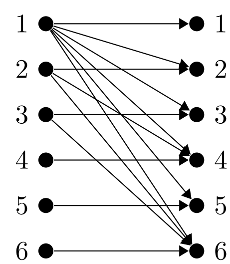
CHAPTER 9
Relations

SECTION 9.1 Relations and Their Properties

2. **a)** $(1,1)$, $(1,2)$, $(1,3)$, $(1,4)$, $(1,5)$, $(1,6)$, $(2,2)$, $(2,4)$, $(2,6)$, $(3,3)$, $(3,6)$, $(4,4)$, $(5,5)$, $(6,6)$

b) We draw a line from a to b whenever a divides b, using separate sets of points; an alternate form of this graph would have just one set of points.



c) We put an $\times$ in the i^{th} row and j^{th} column if and only if i divides j.

R	1	2	3	4	5	6
1	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$
2		$\times$		$\times$		$\times$
3			$\times$			$\times$
4				$\times$		
5					$\times$	
6						$\times$

4. **a)** Being taller than is not reflexive (I am not taller than myself), nor symmetric (I am taller than my daughter, but she is not taller than I). It is antisymmetric (vacuously, since we never have A taller than B, and B taller than A, even if $A = B$). It is clearly transitive.

b) This is clearly reflexive, symmetric, and transitive (it is an equivalence relation—see Section 9.5). It is not antisymmetric, since twins, for example, are unequal people born on the same day.

c) This has exactly the same answers as part (b), since having the same first name is just like having the same birthday.

d) This is clearly reflexive and symmetric. It is not antisymmetric, since my cousin and I have a common grandparent, and I and my cousin have a common grandparent, but I am not equal to my cousin. This relation is not transitive. My cousin and I have a common grandparent; my cousin and her cousin on the other side of her family have a common grandparent. My cousin's cousin and I do not have a common grandparent.

6. **a)** Since $1+1 \neq 0$, this relation is not reflexive. Since $x+y = y+x$, it follows that $x+y=0$ if and only if $y+x=0$, so the relation is symmetric. Since $(1,-1)$ and $(-1,1)$ are both in R, the relation is not antisymmetric. The relation is not transitive; for example, $(1,-1) \in R$ and $(-1,1) \in R$, but $(1,1) \notin R$.

b) Since $x = \pm x$ (choosing the plus sign), the relation is reflexive. Since $x = \pm y$ if and only if $y = \pm x$, the relation is symmetric. Since $(1,-1)$ and $(-1,1)$ are both in R, the relation is not antisymmetric. The relation is transitive, essentially because the product of 1's and -1's is ± 1.

c) The relation is reflexive, since $x - x = 0$ is a rational number. The relation is symmetric, because if $x - y$ is rational, then so is $-(x-y) = y - x$. Since $(1,-1)$ and $(-1,1)$ are both in R, the relation is not antisymmetric. To see that the relation is transitive, note that if $(x, y) \in R$ and $(y, z) \in R$, then $x - y$ and $y - z$ are rational numbers. Therefore their sum $x - z$ is rational, and that means that $(x, z) \in R$.

d) Since $1 \neq 2 \cdot 1$, this relation is not reflexive. It is not symmetric, since $(2, 1) \in R$, but $(1, 2) \notin R$. To see that it is antisymmetric, suppose that $x = 2y$ and $y = 2x$. Then $y = 4y$, from which it follows that $y = 0$ and hence $x = 0$. Thus the only time that (x, y) and (y, x) are both is R is when $x = y$ (and both are 0). This relation is clearly not transitive, since $(4, 2) \in R$ and $(2, 1) \in R$, but $(4, 1) \notin R$.

e) This relation is reflexive since squares are always nonnegative. It is clearly symmetric (the roles of x and y in the statement are interchangeable). It is not antisymmetric, since $(2, 3)$ and $(3, 2)$ are both in R. It is not transitive; for example, $(1, 0) \in R$ and $(0, -2) \in R$, but $(1, -2) \notin R$.

f) This is not reflexive, since $(1, 1) \notin R$. It is clearly symmetric (the roles of x and y in the statement are interchangeable). It is not antisymmetric, since $(2, 0)$ and $(0, 2)$ are both in R. It is not transitive; for example, $(1, 0) \in R$ and $(0, -2) \in R$, but $(1, -2) \notin R$.

g) This is not reflexive, since $(2, 2) \notin R$. It is not symmetric, since $(1, 2) \in R$ but $(2, 1) \notin R$. It is antisymmetric, because if $(x, y) \in R$ and $(y, x) \in R$, then $x = 1$ and $y = 1$, so $x = y$. It is transitive, because if $(x, y) \in R$ and $(y, z) \in R$, then $x = 1$ (and $y = 1$, although that doesn't matter), so $(x, z) \in R$.

h) This is not reflexive, since $(2, 2) \notin R$. It is clearly symmetric (the roles of x and y in the statement are interchangeable). It is not antisymmetric, since $(2, 1)$ and $(1, 2)$ are both in R. It is not transitive; for example, $(3, 1) \in R$ and $(1, 7) \in R$, but $(3, 7) \notin R$.

8. If $R = \emptyset$, then the hypotheses of the conditional statements in the definitions of *symmetric* and *transitive* are never true, so those statements are always true by definition. Because $S \neq \emptyset$, the statement $(a, a) \in R$ is false for an element of S, so $\forall a\, (a, a) \in R$ is not true; thus R is not reflexive.

10. We give the simplest example in each case.

a) the empty set on $\{a\}$ (vacuously symmetric and antisymmetric)

b) $\{(a, b), (b, a), (a, c)\}$ on $\{a, b, c\}$

12. Only the relation in part (a) is irreflexive (the others are all reflexive).

14. **a)** not irreflexive, since $(0, 0) \in R$. **b)** not irreflexive, since $(0, 0) \in R$.

c) not irreflexive, since $(0, 0) \in R$. **d)** not irreflexive, since $(0, 0) \in R$.

e) not irreflexive, since $(0, 0) \in R$. **f)** not irreflexive, since $(0, 0) \in R$.

g) not irreflexive, since $(1, 1) \in R$. **h)** not irreflexive, since $(1, 1) \in R$.

16. $\forall x\, ((x, x) \notin R)$

18. The relations in parts (a), (b), and (e) are not asymmetric since they contain pairs of the form (x, x). Clearly the relation in part (c) is not asymmetric. The relation in part (f) is not asymmetric (both $(1, 3)$ and $(3, 1)$ are in the relation). It is easy to see that the relation in part (d) is asymmetric.

20. According to the preamble to Exercise 18, an asymmetric relation is one for which $(a, b) \in R$ and $(b, a) \in R$ can never hold simultaneously, even if $a = b$. Thus R is asymmetric if and only if R is antisymmetric and also irreflexive.

a) This is not asymmetric, since in fact (a, a) is always in R.

b) For any page a with no links, $(a, a) \in R$, so this is not asymmetric.

c) For any page a with links, $(a, a) \in R$, so this is not asymmetric.

d) For any page a that is linked to, $(a, a) \in R$, so this is not asymmetric.

22. An asymmetric relation must be antisymmetric, since the hypothesis of the condition for antisymmetry is false if the relation is asymmetric. The relation $\{(a,a)\}$ on $\{a\}$ is antisymmetric but not asymmetric, however, so the answer to the second question is no. In fact, it is easy to see that R is asymmetric if and only if R is antisymmetric and irreflexive.

24. Of course many answers are possible. The empty relation is always asymmetric (x is never related to y). A less trivial example would be $(a,b) \in R$ if and only if a is taller than b. Clearly it is impossible that both a is taller than b and b is taller than a at the same time.

26. **a)** $R^{-1} = \{\, (b,a) \mid (a,b) \in R \,\} = \{\, (b,a) \mid a < b \,\} = \{\, (a,b) \mid a > b \,\}$
b) $\overline{R} = \{\, (a,b) \mid (a,b) \notin R \,\} = \{\, (a,b) \mid a \not< b \,\} = \{\, (a,b) \mid a \geq b \,\}$

28. **a)** Since this relation is symmetric, $R^{-1} = R$.
b) This relation consists of all pairs (a,b) in which state a does not border state b.

30. These are merely routine exercises in set theory. Note that $R_1 \subseteq R_2$.
a) $\{(1,1),(1,2),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3),(3,4)\} = R_2$ **b)** $\{(1,2),(2,3),(3,4)\} = R_1$
c) $\emptyset$ **d)** $\{(1,1),(2,1),(2,2),(3,1),(3,2),(3,3)\}$

32. Since $(1,2) \in R$ and $(2,1) \in S$, we have $(1,1) \in S \circ R$. We use similar reasoning to form the rest of the pairs in the composition, giving us the answer $\{(1,1),(1,2),(2,1),(2,2)\}$.

34. **a)** The union of two relations is the union of these sets. Thus $R_1 \cup R_3$ holds between two real numbers if R_1 holds or R_3 holds (or both, it goes without saying). Here this means that the first number is greater than the second or vice versa—in other words, that the two numbers are not equal. This is just relation R_6.

b) For (a,b) to be in $R_3 \cup R_6$, we must have $a > b$ or $a = b$. Since this happens precisely when $a \geq b$, we see that the answer is R_2.

c) The intersection of two relations is the intersection of these sets. Thus $R_2 \cap R_4$ holds between two real numbers if R_2 holds and R_4 holds as well. Thus for (a,b) to be in $R_2 \cap R_4$, we must have $a \geq b$ and $a \leq b$. Since this happens precisely when $a = b$, we see that the answer is R_5.

d) For (a,b) to be in $R_3 \cap R_5$, we must have $a < b$ and $a = b$. It is impossible for $a < b$ and $a = b$ to hold at the same time, so the answer is $\emptyset$, i.e., the relation that never holds.

e) Recall that $R_1 - R_2 = R_1 \cap \overline{R_2}$. But $\overline{R_2} = R_3$, so we are asked for $R_1 \cap R_3$. It is impossible for $a > b$ and $a < b$ to hold at the same time, so the answer is $\emptyset$, i.e., the relation that never holds.

f) Reasoning as in part (e), we want $R_2 \cap \overline{R_1} = R_2 \cap R_4$, which is R_5 (this was part (c)).

g) Recall that $R_1 \oplus R_3 = (R_1 \cap \overline{R_3}) \cup (R_3 \cap \overline{R_1})$. We see that $R_1 \cap \overline{R_3} = R_1 \cap R_2 = R_1$, and $R_3 \cap \overline{R_1} = R_3 \cap R_4 = R_3$. Thus our answer is $R_1 \cup R_3 = R_6$ (as in part (a)).

h) Recall that $R_2 \oplus R_4 = (R_2 \cap \overline{R_4}) \cup (R_4 \cap \overline{R_2})$. We see that $R_2 \cap \overline{R_4} = R_2 \cap R_1 = R_1$, and $R_4 \cap \overline{R_2} = R_4 \cap R_3 = R_3$. Thus our answer is $R_1 \cup R_3 = R_6$ (as in part (a)).

36. Recall that the composition of two relations all defined on a common set is defined as follows: $(a,c) \in S \circ R$ if and only if there is some element b such that $(a,b) \in R$ and $(b,c) \in S$. We have to apply this in each case.

a) For (a,c) to be in $R_1 \circ R_1$, we must find an element b such that $(a,b) \in R_1$ and $(b,c) \in R_1$. This means that $a > b$ and $b > c$. Clearly this can be done if and only if $a > c$ to begin with. But that is precisely the statement that $(a,c) \in R_1$. Therefore we have $R_1 \circ R_1 = R_1$. We can interpret (part of) this as showing that R_1 is transitive.

b) For (a,c) to be in $R_1 \circ R_2$, we must find an element b such that $(a,b) \in R_2$ and $(b,c) \in R_1$. This means that $a \geq b$ and $b > c$. Clearly this can be done if and only if $a > c$ to begin with. But that is precisely the statement that $(a,c) \in R_1$. Therefore we have $R_1 \circ R_2 = R_1$.

c) For (a,c) to be in $R_1 \circ R_3$, we must find an element b such that $(a,b) \in R_3$ and $(b,c) \in R_1$. This means that $a < b$ and $b > c$. Clearly this can always be done simply by choosing b to be large enough. Therefore we have $R_1 \circ R_3 = \mathbf{R}^2$, the relation that always holds.

d) For (a,c) to be in $R_1 \circ R_4$, we must find an element b such that $(a,b) \in R_4$ and $(b,c) \in R_1$. This means that $a \le b$ and $b > c$. Clearly this can always be done simply by choosing b to be large enough. Therefore we have $R_1 \circ R_4 = \mathbf{R}^2$, the relation that always holds.

e) For (a,c) to be in $R_1 \circ R_5$, we must find an element b such that $(a,b) \in R_5$ and $(b,c) \in R_1$. This means that $a = b$ and $b > c$. Clearly this can be done if and only if $a > c$ to begin with (choose $b = a$). But that is precisely the statement that $(a,c) \in R_1$. Therefore we have $R_1 \circ R_5 = R_1$. One way to look at this is to say that R_5, the equality relation, acts as an identity for the composition operation (on the right—although it is also an identity on the left as well).

f) For (a,c) to be in $R_1 \circ R_6$, we must find an element b such that $(a,b) \in R_6$ and $(b,c) \in R_1$. This means that $a \ne b$ and $b > c$. Clearly this can always be done simply by choosing b to be large enough. Therefore we have $R_1 \circ R_6 = \mathbf{R}^2$, the relation that always holds.

g) For (a,c) to be in $R_2 \circ R_3$, we must find an element b such that $(a,b) \in R_3$ and $(b,c) \in R_2$. This means that $a < b$ and $b \ge c$. Clearly this can always be done simply by choosing b to be large enough. Therefore we have $R_2 \circ R_3 = \mathbf{R}^2$, the relation that always holds.

h) For (a,c) to be in $R_3 \circ R_3$, we must find an element b such that $(a,b) \in R_3$ and $(b,c) \in R_3$. This means that $a < b$ and $b < c$. Clearly this can be done if and only if $a < c$ to begin with. But that is precisely the statement that $(a,c) \in R_3$. Therefore we have $R_3 \circ R_3 = R_3$. We can interpret (part of) this as showing that R_3 is transitive.

38. Note that these relations all describe the usual inequalities and equality on real numbers, and "less than," "less than or equal to," "greater than," "greater than or equal to," and "equal to" are all transitive relations on real numbers (see, for example, Example 14 or Appendix 1). Consequently, by Theorem 1, for $i = 1,2,3,4,5$, $R_i^2 \subseteq R_i$. On the other hand, we observe that the average of two real numbers is between those numbers (or equal to them if the two numbers are equal). So for $i = 1,2,3,4,5$, if $(a,b) \in R_i$, then $(a, \frac{a+b}{2}), (\frac{a+b}{2}, b) \in R_i$, and thus transitivity implies that $(a,b) \in R_i^2$.

The relation R_6 is different. By definition of the relation, $R_6 \subseteq \mathbf{R}^2$. This is in fact an equality. For $(a,b) \in \mathbf{R}^2$, choose any real number c different from both a and b (which themselves may be the same or different). Since $a \ne c$ and $c \ne b$, $(a,c) \in R_6$ and $(c,b) \in R_6$. Then, by definition of relation composition, $(a,b) \in R_6$ and we conclude $R_6 = \mathbf{R}^2$.

40. For (a,b) to be an element of R^3, we must find people c and d such that $(a,c) \in R$, $(c,d) \in R$, and $(d,b) \in R$. In words, this says that a is the parent of someone who is the parent of someone who is the parent of b. More simply, a is a great-grandparent of b.

42. Note that these two relations are inverses of each other, since a is a multiple of b if and only if b divides a (see the preamble to Exercise 26).

a) The union of two relations is the union of these sets. Thus $R_1 \cup R_2$ holds between two integers if R_1 holds or R_2 holds (or both, it goes without saying). Thus $(a,b) \in R_1 \cup R_2$ if and only if $a \mid b$ or $b \mid a$. There is not a good easier way to state this.

b) The intersection of two relations is the intersection of these sets. Thus $R_1 \cap R_2$ holds between two integers if R_1 holds and R_2 holds. Thus $(a,b) \in R_1 \cap R_2$ if and only if $a \mid b$ and $b \mid a$. This happens if and only if $a = \pm b$ and $a \ne 0$.

c) By definition $R_1 - R_2 = R_1 \cap \overline{R_2}$. Thus this relation holds between two integers if R_1 holds and R_2 does not hold. We can write this in symbols by saying that $(a,b) \in R_1 - R_2$ if and only if $a \mid b$ and $b \nmid a$. This is equivalent to saying that $a \mid b$ and $a \ne \pm b$.

d) By definition $R_2 - R_1 = R_2 \cap \overline{R_1}$. Thus this relation holds between two integers if R_2 holds and R_1 does not hold. We can write this in symbols by saying that $(a,b) \in R_2 - R_1$ if and only if $b \mid a$ and $a \nmid b$. This is equivalent to saying that $b \mid a$ and $a \neq \pm b$.

e) We know that $R_1 \oplus R_2 = (R_1 - R_2) \cup (R_2 - R_1)$, so we look at our solutions to part (c) and part (d). Thus this relation holds between two integers if R_1 holds and R_2 does not hold, or vice versa. This happens if and only if $a \mid b$ or $b \mid a$, but $a \neq \pm b$.

44. These are just the 16 different subsets of $\{(0,0),(0,1),(1,0),(1,1)\}$.

1. $\emptyset$
2. $\{(0,0)\}$
3. $\{(0,1)\}$
4. $\{(1,0)\}$
5. $\{(1,1)\}$
6. $\{(0,0),(0,1)\}$
7. $\{(0,0),(1,0)\}$
8. $\{(0,0),(1,1)\}$
9. $\{(0,1),(1,0)\}$
10. $\{(0,1),(1,1)\}$
11. $\{(1,0),(1,1)\}$
12. $\{(0,0),(0,1),(1,0)\}$
13. $\{(0,0),(0,1),(1,1)\}$
14. $\{(0,0),(1,0),(1,1)\}$
15. $\{(0,1),(1,0),(1,1)\}$
16. $\{(0,0),(0,1),(1,0),(1,1)\}$

46. We list the relations by number as given in the solution above.

a) 8, 13, 14, 16 **b)** 1, 3, 4, 9 **c)** 1, 2, 5, 8, 9, 12, 15, 16

d) 1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 13, 14 **e)** 1, 3, 4 **f)** 1, 2, 3, 4, 5, 6, 7, 8, 10, 11, 13, 14, 16

48. This is similar to Example 16 in this section. A relation on a set S with n elements is a subset of $S \times S$. Since $S \times S$ has n^2 elements, so there are 2^{n^2} relations on S if no restrictions are imposed. One might observe here that the condition that $a \neq b$ is not relevant.

a) Half of these relations contain (a,b) and half do not, so the answer is $2^{n^2}/2 = 2^{n^2-1}$. Looking at it another way, we see that there are $n^2 - 1$ choices involved in specifying such a relation, since we have no choice about (a,b).

b) The analysis and answer are exactly the same as in part (a).

c) Of the n^2 possible pairs to put in R, exactly n of them have a as their first element. We must use none of these, so there are $n^2 - n$ pairs that we are free to work with. Therefore there are 2^{n^2-n} possible choices for R.

d) By part (c) we know that there are 2^{n^2-n} relations that do not contain at least one ordered pair with a as its first element, so all the other relations, namely $2^{n^2} - 2^{n^2-n}$ of them, do contain at least one ordered pair with a as its first element.

e) We reason as in part (c). There are n ordered pairs that have a as their first element, and n more that have b as their second element, although this counts (a,b) twice, so there are a total of $2n-1$ pairs that violate the condition. This means that there are $n^2 - 2n + 1 = (n-1)^2$ pairs that we are free to choose for R. Thus the answer is $2^{(n-1)^2}$. Another way to look at this is to visualize the matrix representing R. The a^{th} row must be all 0's, as must the b^{th} column. If we cross out that row and column we have in effect an $n-1$ by $n-1$ matrix, with $(n-1)^2$ entries. Since we can fill each entry with either a 0 or a 1, there are $2^{(n-1)^2}$ choices for specifying S.

f) This is the opposite condition from part (e). Therefore reasoning as in part (d), we have $2^{n^2} - 2^{(n-1)^2}$ possible relations.

50. **a)** There are two relations on a set with only one element, and they are both transitive.

b) There are 16 relations on a set with two elements, and we saw in Exercise 46f that 13 of them are transitive.

c) For $n = 3$ there are $2^{3^2} = 512$ relations. One way to find out how many of them are transitive is to use a computer to generate them all and check each one for transitivity. If we do this, then we find that 171 of them are transitive. Doing this by hand is not pleasant, since there are many cases to consider.

52. **a)** Since R contains all the pairs (x,x), so does $R \cup S$. Therefore $R \cup S$ is reflexive.

b) Since R and S each contain all the pairs (x,x), so does $R \cap S$. Therefore $R \cap S$ is reflexive.

c) Since R and S each contain all the pairs (x,x), we know that $R \oplus S$ contains none of these pairs. Therefore $R \oplus S$ is irreflexive.

d) Since R and S each contain all the pairs (x,x), we know that $R - S$ contains none of these pairs. Therefore $R - S$ is irreflexive.

e) Since R and S each contain all the pairs (x,x), so does $S \circ R$. Therefore $S \circ R$ is reflexive.

54. By definition, to say that R is antisymmetric is to say that $R \cap R^{-1}$ contains only pairs of the form (a,a). The statement we are asked to prove is just a rephrasing of this.

56. This is immediate from the definition, since R is reflexive if and only if it contains all the pairs (x,x), which in turn happens if and only if $\overline{R}$ contains none of these pairs, i.e., $\overline{R}$ is irreflexive.

58. We just apply the definition each time. We find that R^2 contains all the pairs in $\{1,2,3,4,5\} \times \{1,2,3,4,5\}$ except $(2,3)$ and $(4,5)$; and R^3, R^4, and R^5 contain all the pairs.

60. We prove this by induction on n. There is nothing to prove in the basis step ($n = 1$). Assume the inductive hypothesis that R^n is symmetric, and let $(a,c) \in R^{n+1} = R^n \circ R$. Then there is a $b \in A$ such that $(a,b) \in R$ and $(b,c) \in R^n$. Since R^n and R are symmetric, $(b,a) \in R$ and $(c,b) \in R^n$. Thus by definition $(c,a) \in R \circ R^n$. We will have completed the proof if we can show that $R \circ R^n = R^{n+1}$. This we do in two steps. First, composition of relations is associative, that is, $(R \circ S) \circ T = R \circ (S \circ T)$ for all relations with appropriate domains and codomains. (The proof of this is straightforward applications of the definition.) Second we show that $R \circ R^n = R^{n+1}$ by induction on n. Again the basis step is trivial. Under the inductive hypothesis, then, $R \circ R^{n+1} = R \circ (R^n \circ R) = (R \circ R^n) \circ R = R^{n+1} \circ R = R^{n+2}$, as desired.

62. First note that, given a set A of n elements, there are n^2 ordered pairs in A^2, and thus 2^{n^2} possible relations on A. Given a relation R, we can determine whether it is transitive by checking all ordered triples (x,y,z) of elements of the set A. If $R(x,y)$ and $R(y,z)$ are ever both in the relation but $R(x,z)$ is not, then we know that the relation is not transitive. On the other hand, if the relation passes the test for all triples, then the relation is transitive. Since there are n^3 triples to check for each of the 2^{n^2} relations, the algorithm is $O(2^{n^2} n^3)$.

SECTION 9.3 Representing Relations

2. In each case we use a 4×4 matrix, putting a 1 in position (i,j) if the pair (i,j) is in the relation and a 0 in position (i,j) if the pair (i,j) is not in the relation.

a) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ **b)** $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ **c)** $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$ **d)** $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

4. a) Since the $(1,1)^{\text{th}}$ entry is a 1, $(1,1)$ is in the relation. Since $(1,3)^{\text{th}}$ entry is a 0, $(1,3)$ is not in the relation. Continuing in this manner, we see that the relation contains $(1,1)$, $(1,2)$, $(1,4)$, $(2,1)$, $(2,3)$, $(3,2)$, $(3,3)$, $(3,4)$, $(4,1)$, $(4,3)$, and $(4,4)$.

b) $(1,1)$, $(1,2)$, $(1,3)$, $(2,2)$, $(3,3)$, $(3,4)$, $(4,1)$, and $(1,4)$

c) $(1,2)$, $(1,4)$, $(2,1)$, $(2,3)$, $(3,2)$, $(3,4)$, $(4,1)$, and $(4,3)$

6. An asymmetric relation (see the preamble to Exercise 18 in Section 9.1) is one for which $(a,b) \in R$ and $(b,a) \in R$ can never hold simultaneously, even if $a = b$. In the matrix, this means that there are no 1's on the main diagonal (position m_{ii} for some i), and there is no pair of 1's symmetrically placed around the main diagonal (i.e., we cannot have $m_{ij} = m_{ji} = 1$ for any values of i and j).

8. For reflexivity we want all 1's on the main diagonal; for irreflexivity we want all 0's on the main diagonal; for symmetry, we want the matrix to be symmetric about the main diagonal (equivalently, the matrix equals its transpose); for antisymmetry we want there never to be two 1's symmetrically placed about the main diagonal (equivalently, the meet of the matrix and its transpose has no 1's off the main diagonal); and for transitivity we want the Boolean square of the matrix (the Boolean product of the matrix and itself) to be "less than or equal to" the original matrix in the sense that there is a 1 in the original matrix at every location where there is a 1 in the Boolean square.

a) Since some 1's and some 0's on the main diagonal, this relation is neither reflexive nor irreflexive. Since the matrix is symmetric, the relation is symmetric. The relation is not antisymmetric—look at positions $(1,2)$ and $(2,1)$. Finally, the relation is not transitive; for example, the 1's in positions $(1,2)$ and $(2,3)$ would require a 1 in position $(1,3)$ if the relation were to be transitive.

b) Since there are all 1's on the main diagonal, this relation is reflexive and not irreflexive. Since the matrix is not symmetric, the relation is not symmetric (look at positions $(1,2)$ and $(2,1)$, for example). The relation is antisymmetric since there are never two 1's symmetrically placed with respect to the main diagonal. Finally, the Boolean square of this matrix is not itself (look at position $(1,4)$ in the square), so the relation is not transitive.

c) Since there are all 0's on the main diagonal, this relation is not reflexive but is irreflexive. Since the matrix is symmetric, the relation is symmetric. The relation is not antisymmetric—look at positions $(1,2)$ and $(2,1)$, for example. Finally, the Boolean square of this matrix has a 1 in position $(1,1)$, so the relation is not transitive.

10. Note that the total number of entries in the matrix is $1000^2 = 1{,}000{,}000$.

a) There is a 1 in the matrix for each pair of distinct positive integers not exceeding 1000, namely in position (a,b) where $a \le b$, as well as 1's along the diagonal. Thus the answer is the number of subsets of size 2 from a set of 1000 elements, plus 1000, i.e., $C(1000,2) + 1000 = 499500 + 1000 = 500{,}500$.

b) There two 1's in each row of the matrix except the first and last rows, in which there is one 1. Therefore the answer is $998 \cdot 2 + 2 = 1998$.

c) There is a 1 in the matrix at each entry just above and to the left of the "anti-diagonal" (i.e., in positions $(1,999)$, $(2,998)$, $\ldots$, $(999,1)$. Therefore the answer is 999.

d) There is a 1 in the matrix at each entry on or above (to the left of) the "anti-diagonal." This is the same number of 1's as in part (a), so the answer is again 500,500.

e) The condition is trivially true (since $1 \le a \le 1000$), so all 1,000,000 entries are 1.

12. We take the transpose of the matrix, since we want the $(i,j)^{\text{th}}$ entry of the matrix for R^{-1} to be 1 if and only if the $(j,i)^{\text{th}}$ entry of R is 1.

14. a) The matrix for the union is formed by taking the join: $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$.

b) The matrix for the intersection is formed by taking the meet: $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

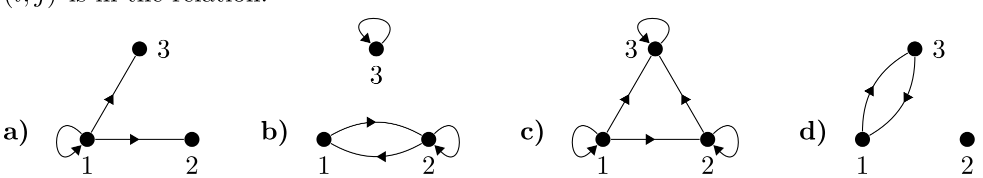
c) The matrix is the Boolean product $\boldsymbol{M}_{R_1} \odot \boldsymbol{M}_{R_2} = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

d) The matrix is the Boolean product $\boldsymbol{M}_{R_1} \odot \boldsymbol{M}_{R_1} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

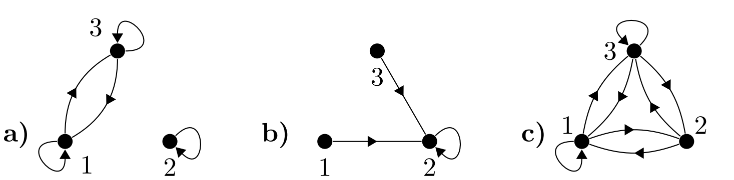
e) The matrix is the entrywise XOR: $\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

16. Since the matrix for R^{-1} is just the transpose of the matrix for R (see Exercise 12), the entries are the same collection of 0's and 1's, so there are k nonzero entries in $\mathbf{M}_{R^{-1}}$ as well.

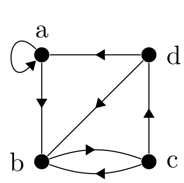
18. We draw the directed graphs, in each case with the vertex set being $\{1,2,3\}$ and an edge from i to j whenever (i,j) is in the relation.



20. In each case we draw a directed graph on three vertices with an edge from a to b for each pair (a,b) in the relation, i.e., whenever there is a 1 in position (a,b) in the matrix. In part (a), for instance, we need an edge from 1 to itself since there is a 1 in position $(1,1)$ in the matrix, and an edge from 1 to 3, but no edge from 1 to 2.



22. We draw the directed graph with the vertex set being $\{a,b,c,d\}$ and an edge from i to j whenever (i,j) is in the relation.



24. We list all the pairs (x,y) for which there is an edge from x to y in the directed graph: $\{(a,a),(a,c),(b,a),(b,b),(b,c),(c,c)\}$.

26. We list all the pairs (x,y) for which there is an edge from x to y in the directed graph: $\{(a,a),(a,b),(b,a),(b,b),(c,a),(c,c),(c,d),(d,d)\}$.

28. We list all the pairs (x,y) for which there is an edge from x to y in the directed graph: $\{(a,a),(a,b),(b,a),(b,b),(c,c),(c,d),(d,c),(d,d)\}$.

30. Clearly R is irreflexive if and only if there are no loops in the directed graph for R.

32. Recall that the relation is reflexive if there is a loop at each vertex; irreflexive if there are no loops at all; symmetric if edges appear only in **antiparallel** pairs (edges from one vertex to a second vertex and from the second back to the first); antisymmetric if there is no pair of antiparallel edges; asymmetric if is both antisymmetric and irreflexive; and transitive if all paths of length 2 (a pair of edges (x,y) and (y,z)) are accompanied by the corresponding path of length 1 (the edge (x,z)). The relation drawn in Exercise 26 is reflexive but not irreflexive since there are loops at each vertex. It is not symmetric, since, for instance, the edge (c,a) is present but not the edge (a,c). It is not antisymmetric, since both edges (a,b) and (b,a) are present. So it is not asymmetric either. It is not transitive, since the path $(c,a),(a,b)$ from c to b is not accompanied by the edge (c,b). The relation drawn in Exercise 27 is neither reflexive nor irreflexive since there are some loops but not a loop at each vertex. It is symmetric, since the edges appear in antiparallel pairs. It is not antisymmetric, since, for instance, both edges (a,b) and (b,a) are present. So it is not asymmetric either. It is not transitive, since edges (c,a) and (a,c) are present, but not (c,c). The relation drawn in Exercise 28 is reflexive and not irreflexive since there are loops at all vertices. It is symmetric but not antisymmetric or asymmetric. It is transitive; the only nontrivial paths of length 2 have the necessary loop shortcuts.

34. For each pair (a,b) of vertices (including the pairs (a,a) in which the two vertices are the same), if there is an edge from a to b, then erase it, and if there is no edge from a to b, put add it in.

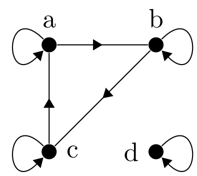
36. We assume that the two relations are on the same set. For the union, we simply take the union of the directed graphs, i.e., take the directed graph on the same vertices and put in an edge from i to j whenever there is an edge from i to j in either of them. For intersection, we simply take the intersection of the directed graphs, i.e., take the directed graph on the same vertices and put in an edge from i to j whenever there are edges from i to j in both of them. For symmetric difference, we simply take the symmetric difference of the directed graphs, i.e., take the directed graph on the same vertices and put in an edge from i to j whenever there is an edge from i to j in one, but not both, of them. Similarly, to form the difference, we take the difference of the directed graphs, i.e., take the directed graph on the same vertices and put in an edge from i to j whenever there is an edge from i to j in the first but not the second. To form the directed graph for the composition $S \circ R$ of relations R and S, we draw a directed graph on the same set of vertices and put in an edge from i to j whenever there is a vertex k such that there is an edge from i to k in R, and an edge from k to j in S.

SECTION 9.4 Closures of Relations

2. When we add all the pairs (x,x) to the given relation we have all of $\mathbf{Z}\times\mathbf{Z}$; in other words, we have the relation that always holds.

4. To form the reflexive closure, we simply need to add a loop at each vertex that does not already have one.

6. We form the reflexive closure by taking the given directed graph and appending loops at all vertices at which there are not already loops.



8. To form the digraph of the symmetric closure, we simply need to add an edge from x to y whenever this edge is not already in the directed graph but the edge from y to x is.

10. The symmetric closure was found in Example 2 to be the "is not equal to" relation. If we now make this relation reflexive as well, we will have the relation that always holds.

12. $\boldsymbol{M}_R\vee\boldsymbol{I}_n$ is by definition the same as $\boldsymbol{M}_R$ except that it has all 1's on the main diagonal. This must represent the reflexive closure of R, since this closure is the same as R except for the addition of all the pairs (x,x) that were not already present.

14. Suppose that the closure C exists. We must show that C is the intersection I of all the relations S that have property $\boldsymbol{P}$ and contain R. Certainly $I\subseteq C$, since C is one of the sets in the intersection. Conversely, by definition of closure, C is a subset of every relation S that has property $\boldsymbol{P}$ and contains R; therefore C is contained in their intersection.

16. In each case, the sequence is a path if and only if there is an edge from each vertex in the sequence to the vertex following it.

a) This is a path. **b)** This is not a path (there is no edge from e to c). **c)** This is a path.

d) This is not a path (there is no edge from d to a). **e)** This is a path.

f) This is not a path (there is no loop at b).

18. In the language of Chapter 10, this digraph is strongly connected, so there will be a path from every vertex to every other vertex.

a) One path is a,b. **b)** One path is b,e,a. **c)** One path is b,c,b; a shorter one is just b.

d) One path is a,b,e. **e)** One path is b,e,d. **f)** One path is c,e,d.

g) One path is d,e,d. Another is the path of length 0 from d to itself.

h) One path is e,a. Another is e,a,b,e,a,b,e,a,b,e,a. **i)** One path is e,a,b,c.

20. **a)** The pair (a,b) is in R^2 precisely when there is a city c such that there is a direct flight from a to c and a direct flight from c to b—in other words, when it is possible to fly from a to b with a scheduled stop (and possibly a plane change) in some intermediate city.

b) The pair (a,b) is in R^3 precisely when there are cities c and d such that there is a direct flight from a to c, a direct flight from c to d, and a direct flight from d to b—in other words, when it is possible to fly from a to b with two scheduled stops (and possibly a plane change at one or both) in intermediate cities.

c) The pair (a,b) is in R^* precisely when it is possible to fly from a to b.

22. Since $R \subseteq R^*$, clearly if $\Delta \subseteq R$, then $\Delta \subseteq R^*$.

24. It is certainly possibly for R^2 to contain some pairs (a,a). For example, let $R = \{(1,2),(2,1)\}$.

26. **a)** We show the various matrices that are involved. First,

$$\boldsymbol{A} = \begin{bmatrix} 0&0&1&0&0\\ 0&0&0&1&0\\ 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&1&0 \end{bmatrix}, \quad \boldsymbol{A}^{[2]} \begin{bmatrix} 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&1&0&0&0 \end{bmatrix}, \text{ and } \boldsymbol{A}^{[3]} = \begin{bmatrix} 0&0&1&0&0\\ 0&0&0&1&0\\ 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&1&0 \end{bmatrix} = \boldsymbol{A}.$$

It follows that $\boldsymbol{A}^{[4]} = \boldsymbol{A}^{[2]}$ and $\boldsymbol{A}^{[5]} = \boldsymbol{A}^{[3]}$. Therefore the answer $\boldsymbol{B}$, the meet of all the $\boldsymbol{A}$'s, is $\boldsymbol{A} \vee \boldsymbol{A}^{[2]}$, namely

$$\begin{bmatrix} 1&0&1&0&0\\ 0&1&0&1&0\\ 1&0&1&0&0\\ 0&1&0&1&0\\ 0&1&0&1&0 \end{bmatrix}.$$

b) For this and the remaining parts we just exhibit the matrices that arise.

$$\boldsymbol{A} = \begin{bmatrix} 0&0&0&0&0\\ 0&0&1&0&1\\ 0&0&0&0&1\\ 1&0&0&0&0\\ 0&1&1&0&0 \end{bmatrix} \quad \boldsymbol{A}^{[2]} = \begin{bmatrix} 0&0&0&0&0\\ 0&1&1&0&1\\ 0&1&1&0&0\\ 0&0&0&0&0\\ 0&0&1&0&1 \end{bmatrix} \quad \boldsymbol{A}^{[3]} = \begin{bmatrix} 0&0&0&0&0\\ 0&1&1&0&1\\ 0&0&1&0&1\\ 0&0&0&0&0\\ 0&1&1&0&1 \end{bmatrix}$$

$$\boldsymbol{A}^{[4]} = \begin{bmatrix} 0&0&0&0&0\\ 0&1&1&0&1\\ 0&1&1&0&1\\ 0&0&0&0&0\\ 0&1&1&0&1 \end{bmatrix} = \boldsymbol{A}^{[5]} \qquad \boldsymbol{B} = \begin{bmatrix} 0&0&0&0&0\\ 0&1&1&0&1\\ 0&1&1&0&1\\ 1&0&0&0&0\\ 0&1&1&0&1 \end{bmatrix}$$

c) $$\boldsymbol{A} = \begin{bmatrix} 0&1&1&0&1\\ 1&0&1&0&0\\ 1&1&0&0&0\\ 1&0&0&0&0\\ 0&0&0&1&0 \end{bmatrix} \quad \boldsymbol{A}^{[2]} = \begin{bmatrix} 1&1&1&1&0\\ 1&1&1&0&1\\ 1&1&1&0&1\\ 0&1&1&0&1\\ 1&0&0&0&0 \end{bmatrix} \quad \boldsymbol{A}^{[3]} = \begin{bmatrix} 1&1&1&0&1\\ 1&1&1&1&1\\ 1&1&1&1&1\\ 1&1&1&1&0\\ 0&1&1&0&1 \end{bmatrix}$$

$$\boldsymbol{A}^{[4]} = \begin{bmatrix} 1&1&1&1&1\\ 1&1&1&1&1\\ 1&1&1&1&1\\ 1&1&1&0&1\\ 1&1&1&1&0 \end{bmatrix} \quad \boldsymbol{A}^{[5]} = \begin{bmatrix} 1&1&1&1&1\\ 1&1&1&1&1\\ 1&1&1&1&1\\ 1&1&1&1&1\\ 1&1&1&0&1 \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} 1&1&1&1&1\\ 1&1&1&1&1\\ 1&1&1&1&1\\ 1&1&1&1&1\\ 1&1&1&1&1 \end{bmatrix}$$

d) $$\boldsymbol{A} = \begin{bmatrix} 0&0&0&0&1\\ 1&0&0&1&0\\ 0&0&0&1&0\\ 1&0&1&0&0\\ 1&1&1&0&1 \end{bmatrix} \quad \boldsymbol{A}^{[2]} = \begin{bmatrix} 1&1&1&0&1\\ 1&0&1&0&1\\ 1&0&1&0&0\\ 0&0&0&1&1\\ 1&1&1&1&1 \end{bmatrix} \quad \boldsymbol{A}^{[3]} = \begin{bmatrix} 1&1&1&1&1\\ 1&1&1&1&1\\ 0&0&0&1&1\\ 1&1&1&0&1\\ 1&1&1&1&1 \end{bmatrix}$$

$$\boldsymbol{A}^{[4]} = \begin{bmatrix} 1&1&1&1&1\\ 1&1&1&1&1\\ 1&1&1&0&1\\ 1&1&1&1&1\\ 1&1&1&1&1 \end{bmatrix} \quad \boldsymbol{A}^{[5]} = \begin{bmatrix} 1&1&1&1&1\\ 1&1&1&1&1\\ 1&1&1&1&1\\ 1&1&1&1&1\\ 1&1&1&1&1 \end{bmatrix} = \boldsymbol{B}$$

28. We compute the matrices $\boldsymbol{W}_i$ for $i = 0, 1, 2, 3, 4, 5$, and then $\boldsymbol{W}_5$ is the answer.

a) $\boldsymbol{W}_0 = \begin{bmatrix} 0&0&1&0&0\\ 0&0&0&1&0\\ 1&0&0&0&0\\ 0&1&0&0&0\\ 0&0&0&1&0 \end{bmatrix}$ $\quad\boldsymbol{W}_1 = \begin{bmatrix} 0&0&1&0&0\\ 0&0&0&1&0\\ 1&0&1&0&0\\ 0&1&0&0&0\\ 0&0&0&1&0 \end{bmatrix}$ $\quad\boldsymbol{W}_2 = \begin{bmatrix} 0&0&1&0&0\\ 0&0&0&1&0\\ 1&0&1&0&0\\ 0&1&0&1&0\\ 0&0&0&1&0 \end{bmatrix}$

$\boldsymbol{W}_3 = \begin{bmatrix} 1&0&1&0&0\\ 0&0&0&1&0\\ 1&0&1&0&0\\ 0&1&0&1&0\\ 0&0&0&1&0 \end{bmatrix}$ $\quad\boldsymbol{W}_4 = \begin{bmatrix} 1&0&1&0&0\\ 0&1&0&1&0\\ 1&0&1&0&0\\ 0&1&0&1&0\\ 0&1&0&1&0 \end{bmatrix} = \boldsymbol{W}_5$

b) $\boldsymbol{W}_0 = \begin{bmatrix} 0&0&0&0&0\\ 0&0&1&0&1\\ 0&0&0&0&1\\ 1&0&0&0&0\\ 0&1&1&0&0 \end{bmatrix} = \boldsymbol{W}_1$ $\quad\boldsymbol{W}_2 = \begin{bmatrix} 0&0&0&0&0\\ 0&0&1&0&1\\ 0&0&0&0&1\\ 1&0&0&0&0\\ 0&1&1&0&1 \end{bmatrix} = \boldsymbol{W}_3 = \boldsymbol{W}_4$

$\boldsymbol{W}_5 = \begin{bmatrix} 0&0&0&0&0\\ 0&1&1&0&1\\ 0&1&1&0&1\\ 1&0&0&0&0\\ 0&1&1&0&1 \end{bmatrix}$

c) $\boldsymbol{W}_0 = \begin{bmatrix} 0&1&1&0&1\\ 1&0&1&0&0\\ 1&1&0&0&0\\ 1&0&0&0&0\\ 0&0&0&1&0 \end{bmatrix}$ $\quad\boldsymbol{W}_1 = \begin{bmatrix} 0&1&1&0&1\\ 1&1&1&0&1\\ 1&1&1&0&1\\ 1&1&1&0&1\\ 0&0&0&1&0 \end{bmatrix}$ $\quad\boldsymbol{W}_2 = \begin{bmatrix} 1&1&1&0&1\\ 1&1&1&0&1\\ 1&1&1&0&1\\ 1&1&1&0&1\\ 0&0&0&1&0 \end{bmatrix} = \boldsymbol{W}_3$

$\boldsymbol{W}_4 = \begin{bmatrix} 1&1&1&0&1\\ 1&1&1&0&1\\ 1&1&1&0&1\\ 1&1&1&0&1\\ 1&1&1&1&1 \end{bmatrix}$ $\quad\boldsymbol{W}_5 = \begin{bmatrix} 1&1&1&1&1\\ 1&1&1&1&1\\ 1&1&1&1&1\\ 1&1&1&1&1\\ 1&1&1&1&1 \end{bmatrix}$

d) $\boldsymbol{W}_0 = \begin{bmatrix} 0&0&0&0&1\\ 1&0&0&1&0\\ 0&0&0&1&0\\ 1&0&1&0&0\\ 1&1&1&0&1 \end{bmatrix}$ $\quad\boldsymbol{W}_1 = \begin{bmatrix} 0&0&0&0&1\\ 1&0&0&1&1\\ 0&0&0&1&0\\ 1&0&1&0&1\\ 1&1&1&0&1 \end{bmatrix}$ $\quad\boldsymbol{W}_2 = \begin{bmatrix} 0&0&0&0&1\\ 1&0&0&1&1\\ 0&0&0&1&0\\ 1&0&1&0&1\\ 1&1&1&1&1 \end{bmatrix}$

$\boldsymbol{W}_3 = \begin{bmatrix} 0&0&0&0&1\\ 1&0&0&1&1\\ 0&0&0&1&0\\ 1&0&1&1&1\\ 1&1&1&1&1 \end{bmatrix}$ $\quad\boldsymbol{W}_4 = \begin{bmatrix} 0&0&0&0&1\\ 1&0&1&1&1\\ 1&0&1&1&1\\ 1&0&1&1&1\\ 1&1&1&1&1 \end{bmatrix}$ $\quad\boldsymbol{W}_5 = \begin{bmatrix} 1&1&1&1&1\\ 1&1&1&1&1\\ 1&1&1&1&1\\ 1&1&1&1&1\\ 1&1&1&1&1 \end{bmatrix}$

30. Let m be the length of the shortest path from a to b, and let $a = x_0, x_1, \ldots, x_{m-1}, x_m = b$ be such a path. If $m > n-1$, then $m \geq n$, so $m+1 \geq n+1$, which means that not all of the vertices x_0, x_1, x_2, $\ldots$, x_m are distinct. Thus $x_i = x_j$ for some i and j with $0 \leq i < j \leq m$ (but not both $i = 0$ and $j = m$, since $a \neq b$). We can then excise the circuit from x_i to x_j, leaving a shorter path from a to b, namely $x_0, \ldots, x_i, x_{j+1}, \ldots, x_m$. This contradicts the choice of m. Therefore $m \leq n-1$, as desired.

32. Warshall's algorithm determines the existence of paths. If instead we keep track of the lengths of paths, then we can get the desired information. Thus we make the following changes in Algorithm 2. First, instead of initializing $\boldsymbol{W}$ to be $\boldsymbol{M}_R$, we initialize it to be $\boldsymbol{M}_R$ with each 0 replaced by ∞. Second, the computational step becomes $w_{ij} := \min(w_{ij}, w_{ik} + w_{kj})$.

34. All we need to do is make sure that all the pairs (x,x) are included. An easy way to accomplish this is to add them at the end, by setting $\boldsymbol{W} := \boldsymbol{W} \vee \boldsymbol{I}_n$.

36. Consider $R = \{(a,b),(c,b)\}$. The transitive closure of R is again R. The reflexive closure of the transitive closure is $\{(a,a),(a,b),(b,b),(c,b),(c,c)\}$. The symmetric closure of that is

$$\{(a,a),(a,b),(b,a),(b,b),(b,c),(c,b),(c,c)\}.$$

This is not transitive because it contains both (a,b) and (b,c), but not (a,c).

SECTION 9.5 Equivalence Relations

2. a) This is an equivalence relation by Exercise 9 ($f(x)$ is x's age).

b) This is an equivalence relation by Exercise 9 ($f(x)$ is x's parents).

c) This is not an equivalence relation, since it need not be transitive. (We assume that biological parentage is at issue here, so it is possible for A to be the child of W and X, B to be the child of X and Y, and C to be the child of Y and Z. Then A is related to B, and B is related to C, but A is not related to C.)

d) This is not an equivalence relation since it is clearly not transitive.

e) Again, just as in part (c), this is not transitive.

4. One relation is that a and b are related if they were born in the same U.S. state (with "not in a state of the U.S." counting as one state). Here the equivalence classes are the nonempty sets of students from each state. Another example is for a to be related to b if a and b have lived the same number of complete decades. The equivalence classes are the set of all 10-to-19 year-olds, the set of all 20-to-29 year-olds, and so on (the sets among these that are nonempty, that is). A third example is for a to be related to b if 10 is a divisor of the difference between a's age and b's age, where "age" means the whole number of years since birth, as of the first day of class. For each $i = 0, 1, \ldots, 9$, there is the equivalence class (if it is nonempty) of those students whose age ends with the digit i.

6. One way to partition the classes would be by level. At many schools, classes have three-digit numbers, the first digit of which is approximately the level of the course, so that courses numbered 100–199 are taken by freshman, 200–299 by sophomores, and so on. Formally, two classes are related if their numbers have the same digit in the hundreds column; the equivalence classes are the set of all 100-level classes, the set of all 200-level classes, and so on. A second example would focus on department. Two classes are equivalent if they are offered by the same department; for example, MATH 154 is equivalent to MATH 372, but not to EGR 141. The equivalence classes are the sets of classes offered by each department (the set of math classes, the set of engineering classes, and so on). A third—and more egocentric—classification would be to have one equivalence class be the set of classes that you have completed successfully and the other equivalence class to be all the other classes. Formally, two classes are equivalent if they have the same answer to the question, "Have I completed this class successfully?"

8. Recall (Definition 1 in Section 2.5) that two sets have the same cardinality if there is a bijection (one-to-one and onto function) from one set to the other. We must show that R is reflexive, symmetric, and transitive. Every set has the same cardinality as itself because of the identity function. If f is a bijection from S to T, then f^{-1} is a bijection from T to S, so R is symmetric. Finally, if f is a bijection from S to T and g is a bijection from T to U, then $g \circ f$ is a bijection from T to U, so R is transitive (see Exercise 33 in Section 2.3).

The equivalence class of $\{1,2,3\}$ is the set of all three-element sets of real numbers, including such sets as $\{4,25,1948\}$ and $\{e,\pi,\sqrt{2}\}$. Similarly, $[\mathbf{Z}]$ is the set of all infinite countable sets of real numbers (see Section 2.5), such as the set of natural numbers, the set of rational numbers, and the set of the prime numbers, but not including the set $\{1,2,3\}$ (it's too small) or the set of all real numbers (it's too big). See Section 2.5 for more on countable sets.

10. The function that sends each $x \in A$ to its equivalence class $[x]$ is obviously such a function.

12. This follows from Exercise 9, where f is the function that takes a bit string of length $n \geq 3$ to its last $n-3$ bits.

14. This follows from Exercise 9, where f is the function that takes a string of uppercase and lowercase English letters and changes all the lower case letters to their uppercase equivalents (and leaves the uppercase letters unchanged).

16. This follows from Exercise 9, where f is the function from the set of pairs of positive integers to the set of positive rational numbers that takes (a,b) to a/b, since clearly $ad = bc$ if and only if $a/b = c/d$.

If we want an explicit proof, we can argue as follows. For reflexivity, $((a,b),(a,b)) \in R$ because $a \cdot b = b \cdot a$. If $((a,b),(c,d)) \in R$ then $ad = bc$, which also means that $cb = da$, so $((c,d),(a,b)) \in R$; this tells us that R is symmetric. Finally, if $((a,b),(c,d)) \in R$ and $((c,d),(e,f)) \in R$ then $ad = bc$ and $cf = de$. Multiplying these equations gives $acdf = bcde$, and since all these numbers are nonzero, we have $af = be$, so $((a,b),(e,f)) \in R$; this tells us that R is transitive.

18. **a)** This follows from Exercise 9, where the function f from the set of polynomials to the set of polynomials is the operator that takes the derivative n times—i.e., f of a function g is the function $g^{(n)}$. The best way to think about this is that any relation defined by a statement of the form "a and b are equivalent if they have the same whatever" is an equivalence relation. Here "whatever" is "n^{th} derivative"; in the general situation of Exercise 9, "whatever" is "function value under f."

b) The third derivative of x^4 is $24x$. Since the third derivative of a polynomial of degree 2 or less is 0, the polynomials of the form $x^4 + ax^2 + bx + c$ have the same third derivative. Thus these are the functions in the same equivalence class as f.

20. This follows from Exercise 9, where the function f from the set of people to the set of web-traversing behaviors starting at the given particular web page takes the person to the behavior that person exhibited.

22. We need to observe whether the relation is reflexive (there is a loop at each vertex), symmetric (every edge that appears is accompanied by its antiparallel mate—an edge involving the same two vertices but pointing in the opposite direction), and transitive (paths of length 2 are accompanied by the path of length 1—i.e., edge—between the same two vertices in the same direction). We see that this relation is an equivalence relation, satisfying all three properties. The equivalence classes are $\{a,d\}$ and $\{b,c\}$.

24. **a)** This is not an equivalence relation, since it is not symmetric.

b) This is an equivalence relation; one equivalence class consists of the first and third elements, and the other consists of the second and fourth elements.

c) This is an equivalence relation; one equivalence class consists of the first, second, and third elements, and the other consists of the fourth element.

26. Only part (a) and part (c) are equivalence relations. In part (a) each element is in an equivalence class by itself. In part (c) the elements 1 and 2 are in one equivalence class, and 0 and 3 are each in their own equivalence class.

28. Only part (a) and part (d) are equivalence relations. In part (a) there is one equivalence class for each $n \in \mathbf{Z}$, and it contains all those functions whose value at 1 is n. In part (d) there really is no good way to describe the equivalence classes. For one thing, the set of equivalence classes is uncountable. For each function $f : \mathbf{Z} \to \mathbf{Z}$, there is the equivalence class consisting of all those functions g for which there is a constant C such that $g(n) = f(n) + C$ for all $n \in \mathbf{Z}$.

30. **a)** all the strings whose first three bits are 010 **b)** all the strings whose first three bits are 101
c) all the strings whose first three bits are 111 **d)** all the strings whose first three bits are 010

32. Since two bit strings are related if and only if they agree in their first and third bits, the equivalence class of a bit string $xyzt$, where x, y, and z are bits and t is a bit string, is the set of all bit strings of the form $xy'zt'$, where y' is any bit and t' is any bit string.

a) the set of all bit strings that start 010 or 000

b) the set of all bit strings that start 101 or 111

c) the set of all bit strings that start 101 or 111

d) the set of all bit strings that start 000 or 010

34. **a)** Since this string has length less than 5, its equivalence class consists only of itself.

b) This is similar to part (a): $[1011]_{R_5} = \{1011\}$.

c) Since this string has length 5, its equivalence class consists of all strings that start 11111.

d) This is similar to part (c): $[01010101]_{R_5} = \{\, 01010s \mid s \text{ is any bit string} \,\}$.

36. In each case, the equivalence class of 4 is the set of all integers congruent to 4, modulo m.

a) $\{\, 4 + 2n \mid n \in \mathbf{Z} \,\} = \{\ldots, -2, 0, 2, 4, \ldots\}$ **b)** $\{\, 4 + 3n \mid n \in \mathbf{Z} \,\} = \{\ldots, -2, 1, 4, 7, \ldots\}$
c) $\{\, 4 + 6n \mid n \in \mathbf{Z} \,\} = \{\ldots, -2, 4, 10, 16, \ldots\}$ **d)** $\{\, 4 + 8n \mid n \in \mathbf{Z} \,\} = \{\ldots, -4, 4, 12, 20, \ldots\}$

38. In each case we need to allow all strings that agree with the given string if we ignore the case in which the letters occur.

a) $\{NO, No, nO, no\}$

b) $\{\,YES, YEs, YeS, Yes, yES, yEs, yeS, yes\}$

c) $\{HELP, HELp, HElP, HElp, HeLP, HeLp, HelP, Help, hELP, hELp, hElP, hElp, heLP, heLp, helP, help\}$

40. **a)** By our observation in the solution to Exercise 16, the equivalence class of $(1, 2)$ is the set of all pairs (a, b) such that the fraction a/b equals $1/2$.

b) Again by our observation, the equivalence classes are the positive rational numbers. (Indeed, this is the way one can rigorously define what a rational number is, and this is why fractions are so difficult for children to understand.)

42. **a)** This is a partition, since it satisfies the definition.

b) This is not a partition, since the subsets are not disjoint.

c) This is a partition, since it satisfies the definition.

d) This is not a partition, since the union of the subsets leaves out 0.

44. **a)** This is clearly a partition. **b)** This is not a partition, since 0 is in neither set.

c) This is a partition by the division algorithm.

d) This is a partition, since the second set mentioned is the set of all number between -100 and 100, inclusive.

e) The first two sets are not disjoint (4 is in both), so this is not a partition.

46. **a)** This is a partition, since it satisfies the definition.

b) This is a partition, since it satisfies the definition.

c) This is not a partition, since the intervals are not disjoint (they share endpoints).

d) This is not a partition, since the union of the subsets leaves out the integers.

e) This is a partition, since it satisfies the definition.

f) This is a partition, since it satisfies the definition. Each equivalence class consists of all real numbers with a fixed fractional part.

48. In each case, we need to list all the pairs we can where both coordinates are chosen from the same subset. We should proceed in an organized fashion, listing all the pairs corresponding to each part of the partition.

a) $\{(a,a),(a,b),(b,a),(b,b),(c,c),(c,d),(d,c),(d,d),(e,e),(e,f),(e,g),(f,e),(f,f),(f,g),(g,e),(g,f),(g,g)\}$

b) $\{(a,a),(b,b),(c,c),(c,d),(d,c),(d,d),(e,e),(e,f),(f,e),(f,f),(g,g)\}$

c) $\{(a,a),(a,b),(a,c),(a,d),(b,a),(b,b),(b,c),(b,d),(c,a),(c,b),(c,c),(c,d),(d,a),(d,b),(d,c),(d,d),$ $(e,e),(e,f),(e,g),(f,e),(f,f),(f,g),(g,e),(g,f),(g,g)\}$

d) $\{(a,a),(a,c),(a,e),(a,g),(c,a),(c,c),(c,e),(c,g),(e,a),(e,c),(e,e),(e,g),(g,a),(g,c),(g,e),(g,g),$ $(b,b),(b,d),(d,b),(d,d),(f,f)\}$

50. We need to show that every equivalence class consisting of people living in the same county (or parish) and same state is contained in an equivalence class of all people living in the same state. This is clear. The equivalence class of all people living in county c in state s is a subset of the set of people living in state s.

52. We are asked to show that every equivalence class for R_4 is a subset of some equivalence class for R_3. Let $[y]_{R_4}$ be an arbitrary equivalence class for R_4. We claim that $[y]_{R_4} \subseteq [y]_{R_3}$; proving this claim finishes the proof. To show that one set is a subset of another set, we choose an arbitrary bit string x in the first set and show that it is also an element of the second set. In this case since $y \in [x]_{R_4}$, we know that y is equivalent to x under R_4, that is, that either $y = x$ or y and x are each at least 4 bits long and agree on their first 4 bits. Because strings that are at least 4 bits long and agree on their first 4 bits perforce are at least 3 bits long and agree on their first 3 bits, we know that either $y = x$ or y and x are each at least 3 bits long and agree on their first 3 bits. This means that y is equivalent to x under R_3, that is, that $y \in [x]_{R_3}$.

54. First, suppose that $R_1 \subseteq R_2$. We must show that P_1 is a refinement of P_2. Let $[a]_{R_1}$ be an equivalence class in P_1. We must show that $[a]_{R_1}$ is contained in an equivalence class in P_2. In fact, we will show that $[a]_{R_1} \subseteq [a]_{R_2}$. To this end, let $b \in [a]_{R_1}$. Then $(a,b) \in R_1 \subseteq R_2$. Therefore $b \in [a]_{R_2}$, as desired.

Conversely, suppose that P_1 is a refinement of P_2. Since $a \in [a]_{R_2}$, the definition of "refinement" forces $[a]_{R_1} \subseteq [a]_{R_2}$ for all $a \in A$. This means that for all $b \in A$ we have $(a,b) \in R_1 \to (a,b) \in R_2$; in other words, $R_1 \subseteq R_2$.

56. **a)** This need not be an equivalence relation, since it need not be transitive.

b) Since the intersection of reflexive, symmetric, and transitive relations also have these properties (see Section 9.1), the intersection of equivalence relations is an equivalence relation.

c) This will never be an equivalence relation on a nonempty set, since it is not reflexive.

58. This exercise is very similar to Exercise 59, and the reader should look at the solution there for details.

a) As in Exercise 59, the motions of the bracelet form a dihedral group, in this case consisting of six motions: rotations of 0^o, 120^o, and 240^o, and three reflections, each keeping one bead fixed and interchanging the other two. The composition of any two of these operations is again one of these operations. The 0^o rotation plays the role of the identity, which says that the relation is reflexive. Each operation has an inverse (reflections are their own inverses, the 0^o rotation is its own inverse, and the 120^o and 240^o rotations are inverses of each other); this proves symmetry. And transitivity follows from the group table.

b) The equivalence classes are the indistinguishable bracelets. If we denote a bracelet by the colors of its beads, then these classes can be described as RRR, WWW, BBB, RRW, RRB, WWR, WWB, BBR, BBW, and RWB. Note that once we specify the colors, then every two bracelets with those colors are equivalent. This would not be the case if there were four or more beads, however. For example, in a 4-bead bracelet with two reds and two whites, the bracelet in which the red beads are adjacent is not equivalent to the one in which they are not.

60. **a)** In Exercise 31 of Section 3.2, we showed that $f(x)$ is $\Theta(g(x))$ if and only if $f(x)$ is $O(g(x))$ and $g(x)$ is $O(f(x))$. To show that R is reflexive, we need to show that $f(x)$ is $O(f(x))$, which is clear by taking $C=1$ and $k=1$ in the definition. Symmetry is immediate from the definition, since if $f(x)$ is $O(g(x))$ and $g(x)$ is $O(f(x))$, then $g(x)$ is $O(f(x))$ and $f(x)$ is $O(g(x))$. Finally, transitivity follows immediately from the transitiveity of the "is big-O of" relation, which was proved in Exercise 17 of Section 3.2.

b) This is the class of all functions that asymptotically (i.e., as $n \to \infty$) grow just as fast as a multiple of $f(n) = n^2$. So, for example, functions such as $g(n) = 5n^2 + \log n$, or $g(n) = (n^3 - 17)/(100n + 10^{10})$ belong to this class, but $g(n) = n^{2.01}$ does not (it grows too fast), and $g(n) = n^2/\log n$ does not (it grows too slowly). Another way to express this class is to say that it is the set of all functions g such that there exist constants positive C_1 and C_2 such that the ratio $f(n)/g(n)$ always lies between C_1 and C_2.

62. We will count partitions instead, since equivalence relations are in one-to-one correspondence with partitions. Without loss of generality let the set be $\{1,2,3,4\}$. There is 1 partition in which all the elements are in the same set, namely $\{\{1,2,3,4\}\}$. There are 4 partitions in which the sizes of the sets are 1 and 3, namely $\{\{1\},\{2,3,4\}\}$ and three more like it. There are 3 partitions in which the sizes of the sets are 2 and 2, namely $\{\{1,2\},\{3,4\}\}$ and two more like it. There are 6 partitions in which the sizes of the sets are 2, 1, and 1, namely $\{\{1,2\},\{3\},\{4\}\}$ and five more like it. Finally, there is 1 partition in which all the elements are in separate sets. This gives a total of 15. To actually list the 15 relations would be tedious.

64. No. Here is a counterexample. Start with $\{(1,2),(3,2)\}$ on the set $\{1,2,3\}$. Its transitive closure is itself. The reflexive closure of that is $\{(1,1),(1,2),(2,2),(3,2),(3,3)\}$. The symmetric closure of that is $\{(1,1),(1,2),(2,1),(2,2),(2,3),(3,2),(3,3)\}$. The result is not transitive; for example, $(1,3)$ is missing. Therefore this is not an equivalence relation.

66. We end up with the original partition P.

68. We will develop this recurrence relation in the context of partitions of the set $\{1,2,\ldots,n\}$. Note that $p(0)=1$, since there is only one way to partition the empty set (namely, into the empty collection of subsets). For warm-up, we also note that $p(1)=1$, since $\{\{1\}\}$ is the only partition of $\{1\}$; that $p(2)=2$, since we can partition $\{1,2\}$ either as $\{\{1,2\}\}$ or as $\{\{1\},\{2\}\}$; and that $p(3)=5$, since there are the following partitions: $\{\{1,2,3\}\}$, $\{\{1,2\},\{3\}\}$, $\{\{1,3\},\{2\}\}$, $\{\{2,3\},\{1\}\}$, $\{\{1\},\{2\},\{3\}\}$. Now to partition $\{1,2,\ldots,n\}$, we first decide how many other elements of this set will go into the same subset as n goes into. Call this number j, and note that j can take any value from 0 through $n-1$. Once we have determined j, we can specify the partition by deciding on the subset of j elements from $\{1,2,\ldots,n-1\}$ that will go into the same subset as n (and this can be done in $C(n-1,j)$ ways), and then we need to decide how to partition the remaining $n-1-j$ elements (and this can be done in $p(n-j-1)$ ways). The given recurrence relation now follows.

SECTION 9.6 Partial Orderings

2. The question in each case is whether the relation is reflexive, antisymmetric, and transitive. Suppose the relation is called R.

a) This relation is not reflexive because 1 is not related to itself. Therefore R is not a partial ordering. The relation is antisymmetric, because the only way for a to be related to b is for a to equal b. Similarly, the relation is transitive, because if a is related to b, and b is related to c, then necessarily $a = b = c \neq 1$ so a is related to c.

b) This is a partial ordering, because it is reflexive and the pairs $(2,0)$ and $(2,3)$ will not introduce any violations of antisymmetry or transitivity.

c) This is not a partial ordering, because it is not transitive: $3\,R\,1$ and $1\,R\,2$, but 3 is not related to 2. It is reflexive and the pairs $(1,2)$ and $(3,1)$ will not introduce any violations of antisymmetry.

d) This is not a partial ordering, because it is not transitive: $1\,R\,2$ and $2\,R\,0$, but 1 is not related to 0. It is reflexive and the nonreflexive pairs will not introduce any violations of antisymmetry.

e) The relation is clearly reflexive, but it is not antisymmetric ($0\,R\,1$ and $1\,R\,0$, but $0 \neq 1$) and not transitive ($2\,R\,0$ and $0\,R\,1$, but 2 is not related to 1).

4. The question in each case is whether the relation is reflexive, antisymmetric, and transitive.

a) Since there surely are unequal people of the same height (to whatever degree of precision heights are measured), this relation is not antisymmetric, so (S,R) cannot be a poset.

b) Since nobody weighs more than herself, this relation is not reflexive, so (S,R) cannot be a poset.

c) This is a poset. The equality clause in the definition of R guarantees that R is reflexive. To check antisymmetry and transitivity it suffices to consider unequal elements (these rules hold for equal elements trivially). If a is a descendant of b, then b cannot be a descendant of a (for one thing, a descendant needs to be born after any ancestor), so the relation is vacuously antisymmetric. If a is a descendant of b, and b is a descendant of c, then by the way "descendant" is defined, we know that a is a descendant of c; thus R is transitive.

d) This relation is not reflexive, because anyone and himself have a common friend.

6. The question in each case is whether the relation is reflexive, antisymmetric, and transitive.

a) The equality relation on any set satisfies all three conditions and is therefore a partial order. (It is the smallest partial order; reflexivity insures that every partial order contains at least all the pairs (a,a).)

b) This is not a poset, since the relation is not reflexive, although it is antisymmetric and transitive. Any relation of this sort can be turned into a partial ordering by adding in all the pairs (a,a).

c) This is a poset, very similar to Example 1.

d) This is not a poset, since the relation is not reflexive, not antisymmetric, and not transitive (the absence of one of these properties would have been enough to give a negative answer).

8. **a)** This relation is $\{(1,1),(1,3),(2,1),(2,2),(3,3)\}$. It is clearly reflexive and antisymmetric. The only pairs that might present problems with transitivity are the nondiagonal pairs, $(2,1)$ and $(1,3)$. If the relation were to be transitive, then we would also need the pair $(2,3)$ in the relation. Since it is not there, the relation is not a partial order.

b) Reasoning as in part (a), we see that this relation is a partial order, since the pair $(3,1)$ can cause no problem with transitivity.

c) A little trial and error shows that this relation is not transitive ($(1,3)$ and $(3,4)$ are present, but not $(1,4)$) and therefore not a partial order.

10. This relation is not transitive (there is no arrow from c to b), so it is not a partial order.

12. This follows immediately from the definition. Clearly R^{-1} is reflexive if R is. For antisymmetry, suppose that $(a,b) \in R^{-1}$ and $a \neq b$. Then $(b,a) \in R$, so $(a,b) \notin R$, whence $(b,a) \notin R^{-1}$. Finally, if $(a,b) \in R^{-1}$ and $(b,c) \in R^{-1}$, then $(b,a) \in R$ and $(c,b) \in R$, so $(c,a) \in R$ (since R is transitive), and therefore $(a,c) \in R^{-1}$; thus R^{-1} is transitive.

14. **a)** These are comparable, since $5 \mid 15$.

b) These are not comparable since neither divides the other.

c) These are comparable, since $8 \mid 16$.

d) These are comparable, since $7 \mid 7$.

16. **a)** We need either a number less than 2 in the first coordinate, or a 2 in the first coordinate and a number less than 3 in the second coordinate. Therefore the answer is $(1,1)$, $(1,2)$, $(1,3)$, $(1,4)$, $(2,1)$, and $(2,2)$.

b) We need either a number greater than 3 in the first coordinate, or a 3 in the first coordinate and a number greater than 1 in the second coordinate. Therefore the answer is $(4,1)$, $(4,2)$, $(4,3)$, $(4,4)$, $(3,2)$, $(3,3)$, and $(3,4)$.

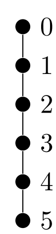
c) The Hasse diagram is a straight line with 16 points on it, since this is a total order. The pair $(4,4)$ is at the top, $(4,3)$ beneath it, $(4,2)$ beneath that, and so on, with $(1,1)$ at the bottom. To save space, we will not actually draw this picture.

18. **a)** The string *quack* comes first, since it is an initial substring of *quacking*, which comes next (since the other three strings all begin *qui*, not *qua*). Similarly, these last three strings are in the order *quick*, *quicksand*, *quicksilver*.

b) The order is *open*, *opened*, *opener*, *opera*, *operand*.

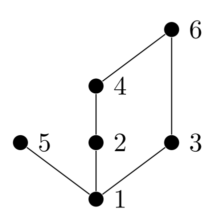
c) The order is *zero*, *zoo*, *zoological*, *zoology*, *zoom*.

20. The Hasse diagram for this total order is a straight line, as shown, with 0 at the top (it is the "largest" element under the "is greater than or equal to" relation) and 5 at the bottom.

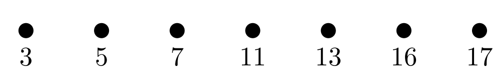


22. In each case we put a above b and draw a line between them if $b \mid a$ but there is no element c other than a and b such that $b \mid c$ and $c \mid a$.

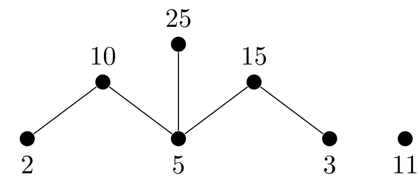
a) Note that 1 divides all numbers, so the numbers on the second level from the bottom are the primes.



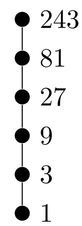
b) In this case these numbers are pairwise relatively prime, so there are no lines in the Hasse diagram.



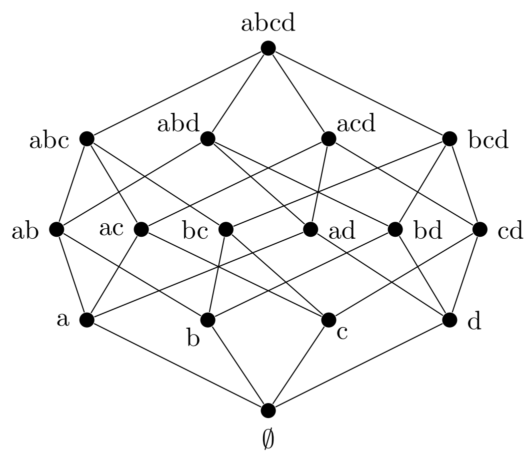
c) Note that we can place the points as we wish, as long as a is above b when $b \mid a$.



d) In this case these numbers each divide the next, so the Hasse diagram is a straight line.



24. This picture is a four-dimensional cube. We draw the sets with k elements at level k: the empty set at level 0 (the bottom), the entire set at level 4 (the top).



26. The procedure is the same as in Exercise 25: $\{(a,a),(a,b),(a,c),(a,d),(a,e),(b,b),(b,d),(b,e),(c,c),(c,d),(d,d),(e,e)\}$

28. In this problem $a \preceq b$ when $a \mid b$. For (a,b) to be in the covering relation, we need a to be a proper divisor of b but we also must have no element in our set $\{1, 2, 3, 4, 6, 12\}$ being a proper multiple of a and a proper divisor of b. For example, $(2,12)$ is not in the covering relation, since $2 \mid 6$ and $6 \mid 12$. With this understanding it is easy to list the pairs in the covering relation: $(1,2)$, $(1,3)$, $(2,4)$, $(2,6)$, $(3,6)$, $(4,12)$, and $(6,12)$.

30. This poset has 32 elements, consisting of all pairs (A,C) where A is one of 0, 1, 2, and 3 (here representing unclassified, confidential, secret, and top secret) and C is one of the eight subsets of $\{s,m,d\}$ (where these letters represent spies, moles, and double agents). The following list gives the covering relation: $(0,\emptyset) \prec (0,\{s\})$, $(0,\emptyset) \prec (0,\{m\})$, $(0,\emptyset) \prec (0,\{d\})$, $(0,\{s\}) \prec (0,\{s,m\})$, $(0,\{s\}) \prec (0,\{s,d\})$, $(0,\{m\}) \prec (0,\{s,m\})$, $(0,\{m\}) \prec (0,\{m,d\})$, $(0,\{d\}) \prec (0,\{s,d\})$, $(0,\{d\}) \prec (0,\{m,d\})$, $(0,\{s,m\}) \prec (0,\{s,m,d\})$, $(0,\{s,d\}) \prec (0,\{s,m,d\})$, $(0,\{m,d\}) \prec (0,\{s,m,d\})$, and 36 more of this form with 0 replaced successively by 1, 2, and 3, together with 8 statements of each of the forms $(0,C) \prec (1,C)$, $(1,C) \prec (2,C)$, and $(2,C) \prec (3,C)$ where $C \subseteq \{s,m,d\}$. In all, the covering relation has 72 pairs.

32. a) The maximal elements are the ones with no other elements above them, namely l and m.

b) The minimal elements are the ones with no other elements below them, namely a, b, and c.

c) There is no greatest element, since neither l nor m is greater than the other.

d) There is no least element, since neither a nor b is less than the other.

e) We need to find elements from which we can find downward paths to all of a, b, and c. It is clear that k, l, and m are the elements fitting this description.

f) Since k is less than both l and m, it is the least upper bound of a, b, and c.

g) No element is less than both f and h, so there are no lower bounds.

h) Since there are no lower bounds, there can be no greatest lower bound.

34. The reader should draw the Hasse diagram to aid in answering these questions.

a) Clearly the numbers 27, 48, 60, and 72 are maximal, since each divides no number in the list other than itself. All of the other numbers divide 72, however, so they are not maximal.

b) Only 2 and 9 are minimal. Every other element is divisible by either 2 or 9.

c) There is no greatest element, since, for example, there is no number in the set that both 60 and 72 divide.

d) There is no least element, since there is no number in the set that divides both 2 and 9.

e) We need to find numbers in the list that are multiples of both 2 and 9. Clearly 18, 36, and 72 are the numbers we are looking for.

f) Of the numbers we found in the previous part, 18 satisfies the definition of the least upper bound, since it divides the other two upper bounds.

g) We need to find numbers in the list that are divisors of both 60 and 72. Clearly 2, 4, 6, and 12 are the numbers we are looking for.

h) Of the numbers we found in the previous part, 12 satisfies the definition of the greatest lower bound, since the other three lower bounds divide it.

36. a) One example is the natural numbers under "is less than or equal to." Here 1 is the (only) minimal element, and there are no maximal elements.

b) Dual to part (a), the answer is the natural numbers under "is greater than or equal to."

c) Combining the answers for the first two parts, we look at the set of integers under "is less than or equal to." Clearly there are no maximal or minimal elements.

38. Reflexivity is clear from the definition. To show antisymmetry, suppose that $a_1 \dots a_m < b_1 \dots b_n$, and let $t = \min(m,n)$. This means that either $a_1 \dots a_t = b_1 \dots b_t$ and $m < n$, so that $b_1 \dots b_n \not< a_1 \dots a_m$, or else $a_1 \dots a_t < b_1 \dots b_t$, so that $b_1 \dots b_t \not< a_1 \dots a_t$ and hence again $b_1 \dots b_n \not< a_1 \dots a_m$. Finally for transitivity, suppose that $a_1 \dots a_m < b_1 \dots b_n < c_1 \dots c_p$. Let $t = \min(m,n)$, $r = \min(n,p)$, $s = \min(m,p)$, and $l = \min(m,n,p)$. Now if $a_1 \dots a_l < b_1 \dots b_l < c_1 \dots c_l$, then clearly $a_1 \dots a_m < c_1 \dots c_p$. Otherwise, without loss of generality we may assume that $a_1 \dots a_l = b_1 \dots b_l$. If $l = t$, then $m < n$ and $m \le p$. Furthermore, either $b_1 \dots b_r < c_1 \dots c_r$, or $b_1 \dots b_r = c_1 \dots c_r$ and $n < p$. In the former case, if $r > l$, then since $p > m$ we have $a_1 \dots a_m < c_1 \dots c_p$, whereas if $r = l$, then $a_1 \dots a_l < c_1 \dots c_l$. In the latter case, $a_1 \dots a_s = c_1 \dots c_s$ and $m < p$, so again $a_1 \dots a_m < c_1 \dots c_p$. If $l < t$, then we must have $b_1 \dots b_l < c_1 \dots c_l$, whence $a_1 \dots a_l < c_1 \dots c_l$.

40. a) If x and y are both greatest elements, then by definition, $x \preceq y$ and $y \preceq x$, whence $x = y$.

b) This is dual to part (a). If x and y are both least elements, then by definition, $x \preceq y$ and $y \preceq x$, whence $x = y$.

42. a) If x and y are both least upper bounds, then by definition, $x \preceq y$ and $y \preceq x$, whence $x = y$.

b) This is dual to part (a). If x and y are both greatest lower bounds, then by definition, $x \preceq y$ and $y \preceq x$, whence $x = y$.

44. In each case, we need to decide whether every pair of elements has a least upper bound and a greatest lower bound.

a) This is not a lattice, since the elements 6 and 9 have no upper bound (no element in our set is a multiple of both of them).

b) This is a lattice; in fact it is a linear order, since each element in the list divides the next one. The least upper bound of two numbers in the list is the larger, and the greatest lower bound is the smaller.

c) Again, this is a lattice because it is a linear order. The least upper bound of two numbers in the list is the smaller number (since here "greater" really means "less"!), and the greatest lower bound is the larger of the two numbers.

d) This is similar to Example 24, with the roles of subset and superset reversed. Here the g.l.b. of two subsets A and B is $A \cup B$, and their l.u.b. is $A \cap B$.

46. By the duality in the definitions, the greatest lower bound of two elements of S under R is their least upper bound under R^{-1}, and their least upper bound under R is their greatest lower bound under R^{-1}. Therefore, if (S, R) is a lattice (i.e., all the l.u.b.'s and g.l.b.'s exist), then so is (S, R^{-1}).

48. We need to verify the various defining properties of a lattice. First, we need to show that S is a poset under the given $\preceq$ relation. Clearly $(A, C) \preceq (A, C)$, since $A \leq A$ and $C \subseteq C$; thus we have established reflexivity. For antisymmetry, suppose that $(A_1, C_1) \preceq (A_2, C_2)$ and $(A_2, C_2) \preceq (A_1, C_1)$. This means that $A_1 \leq A_2$, $C_1 \subseteq C_2$, $A_2 \leq A_1$, and $C_2 \subseteq C_1$. By the properties of $\leq$ and $\subseteq$ it immediately follows that $A_1 = A_2$ and $C_1 = C_2$, so $(A_1, C_1) = (A_2, C_2)$. Transitivity is proved in a similar way, using the transitivity of $\leq$ and $\subseteq$. Second, we need to show that greatest lower bounds and least upper bounds exist. Suppose that (A_1, C_1) and (A_2, C_2) are two elements of S; we claim that $(\min(A_1, A_2), C_1 \cap C_2)$ is their greatest lower bound. Clearly $\min(A_1, A_2) \leq A_1$ and $\min(A_1, A_2) \leq A_2$; and $C_1 \cap C_2 \subseteq C_1$ and $C_1 \cap C_2 \subseteq C_2$. Therefore $(\min(A_1, A_2), C_1 \cap C_2) \preceq (A_1, C_1)$ and $(\min(A_1, A_2), C_1 \cap C_2) \preceq (A_2, C_2)$, so this is a lower bound. On the other hand, if (A, C) is any lower bound, then $A \leq A_1$, $A \leq A_2$, $C \subseteq C_1$, and $C \subseteq C_2$. It follows from the properties of $\leq$ and $\subseteq$ that $A \leq \min(A_1, A_2)$ and $C \subseteq C_1 \cap C_2$. Therefore $(A, C) \preceq (\min(A_1, A_2), C_1 \cap C_2)$. This means that $(\min(A_1, A_2), C_1 \cap C_2)$ is the greatest lower bound. The proof that $(\max(A_1, A_2), C_1 \cup C_2)$ is the least upper bound is exactly dual to this argument.

50. This issue was already dealt with in our solution to Exercise 44, parts (b) and (c). If $(S, \leq)$ is a total (linear) order, then the least upper bound of two elements is the larger one, and their greatest lower bound is the smaller.

52. By Exercise 50, we can try to choose our examples from among total orders, such as subsets of $\mathbf{Z}$ under $\leq$.
a) $(\mathbf{Z}, \leq)$ **b)** $(\mathbf{Z}^+, \leq)$ **c)** $(\mathbf{Z}^-, \leq)$, where $\mathbf{Z}^-$ is the set of negative integers **d)** $(\{1\}, \leq)$

54. In each case, the issue is whether every nonempty subset contains a least element.

a) The is well-ordered, since the minimum element in any nonempty subset is its smallest element.

b) This is not well-ordered. For example, the set $\{\frac{1}{n} \mid n \in \mathbf{N}\}$ contains no minimum element.

c) Note that $S = \{\frac{1}{2}, 1, \frac{3}{2}, 2, \frac{5}{2}, \ldots\}$. This is well-ordered, since the minimum element in any nonempty subset is its smallest element.

d) This is well-ordered, since it has the same structure as the positive integers under $\leq$, because $x \geq y$ if and only if $-x \leq -y$. Thus the minimum element in any nonempty subset is its largest element.

56. Let x_0 and x_1 be two elements in the dense poset, with $x_0 \prec x_1$ (guaranteed by the conditions stated). By density, there is an element x_2 between x_0 and x_1, i.e., with $x_0 \prec x_2 \prec x_1$. Again by density, there is an element x_3 between x_0 and x_2, i.e., with $x_0 \prec x_3 \prec x_2$. We continue in this manner and have produced an infinite decreasing sequence: $\cdots \prec x_4 \prec x_3 \prec x_2 \prec x_1$. Thus the poset is not well-founded.

58. It is not well-founded because of the infinite decreasing sequence $\cdots \prec aaab \prec aab \prec ab \prec b$. It is not dense, because there is no element between a and aa in this order.

60. This is dual to Lemma 1. We can simply copy the proof, changing every "minimal" to "maximal" and reversing each inequality.

62. Since a larger number can never divide a smaller one, the "is less than or equal to" relation on any set is a compatible total order for the divisibility relation. This gives $1 \prec_t 2 \prec_t 3 \prec_t 6 \prec_t 8 \prec_t 12 \prec_t 24 \prec_t 36$.

64. Clearly g must go in the middle, with any of the six permutations of $\{a,b,c\}$ before g and any of the six permutations of $\{d,e,f\}$ following g. Thus there are 36 compatible total orderings for this poset, such as $a \prec b \prec c \prec g \prec d \prec e \prec f$ and $b \prec a \prec c \prec g \prec f \prec e \prec d$.

66. There are many compatible total orders here. We just need to work from the bottom up. One answer is to take Foundation $\prec$ Framing $\prec$ Roof $\prec$ Exterior siding $\prec$ Wiring $\prec$ Plumbing $\prec$ Flooring $\prec$ Wall $-$ board $\prec$ Exterior painting $\prec$ Interior painting $\prec$ Carpeting $\prec$ Interior fixtures $\prec$ Exterior fixtures $\prec$ Completion.

SUPPLEMENTARY EXERCISES FOR CHAPTER 9

2. In each case we will construct a simplest such relation.

a) $\{(a,a),(b,b),(c,c),(a,b),(b,a),(b,c),(c,b),(d,d)\}$ **b)** $\emptyset$ **c)** $\{(a,b),(b,c)\}$

d) $\{(a,a),(b,b),(c,c),(a,b),(b,a),(c,a),(c,b),(d,d)\}$ **e)** $\{(a,b),(b,a),(c,c),(c,a)\}$

4. Suppose that $R_1 \subseteq R_2$ and that R_2 is antisymmetric. We must show that R_1 is also antisymmetric. Let $(a,b) \in R_1$ and $(b,a) \in R_1$. Since these two pairs are also both in R_2, we know that $a = b$, as desired.

6. Since $(a,a) \in R_1$ and $(a,a) \in R_2$ for all $a \in A$, it follows that $(a,a) \notin R_1 \oplus R_2$ for all $a \in A$.

8. Under this hypothesis, $\overline{R}$ must also be symmetric, for if $(a,b) \in \overline{R}$, then $(a,b) \notin R$, whence (b,a) cannot be in R, either (by the symmetry of R); in other words, (b,a) is also in $\overline{R}$.

10. First suppose that R is reflexive and circular. We need to show that R is symmetric and transitive. Let $(a,b) \in R$. Since also $(b,b) \in R$, it follows by circularity that $(b,a) \in R$; this proves symmetry. Now if $(a,b) \in R$ and $(b,c) \in R$, then by circularity $(c,a) \in R$ and so by symmetry $(a,c) \in R$; thus R is transitive. Conversely, transitivity and symmetry immediately imply circularity, so every equivalence relation is reflexive and circular.

12. A primary key in the first relation need not be a primary key in the join. Let the first relation contain the pairs $(\text{John}, \text{boy})$ and $(\text{Mary}, \text{girl})$; and let the second relation contain the pairs $(\text{boy}, \text{vain})$, $(\text{girl}, \text{athletic})$, and $(\text{girl}, \text{smart})$. Clearly *Name* is a primary key for the first relation. If we take the join on the *Sex* column, then we obtain the relation containing the pairs $(\text{John}, \text{boy}, \text{vain})$, $(\text{Mary}, \text{girl}, \text{athletic})$, and $(\text{Mary}, \text{girl}, \text{smart})$; in this relation *Name* is not a primary key.

14. a) Two mathematicians are related under R^2 if and only if each has written a joint paper with some mathematician c.

b) Two mathematicians are related under R^* if there is a finite sequence of mathematicians $a = c_0$, c_1, c_2, $\ldots$, c_{m-1}, $c_m = b$, with $m \geq 1$, such that for each i from 1 to m, mathematician c_i has written a joint paper with mathematician c_{i-1}.

c) The Erdős number of a is the length of a shortest path in R from a to Erdős, if such a path exists. (Some mathematicians have no Erdős number.)

16. We assume that the notion of calling is a potential one—subroutine $\boldsymbol{P}$ is related to subroutine $\mathbf{Q}$ if it might be possible for $\boldsymbol{P}$ to call $\mathbf{Q}$ during its execution (in other words, there is a call to $\mathbf{Q}$ as one of the steps in the subroutine $\boldsymbol{P}$). Otherwise this exercise would not be well-defined, since *actual* calls are unpredictable—they depend on what actually happens as the programs execute.

a) Let $\boldsymbol{P}$ and $\mathbf{Q}$ be subroutines. Then $\boldsymbol{P}$ is related to $\mathbf{Q}$ under the transitive closure of R if and only if at some time during an active invocation of $\boldsymbol{P}$ it might be possible for $\mathbf{Q}$ to be called.

b) Routines such as this are usually called recursive—it might be possible for $\boldsymbol{P}$ to be called again while it is still active.

c) The reflexive closure of the transitive closure of any relation is just the transitive closure (see part (a)) with all the loops adjoined.

18. We can prove this symbolically, since the symmetric closure of a relation is the union of the relation and its inverse. Thus we have $(R \cup S) \cup (R \cup S)^{-1} = R \cup S \cup R^{-1} \cup S^{-1} = (R \cup R^{-1}) \cup (S \cup S^{-1})$.

20. a) This is an equivalence relation by Exercise 9 in Section 9.5, letting $f(x)$ be the sign of the zodiac under which x was born.

b) This is an equivalence relation by Exercise 9 in Section 9.5, letting $f(x)$ be the year in which x was born.

c) This is not an equivalence relation (it is not transitive).

22. This relation is reflexive, since $x - x = 0 \in \mathbf{Q}$. To see that it is symmetric, suppose that $x - y \in \mathbf{Q}$. Then $y - x = -(x - y)$ is again a rational number. For transitivity, if $x - y \in \mathbf{Q}$ and $y - z \in \mathbf{Q}$, then their sum, namely $x - z$, is also rational (the rational numbers are closed under addition). The equivalence class of 1 and of $1/2$ are both just the set of rational numbers. The equivalence class of π is the set of real numbers that differ from π by a rational number; in other words it is $\{\, \pi + r \mid r \in \mathbf{Q} \,\}$.

24. Let S be the transitive closure of the symmetric closure of the reflexive closure of R. Then by Exercise 23 in Section 9.4, S is symmetric. Since it is also clearly transitive and reflexive, S is an equivalence relation. Furthermore, every element added to R to produce S was forced to be added in order to insure reflexivity, symmetry, or transitivity; therefore S is the smallest equivalence relation containing R.

26. This follows from the fact (Exercise 54 in Section 9.5) that two partitions are related under the refinement relation if and only if their corresponding equivalence relations are related under the $\subseteq$ relation, together with the fact that $\subseteq$ is a partial order on every collection of sets.

28. A subset of a chain is again a chain, so we list only the maximal chains.

a) $\{a, b, c\}$ and $\{a, b, d\}$ **b)** $\{a, b, e\}$, $\{a, b, d\}$, and $\{a, c, d\}$

c) In this case there are 9 maximal chains, each consisting of one element from the top row, the element in the middle, and one element in the bottom row.

30. The vertices are arranged in three columns. Each pair of vertices in the same column are clearly comparable. Therefore the largest antichain can have at most three elements. One such antichain is $\{a, b, c\}$.

32. This result is known as Dilworth's theorem. For a proof, see, for instance, page 58 of *Graph Theory* by Béla Bollobás (Springer-Verlag, 1979).

34. Let x be a minimal element in S. Then the hypothesis $\forall y(y \prec x \to P(y))$ is vacuously true, so the conclusion $P(x)$ is true, which is what we wanted to show.

36. Reflexivity is the statement that f is $O(f)$. This is trivial, by taking $C = 1$ and $k = 1$ in the definition of the big-O relation. Transitivity was proved in Exercise 17 of Section 3.2.

38. It was proved in Exercise 37 that $R \cap R^{-1}$ is an equivalence relation whenever R is a quasi-ordering on a set A. Therefore it makes sense to speak of the equivalence classes of $R \cap R^{-1}$, and the relation S is well-defined from its syntax. To show that S is a partial order, we must show that it is reflexive, anti-symmetric, and transitive. For the first of these, we need to show that (C,C) belongs to S, which means that there are elements $c \in C$ and $d \in C$ such that (c,d) belongs to R. By the definition of equivalence class, C is not empty, so let c be any element of C, and let $d = c$. Then (c,c) belongs to R by the reflexivity of R. Next, for antisymmetry, suppose that (C,D) and (D,C) both belong to S; we must show that $C = D$. We have that (c,d) belongs to R for some $c \in C$ and $d \in D$; and we have that (d',c') belongs to R for some $d' \in D$ and $c' \in C$. If we show that (c,d) also belongs to R^{-1}, then we will know that c and d are in the same equivalence class of $R \cap R'$, and therefore that $C = D$. To do this, we need to show that (d,c) belongs to R. Since d and d' are in the same equivalence class, we know that (d,d') belongs to R; we already mentioned that (d',c') belongs to R; and since c' and c are in the same equivalence class, we know that (c',c) belongs to R. Applying the transitivity of R three times, we conclude that (d,c) belongs to R, as desired.

Finally, to show the transitivity of S, we must show that if (C,D) belongs to S and (D,E) belongs to S, then (C,E) belongs to S. The hypothesis tells us that (c,d) belongs to R for some $c \in C$ and $d \in D$, and that (d',e) belongs to R for some $d' \in D$ and $e \in E$. As in the previous paragraph, we know that (d,d') belongs to R. Therefore by the transitivity of R (thrice), (c,e) belongs to R, and our proof is complete.

40. This follows in essentially one step from part (c) of Exercise 39. Suppose that $x \vee y = y$. Then by the first absorption law, $x = x \wedge (x \vee y) = x \wedge y$. Conversely, if $x \wedge y = x$, then by the second absorption law (with the roles of x and y reversed), $y = y \vee (x \wedge y) = y \vee x$. (We are using the commutative law as well, of course.)

42. By Exercise 51 in Section 9.6, every finite lattice has a least element and a greatest element. These elements are the 0 and 1, respectively, discussed in the preamble to this exercise.

44. We learned in Example 24 of Section 9.6 that the meet and join in this lattice are $\cap$ and $\cup$. We know from Section 2.2 (see Table 1) that these operations are distributive over each other. There is nothing more to prove.

46. Here is one example. The reader should draw the Hasse diagram to see it more vividly. The elements in the lattice are 0, 1, a, b, c, d, and e. The relations are that 0 precedes all other elements; all other elements precede 1; b, d, and e precede c; and b precedes a. Then both d and e are complements of a, but b has no complement (since $b \vee x \neq 1$ unless $x = 1$).

48. This can be proved by playing around with the symbolism. Suppose that a and b are both complements of x. This means that $x \vee a = 1$, $x \wedge a = 0$, $x \vee b = 1$, and $x \wedge b = 0$. Now using the various identities in Exercises 39 and 41 and the preamble to Exercise 43, we have $a = a \wedge 1 = a \wedge (x \vee b) = (a \wedge x) \vee (a \wedge b) = 0 \vee (a \wedge b) = a \wedge b$. By the same argument, we can also show that $b = a \wedge b$. By transitivity of equality, it follows that $a = b$.

50. Actually all finite games have a winning strategy for one player or the other; one can see this by writing down the game tree and analyzing it from the bottom up, as shown in Section 11.2. What we can show in this case is that the player who goes first has a winning strategy. We give a proof by contradiction.

By the remark above, if the first player does not have a winning strategy, then the second player does. In particular, the second player has a winning response and strategy if the first player chooses b as her first move. Suppose that c is the first move of that winning strategy of the second player. But because $c \preceq b$, if the first player makes the move c at her first turn, then play can proceed exactly as if the first player had chosen b and then the second player had chosen c (because element b would be removed anyway when c is chosen). Thus the first player can win by adopting the strategy that the second player would have adopted. This is a contradiction, because it is impossible for both players to have a winning strategy. Therefore we can conclude that our assumption that the first player does not have a winning strategy is wrong, and therefore the first player does have a winning strategy.