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RESEARCH ARTICLE

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Evasion policies for a vessel being chased by pirate skiffs

Yu Wang | Xiangtong Qi

Department of Industrial Engineering and Logistics Management, The Hong Kong University of Science and Technology, Kowloon, Hong Kong

Correspondence
Xiangtong Qi, Department of Industrial Engineering and Logistics Management, The Hong Kong University of Science and Technology, Clear Water Bay, Kowloon, Hong Kong.
Email: ieemqi@ust.hk

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Abstract

Piracy attack is a serious safety problem for maritime transport worldwide. Whilst various strategic actions can be taken, such as rerouting vessels and strengthening navy patrols, this still cannot completely eliminate the possibility of a piracy attack. It is therefore important for a commercial vessel to be equipped with operational solutions in case of piracy attacks. In particular, the choice of a direction for rapidly fleeing is a critical decision for the vessel. In this article, we formulate such a problem as a nonlinear optimal control problem. We consider various policies, such as maintaining a straight direction or making turns, develop algorithms to optimize the policies, and derive conditions under which these policies are effective and safe. Our work can be used as a real-time decision making tool that enables a vessel master to evaluate different scenarios and quickly make decisions.

KEYWORDS
optimal control, maritime shipping, piracy attack, transportation risk management

1 | INTRODUCTION

Piracy has been a constant threat to maritime transport for hundreds of years. Over the past decade, piracy attacks have again hit the news headlines in the media, mainly due to the sudden spread of attacks originating in the Somali area. Following the tremendous efforts put forth by the international community, the situation with Somali piracy seems to be much improved as of today. However, the extent of the piracy risk is now beyond what happened around the Somali area. According to ICC Commercial Crime Services (Table 1), there were 246 reported attacks worldwide in 2015, of which none occurred in the Somali area.

A modern pirate group often operates by using motherships and skiffs (high speed small boats). A mother ship is a large slow ship which carries the pirates, food, and fuel, and enables the pirates to extend their territory to a much larger area than before. Attack skiffs are towed behind the mother ships. Once a commercial vessel is identified as a target, the pirates use one or two skiffs to approach the vessel at a high speed. The chase may result in gun fire when the skiffs are close enough to the vessel, and pirates will start climbing up the vessel if the skiff manages to catch up with it. Due to the limited fuel supply of a skiff, the pirates will stop chasing after a certain time if they cannot get close to the vessel. The chase may last 1–2 hours (IMB, 2017; MarEx, 2017).

TABLE 1 Number of actual and attempted piracy attacks, 2010–2015

Locations	2010	2011	2012	2013	2014	2015
South-East Asia	70	80	104	128	141	147
Indian Subcontinent	23	10	11	12	21	11
Americas	40	25	17	18	4	8
Somalia/Gulf of Aden	219	236	75	15	11	0
Nigeria	19	10	27	31	18	14
Other Africa	17	47	48	32	26	21
Rest of World	57	31	15	28	24	45
Total	445	439	297	264	245	246

(source: https://www.icc-ccs.org).

To address the issue of piracy attacks, the Maritime Safety Committee of the IMO made a series of recommendations to ocean carriers in their Best Management Practices (BMP, 2011). According to the BMP, carriers should consider preventive actions such as avoiding certain high-risk areas, joining group transit schemes with a military or independent convoy, strengthening ship protection measures such as making it difficult for the ship to be climbed, and so on. In practice, however, preventive actions cannot completely eliminate the risk of being attacked when a commercial vessel is sailing on the open sea. If a pirate skiff is found to be approaching in real time, a commercial vessel can only run away, as suggested in the BMP:

> *"One of the most effective ways to defeat a pirate attack is by using speed to try to outrun the attackers and/or make it difficult to board"* and *"try to steer a straight course to maintain a maximum speed."*

Intuitively, running away at maximum speed seems to be a straightforward action to take, so the solution appears rather simple. However, there are indeed other factors to consider, for example, the direction in which the vessel sails. Note that each vessel has a planned route to its destination. It would be ideal if the vessel could evade the chase by speeding along the planned route, a policy referred to hereafter as a direct heading. Unfortunately, a direct heading is not always safe or feasible, depending on the positions of the pirate skiffs. Therefore, the vessel needs to consider changing its sailing direction (by altering its course), causing the vessel to deviate from its planned route. The most conservative choice is changing direction to exactly opposite that of the chasing skiffs, but that may deviate significantly from the planned route. After hours of evasion at maximum speed, the vessel may end up at a safe place, but dozens of nautical miles away from the planned route, which will then cost the vessel additional fuel to return to the planned route. However, it may not be necessary for the vessel to take such conservative action. There should be a direction that keeps the vessel safe and at the same time keeps the vessel as close as possible to the planned route.

In this article, we aim to study such an evasion problem for a commercial vessel that is being chased by pirate skiffs. The purpose is to design a cost-effective and safe strategy by optimizing the steering direction of the vessel. We highlight our main results and contribution as follows.

We properly formulate the problem by identifying the key factors in the chase process. We first present a dynamic differential game model with a pirate skiff as pursuer and the commercial vessel as evader. Assuming that the pirate skiff takes a greedy policy referred to as the pure pursuit guidance law, we then transform the game into an optimal control problem for the vessel.

We focus on three practical policies for the vessel, namely, a direct heading, making one turn, and making two turns. Please note that such policies are consistent with the BMP recommendation because making too many turns will reduce the speed of the vessel. For each policy, we are able to derive a few structural properties, and then develop algorithms to efficiently compute the optimal decisions.

We conduct extensive simulation to validate our policies. The results show that our policies can lead to safe and cost-effective decisions for commercial vessels. Our model can generate a set of Pareto optimal solutions, making it possible for the vessel to evaluate and make decisions by considering different scenarios.

The rest of the article is organized as follows. In Section 2, we review related literature. In Section 3, we introduce our model and the problem formulation. In Sections 4 and 5, we study different policies when there is a single skiff chasing, and then we discuss in Section 6 how to cope with the case of multiple skiffs chasing. The conclusion of the article is in Section 7.

2 | LITERATURE REVIEW

Fighting the piracy attack is an old problem that has been discussed in different areas. For example, Onuoha (2009) and Percy and Shortland (2013) study the activities of Somali piracy. Fu et al. (2010) address the impact of piracy on the global economy. Martínez-Zarzoso and Bensassi (2013) study how the piracy risk may cause extra ocean transportation costs, and Helmick (2015) discusses how the global supply chain may be disrupted. An extended literature review on maritime safety and security can be found in Lee and Song (2016). The work on quantitative operations models of the battle with piracy now attracts more attention, but the results are still scarce. Vaněk et al. (2013) use simulation to evaluate the risk level of a specific water area, Varol and Gunal (2015) use simulation to compare a variety of prevention operations, and Vaněk et al. (2014) study how to improve the group transit schemes. All such work belongs to proactive actions which are important to take in advance. However, proactive actions also mean a huge cost to the supply chain. Our work is different in that we focus on reactive actions to be taken in real time.

From a broader point of view, the pirate chasing problem that we study here belongs to the general pursuit/evasion problem that has been studied in other situations, especially in military applications such as the cruise missile attack. The related research can be classified into three categories, the pursuit problem, evasion problem, and the pursuit-evasion game. In what follows, we give an overview of such problems. We will explain the uniqueness of our problem in the next section after we present the details of our problem.

The pursuit problem studies the pursuer's strategy to chase or intercept a moving target. There are three classical guidance laws applicable for planar motion control, line of sight (LOS) law, pure pursuit (PP) law, and proportional navigation

(PN) law. We refer to Breivik et al. (2008) for details of them. Roughly speaking, under the principal of LOS law, for example, Shneydor (1998), the control is to guide the pursuer on a LOS course of a fixed ground station and the evader; under the PP law, for example, Shneydor (1998) and Yamasaki et al. (2009), the pursuer always aligns its velocity along the LOS angle between the evader and itself; under the PN law, for example, Shneydor (1998), Cliff and Ben-Asher (1989), Imado and Miwa (1986), and Imado and Miwa (1989), the pursuer selects the rotation rate of its velocity directly proportional to the rotation rate of the LOS angle between the pursuer and the evader. Among the three, the PN law has the best performance in missile attacking, especially for high-speed targets. For low-speed targets, the PP law also performs well and guarantees that an intercept will occur if the pursuer has a higher speed than the evader and the chasing time is long enough.

Modern guidance laws are mainly based upon the optimal control theory, including the improved PN laws, predictive guidance and even the differential game. However, it is quite computationally intensive to implement these guidance laws. As the first step to study the pirate-chasing problem, we assume that the pirate adopts the PP law in this article.

The evasion problem, on the other hand, focuses on the evader's strategy to get rid of the chasing, for example, Cliff and Ben-Asher (1989), Karelahti et al. (2007), Shinar and Steinberg (1977) and Nahin (2012). Under a given strategy of the pursuer, the goal of the evader is to find the safest strategy by maximizing the capture time or to find an effective strategy with the least fuel cost. There are two approaches in the literature, direct method and indirect method. The direct method is to reformulate the problem as a nonlinear program by the discretization method, and then solve the nonlinear program directly, for example, Karelahti et al. (2007), Ehtamo and Raivio (2001) and Raivio and Ehtamo (2000). The indirect method focuses on identifying optimality conditions for the problem, for example, Cliff and Ben-Asher (1989), Shinar and Steinberg (1977). Then algorithms can be developed based on optimality conditions. In this article, we follow the indirect approach where we can obtain not only some algorithms to solve the problem, but also the certain general guidance that will be revealed by the optimality conditions.

The pursuit-evasion game studies the problem by simultaneously considering the pursuer and the evader's strategies, for example, Chernousko and Zak (1985), Ehtamo and Raivio (2001), Raivio and Ehtamo (2000), Lewin and Olsder (1979), and Lachner (1997). Due to the complexity of the problem, such games are usually solved computationally. Our problem can also be regarded as a game, where the evader has an objective function that has not been investigated in the literature.

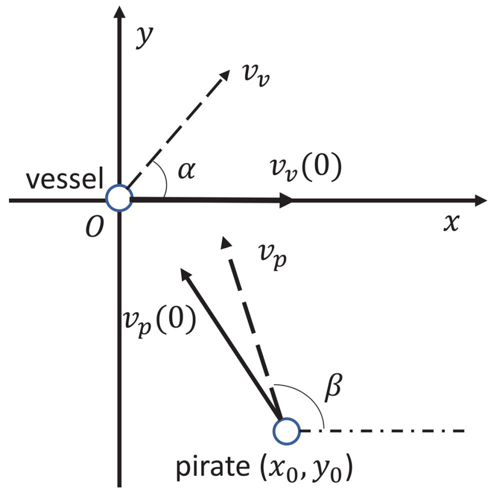

FIGURE 1 The initial situation [Color figure can be viewed at wileyonlinelibrary.com]

3 | PROBLEM FORMULATION

We consider a pirate chasing situation, where a commercial vessel finds itself being chased by a pirate skiff at time zero. Consequently, the commercial vessel needs to decide its sailing policy, including its sailing direction and speed such that it can escape from the chasing; at the same time, the pirate skiff wants to catch the commercial vessel as early as possible. Hereafter we will use the vessel to refer to the commercial vessel, and the pirate to refer to the pirate skiff.

We can use a Cartesian coordinate system to describe the problem. As depicted in Figure 1, the initial position of the vessel is at the origin $(0, 0)$, and the initial sailing direction is toward the x -axis. Without loss of generality, assume that the initial position (x_0, y_0) of the pirate skiff is below or on the x -axis, that is, $y_0 \leq 0$. At any time t, let $(x_v(t), y_v(t))$ and $(x_p(t), y_p(t))$ denote the positions of the vessel and the pirate, respectively. Let $v_v(t)$ denote the vessel's speed and $\alpha(t)$, an angle formed with the x-axis, denote its sailing direction. Similarly, let $v_p(t)$ and $\beta(t)$ denote the sailing speed and direction of the pirate, respectively.

The dynamic process of the system can be described by the following state equations,

$$\begin{cases} \dfrac{dx_v(t)}{dt} = v_v(t)\cos\alpha(t), \\ \dfrac{dy_v(t)}{dt} = v_v(t)\sin\alpha(t), \\ \dfrac{dx_p(t)}{dt} = v_p(t)\cos\beta(t), \\ \dfrac{dy_p(t)}{dt} = v_p(t)\sin\beta(t), \end{cases} \tag{1}$$

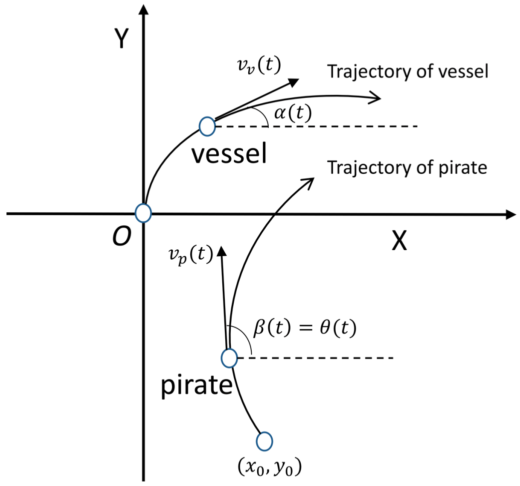

FIGURE 2 Dynamic process under PP law [Color figure can be viewed at wileyonlinelibrary.com]

with the initial condition $(x_v(0), y_v(0)) = (0,0), (x_p(0), y_p(0)) = (x_0, y_0)$.

The above equations define a general pursuit-evasion game where both the vessel and the pirate have to determine their policies. In our problem, one key concern is the distance between them. Let $r(t) = \sqrt{(x_v(t) - x_p(t))^2 + (y_v(t) - y_p(t))^2}$ denote their distance at time t, and $r_0 = \sqrt{x_0^2 + y_0^2}$ denote the initial distance. The following assumptions streamline the definition of our problem.

Assumption 1 The vessel is safe if and only if $r(t) \geq R$ for $t \in [0, T]$, where R and T are given positive constants.

The first constant R indicates a minimum safety distance. For example, R can be set as the longest distance at which the pirate may make an effective shooting. As long as the vessel keeps away from the pirate at a distance no less than R, the vessel can be assumed to be safe. To avoid the trivial case, we assume $r_0 \geq R$. The second constant T denotes the maximum time before which the pirate gives up chasing the vessel due to the limited fuel on the skiff. For example, it is reported that the pirates will chase for up to two hours in many cases (IMB, 2017; MarEx, 2017).

We first consider how the pirate may determine $\beta(t)$, the sailing direction of the pirate. We assume that the pirate will always approach directly toward the vessel, which is known as the PP guidance law. Specifically, at any time t, we use $\theta(t)$ to denote the angle of the pirate's LOS toward the vessel with respect to the x -axis, as shown in Figure 2. Then the pirate will set the sailing direction $\beta(t)$ as $\theta(t)$. Note that, by definition, for $r(t) > 0$ we have

$$\cos\theta(t) = \frac{x_v(t) - x_p(t)}{r(t)} \quad \text{and}$$
$$\sin\theta(t) = \frac{y_v(t) - y_p(t)}{r(t)}. \tag{2}$$

Assumption 2 The pirate will take the pure pursuit guidance law in which $\beta(t) = \theta(t)$ for $t \in [0, T]$.

Assuming that the pirate wants to catch the vessel as early as possible, then we can partially justify the PP guidance law by the following lemma. The lemma implies that taking the pure pursuit guidance law is a locally optimal policy for the pirate because it leads to the steepest descent of $r(t)$ at any time t. Note that (2) is given under the condition $r(t) > 0$, which is the meaningful case for our problem. In what follows, our analysis is also applied to the case of $r(t) > 0$ unless otherwise specified, though we will denote the time range as $t \in [0, T]$. For simplicity, we will not state the condition $r(t) > 0$ again.

Lemma 1 *At any time t, $\frac{dr(t)}{dt}$ is minimized by $\beta(t) = \theta(t)$.*

All proofs can be found in Appendix. Next, we consider the speed decisions of both the pirate and the vessel.

Assumption 3 The speeds of both the vessel and the pirate are constant, i.e., $v_v(t) \equiv v_v$, $v_p(t) \equiv v_p$, for $t \in [0, T]$.

This assumption is consistent with some practical evidence. In particular, BMP suggests that a vessel steers a straight course to maintain a maximum speed. At the same time, the pirate also has a reasonable incentive to use the maximum speed in order to catch the vessel as soon as possible, for example, before any rescue team arrives. Thus we consider the common case where both the pirate and vessel take their respective maximum speed. Actually our analysis can also be used for the vessel to evaluate the option of taking a slower speed as long as the speed does not vary over time.

For notational convenience, let $\gamma = \frac{v_p}{v_v}$ denote the ratio of speeds between the pirate and the vessel. Note that $\gamma > 1$ means that the pirate has a higher speed over the vessel, the more often case resulting in a successful hijack. When $\gamma = 1$, that is, the pirate and vessel have the same speed, we then use v to denote the speed.

Given the above assumptions, we can simplify the system formulation (1) by replacing $v_v(t)$ by v_v, $v_p(t)$ by v_p, and $\beta(t)$ by $\theta(t)$. In addition, for the position of the pirate, we use $r(t)$ and $\theta(t)$ to characterize its relative position to the vessel, thus eliminating the notion of $x_p(t)$ and $y_p(t)$. Now consider the dynamic process of $\theta(t)$. Differentiating on both sides of (2), we have

$$-\sin\theta(t)\frac{d\theta(t)}{dt} = \frac{\frac{dx_v(t)}{dt} - \frac{dx_p(t)}{dt}}{r(t)} - \frac{(x_v(t) - x_p(t))\frac{dr(t)}{dt}}{r^2(t)}.$$

Substituting (1) and (2) into the above equation and using the result from Lemma 1, we have

$$\frac{d\theta(t)}{dt} = \frac{v_v \sin(\alpha(t) - \theta(t))}{r(t)}.$$

Thereafter, the dynamic process of the pirate can be reformulated as follows,

$$\begin{cases} \dfrac{dr(t)}{dt} = v_v(\cos(\alpha(t) - \theta(t)) - \gamma), \\ \dfrac{d\theta(t)}{dt} = v_v \dfrac{\sin(\alpha(t) - \theta(t))}{r(t)}, \end{cases} \tag{3}$$

with the safety constraint:

$$r(t) \geq R, \forall t \in [0, T]. \tag{4}$$

In the new formulation, the only decision is $\alpha(t)$, the sailing direction of the vessel. The problem becomes an optimal control problem where a feasible control policy $\alpha(t), t \in [0, T]$ is needed to optimize some objective function.

While any policy $\alpha(t)$ satisfying (3) and (4) is feasible, the vessel may choose one that is also effective with respect to cost. For example, the vessel may simply choose $\alpha(t) = \theta(t)$, sailing just opposite to the pirate, but that may lead the vessel to a position very far away from the original route, and after the chasing, the vessel needs to sail an additional voyage to return. A more reasonable policy for the vessel should be the one ending up at the position close to the original route. To this end, we consider the end position $(x_v(T), y_v(T))$, where $x_v(T)$ measures the movement along the original direction, and $y_v(T)$ gives the deviated distance. In general, we hope $x_v(T)$ to be as large as possible, and $y_v(T)$ to be close to zero as much as possible.

Definition 1 For any two feasible policies $\alpha(t)$ and $\alpha'(t)$ on $t \in [0, T]$, we say $\alpha(t)$ dominates $\alpha'(t)$ if and only if one of the following two statements is true:

1. $x_v(T) > x'_v(T)$ for the case of $x'_v(T) < 0$, or
2. $x_v(T) \geq x'_v(T)$ and $|y_v(T)| \leq |y'_v(T)|$ for the case of $x'_v(T) \geq 0$, and at least one inequality is not tight.

The meaning of dominance can be explained as follows. The common condition $x_v(T) > x'_v(T)$ implies that under $\alpha(t)$ the vessel moves a longer distance along the planned direction, which is preferred. For the case $x'_v(T) > 0$, we may also prefer a smaller deviation from the planned direction, so we add another condition on $y_v(T)$. For the case $x'_v(T) < 0$, it is reasonable to give priority to shorter backward sailing, so the condition on $y_v(T)$ is not added. By definition, a vessel does not need to consider a policy $\alpha'(t)$ if it is dominated by another policy $\alpha(t)$.

Definition 2 A feasible control policy $\alpha(t)$ is Pareto optimal if and only if $\alpha(t)$ is not dominated by any other feasible control policy. If all Pareto optimal controls have the same end position $(x_v(T), y_v(T))$, then they are globally optimal.

Any Pareto optimal policy has certain advantages against another one, with respect to either sailing a longer distance along the x -axis or having a smaller deviation along the y -axis. In this article, our goal is to characterize the frontier of the set of Pareto optimal policies. This enables the vessel master to understand and evaluate all possible choices for making a decision.

The policies that we will study can be described as follows. Given a maximum chasing time T, we introduce k decision points $0 = t_1 < t_2 < \cdots < t_k < T$. Also denote $t_{k+1} = T$. At each decision point t_i, for $i = 1, 2, \cdots, k$, we decide a sailing direction, α_i, for the vessel during time interval $[t_i, t_{i+1}]$. The goal is to find a set of Pareto optimal controls characterized by the turn time points $(t_1, t_2, \cdots, t_k)$ and the corresponding $(\alpha_1, \alpha_2, \cdots, \alpha_k)$.

Theoretically speaking, the vessel can take the above k -turn policy for any non-negative k. However, making too many turns is not practical because the vessel may sacrifice certain time and speed to make a turn. So it is reasonable for the vessel to consider making as few turns as possible. In fact, BMP suggests a vessel "try to steer a straight course". In what follows, we will investigate three cases in detail, not making turns, making one turn (Figure 3) and making two turns (Figure 4). The case of making more turns can be analyzed in a similar approach as making two turns.

It is obvious that the policy of not making turns, referred to as direct heading, can be regarded as making one turn with $\alpha_1 = 0$. So direct heading is actually a special case of the one-turn policy. Similarly, the one-turn policy can be regarded as a special case of two-turn policy. We will study the direct heading policy first. The results of direct heading will be used to handle one-turn and two-turn policies.

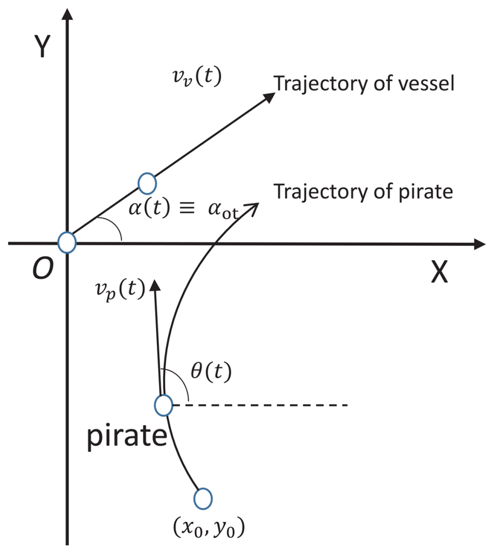

FIGURE 3 Dynamic process under one-turn policy [Color figure can be viewed at wileyonlinelibrary.com]

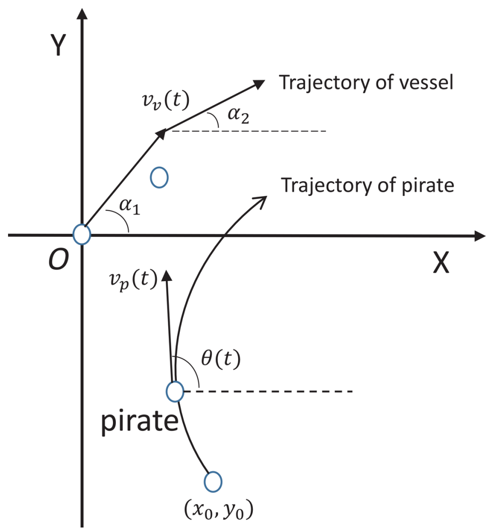

FIGURE 4 Dynamic process under two-turn policy [Color figure can be viewed at wileyonlinelibrary.com]

4 | DIRECT HEADING

In direct heading policy, we have $\alpha(t) \equiv 0$ for $t \in [0, T]$, leading to $(x_v(T), y_v(T)) = (v_vT, 0)$. This means that direct heading will be the unique globally optimal policy if it satisfies (3) and (4) because $x_v(T)$ achieves its maximum value and $|y_v(T)| = 0$ is at its minimum value. So the issue related to direct heading is testing its feasibility, especially its dependence on the initial position of the pirate (r_0, θ_0).

4.1 | Feasibility test for direct heading

The feasibility test is to check if $r(t) \geq R$, $\forall t \in [0, T]$. We first discuss two special cases with respect to θ_0, $\theta_0 = 0$, and $\theta_0 = \pi$. For the case of $\theta_0 = 0$, that is, the pirate appears directly behind the vessel, the pirate will also keep its direction because $\theta(t) = 0$. So direct heading is feasible as long as $\gamma \leq 1$, or $\gamma > 1$ but $(v_p - v_v)T \leq r_0 - R$. For the case of $\theta_0 = \pi$, that is, the pirate appears directly in front of the vessel, direct heading means that the vessel sails directly toward to the pirate, which is not a reasonable choice at all. In the following analysis, we will exclude these two special cases and assume $\theta_0 \in (0, \pi)$, that is, the pirate does not appear exactly on the vessel's sailing direction.

Under the direct heading policy where $\alpha(t) \equiv 0$, the processes of $r(t)$ and $\theta(t)$, before $r(t) = 0$, can be simplified into

$$\frac{dr(t)}{dt} = v_v(\cos\theta(t) - \gamma), \tag{5}$$

$$\frac{d\theta(t)}{dt} = -v_v\frac{\sin\theta(t)}{r(t)}. \tag{6}$$

An analytical solution of $r(t)$, as a function of $\theta(t)$, is given in Shneydor (1998). Specially, when $\theta(t)$ is known to be in $(0, \pi)$, we have

$$r(t) = C_0\frac{\tan^\gamma\frac{\theta(t)}{2}}{\sin\theta(t)}, \tag{7}$$

and a revised version of the implicit solution of $\theta(t)$ is

$$-\frac{1}{2(1+\cos\theta(t))} + \frac{1}{4}\ln\left(\frac{1+\cos\theta(t)}{1-\cos\theta(t)}\right) = \frac{v}{C_0}t + C_1, \text{ if } \gamma = 1, \tag{8}$$

$$\frac{1}{2(\gamma-1)}\tan^{\gamma-1}\frac{\theta(t)}{2} + \frac{1}{2(\gamma+1)}\tan^{\gamma+1}\frac{\theta(t)}{2} = -\frac{v_v}{C_0}t + C_2, \text{ if } \gamma \neq 1, \tag{9}$$

where $C_0 = \frac{r_0\sin\theta_0}{\tan^\gamma\frac{\theta_0}{2}}$, $C_1 = -\frac{1}{2(1+\cos\theta_0)} + \frac{1}{4}\ln\left(\frac{1+\cos\theta_0}{1-\cos\theta_0}\right)$, and $C_2 = \frac{1}{2(\gamma-1)}\tan^{\gamma-1}\frac{\theta_0}{2} + \frac{1}{2(\gamma+1)}\tan^{\gamma+1}\frac{\theta_0}{2}$ are three parameters depending on the initial position of the pirate (r_0, θ_0) and the speed ratio γ of the two objects.

To make the solution (7)-(9) valid to our problem, we need to guarantee $\theta(t) \in (0, \pi)$ before $r(t) = 0$. From (6), we can see that $\theta(t)$ is non-increasing on t for $\theta(t) \in [0, \pi]$ because $\frac{d\theta(t)}{dt} \leq 0$. This means that, under the initial condition $\theta_0 \in (0, \pi)$, $\theta(t)$ is strictly decreasing until possibly at a certain time $t = \tau$, we have $\theta(\tau) = 0$, and then $\theta(t) = 0$ for $t > \tau$. The next lemma discusses the existence of such a τ.

Lemma 2 *If $\gamma > 1$, there exists a time $\tau = C_0C_2/v_v$ such that $\theta(\tau) = 0$, $\theta(t)$ is strictly* *decreasing on t and $\theta(t) > 0$ for $t \in [0, \tau)$, and furthermore, we have $r(t) > 0$ for $t \in [0, \tau)$ and $r(\tau) = 0$. If $\gamma \leq 1$, $\theta(t)$ is strictly decreasing on t and $\theta(t) > 0$ for all $t \in [0, +\infty)$, i.e., such a time τ does not exist.*

Lemma 2 implies that the solution given in (7)-(9) fully characterizes the process that we are interested in. Based on that, we can calculate $r(t)$ and $\theta(t)$ for a given time t. Note that there is no closed-form solution of $\theta(t)$ from (8) or (9), but it is not difficult to solve it computationally. In fact, if we treat $\theta(t)$ as the unknown variable in the left-hand-side (LHS) of (8) or (9), then it is easy to show that the LHS of (8) or (9) will be monotone on $\theta(t)$. As a result, we can use a bisection search or Newton's method to find a $\theta(t)$ that solves (8) or (9). To summarize, we have Algorithm 1 to calculate the pirate's relative position when the vessel takes the direct heading policy.

Although we can calculate $r(t)$ for any given t, it is still not clear how to check the feasibility of direct heading because we need to ensure $r(t) \geq R$ for $t \in [0, T]$. To do the feasibility test efficiently, we introduce a new term, the capture time.

Algorithm 1 Calculating $r(t)$ and $\theta(t)$ for direct heading

Input: (r_0, θ_0, γ) and t, Output: $r(t), \theta(t)$
If $\gamma > 1$ and $t \geq C_0 C_2 / v_v$,
 the vessel has been caught before t. Stop.
Solve (8) or (9) with $\theta(t)$ as the unknown variable.
Calculate $r(t)$ from (7).

Definition 3 The capture time, denoted by T_c, is the earliest time when the distance $r(t)$ between the vessel and the pirate decreases to R, that is, $r(t) > R$ for $t \in [0, T_c)$ and $r(T_c) = R$.

Note that T_c may or may not exist. If T_c does not exist, the vessel will always be safe regardless of the chasing time limit T. If T_c exists and $T_c > T$, the vessel is still safe before T. So the feasibility test can be done by checking the existence of T_c and calculating its value if existing. By definition, if T_c exists, it should satisfy $r(T_c) = R$ together with (8) for the case of $\gamma = 1$, and satisfy $r(T_c) = R$ together with (9) for the case of $\gamma \neq 1$.

We first consider the case of $\gamma = 1$, that is, the pirate and the vessel have the same speed. In this case, a closed-form solution is available. It gives the necessary and sufficient condition for the direct heading policy to be safe to the vessel.

Proposition 1 *Consider the case of $\gamma = 1$, if $r_0(1 + \cos\theta_0) \geq 2R$, the capture time T_c does not exist and direct heading is feasible. If $r_0(1 + \cos\theta_0) < 2R$, the capture time*

$$T_c = \frac{r_0 - R}{2v_v} + \frac{r_0(1 + \cos\theta_0)}{4v_v} \ln \frac{r_0(1 - \cos\theta_0)}{2R - r_0(1 + \cos\theta_0)},$$

and direct heading is feasible if $T_c \geq T$.

Next, we consider the case of $\gamma \neq 1$. In this case we cannot obtain a closed-form solution for the capture time T_c. So we need to computationally find T_c, which is a solution of the following nonlinear equations of t.

$$\begin{cases} C_0 \dfrac{\tan^\gamma \frac{\theta(t)}{2}}{\sin\theta(t)} = R, \\ \dfrac{1}{2(\gamma-1)} \tan^{\gamma-1} \dfrac{\theta(t)}{2} + \dfrac{1}{2(\gamma+1)} \tan^{\gamma+1} \dfrac{\theta(t)}{2} = -\dfrac{v_v}{C_0} t + C_2. \end{cases} \tag{10}$$

Before presenting our algorithm to solve (10), we first discuss the existence, uniqueness, and possible range of the solution. The case is relatively simple if the pirate skiff has a higher speed, that is, $\gamma > 1$, where a finite upper bound of T_c can be given.

Proposition 2 *When $\gamma > 1$, (10) has a unique solution T_c and $T_c \in (0, \frac{r_0(\gamma + \cos\theta_0)}{v_v(\gamma^2 - 1)})$.*

The proof of Proposition 2 relies on a key fact that the distance function $r(t)$ is decreasing on t until $r(t) = 0$. When $\gamma < 1$, that is, the pirate has a lower speed, the distance function $r(t)$ may not be monotone on t. It may be decreasing first, then become increasing. So the situation is more complicated. To characterize the shape of $r(t)$, we define three critical parameters that can be calculated directly for any given r_0, θ_0, and γ.

$$\overline{\theta}(\gamma) \triangleq \arccos\gamma, \text{ where } \overline{\theta}(\gamma) \in (0, \pi/2);$$

$$\bar{r}(\gamma) = C_0 \frac{\tan^\gamma \frac{\overline{\theta}(\gamma)}{2}}{\sin\overline{\theta}(\gamma)}, \ \bar{t}(\gamma) = \frac{r_0(\gamma + \cos\theta_0) - 2\gamma\bar{r}(\gamma)}{v_v(\gamma^2 - 1)}.$$

The first parameter $\overline{\theta}(\gamma)$ gives a threshold value on θ_0 for the existence of T_c. Specifically, when $\theta_0 \leq \overline{\theta}(\gamma)$, T_c does not exist for any $r_0 > R$. The second parameter $\bar{r}(\gamma)$ is the minimum value of the distance function $r(t)$, giving a sufficient and necessary condition for the existence of T_c when $\theta_0 > \overline{\theta}(\gamma)$. The third parameter $\bar{t}(\gamma)$ is an upper bound of T_c when T_c does exist. In fact, $r(\bar{t}(\gamma)) = \bar{r}(\gamma)$. These are formally given in the following proposition.

Algorithm 2 Calculating the capture time T_c for direct heading

Input: (r_0, θ_0, γ), Output: T_c

Step 1. If $\gamma = 1$, calculate T_c from Proposition 1. Stop.

Step 2. If $\gamma < 1$ and one of the follows is true

$\theta_0 \leq \overline{\theta}(\gamma)$, or

$\theta_0 > \overline{\theta}(\gamma)$ and $\bar{r}(\gamma) > R$

T_c does not exist. Stop.

Step 3. Solve $C_0 \frac{\tan^{\gamma} \frac{\theta(t)}{2}}{\sin\theta(t)} = R$ with respect to $\theta(t)$,

for $\theta(t) \in (0, \theta_0)$ when $\gamma > 1$, and

for $\theta(t) \in [\overline{\theta}(\gamma), \theta_0)$ when $\gamma < 1$.

Step 4. Given the solution $\theta(t)$ found in Step 3, calculate t from (9). Then $T_c = t$.

Proposition 3 *When $\gamma < 1$,*

1) *$\bar{r}(\gamma)$ is the globally minimum value of $r(t)$,*
2) *if $\theta_0 \leq \overline{\theta}(\gamma)$, or $\theta_0 > \overline{\theta}(\gamma)$ but $\bar{r}(\gamma) > R$, T_c does not exist, and*
3) *if $\theta_0 \leq \overline{\theta}(\gamma)$ and $\bar{r}(\gamma) \leq R$, T_c exists in $(0, \bar{t}(\gamma)]$.*

From Propositions 2 and 3, we know that, when $\gamma > 1$, T_c is the unique solution to (10) within the interval $t \in (0, \frac{r_0(\gamma+\cos\theta_0)}{v_v(\gamma^2-1)})$, and when $\gamma < 1$, T_c, if existing, is the unique solution to (10) within the interval $t \in (0, \bar{t}(\gamma)]$.

We note that (10) can be solved in a similar way to Algorithm 1. Specifically, we can first treat $\theta(t)$ as the unknown variable, from which we can obtain t. Details are given in Algorithm 2.

Step 3 in Algorithm 2 is to solve a standard unconstrained nonlinear problem, which can be done either by a bisection search or Newton's method within the specified interval. Converging to the unique solution is guaranteed by Propositions 2 and 3.

4.2 | Infeasible region

Given any initial position of the pirate, (r_0, θ_0), we are able to use Algorithm 2 to check the feasibility of direct heading. Based on that, we can characterize the entire set of initial positions under which direct heading is infeasible, where we refer to the set as *Infeasible Region*. It is helpful to construct the Infeasible Region in advance. For example, the vessel should strengthen the surveillance over the Infeasible Region. In addition, when a pirate skiff is found with an uncertain speed, the vessel can evaluate a number of scenarios by checking the Infeasible Regions with different parameters, which enables the vessel to make a decision in real time.

Algorithm 3 Finding the border of Infeasible Region for a given direction θ_0.

Input: (θ_0, γ), Output: r_0^*.

Initialization: $\theta(T)_u = \theta_0$, $\theta(T)_l = 0$.

Repeat the following steps.

Let $\theta(T) = \frac{\theta(T)_l + \theta(T)_u}{2}$, and

if $\gamma = 1$, solve (8) with r_0 as unknown;

if $\gamma \neq 1$, solve (9) with r_0 as unknown.

With the solved r_0, calculate $r(T, r_0, \theta_0)$

When $\gamma \geq 1$, $r_c = r(T, r_0, \theta_0))$;

When $\gamma < 1$, calculate $\bar{t}(\gamma)$ and $\bar{r}(\gamma, r_0, \theta_0)$. $r_c = \begin{cases} r(T, r_0, \theta_0) & \text{if } \bar{t}(\gamma) > T \\ \bar{r}(\gamma, r_0, \theta_0) & \text{if } \bar{t}(\gamma) \leq T \end{cases}$;

If $r_c = R$, stop; otherwise

if $r_c > R$, let $\theta(T)_u = \theta(T)_k$;

if $r_c < R$, let $\theta(T)_l = \theta(T)_k$.

We first study how the Infeasible Region depends on the initial position (r_0, θ_0). To this end, we slightly modify the notation $r(t)$ to $r(t, r_0, \theta_0)$, and $\bar{r}(\gamma)$ to $\bar{r}(\gamma, r_0, \theta_0)$, in order to explicitly show their dependence on (r_0, θ_0). From the previous analysis, we know that the Infeasible Region is given by $\{(r_0, \theta_0) : r(T, r_0, \theta_0) < R\}$ when $\gamma \geq 1$, because $r(t)$ is decreasing on t. When $\gamma < 1$, if $T < \bar{t}(\gamma)$, $r(t)$ is decreasing on t, so the Infeasible Region is given by $\{(r_0, \theta_0) : r(T, r_0, \theta_0) < R\}$; if $T \geq \bar{t}(\gamma)$, $r(t)$ is not monotone and its minimum value is $\bar{r}(\gamma, r_0, \theta_0)$, so the Infeasible Region is given by $\{(r_0, \theta_0) : \bar{r}(\gamma, r_0, \theta_0) < R\}$.

The following proposition shows that the range of Infeasible Region has certain monotonicity in terms of the pirate's initial position.

Proposition 4 *For any given chasing time T and speed ratio γ, if direct heading is feasible with respect to an initial position of pirate, (r_0, θ_0), then direct heading is feasible with respect to any initial position (r, θ_0) where $r \geq r_0$ and any initial position (r_0, θ) where $\theta \leq \theta_0$.*

Proposition 4 shows that when the pirate is relatively behind the vessel's sailing direction for a given distance r_0 or is far away from the vessel along a given LOS angle θ_0, the vessel is more likely to be safe by simply sailing straightforward. Although this conclusion is consistent with intuition, the justification is not simple. One may assume that we only need to prove the monotonicity of the distance function $r(T, r_0, \theta_0)$. However, when $\gamma < 1$, there exist cases where $r(T, r_0, \theta_0)$ is decreasing on r_0; in such a case the proposition still holds because it is the minimum distance function $\bar{r}(\gamma)$ that matters, rather than $r(T, r_0, \theta_0)$.

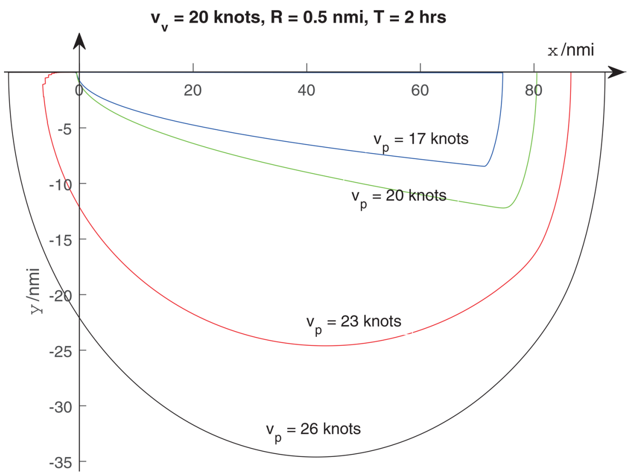

FIGURE 5 Infeasible Regions for direct heading under fixed v_v [Color figure can be viewed at wileyonlinelibrary.com]

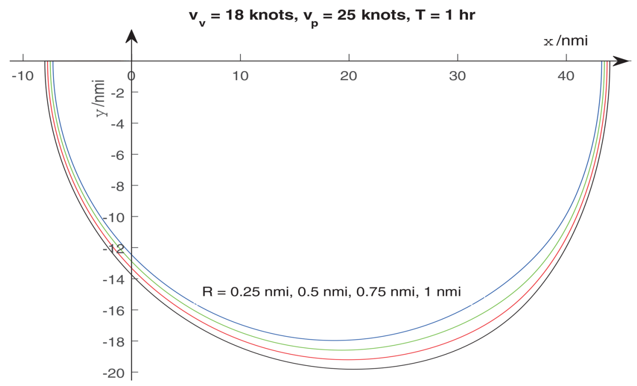

FIGURE 7 Infeasible Regions for direct heading under different R, when pirate is faster [Color figure can be viewed at wileyonlinelibrary.com]

Proposition 4 also implies that the Infeasible Region is a compact set with respect to the initial position (r_0, θ_0). Thus, to construct the Infeasible Region, we can find the border of the region. Specifically, we can enumerate over a set of θ_0 values such as $\theta_0 \in \{1^o, 2^o, \dots\}$, and for a given θ_0, a bisection search can be used to find an initial distance r_0^* that $r(T, r_0^*, \theta_0) = R$ when $\gamma \geq 1$ or $\gamma < 1$ and $T < \bar{t}(\gamma)$, or $\bar{r}(\gamma, r_0^*, \theta_0) = R$ when $\gamma < 1$ and $T \geq \bar{t}(\gamma)$, which means that r_0^* is on the border of the Infeasible Region along the direction of θ_0. Details are given in Algorithm 3. The algorithm is based on a fact that $\theta(T)$ is a monotone function on r_0. Note that in the calculation, C_0 is a function of r_0.

We now illustrate some Infeasible Regions for additional insights. We first show how Infeasible Regions may change with the speeds in Figures 5 and 6, where the safety distance $R = 0.5$ nmi and the chasing time $T = 2$ hours. Figure 5 is for the case when the speed of the pirate varies with the vessel speed v_v fixed at 20 knots, and Figure 6 is for the case when the vessel speed varies with the pirate speed v_p fixed at 20 knots. Both figures show that the Infeasible Region expands quickly when the pirate speed increases. This underscores the importance of maintaining a high speed of the vessel, consistent with the BMP guidance.

The range of Infeasible Regions also depends on the value of R, the safety distance. Typically it should be no less than the effective range of the weapons that the pirate may have, for example, a few hundred meters for a rifle, and up to 2000m for a machine gun (from www.wikipedia.com). The choice can be made by the vessel based on the experience and available information, such as previously reported cases of pirate attacks in the nearby area.

In Figures 7 and 8, we give the Infeasible Regions under different R values for two cases, respectively, when the pirate has a higher speed and when the vessel has a higher speed. The two figures show that the Infeasible Region is less sensitive to the safety distance when the pirate speed is higher, but more sensitive when the vessel speed is higher.

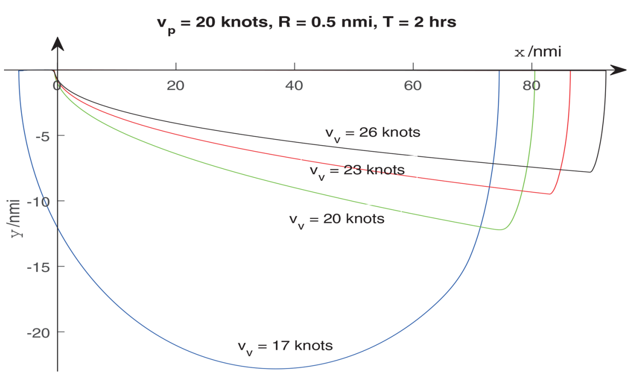

FIGURE 6 Infeasible Regions for direct heading under fixed v_p [Color figure can be viewed at wileyonlinelibrary.com]

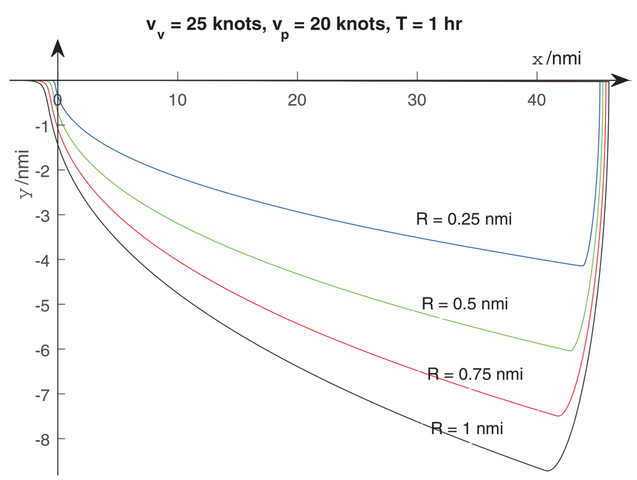

FIGURE 8 Infeasible Regions for direct heading under different R, when vessel is faster [Color figure can be viewed at wileyonlinelibrary.com]

5 | POLICIES WITH ONE OR TWO TURNS

Although direct heading is an optimal policy if it is feasible, it is often infeasible when the pirate skiff is not found early enough. The previous numerical examples show that the Infeasible Region of the direct heading policy is quite large, especially when the pirate has a higher speed than the vessel. Therefore, the vessel has to change its sailing direction.

5.1 | One-turn policy

Under the one-turn policy, the vessel will turn its sailing direction to a specific α immediately after it finds it is being chased, and stay at this direction until T. We will refer to it as policy-α. There is another option of making only one turn where the vessel keeps the current sailing direction until time t, then turns to a new direction α and maintains the direction until time T. We will treat this option as a special case of the two-turn policy (α_1, α_2) where $\alpha_1 = 0$ and $\alpha_2 = \alpha$.

It is easy to realize that any policy-α can be regarded as direct heading if we rotate the coordinate system anticlockwise by a degree of α. Such a view enables us to directly check the feasibility of the one-turn policy for any given α. We note that the range of α can be pretty large. However, we can only consider $\alpha \in [0, \theta_0]$. Because of the symmetry, for any $\alpha \in [0, \theta_0]$, policy-α and policy-$(2\theta_0 - \alpha)$ will result in the same $r(t)$ during $[0, T]$. Figure 9 shows this symmetry of the processes under policy-α and policy-$(2\theta_0 - \alpha)$.

Furthermore, for any $\alpha \in (0, \theta_0]$, we can also see that policy-α will dominate policy-$(2\theta_0 - \alpha)$ because, when $\theta_0 \in [0, \pi)$ as we have assumed, the policy-α always results in the vessel sailing a longer distance along the original direction. Therefore, we only need to focus on the range where $\alpha \in (0, \theta_0]$. From Lemma 2, the pirate will stay below the rotated x -axis, which means $\theta(t) > \alpha$, before their distance becomes zero.

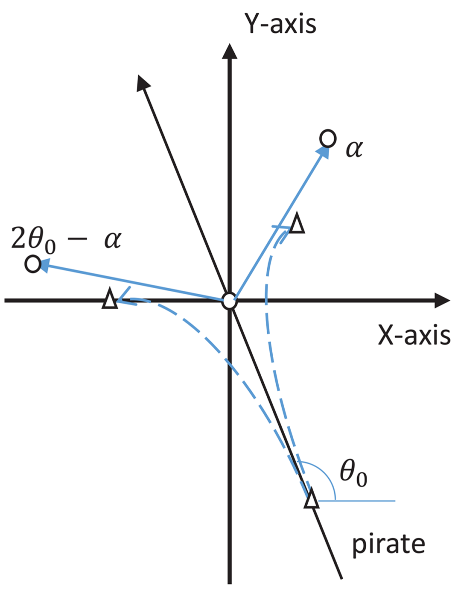

FIGURE 9 Symmetry of policy-α and policy-$(2\theta_0 - \alpha)$ [Color figure can be viewed at wileyonlinelibrary.com]

Feasibility test of a policy-α can be done by applying Algorithm 2 on the rotated coordinate system. Specifically, we only need to input $(r_0, \theta_0 - \alpha, \gamma)$ rather than (r_0, θ_0, γ). Then a capture time T_c can be obtained for policy α.

We now discuss the existence of a unique optimal policy-α. Consider any two feasible policies: policy-α and policy-α' where $0 \le \alpha < \alpha' \le \theta_0$. The final positions of the vessel under the two policies are $(x_v(T), y_v(T)) = (v_v T \cos\alpha, v_v T \sin\alpha)$ and $(x_v'(T), y_v'(T)) = (v_v T \cos\alpha', v_v T \sin\alpha')$, respectively. When $\alpha' \le \pi/2$, we always have $x_v(T) > x_v'(T) \ge 0$ and $y_v'(T) > y_v(T) \ge 0$; when $\alpha' > \pi/2$, we have $x_v'(T) < 0$ and $x_v(T) > x_v'(T)$. According to Definition 1, α will dominate α'. This implies that there exists a unique optimal one-turn policy-α^* that dominates all other one-turn policies; specifically, α^* is the smallest among all feasible α 's. The next lemma enables us to find the optimal α^*.

Algorithm 4 Finding the optimal one-turn policy α^*

Input: (r_0, θ_0, γ), output: α^*
Initialization: $\alpha_u = \theta_0$, $\alpha_l = 0$. Let $\alpha = \frac{\alpha_u + \alpha_l}{2}$
While $\alpha_u - \alpha_l \ge \varepsilon$, do
  Use Algorithm 2 to find Capture time T_c with input $(r_0, \theta_0 - \alpha, \gamma)$. Calculate $\bar{t}(\gamma)$ if $\gamma < 1$.
  If $T_c = \bar{t}(\gamma)$, stop; Else,
  If T_c does not exist or $T_c > T$, let $\alpha_u = \alpha$.
  If $T_c < T$, let $\alpha_l = \alpha$.

Lemma 3 *Consider two one-turn policies α and α' with $0 \le \alpha < \alpha' < \theta_0$. At any time t before $r(t, \alpha) = 0$, we have $r(t, \alpha) < r(t, \alpha')$.*

This lemma shows that if a policy-α is feasible, then any policy-α', $\alpha' \in [\alpha, \theta_0]$ is feasible. Hence, the optimal policy-α^* partitions the entire set $[0, \theta_0]$ into the feasible set $[\alpha^*, \theta_0]$ and infeasible set $[0, \alpha^*)$. Therefore, α^* can be found by a bisection search in $[0, \theta_0]$, details are given in Algorithm 4 where ε is the desired accuracy.

5.2 | Two-turn policy

We now discuss the policy of making two turns. We first point out that one prior condition for any feasible two-turn, even multi-turn, policy to exist is the existence of feasible one-turn policies because one-turn policies include the safest and most conservative policy $\alpha = \theta_0$. However, sometimes even the optimal one-turn policy deviates from the planned route too much. Hence we hope to find two-turn policies that dominate the optimal one-turn policy.

In a two-turn policy, the whole period $[0, T]$ is divided into two intervals $[0, \tau]$ and $[\tau, T]$, and the vessel has two sailing directions α_1 and α_2, as shown in Figure 4, such that

$$\alpha(t) = \begin{cases} \alpha_1, & \text{if } 0 \le t \le \tau \\ \alpha_2, & \text{if } \tau < t \le T. \end{cases} \tag{11}$$

Note that τ, α_1, and α_2 are all decision variables.

Given a two-turn policy $(\tau, \alpha_1, \alpha_2)$, we can regard it as two connected one-turn policies. Similarly to the one-turn policy, we can conclude $\alpha(t) \le \theta(t)$ for $t \in [0, T]$ due to the symmetric property of the process. Hence, we only need to consider the region where $\alpha_1 \in [\theta_0 - \pi, \theta_0]$ in the first stage and $\alpha_2 \in [\theta(\tau) - \pi, \theta(\tau)]$ in the second stage.

Algorithm 5 Feasibility test for a two-turn policy

Input: $(\tau, \alpha_1, \alpha_2)$ and (r_0, θ_0, γ), Output: feasibility of $(\tau, \alpha_1, \alpha_2)$

Calculate Capture time T_{c1} by Algorithm 2 with input $(r_0, \theta_0 - \alpha_1, \gamma)$.

If T_{c1} exists and $T_{c1} < \bar{t}_1(\gamma)$ when $\gamma < 1$, $T_{c1} < \tau$

$(\tau, \alpha_1, \alpha_2)$ is infeasible. Stop.

Calculate $r(\tau)$ and $\theta(\tau)$ by Algorithm 1 with input $(r_0, \theta_0 - \alpha_1, \gamma, \tau)$.

Calculate Capture time T_{c2} by Algorithm 2 with input $(r(\tau), \theta(\tau) - \alpha_2 + \alpha_1, \gamma)$.

If T_{c2} exists and $T_{c2} < \bar{t}_2(\gamma)$ when $\gamma < 1$, $T_{c2} < T - \tau$

$(\tau, \alpha_1, \alpha_2)$ is infeasible. Stop.

Claim $(\tau, \alpha_1, \alpha_2)$ is feasible. Stop.

The final state under a two-turn policy, taking $\gamma \neq 1$ as an example, satisfies the following nonlinear equations.

$$\begin{cases} r(\tau) = C_{10} \dfrac{\tan^{\gamma} \frac{\theta(\tau) - \alpha_1}{2}}{\sin(\theta(\tau) - \alpha_1)} \\ \dfrac{1}{2(\gamma - 1)} \tan^{\gamma - 1} \dfrac{\theta(\tau) - \alpha_1}{2} + \dfrac{1}{2(\gamma + 1)} \tan^{\gamma + 1} \dfrac{\theta(\tau) - \alpha_1}{2} \\ \quad = -\dfrac{v_v}{C_{10}} \tau + C_{11}, \end{cases} \tag{12}$$

$$\begin{cases} r(T) = C_{20} \dfrac{\tan^{\gamma} \frac{\theta(T) - \alpha_2}{2}}{\sin(\theta(T) - \alpha_2)} \\ \dfrac{1}{2(\gamma - 1)} \tan^{\gamma - 1} \dfrac{\theta(T) - \alpha_2}{2} + \dfrac{1}{2(\gamma + 1)} \tan^{\gamma + 1} \dfrac{\theta(T) - \alpha_2}{2} \\ \quad = -\dfrac{v_v}{C_{20}} (T - \tau) + C_{21}, \end{cases} \tag{13}$$

where

$$C_{10} = r_0 \frac{\sin(\theta_0 - \alpha_1)}{\tan^{\gamma} \frac{\theta_0 - \alpha_1}{2}}, C_{11} = \frac{1}{2(\gamma - 1)} \tan^{\gamma - 1} \frac{\theta_0 - \alpha_1}{2} + \frac{1}{2(\gamma + 1)} \tan^{\gamma + 1} \frac{\theta_0 - \alpha_1}{2},$$

$$C_{20} = r(\tau) \frac{\sin(\theta(\tau) - \alpha_2)}{\tan^{\gamma} \frac{\theta(\tau) - \alpha_2}{2}}, C_{21} = \frac{1}{2(\gamma - 1)} \tan^{\gamma - 1} \frac{\theta(\tau) - \alpha_2}{2} + \frac{1}{2(\gamma + 1)} \tan^{\gamma + 1} \frac{\theta(\tau) - \alpha_2}{2}.$$

The feasibility test of a two-turn policy can be done by checking each one-turn policy separately. Details are given in Algorithm 5.

In general, given (α_1, α_2), there may be many feasible two-turn policies $(\tau, \alpha_1, \alpha_2)$ with different turn times t. Next we discuss how to find the best τ.

We can first rule out some cases. Suppose that the optimal one-turn policy α^* has been found. Recall that α^* is the smallest feasible turning angle. Then we can eliminate some two-turn policies. First, any two-turn policy $(\tau, \alpha_1, \alpha_2)$ with $\alpha_1 < \alpha^*$ and $\alpha_2 < \alpha^*$ is infeasible regardless of the turning time τ. By using Lemma 3 twice, we can see that such a two-turn policy will lead to a shorter distance $r(t)$ at any time t than policy α^* does. Hence the two-turn policy is infeasible. Second, for the same reason, any two-turn policy $(\tau, \alpha_1, \alpha_2)$ with $\alpha_1 > \alpha^*$ and $\alpha_2 > \alpha^*$ is feasible but dominated by one-turn policy-α^* regardless of the turning time τ. Based on the above analysis, we need to consider two cases: $\alpha_1 > \alpha^* > \alpha_2$, and $\alpha_1 < \alpha^* < \alpha_2$.

To determine a best time τ for given α_1 and α_2, we need to study how the distance at time T, denoted by $r(T; \tau, \alpha_1, \alpha_2)$, changes with τ.

Lemma 4 *Given a two-turn policy $(\tau, \alpha_1, \alpha_2)$, the final distance $r(T; \tau, \alpha_1, \alpha_2)$ is increasing on τ when $\alpha_1 > \alpha_2$ and decreasing on τ when $\alpha_1 < \alpha_2$ under the condition $\gamma \ge \cos(\theta(T; \tau, \alpha_1, \alpha_2) - \alpha_2)$.*

Lemma 4 focuses on the feasibility in terms of the final distance. Under some condition, if τ is a feasible turn time, then any τ' in the interval $[\tau, T]$ is also feasible when $\alpha_1 > \alpha_2$, and any τ' in the interval $[0, \tau]$ is also feasible when $\alpha_1 < \alpha_2$.

When $\gamma \ge 1$, the relative distance $r(t)$ is strictly decreasing on $t \in [0, T]$. This means that the feasibility of a two-turn policy $(\tau, \alpha_1, \alpha_2)$ only depends on whether $r(T; \tau, \alpha_1, \alpha_2) \ge R$. So Lemma 4 is sufficient to characterize the range of feasible turn times for given α_1, α_2.

However, when $\gamma < 1$, the minimum distance $r(t)$ may occur at a time t within the interval $[0, T)$, instead of $r(T)$.

Therefore, the feasibility of a turn time τ depends on the real minimum relative distance. During $[0, \tau]$, if $\alpha_1 > \alpha^*$, the vessel will never be caught for $\forall \tau \in (0, T]$, and if $\alpha_1 < \alpha^*$, when the vessel is caught during $[0, \tau]$, a bigger τ' will also lead to the vessel being caught. Let $\bar{r}_2(\gamma; \tau, \alpha_1, \alpha_2)$ denote the minimum relative distance given pirate's position being $(r(\tau), \theta(\tau))$, which is strongly dependent on the turn time τ. If $\bar{r}_2(\gamma; \tau, \alpha_1, \alpha_2)$ does not occur during $[\tau, T)$, the final relative distance is the smallest distance during $[\tau, T]$ and Lemma 4 could be used to check the range of feasible turn times. Otherwise, $\bar{r}_2(\gamma; \tau, \alpha_1, \alpha_2)$ determines the feasibility of $(\tau, \alpha_1, \alpha_2)$. The following lemma shows the relationship between the τ and $\bar{r}_2(\gamma; \tau, \alpha_1, \alpha_2)$.

Lemma 5 *When $\gamma < 1$, given a two-turn policy $(\tau, \alpha_1, \alpha_2)$, $\bar{r}_2(\gamma; \tau, \alpha_1, \alpha_2)$ is increasing on τ when $\alpha_1 > \alpha_2$ and decreasing on τ when $\alpha_1 < \alpha_2$.*

With Lemmas 4 and 5, we are able to conclude the interval of the feasible turn time for given α_1 and α_2.

Proposition 5 *For any given α_1 and α_2, assume that the two-turn policy $(\tau, \alpha_1, \alpha_2)$ is feasible. Then any policy $(\tau', \alpha_1, \alpha_2)$ is feasible, for $\tau' > \tau$ and $\alpha_1 > \alpha^* > \alpha_2$, and for $\tau' < \tau$ and $\alpha_1 < \alpha^* < \alpha_2$.*

Proposition 5 implies that there exists a threshold value on the turn time τ, denoted by $\tau^*(\alpha_1, \alpha_2)$, for any given α_1 and α_2. The interval of feasible turn time is $[\tau^*(\alpha_1, \alpha_2), T]$ if $\alpha_1 > \alpha^* > \alpha_2$, and $[0, \tau^*(\alpha_1, \alpha_2)]$ if $\alpha_1 < \alpha^* < \alpha_2$.

In fact, we can further claim that some two-turn policies cannot be Pareto optimal policy, turn time not being $\tau^*(\alpha_1, \alpha_2)$ and two-turn policy with $y_v(T) < 0$.

Firstly, consider the case where $\alpha_1 > \alpha_2$. If $y_v(T) < 0$, it is easy to see that two-turn policy $(\tau^*(\alpha_1, \alpha_2), \alpha_1, \alpha_2')$ with $\alpha_2' > \alpha_2$ and $y_v'(T) = 0$ is feasible as well and dominates $(\tau^*(\alpha_1, \alpha_2), \alpha_1, \alpha_2)$. Otherwise, if $\alpha_2 > -\alpha_1$, $x_v(T)$ is decreasing and $y_v(T)$ is increasing on τ, and thus $(\tau^*(\alpha_1, \alpha_2), \alpha_1, \alpha_2)$ will dominate any two-turn policy $(\tau', \alpha_1, \alpha_2)$ with $\tau' > \tau$. If $\alpha_2 < -\alpha_1$, $x_v(T)$ will be increasing on τ, which implies that $(\tau^*(\alpha_1, \alpha_2), \alpha_1, \alpha_2)$ cannot dominate $(\tau', \alpha_1, \alpha_2)$. Figure 10 shows a better two-turn policy $(\tau^*(\alpha_1, \alpha_2), \alpha_1, \alpha_2')$ than $(\tau', \alpha_1, \alpha_2)$ with $\alpha_2' > \alpha_2$ and $y_v'(T) = y_v(T)$.

Secondly, $\alpha_1 < \alpha_2$. If $y_v(T) < 0$, $x_v(T)$ and $y_v(T)$ are both increasing on α_1, and hence $(\tau^*(\alpha_1, \alpha_2), \alpha_1', \alpha_2)$ will dominate $(\tau^*(\alpha_1, \alpha_2), \alpha_1, \alpha_2)$ with $\alpha_1' > \alpha_1$, see Figure 11. Otherwise, if $\alpha_1 > -\alpha_2$, $(\tau^*(\alpha_1, \alpha_2), \alpha_1, \alpha_2)$ can dominate any other two-turn policy $(\tau', \alpha_1, \alpha_2)$ with $\tau' < \tau^*$ due to the monotonicity. If $\alpha_1 < -\alpha_2$, $(\tau^*(\alpha_1, \alpha_2), \alpha_1, \alpha_2)$ cannot dominate $(\tau', \alpha_1, \alpha_2)$.

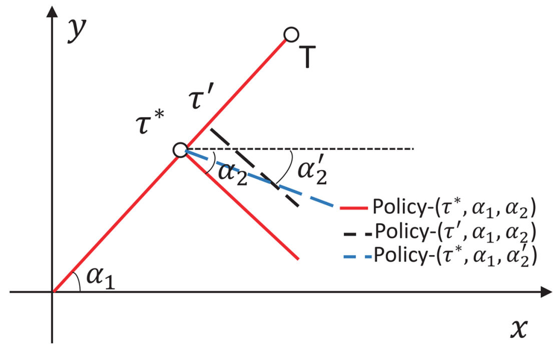

FIGURE 10 Counterexample of turn time not being $\tau^*(\alpha_1, \alpha_2)$ when $\alpha_1 > \alpha_2$ [Color figure can be viewed at wileyonlinelibrary.com]

However, $(\tau', \alpha_1, \alpha_2)$ will be dominated by $(\tau', \alpha_1, \alpha_2')$ with $\alpha_2' < \alpha_2$.

Corollary 1 *For given $\alpha_1 < \alpha^* < \alpha_2$ or $\alpha_2 < \alpha^* < \alpha_1$, only $\tau^*(\alpha_1, \alpha_2)$ can be Pareto optimal over all two-turn policies $(\tau, \alpha_1, \alpha_2)$ and $y_v(T; \tau^*(\alpha_1, \alpha_2), \alpha_1, \alpha_2) \geq 0$.*

According to Proposition 5, we are able to find $\tau^*(\alpha_1, \alpha_2)$ by a bisection search for any given α_1 and α_2. The detailed procedure is given in Algorithm 6 with the desired accuracy ε.

Algorithm 6 Finding the turn time $\tau^*(\alpha_1, \alpha_2)$

Input: $(\alpha_1, \alpha_2, r_0, \theta_0, \gamma)$, Output: $\tau^*(\alpha_1, \alpha_2)$
Initialization: $\tau_l = 0$, $\tau_u = T$.
Repeat the following steps until $\tau_u - \tau_l < \varepsilon$.
 Let $\tau = (\tau_l + \tau_u)/2$
 Check the feasibility of policy $(\tau, \alpha_1, \alpha_2)$ by Algorithm 5.
 If $(\tau, \alpha_1, \alpha_2)$ is feasible for $\alpha_1 > \alpha_2$, or $(\tau, \alpha_1, \alpha_2)$ is infeasible for $\alpha_1 < \alpha_2$,
  Let $\tau_u = \tau$
 If $(\tau, \alpha_1, \alpha_2)$ is infeasible for $\alpha_1 > \alpha_2$ or $(\tau, \alpha_1, \alpha_2)$ is feasible for $\alpha_1 < \alpha_2$,
  Let $\tau_l = \tau$

5.3 | Set of Pareto optimal policies

Given α_1 and α_2, we can use Algorithm 6 to find the optimal turn time $\tau^*(\alpha_1, \alpha_2)$ and a two-turn policy $(\tau^*(\alpha_1, \alpha_2), \alpha_1, \alpha_2)$ that dominates other two-turn policies $(\tau', \alpha_1, \alpha_2)$. By enumerating α_1 and α_2, we can identify the set of all Pareto optimal two-turn policies computationally, making it possible for the vessel to evaluate alternative policies.

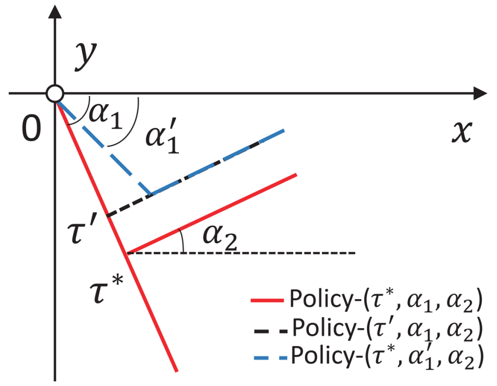

FIGURE 11 Counterexample of turn time not being $\tau^*(\alpha_1, \alpha_2)$ when $\alpha_1 < \alpha_2$ [Color figure can be viewed at wileyonlinelibrary.com]

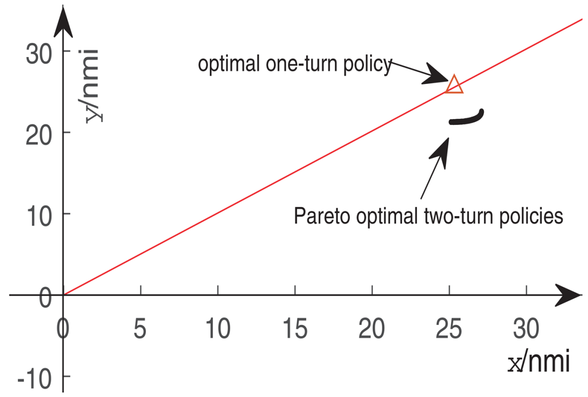

FIGURE 13 An example of two-turn policies with the pirate relatively in front of the vessel [Color figure can be viewed at wileyonlinelibrary.com]

We now use some examples to demonstrate the results. First we compare the vessel's final positions $(x_v(T), y_v(T))$ under different policies, showing how two-turn policies may improve the optimal one-turn policy. In all examples, we set $v_v = 18$ knots, $v_p = 25$ knots, $R = 0.5$ nmi, and $T = 2$ hours.

Figure 12 shows the set of final positions $(x_v(T), y_v(T))$ of the vessel when the pirate is found at position $(-10, -15)$ initially, that is, the pirate is behind the vessel. The red straight line is the trace of the optimal one-turn policy where $\alpha^* = 0.2324$ or 13.3^o, and the small triangle is the final position. The black dots under the one-turn trace are the final positions under all Pareto optimal two-turn policies. In this example, we see that there are indeed some two-turn policies dominating the one-turn policy-α^* with higher $x_v(T)$ and smaller $y_v(T)$, which verifies the advantage of two-turn policies. At the same time, we find that the range of different $x_v(T)$ values is quite narrow compared to the range of different $y_v(T)$ values. This shows that, in this case, the benefit of the two-turn policies is a higher chance of minimizing the deviation from the original sailing direction.

Figure 13 shows the set of the vessel's final positions when the pirate is found at position $(10, -20)$ initially, that is, the pirate is relatively in front of the vessel. The optimal one-turn policy is $\alpha^* = 0.7879$ or 45^o. Similar to Figure 12, we see the existence of two-turn policies dominating the optimal one-turn policy. There is a different observation. In Figure 13, the range of $x_v(T)$ values under all Pareto optimal two-turn policies is now much bigger than the range of $y_v(T)$ values. In this case, the benefit of the two-turn policies is a higher chance of sailing a farther distance along the original sailing direction.

To further investigate the two-turn policies, in Figures 14 and 15, we plot the set of Pareto optimal policies, represented by (α_1, α_2), for the two examples. Recall that we only need to consider two cases, $\alpha_1 > \alpha^* > \alpha_2$ and $\alpha_2 > \alpha^* > \alpha_1$. In the first case the vessel tends to evade chasing by making a larger turn in the first stage, and then tries to return to the original direction; and in the second case, the vessel tends to

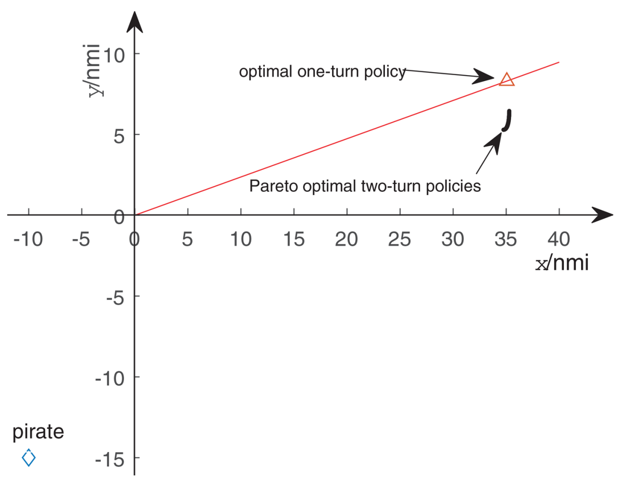

FIGURE 12 An example of two-turn policies with the pirate relatively behind the vessel [Color figure can be viewed at wileyonlinelibrary.com]

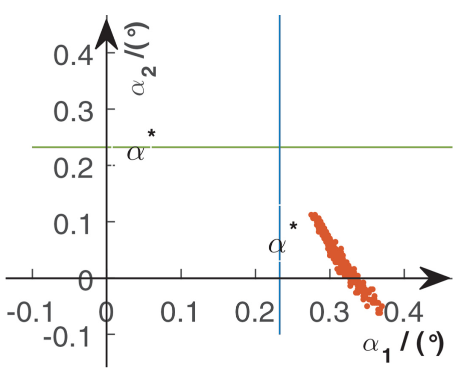

FIGURE 14 The set of Pareto optimal policies for the example in Figure 12 [Color figure can be viewed at wileyonlinelibrary.com]

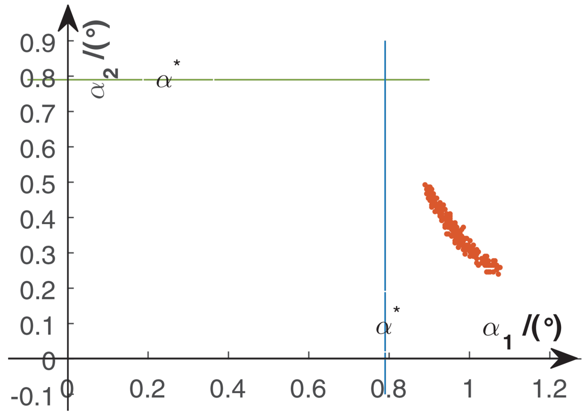

FIGURE 15 The set of Pareto optimal policies for the example in Figure 13 [Color figure can be viewed at wileyonlinelibrary.com]

stick on the original direction with a smaller turn in the first stage and then tries to evade chasing. For these two examples, we can see that $\alpha_1 > \alpha_2$ for all the Pareto optimal policies, that is, any two-turn policy with $\alpha_1 < \alpha_2$ is dominated by some other two-turn policy with $\alpha_1 > \alpha_2$. In fact, the computational details reveal an even stronger result. We have observed that $(\tau, \alpha_1, \alpha_2)$ is always safer than $(T-\tau, \alpha_2, \alpha_1)$ with a bigger smallest relative distance during $[0, T]$, for any $\alpha_1 > \alpha_2$ and any τ. This may be due to the fact that making a larger turn in the first stage will help the vessel maintain a farther distance from the pirate.

6 | THE CASE OF MULTIPLE CHASING SKIFFS

We now discuss how to apply the results for dealing with one chasing skiff to the case of multiple skiffs approaching from different directions. Assuming that each skiff takes its own pursuit guidance law, we will discuss how to check the feasibility of the one-turn and two-turn policies, and how to find the Pareto optimal policy. To simplify the presentation, we consider the case of two chasing skiffs, but our discussion can be generalized to the case of more than two skiffs. Specifically, the generalization for the one-turn policy can be done straightforwardly, and the generalization for the two-turn policy will cause additional complexity with respect to the number of skiffs to consider.

Let the initial positions of the skiffs be denoted by (x_1, y_1) and (x_2, y_2), or (r_1, θ_1) and (r_2, θ_2), respectively. As indicated by Figures 16 and 17, there are two cases in terms of the skiffs' positions. The two skiffs may be on the two sides of the vessel where we assume $y_1 \geq 0$ and $y_2 < 0$; they may be on the same side of the vessel where we assume $y_i \leq 0$, $i = 1, 2$.

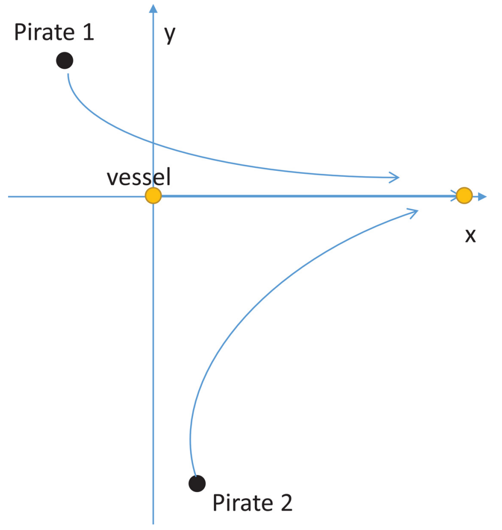

FIGURE 16 Two pirate skiffs on two sides [Color figure can be viewed at wileyonlinelibrary.com]

6.1 | One-turn policy

We first discuss the one-turn policy, including direct heading as a special case. For any given turning angle α, one-turn policy-α is feasible means that it is feasible to both skiffs. This can be checked by applying Algorithm 4 twice, once to each pirate.

To find the optimal one-turn policy-α^* for the vessel, we need to know the set of all feasible α values, denoted by A. Clearly, $A = A_1 \cap A_2$, where A_i is the set of α values feasible to evading pirate i, $i = 1, 2$. Recall that in the case of having a single chasing skiff, we point out that we only need to consider searching in $[0, \theta_0]$ to find the optimal turning angle, because any α not in $[0, \theta_0]$, even if feasible, is dominated by

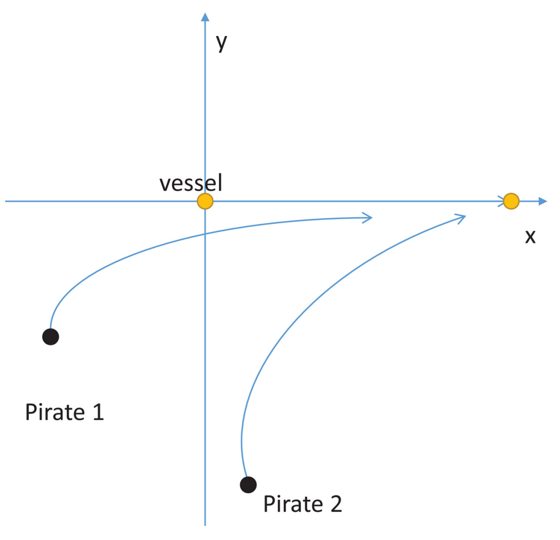

FIGURE 17 Two pirate skiffs on one side [Color figure can be viewed at wileyonlinelibrary.com]

another counterpart feasible α' in $[0, \theta_0]$. However, now we have to consider A_i in a larger range because, though some α is dominated by its counterpart α', α' may be infeasible to the other chasing skiff. So we need to find the complete set of feasible α values in $[-\pi, \pi]$, which can be done as follows. Without loss of generality, assume $y_i < 0$.

There are two cases with a slight difference in determining A_i, depending on whether direct heading is feasible to skiff i.

If direct heading is infeasible to skiff i, we have Algorithm 4 to find the optimal one-turn direction α^i. From symmetry, we have $A_i = [\alpha^i, 2\theta_i - \alpha^i]$. In case $2\theta_i - \alpha^i > \pi$, we rewrite it as $A_i = [\alpha^i, \pi] \cup (-\pi, 2\theta_i - \alpha^i - 2\pi]$ such that $A_i \subset [-\pi, \pi]$.

If direct heading is feasible to skiff i, that is, $\alpha^i = 0$. there exists some feasible one-turn policy with $\alpha < 0$ because $r(t, \alpha)$ is increasing on α. Let $\overline{\alpha}_i$ denote the minimum feasible α. We are able to find $\overline{\alpha}_i$ by applying Algorithm 3 with the lower bound being $\theta_i - \pi$ and upper bound being 0. The set of feasible α is now $A_i = [\overline{\alpha}_i, 2\theta_i - \overline{\alpha}_i]$ when $2\theta_i - \overline{\alpha}_i \leq \pi$ or $A_i = [\overline{\alpha}_i, \pi] \cup (-\pi, 2\theta_i - \overline{\alpha}_i - 2\pi)$ when $2\theta_i - \overline{\alpha}_i > \pi$.

Given each A_i, we can obtain A, the set of all feasible one-turn policies. Then the optimal one-turn policy-α^* can be found in A as $|\alpha^*| = \min_{\alpha \in A} |\alpha|$. This applies to any number of chasing skiffs.

There are some special cases where the optimal α^* can be identified as one α_i, the optimal one-turn policy to one of the skiffs. For example, when the two skiffs are on the different sides, if $\overline{\alpha}_2 \leq \alpha^1$, the optimal one-turn policy $\alpha^* = \alpha^1$. When the two skiffs are on the same side, if $\max\left\{\alpha^1, \alpha^2\right\} \leq \min\left\{2\theta_1 - \alpha^1, 2\theta_2 - \alpha^2\right\}$, then the optimal one-turn policy is $\alpha^* = \max\left\{\alpha^1, \alpha^2\right\}$; otherwise, there is no feasible one-turn policy.

6.2 | Two-turn policy

When there are two skiffs, analytical properties of the two-turn policy are hard to derive because there are various cases to discuss. The complexity against the single-skiff case can be shown by the existence of feasible policies. For the problem with a single skiff, if there is no feasible one-turn policy, then there is no feasible two-turn policy either, because the one-turn policy $\alpha = \theta_0$ is the safest policy. For the problem with two skiffs, the non-existence of feasible one-turn policies does not necessarily imply there are no feasible two-turn policies. When there is no feasible one-turn policy with respect to two skiffs, there are actually two scenarios. First, there is no feasible one-turn policy with respect to one of the skiffs; in this case, there exists no two-turn policy with respect to the two skiffs. Second, there exists a set of feasible one-turn policies with respect to each skiff separately, but there is no overlap between the two sets; in this case, it is possible that there exist feasible two-turn policies.

The analysis of two-turn policy for the two-skiff case is still based on the two-turn policy for the one-skiff case, which includes finding an optimal turning time for a pair of turning directions and the range of feasible turning directions. We first discuss the turning time.

Given (α_1, α_2) as the two turning directions of a two-turn policy, we can find the earliest/latest turning time $\tau^i(\alpha_1, \alpha_2)$ with respect to each single skiff i, according to Proposition 5. The set of feasible turning time to skiff i, denoted by T^i, is either $[0, \tau^i(\alpha_1, \alpha_2)]$ or $[\tau^i(\alpha_1, \alpha_2), T]$, depending on the specific values of (α_1, α_2). When there is only one skiff i, we argue that only the two-turn policy with turn time being $\tau^i(\alpha_1, \alpha_2)$ could be Pareto-optimal two-turn policy. The turn time $\tau^*(\alpha_1, \alpha_2)$ in the Pareto-optimal two-turn policies can be determined by checking the intersection of T^1 and T^2. If the intersection is empty, then there is no feasible turn time given the turning directions; otherwise, only the two-turn policies with turn time being $\tau^i(\alpha_1, \alpha_2)$ could be the Pareto-optimal two-turn policies.

We can determine the range of feasible turning directions as follows. Consider the case where the two skiffs are on the same side of the vessel as shown in Figure 17. Recall that we use A_i to denote the set of feasible turning directions of the one-turn policy with respect to skiff i, where $A_i \subset (-\pi, \pi)$ and $i = 1, 2$. We assume that $\theta_2 > \theta_1$. Then Table 2 shows how to identify the potential sets of (α_1, α_2) of the Pareto optimal two-turn policies. The first condition implies whether one-turn policy exists or not, in which *"No"* means one-turn policy does not exist and *"Yes"* otherwise. The other three conditions reflect the relative position of α^1 and α^2, which somehow implies which skiff is more dangerous.

Now we will give a brief explanation of some cases, taking the first three cases as examples. In the first case, $\alpha^1 + \alpha^2 > 2\theta_1$, that is, $\alpha_2 > 2\theta_1 - \alpha_1$, it implies that there is no feasible one-turn policy as we assume $\theta_2 > \theta_1$. To find a feasible two-turn policy, the vessel needs to choose one direction from each A_i, $i = 1, 2$. However, any feasible turn policy that contains a turn direction belonging to $[\theta_2, 2\theta_2 - \alpha^2]$ will be dominated by a smaller turn angle belonging to $[\alpha^2, \theta_2]$. Hence, the potential sets of (α_1, α_2) in the Pareto-optimal policy is (1) $\alpha_1 \in A_1$, $\alpha_2 \in [\alpha^2, \theta_2]$; (2) $\alpha_1 \in [\alpha^2, \theta_2]$, $\alpha_2 \in A_1$. Meanwhile, in the second and third case, one-turn policy exists based on the first condition. The second condition implies that $A_1 \subset A_2$. The optimal one-turn policy is α^1. At this time, pirate 1 is relatively more dangerous than pirate 2. The Pareto-optimal policy mainly relies on the position of pirate 1, and the potential set is $\alpha_1 \in [\alpha^1, \theta_2]$, $\alpha_2 \in [-\pi/2, \alpha^1]$ in the second case, and (1) $\alpha_1 \in [2\theta_1 - \alpha^1, \theta_2], \alpha_2 \in A_1$; (2) $\alpha_1 \in [\alpha^1, 2\theta_1 - \alpha^1], \alpha_2 \in [-\pi/2, \alpha^1]$ in the third case, respectively.

Secondly, the two pirate skiffs are on the two sides of the vessel. Let α^1 and α^2 denote the optimal one-turn policy

TABLE 2 Potential set of (α_1, α_2) in Pareto optimal two-turn policies when the two skiffs are at the same side of the vessel

$\alpha^1 + \alpha^2 < 2\theta_1$	$\alpha^1 > \alpha^2$	$\alpha^2 > \theta_1$	$\theta_2 > 2\theta_1 - \alpha^1$	Potential sets
No	/	/	/	1) $\alpha_1 \in A_1, \alpha_2 \in [\alpha^2, \theta_2]$,
				2) $\alpha_1 \in [\alpha^2, \theta_2], \alpha_2 \in A_1$
Yes	Yes	/	No	1) $\alpha_1 \in [\alpha^1, \theta_2], \alpha_2 \in [-\pi/2, \alpha^1]$
Yes	Yes	/	Yes	1) $\alpha_1 \in [2\theta_1 - \alpha^1, \theta_2], \alpha_2 \in A_1$
				2) $\alpha_1 \in [\alpha^1, 2\theta_1 - \alpha^1], \alpha_2 \in [-\pi/2, \alpha^1]$
Yes	No	No	No	1) $\alpha_1 \in [\alpha^2, \theta_2], \alpha_2 \in [-\pi/2, \alpha^2]$
Yes	No	No	Yes	1) $\alpha_1 \in [2\theta_1 - \alpha^1, \theta_2], \alpha_2 \in A_1$;
				2) $\alpha_1 \in [\alpha^2, 2\theta_1 - \alpha^1], \alpha_2 \in [-\pi/2, \alpha^2]$
Yes	No	Yes	No	1) $\alpha_1 \in [2\theta_1 - \alpha^1, \theta_2], \alpha_2 \in A_1$;
				2) $\alpha_1 \in [\alpha^2, 2\theta_1 - \alpha^1], \alpha_2 \in [-\pi/2, \alpha^2]$;
				3)$\alpha_1 \in [\alpha^1, \alpha^2], \alpha_2 \in [2\theta_1 - \alpha^1, \theta_2]$
Yes	No	Yes	Yes	1) $\alpha_1 \in [\alpha^1, \alpha^2], \alpha_2 \in [-\pi/2, \alpha^2]$

*"/" means no requirement on the corresponding condition.

for pirate 1 and 2, respectively. And $\overline{\alpha}_1$ denote maximum one-turn policy when direct heading policy is feasible for the vessel to evade pirate 1. In the first case, the vessel will not evade the chasing from neither pirate 1 nor pirate 2 if it adopts direct heading policy. However, there might still exist one-turn policy for the vessel. At this time, it will be complex to reduce the potential region of two-turn policies. To find the feasible two-turn policies, we need to check all possible combinations by choosing one direction from each $A_i, i = 1, 2$.

The other three cases occur when direct heading policy is assumed to be feasible to skiff 1. In the second case, the optimal one-turn policy is α^2. This will degenerate to the problem to evade the chasing from pirate 2 only, and thus the potential set of Pareto optimal policy is $\alpha_1 \in [\alpha^2, \theta_2]$, $\alpha_2 \in [-\pi/2, \alpha^2]$. In the third case, the optimal one-turn policy is still α^2. The whole region can be divided into three intervals: $[-\pi/2, \alpha^2]$, $[\alpha^2, \overline{\alpha}_1]$ and $[\overline{\alpha}_1, \theta_2]$. When $\alpha_1 \in [-\pi/2, \alpha^2]$, any $\alpha_2 \in [\alpha^2, \overline{\alpha}_1]$ will be dominated by replacing the turn angles. Hence, we only need to consider $\alpha_2 \in [\overline{\alpha}_1, \theta_1]$. When $\alpha_1 \in [\alpha^2, \overline{\alpha}_1]$, we only need $\alpha_2 \in [-\pi/2, \alpha^2]$. When $\alpha_1 \in [\overline{\alpha}_1, \theta_2]$, the vessel needs to choose a turn angle from A_1. If $\alpha_2 \in [\alpha^2, \overline{\alpha}_1]$, such a two-turn policy will be dominated by replacing the sequence of α_1, α_2, which is definitely dominated by the optimal one-turn policy. Therefore, we only need $\alpha_2 \in [-\pi/2, \alpha^2]$. Thus we can conclude the potential sets as in Table 3.

Above all, we are able to conduct the computational experiments for any given two initial positions of the pirate skiffs. In fact, when we are checking the feasibility of a two-turn policy where the two pirates are on the different sides of the vessel, we may rotate the coordinate system to make the pirates on the same side of the vessel.

The following computational experiment helps illustrate the unique feature of the two-turn policies with multiple skiffs.

In Figure 18, we fix the position of pirate 2 and change the position of pirate 1. We use the two-turn policies which maximizes $x(T)$ or minimizes $|y(T)|$ as an example. When pirate 1 is at $(-5, 20)$, the vessel will be caught by both pirate 1 and pirate 2 if direct heading policy is adopted. Besides, there is no feasible one-turn policy and the computational result shows that there exist no feasible two-turn policies. When pirate 1 is at $(-13.5, 18)$, direct heading policy is feasible with respect to only pirate 1. However, there are still no feasible one-turn policies to evade the chasing from the two pirate skiffs. However, feasible two-turn policies can be found, and the Pareto optimal two-turn policies can be found. When pirate 1 is at $(-20, 25)$, direct heading is feasible with respect to pirate 1 and there are feasible one-turn policies to evade the

TABLE 3 Potential sets of (α_1, α_2) in Pareto optimal two-turn policies when the two skiffs are at the different sides of the vessel

Conditions on $\alpha^1, \alpha^2, \overline{\alpha}_1$	Potential sets
$\alpha^2 > 0 > \alpha^1$	1) $\alpha_1 \in A_1, \alpha_2 \in A_2$; 2) $\alpha_1 \in A_2, \alpha_2 \in A_1$
$\overline{\alpha}_1 > \theta_2$	1) $\alpha_1 \in [\alpha^2, \theta_2], \alpha_2 \in [-\pi/2, \alpha^2]$ for pirate 2 only
$\alpha^2 \le \overline{\alpha}_1 \le \theta_2$	1) $\alpha_1 \in [\alpha^2, \theta_2], \alpha_2 \in [-\pi/2, \alpha^2]$; 2) $\alpha_1 \in [-\pi/2, \alpha^2], \alpha_2 \in [\overline{\alpha}_1, \theta_2]$
$\overline{\alpha}_1 < \alpha^2$	1) $\alpha_1 \in [\alpha^2, \theta_2], \alpha_2 \in [-\pi/2, \overline{\alpha}_1]$; 2) $\alpha_1 \in [-\pi/2, \overline{\alpha}_1], \alpha_2 \in [\alpha^2, \theta_2]$

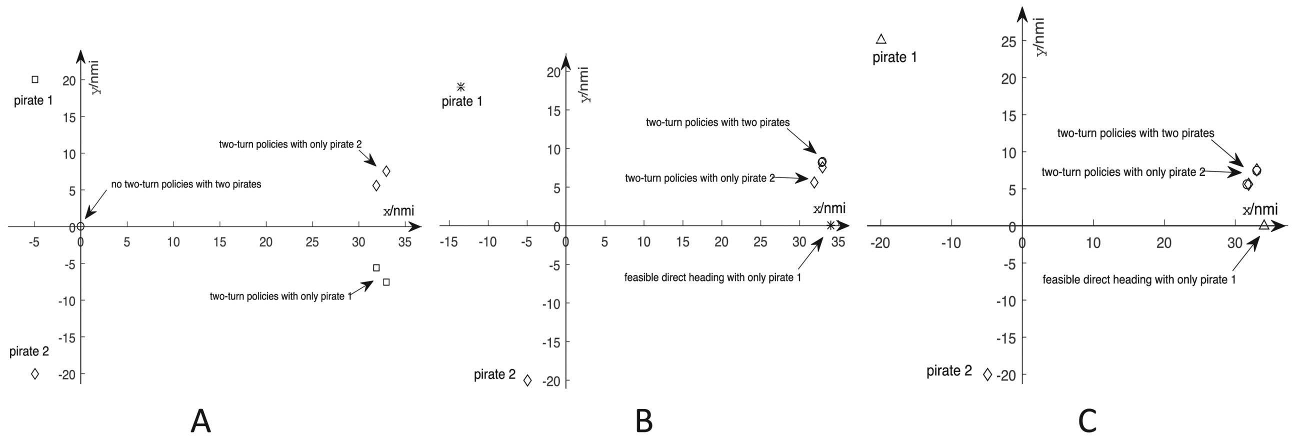

FIGURE 18 Influence of pirate's position on two-turn policies [Color figure can be viewed at wileyonlinelibrary.com]

two pirate skiffs. The Pareto optimal two-turn policies are almost the same as the case with only pirate 2.

7 | CONCLUSION

We have studied a pirate chasing problem for a commercial vessel in this paper. To the best of our knowledge, this is the first analytical model for pirate chasing at the operational level. We give feasibility conditions for the vessel to evade the chasing from pirate skiffs, under different policies. The results not only can be used as an operational tool for decision making in real time, but also confirm the effectiveness of the practical guidance issued by IMO.

As the first work, we have made some relatively simple assumptions in this paper. In what follows, we plan to work on more sophisticated models for more practical results. There are two important issues to consider. First, the pirate may take some wiser strategies than PP, and second, the vessel may lose some speed and time when making a turn. These will make the analysis more complicated. Finally, it is also interesting to consider the k -turn policy with k approaching infinity, or the case of continuous turning policy.

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ORCID

Xiangtong Qi http://orcid.org/0000-0003-4573-0157

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APPENDIX

Proof of Lemma 1

Proof. From (1), the dynamic process of $r(t)$ can be described by the following differential equation

$$\frac{dr(t)}{dt} = \frac{\begin{array}{c} 2(x_v(t) - x_p(t))\left(\frac{dx_v(t)}{dt} - \frac{dx_p(t)}{dt}\right) \\ +2(y_v(t) - y_p(t))\left(\frac{dy_v(t)}{dt} - \frac{dy_p(t)}{dt}\right) \end{array}}{2r(t)}.$$

With (1) and (2), we can simplify it to $\frac{dr(t)}{dt} = v_v(t)\cos(\alpha(t) - \theta(t)) - v_p(t)\cos(\beta(t) - \theta(t))$. Therefore, with respect to $\beta(t)$, $\frac{dr(t)}{dt}$ is minimized when $\beta(t) = \theta(t)$. ■

Proof of Lemma 2

Proof. We first consider the case $\gamma > 1$.

Consider (9). Since $\lim_{t\to\left(\frac{C_0C_2}{v_v}\right)^-} rhs \text{ of}(9) = 0$, we have $\lim_{t\to\left(\frac{C_0C_2}{v_v}\right)^-} lhs \text{ of}(9) = 0$. This implies $\lim_{t\to\left(\frac{C_0C_2}{v_v}\right)^-} \theta(t) = 0$ because the two terms of the *lhs* of (9) are both non-negative. Moreover, $\lim_{t\to\left(\frac{C_0C_2}{v_v}\right)^-} r(t) = \lim_{t\to\left(\frac{C_0C_2}{v_v}\right)^-} C_0\frac{\tan^\gamma\frac{\theta(t)}{2}}{\sin\theta(t)} = \lim_{t\to\left(\frac{C_0C_2}{v_v}\right)^-} C_0\frac{\sin^{\gamma-1}\theta(t)}{(1+\cos\theta(t))^\gamma} = 0$. Besides, both $r(t)$ and $\theta(t)$ are strictly decreasing when $\theta(t) > 0$, and $\theta(t) \equiv 0$ after it decreases to 0. Therefore, $\theta(t) > 0$ for $t < \frac{C_0C_2}{v_v}$ and $\theta(t) = 0$ for $t \geq \frac{C_0C_2}{v_v}$. Hence, we conclude $\tau = \frac{C_0C_2}{v_v}$ by the definition of τ.

Next, we will prove the non-existence of τ when $\gamma \leq 1$ by a contradiction.

We first assume such a finite τ exists for any given $\gamma \leq 1$, and the corresponding relative distance is denoted by $r(\tau)$. As τ is finite, the reduced relative distance will be no more than $v_v\tau(\gamma - \cos\theta_0)$, which implies $r(\tau)$ must be finite as well.

When $\gamma = 1$, we prove a contradiction by considering the continuity of $\theta(t)$. Note that $\lim_{t\to\tau^-}\theta(t) = 0$ implies $\lim_{t\to\tau^-} lhs \text{ of}(8) = +\infty$. On the other hand, $\lim_{t\to\tau^-} rhs \text{ of}(8) = \lim_{t\to\tau^-}\frac{v_vt}{C_0} + C_1 = \frac{v_v\tau}{C_0} + C_1$. Since τ is finite, $\frac{v_v\tau}{C_0} + C_1$ must be finite. Therefore, $\lim_{t\to\tau^-} rhs \text{ of}(8) \neq \lim_{t\to\tau^-} lhs \text{ of}(8)$, which implies a contradiction and thus such a finite τ doesn't exist.

When $\gamma < 1$, we prove a contradiction by considering the continuity of $r(t)$. As $\lim_{t\to\tau^-}\theta(t) = 0$, $\lim_{t\to\tau^-} r(t) = \lim_{t\to\tau^-} C_0\frac{\tan^\gamma\frac{\theta(t)}{2}}{\sin\theta(t)} = \lim_{t\to\tau^-} C_0\frac{\sin^{\gamma-1}\theta(t)}{(1+\cos\theta(t))^\gamma} = +\infty$, while we know $r(\tau)$ is finite. Therefore, $\lim_{t\to\tau^-} r(t) \neq r(\tau)$, which implies a contradiction and thus finite τ does not exist. ∎

Proof of Proposition 1

Proof. When $\gamma = 1$, (7) becomes

$$r(t) = C_0 \frac{\tan\frac{\theta(t)}{2}}{\sin\theta(t)} = C_0 \frac{\frac{\sin\theta(t)}{1+\cos\theta(t)}}{\sin\theta(t)} = \frac{C_0}{1+\cos\theta(t)}. \quad \text{(A1)}$$

Therefore, when $C_0 = r_0(1+\cos\theta_0) \geq 2R$, we have $r(t) > \frac{r_0(1+\cos\theta_0)}{2} \geq R$ for $t \in [0, +\infty)$, that is, T_c does not exist.

When $C_0 < 2R$, we can verify that

$$t = T_c = \frac{r_0 - R}{2v_v} + \frac{r_0(1+\cos\theta_0)}{4v_v} \ln \frac{r_0(1-\cos\theta_0)}{2R - r_0(1+\cos\theta_0)} \quad \text{(A2)}$$

is a solution of (A1) and (8). It is also the unique solution because (5) implies $r(t)$ is strictly decreasing, that is, $r(t) > R$ for $t \in [0, T_c)$, and $r(T_c) = R$. ∎

Proof of Proposition 2

Proof. When $\gamma > 1$, $\frac{dr(t)}{dt} = v_v(\cos\theta(t) - \gamma) < 0$ for any $t \geq 0$, which means the distance $r(t)$ between the vessel and the pirate will keep decreasing until $r(t) = 0$. Let $T_u \triangleq \frac{r_0(\gamma+\cos\theta_0)}{v_v(\gamma^2-1)}$. From (9), we have $\lim\limits_{t\to T_u^-} \theta(t) = 0$. Since $\gamma > 1$, $\lim\limits_{t\to T_u^-} r(t) = \lim\limits_{t\to T_u^-} C_0 \frac{\tan^\gamma \frac{\theta(t)}{2}}{\sin\theta(t)} = \lim\limits_{t\to T_u^-} C_0 \frac{\sin^{\gamma-1}\theta(t)}{(1+\cos\theta(t))^\gamma} = 0$. Hence we can claim that $T_c \in (0, \frac{r_0(\gamma+\cos\theta_0)}{v_v(\gamma^2-1)})$. ∎

Proof of Proposition 3

Proof.

1. Since $r(t)$ is a convex function on t, we can solve the first-order condition $\frac{dr(t)}{dt} = v_v(\cos\theta(t) - \gamma) = 0$, which shows $r(t)$ is minimized at $\cos\theta(t) = \gamma$, or equivalently, $\theta(t) = \overline{\theta}(\gamma)$. Substituting $\theta(t)$ by $\overline{\theta}(\gamma)$ in (9), we can get

$$\begin{aligned}\bar{t}(\gamma) = &-\frac{C_0}{v_v}\left[\frac{1}{2(\gamma-1)}\tan^{\gamma-1}\frac{\overline{\theta}(\gamma)}{2}\right. \\ &\left. + \frac{1}{2(\gamma+1)}\tan^{\gamma+1}\frac{\overline{\theta}(\gamma)}{2} - C_2\right] \\ = &\frac{r_0(\gamma+\cos\theta_0) - 2\gamma\bar{r}(\gamma)}{v_v(\gamma^2-1)},\end{aligned}$$

which is the only solution to $\cos\theta(t) = \gamma$ due to the strict monotonicity of $\theta(t)$ on t. Therefore, $r(t)$ achieves the global minimum at $\bar{t}(\gamma)$, and from (7), we can get the minimum distance $r(\bar{t}(\gamma)) = \bar{r}(\gamma)$.

2. If $\theta_0 \leq \overline{\theta}(\gamma)$, we have $\theta(t) \leq \theta_0$ and $\cos\theta(t) \geq \cos\theta_0 \geq \gamma, \forall t \in [0, +\infty)$ because $\theta(t)$ is a decreasing function on t. Therefore, $r(t)$ will be increasing on $t \in [0, +\infty)$. Due to the assumption $r_0 \geq R$, T_c does not exist.

   If $\theta_0 > \overline{\theta}(\gamma)$ but $\bar{r}(\gamma) \geq R$, from (1) we know T_c does not exist because $\bar{r}(\gamma)$ is the minimum of $r(t)$.

3. Also from (1), if $\theta_0 \leq \overline{\theta}(\gamma)$ and $\bar{r}(\gamma) < R$, we have $T_c < \bar{t}(\gamma)$. ∎

Proof of Proposition 4

Proof. To simplify the notation, we replace $r(T, r_0, \theta_0)$ and $\theta(T, r_0, \theta_0)$ by $r(T)$ and $\theta(T)$, respectively, in the proof.

Part One: the case of $\gamma \geq 1$. In this case, the sufficient and necessary condition for the vessel to be safe is $r(T) \geq R$. Thus, we need to prove that $r(T)$ is nondecreasing on r_0 with fixed θ_0 and nonincreasing on θ_0 with fixed r_0, that is, $\frac{dr(T)}{d\theta_0} \leq 0$ and $\frac{dr(T)}{dr_0} \geq 0$. We provide the details of proof for $\gamma > 1$. Similar analysis can be done for $\gamma = 1$.

The implicit function of $\theta(T)$ is

$$\begin{aligned}&\frac{1}{2(\gamma-1)}\tan^{\gamma-1}\frac{\theta(T)}{2} + \frac{1}{2(\gamma+1)}\tan^{\gamma+1}\frac{\theta(T)}{2} \\ &= -\frac{v_v T \tan^\gamma \frac{\theta_0}{2}}{r_0 \sin\theta_0} + \frac{1}{2(\gamma-1)}\tan^{\gamma-1}\frac{\theta_0}{2} + \frac{1}{2(\gamma+1)}\tan^{\gamma+1}\frac{\theta_0}{2}\end{aligned}$$

Taking derivative with respect to θ_0 on both sides, we have

$$\frac{\tan^\gamma \frac{\theta(T)}{2}}{\sin^2\theta(T)} \frac{d\theta(T)}{d\theta_0} = \frac{\tan^\gamma \frac{\theta_0}{2}}{\sin^2\theta_0} - \frac{v_v T(\gamma - \cos\theta_0)}{r_0} \frac{\tan^\gamma \frac{\theta_0}{2}}{\sin^2\theta_0}$$

With regard to $r(t)$,

$$r(T) = \frac{r_0 \sin\theta_0}{\tan^\gamma \frac{\theta_0}{2}} \frac{\tan^\gamma \frac{\theta(T)}{2}}{\sin\theta(T)}$$

Similarly, taking derivative with respect to θ_0 on both sides, we have

$$\frac{dr(T)}{d\theta_0} = r_0\frac{\cos\theta_0-\gamma}{\tan^\gamma\frac{\theta_0}{2}}\frac{\tan^\gamma\frac{\theta(T)}{2}}{\sin\theta(T)} + r_0\frac{\sin\theta_0}{\tan^\gamma\frac{\theta_0}{2}}\frac{(\gamma-\cos\theta(T))\tan^\gamma\frac{\theta(T)}{2}}{\sin^2\theta(T)}\frac{d\theta(T)}{d\theta_0}$$

Substituting $\frac{d\theta(T)}{d\theta_0}$ into $\frac{dr(T)}{d\theta_0}$, we can rewrite $\frac{dr(T)}{d\theta_0}$ as follows.

$$\frac{dr(T)}{d\theta_0} = \frac{r_0(\gamma-\cos\theta(T)) - r(T)(\gamma-\cos\theta_0) - v_vT(\gamma-\cos\theta_0)(\gamma-\cos\theta(T))}{\sin\theta_0}$$

With the same process, we can obtain

$$\frac{dr(T)}{dr_0} = \frac{r(T)+v_vT(\gamma-\cos\theta(T))}{r_0}$$

When $\gamma > 1$, we can rewrite the $\frac{dr(T)}{d\theta_0}$ as following:

$$\frac{dr(T)}{d\theta_0} = \frac{(r_0-r(T)-v_vT(\gamma-\cos\theta_0))(\gamma-\cos\theta(T)) + r(T)(\cos\theta_0-\cos\theta(T))}{\sin\theta_0}$$

We can expect $r_0 \le r(T)+v_vT(\gamma-\cos\theta_0)$ since $0 < \theta(t) < \theta_0$, $\forall t \in (0,T]$. Besides, $\cos\theta_0 < \cos\theta(T)$ as well. Hence, $\frac{dr(T)}{d\theta_0} < 0$. On the other hand, $\frac{dr(T)}{dr_0} > 0$, directly from $\gamma > 1$.

Part Two: the case of $\gamma < 1$. There are two scenarios, $\bar{t}(\gamma) < T$ with $\bar{r}(\gamma) \ge R$, and $\bar{t}(\gamma) \ge T$ with $r(T) \ge R$.

1) $\bar{t}(\gamma) < T$, which means $\cos\theta(T)-\gamma > 0$.

Since $\bar{r}(\gamma, r_0, \theta_0) = r_0\frac{\sin\theta_0}{\tan^\gamma\frac{\theta_0}{2}}\frac{\tan^\gamma\frac{\theta_\gamma}{2}}{\sin\theta_\gamma}$, we can derive

$$\frac{d\bar{r}(\gamma,r_0,\theta_0)}{dr_0} = \frac{\sin\theta_0}{\tan^\gamma\frac{\theta_0}{2}}\frac{\tan^\gamma\frac{\theta_\gamma}{2}}{\sin\theta_\gamma},$$

$$\frac{d\bar{r}(\gamma,r_0,\theta_0)}{d\theta_0} = \frac{r_0(\cos\theta_0-\gamma)}{\tan^\gamma\frac{\theta_0}{2}}\frac{\tan^\gamma\frac{\theta_\gamma}{2}}{\sin\theta_\gamma}.$$

Thus, $\frac{d\bar{r}(\gamma,r_0,\theta_0)}{dr_0} > 0$ and $\frac{d\bar{r}(\gamma,r_0,\theta_0)}{d\theta_0} < 0$. Note that the case $\cos\theta_0-\gamma \ge 0$ is excluded since $\bar{r}(\gamma,r_0,\theta_0)$ will have no meaning at that time.

2) $\bar{t}(\gamma) \ge T$, which means $\cos\theta(T)-\gamma < 0$.

$$\frac{dr(T)}{dr_0} = \frac{r(T)+v_vT(\gamma-\cos\theta(T))}{r_0} > 0$$

We only need to prove $\frac{dr(T)}{d\theta_0} < 0$. We consider the following three cases:

a) $\cos\theta_0 > \gamma$, that is, $\theta_0 < \arccos\gamma$

Since $\theta(t)$ is decreasing, $\cos\theta(t) > \gamma$, $t \in (0,T]$ and $r(t)$ is thus increasing function on $t \in [0,T]$. At this time, we will have $r(T) < r_0 + v_vT(\cos\theta(T)-\gamma)$ instead. Rewrite $\frac{dr(T)}{d\theta_0}$ as follows.

$$\frac{dr(T)}{d\theta_0} = \frac{(r(T)-r_0-v_vT(\cos\theta(T)-\gamma))(\cos\theta_0-\gamma) + r_0(\cos\theta_0-\cos\theta(T))}{\sin\theta_0} < 0$$

b) $\cos\theta_0 \le \gamma$ & $\cos\theta(T) \le \gamma$

In this case, $r(t)$ will be monotone decreasing function. All the constraints are the same as the case where $\gamma > 1$. Thus, same proof can be applied.

c) $\cos\theta_0 \le \gamma$ & $\cos\theta(T) > \gamma$

At this time, $r(t)$ will be a convex function on t. It will be first decreasing and then increasing. Consider $\frac{d\theta(T)}{d\theta_0}$ at first. Note that $\theta(T)$ is increasing when $r_0 - v_vT(\cos\theta_0-\gamma) > 0$, that is, $\theta_0 < \arccos\left(\gamma - \frac{r_0}{v_vT}\right)$ and decreasing otherwise.

If $\theta_0 \in [\arccos\gamma, \arccos\left(\gamma-\frac{r_0}{v_vT}\right)]$, we have $\frac{d\theta(T)}{d\theta_0} > 0$ and thus $\frac{dr(T)}{d\theta_0} < 0$.

If $\theta_0 \in (\arccos\left(\gamma-\frac{r_0}{v_vT}\right), \pi)$, $\frac{d\theta(T)}{d\theta_0} < 0$. Rewrite the relationship between $r(T)$ and $\theta(T)$ as follows:

$$T = \frac{r_0(\gamma+\cos\theta_0) - r(T)(\gamma+\cos\theta(T))}{v_v(\gamma^2-1)}$$

or

$$v_vT(\gamma^2-1) = r_0(\gamma+\cos\theta_0) - r(T)(\gamma+\cos\theta(T))$$

Taking derivative with respect to θ_0 on both sides, we have

$$0 = -r_0\sin\theta_0 - \left(\frac{dr(T)}{d\theta_0}(\gamma+\cos\theta(T)) - r(T)\sin\theta(T)\frac{d\theta(T)}{d\theta_0}\right)$$

Therefore,

$$\frac{dr(T)}{d\theta_0}(\gamma+\cos\theta(T)) = -r_0\sin\theta_0 + r(T)\sin\theta(T)\frac{d\theta(T)}{d\theta_0}$$

Since $\frac{d\theta(T)}{d\theta_0} < 0, -r_0 \sin\theta_0 + r(T)\sin\theta(T)\frac{d\theta(T)}{d\theta_0} < 0$, and thus $\frac{dr(T)}{d\theta_0} < 0$ since $\gamma + \cos\theta(T) > 0$.

Above all, under the two scenarios, we have proved the result in Proposition 4. ∎

Proof of Lemma 4

Proof. To simplify the notation, let $r(\tau)$ and $\theta(\tau)$ denote the relative distance and LOS angle at τ. And $r(T)$, $\theta(T)$ denote the relative distance and LOS angle at T when two-turn policy $(\tau, \alpha_1, \alpha_2)$ is applied. In the first stage where $t \in [0, \tau]$, the dynamic process is the same as one-turn policy. Thus $\frac{dr(\tau)}{d\tau}$ and $\frac{d\theta(\tau)}{d\tau}$ can be obtained from Equations (5) and (6):

$$\frac{dr(\tau)}{d\tau} = -v_v(\gamma - \cos(\theta(\tau) - \alpha_1)),$$
$$\frac{d\theta(\tau)}{d\tau} = -\frac{v_v \sin(\theta(\tau) - \alpha_1)}{r(\tau)}. \tag{A3}$$

Now given $r(\tau)$ and $\theta(\tau)$, the derivatives of $\theta(T)$ and $r(T)$ on τ will be as follows:

$$\begin{aligned}\frac{\tan^\gamma\frac{\theta(T)-\alpha_2}{2}}{\sin^2(\theta(T)-\alpha_2)}\frac{d\theta(T)}{d\tau} &= \frac{\tan^\gamma\frac{\theta(\tau)-\alpha_2}{2}}{\sin^2(\theta(\tau)-\alpha_2)}\frac{d\theta(\tau)}{d\tau}\\ &- \frac{v_v r(\tau) - v_1(T-\tau)}{r(\tau)^2}\frac{\tan^\gamma\frac{\theta(\tau)-\alpha_2}{2}}{\sin(\theta(\tau)-\alpha_2)}\\ &- \frac{v_v(T-\tau)}{r(\tau)}\frac{(\gamma-\cos(\theta(\tau)-\alpha_2))\tan^\gamma\frac{\theta(\tau)-\alpha_2}{2}}{\sin^2(\theta(\tau)-\alpha_2)}\frac{d\theta(\tau)}{d\tau}\end{aligned} \tag{A4}$$

$$\begin{aligned}\frac{dr(T)}{d\tau} &= \frac{\sin(\theta(\tau)-\alpha_2)}{\tan^\gamma\frac{\theta(\tau)-\alpha_2}{2}}\frac{\tan^\gamma\frac{\theta(T)-\alpha_2}{2}}{\sin(\theta(T)-\alpha_2)}\frac{dr(\tau)}{d\tau}\\ &+ r(\tau)\frac{\tan^\gamma\frac{\theta(T)-\alpha_2}{2}}{\sin(\theta(T)-\alpha_2)}\frac{\cos(\theta(\tau)-\alpha_2)-\gamma}{\tan^\gamma\frac{\theta(\tau)-\alpha_2}{2}}\frac{d\theta(\tau)}{d\tau}\\ &+ r(\tau)\frac{\sin(\theta(\tau)-\alpha_2)}{\tan^\gamma\frac{\theta(\tau)-\alpha_2}{2}}\frac{(\gamma-\cos(\theta(T)-\alpha_2))\tan^\gamma\frac{\theta(T)-\alpha_2}{2}}{\sin^2(\theta(T)-\alpha_2)}\\ &\times\frac{d\theta(T)}{d\tau}\end{aligned} \tag{A5}$$

Substituting (A3) and (A4) into (A5),

$$\begin{aligned}&\frac{dr(T)}{d\tau}\\ &= K_0\{r(T)\sin(\theta(\tau)-\alpha_1)[\gamma-\cos(\theta(\tau)-\alpha_2)]\\ &\quad - r(T)\sin(\theta(\tau)-\alpha_2)[\gamma-\cos(\theta(\tau)-\alpha_1)]\\ &\quad - r(\tau)\sin(\theta(\tau)-\alpha_1)[\gamma-\cos(\theta(T)-\alpha_2)]\\ &\quad + r(\tau)\sin(\theta(\tau)-\alpha_2)[\gamma-\cos(\theta(T)-\alpha_2)]\\ &\quad - v_v(T-\tau)\sin(\theta(\tau)-\alpha_2)[\gamma-\cos(\theta(T)-\alpha_2)]\\ &\quad \times[\gamma-\cos(\theta(\tau)-\alpha_1)]\\ &\quad + v_v(T-\tau)\sin(\theta(\tau)-\alpha_1)[\gamma-\cos(\theta(T)-\alpha_2)]\\ &\quad \times[\gamma-\cos(\theta(\tau)-\alpha_2)]\}\\ &= K_0\{K_1\sin(\theta_1-\alpha_2) - K_2\sin(\theta_1-\alpha_1)\}\end{aligned}$$

where $K_0 = \frac{v_v}{r(\tau)\sin(\theta(\tau)-\alpha_2)}$, $K_1 = r(\tau)(\gamma - \cos(\theta(T)-\alpha_2)) - v_v(T-\tau)(\gamma-\cos(\theta(\tau)-\alpha_1))(\gamma-\cos(\theta(T)-\alpha_2)) - r(T)(\gamma-\cos(\theta(\tau)-\alpha_1))$, and $K_2 = r(\tau)(\gamma-\cos(\theta(T)-\alpha_2)) - v_v(T-\tau)(\gamma-\cos(\theta(\tau)-\alpha_2))(\gamma-\cos(\theta(T)-\alpha_2)) - r(T)(\gamma-\cos(\theta(\tau)-\alpha_2))$.

As we assumed that $\alpha_2 \in [\theta_0-\pi, \theta(\tau))$, that is, $\theta(\tau)-\alpha_2 \in [0,\pi]$, hence $\sin(\theta(\tau)-\alpha_2) \geq 0$, which implies $K_0 > 0$.

Consider $F \triangleq \frac{K_1\sin(\theta(\tau)-\alpha_2) - K_2\sin(\theta(\tau)-\alpha_1)}{r(\tau)}$, which can be simplified into:

$$\begin{aligned}F = 2\sin\frac{\alpha_1-\alpha_2}{2}\Big[&(A+\gamma-A\gamma-\cos(\theta(T)-\alpha_2))\cos\\ &\times\frac{\theta(\tau)-\alpha_1}{2}\cos\frac{\theta(\tau)-\alpha_2}{2}\\ &+(A-\gamma+A\gamma+\cos(\theta(T)-\alpha_2))\sin\frac{\theta(\tau)-\alpha_1}{2}\sin\\ &\times\frac{\theta(\tau)-\alpha_2}{2}\Big]\end{aligned}$$

where $A = \frac{r(T)+v_v(T-\tau)(\gamma-\cos(\theta(T)-\alpha_2))}{r(\tau)}$. With $\gamma - \cos(\theta(T)-\alpha_2) > 0$, we have $A > 0$. Besides, noting that $\gamma-\cos(\theta(t)-\alpha_2)$ is decreasing since $\theta(t)$ is decreasing and $\frac{dr(t)}{dt} = v_v(\cos(\theta(t)-\alpha_2)-\gamma)$ on $t \in [\tau, T]$, we can derive that $r(\tau) > r(T)+v_v(T-\tau)(\gamma-\cos(\theta(T)-\alpha_2))$ with mean value theorem, and thus $A < 1$. Above all, here we have $A \in [0,1]$.

Then we have to prove that both $A-\gamma+A\gamma+\cos(\theta(T)-\alpha_2)$ and $A+\gamma-A\gamma-\cos(\theta(T)-\alpha_2)$ are nonnegative.

1) $A+\gamma-A\gamma-\cos(\theta(T)-\alpha_2) > 0$.

$$\begin{aligned}&A+\gamma-A\gamma-\cos(\theta(T)-\alpha_2)\\ &\quad = (1-A)\gamma - \cos(\theta(T)-\alpha_2) + A\cos(\theta(T)-\alpha_2)\\ &\qquad + A(1-\cos(\theta(T)-\alpha_2))\\ &\quad = (1-A)(\gamma-\cos(\theta(T)-\alpha_2))\\ &\qquad + A(1-\cos(\theta(T)-\alpha_2))\end{aligned}$$

Since $0 < A < 1$, $\cos(\theta(T) - \alpha_2) < 1$ and $\gamma - \cos(\theta(T) - \alpha_2) > 0, A + \gamma - A\gamma - \cos(\theta(T) - \alpha_2) > 0$.

2) $A - \gamma + A\gamma + \cos(\theta(T) - \alpha_2) \geq 0$.

$$
\begin{aligned}
&A - \gamma + A\gamma + \cos(\theta(T) - \alpha_2)\\
&= (1+\gamma)\frac{r(T) + v_v(T-\tau)(\gamma - \cos(\theta(T) - \alpha_2))}{r(\tau)}\\
&\quad + (\cos(\theta(T) - \alpha_2) - \gamma)\\
&= (\gamma+1)\frac{r(T) + v_v(T-\tau)(\gamma - \cos(\theta(T) - \alpha_2))}{r(\tau)}\\
&\quad - (\gamma - \cos(\theta(T) - \alpha_2))\\
&= \frac{(\gamma+1)r(T) + (v_v(T-\tau)(\gamma+1) - r(\tau)) \times (\gamma - \cos(\theta(T) - \alpha_2))}{r(\tau)}\\
&= \frac{(1 + \cos(\theta(T) - \alpha_2))r(T) + (r(T) + v_v(T-\tau)(\gamma+1) - r(\tau))(\gamma - \cos(\theta(T) - \alpha_2))}{r(\tau)}
\end{aligned}
$$

As $r(T) + v_v(T-\tau)(\gamma+1) - r(\tau) \geq r(T) + v_v(T-\tau)(\gamma - \cos(\theta(\tau) - \alpha_2)) - r(\tau) > 0$ and $\gamma - \cos(\theta(T) - \alpha_2) > 0$, thus $A - \gamma + A\gamma + \cos(\theta(T) - \alpha_2) > 0$.Above all, $F > 0$ when $\alpha_1 > \alpha_2$, and $F < 0$, otherwise, and so does $\frac{dr(T)}{d\tau}$. ■

Proof of Lemma 5

Proof. As we know, $\bar{r}_2(\gamma; \tau, \alpha_1, \alpha_2)$ occurs when $\theta_2(t) - \alpha_2 = \arccos\gamma$, which means

$$
\bar{r}(\gamma; \tau, \alpha_1, \alpha_2) = \frac{r(\tau)\sin(\theta(\tau) - \alpha_2)}{\tan^{\gamma}\frac{\theta(\tau)-\alpha_2}{2}} \frac{\tan^{\gamma}\frac{\arccos\gamma}{2}}{\sin(\arccos\gamma)}
$$

Let $c \triangleq \frac{\tan^{\gamma}\frac{\arccos\gamma}{2}}{\sin(\arccos\gamma)}$,

$$
\begin{aligned}
&\frac{d\bar{r}(\gamma)}{d\tau}\\
&= c\frac{\sin(\theta(\tau) - \alpha_2)}{\tan^{\gamma}\frac{\theta(\tau)-\alpha_2}{2}} \frac{dr(\tau)}{d\tau}\\
&\quad + cr(\tau)\frac{\cos(\theta(\tau) - \alpha_2) - \gamma}{\tan^{\gamma}\frac{\theta(\tau)-\alpha_2}{2}} \frac{d\theta(\tau)}{d\tau}\\
&= c\left[\frac{\sin(\theta_1 - \alpha_2)}{\tan^{\gamma}\frac{\theta_1-\alpha_2}{2}} v_v(\cos(\theta(\tau) - \alpha_1) - \gamma)\right.\\
&\quad \left. + r(\tau)\frac{\cos(\theta(\tau) - \alpha_2) - \gamma}{\tan^{\gamma}\frac{\theta(\tau)-\alpha_2}{2}} v_v \frac{\sin(\alpha_1 - \theta_1)}{r(\tau)}\right]\\
&= \frac{v_v c}{\tan^{\gamma}\frac{\theta(\tau)-\alpha_2}{2}}[(\gamma - \cos(\theta(\tau) - \alpha_2))\sin(\theta(\tau) - \alpha_1)\\
&\quad - (\gamma - \cos(\theta(\tau) - \alpha_1))\sin(\theta(\tau) - \alpha_2)]\\
&= \frac{v_v c}{\tan^{\gamma}\frac{\theta(\tau)-\alpha_2}{2}}\left[2\sin\frac{\alpha_1 - \alpha_2}{2}\left(\cos\frac{\alpha_1 - \alpha_2}{2}\right.\right.\\
&\quad \left.\left. - \gamma\cos\left(\theta(\tau) - \frac{\alpha_1 + \alpha_2}{2}\right)\right)\right]\\
&= \frac{2v_v c \sin\frac{\alpha_1-\alpha_2}{2}}{\tan^{\gamma}\frac{\theta(\tau)-\alpha_2}{2}}\left[\cos\left(\frac{\theta(\tau) - \alpha_2}{2} - \frac{\theta(\tau) - \alpha_1}{2}\right)\right.\\
&\quad \left. - \gamma\cos\left(\frac{\theta(\tau) - \alpha_1}{2} + \frac{\theta(\tau) - \alpha_2}{2}\right)\right]\\
&= \frac{2v_v c \sin\frac{\alpha_1-\alpha_2}{2}}{\tan^{\gamma}\frac{\theta(\tau)-\alpha_2}{2}}\left[(1-\gamma)\cos\frac{\theta(\tau) - \alpha_1}{2}\cos\frac{\theta(\tau) - \alpha_2}{2}\right.\\
&\quad \left. + (1+\gamma)\sin\frac{\theta(\tau) - \alpha_1}{2}\sin\frac{\theta(\tau) - \alpha_2}{2}\right]
\end{aligned}
$$

As all the items are nonnegative, we can conclude $\frac{d\bar{r}(\gamma)}{d\tau} \geq 0$ when $\alpha_1 > \alpha_2$ and $\frac{d\bar{r}(\gamma)}{d\tau} < 0$ when $\alpha_1 < \alpha_2$. ■

Proof of Proposition 5

Proof. When $\gamma > 1$, $r(t)$ is strictly decreasing during $t \in [0, T]$, and the sufficient and necessary condition for a two-turn policy being feasible is $r(T; \tau, \alpha_1, \alpha_2) \geq R$. With Lemma 4, we can directly conclude the result.

When $\gamma < 1$, $r(t)$ may be strictly decreasing during $t \in [0, T]$, or first decreasing and then increasing on each stage. The condition is divided into two scenarios.

If $\gamma - \cos(\theta(T; \tau, \alpha_1, \alpha_2) - \alpha_2) \geq 0$, $r(t)$ will be decreasing on $t \in [\tau, T]$. If $\alpha_1 > \alpha^* > \alpha_2$, we have $r(t) > R$ during $t \in [0, \tau]$ for any turn time τ. Therefore, the sufficient and necessary condition is still $r(T; \tau, \alpha_1, \alpha_2) \geq R$. The result holds in the same way as the case $\gamma > 1$. If $\alpha_1 < \alpha^* < \alpha_2$, notice that it is impossible that $\gamma - \cos(\theta(\tau; \tau, \alpha_1, \alpha_2) - \alpha_1) < 0$. As $\gamma - \cos(\theta(\tau; \tau, \alpha_1, \alpha_2) - \alpha_1) \geq 0$, which implies $r(t)$ is decreasing during $t \in [0, T]$, thus the sufficient and necessary condition will be $r(T; \tau, \alpha_1, \alpha_2) \geq R$. With Lemma 4, we can conclude that the vessel is safe before a specific τ and Proposition 5 holds.

On the other hand, if $\gamma - \cos(\theta(T; \tau, \alpha_1, \alpha_2) - \alpha_2) < 0$, which implies $r(T; \tau, \alpha_1, \alpha_2)$ is not critical for the vessel to be safe. If $\alpha_1 > \alpha^* > \alpha_2$, the vessel is always safe during $t \in [0, \tau]$. Now that $(\tau, \alpha_1, \alpha_2)$ is feasible, $(\tau', \alpha_1, \alpha_2)$ is feasible as well since $\bar{r}_2(\gamma, \tau, \alpha_1, \alpha_2)$ is increasing on τ. For the case of $\alpha_1 < \alpha^* < \alpha_2$, $r(t)$ must be decreasing on $t \in [0, \tau]$. Therefore, $r(\tau') > r(\tau)$ if $\tau' < \tau$. Meanwhile, we have proved that $\bar{r}_2(\gamma; \tau, \alpha_1, \alpha_2)$ is decreasing on τ. If $\bar{r}_2(\gamma; \tau, \alpha_1, \alpha_2)$ is not achieved during $[\tau, T]$, it implies that $r(t)$ is increasing on $t \in [\tau, T]$ and the feasibility condition only depends on $r(\tau)$ and the result holds definitely. Otherwise, $\bar{r}_2(\gamma; \tau, \alpha_1, \alpha_2)$ is the smallest $r(t)$ over $[0, T]$ and the result can be concluded due to the monotonicity of $\bar{r}_2(\gamma; \tau, \alpha_1, \alpha_2)$ from Lemma 5.

∎