Answer

Q1

a) Assume that the x sequence is already sorted into nonincreasing order. There are only a finite number of possible orderings for the y sequences. If there are y_i and y_j which are out of order (i.e., $i < j$ but $y_i > y_j$), then we have

$$x_i y_i + x_j y_j - (x_i y_j + x_j y_i) = (x_i - x_j)(y_i - y_j) \leq 0 \iff x_i y_i + x_j y_j \leq (x_i y_j + x_j y_i)$$

That is, we can switch the y_i and y_j to make the sum larger (at least not lower).

b) Similar to a)

Q2

Amount the integers $1, 2, \cdots, a_n$, where a_n is the nth positive integer not a perfect square, the nonsquares are a_1, a_2, $\cdots$, a_n, and the squares are 1^2, 2^2, $\cdots$, k^2, where k is the integer with $k^2 < n + k < (k+1)^2$.

Consequently, $a_n = n + k$, where $k^2 < a_n < (k+1)^2$.

To find k, first note that

$$k^2 < n + k < (k+1)^2 \iff k^2 + 1 \leq n + k \leq (k+1)^2 - 1$$

Hence,

$$(k - \frac{1}{2})^2 + \frac{3}{4} = k^2 - k + 1 \leq n \leq k^2 + k = (k + \frac{1}{2})^2 - \frac{1}{4}$$

If follows that,

$$k - \frac{1}{2} < \sqrt{n} < k + \frac{1}{2}$$

So $k = \{\sqrt{n}\}$ and $a_n = n + k = n + \{\sqrt{n}\}$

Q3

There are a finite number of bit strings of length m, namely, 2^m. The set of all bit strings is the union of the sets of bit strings of length m for $m = 0, 1, 2, \cdots$. The union of a countable number of countable sets is countable, there are a countable number of bit strings.

Note that the map from the set of finite bit strings to binary unsigned integers are not a bijection.

Q4

$$\frac{1}{2} \times \frac{n(n+1)/2}{n} + \frac{1}{2} \times (n+1) = \frac{3n+1}{4}$$

Q5

Since $f(x)$ is $O(g(x))$, there are constants C and k such $|f(x)| \leq C|g(x)|$ for $x > k$. Hence $|f^n(x)| \leq C^n |g^n(x)|$ for $x > k$. So $f^n(x)$ is $O(g^n(x))$ by taking the constant to be C.

Q6

Omitted.

Q7

$$23 + 30k, k \in \mathbb{Z}$$

Q8

Suppose that $x^2 \equiv 1 \pmod{p}$.

Then p divides $x^2 - 1 = (x+1)(x-1)$. It follows that $p \mid x + 1$ or $p \mid x - 1$.

So $x \equiv 1 \pmod{p}$ or $x \equiv -1 \pmod{p}$.

Q9

Let $P(n, k)$ be the claim that that a $2n \times k$ checkerboard missing a white and a black cell can be covered by dominoes, where $n \geq 1$ and $k \geq 2$.

First, $P(1, 1)$ is true since without a black and a white cell the checkboard is only possible to be a 1×2 rectangle.

Assume that $P(1, k)$ is true. Then for $P(1, k + 1)$.

Assume that $P(n, k)$ is true for all $k \geq 2$. Then for $P(n + 1, k)$, we can divide the checkerboard into to $2(n - 1) \times k$ and $2 \times k$.

- If the two missing cells lie in the same subboard, we can use the induction hypothesis to show that $P(n, k + 1)$ is true.
- Otherwise, we can remove two cells with indices $(2, i)$ and $(3, i)$ where the cell indexed by $(2, i)$ has a different color with the missing cell in the subboard shaped of $2 \times k$. Then from the induction hypothesis we can know that $P(n, k + 1)$ is true.

Q10

Omitted.