ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1146. *Proposed by P. B. Johnson, Occidental College and Haverford College*

Show that any rectangle whose edges and diagonal are measured in integers can be made the base of a rectangular parallelopiped whose three edges and main diagonal are measured in integers.

E 1147. *Proposed by E. P. Starke, Rutgers University*

If $\cos \alpha$ is rational $(0 < \alpha < \pi)$, prove there are infinitely many triangles with integer sides having α as one angle. In particular, given $\cos \alpha = r/s$, find a three-parameter solution for the sides a, b, c.

E 1148. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Let a, b, c be arbitrary points on the sides BC, CA, AB of triangle ABC, and let A', B', C' be the reflections of A, B, C in the midpoints of the segments bc, ca, ab. Show that triangles abc and $A'B'C'$ have equal areas.

E 1149. *Proposed by A. S. Gregory, University of Illinois*

For each $n = 1, 2, \cdots$ find the least positive integer which when added to 2^n yields a perfect square.

E 1150. *Proposed by Frank Hawthorne, Hofstra College*

If three points are selected at random in a rectangle $A \times 2A$, what is the probability that the triangle so determined is obtuse?

SOLUTIONS

The Counterfeiters of Lower Slobbovia

E 1096 [1954, 46, 472]. *Proposed by L. R. Ford, Jr., The RAND Corporation, Santa Monica, Calif.*

As is well known, Lower Slobbovia is too poor a country to afford its own mint. There are N coiners engaged in making Rasbuckniks, the local currency, to government specifications. However it is suspected that some of them may be counterfeiting by introducing some base metal into the alloy. Any pair of counterfeits will weigh the same, although slightly different from the weight of a good coin. Each coiner produces either all good coins or all counterfeits. With one guaranteed good coin, a set of infinitely refinable weights, a beam balance, and as many coins from each coiner as may be needed, determine in three weighings whether any of the coiners is dishonest, and which ones.

II. *Solution by John Selfridge, University of California at Los Angeles.* The recent solution [1954, 473] of this problem will give the correct answer if all the dishonest coiners are among the first 6, or if it is known that there are at most 3 dishonest coiners. If the ith coiner is dishonest for $i=1$, 3 and 7, or for $i=3$, 4, 5, and 7, or for $i=1$, 2, 5, 6, and 7, then the ratio D'/D is 23; that is, in general the integer S is *not* unique.

To correct this, proceed as in the above solution for the first two weighings. Then if $D\neq 0$ take $(i-1)(i-1)!$ coins from the ith coiner, determine the total weight T', and compute $D'=T'-(N!-1)W_g$. If $D'=0$ only the first coiner is dishonest. If $D'\neq 0$ select the largest n such that $(n-1)!-(1/n)\leqq D'/D$, and then find the largest k such that $D'/D\leqq(n!-1)/k$. It follows that there are k dishonest coiners, the number of coins taken from these for the third weighing was kD'/D, and this is a unique sum of k numbers of the form $(i-1)(i-1)!$ so that we tell which coiners are dishonest.

III. *Solution by Leonard Carlitz, Duke University.* Number the coiners $1, 2, \cdots, N$ in some manner and assume the counterfeiters are numbered $i_1< \cdots <i_m$, $m\leqq N$. Let the weight of the good coin be 1 (first weighing). Second, take one coin from each coiner; if we let $1-\epsilon$ be the weight of a counterfeit coin, then this weighing gives $N-m\epsilon$ and the discrepancy is $D=m\epsilon$. Next, if $p>N$, but otherwise arbitrary, take p^i coins from the ith coiner; this weighing gives the discrepancy

$$D' = (p^{i_1} + \cdots + p^{i_m})\epsilon.$$

Now the equation

$$k = D'/D = (p^{i_1} + \cdots + p^{i_m})/m, \qquad (i_1< \cdots < i_m,\ m < p),$$

determines $m, i_1, \cdots, i_m$ uniquely for fixed k. For assume

$$(p^{i_1} + \cdots + p^{i_m})/m = (p^{j_1} + \cdots + p^{j_n})/n,$$

with $n\leqq N<p$ and $j_1< \cdots <j_n$. Then

$$n(p^{i_1} + \cdots + p^{i_m}) = m(p^{j_1} + \cdots + p^{j_n}).$$

Since the decimal representation of an integer to the base p is unique it follows that $m=n$, $i_1=j_1, \cdots, i_m=j_m$.

Concerning Pandiagonal Heterosquares

E 1116 [1954, 345]. *Proposed by C. W. Trigg, Los Angeles City College*

We define a pandiagonal heterosquare as a square array of the first n^2 positive integers, so arranged that no two of the rows, columns, and diagonals