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A SAMPLING INSPECTION PROBLEM IN ARMS CONTROL AGREEMENTS: A GAME-THEORETIC ANALYSIS

Melvin Dresher

PREPARED FOR:
ADVANCED RESEARCH PROJECTS AGENCY

The RAND Corporation
SANTA MONICA • CALIFORNIA

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1. INTRODUCTION

Statistical investigations in science, industry, or census taking generally involve sampling a universe that has not been modified to mislead the investigators. In sampling for disarmament (or arms-control) verification, we must assume that the statistical universe can be tampered with to conceal a violation. The usual sampling theory, therefore, is not sufficiently applicable to sampling problems for disarmament verification. In many situations, sampling to monitor disarmament agreements can be formulated as a zero-sum two-person game.

Another problem in which a statistical universe may be tampered with occurs in certain weather-control and medical experiments. Blackwell and Hodges have analyzed this problem in Annals of Mathematical Statistics, Vol. 28, 1957, pp. 449-460.

This study analyzes as a game the sampling problem of selecting a fixed number of items for close inspection out of a series of items. This problem would arise, for example, in the following arms-control or disarmament agreements:

(1) A ban on underground nuclear tests verified by on-site inspections of a limited number of those seismic shocks that cannot be identified by seismology to be earthquakes.

(2) A demilitarized zone such that its continued demilitarization would, in part, be verified by a fixed number of inspections of incoming vehicles (railroad cars, trucks, planes).

(3) An agreement to prohibit weapons of mass destruction in space, to be verified by inspecting a fraction of all space launchings.

(4) A prohibition of the production of certain military equipment to be verified by a limited number of inspections per year of the production of related civilian equipment.

In each of the above cases, the inspector must find a compromise between two conflicting arguments. The inspections must not be early, on account of possible later violations; nor must they be late, on account of possible early violations. The potential violator also must compromise between two opposing arguments. Stated in this form, the design of an inspection system is a problem in the timing of decisions in a competitive environment, or a game of timing.

In this paper we formulate and analyze this game of timing in a simplified form: the violator chooses a time for a single violation, with the option of refraining from any violation; and the inspector chooses times for on–site inspections.

In order to formulate the problem as a zero–sum game, in the terminology of game theory, we must assign relative weights for the advantage to the violator of an undetected violation, as compared with the advantage to the inspector of detecting a violation. We give a complete solution in case these advantages are equal, and a partial solution if they are unequal. We also formulate, but do not solve, a game in which the evader may try more than one violation.

Objections may be made that the problem is oversimplified, but it will be seen that the problem, as stated, does introduce many essential elements. For a qualitative discussion of the results see Sec. 9.

2. FORMULATION OF PROBLEM

Suppose that n consecutive events are expected during a time period. Each of these n events could contain or conceal a violation of an agreement. Only by inspection can it be determined whether the event contains a violation. It is impossible, however, to inspect every event. The number of inspections allowed may be much smaller than the number of events that might conceal a violation.

Suppose that m inspections are available, where $m \leq n$. How should the Inspector select these m inspections from the n events?

Suppose that at most one of these n events will contain or conceal a violation. How should the Violator place this violation, if any, among the n events?

On our assumptions, the Violator wishes to pick the best time (i.e., the best one of n events) for his violation, and the Inspector wishes to pick the best m of the n events for his inspections.

Now the object of the inspections is to detect a violation, if any occurs. Further, we assume that the Violator's object is to evade these inspections. Hence we may define the payoff to the Inspector as follows:

$$(1) \qquad \begin{cases} +1, & \text{if the Inspector detects a violation,} \\ -1, & \text{if the Inspector fails to detect a violation that occurs,} \\ 0, & \text{if there is no violation.} \end{cases}$$

The payoff to the Violator will be the negative of (1).

The assumption that the cost of an inspection that does not detect a violation is the same as the gain if a violation occurs is, of course, arbitrary. Other values could be assigned to these outcomes, as will be seen in Sec. 8.

We also need to specify the information available to the two sides. First, we shall assume that both sides know m and n at the start of the game. Further, at all times the sides know the state of the game — i.e., they know the number of events and inspections remaining for the rest of the period.

3. VALUE AND OPTIMAL STRATEGIES OF GAME

We shall analyze the following game with n moves, each move being a simultaneous choice by both sides. At the start of the game the Inspector has available m on–site inspections to detect a violation during the n moves. A move by the Inspector is a choice of whether or not to inspect a specific event. A move by the Violator is a choice of whether to introduce a violation during that time defined by the event. These choices are made simultaneously by the two sides. After the choices are made by the sides, and the move is completed, the Violator knows the Inspector's choice for that move. It is assumed that if a violation occurs and the Inspector inspects, he detects it. This assumption could be modified.

It is evident that the value of the game and optimal strategies will depend on the number of inspections and the number of moves available. Let us then define the function $v(m, n)$ by

$v(m, n)$ = the value of the game in which the Inspector has m inspections, and the Violator has one violation during n events.

Then, using (1), $v(m, n)$ is the value of the 2×2 matrix game

$$(2) \qquad \begin{array}{ccc} & & \text{Violator} \\ & & \begin{array}{cc} \text{Violate} & \text{Not Violate} \end{array} \\ \text{Inspector} & \begin{array}{c} \text{Inspect} \\ \text{Not Inspect} \end{array} & \begin{pmatrix} +1 & v(m-1, n-1) \\ -1 & v(m, n-1) \end{pmatrix} \end{array},$$

with initial conditions

$$(3) \qquad \begin{aligned} v(0, n) &= -1, & n &\geq 1, \\ v(m, n) &= 0, & m &\geq n \geq 0. \end{aligned}$$

From the description of the payoff in this game, it follows that

$$(4) \qquad 1 \geq v(m, n-1) \geq v(m-1, n-1) \geq -1.$$

Therefore, the above 2 x 2 matrix does not have a saddle-point except when $m = n$. Solving the game defined by (2), we obtain $v(m, n)$ as a solution of the functional equation

$$(5) \qquad v(m, n) = \frac{v(m, n-1) + v(m-1, n-1)}{v(m, n-1) - v(m-1, n-1) + 2} .$$

Figure 1 shows how $v(m, n)$ changes with m for $n = 4, 5, 6, 10, 20, 50, 100$.

Solving the game defined by (2),we also obtain the following optimal strategy vectors:

$$\frac{v(m, n)}{v(m, n-1) + v(m-1, n-1)} \begin{pmatrix} 1 + v(m, n-1) \\ 1 - v(m-1, n-1) \end{pmatrix}, \quad \text{for Inspector, if } n > m;$$

$$(6) \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{for Inspector, if } n = m;$$

$$\frac{v(m, n)}{v(m, n-1) + v(m-1, n-1)} \begin{pmatrix} v(m, n-1) - v(m-1, n-1) \\ 2 \end{pmatrix}, \quad \text{for Violator.}$$

Now in the special case in which $n = m$, the payoff matrix (2) becomes

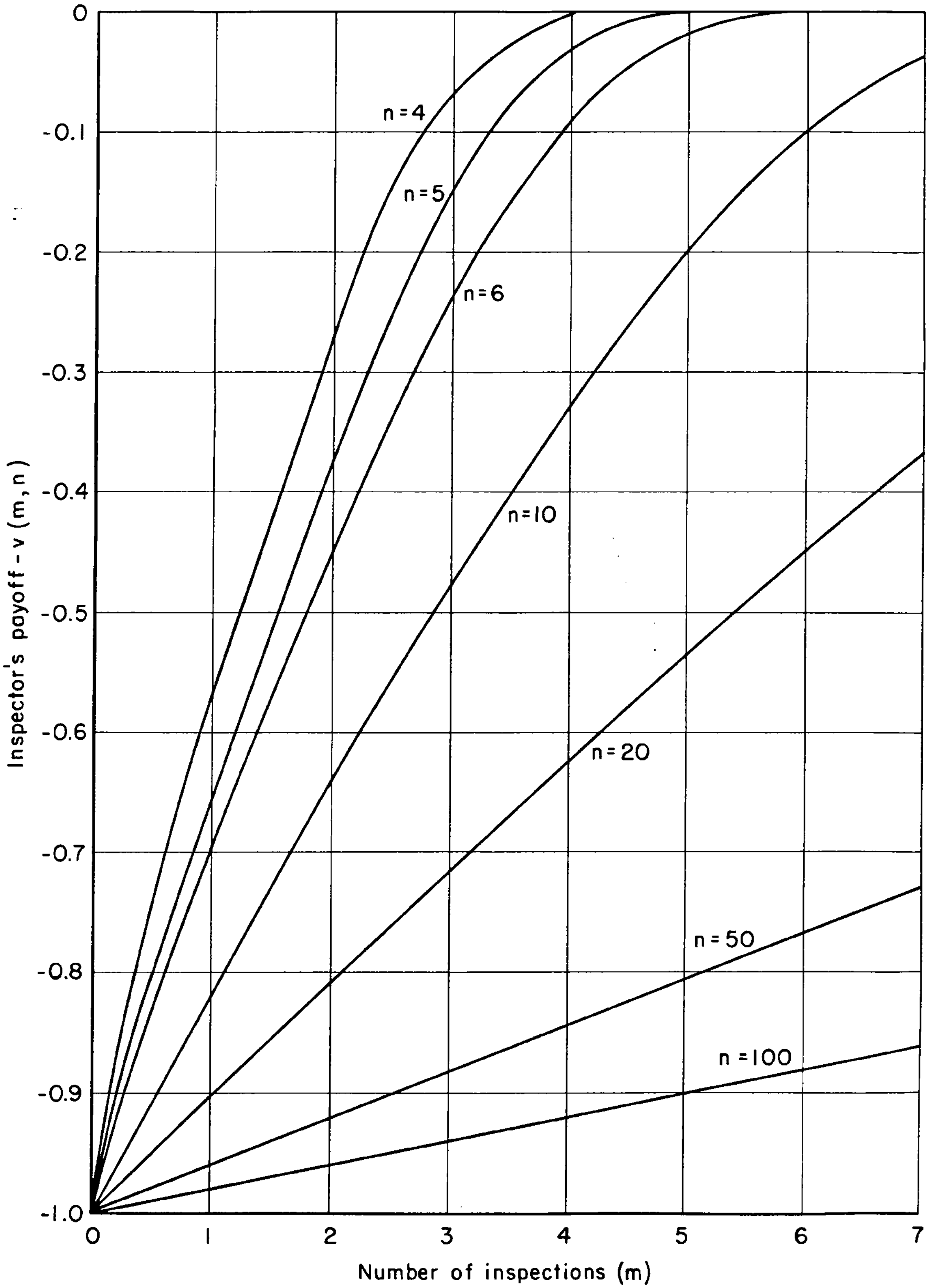

Fig. 1 — Expected payoff to Inspector

$$\begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix} .$$

The Violator thus has a unique optimal strategy: not violate. However, the Inspector has an infinite number of optimal strategies; in fact,any strategy of the form

$$\begin{pmatrix} \frac{1}{2} + \epsilon \\ \frac{1}{2} - \epsilon \end{pmatrix} ,$$

where $0 \le \epsilon \le \frac{1}{2}$, is optimal. But since $m = n$, the Inspector has an inspection for each event. It seems good sense for the Inspector to use the inspection at each event. Thus if $n = m$, the "preferred" optimal strategy for the Inspector is

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} .$$

Figure 2 shows how the probability of inspection (and hence the probability of detecting a violation if it occurs) changes with n for $m = 3, 6, 11$.

4. SOLUTION FOR $m = 1$

Before solving (5) in general, let us first solve this equation for $m = 1$. We are seeking a solution of

$$\begin{aligned} v(1, n) &= \frac{v(1, n-1) + v(0, n-1)}{v(1, n-1) - v(0, n-1) + 2} \\ &= \frac{v(1, n-1) - 1}{v(1, n-1) + 3} . \end{aligned} \tag{7}$$

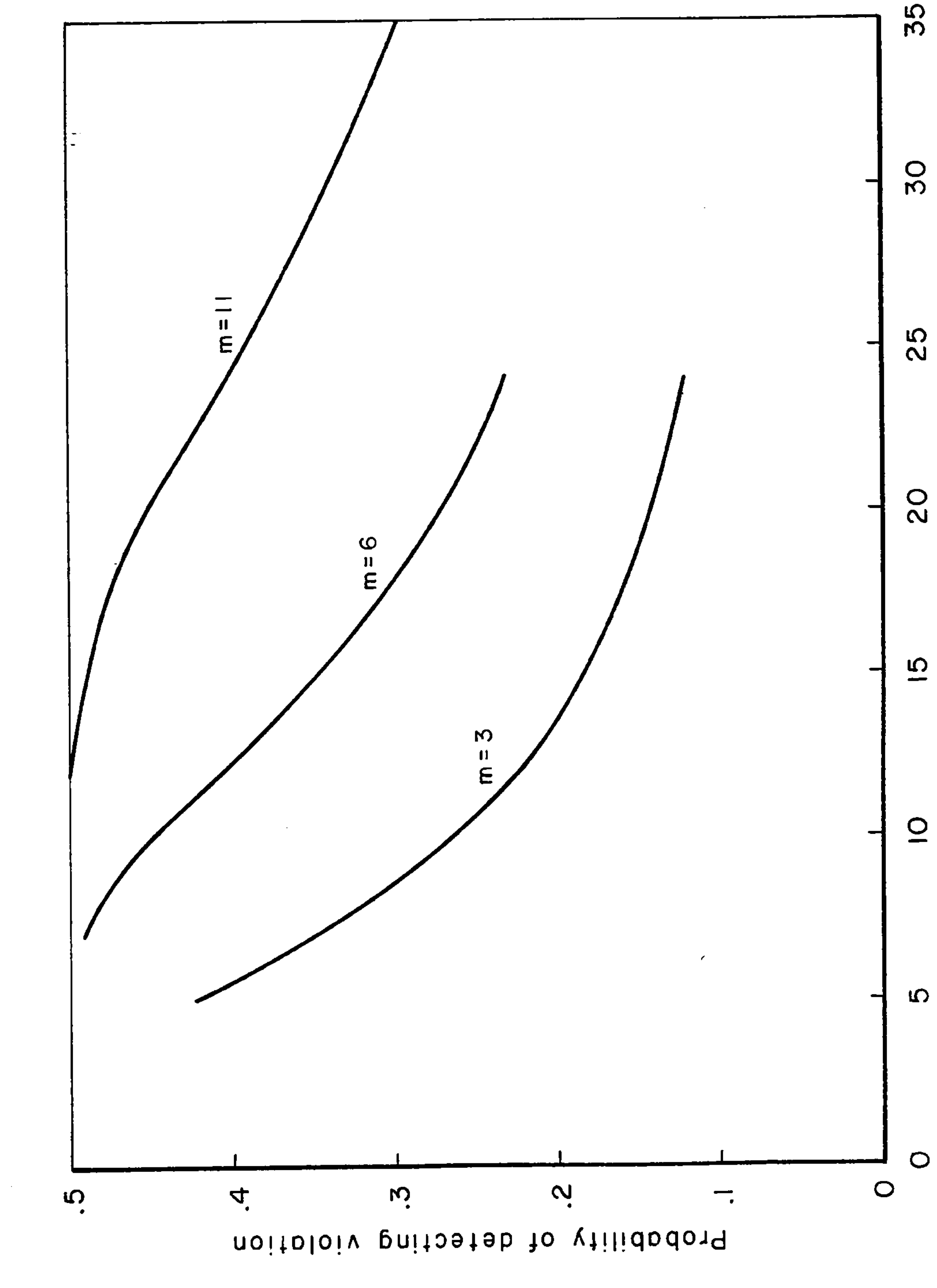

Fig. 2 — Optimal strategy of Inspector

Let

$$w_n = \frac{1}{v(1,n) + 1},$$

or

$$v(1,n) = \frac{1}{w_n} - 1.$$

In terms of the new variable, (7) may be written as

$$\frac{1}{w_n} - 1 = \frac{\frac{1}{w_{n-1}} - 2}{\frac{1}{w_{n-1}} + 2}.$$

Simplifying, we get the recursion equation

$$w_n = \frac{1}{2} + w_{n-1}.$$

This yields as a solution

$$w_n = \frac{1}{2} + \frac{1}{2} + \ldots + \frac{1}{2} = \frac{n+1}{2},$$

or

$$(8) \qquad v(1,n) = \frac{1}{w_n} - 1 = -\frac{n-1}{n+1}.$$

In a similar manner, using (8), we get

$$v(2,n) = -\frac{(n-1)(n-2)}{n^2 + n + 2},$$

and

$$v(3,n) = -\frac{(n-1)(n-2)(n-3)}{n^3 + 5n + 6} .$$

5. GENERAL SOLUTION

We shall now prove that the general solution of (5), for all m and n, is given by

$$v(m,n) = -\frac{\binom{n-1}{m}}{\sum_{i=0}^{m}\binom{n}{i}} . \tag{9}$$

Our method of proof consists of verifying that $v(m,n)$, as defined by (9), satisfies the functional equation given by (5). The recursion relation (5) shows that the solution is unique.

First, let us evaluate $v(m,n+1)$ in terms of $v(m,n)$. From (9) we obtain

$$v(m,n+1) = -\frac{\binom{n}{m}}{\sum_{i=0}^{m}\binom{n+1}{i}}. \tag{10}$$

Now by Pascal's rule we have

$$\binom{n+1}{i} = \binom{n}{i-1} + \binom{n}{i} .$$

Hence we obtain

$$\sum_{i=0}^{m} \binom{n+1}{i} = \sum_{i=0}^{m} \binom{n}{i-1} + \sum_{i=0}^{m} \binom{n}{i}$$

$$= \sum_{i=0}^{m-1} \binom{n}{i} + \sum_{i=0}^{m} \binom{n}{i}$$

$$= 2 \sum_{i=0}^{m} \binom{n}{i} - \binom{n}{m} .$$

Substituting the last expression in (10) and using (9), we get

$$v(m, n+1) = \frac{\binom{n}{m}}{2 \sum_{i=0}^{m} \binom{n}{i} - \binom{n}{m}} = \frac{\binom{n}{m} \Big/ \binom{n-1}{m}}{\frac{2}{v(m, n)} + \binom{n}{m} \Big/ \binom{n-1}{m}}$$

But since

$$\binom{n}{m} \Big/ \binom{n-1}{m} = \frac{n}{n-m} ,$$

this reduces to

$$v(m, n+1) = \frac{\frac{n}{n-m}}{\frac{2}{v(m, n)} + \frac{n}{n-m}} .$$

Simplifying the last expression, we get

$$\text{(11)} \qquad v(m, n+1) = \frac{nv(m, n)}{2(n-m) + nv(m, n)} .$$

Now let us obtain an expression for $v(m+1, n)$ in terms of $v(m, n)$. Using (9), we have

$$v(m+1, n) = -\frac{\binom{n-1}{m+1}}{\sum_{i=0}^{m+1} \binom{n}{i}}$$

$$= -\frac{\binom{n-1}{m+1}}{\sum_{i=0}^{m} \binom{n}{i} + \binom{n}{m+1}}$$

$$= \frac{-\binom{n-1}{m+1} \Big/ \binom{n-1}{m}}{\sum_{0}^{m} \binom{n}{i} \Big/ \binom{n-1}{m} + \binom{n}{m+1} \Big/ \binom{n-1}{m}}$$

$$= \frac{\binom{n-1}{m+1} \Big/ \binom{n-1}{m}}{\dfrac{1}{v(m, n)} - \binom{n}{m+1} \Big/ \binom{n-1}{m}} .$$

But

$$\binom{n-1}{m+1} \Big/ \binom{n-1}{m} = \frac{n - m - 1}{m + 1} ,$$

$$\binom{n}{m+1} \Big/ \binom{n-1}{m} = \frac{n}{m + 1} ,$$

whence

$$v(m+1, n) = \frac{\dfrac{n - m - 1}{m + 1}}{\dfrac{1}{v(m, n)} - \dfrac{n}{m + 1}} .$$

Simplifying, we get

$$v(m+1, n) = \frac{(n-m-1)\ v(m, n)}{(m+1) - nv(m, n)} \ . \tag{12}$$

6. VERIFICATION OF SOLUTION

We are now prepared to show that (9) satisfies (5). We may write (11) as follows:

$$v(m, n) = \frac{(n-1)v(m, n-1)}{2(n-1-m) + (n-1)v(m, n-1)} \ . \tag{13}$$

We may write (12) as follows:

$$\begin{aligned} v(m, n-1) &= \frac{(n-m-1)\ v(m-1, n-1)}{m - (n-1)\ v(m-1, n-1)} \\ &= \frac{\left(\frac{n-1}{m} - 1\right) v(m-1, n-1)}{1 - \frac{n-1}{m}\ v(m-1, n-1)} \ . \end{aligned} \tag{14}$$

From the last expression we have

$$v(m, n-1) - \frac{n-1}{m}\ v(m, n-1)\ v(m-1, n-1) =$$

$$\frac{n-1}{m}\ v(m-1, n-1) - v(m-1, n-1).$$

Solving this for $\frac{n-1}{m}$, we obtain

$$\frac{n-1}{m} = \frac{v(m-1, n-1) + v(m, n-1)}{v(m-1, n-1)\,[\,v(m, n-1) + 1\,]} \,.$$

Now from (13) we have

$$v(m, n) = \frac{\frac{n-1}{m}\; v(m, n-1)}{2\,\frac{n-1}{m} - 2 + \frac{n-1}{m}\; v(m, n-1)}$$

$$(15) \qquad = \frac{v(m, n-1)}{2 - 2\,\frac{m}{n-1} + v(m, n-1)}$$

$$= \frac{v(m, n-1)}{2 - \frac{2\,v(m-1, n-1)\,[\,v(m, n-1) + 1\,]}{v(m-1, n-1) + v(m, n-1)} + v(m, n-1)} \,.$$

For convenience in writing, let us set

$$a = v(m-1, n-1),$$

$$b = v(m, n-1) \,.$$

Then (15) becomes

$$v(m, n) = \frac{b}{2 - \frac{2a(b+1)}{a + b} + b}$$

$$= \frac{b(a + b)}{(2 + b)\,(a + b) - 2a(b+1)}$$

$$= \frac{b(a + b)}{b(b - a + 2)} \, .$$

Hence we have

$$v(m, n) = \frac{v(m-1, n-1) + v(m, n-1)}{v(m, n-1) - v(m-1, n-1) + 2} \, ,$$

which agrees with (5). We have thus proved that (9) is a solution of (5).

Using (9), we can also express the optimal strategies, (6), as functions of m and n, as given in Sec. 3.

7. SOME EXAMPLES OF OPTIMAL STRATEGIES

Table 1 gives the optimal strategy of the Inspector for five values of n (9, 10, 20, 50, 100) and five values of m (1–5). For example, the entry at $m = 2$, $n = 20$, means that whenever there are 20 periods to go and the Inspector has two inspections left, he should inspect at the next period with probability 0.095.

Table 1 — Optimal Strategy of Inspector
(Probability of inspecting the next event if the Inspector has m inspections left and there are n periods to go)

m	n: 9	10	20	50	100
1	0.100	0.091	0.048	0.019	0.010
2	0.195	0.179	0.095	0.039	0.020
3	0.285	0.261	0.141	0.059	0.030
4	0.363	0.337	0.187	0.078	0.039
5	0.427	0.401	0.232	0.098	0.049

Table 2 gives the optimal strategy of the Violator for the same values of m and n as in Table 1. For example, the entry at $m = 3$, $n = 10$, means that whenever there are ten periods to go and the Inspector has three inspections left, the Violator should schedule a violation at the next period with probability 0.082, provided no violation has yet occurred.

Table 2 — Optimal Strategy of Violator
(Probability of scheduling a violation during the n–th event if the Inspector has m inspections left and if no violation has yet occurred)

	n				
m	9	10	20	50	100
1	0.100	0.091	0.048	0.020	0.010
2	0.095	0.087	0.047	0.020	0.010
3	0.087	0.082	0.047	0.020	0.010
4	0.075	0.073	0.046	0.019	0.009
5	0.056	0.060	0.045	0.019	0.009

8. GENERALIZATION OF THE MODEL

As already stated, we have studied an extremely simplified model of the game of sampling in a competitive environment, a model chosen only to illustrate the principles involved. No great difficulty arises in carrying out analogous arguments for more general games of sampling.

In the simple model the payoff function implies that the payoff to the Inspector if he detects a violation is the same as the payoff to the Violator if a violation is not detected. We can remove this implication by making the payoffs different, in which case $v(m, n)$ is obtained as the value of the following matrix game :

$$\begin{pmatrix} c & v(m-1, n-1) \\ -1 & v(m, n-1) \end{pmatrix},$$

where $c > 0$ is the payoff to the Inspector if he detects a violation. The value of the game satisfies the functional equation

$$v(m, n) = \frac{cv(m, n-1) + v(m-1, n-1)}{v(m, n-1) - v(m-1, n-1) + 1 + c} .$$

In the special case $m = 1$ (one inspection available) we can evaluate v explicitly, obtaining

$$v(1, n) = -\frac{n-1}{n + c} . \tag{16}$$

The optimal strategies in this case are

$$\frac{1}{n + c} \begin{pmatrix} 1 \\ n-1 + c \end{pmatrix}, \text{ for the Inspector, if } n \geq 2,$$

(17)

$$\frac{1}{n + c} \begin{pmatrix} 1 \\ n-1 + c \end{pmatrix}, \text{ for the Violator, if } n \geq 2.$$

If $n = 1$, however, the optimal strategies are

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ for the Inspector,}$$

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}, \text{ for the Violator.}$$

If $c = 0$ (i.e., the Violator suffers no penalty if a violation is detected) we can evaluate $v(m, n)$ explicitly. In this case, we have

$$v(m, n) = -\frac{n-m}{n}.$$

The optimal strategies are

$$\frac{1}{n} \begin{pmatrix} m \\ n-m \end{pmatrix}, \text{ for the Inspector,}$$

$$\frac{1}{n} \begin{pmatrix} 1 \\ n-1 \end{pmatrix}, \text{ for the Violator.}$$

The simple model further assumes, in effect, that the Violator is capable of at most one violation during the n events. Suppose we generalize this by assuming that the Violator is capable of n violation attempts, one at each of the n events. Thus if a violation is not detected, the Violator can have additional violations at later periods. The game terminates, however, when a violation is detected. Now $v(m, n)$ is the value of the following 2×2 matrix game:

$$\begin{pmatrix} 1 & v(m-1, n-1) \\ -1 + v(m, n-1) & v(m, n-1) \end{pmatrix},$$

or

$$v(m, n) = \frac{v(m, n-1) + v(m-1, n-1) - v(m, n-1)\ v(m-1, n-1)}{2 - v(m-1, n-1)}.$$

Further, we have assumed a fixed number of inspections to be used in a fixed time period. It turns out that if the Inspector uses an optimal strategy he may, with probability larger than zero, use all m inspections earlier than n periods. The Violator can then be sure of success if he acts after the m–th inspection, and before the n periods have expired. One way of rectifying this situation is by declaring arbitrarily that whenever $m = 0$ and $n > 0$, there is a probability that an additional inspection will be allowed. This will yield a different game and different optimal strategies.

9. DISCUSSION OF RESULTS

It is of interest to compare the results of using an optimal strategy with other possible strategies. We can readily make this comparison for the case in which just one inspection and one violation are possible. Let us do this for the more general model in which the Violator gains one unit by an undetected violation and the Inspector gains c units if he detects a violation.

We recall that in this case the value of the game (the expected loss by the Inspector) is $(n-1)/(n+c)$, where n is the number of events.

Let us compare the above with the outcome if the Inspector uses the strategy of allotting equal probabilities to all events, and the Violator uses the strategy of scheduling a violation one period after the inspection. Then the expected loss by the Inspector for this strategy is $(n-1)/n$, as compared with $(n-1)/(n+c)$ for the optimal strategy.

If c is small compared with n, then the two results (expected losses) are both the same. If c is comparable in size to n, however, then the optimal strategy gives a substantial improvement. Thus if there is a great loss to the Violator in being detected, then the Inspector gains much by following an optimal strategy.

From (17) we observe that, for optimal inspection, the larger the value of c the more likely the Inspector will defer his inspection to the later periods. Further, the probability is $(1+c)/(n+c)$ that the Inspector defers his inspection to the last period.