Quiz 1

Name: ________________ Student Id: ________________ Score: ________________

Q1 (8pt) Prove that if x^3 is irrational, then x is irrational.

Q2 (8pt) Determine whether each of these compound propositions is satisfiable:

a) $(p \vee q) \wedge (\neg p \vee q) \wedge (\neg p \vee \neg q)$
b) $(p \to q) \wedge (p \to \neg q) \wedge (\neg p \to q) \wedge (\neg p \to \neg q)$
c) $(p \leftrightarrow q) \wedge (\neg p \leftrightarrow q)$

Q3 (8pt) Show that $(p \to q) \to r$ and $p \to (q \to r)$ are not logically equivalent.

Q4 (8pt) Let $P(x)$, $Q(x)$, and $R(x)$ be the statements "x is a professor", "x is ignorant" and "x is vain", respectively. Express each of these statements using quantifiers; logical connectives; and $P(x)$, $Q(x)$, and $R(x)$, where the domain consists of all people.

a) No professors are ignorant.
b) All ignorant people are vain.
c) No professors are vain.
d) Does (c) follow from (a) and (b)?

Q5 (8pt) Use a proof by contradiction to show that there is no rational number r for which $r^3 + r + 1 = 0$.

Q6 (8pt) Use a proof by exhaustion to show that a tiling using dominoes of a 4×4 checkerboard with opposite corners removed does not exist.

Q7 (8pt) Prove that between every two rational numbers there is an irrational number.

Q8 (8pt) Prove that if x is positive real number, then

a) $\lfloor \sqrt{\lfloor x \rfloor} \rfloor = \lfloor \sqrt{x} \rfloor$
b) $\lceil \sqrt{\lceil x \rceil} \rceil = \lceil \sqrt{x} \rceil$

Q9 (8pt) Let f be a funcion from the set A to the set B. Let S and T be subsets of A. Show that

a) $f(S \cup T) = f(S) \cup f(T)$
b) $f(S \cap T) \subseteq f(S) \cap f(T)$

Q10 (8pt) Find $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$ if for every positive integer i,

a) $A_i = \{-i, -i+1, \cdots, -1, 0, 1, \cdots, i-1, i\}$.
b) $A_i = \{-i, i\}$.
c) $A_i = [-i, i]$, that is, the set of real numbers x with $-i \leq x \leq i$.
d) $A_i = [i, \infty)$, that is, the set of real numbers x with $x \geq i$.

Q11 (10pt) Show that the set of functions from the positive integers to the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ is uncountable.

Q12 (10pt) Describe an algorithm for finding the smallest integer in a finite sequence of natural numbers.

Answer

Q1 This proposition is equivalent to "if x is rational, then x^3 is rational".

Q2

a) Satisfiable
b) Not satisfiable
c) Not satisfiable

Q3 When p, q, r are all false, $(p \to q) \to r$ is false, but $p \to (q \to r)$ is true.

Q4

a) $\forall x(P(x) \to \neg Q(x))$
b) $\forall x(Q(x) \to R(x))$
c) $\forall x(P(x) \to \neg R(x))$
d) The conclusion does not follow. There may be vain professors, because the premises do not rule out the possibility that there are other vain people besides ignorant ones.

Q5 Suppose by way of contradiction that a/b is a rational root, where a and b are integers and this fraction is in lowest terms (that is, a and b have no common divisor greater than 1).

Plug this proposed root into the equation to obtain $a^3/b^3 + a/b + 1 = 0$. Multiply through by b^3 to obtain $a^3 + ab^2 + b^3 = 0$. If a and b are both odd, then the left-hand side is the sum of three odd numbers and therefore must be odd.

If a is odd and b is even, then the left-hand side is odd+even+even, which is again odd.

Similarly, if a is even and b is odd, then the left-hand side is even+even+odd, which is again odd. Because the fraction a/b is in simplest terms, it cannot happen that both a and b are even. Thus, in all cases, the left-hand side is odd, and therefore cannot equal 0.

This contradiction shows that no such root exists.

Q6 We can rotate the board if necessary to make the removed squares be 1 and 16. Square 2 must be covered by a domino. If that domino is placed to cover squares 2 and 6, then the following domino placements are forced in succession: 5-9, 13-14, and 10-11, at which point there is no way to cover square 15.

Otherwise, square 2 must be covered by a domino placed at 2-3. Then the following domino placements are forced: 4-8, 11-12, 6-7, 5-9, and 10-14, and again there is no way to cover square 15.

Q7 By finding a common denominator, we can assume that the given rational numbers are a/b and c/b, where b is a positive integer and a and c are integers with $a < c$.

In particular, $(a+1)/b \le c/b$. Thus, $x = (a + \sqrt{2}/2)/b$. is between the two given rational numbers, because $0 < \sqrt{2} < 2$.

Furthermore, x is irrational, because if x were rational, then $2(bx - a) = \sqrt{2}$ would be as well.

Q8

a) If x is a positive integer, then the two sides are equal. So suppose that $x = n^2 + m + \epsilon$, where n^2 is the largest perfect square less than x, m is a nonnegative integer, and $0 < \epsilon \le 1$.
Then both $\sqrt{x}$ and $\sqrt{\lfloor x \rfloor} = \sqrt{n^2 + m}$ are between n and $n+1$, so both sides equal n.
b) If x is a positive integer, then the two sides are equal. So suppose that $x = n^2 - m - \epsilon$, where n^2 is the smallest perfect square greater than x, m is a nonnegative integer, and $0 < \epsilon \le 1$.
Then both $\sqrt{x}$ and $\sqrt{\lceil x \rceil} = \sqrt{n^2 - m}$ are between $n-1$ and n, so both sides equal n. Therefore, both sides of the equation equal n.

Q9

a) For any $x \in S \cup T$, there is either $x \in S$ or $x \in T$. Hence $f(x)$ is in either $f(S)$ or $f(T)$, that is, $f(S \cup T) \subseteq f(S) \cup f(T)$.
For any $y \in f(S) \cup f(T)$, there is either $y \in f(S)$ or $y \in f(T)$. Hence there should be some $x \in S$ or $x \in T$ that satisfies $f(x) = y$, that is, $f(S \cup T) \supseteq f(S) \cup f(T)$.
b) For any $x \in S \cap T$, there is $f(x) \in f(S)$ and $f(x) \in f(T)$. Hence $f(S \cap T) \subseteq f(S) \cap f(T)$.

Q10

a) $\mathbb{Z}$, $\{-1, 0, 1\}$
b) $\mathbb{Z} - \{0\}$, $\varnothing$
c) $\mathbb{R}$, $[-1, 1]$
d) $[1, \infty)$, $\varnothing$

Q11 Let S be the set of functions from positive integer to the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.
Then there is a bijection between S and $[0, 1)$. Since $[0, 1)$ is uncountable, S is uncountable as well.

Q12 In C language:

```c
int find_smallest_integer(int a[], size_t length){
    int min = INT_MAX;
    for(int i=0; i<length; i++){
        min = a[i] < min ? a[i] : min;
    }
    return min;
}
```