}{\Delta y} + \rho g \sin\theta = 0$$

求极限，有 $$\frac{d}{dy}\tau_{yx} + \rho g \sin\theta = 0$$

对y求积分，有 $$\tau_{yx} = -\rho g \sin\theta\, y + C_1$$

利用$y=L$处为自由面，剪应力为零的条件，得

$$\tau_{yx} = \rho g L \sin\theta\left[1 - \frac{y}{L}\right]$$

牛顿流体存在剪应力与剪应变存在线性关系，上式可变为 $$\frac{dv_x}{dy} = \frac{\rho g L \sin\theta}{\mu}\left[1 - \frac{y}{L}\right]$$

对上式再一次积分，有 $$v_x = \frac{\rho g L \sin\theta}{\mu}\left[y - \frac{y^2}{2L}\right] + C_2$$

$$\frac{dv_x}{dy} = \frac{\rho g L \sin\theta}{\mu}\left[1 - \frac{y}{L}\right]$$

考虑到 *y=0* 壁面处的无滑移速度条件，得 $C_2 = 0$

得到抛物型速度分布 $$v_x = \frac{\rho g L^2 \sin\theta}{\mu}\left[\frac{y}{L} - \frac{1}{2}\left(\frac{y}{L}\right)^2\right]$$

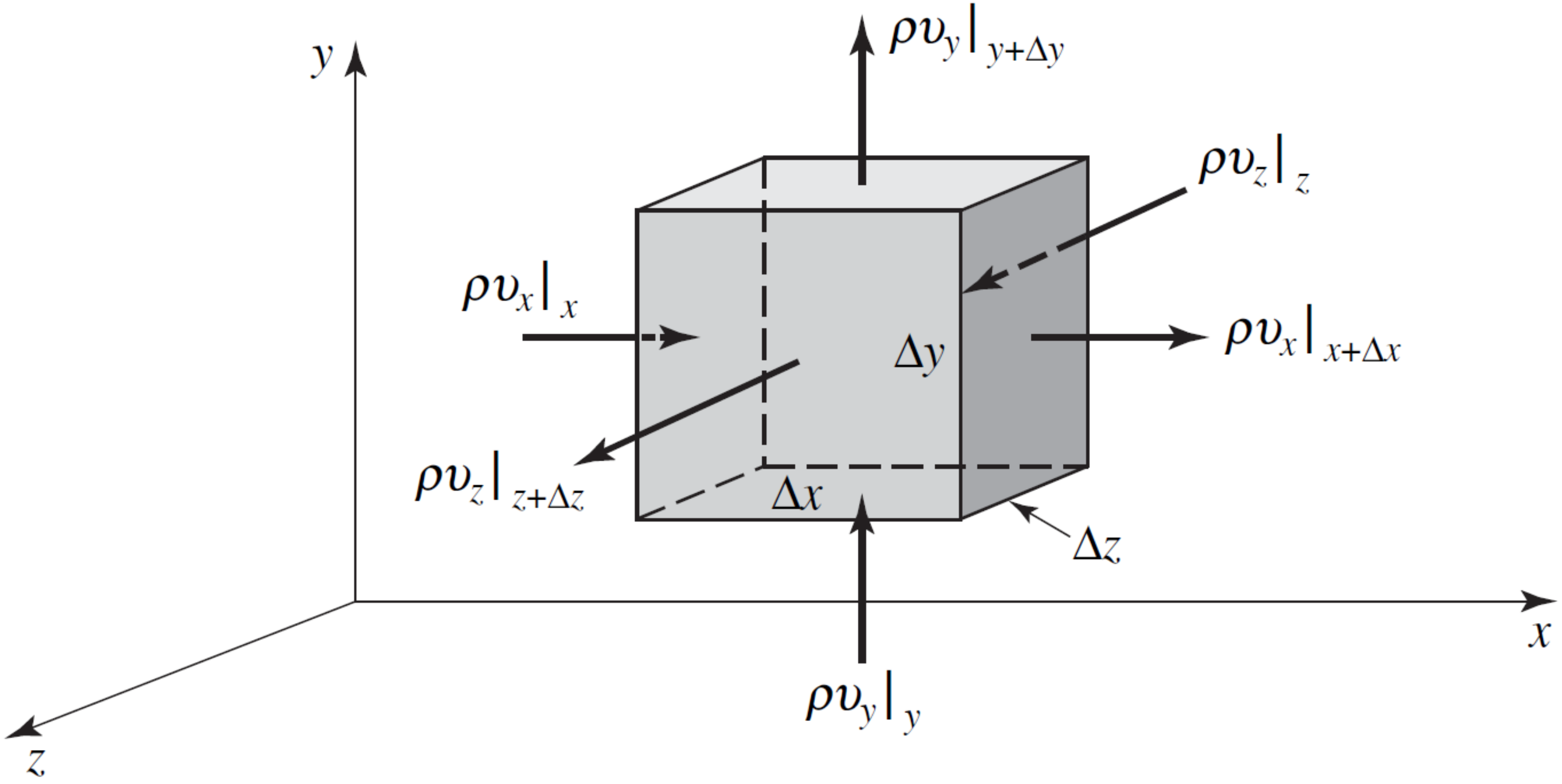
y=L 自由面处速度最大，有 $$v_{\max} = \frac{\rho g L^2 \sin\theta}{2\mu}$$

简单积分比较，可以得到 $$v_{avg} = \frac{2v_{max}}{3}$$

控制体下的质量守恒形式

$$\iint \rho(\mathbf{v}\cdot\mathbf{n})dA+\frac{\partial}{\partial t}\iiint \rho\, dV=0$$

$$\left\{\begin{matrix}\text{控制体的}\\ \text{质量净流出率}\end{matrix}\right\}+\left\{\begin{matrix}\text{控制体的}\\ \text{质量增长率}\end{matrix}\right\}=\mathbf{0}$$



$$\left\{\begin{matrix}\text{控制体的}\\ \text{质量增长率}\end{matrix}\right\} = \frac{\partial}{\partial t}(\rho\ \Delta x\ \Delta y\ \Delta z)$$

x方向的净质量流出率： $\left(\rho v_x|_{x+\Delta x} - \rho v_x|_x\right)\Delta y\,\Delta z$

y方向的净质量流出率： $\left(\rho v_y|_{y+\Delta y} - \rho v_y|_y\right)\Delta x\,\Delta z$

z方向的净质量流出率： $\left(\rho v_z|_{z+\Delta z} - \rho v_z|_z\right)\Delta x\,\Delta y$

代入质量守恒方程：

$$\left(\rho v_x|_{x+\Delta x} - \rho v_x|_x\right)\Delta y\,\Delta z + \left(\rho v_y|_{y+\Delta y} - \rho v_y|_y\right)\Delta x\,\Delta z + \left(\rho v_z|_{z+\Delta z} - \rho v_z|_z\right)\Delta x\,\Delta y + \frac{\partial}{\partial t}(\rho\,\Delta x\,\Delta y\,\Delta z) = 0$$

各项除以$\Delta x\Delta y\Delta z$，并求极限：

$$\frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) + \frac{\partial \rho}{\partial t} = 0$$

微分形式的连续性方程

引入散度（divergence）： $\operatorname{div}\mathbf{F} = \nabla\cdot\mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$ $\mathbf{F} = M(x,y,z)\mathbf{i} + N(x,y,z)\mathbf{j} + P(x,y,z)\mathbf{k}$

连续性方程变成: $\nabla\cdot\rho\mathbf{v} + \frac{\partial\rho}{\partial t} = 0 \Longleftrightarrow \frac{\partial}{\partial x}(\rho v_x) + \frac{\partial}{\partial y}(\rho v_y) + \frac{\partial}{\partial z}(\rho v_z) + \frac{\partial\rho}{\partial t} = 0$

连续性方程各项求导，可变成: $\frac{\partial\rho}{\partial t} + v_x\frac{\partial\rho}{\partial x} + v_y\frac{\partial\rho}{\partial y} + v_z\frac{\partial\rho}{\partial z} + \rho\left(\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z}\right) = 0$

引入随体导数概念:
(substantial derivative)

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}$$

物理量沿流体微元运动途径的时间变化率

随体导数　局部导数　控制体输出的输运量

连续性方程的另一种形式: $\frac{D\rho}{Dt} + \rho\nabla\cdot\mathbf{v} = 0$

不可压缩流动条件下 $\nabla\cdot\mathbf{v} = 0$

纳维-斯托克斯方程 (Navier-Stokes equations)

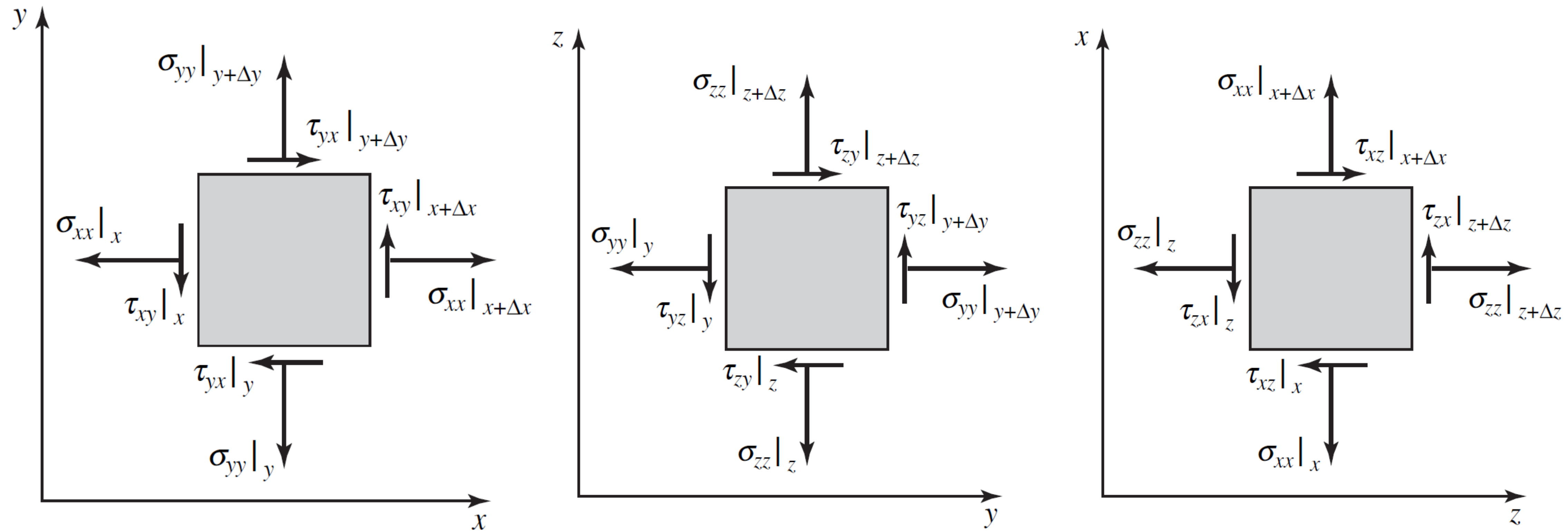
控制体下的动量守恒形式

$$\sum \mathbf{F} = \iint_{\text{c.s.}} \rho \mathbf{v}(\mathbf{v} \cdot \mathbf{n}) dA + \frac{\partial}{\partial t} \iiint_{\text{c.v.}} \rho \mathbf{v} dV$$

$$\begin{Bmatrix} \text{作用在控制体上的} \\ \text{总外力} \end{Bmatrix} = \begin{Bmatrix} \text{控制体的} \\ \text{动量净流出率} \end{Bmatrix} + \begin{Bmatrix} \text{控制体内的} \\ \text{动量变化率} \end{Bmatrix}$$

各项除以$\Delta x \Delta y \Delta z$，有：

$$\lim_{\Delta x, \Delta y, \Delta z \to 0} \frac{\sum \mathbf{F}}{\Delta x \, \Delta y \, \Delta z} = \lim_{\Delta x, \Delta y, \Delta z \to 0} \frac{\iint \rho \mathbf{v}(\mathbf{v} \cdot \mathbf{n}) \, dA}{\Delta x \, \Delta y \, \Delta z} + \lim_{\Delta x, \Delta y, \Delta z \to 0} \frac{\partial / \partial t \iiint \rho \mathbf{v} \, dV}{\Delta x \, \Delta y \, \Delta z}$$



x方向上，有：

$$\sum F_x = (\sigma_{xx}|_{x+\Delta x} - \sigma_{xx}|_{x})\Delta y\,\Delta z + (\tau_{yx}|_{y+\Delta y} - \tau_{yx}|_{y})\Delta x\,\Delta z + (\tau_{zx}|_{z+\Delta z} - \tau_{zx}|_{z})\Delta x\,\Delta y + g_x\rho\,\Delta x\,\Delta y\,\Delta z$$

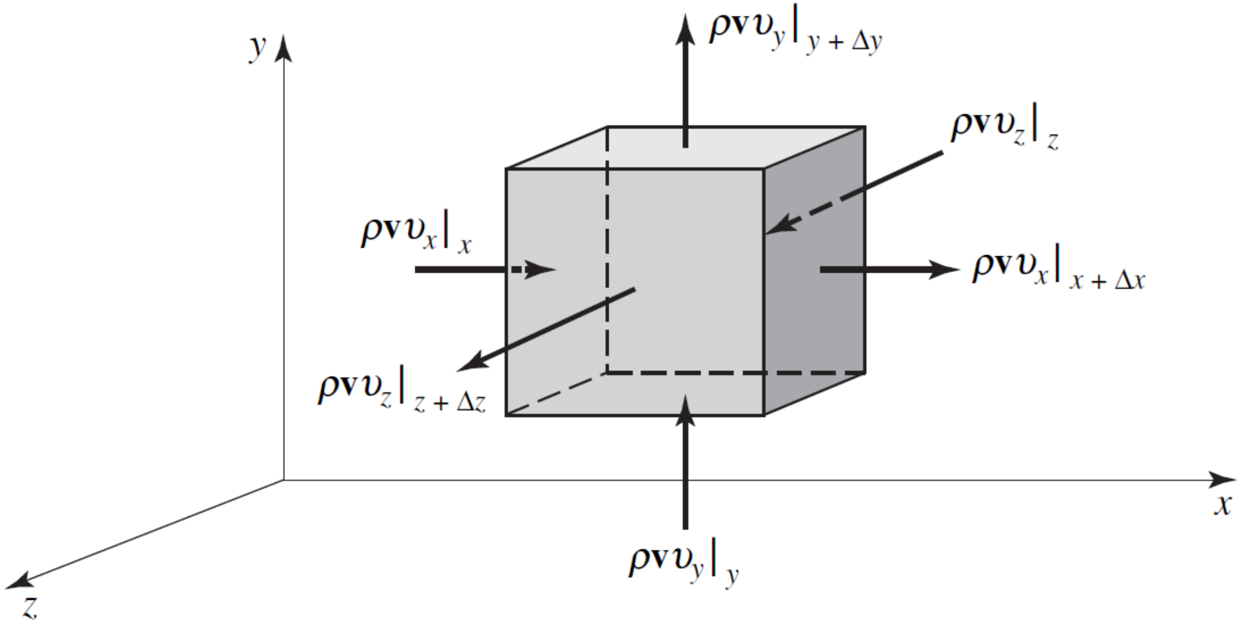
各项除以ΔxΔyΔz，有：

$$\lim_{\Delta x,\Delta y,\Delta z\to 0}\frac{\sum F_x}{\Delta x\,\Delta y\,\Delta z}=\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\tau_{yx}}{\partial y}+\frac{\partial\tau_{zx}}{\partial z}+\rho g_x$$

控制体的动量净流出率：

$$\lim_{\Delta x,\Delta y,\Delta z\to 0}\frac{\iint\rho\mathbf{v}(\mathbf{v}\cdot\mathbf{n})dA}{\Delta x\,\Delta y\,\Delta z}=\lim_{\Delta x,\Delta y,\Delta z\to 0}\left[\frac{\left(\rho\mathbf{v}v_x|_{x+\Delta x}-\rho\mathbf{v}v_x|_x\right)\Delta y\,\Delta z}{\Delta x\,\Delta y\,\Delta z}+\frac{\left(\rho\mathbf{v}v_y|_{y+\Delta y}-\rho\mathbf{v}v_y|_y\right)\Delta x\,\Delta z}{\Delta x\,\Delta y\,\Delta z}+\frac{\left(\rho\mathbf{v}v_z|_{z+\Delta z}-\rho\mathbf{v}v_z|_z\right)\Delta x\,\Delta y}{\Delta x\,\Delta y\,\Delta z}\right.$$

$$=\frac{\partial}{\partial x}(\rho\mathbf{v}v_x)+\frac{\partial}{\partial y}(\rho\mathbf{v}v_y)+\frac{\partial}{\partial z}(\rho\mathbf{v}v_z)$$



括弧内微分，可以变成：$$\lim_{\Delta x,\Delta y,\Delta z\to 0}\frac{\iint \rho\mathbf{v}(\mathbf{v}\cdot\mathbf{n})dA}{\Delta x\,\Delta y\,\Delta z}=\mathbf{v}\left[\frac{\partial}{\partial x}(\rho v_x)+\frac{\partial}{\partial y}(\rho v_y)+\frac{\partial}{\partial z}(\rho v_z)\right]+\rho\left[v_x\frac{\partial\mathbf{v}}{\partial x}+v_y\frac{\partial\mathbf{v}}{\partial y}+v_z\frac{\partial\mathbf{v}}{\partial z}\right]$$

把连续性方程 $\frac{\partial\rho}{\partial t}+\frac{\partial}{\partial x}(\rho v_x)+\frac{\partial}{\partial y}(\rho v_y)+\frac{\partial}{\partial z}(\rho v_z)=0$ 代入上式，有

$$\lim_{\Delta x,\Delta y,\Delta z\to 0}\frac{\iint \rho\mathbf{v}(\mathbf{v}\cdot\mathbf{n})dA}{\Delta x\,\Delta y\,\Delta z}=-\mathbf{v}\frac{\partial\rho}{\partial t}+\rho\left[v_x\frac{\partial\mathbf{v}}{\partial x}+v_y\frac{\partial\mathbf{v}}{\partial y}+vz\frac{\partial\mathbf{v}}{\partial z}\right]$$

控制体内的动量变化率：$$\lim_{\Delta x,\Delta y,\Delta z\to 0}\frac{\partial/\partial t\iiint \mathbf{v}\rho\,dV}{\Delta x\,\Delta y\,\Delta z}=\frac{(\partial/\partial t)\rho\mathbf{v}\,\Delta x\,\Delta y\,\Delta z}{\Delta x\,\Delta y\,\Delta z}=\frac{\partial}{\partial t}\rho\mathbf{v}=\rho\frac{\partial\mathbf{v}}{\partial t}+\mathbf{v}\frac{\partial\rho}{\partial t}$$

$$\sum \mathbf{F} = \iint_{\text{c.s.}} \rho \mathbf{v}(\mathbf{v} \cdot \mathbf{n}) dA + \frac{\partial}{\partial t} \iiint_{\text{c.v.}} \rho \mathbf{v} dV$$

动量方程的各项为：

① $$\lim_{\Delta x, \Delta y, \Delta z \to 0} \frac{\sum \mathbf{F}}{\Delta x\, \Delta y\, \Delta z} = \left\{ \begin{aligned} &\left(\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho g_x \right) \mathbf{e}_x \\ &\left(\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho g_y \right) \mathbf{e}_y \\ &\left(\frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + \rho g_z \right) \mathbf{e}_z \end{aligned} \right\}$$

② $$\lim_{\Delta x, \Delta y, \Delta z \to 0} \frac{\iint \rho \mathbf{v}(\mathbf{v} \cdot \mathbf{n}) dA}{\Delta x\, \Delta y\, \Delta z} = -\mathbf{v} \frac{\partial \rho}{\partial t} + \rho \left(v_x \frac{\partial \mathbf{v}}{\partial x} + v_y \frac{\partial \mathbf{v}}{\partial y} + v_z \frac{\partial \mathbf{v}}{\partial z} \right)$$

③ $$\lim_{\Delta x, \Delta y, \Delta z \to 0} \frac{\partial / \partial t \iiint \rho \mathbf{v} dV}{\Delta x\, \Delta y\, \Delta z} = \rho \frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \frac{\partial \rho}{\partial t}$$

可以写出三个方向上的动量方程：

$$\rho\left(\frac{\partial v_x}{\partial t}+v_x\frac{\partial v_x}{\partial x}+v_y\frac{\partial v_x}{\partial y}+v_z\frac{\partial v_x}{\partial z}\right)=\rho g_x+\frac{\partial \sigma_{xx}}{\partial x}+\frac{\partial \tau_{yx}}{\partial y}+\frac{\partial \tau_{zx}}{\partial z}$$

$$\rho\left(\frac{\partial v_y}{\partial t}+v_x\frac{\partial v_y}{\partial x}+v_y\frac{\partial v_y}{\partial y}+v_z\frac{\partial v_y}{\partial z}\right)=\rho g_y+\frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_{yy}}{\partial y}+\frac{\partial \tau_{zy}}{\partial z}$$

$$\rho\left(\frac{\partial v_z}{\partial t}+v_x\frac{\partial v_z}{\partial x}+v_y\frac{\partial v_z}{\partial y}+v_z\frac{\partial v_z}{\partial z}\right)=\rho g_z+\frac{\partial \tau_{xz}}{\partial x}+\frac{\partial \tau_{yz}}{\partial y}+\frac{\partial \sigma_{zz}}{\partial z}$$

$$\underbrace{\frac{\partial v_x}{\partial t}}\quad + \underbrace{v_x\frac{\partial v_x}{\partial x} + v_y\frac{\partial v_x}{\partial y} + v_z\frac{\partial v_x}{\partial z}} = \left(\frac{\partial}{\partial t} + v_x\frac{\partial}{\partial x} + v_y\frac{\partial}{\partial y} + v_z\frac{\partial}{\partial z}\right)v_x$$

当地加速度
local acceleration

对流加速度/迁移加速度
convective acceleration

随体导数
Substantial derivative

或者

全导数
Total derivative

用随体导数表示，有:

$$\rho\frac{Dv_x}{Dt} = \rho g_x + \frac{\partial\sigma_{xx}}{\partial x} + \frac{\partial\tau_{yx}}{\partial y} + \frac{\partial\tau_{zx}}{\partial z}$$

$$\rho\frac{Dv_y}{Dt} = \rho g_y + \frac{\partial\tau_{xy}}{\partial x} + \frac{\partial\sigma_{yy}}{\partial y} + \frac{\partial\tau_{zy}}{\partial z}$$

$$\rho\frac{Dv_z}{Dt} = \rho g_z + \frac{\partial\tau_{xz}}{\partial x} + \frac{\partial\tau_{yz}}{\partial y} + \frac{\partial\sigma_{zz}}{\partial z}$$

有关随体导数（全导数）的概念

分析：

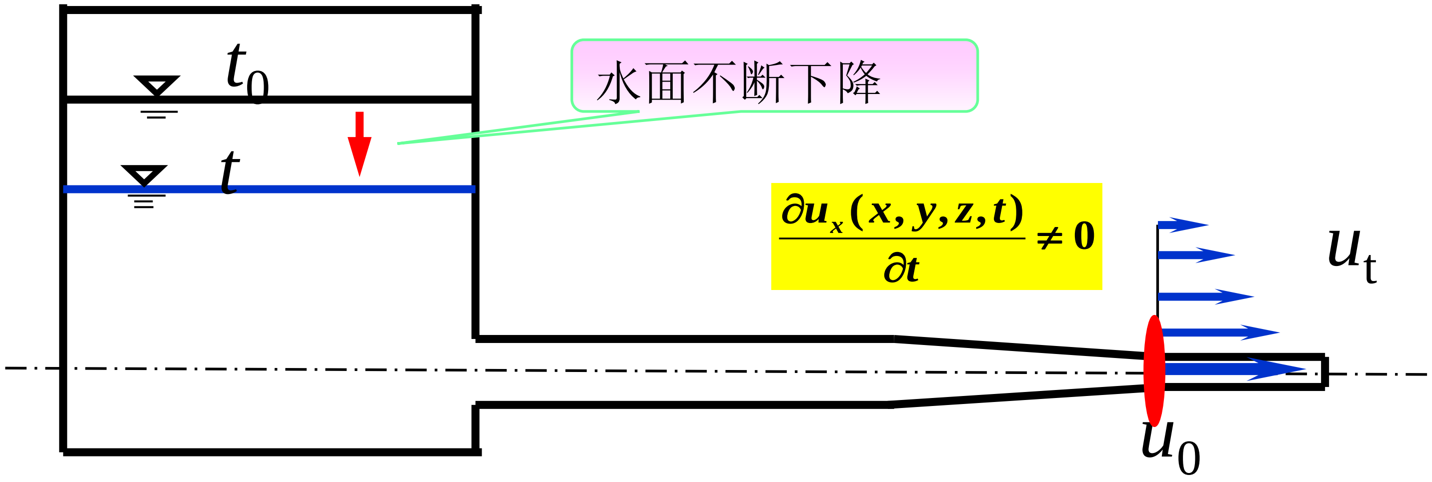
- 从欧拉法看，同一空间点上，因时间先后不同，流速可不同；
- 不同空间位置上，流体的流速也可以不同；
- 因此，加速度分

- **时变加速度（当地加速度）**：同一空间点，不同时刻上因流速不同，而产生的加速度。
- **位变加速度（迁移加速度）**：同一时刻，不同空间点上流速不同，而产生的加速度。

$$\mathbf{a}=\frac{d\mathbf{V}}{dt}=\underbrace{\frac{\partial\mathbf{V}}{\partial t}}_{\text{当地加速度}}+\underbrace{\left(u\frac{\partial\mathbf{V}}{\partial x}+v\frac{\partial\mathbf{V}}{\partial y}+w\frac{\partial\mathbf{V}}{\partial z}\right)}_{\text{迁移加速度}}=\frac{\partial\mathbf{V}}{\partial t}+(\mathbf{V}\cdot\nabla)\mathbf{V}$$

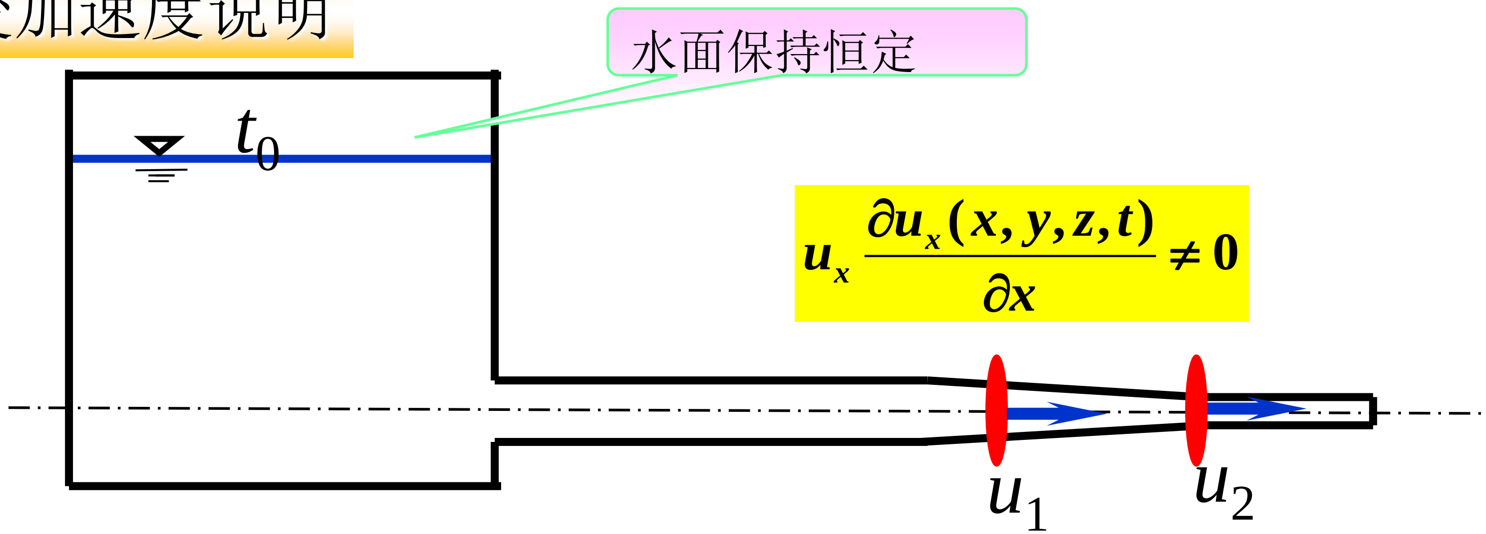
示例：水箱放水

时变加速度产生说明



$$\frac{\partial u_x(x,y,z,t)}{\partial t} \neq 0$$

位变加速度说明



$$u_x \frac{\partial u_x(x,y,z,t)}{\partial x} \neq 0$$

其他流体的物理量，比如压强P的变化，也可以表示为

$$dP=\frac{\partial P}{\partial t}dt+\frac{\partial P}{\partial x}dx+\frac{\partial P}{\partial y}dy+\frac{\partial P}{\partial z}dz$$

$$\frac{dP}{dt}=\frac{\partial P}{\partial t}+\frac{dx}{dt}\frac{\partial P}{\partial x}+\frac{dy}{dt}\frac{\partial P}{\partial y}+\frac{dz}{dt}\frac{\partial P}{\partial z}$$

$$\frac{dP}{dt}=\frac{DP}{Dt}=\underbrace{\frac{\partial P}{\partial t}}_{\substack{\text{local}\\ \text{rate of}\\ \text{change of}\\ \text{pressure}}}+\underbrace{v_x\frac{\partial P}{\partial x}+v_y\frac{\partial P}{\partial y}+v_z\frac{\partial P}{\partial z}}_{\substack{\text{rate of change}\\ \text{of pressure}\\ \text{due to motion}}}$$

对牛顿流体，
有第7章的关系式:

$$\tau_{xy} = \tau_{yx} = \mu\left(\frac{\partial v_x}{\partial y} + \frac{\partial v_y}{\partial x}\right) \qquad \sigma_{xx} = \mu\left(2\frac{\partial v_x}{\partial x} - \frac{2}{3}\nabla\cdot\mathbf{v}\right) - P$$

$$\tau_{yz} = \tau_{zy} = \mu\left(\frac{\partial v_y}{\partial z} + \frac{\partial v_z}{\partial y}\right) \qquad \sigma_{yy} = \mu\left(2\frac{\partial v_y}{\partial y} - \frac{2}{3}\nabla\cdot\mathbf{v}\right) - P$$

$$\tau_{zx} = \tau_{xz} = \mu\left(\frac{\partial v_z}{\partial x} + \frac{\partial v_x}{\partial z}\right) \qquad \sigma_{zz} = \mu\left(2\frac{\partial v_z}{\partial z} - \frac{2}{3}\nabla\cdot\mathbf{v}\right) - P$$

动量方程可以表达为:

$$\rho\frac{Dv_x}{Dt} = \rho g_x - \frac{\partial P}{\partial x} - \frac{\partial}{\partial x}\left(\frac{2}{3}\mu\nabla\cdot\mathbf{v}\right) + \nabla\cdot\left(\mu\frac{\partial\mathbf{v}}{\partial x}\right) + \nabla\cdot(\mu\nabla v_x)$$

$$\rho\frac{Dv_y}{Dt} = \rho g_y - \frac{\partial P}{\partial y} - \frac{\partial}{\partial y}\left(\frac{2}{3}\mu\nabla\cdot\mathbf{v}\right) + \nabla\cdot\left(\mu\frac{\partial\mathbf{v}}{\partial y}\right) + \nabla\cdot(\mu\nabla v_y)$$

$$\rho\frac{Dv_z}{Dt} = \rho g_z - \frac{\partial P}{\partial z} - \frac{\partial}{\partial z}\left(\frac{2}{3}\mu\nabla\cdot\mathbf{v}\right) + \nabla\cdot\left(\mu\frac{\partial\mathbf{v}}{\partial z}\right) + \nabla\cdot(\mu\nabla v_z)$$

最复杂的偏微分方程（**PDE**）：牛顿流体的纳维-斯托克斯方程

(Navier-Stokes equations for Newtonian fluid)

千禧年七大数学难题之一 **Navier-Stokes existence and smoothness**

对不可压缩流动 $\nabla \cdot \mathbf{v} = 0$

NS方程化简为：

$$\rho \frac{Dv_x}{Dt} = \rho g_x - \frac{\partial P}{\partial x} + \mu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right)$$

$$\rho \frac{Dv_y}{Dt} = \rho g_y - \frac{\partial P}{\partial y} + \mu \left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \right)$$

$$\rho \frac{Dv_z}{Dt} = \rho g_z - \frac{\partial P}{\partial z} + \mu \left(\frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \right)$$

写成矢量形式：

$$\rho \frac{D\mathbf{v}}{Dt} = \rho \mathbf{g} - \nabla P + \mu \nabla^2 \mathbf{v}$$

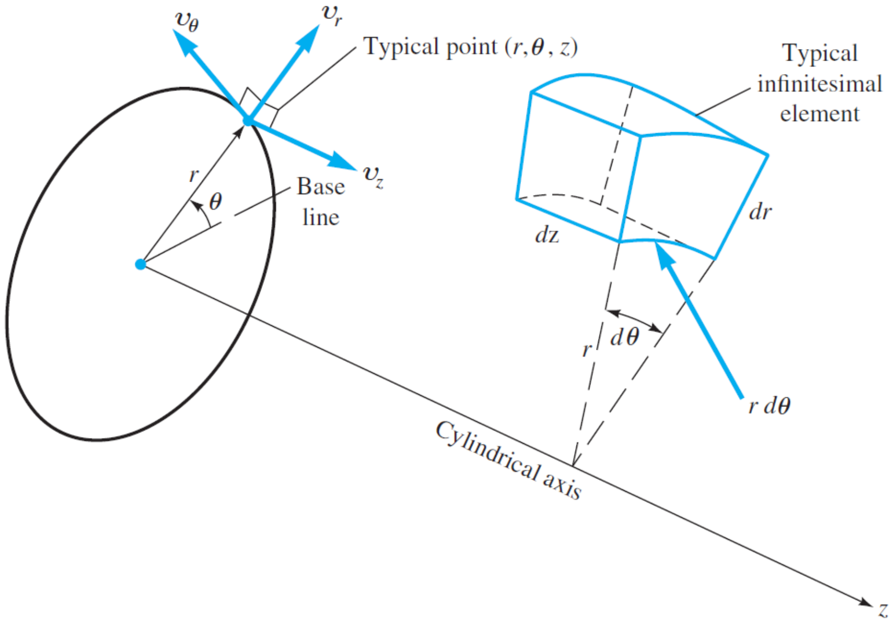
$\longleftrightarrow$

1. incompressible flow,
2. constant viscosity,
3. laminar flow.[2]

如果无黏性，则变为欧拉方程 (Euler’s equation)：

$$\rho \frac{D\mathbf{v}}{Dt} = \rho \mathbf{g} - \nabla P$$

柱坐标系不可压缩牛顿流体的N-S方程



r direction

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta^2}{r}+v_z\frac{\partial v_r}{\partial z}\right)$$
$$=-\frac{\partial P}{\partial r}+\rho g_r+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_r)\right)+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2}-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}+\frac{\partial^2 v_r}{\partial z^2}\right]$$

θ direction

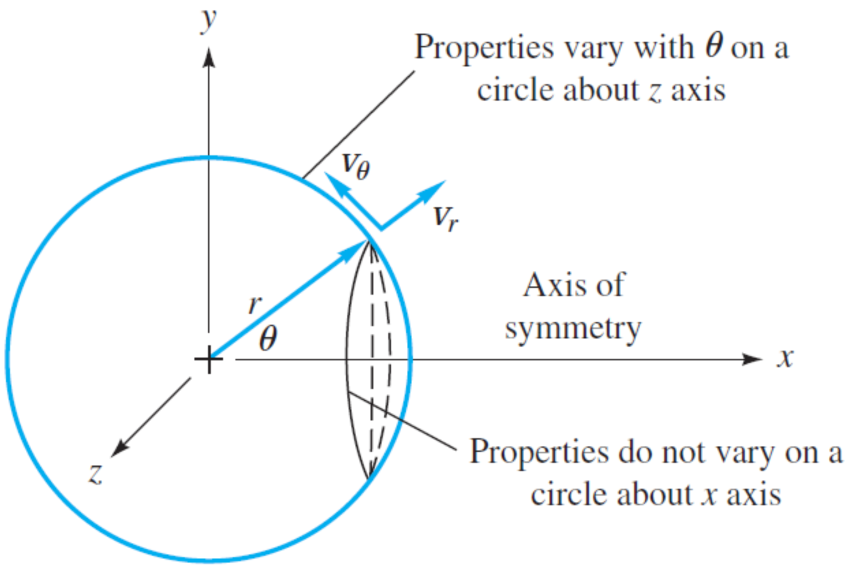
$$\rho\left(\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_r v_\theta}{r}+v_z\frac{\partial v_\theta}{\partial z}\right)=-\frac{1}{r}\frac{\partial P}{\partial \theta}+\rho g_\theta$$
$$+\mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r v_\theta)\right)+\frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2}+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}+\frac{\partial^2 v_\theta}{\partial z^2}\right]$$

z direction

$$\rho\left(\frac{\partial v_z}{\partial t}+v_r\frac{\partial v_z}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta}+v_z\frac{\partial v_z}{\partial z}\right)$$
$$=-\frac{\partial P}{\partial z}+\rho g_z+\mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2}+\frac{\partial^2 v_z}{\partial z^2}\right]$$

教材附录E

球坐标系不可压缩牛顿流体的N-S方程



r direction

$$\rho\left(\frac{\partial v_r}{\partial t}+v_r\frac{\partial v_r}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_r}{\partial \phi}-\frac{v_\phi^2}{r}-\frac{v_\theta^2}{r}\right)$$

$$=-\frac{\partial P}{\partial r}+\rho g_r+\mu\left[\boldsymbol{\nabla}^2 v_r-\frac{2}{r^2}v_r-\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}-\frac{2}{r^2}v_\theta\cot\theta-\frac{2}{r^2\sin\theta}\frac{\partial v_\phi}{\partial \phi}\right]$$

θ direction

$$\rho\left[\frac{\partial v_\theta}{\partial t}+v_r\frac{\partial v_\theta}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\theta}{\partial \phi}+\frac{v_r v_\theta}{r}-\frac{\partial v_\phi^2\cot\theta}{r}\right]$$

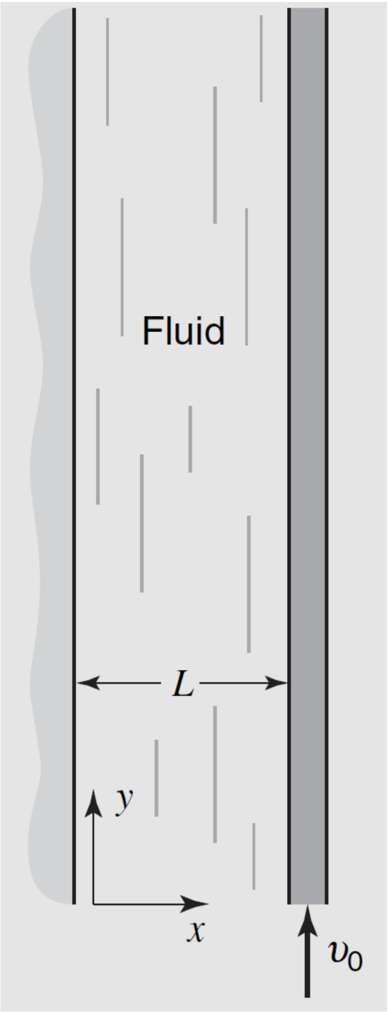
$$=-\frac{1}{r}\frac{\partial P}{\partial \theta}+\rho g_\theta+\mu\left[\boldsymbol{\nabla}^2 v_\theta+\frac{2}{r^2}\frac{\partial v_r}{\partial \theta}-\frac{v_\theta}{r^2\sin^2\theta}-\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\phi}{\partial \phi}\right]$$

ϕ direction

$$\rho\left(\frac{\partial v_\phi}{\partial t}+v_r\frac{\partial v_\phi}{\partial r}+\frac{v_\theta}{r}\frac{\partial v_\phi}{\partial \theta}+\frac{v_\phi}{r\sin\theta}\frac{\partial v_\phi}{\partial \phi}+\frac{v_\phi v_r}{r}+\frac{v_\theta v_\phi}{r}\cot\theta\right)$$

$$=-\frac{1}{r\sin\theta}\frac{\partial P}{\partial \phi}+\rho g_\phi+\mu\left[\nabla^2 v_\phi-\frac{v_\phi}{r^2\sin^2\theta}+\frac{2}{r^2\sin\theta}\frac{\partial v_r}{\partial \phi}+\frac{2\cos\theta}{r^2\sin^2\theta}\frac{\partial v_\theta}{\partial \phi}\right]$$

例3：垂直平板间的不可压缩牛顿流体定常流动分析



求解不可压缩NS方程

$$\boxed{\rho\frac{Dv_y}{Dt} = \rho g_y - \frac{\partial P}{\partial y} + \mu\left(\frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2}\right)}$$

各项表达式

$$\rho\frac{D\mathbf{v}}{Dt} = \rho\left\{\frac{\partial \mathbf{v}}{\partial t} + v_x\frac{\partial \mathbf{v}}{\partial x} + v_y\frac{\partial \mathbf{v}}{\partial y} + v_z\frac{\partial \mathbf{v}}{\partial z}\right\} = 0$$ 为什么?

$$\rho\mathbf{g} = -\rho g\mathbf{e}_y$$

$$\nabla P = \frac{dP}{dy}\mathbf{e}_y$$

$\frac{dP}{dy}$ 为常数

$$\mu\nabla^2\mathbf{v} = \mu\frac{d^2 v_y}{dx^2}\mathbf{e}_y$$

只有在y方向存在速度分量

得到控制方程
(governing equations)

$$0 = -\rho g - \frac{dP}{dy} + \mu\frac{d^2 v_y}{dx^2}$$

$$0 = -\rho g - \frac{dP}{dy} + \mu \frac{d^2 v_y}{dx^2}$$

对x积分一次，得到

$$\frac{dv_y}{dx} + \frac{x}{\mu}\left\{-\rho g - \frac{dP}{dy}\right\} = C_1$$

再对x积分一次，得到

$$v_y + \frac{x^2}{2\mu}\left\{-\rho g - \frac{dP}{dy}\right\} = C_1 x + C_2$$

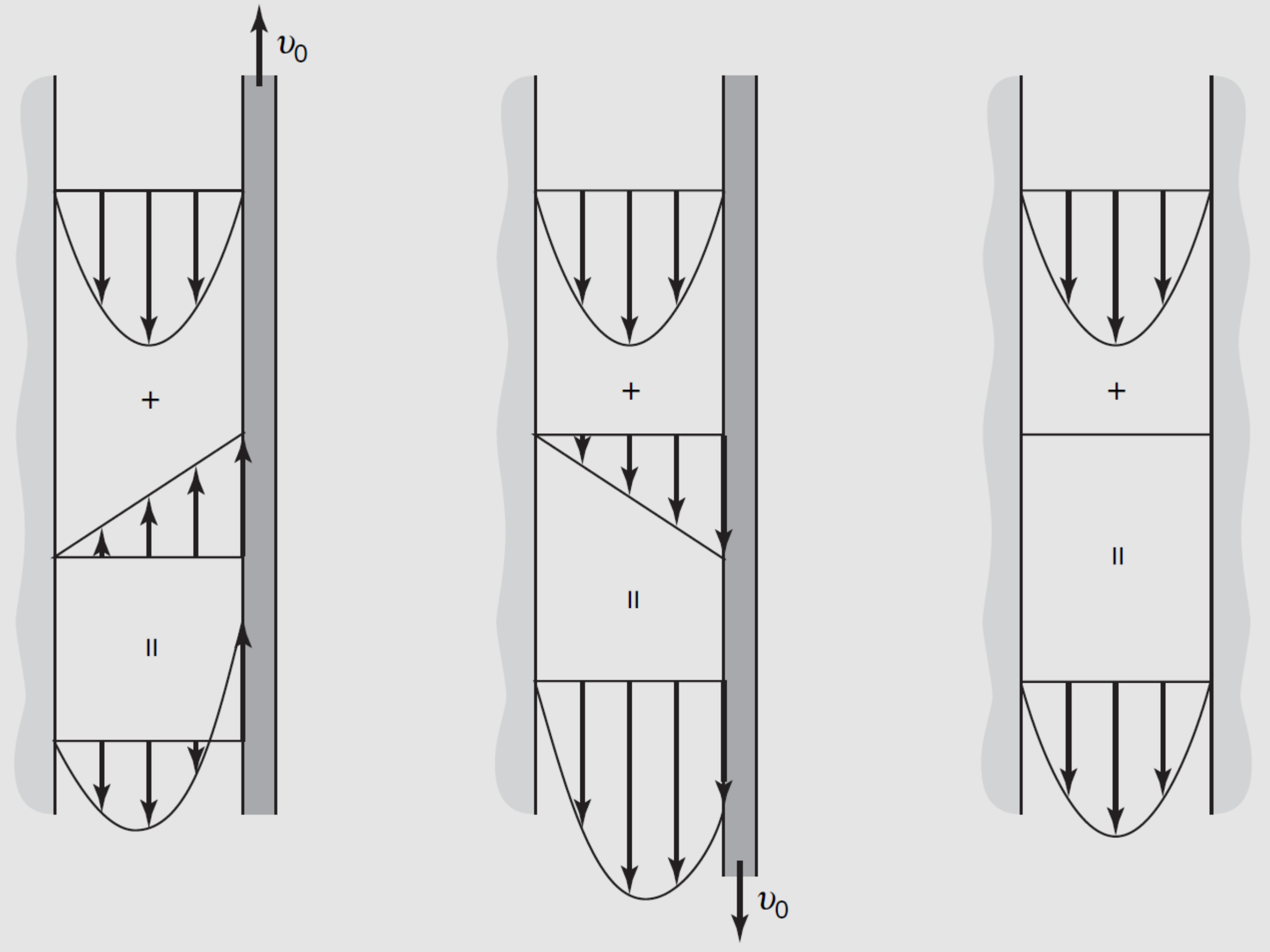
利用平板上的两个边界条件 $x = 0, v_y = 0$ 以及 $x = L, v_y = v_0$ ，可以确定两个积分常数，

$$C_1 = \frac{v_0}{L} + \frac{L}{2\mu}\left\{-\rho g - \frac{dP}{dy}\right\} \quad \text{and} \quad C_2 = 0$$

最终的速度表达式为

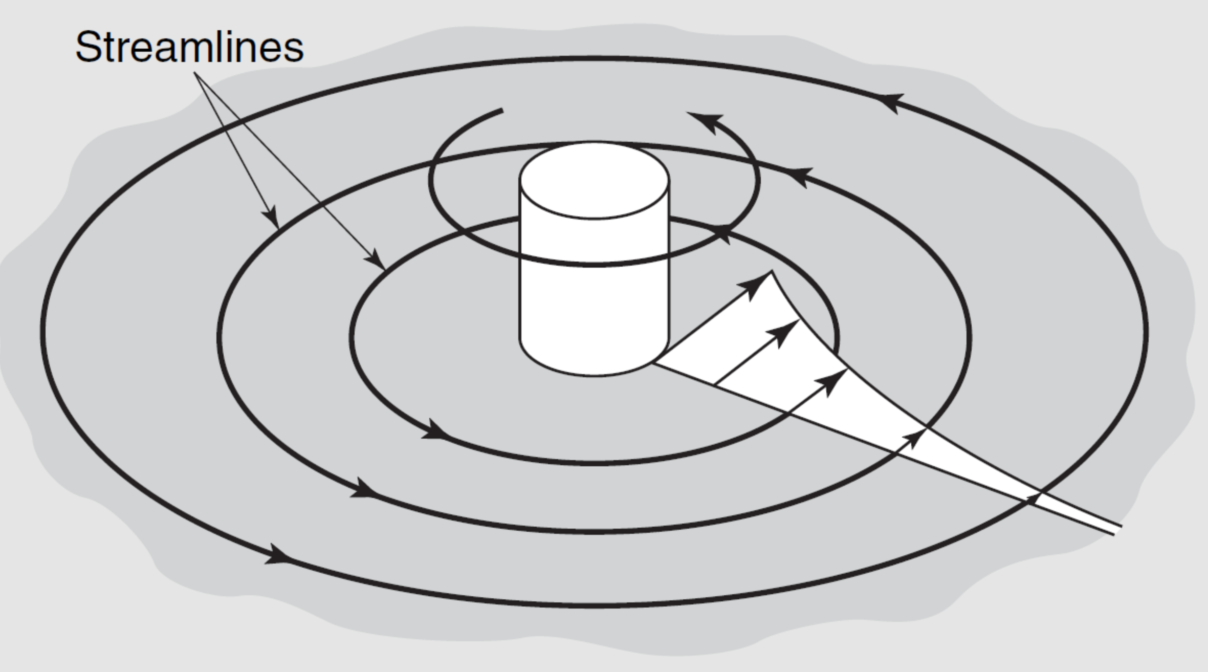
$$v_y = \underbrace{\frac{1}{2\mu}\left\{-\rho g - \frac{dP}{dy}\right\}\{Lx - x^2\}}_{①} + \underbrace{v_0 \frac{x}{L}}_{②}$$

两个速度的线性叠加！
为什么？



不同运动速度的线性叠加

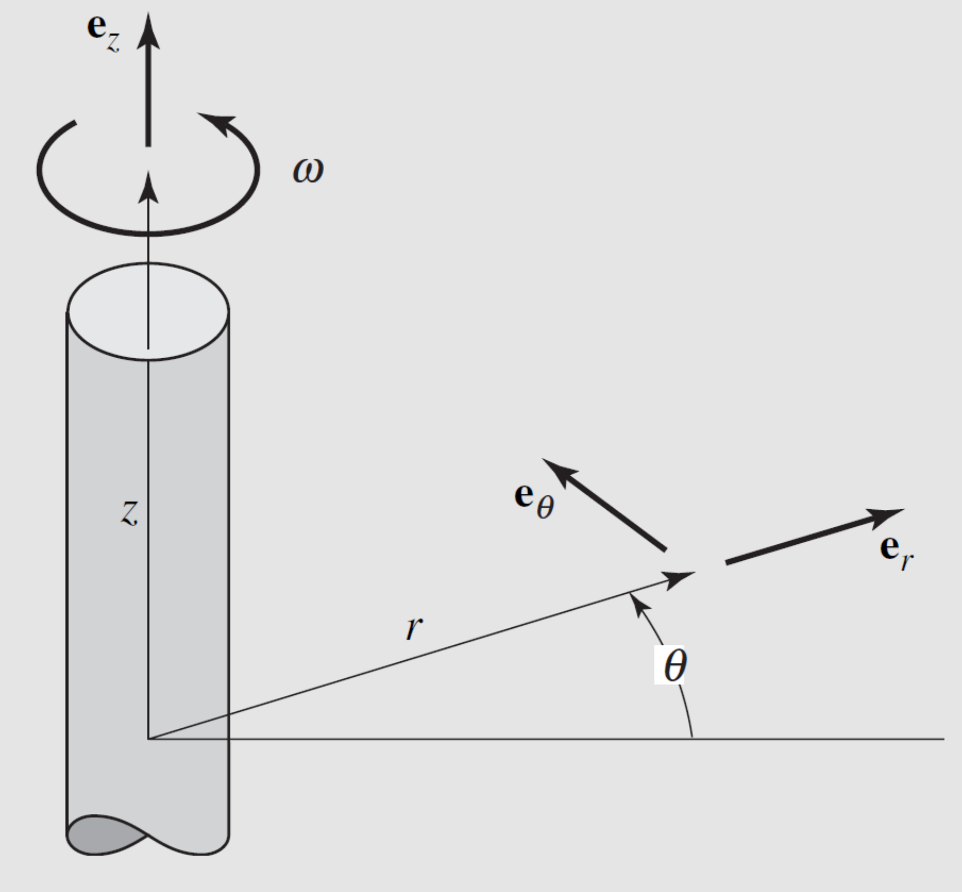
例4：无黏流体旋转流动的自由面(free surface)分析



已知：1. 流动速度与距离成反比

2. 无黏

求：自由面怎么变化?



柱坐标

出发点：压强沿自由面为常数，因此如果压强确定了，自由面变化也就确定了。

控制方程为欧拉方程 $$\nabla P = \rho \mathbf{g} - \rho \frac{D\mathbf{v}}{Dt}$$ 这里 $\mathbf{v} = A\mathbf{e}_\theta / r$

在转子和流体的接触面上，流体速度与转子速度相同 $v(R) = \omega R = \frac{A}{R}$ 得到 $A = \omega R^2$

代入 $\mathbf{v} = A\mathbf{e}_\theta / r$ 得到 $$\mathbf{v} = \frac{\omega R^2}{r}\mathbf{e}_\theta$$

求全导数 $$\frac{d\mathbf{v}}{dt} = -\frac{\omega R^2}{r^2}\mathbf{e}_\theta \dot{r} + \frac{\omega R^2}{r}\frac{d\mathbf{e}_\theta}{dt}$$

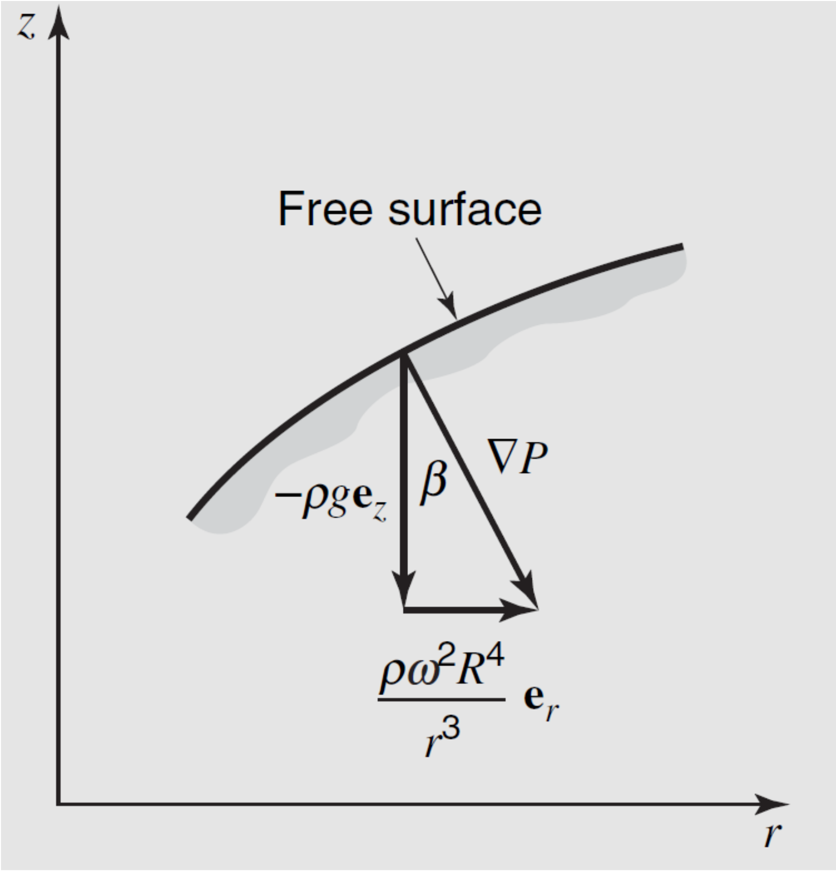
考虑到 $d\mathbf{e}_\theta/dt = -\dot{\theta}\mathbf{e}_r$，上式可变成 $$\frac{d\mathbf{v}}{dt} = -\frac{\omega R^2}{r^2}\dot{r}\mathbf{e}_\theta - \frac{\omega R^2}{r}\dot{\theta}\mathbf{e}_r$$

柱坐标关系 $$\mathbf{e}_\theta = -\mathbf{e}_x \sin\theta + \mathbf{e}_y \cos\theta$$ $$\mathbf{e}_r = \mathbf{e}_x \cos\theta + \mathbf{e}_y \sin\theta$$

由于径向速度为零，以及 $\dot{\theta} = v/r$ ，所以 $\left(\frac{d\mathbf{v}}{dt}\right)_{\text{fluid}} = \frac{D\mathbf{v}}{Dt} = -\frac{\omega R^2}{r^2} v\mathbf{e}_r = -\frac{\omega^2 R^4}{r^3}\mathbf{e}_r$

压强梯度表示为 $$\nabla P = -\rho g\mathbf{e}_z + \rho\frac{\omega^2 R^4 \mathbf{e}_r}{r^3}$$

自由面形成一个角度 $$\tan\beta = \frac{\rho\omega^2 R^4}{r^3\rho g} = \frac{\omega^2 R^4}{g r^3}$$

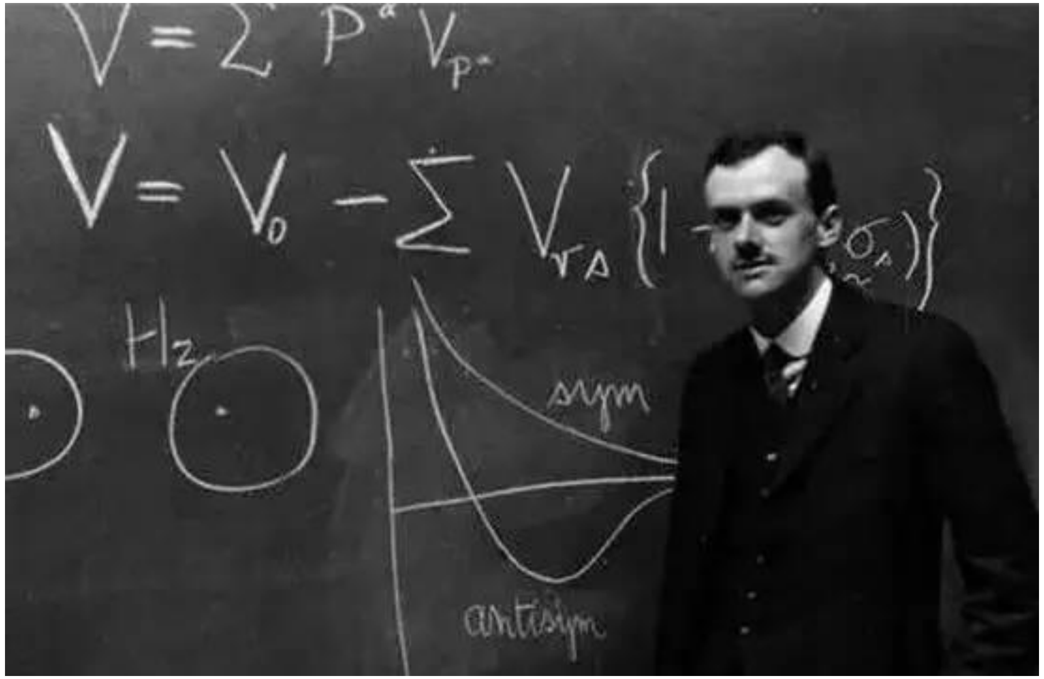


一点引申

有无更直接的方法?

$$\nabla P = -\rho g \mathbf{e}_z + \rho \frac{\omega^2 R^4 \mathbf{e}_r}{r^3}$$

第二项其实来自于离心加速度

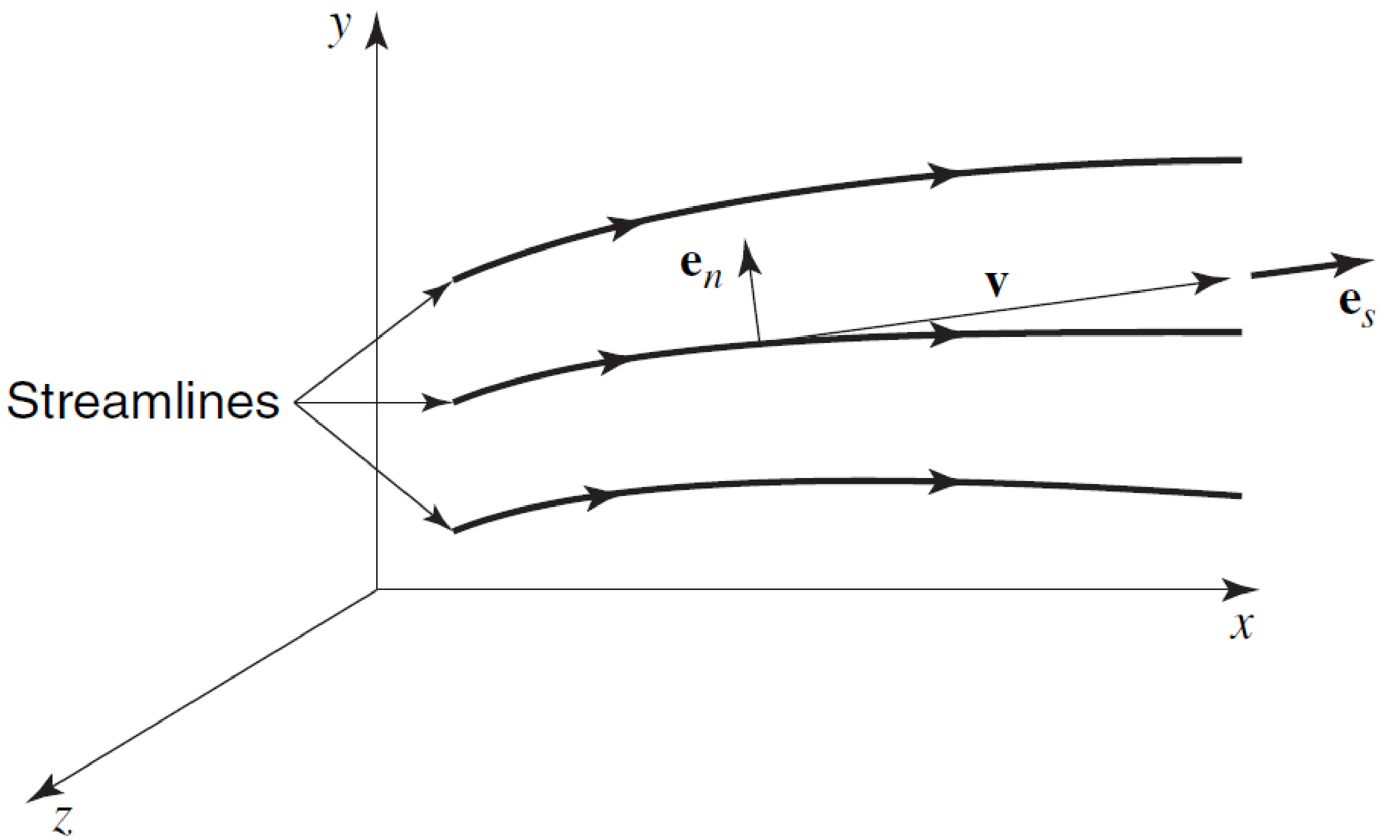


保罗· 狄拉克 **Paul Dirac**

"原先，我只对完全正确的方程感兴趣。然而我所接受的工程训练教导我要容许近似，有时候我能够从这些理论中发现惊人的美，即使它是以近似为基础...如果没有这些来自工程学的训练，我或许无法在后来的研究作出任何成果...我持续在之后的工作运用这些不完全严谨的工程数学，我相信你们可以从我后来的文章中看出来...那些要求所有计算推导上完全精确的数学家很难在物理上走得很远。"

"我认为，未来的正确路线在于不要力求数学的严密，而是要在实际例子中去获取方法。"

伯努利方程



流线坐标
（法向和切向）

$$\mathbf{v} = \mathbf{v}(s, n, t) \qquad P = P(s, n, t)$$

速度沿流线的随体导数

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + \dot{s}\frac{\partial \mathbf{v}}{\partial s} + \dot{n}\frac{\partial \mathbf{v}}{\partial n}$$

$$\rho\frac{D\mathbf{v}}{Dt} = \rho\mathbf{g} - \boldsymbol{\nabla}P$$

由于

$$\dot{s} = v, \dot{n} = 0$$

得到

$$\frac{d\mathbf{v}}{dt} = \frac{\partial \mathbf{v}}{\partial t} + v\frac{\partial \mathbf{v}}{\partial s}$$

流线坐标上的压强梯度

$$\boldsymbol{\nabla}P = \frac{\partial P}{\partial s}\mathbf{e}_s + \frac{\partial P}{\partial n}\mathbf{e}_n$$

根据欧拉方程，写出流线切向方向s的方程

$$\rho\left(\frac{\partial \mathbf{v}}{\partial t}\cdot\ \mathbf{e}_s\,ds + v\frac{\partial \mathbf{v}}{\partial s}\cdot\ \mathbf{e}_s\,ds\right) = \rho\mathbf{g}\cdot\mathbf{e}_s\,ds - \left(\frac{\partial P}{\partial s}\mathbf{e}_s + \frac{\partial P}{\partial n}\mathbf{e}_n\right)\cdot\mathbf{e}_s\,ds$$

由于 $\partial \mathbf{v}/\partial s \cdot \mathbf{e}_s = \partial/\partial s(\mathbf{v} \cdot \mathbf{e}_s) = \partial v/\partial s$

得到 $$\rho\left(\frac{\partial \mathbf{v}}{\partial t} \cdot \mathbf{e}_s\, ds + \frac{\partial}{\partial s}\left\{\frac{v^2}{2}\right\} ds\right) = \rho \mathbf{g} \cdot \mathbf{e}_s\, ds - \frac{\partial P}{\partial s}\, ds$$

让重力作用于y方向向下，则有 $\mathbf{g} \cdot \mathbf{e}_s\, ds = -g\, dy$

对稳态流动，积分为常数，即 $$\frac{v^2}{2} + gy + \frac{P}{\rho} = \text{constant}$$

使用条件

1. inviscid flow,

2. steady flow,

3. incompressible flow,

4. the equation applies along a streamline.

总结

- 掌握流体微元概念和应用
- 熟悉微分形式的连续性方程、纳维-斯托克斯方程、欧拉方程。不同坐标下的方程要了解
- 应用微分方程求解具体流动问题的解析解
- 流线上伯努利方程的推导和含义

课后作业

8.2、8.12、9.9

复习 Navier-Stokes 方程的推导过程