Mathematical Logic
(The Berkeley undergraduate course)

Theodore A. Slaman
W. Hugh Woodin

Contents

1
Propositional logic

Propositional logic governs the way by which propositions are combined in compound sentences.

Informally, a proposition is a declarative sentence, such as any of the following.

- Life is nothing but a competition to be the criminal rather than the victim. (B. Russell)
- Life is as tedious as a twice-told tale. (W. Shakespeare)
- Life is a dead-end street. (H. L. Menken)
- Life is too short to learn German. (R. Porson)

Propositions may be combined by logical connectives to form more complicated statements, such as "If life is a dead-end street, then life is too short to learn German" or "If it is not the case that life is too short to learn German, then life is as tedious as a twice told tale". The truth or falsity of a compound statement depends solely on that of its parts. Understanding propositional logic is just understanding that dependence.

In the next section, we will introduce symbols A_n for propositions, $\neg$ for negation, and $\to$ for implication. Using $A_1, \dots, A_4$ to denote the above propositions, our two compound sentences would be denoted by $(A_3 \to A_4)$ and $((\neg A_4) \to A_2)$.

1.1 The language

Our language for propositional logic consists of (certain) finite sequences of symbols.

Definition 1.1
- The *logical symbols* are the following symbols.

$$(\quad) \quad \neg \quad \to$$

- The *propositional symbols* are A_n, for n in $\mathbb{N}$. ($\mathbb{N}$ is the set of non-negative integers; i.e. the set of *natural numbers*)

Definition 1.2 If $s = \langle s_1, \dots, s_n \rangle$ and $t = \langle t_1, \dots, t_m \rangle$ are finite sequences, we let $s + t$ denote the finite sequence

$$s + t = \langle s_1, \dots, s_n, t_1, \dots, t_m \rangle.$$

Definition 1.3 The *propositional language* $\mathcal{L}_0$ is the smallest set L of finite sequences of the above symbols satisfying the following properties.

(1) For each propositional symbol A_n with $n \in \mathbb{N}$,

$$A_n \in L.$$

(2) For each pair of finite sequences s and t, if s and t belong to L, then

$$(\neg s) \in L$$

and

$$(s \to t) \in L.$$

For the duration of Chapter 1, we will use *propositional formula* or just *formula* to refer to an element of $\mathcal{L}_0$.

In Definition 1.3, we defined the propositional language $\mathcal{L}_0$ as the smallest set which is closed under Conditions 1 and 2. In the following, we show that $\mathcal{L}_0$ is well defined.

Theorem 1.4 *$\mathcal{L}_0$ is the intersection of all of the sets which satisfy the two conditions of Definition 1.3.*

Proof. Let L_0 be the intersection of all of the sets which satisfy the two conditions of Definition 1.3. There is at least one such set, since the set of all finite sequences of symbols does satisfy the two conditions. We claim that L_0 is a set which satisfies those two conditions.

For each $n \in \mathbb{N}$, A_n is an element of every set which satisfies Condition 1. Consequently, A_n is an element of the intersection of all such sets, and thus it is an element of L_0.

Now, suppose that s and t belong to L_0. Then they belong to every set which satisfies Conditions 1 and 2. But then, for every such set, we can apply Condition 2 to conclude that $(\neg s)$ and $(s \to t)$ also belong to that set. Therefore, $(\neg s)$ and $(s \to t)$ belong to the intersection of all such sets, and thus belong to L_0.

Hence, L_0 satisfies Conditions 1 and 2. Since it is contained in every set which also satisfies those conditions, it must the smallest such set. Consequently, $\mathcal{L}_0$ is equal to L_0. □

1.1.1 Subformulas

Definition 1.5 Suppose that $s = \langle s_1, \ldots, s_n \rangle$ is a finite sequence. A finite sequence t is a *block-subsequence* of s if there exist non-negative integers i and j such that

(1) $i + j \leq n$,
(2) $t = \langle s_i, s_{i+1}, \ldots, s_{i+j} \rangle$.

Example 1.6 (1) $\langle 3 \rangle$ is a block-subsequence of $\langle 1, 2, 3, 4, 5, 6 \rangle$.

(2) $\langle 3,4,5\rangle$ is a block-subsequence of $\langle 1,2,3,4,5,6\rangle$.
(3) $\langle 1,6\rangle$ is not a block-subsequence of $\langle 1,2,3,4,5,6\rangle$.
(4) If s is a finite sequence and s has length n, then there are at most

$$\sum_{i=1}^{n}(n-i)+1 = n^2+n-(1/2)n(n+1) = 1/2(n+1)n$$

block-subsequences of s.

Definition 1.7 (1) A sequence $\langle a_1,\ldots,a_k\rangle$ is an *initial segment* of another sequence $\langle b_1,\ldots,b_m\rangle$ if and only if k is less than or equal to m and for all $i\leq k$, $a_i=b_i$. In other words, $\langle b_1,\ldots,b_m\rangle$ is equal to $\langle a_1,\ldots,a_k\rangle+\langle b_{k+1},\ldots,b_m\rangle$, where $\langle b_{k+1},\ldots,b_m\rangle$ could be the empty sequence.
(2) When m is greater than k, we say that $\langle a_1,\ldots,a_k\rangle$ is a *proper initial segment* of $\langle a_1,\ldots,a_k,b_{k+1},\ldots,b_m\rangle$.

Definition 1.8 Suppose that φ is a formula. A formula ψ is a *subformula* of φ if ψ is a block-subsequence of φ.

The subformulas of φ are precisely the formulas which arise in the definition of φ.

Definition 1.9 Suppose that

$$\varphi=\langle a_1,\ldots,a_n\rangle$$

is a formula.

Suppose s is a finite sequence. An *occurrence* of s in φ is an interval

$$[j_1,j_2]$$

such that $s=\langle a_{j_1},\ldots,a_{j_2}\rangle$.

Remark 1.10 Suppose that φ is a formula. A formula ψ is a subformula of φ if and only if ψ has an occurrence in φ.

Lemma 1.11 (Readability) *Suppose that φ is a formula in $\mathcal{L}_0$. Then exactly one of the following conditions applies.*
(1) There is an n such that $\varphi=\langle A_n\rangle$.
(2) There is a $\psi\in\mathcal{L}_0$ such that $\varphi=(\neg\psi)$.
(3) There are ψ_1 and ψ_2 in $\mathcal{L}_0$ such that $\varphi=(\psi_1\rightarrow\psi_2)$.

Proof. Consider the subset L of $\mathcal{L}_0$ which consists of those formulas which satisfy the above three clauses. By the first clause, if $n\in\mathbb{N}$, then $\langle A_n\rangle\in L$. Consequently, L satisfies Condition 1 of Definition 1.3. Secondly, if ψ is in L, then $\psi\in\mathcal{L}_0$ and so $(\neg\psi)\in\mathcal{L}_0$. But then $(\neg\psi)$ is an element of $\mathcal{L}_0$ which satisfies the second of the above clauses, and hence $(\neg\psi) \in L$. Similarly, if ψ_1 and ψ_2 belong to L, then so does $(\psi_1 \to \psi_2)$. Thus, L satisfies Condition 2 of Definition 1.3. It follows that $\mathcal{L}_0 \subseteq L$, and so $\mathcal{L}_0 = L$.

It remains to show that the three possibilities are mutually exclusive.

Clearly, the first case excludes the other two since both of the formulas in the latter two cases begin with the symbol (. Now, if $\varphi = (\neg\psi)$, then the second symbol in φ is $\neg$. However, if $\varphi = (\psi_1 \to \psi_2)$, then the second symbol in φ is the first symbol in ψ_1, which by the above is either an A_n or (. Consequently, these two cases are mutually exclusive. $\square$

We now prove a technical lemma which we will apply to show that the subformulas of φ mentioned above are uniquely determined.

Lemma 1.12 *If $\varphi \in \mathcal{L}_0$, then no proper initial segment of φ is an element of $\mathcal{L}_0$.*

Proof. We prove Lemma 1.12 by induction on n.

If φ has length 1 then the only subsequence to be considered is the empty sequence, which by Lemma 1.11 is not an element of $\mathcal{L}_0$.

Now suppose that $\varphi \in \mathcal{L}_0$ has length n, $n > 1$, and Lemma 1.12 holds for all m less than n. By Lemma 1.11, since φ has length greater than 1, φ has one of two forms: $(\neg\psi)$ or $(\psi_1 \to \psi_2)$.

Suppose that φ is $(\neg\psi)$. For a contradiction, suppose that $\theta \in \mathcal{L}_0$ is a proper initial segment of $(\neg\psi)$. Then the first symbol in θ is (, so θ is not of the form $\langle A_i \rangle$, and by Lemma 1.11 the length of θ is greater than one. Thus, the second symbol in θ is $\neg$, which by Lemma 1.11 is not the first symbol of any element of $\mathcal{L}_0$, and so θ cannot be of the form $(\theta_1 \to \theta_2)$. Consequently, there is a $\theta_1 \in \mathcal{L}_0$ such that θ is equal to $(\neg\theta_1)$. But then $(\neg\psi)$ has $(\neg\theta_1)$ as a proper initial segment, and so ψ has θ_1 as a proper initial segment, contradiction to the induction hypothesis.

Finally, suppose that φ is $(\psi_1 \to \psi_2)$ and that $\theta \in \mathcal{L}_0$ is a proper initial segment of φ. We can apply Lemma 1.11 and argue as in the previous paragraph that there are θ_1 and θ_2 in $\mathcal{L}_0$ such that $\theta = (\theta_1 \to \theta_2)$. But then either θ_1 is a proper initial segment of ψ_1 (a contradiction), ψ_1 is a proper initial segment of θ_1 (a contradiction), or $\psi_1 = \theta_1$ and θ_2 is a proper initial segment of ψ_2 (a contradiction).

In either case, φ has no proper initial segment in $\mathcal{L}_0$. $\square$

Theorem 1.13 (Unique Readability) *Suppose that φ is a formula in $\mathcal{L}_0$. Then exactly one of the following conditions applies.*

(1) There is an n such that $\varphi = \langle A_n \rangle$.

(2) There is a $\psi \in \mathcal{L}_0$ such that $\varphi = (\neg\psi)$.

(3) There are ψ_1 and ψ_2 in $\mathcal{L}_0$ such that $\varphi = (\psi_1 \to \psi_2)$.

Further, in cases (2) and (3), the formulas ψ, and ψ_1 and ψ_2 are unique, respectively.

Proof. By Lemma 1.11, it is enough to check the claim of uniqueness.

First, suppose that $\varphi = (\neg\psi)$ and $\varphi = (\neg\theta)$. Thus, the sequence of symbols φ can be read as $\langle (, \neg\rangle + \psi + \langle)\rangle$ and as $\langle (, \neg\rangle + \theta + \langle)\rangle$. The occurrences of ψ and θ within φ have the same length and the same elements, and therefore are equal.

Finally, suppose that $\varphi = (\psi_1 \to \psi_2)$ and $\varphi = (\theta_1 \to \theta_2)$. Since both ψ_1 and θ_1 belong to $\mathcal{L}_0$, by Lemma 1.12 neither can be a proper initial segment of the other. Since they are initial segments of each other, they must be equal. As in the case of negation, it follows that ψ_2 and θ_2 are also equal. $\square$

1.1.2 Exercises

(1) Give three examples of elements of $\mathcal{L}_0$ with at least 15 symbols each. The examples should have an interesting structure. For one of these examples, give a meaningful sentence which has the same structure.

(2) For which natural numbers n are there elements of $\mathcal{L}_0$ of length n?

(3) Show that a sequence φ is an element of $\mathcal{L}_0$ if and only if there is a finite sequence of sequences $\langle \varphi_1, \dots, \varphi_n\rangle$ such that $\varphi_n = \varphi$, and for each if i less than or equal to n either there is an m such that $\varphi_i = \langle A_m\rangle$, or there is a j less than i such that φ_i is equal to $(\neg\varphi_j)$, or there are j_1 and j_2 less than i such that φ_i is equal to $(\varphi_{j_1} \to \varphi_{j_2})$.

(4) Give an algorithm (suitable to be programmed for a computer) to determine whether a given finite sequence belongs to $\mathcal{L}_0$.

(5) Consider the set of symbols $*$ and $\#$. Let $\mathcal{L}^*$ be the smallest set L of sequences of these symbols with the following properties.
   a) The length one sequences $\langle *\rangle$ and $\langle \#\rangle$ belong to L.
   b) If σ and τ belong to L, then so do $\langle *\rangle + \sigma + \langle \#\rangle$ and $\langle *\rangle + \sigma + \tau + \langle \#\rangle$.
   State Readability and Unique Readability for $\mathcal{L}^*$ and determine for each whether it holds.

(6) (Polish Notation) Let $\mathcal{P}_0$ be the smallest set of sequences P such that the following conditions hold.
   a) For each n, $\langle A_n\rangle \in P$.
   b) If ψ_1 and ψ_2 belong to P, then so do $\neg\psi_1 = \langle\neg\rangle + \psi_1$ and $\to\psi_1\psi_2 = \langle\to\rangle + \psi_1 + \psi_2$.
   State and prove the unique readability theorem for $\mathcal{P}_0$. Note, the Polish system of notation does away with parentheses.

1.2 Truth assignments

We can now describe the semantics for propositional logic.

Definition 1.14 A *truth assignment for* $\mathcal{L}_0$ is a function ν from the set of propositional symbols $\{A_n : n \in \mathbb{N}\}$ into the set $\{T, F\}$.

Now, $(\neg\psi)$ should have the opposite truth value from that of ψ and the truth value of $(\psi_1 \to \psi_2)$ should reflect whether, if ψ_1 has truth value T, then ψ_2 has truth value T.

Theorem 1.15 *Suppose that ν is a truth assignment for $\mathcal{L}_0$. Then there is a unique function $\overline{\nu}$ defined on $\mathcal{L}_0$ with the following properties.*

(1) For all n, $\overline{\nu}(\langle A_n \rangle) = \nu(A_n)$.
(2) For all $\psi \in \mathcal{L}_0$,

$$\overline{\nu}((\neg\psi)) = \begin{cases} T, & \textit{if } \overline{\nu}(\psi) = F; \\ F, & \textit{otherwise.} \end{cases}$$

(3) For all ψ_1 and ψ_2 in $\mathcal{L}_0$,

$$\overline{\nu}((\psi_1 \to \psi_2)) = \begin{cases} F, & \textit{if } \overline{\nu}(\psi_1) = T \textit{ and } \overline{\nu}(\psi_2) = F; \\ T, & \textit{otherwise.} \end{cases}$$

Proof. We can define $\overline{\nu}$ by recursion on the natural numbers greater than or equal to 1. During the $s+1$st step of the recursion, we may assume that $\overline{\nu}$ is already defined on all elements of $\mathcal{L}_0$ which have length less than or equal to s.

Base step. For each $n \in \mathbb{N}$, define $\overline{\nu}(\langle A_n \rangle) = \nu(A_n)$.

Recursion step. Suppose that $s \geq 1$, that $\overline{\nu}$ is defined on all sequences from $\mathcal{L}_0$ of length less than or equal to s, and that φ is an element of $\mathcal{L}_0$ of length $s+1$.

If $\varphi = (\neg\psi)$, we define $\overline{\nu}(\varphi)$ as in (2); if $\varphi = (\psi_1 \to \psi_2)$, we define $\overline{\nu}(\varphi)$ as in (3).

It remains to show that $\overline{\nu}$ is well defined on $\mathcal{L}_0$, it satisfies (1), (2), and (3), and that it is the unique such function—existence and uniqueness.

We will prove the first two claims (existence) by induction on the natural numbers greater than or equal to 1.

Clearly, $\overline{\nu}$ is well defined on the elements of $\mathcal{L}_0$ of length 1 and satisfies (1).

Suppose that $s \geq 1$ and, by induction, that $\overline{\nu}$ is well defined on the set of elements of $\mathcal{L}_0$ of length less than or equal to s. Suppose that $\varphi \in \mathcal{L}_0$ and φ has length $s+1$. By the Unique Readability Theorem 1.13, either there is a $\psi \in \mathcal{L}_0$ such that $\varphi = (\neg\psi)$, or there are ψ_1 and ψ_2 such that $\varphi = (\psi_1 \to \psi_2)$, the two cases are mutually exclusive, and, in either case, the subformulas of φ which appear in the way described are unique. Thus, the condition to define $\overline{\nu}(\varphi)$ is unambiguous, showing that $\overline{\nu}$ is well defined at φ. Further, $\overline{\nu}$ is defined at φ so as to satisfy whichever of (2) or (3) is relevant.

By induction, $\overline{\nu}$ is well defined on $\mathcal{L}_0$ and satisfies (1), (2), and (3).

Now, we verify uniqueness. Suppose that $\overline{\nu}^* : \mathcal{L}_0 \to \{T, F\}$ and satisfies (1), (2), and (3). For the sake of a contradiction, suppose that $\overline{\nu}^*$ is not equal to $\overline{\nu}$. Fix φ so that $\overline{\nu}^*(\varphi) \neq \overline{\nu}(\varphi)$ and so that there is no $\psi \in \mathcal{L}_0$ such that ψ is strictly shorter than φ and $\overline{\nu}^*(\psi) \neq \overline{\nu}(\psi)$.

Since $\overline{\nu}^*$ satisfies (1), for every n, $\overline{\nu}^*(\langle A_n \rangle) = \nu(A_n)$. By definition, $\overline{\nu}^(\langle A_n \rangle) = \nu(A_n)$. Hence, for every n, $\overline{\nu}^*(\langle A_n \rangle) = \overline{\nu}(\langle A_n \rangle)$.

Consequently, the length of φ must be greater than 1. By the Unique Readability Theorem 1.13, φ is either a negation $(\neg\psi)$ or an implication $(\psi_1 \to \psi_2)$

and uniquely so. In the first case, since $\overline{\nu}^*$ satisfies (2), $\overline{\nu}^*((\neg\psi))$ has the opposite value from $\overline{\nu}^*(\psi)$. Since ψ is shorter than φ, $\overline{\nu}^*(\psi) = \overline{\nu}(\psi)$. By definition, $\overline{\nu}((\neg\psi))$ has the opposite value from $\overline{\nu}(\psi)$. It follows that $\overline{\nu}^*((\neg\psi))$ and $\overline{\nu}((\neg\psi))$ are equal. Thus, φ cannot be a negation. An analogous argument shows that φ cannot be an implication. This is a contradiction to the Readability Lemma 1.11. Consequently, $\overline{\nu}^*$ is equal to $\overline{\nu}$, which completes the proof of the theorem. $\square$

Theorem 1.16 *Suppose that $\varphi \in \mathcal{L}_0$ and that ν and μ are truth assignments which agree on the propositional symbols which occur in φ. Then $\overline{\nu}(\varphi) = \overline{\mu}(\varphi)$.*

Proof. Proceed just as in the uniqueness part of the proof of Theorem 1.13. Show that there cannot be a shortest subformula of φ where $\overline{\nu}$ and $\overline{\mu}$ disagree. $\square$

Definition 1.17 (1) A truth assignment ν *satisfies* a formula φ if and only if $\overline{\nu}(\varphi) = T$. Similarly, ν *satisfies* a set of formulas Γ if and only if it satisfies all of the elements of Γ.

(2) φ is a *tautology* if and only if every truth assignment satisfies φ.

(3) $\varphi \in \mathcal{L}_0$ or $\Gamma \subset \mathcal{L}_0$ are *satisfiable* if and only if there is a truth assignment which satisfies φ or Γ, respectively.

(4) φ is a *contradiction* if and only if there is no truth assignment which satisfies φ.

To give an example, consider the formula $(\neg((\neg A_1) \to A_2))$ and a truth assignment ν such that $\nu(A_1) = \nu(A_2) = F$. By Theorem 1.16, the values of ν on A_1 and A_2 determine the value of $\overline{\nu}$ on $(\neg((\neg A_1) \to A_2))$. In Figure 1.1, we show the values of $\overline{\nu}$ on $(\neg((\neg A_1) \to A_2))$ and its subformulas.

A_1	A_2	$(\neg A_1)$	$((\neg A_1) \to A_2)$	$(\neg((\neg A_1) \to A_2))$
F	F	T	F	T

Fig. 1.1 Extending a truth assignment

We can expand the table to systematically examine all possible truth assignments on $(\neg((\neg A_1) \to A_2))$, as in Figure 1.2.

A_1	A_2	$(\neg A_1)$	$((\neg A_1) \to A_2)$	$(\neg((\neg A_1) \to A_2))$
T	T	F	T	F
T	F	F	T	F
F	T	T	T	F
F	F	T	F	T

Fig. 1.2 The truth table for $(\neg((\neg A_1) \to A_2))$

Truth tables, such as the one in Figure 1.2, provide a systematic method to examine all the possible truth assignments for a given formula. Given a formula φ, we generate a truth table for φ as follows.

(1) The top row of the table consists of a list $\psi_1, \psi_2, \ldots, \psi_n = \varphi$ consisting of the subformulas of φ, ordered from left to right as follows.
  a) The subformulas of φ of the form $\langle A_m \rangle$ appear in the list without repetition before any of the other subformulas of φ.
  b) For each $i \leq n$ all of the proper subformulas of ψ_i appear in the list $\psi_1, \psi_2, \ldots, \psi_{i-1}$.
  c) The last element of the list is φ.

(2) Letting k be the number of subformulas of φ of the form $\langle A_m \rangle$, we consider all of the 2^k possible truth assignments for their propositional symbols. We use a row in the table for each such truth assignment ν, and we fill in the cell below $\langle A_m \rangle$ in that row with the value of ν at A_m.

(3) Finally, we work our way across each row and fill in the values of $\overline{\nu}$ at ψ_i as determined by the values already filled in for its subformulas.

We give another example in Figure 1.3. This time we have chosen the tautology expressing the principle that if A_1 implies A_2, then the contrapositive implication from $(\neg A_2)$ to $(\neg A_1)$ also holds.

A_1	A_2	$(A_1 \to A_2)$	$(\neg A_1)$	$(\neg A_2)$	$((\neg A_2) \to (\neg A_1))$	$((A_1 \to A_2) \to ((\neg A_2) \to (\neg A_1)))$
T	T	T	F	F	T	T
T	F	F	F	T	F	T
F	T	T	T	F	T	T
F	F	T	T	T	T	T

Fig. 1.3 The truth table for $((A_1 \to A_2) \to ((\neg A_2) \to (\neg A_1)))$

Theorem 1.18 *There are algorithms to determine whether a propositional formula φ is a tautology, satisfiable, or a contradiction.*

Proof. Starting with a formula φ, we can systematically generate its truth table. Then φ is a tautology if and only if every entry in the last column of its truth table is equal to T. It is satisfiable if and only if there is an entry in the last column of its truth table which is equal to T. It is a contradiction if and only if every entry in the last column of its truth table is equal to F.

Remark 1.19 Roughly speaking, if φ has n many symbols, then the analysis of φ by the method of truth tables involves 2^n many steps. A question which has received a considerable amount of attention is whether there is a more efficient method which when given φ determines whether φ is satisfiable. For more information on this problem, known as the $P = NP$ problem, and even a cash prize, see the following web site.

http://www.claymath.org/millennium/P_vs_NP/

1.2.1 Truth functions

Definition 1.20 An n-place *truth function* is a function whose domain is the set of sequences of T's and F's of length n, written $\{T,F\}^n$ and whose range is contained in $\{T,F\}$.

If φ is a formula in $\mathcal{L}_0$ and the propositional symbols which occur in φ are contained in the set $\{A_0,\dots,A_{n-1}\}$, then we can define the truth function f_φ derived from φ. Given $\sigma \in \{T,F\}^n$, we let ν be the truth assignment on $\{A_0,\dots,A_{n-1}\}$ such that $\nu(A_{i-1})$ is equal to the ith element of σ, and we define $f_\varphi(\sigma)$ to be $\overline{\nu}(\varphi)$.

In the next theorem, we show that $\mathcal{L}_0$ is as expressive as is possible. By this, we mean that every truth function is represented by a formula in $\mathcal{L}_0$.

Theorem 1.21 *Suppose that $f:\{T,F\}^n \to \{T,F\}$ is a truth function. Then there is a formula φ such that $f_\varphi = f$.*

Proof. We build up to the formula φ by a sequence of smaller steps.

For $\sigma \in \{T,F\}^n$, define $\theta_{\sigma,i}$ so that

$$\theta_{\sigma,i} = \begin{cases} A_{i-1}, & \text{if } \sigma(i) = T; \\ (\neg A_{i-1}), & \text{if } \sigma(i) = F. \end{cases}$$

Given two formulas ψ_1 and ψ_2, we define the conjunction of ψ_1 and ψ_2 to be the formula $(\neg(\psi_1 \to (\neg\psi_2)))$. As is seen in Figure 1.2.1, a truth assignment satisfies the conjunction of ψ_1 and ψ_2 if and only if it satisfies both ψ_1 and ψ_2.

ψ_1	ψ_2	$(\neg\psi_2)$	$(\psi_1 \to (\neg\psi_2))$	$(\neg(\psi_1 \to (\neg\psi_2)))$
T	T	F	F	T
T	F	T	T	F
F	T	F	T	F
F	F	T	T	F

Fig. 1.4 The conjunction of ψ_1 and ψ_2.

Given more than two formulas $\psi_1,\dots,\psi_n$, we use recursion and define their conjunction to be the conjunction of ψ_1 with the conjunction of $\psi_2,\dots,\psi_n$. For example, the conjunction of ψ_1, ψ_2, and ψ_3 is the formula

$$(\neg(\psi_1 \to (\neg(\neg(\psi_2 \to (\neg\psi_3)))))).$$

By induction, if ν is a truth assignment, then $\overline{\nu}$ maps the conjunction of $\psi_1,\dots,\psi_n$ to T if and only if $\overline{\nu}$ maps each of $\psi_1,\dots,\psi_n$ to T.

For $\sigma \in \{T,F\}^n$, we let ψ_σ be the conjunction of the formulas $\theta_{\sigma,i}$ for i less than or equal to n. The only truth assignments that satisfy ψ_σ are those which assign $\sigma(i)$ to A_{i-1}.

Given two formulas ψ_1 and ψ_2, we define the disjunction of ψ_1 and ψ_2 to be the formula $((\neg\psi_1) \to \psi_2)$. As is seen in Figure 1.2.1, a truth assignment satisfies

ψ_1	ψ_2	$(\neg\psi_1)$	$((\neg\psi_1) \to \psi_2)$
T	T	F	T
T	F	F	T
F	T	T	T
F	F	T	F

Fig. 1.5 The disjunction of ψ_1 and ψ_2.

the conjunction of ψ_1 and ψ_2 if and only if it satisfies at least one of ψ_1 or ψ_2. As above, when n is greater than two, we define the disjunction of $\psi_1, \dots, \psi_n$ to be the disjunction of ψ_1 with the disjunction of $\psi_2, \dots, \psi_n$. By another induction, if ν is a truth assignment, then $\overline{\nu}$ maps the disjunction of $\psi_1, \dots, \psi_n$ to T if and only if it maps at least one of $\psi_1, \dots, \psi_n$ to T.

Now we let φ_f be the disjunction of the set of formulas ψ_σ for which $f(\sigma) = T$. By construction, if ν is a truth assignment that satisfies φ_f, then there is a σ such that $f(\sigma) = T$ and for all i less than or equal to n, $\nu(A_{i-1})$ is equal to the ith element of σ. Consequently, f is equal to f_{φ_f}, as required. □

Remark 1.22 It is not unusual to include symbols $\wedge$ for conjunction, $\vee$ for disjunction, and $\leftrightarrow$ for "if and only if". By Theorem 1.21, these and all other logical connectives can be expressed in the language with only $\neg$ and $\to$.

Of course, the fewer symbols there are in the language, the fewer the number of cases there are in proofs by induction, so we decided in favor a small number of logical symbols. Occasionally, we pay a price for that decision: for example, with the lengths of the formulas that appeared in the proof of Theorem 1.21.

Remark 1.23 In some applications, it important to the best possible representative of a truth function f. Best possible could mean having the shortest length or having the fewest logical connectives of a certain type. When n is large, it is computationally prohibitive to generate the truth tables for all of the possible formulas with the desired functionality. Finding the optimal φ for a specified f remains an interesting problem.

1.3 A proof system for $\mathcal{L}_0$

Suppose that Γ is a subset of $\mathcal{L}_0$ so that Γ is a set of propositional formulas. We shall define a formal notion of proof. Intuitively a proof from Γ will be a finite sequence,

$$\langle \varphi_1, \dots, \varphi_n \rangle$$

of propositional formulas which satisfies certain conditions. In order to make the definition precise we need to first define the set of *Logical Axioms*.

Definition 1.24 Suppose that φ_1, φ_2 and φ_3 are propositional formulas. Then each of the following propositional formulas is a logical axiom:
(Group I axioms)

(1) $((\varphi_1 \to (\varphi_2 \to \varphi_3)) \to ((\varphi_1 \to \varphi_2) \to (\varphi_1 \to \varphi_3)))$
(2) $(\varphi_1 \to \varphi_1)$
(3) $(\varphi_1 \to (\varphi_2 \to \varphi_1))$

(Group II axioms)

(1) $(\varphi_1 \to ((\neg\varphi_1) \to \varphi_2))$

(Group III axioms)

(1) $(((\neg\varphi_1) \to \varphi_1) \to \varphi_1)$

(Group IV axioms)

(1) $((\neg\varphi_1) \to (\varphi_1 \to \varphi_2))$
(2) $(\varphi_1 \to ((\neg\varphi_2) \to (\neg(\varphi_1 \to \varphi_2))))$ □

It is easily verified that each logical axiom is a tautology.

Definition 1.25 Suppose that $\Gamma \subseteq \mathcal{L}_0$.

(1) Suppose that

$$s = \langle \varphi_1, \ldots, \varphi_n \rangle$$

is a finite sequence of propositional formulas. The finite sequence s is a Γ-proof if for each $i \leq n$ at least one of
  a) $\varphi_i \in \Gamma$; or
  b) φ_i is a logical axiom; or
  c) there exist $j_1 < i$ and $j_2 < i$ such that

$$\varphi_{j_2} = (\varphi_{j_1} \to \varphi_i).$$

(2) $\Gamma \vdash \varphi$ if and only if there exists a finite sequence

$$s = \langle \varphi_1, \ldots, \varphi_n \rangle$$

such that s is a Γ-proof and such that $\varphi_n = \varphi$. □

Notice that if $s = \langle \varphi_1, \ldots, \varphi_n \rangle$ is a Γ-proof and if $t = \langle \psi_1, \ldots, \psi_m \rangle$ is a Γ-proof then so is $s + t = \langle \varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_m \rangle$.

We shall prove a sequence of simple lemmas about the proof system. For each lemma we shall note which logical axioms are actually used.

The first lemma, which concerns inference, requires no logical axioms whatsoever.

Lemma 1.26 (Inference) *Suppose that $\Gamma \subseteq \mathcal{L}_0$, φ is a formula and that ψ is a formula. Suppose that $\Gamma \vdash \psi$ and that $\Gamma \vdash (\psi \to \varphi)$.*

Then $\Gamma \vdash \varphi$.

Proof. Let $\langle \varphi_1, \ldots, \varphi_n \rangle$ be a Γ-proof of ψ, thus $\varphi_n = \psi$. Let $\langle \psi_1, \ldots, \psi_m \rangle$ be a Γ-proof of $(\psi \to \varphi)$. Then, $\langle \varphi_1, \ldots, \varphi_n, \psi_1, \ldots, \psi_m, \varphi \rangle$ is a Γ-proof and $\Gamma \vdash \varphi$. □

The second lemma of our series is the Soundness Lemma, this also is independent of the choice of logical axioms, provided that every logical axiom is a tautology.

Lemma 1.27 (Soundness) *Suppose that* $\Gamma \subseteq \mathcal{L}_0$*,* φ *is a formula and that* $\Gamma \vdash \varphi$*. Suppose that*

$$\nu : \{A_1, \dots, A_n, \dots\} \to \{T, F\}$$

is a truth assignment such that $\overline{\nu}(\psi) = T$ *for all* $\psi \in \Gamma$*.*

Then $\overline{\nu}(\varphi) = T$*.*

Proof. We leave the proof to the reader, but one argues by induction on the length of the Γ-proof.

The next lemma is the Deduction Lemma. This lemma requires the logical axioms from Group I.

Lemma 1.28 (Deduction) *Suppose that* $\Gamma \subseteq \mathcal{L}_0$*,* φ *is a formula,* ψ *is a formula and*

$$\Gamma \cup \{\varphi\} \vdash \psi.$$

Then $\Gamma \vdash (\varphi \to \psi)$*.*

Proof. Let

$$\langle \psi_1, \dots, \psi_n \rangle$$

be a $(\Gamma \cup \{\varphi\})$-proof of ψ. We prove by induction on $i \leq n$ that

$$\Gamma \vdash (\varphi \to \psi_i).$$

First we consider the case $i = 1$. Either $\psi_1 \in \Gamma \cup \{\varphi\}$ or ψ_1 is a logical axiom (possibly both). So there are three subcases of this case.

Subcase 1.1: $\psi_1 \in \Gamma$. So we must show that $\Gamma \vdash (\varphi \to \psi_1)$. However

$$\Gamma \vdash (\psi_1 \to (\varphi \to \psi_1))$$

since $(\psi_1 \to (\varphi \to \psi_1))$ is a logical axiom. Further

$$\Gamma \vdash \psi_1$$

since $\psi_1 \in \Gamma$. Therefore by the Inference Lemma 1.26, $\Gamma \vdash (\varphi \to \psi_1)$.

Subcase 1.2: $\psi_1 = \varphi$. Note that $(\varphi \to \varphi)$ is a logical axiom and so

$$\Gamma \vdash (\varphi \to \varphi).$$

Subcase 1.3: ψ_1 is a logical axiom. This is just like subcase 1.1; $(\psi_1 \to (\varphi \to \psi_1))$ is a logical axiom and so

$$\Gamma \vdash (\psi_1 \to (\varphi \to \psi_1)).$$

Since ψ_1 is a logical axiom, $\Gamma \vdash \psi_1$. Therefore by the inference Lemma, $\Gamma \vdash (\varphi \to \psi_1)$.

We now suppose that $k \leq n$ and assume as an induction hypothesis that for all $i < k$,

$$\Gamma \vdash (\varphi \to \psi_i).$$

There are two subcases.

Subcase 2.1: $\psi_k \in \Gamma \cup \{\varphi\}$ or ψ_k is a logical axiom. But then exactly as in the case of ψ_1, $\Gamma \vdash (\varphi \to \varphi_k)$.

Subcase 2.2: There exist $j_1 < k$ and $j_2 < k$ such that $\psi_{j_2} = (\psi_{j_1} \to \psi_k)$.

By the induction hypothesis; $\Gamma \vdash (\varphi \to \psi_{j_1})$ and $\Gamma \vdash (\varphi \to \psi_{j_2})$. Now we use the logical axiom

$$((\varphi \to (\psi_{j_1} \to \psi_k)) \to ((\varphi \to \psi_{j_1}) \to (\varphi \to \psi_k))).$$

By the induction hypothesis,

$$\Gamma \vdash (\varphi \to (\psi_{j_1} \to \psi_k)),$$

and so by the Inference Lemma,

$$\Gamma \vdash ((\varphi \to \psi_{j_1}) \to (\varphi \to \psi_k)).$$

Again by the induction hypothesis,

$$\Gamma \vdash (\varphi \to \psi_{j_1}),$$

and so by the Inference Lemma one last time,

$$\Gamma \vdash (\varphi \to \psi_k).$$

This completes the induction and so $\Gamma \vdash (\varphi \to \psi)$. Finally we note that only Group I logical axioms were used. □

Definition 1.29 Suppose that $\Gamma \subseteq \mathcal{L}_0$.

(1) Γ is *inconsistent* if for some formula φ, $\Gamma \vdash \varphi$ and $\Gamma \vdash (\neg\varphi)$.
(2) Γ is *consistent* if Γ is not inconsistent. □

If Γ is an inconsistent set of formulas then $\Gamma \vdash \psi$ for *every* formula ψ. This is the content of the next lemma the proof of which appeals to the Deduction Lemma and logical axioms in Group II. Therefore only logical axioms from Groups I and II are needed.

Lemma 1.30 *Suppose that $\Gamma \subseteq \mathcal{L}_0$ and that Γ is inconsistent. Suppose that ψ is a formula.*

Then $\Gamma \vdash \psi$.

Proof. Since Γ is inconsistent there exists a formula φ such that

$$\Gamma \vdash \varphi$$

and $\Gamma \vdash (\neg\varphi)$.

But

$$\Gamma \vdash (\varphi \to ((\neg\varphi) \to \psi))$$

since $(\varphi \to ((\neg\varphi) \to \psi))$ is a logical axiom. Therefore by the Inference Lemma,

$$\Gamma \vdash ((\neg\varphi) \to \psi)$$

and by the Inference Lemma again,

$$\Gamma \vdash \psi.$$

This completes the proof. $\square$

Definition 1.31 Suppose that $\Gamma \subseteq \mathcal{L}_0$ and that Γ is consistent. Then Γ is *maximally consistent* if and only if for each formula ψ if $\Gamma \cup \{\psi\}$ is consistent then $\psi \in \Gamma$. $\square$

Lemma 1.32 *Suppose that $\Gamma \subseteq \mathcal{L}_0$ and that Γ is consistent. Suppose that φ is a formula.*

Then, at least one of $\Gamma \cup \{\varphi\}$ or $\Gamma \cup \{(\neg\varphi)\}$ is consistent, possibly both.

Proof. Suppose that $\Gamma \cup \{(\neg\varphi)\}$ is inconsistent. Therefore, for each formula ψ,

$$\Gamma \cup \{(\neg\varphi)\} \vdash \psi$$

and in particular, $\Gamma \cup \{(\neg\varphi)\} \vdash \varphi$.

Thus by the Deduction Lemma, $\Gamma \vdash ((\neg\varphi) \to \varphi)$. But $(((\neg\varphi) \to \varphi) \to \varphi)$ is a logical axiom (Group III), and so by the Inference Lemma, $\Gamma \vdash \varphi$.

Now assume toward a contradiction that $\Gamma \cup \{\varphi\}$ is inconsistent. By Lemma 1.30, for each formula ψ,

$$\Gamma \cup \{\varphi\} \vdash \psi.$$

By the Deduction Lemma, for each formula ψ,

$$\Gamma \vdash (\varphi \to \psi).$$

But $\Gamma \vdash \varphi$ and so by the Inference Lemma, for each formula ψ, $\Gamma \vdash \psi$. Thus Γ is inconsistent, which is a contradiction. Therefore $\Gamma \cup \{\varphi\}$ is consistent.

So we have proved, assuming the consistency of Γ, that if $\Gamma \cup \{(\neg\varphi)\}$ is inconsistent then $\Gamma \cup \{\varphi\}$ is consistent. $\square$

Corollary 1.33 *Suppose that $\Gamma \subseteq \mathcal{L}_0$ and that Γ is maximally consistent. Suppose that φ is a formula.*

Then:

(1) Either $\varphi \in \Gamma$ or $(\neg\varphi) \in \Gamma$;
(2) If $\Gamma \vdash \varphi$ then $\varphi \in \Gamma$.

Proof. We first prove (1). By Lemma 1.32, either $\Gamma \cup \{\varphi\}$ is consistent or $\Gamma \cup \{(\neg\varphi)\}$ is consistent. Therefore since Γ is maximally consistent (1) must hold.

We finish by proving (2). We are given that $\Gamma \vdash \varphi$. By (1), if $\varphi \notin \Gamma$ then $(\neg\varphi) \in \Gamma$ which implies that $\Gamma \vdash (\neg\varphi)$. But $\Gamma \vdash \varphi$ and so this contradicts the consistency of Γ. □

We now use the logical axioms in Group IV.

Lemma 1.34 *Suppose that $\Gamma \subseteq \mathcal{L}_0$ and that Γ is maximally consistent. Suppose that φ_1 and φ_2 are formulas.*

Then $(\varphi_1 \to \varphi_2) \in \Gamma$ if and only if at least one of $\varphi_1 \notin \Gamma$ or $\varphi_2 \in \Gamma$.

Proof. We first suppose that $\varphi_1 \notin \Gamma$. We must show that $(\varphi_1 \to \varphi_2) \in \Gamma$.

Since $\varphi_1 \notin \Gamma$, by Corollary 1.33, $(\neg\varphi_1) \in \Gamma$.

Thus $\Gamma \vdash (\neg\varphi_1)$. But

$$\Gamma \vdash ((\neg\varphi_1) \to (\varphi_1 \to \varphi_2))$$

since $((\neg\varphi_1) \to (\varphi_1 \to \varphi_2))$ is a logical axiom, and so by the Inference Lemma,

$$\Gamma \vdash (\varphi_1 \to \varphi_2).$$

Therefore by Corollary 1.33, $(\varphi_1 \to \varphi_2) \in \Gamma$.

Next we suppose that $\varphi_2 \in \Gamma$. Now

$$(\varphi_2 \to (\varphi_1 \to \varphi_2))$$

is a logical axiom and so $\Gamma \vdash (\varphi_2 \to (\varphi_1 \to \varphi_2))$. By the Inference Lemma, since $\varphi_2 \in \Gamma$,

$$\Gamma \vdash (\varphi_1 \to \varphi_2).$$

Therefore, again by Corollary 1.33, $(\varphi_1 \to \varphi_2) \in \Gamma$.

To finish, we suppose that $\varphi_1 \in \Gamma$ and $\varphi_2 \notin \Gamma$. Now we must show that $(\varphi_1 \to \varphi_2) \notin \Gamma$.

Since $\varphi_2 \notin \Gamma$, by Corollary 1.33, $(\neg\varphi_2) \in \Gamma$.

Thus $\Gamma \vdash \varphi_1$ and $\Gamma \vdash (\neg\varphi_2)$. But

$$\Gamma \vdash (\varphi_1 \to ((\neg\varphi_2) \to (\neg(\varphi_1 \to \varphi_2)))),$$

since $(\varphi_1 \to ((\neg\varphi_2) \to (\neg(\varphi_1 \to \varphi_2))))$ is a logical axiom. Therefore by the Inference Lemma,

$$\Gamma \vdash ((\neg\varphi_2) \to (\neg(\varphi_1 \to \varphi_2))),$$

and by the Inference Lemma once again,

$$\Gamma \vdash (\neg(\varphi_1 \to \varphi_2)).$$

Finally by Corollary 1.33, $(\neg(\varphi_1 \to \varphi_2)) \in \Gamma$ and so $(\varphi_1 \to \varphi_2) \notin \Gamma$ as required.

This completes the proof of the lemma. □

Our goal is to show that if Γ is consistent then Γ is satisfiable. We first consider the special case that Γ is maximally consistent. This case will turn out to be an easy case for Γ uniquely specifies the truth assignment which witnesses that Γ is satisfiable.

Lemma 1.35 *Suppose that* $\Gamma \subseteq \mathcal{L}_0$ *and that* Γ *is maximally consistent. Then* Γ *is satisfiable.*

Proof. Define a truth assignment ν as follows. For each $i \in \mathbb{N}$, let

$$\nu(A_i) = \begin{cases} T, & \text{if } \langle A_i \rangle \in \Gamma; \\ F, & \text{if } \langle A_i \rangle \notin \Gamma. \end{cases}$$

We claim that for each formula φ, $\overline{\nu}(\varphi) = T$ if $\varphi \in \Gamma$ and $\overline{\nu}(\varphi) = F$ if $\varphi \notin \Gamma$. We organize our proof of this claim by induction on the length of φ.

The case that φ has length 1 is immediate.

Suppose that φ has length $n > 1$ and that as induction hypothesis, for all formulas ψ if ψ has length less than n then $\overline{\nu}(\psi) = T$ if $\psi \in \Gamma$ and $\overline{\nu}(\psi) = F$ if $\psi \notin \Gamma$.

There are two cases.

Case 1: $\varphi = (\neg\psi)$. Since Γ is maximally consistent, $\varphi \in \Gamma$ if and only if $\psi \notin \Gamma$. But $\overline{\nu}(\varphi) = T$ if and only if $\overline{\nu}(\psi) = F$. By the induction hypothesis $\overline{\nu}(\psi) = T$ if and only if $\psi \in \Gamma$.

Thus if $\varphi \in \Gamma$ then $\overline{\nu}(\varphi) = T$ and $\overline{\nu}(\varphi) = F$ if $\varphi \notin \Gamma$.

Case 2: $\varphi = (\psi_1 \to \psi_2)$. Since Γ is maximally consistent, $\varphi \in \Gamma$ if and only if at least one of $\psi_1 \notin \Gamma$ or $\psi_2 \in \Gamma$. This is by Lemma 1.34.

By the definition of $\overline{\nu}$, $\overline{\nu}(\varphi) = T$ if and only if either $\overline{\nu}(\psi_1) = F$ or $\overline{\nu}(\psi_2) = T$. Therefore by the induction hypothesis, $\overline{\nu}(\varphi) = T$ if and only if either $\psi_1 \notin \Gamma$ of $\psi_2 \in \Gamma$.

Thus, $\overline{\nu}(\varphi) = T$ if and only if $\varphi \in \Gamma$.

This completes the induction. □

Theorem 1.36 *Suppose that $\Gamma \subseteq \mathcal{L}_0$ and that Γ is consistent.*

Then there exists a set $\Gamma^* \subset \mathcal{L}_0$ such that

$$\Gamma \subseteq \Gamma^*$$

and such that Γ^* is maximally consistent.

Proof. Let $(\varphi_i : i \in \mathbb{N})$ be an enumeration of all of the formulas of $\mathcal{L}_0$. For example, we could enumerate the finitely many length 1 formulas which use only the propositional symbol A_1; then, we could enumerate the finitely many formulas of length less than or equal to 2 which use no propositional symbols other than A_1 and A_2; and in subsequent steps, enumerate the finitely many n formulas of length less than or equal to n which use no propositional symbols other than $A_1, \dots, A_n$.

We construct a sequence of sets $(\Gamma_n : m \in \mathbb{N})$ by recursion on n. To begin, let Γ_0 equal Γ. Given Γ_n, let Γ_{n+1} be defined as follows.

$$\Gamma_{n+1} = \begin{cases} \Gamma_n \cup \{\varphi_n\}, & \text{if } \Gamma_n \cup \{\varphi_n\} \text{ is consistent;} \\ \Gamma_n \cup \{\neg\varphi_n\}, & \text{otherwise.} \end{cases}$$

We check by induction that each Γ_n is consistent. Clearly, Γ_0 is consistent, since we are given that Γ is consistent. Assuming that Γ_n is consistent, we can apply Lemma 1.32 to conclude that at least one of $\Gamma_n \cup \{\varphi_n\}$ or $\Gamma_n \cup \{\neg\varphi_n\}$ is also consistent. But then, Γ_{n+1} is also consistent.

Now, define Γ^* so that

$$\Gamma^* = \cup_{n \in \mathbb{N}} \Gamma_n.$$

For the sake of a contradiction, suppose that Γ^* is not consistent, then there is are Γ^*-proofs of some formula and of its negation. These Γ^*-proofs are finite, and so there is finite set of formulas Γ^*_F from Γ^* which appear in these proofs. But then Γ^*_F must be a subset of some Γ_n, and so one of the Γ_n's must be inconsistent. Since we have already checked that all of the Γ_n's are consistent, this is impossible. Thus Γ^* is consistent.

Γ^* is also maximally consistent: For every formula φ, there is an n such that φ is equal to φ_n. But we chose Γ_n so that either $\varphi_n \in \Gamma_n$ or $(\neg\varphi_n) \in \Gamma_n$. Since $\varphi = \varphi_n$ and $\Gamma_n \subseteq \Gamma^*$, either $\varphi \in \Gamma^*$ or $(\neg\varphi) \in \Gamma^*$, as required for maximality. □

Theorem 1.37 (Completeness; Version I) *Suppose that $\Gamma \subseteq \mathcal{L}_0$ and that Γ is consistent.*

Then Γ is satisfiable.

Proof. By Theorem 1.36 extend Γ to a maximally consistent set, and then apply Lemma 1.35. □

1.4 Logical implication and compactness

Definition 1.38 Let Γ be a subset of $\mathcal{L}_0$ and let φ be an element of $\mathcal{L}_0$. Then, Γ *logically implies* φ if and only if, for every truth assignment ν, if ν satisfies Γ in the sense of Definition 1.17, then ν satisfies φ.

For example, $\{\varphi\}$ logically implies φ, and $\{A_1, (A_1 \to A_2)\}$ logically implies A_2.

If Γ is a set a formulas, we write $\Gamma \models \varphi$ to indicate that Γ logically implies φ.

Definition 1.39 A subset Γ of $\mathcal{L}_0$ is *finitely satisfiable* if and only if for every finite subset Γ_0 of Γ, there is a truth assignment ν such that for all $\psi \in \Gamma_0$, $\overline{\nu}(\psi) = T$.

Theorem 1.40 (Compactness for $\mathcal{L}_0$) *Suppose that $\Gamma \subseteq \mathcal{L}_0$, $\varphi \in \mathcal{L}_0$, and Γ logically implies φ. Then there is a finite set Γ_0 such that $\Gamma_0 \subseteq \Gamma$ and Γ_0 logically implies φ.*

Proof. Since $\Gamma \models \varphi$, $\Gamma \cup \{(\neg\varphi)\}$ is *not satisfiable*. Therefore by the Completeness Theorem, $\Gamma \cup \{(\neg\varphi)\}$ is inconsistent. But this implies that there exists a finite set $\Gamma_0 \subseteq \Gamma$ such that $\Gamma_0 \cup \{(\neg\varphi)\}$ is inconsistent. Therefore by Lemma 1.27, the Soundness Lemma, $\Gamma_0 \cup \{(\neg\varphi)\}$ is not satisfiable and so $\Gamma_0 \models \varphi$. $\square$

We end this chapter by discussing a reformulation of the Completeness Theorem.

Theorem 1.41 (Completeness; Version II) *Suppose that $\Gamma \subseteq \mathcal{L}_0$, φ is a formula and that $\Gamma \models \varphi$.*

Then $\Gamma \vdash \varphi$.

Proof. Since $\Gamma \models \varphi$, $\Gamma \cup \{(\neg\varphi)\}$ is not satisfiable. Therefore by the Completeness Theorem, $\Gamma \cup \{(\neg\varphi)\}$ is inconsistent and so by Lemma 1.30,

$$\Gamma \cup \{(\neg\varphi)\} \vdash \varphi.$$

By Lemma 1.28, the Deduction Lemma,

$$\Gamma \vdash ((\neg\varphi) \to \varphi).$$

But $(((\neg\varphi) \to \varphi) \to \varphi)$ is a logical axiom and so by Lemma 1.26, the Inference Lemma, $\Gamma \vdash \varphi$. $\square$

1.4.1 Exercises

(1) Which of the formulas

$$(((A_1 \to A_1) \to A_2) \to A_2)$$
$$((((A_1 \to A_2) \to A_2) \to A_2) \to A_2)$$

is a tautology?

(2) For $\Gamma \subseteq \mathcal{L}_0$ and φ and ψ in $\mathcal{L}_0$, show that $\Gamma \cup \{\varphi\}$ logically implies ψ if and only if Γ logically implies $(\varphi \to \psi)$.

(3) Two physicists, A and B, and a logician C, are wearing hats, which they know are either black or white but not all white. A can see the hats of B and C; B can see the hats of A and C; C is blind. Each is asked in turn if they know the color of their own hat. The answers are: A:"No." B: "No." C: "Yes." What color is C's hat and how does C know?

(4) For Γ_1 and Γ_2 subsets of $\mathcal{L}_0$, Γ_1 is *logically equivalent* to Γ_2 if and only if, for all $\varphi \in \mathcal{L}_0$, Γ_1 logically implies φ if and only if Γ_2 logically implies φ. For $\Gamma \subseteq \mathcal{L}_0$, Γ is *independent* if it is not logically equivalent to any of its proper subsets. Prove the following.
   a) If Γ is finite, then there is a Γ_0 such that $\Gamma_0 \subseteq \Gamma$, Γ and Γ_0 are logically equivalent, and Γ_0 is independent.
   b) There is an infinite set Γ such that Γ has no independent and logically equivalent subset.
   c) For every $\Gamma \subseteq \mathcal{L}_0$, there is a $\Delta \subseteq \mathcal{L}_0$ such that Δ is independent and logically equivalent to Γ.

(5) Show that the set of logical consequences of

$$\{A_i : i \neq 1 \text{ and } i \in \mathbb{N}\}$$

is consistent but not maximally consistent. Show that the set of logical consequences of

$$\{A_i : i \in \mathbb{N}\}$$

is maximally consistent.

2

First order logic—syntax

First order languages extend propositional ones by adding the apparatus to refer to elements of a structure and to assert their properties.

Our language consists of (certain) finite sequences of symbols, as described below.

- The *logical symbols* are the following.

$$(\quad) \quad \neg \quad \rightarrow \quad \forall$$

- $\hat{=}$ is the *equality symbol*.
- The *variable symbols* are x_i, for $i \in \mathbb{N}$.
- The *constant symbols* are c_i, for $i \in \mathbb{N}$.
- The *function symbols* are F_i, for $i \in \mathbb{N}$.
- The *predicate symbols* are P_i, for $i \in \mathbb{N}$.

We fix a function π mapping the set of function and predicate symbols to $\mathbb{N}$ so that for each $k \geq 1$, each of the sets

$$\{i \in \mathbb{N} \mid \pi(F_i) = k\}$$

and

$$\{i \in \mathbb{N} \mid \pi(P_i) = k\}$$

is infinite. For example, we could define $\pi : F_i \mapsto k$, where the kth prime is the least prime which divides i. (So $e(j) = 2$ if j is even etc.) The purpose of the function π is to specify the number of arguments or *arity* of each function and predicate symbol.

2.1 Terms

Recall our notation; if $s = \langle s_1, \ldots, s_n \rangle$ and $t = \langle t_1, \ldots, t_m \rangle$ are finite sequences of symbols, then $s + t$ denotes the finite sequence $\langle s_1, \ldots, s_n, t_1, \ldots, t_m \rangle$.

Definition 2.1 The set of *terms*, T, is defined as the smallest set of finite sequences T satisfying the following properties.

(1) For each $i \in \mathbb{N}$, the sequences of length one,

$$\langle x_i \rangle$$

and

$$\langle c_i \rangle$$

belong to T.

(2) If F_i is a function symbol, $n = \pi(F_i)$, and $\tau_1, \ldots, \tau_n$ belong to T, then

$$\langle F_i \rangle + \langle (\rangle + \tau_1 + \cdots + \tau_n + \langle) \rangle$$

belongs to T. More briefly, the concatenation $F_i(\tau_1 \ldots \tau_n)$ belongs to T.

We will assume familiarity with the methods of the previous chapter and omit the proof that T is well defined.

Remark 2.2 We shall adopt several notational conventions.

(1) Often we shall say that x_i is a term. Of course we are referring to the sequence of length 1, $\langle x_i \rangle$.

(2) More generally we shall indicate terms informally and use

$$F_i(\tau_1, \ldots, \tau_n)$$

to indicate the term

$$\langle F_i \rangle + \langle (\rangle + \tau_1 + \cdots + \tau_n + \langle) \rangle$$

The elements of T are uniquely readable, as is pointed out in the next sequence of lemmas.

Lemma 2.3 (Readability) *For each term τ in* T, *exactly one of the following conditions applies.*

(1) There is an $i \in \mathcal{N}$ greater than or equal to 1 *such that τ is x_i or τ is c_i.*

(2) There is an $i \in \mathcal{N}$ greater than or equal to 1 *such that $\pi(F_i) = n$ and there are terms $\tau_1, \ldots, \tau_n$ in* T *such that τ is $F_i(\tau_1, \ldots, \tau_n)$.*

Proof. As in the proof of Lemma 1.11, we let T be the subset of T whose elements satisfy one of the above clauses. We observe that T satisfies the closure properties of Definition 2.1. Consequently, $\mathrm{T} \subseteq T$, as required.

The two conditions are mutually exclusive, as the first symbol in τ determines which condition applies. $\square$

Lemma 2.4 *If $\tau \in \mathrm{T}$, then no proper initial segment of τ is an element of* T.

Proof. We proceed by induction on the length of $\tau \in \mathrm{T}$.

If τ is a term of length 1, then the only proper initial segment is the null sequence, which by Lemma 2.3 is not an element of T.

Suppose that τ has length greater than 1 and assume the lemma for all terms of length less than that of τ. By Lemma 2.3, τ is of the form $F_i(\tau_1, \ldots, \tau_n)$. Suppose that σ is a proper initial segment of τ such that $\sigma \in \mathrm{T}$. As above, σ is not the null sequence, so the first symbol in σ is F_i. By Lemma 2.3, σ must have the form $F_i(\sigma_1, \ldots, \sigma_n)$, where each σ_i belongs to T. But then, σ_1 and τ_1 must be identical, since neither can be a proper initial segment of the other. It follows by an induction up to n, that for each i, σ_i is equal to τ_i. But then $\sigma = \tau$, contradicting the choice of σ. Thus, τ has no proper initial segment in T, as required. $\square$

Theorem 2.5 (Unique Readability) *For each term τ in* T*, exactly one of the following conditions applies.*

(1) There is an $i \in \mathcal{N}$ greater than or equal to 1 such that τ is x_i or τ is c_i.
(2) There is an $i \in \mathcal{N}$ greater than or equal to 1 such that $\pi(F_i) = n$ and there are terms $\tau_1, \ldots, \tau_n$ in T *such that τ is $F_i(\tau_1, \ldots, \tau_n)$.*

Further, in (2), the function symbol F_i and the terms $\tau_1, \ldots, \tau_n$ are unique.

Proof. By Lemma 2.3, it will be sufficient to verify the uniqueness of $\tau_1, \ldots, \tau_n$. This follows as in the proof of Lemma 2.4. Suppose that τ could be written as $F_i(\tau_1, \ldots, \tau_n)$ and as $F_j(\sigma_1, \ldots, \sigma_m)$. Then F_i and F_j both occur as the first symbol in τ, and hence are equal. Consequently, $n = m = \pi(F_i)$. Then, τ_1 and σ_1 must also be equal, as neither can be a proper initial segment of the other. By induction on i less than or equal to n, for each i, τ_i is equal to σ_i, as required. $\square$

2.2 Formulas

Definition 2.6 The set of formulas, $\mathcal{L}$, is the smallest set L of finite sequences of symbols as above satisfying the following properties.

(1) If P_i is a predicate symbol, $n = \pi(P_i)$ is the arity of P_i and $\tau_1, \ldots, \tau_n$ are terms, then

$$P_i(\tau_1 \ldots \tau_n)$$

is an element of L

(2) If τ_1 and τ_2 are terms, then

$$(\tau_1 \hat{=} \tau_2)$$

is an element of L.

(3) If $\varphi \in L$, then

$$(\neg\varphi)$$

is an element of L

(4) If φ_1 and φ_2 are elements of L, then

$$(\varphi_1 \to \varphi_2)$$

is an element of L

(5) If $\varphi \in L$ and x_i is a variable symbol, then

$$(\forall x_i \varphi)$$

is an element of L.

As in the case of T, we will not repeat the argument to show that $\mathcal{L}$ is well defined.

Definition 2.7 The *atomic formulas* are the ones indicated in (1) and (2) of Definition 2.6.

2.3 Subformulas

We define the relation ψ is a subformula of φ for formulas in $\mathcal{L}$.

Definition 2.8 Suppose that φ is a formula. A formula ψ is a *subformula* of φ if ψ is a block-subsequence of φ. (See Definition 1.5.)

Definition 2.9 Suppose that $\varphi = \langle a_1, \dots, a_n \rangle$ is a formula and s is a finite sequence. An *occurrence* of s in φ is an interval $[j_1, j_2]$ such that $s = \langle a_{j_1}, \dots, a_{j_2} \rangle$.

Remark 2.10 Suppose that φ is a formula. A formula ψ is a subformula of φ if and only if ψ has an occurrence in φ.

We will give an abbreviated proof that every formula in $\mathcal{L}$ is uniquely readable, as stated in Theorem 2.13. As above, we proceed by proving a readability lemma, a proper initial segment lemma, and then a uniqueness lemma.

Lemma 2.11 (Readability) *Suppose that φ is a formula. Then exactly one of the following conditions applies.*

(1) There is an i and terms $\tau_1, \dots, \tau_n$ are terms, where $n = \pi(P_i)$, such that $\varphi = P_i(\tau_1 \dots \tau_n)$.
(2) There are terms τ_1 and τ_2 such that $\varphi = (\tau_1 \mathrel{\hat{=}} \tau_2)$.
(3) There is a formula ψ such that $\varphi = (\neg\psi)$.
(4) There are formulas ψ_1 and ψ_2 such that $\varphi = (\psi_1 \to \psi_2)$.
(5) There is a formula ψ and a variable x_i such that $\varphi = (\forall x_i \psi)$.

The proof Lemma 2.11 is analogous to that of Lemma 1.11.

Lemma 2.12 *If $\varphi \in \mathcal{L}$, then no proper initial segment of φ is an element of $\mathcal{L}$.*

Proof. We consider the cases of Lemma 2.11.

Suppose that φ is of the form $P_i(\tau_1 \ldots \tau_n)$ and ψ is a proper initial segment of φ which also belongs to $\mathcal{L}$. Then the first symbol in ψ is P_i and so ψ must also be of the form $P_i(\sigma_1 \ldots \sigma_n)$. But then τ_1 must equal σ_1, or they would be a pair of distinct terms for which one is a proper initial segment of the other, contradicting Lemma 2.4. It follows by induction on i less than or equal to n that each σ_i is equal to τ_i, and hence that φ is equal to ψ.

The case when φ is an equality between terms can be analyzed similarly, using Lemma 2.4.

The cases when φ is $(\neg\psi)$ or $(\psi_1 \to \psi_2)$ are analogous to the same cases in the propositional case. See Lemma 1.12.

Finally, consider the case when φ is $(\forall x_i \varphi_1)$. If ψ is an initial segment of φ, then ψ must be of the form $(\forall x_i \psi_1)$, as φ and ψ must have the same first three symbols. But then induction applies to φ_1, and ψ_1 must equal φ_1. It follows that φ is equal to ψ. $\square$

Theorem 2.13 (Unique Readability) *Suppose that φ is a formula. Then exactly one of the following conditions applies.*

(1) There is an i and terms $\tau_1, \ldots, \tau_n$ are terms, where $n = \pi(P_i)$, such that $\varphi = P_i(\tau_1 \ldots \tau_n)$.
(2) There are terms τ_1 and τ_2 such that $\varphi = (\tau_1 \hat{=} \tau_2)$.
(3) There is a formula ψ such that $\varphi = (\neg\psi)$.
(4) There are formulas ψ_1 and ψ_2 such that $\varphi = (\psi_1 \to \psi_2)$.
(5) There is a formula ψ and a variable x_i such that $\varphi = (\forall x_i \psi)$.

Further, in each of the above cases, the terms and/or subformulas which are mentioned in that case are unique.

We leave the proof of Theorem 2.13 to the Exercises.

2.3.1 Exercises

(1) Prove Theorem 2.13.
(2) Consider the set of sequences defined as in Definition 2.6 except that the last clause is changed to read, "If $\varphi \in L$ and x_i is a variable symbol, then $\forall x_i \varphi$ is an element of L" in which the parentheses are omitted. Is this set uniquely readable?
(3) Consider the set of sequences defined as in Definition 2.6 except that the fourth clause is changed to read, "If φ_1 and φ_2 are elements of L, then $\varphi_1 \to \varphi_2$ is an element of L" in which the parentheses are omitted. Is this set uniquely readable?

2.4 Free variables, bound variables

Suppose that φ is a formula and that x_i is a variable. Then each occurrence of $\forall x_i$ in φ defines a unique subformula of φ. This is the content of the next lemma.

Lemma 2.14 *Suppose that φ is a formula, x_i is a variable, s and t are finite sequences, and that*

$$\varphi = s + \langle \forall, x_i \rangle + t.$$

Then there is a finite sequence $\hat{s}$, there is a formula ψ, and there is a finite sequence $\hat{t}$ such that $s = \hat{s} + \langle (\rangle$ and

$$\varphi = \hat{s} + \psi + \hat{t}.$$

Further, ψ is unique.

Proof. Note that the uniqueness of ψ follows by observing that if there were two such formulas, then one would be a proper initial segment of the other and contradict Lemma 2.12.

We prove the existence claims of Lemma 2.14 by induction on the length of φ. There are no formulas of length 1, and so the lemma is true of all length 1 formulas on trivial grounds. Now assume the lemma is true of every formula which is shorter than φ. By Lemma 2.11, we can analyze φ by considering the various cases of the Lemma. If φ is atomic, then φ does not contain an occurrence of $\langle \forall, x_i \rangle$, and again the claim is true on trivial grounds. If φ is $(\neg\theta)$, then any occurrence of $\langle \forall, x_i \rangle$ in φ is also one in θ and by induction there is a ψ contained in θ as required. Similarly, if φ is $(\psi_1 \to \psi_2)$ and there is an occurrence of $\langle \forall, x_i \rangle$ in φ, then it must be contained completely in ψ_1 or in ψ_2 (there is no $\to$ in $\langle \forall, x_i \rangle$) and the induction hypothesis applies. If φ is $(\forall x_j \varphi_1)$, then either the occurrence of $\langle \forall, x_i \rangle$ is the block of the second and third symbols in φ, $\hat{s}$ and $\hat{t}$ are both equal to the empty sequence, and the formula φ is the desired ψ, or the occurrence of $\langle \forall, x_i \rangle$ is entirely contained in φ_1 and the induction hypothesis applies. □

This suggests the following definition.

Definition 2.15 Suppose that $\varphi = \langle a_1, \ldots, a_n \rangle$ is a formula and x_i is a variable.

(1) An occurrence of $\forall x_i$ in φ is an occurrence of $\langle \forall, x_i \rangle$ in φ (as a block-subsequence).
(2) The *scope* of a particular occurrence of $\forall x_i$ in φ is the unique interval $[j_1, j_2]$ with the following properties.
   a) $[j_1 + 1, j_1 + 2]$ is the given occurrence of $\forall x_i$.
   b) $\langle a_{j_1}, \ldots, a_{j_2} \rangle$ is a formula (which of course is a subformula of φ).

Note that Lemma 2.12 implies that the sequence $\langle a_{j_1}, \ldots, a_{j_2} \rangle$ is unique.

Definition 2.16 Suppose that φ is a formula and that x_i is a variable which occurs in φ.

(1) An occurrence of x_i in φ is *free* if and only if it is not within the scope of any occurrence of $\forall x_i$ in φ. Otherwise, the occurrence is *bound*.

(2) x_i is a *free variable* if and only if there is a free occurrence of x_i in φ.
(3) x_i is a *bound variable* of φ if and only if x_i occurs in φ and is not a free variable of φ.

Definition 2.17 (1) If τ is a term, we write $\tau(x_1, \ldots, x_n)$ to indicate that the variables of τ are included in the set $\{x_1, \ldots, x_n\}$.
(2) If φ is a formula, we write $\varphi(x_1, \ldots, x_n)$ to indicate that the free variables of φ are included in the set $\{x_1, \ldots, x_n\}$.

Definition 2.18 A formula φ is a *sentence* if and only if it has no free variables.

3

First order logic—semantics

3.1 Formulas and structures

Suppose that $\mathcal{A}$ contains some of the constant, predicate, and function symbols of our language. A finite sequence φ is an $\mathcal{L}_\mathcal{A}$-formula if and only if

(1) φ is a formula,
(2) and the constant, predicate, and function symbols occurring in φ are all in $\mathcal{A}$.

Definition 3.1 An $\mathcal{L}_\mathcal{A}$-structure is a pair (M, I) as follows.

(1) $M \neq \emptyset$.
(2) I is a function with domain $\mathcal{A}$ such that for each $i \in \mathbb{N}$ the following conditions hold:
   a) If $c_i \in \mathcal{A}$ then $I(c_i) \in M$;
   b) if $F_i \in \mathcal{A}$ then $I(F_i)$ is a function
   $$I(F_i) : M^n \to M$$
   where $n = \pi(F_i)$;
   c) if $P_i \in \mathcal{A}$ then
   $$I(P_i) \subseteq M^n$$
   where $n = \pi(P_i)$.

3.2 The satisfaction relation

3.2.1 Interpreting terms

Definition 3.2 Suppose that $\mathcal{M} = (M, I)$ is an $\mathcal{L}_\mathcal{A}$-structure. A function ν is an $\mathcal{M}$-*assignment* if ν is a map from $\{x_i : i \in \mathbb{N}\}$ to M.

Thus, an $\mathcal{M}$-assignment is simply a function which associates to each variable symbol an element of the universe of the structure $\mathcal{M}$.

Definition 3.3 Suppose that $\mathcal{M} = (M, I)$ is an $\mathcal{L}_{\mathcal{A}}$-structure and that ν is an $\mathcal{M}$-assignment. We define a function

$$\overline{\nu} : \{\tau : \tau \text{ is an } \mathcal{L}_{\mathcal{A}}\text{-term}\} \to M$$

by induction on the length of $\mathcal{L}_{\mathcal{A}}$-terms as follows.

Base step. If τ has length 1, then we define $\overline{\nu}(\tau)$ by whichever equation applies.

$$\begin{aligned} \overline{\nu}(\langle x_i \rangle) &= \nu(x_i) \\ \overline{\nu}(\langle c_i \rangle) &= I(c_i) \end{aligned}$$

Recursion step. If $\tau = F_i(\tau_1, \ldots, \tau_n)$, where $n = \pi(F_i)$, then

$$\overline{\nu}(\tau) = I(F_i)(\overline{\nu}(\tau_1), \ldots, \overline{\nu}(\tau_n)).$$

By Theorem 2.5, unique readability for terms, $\overline{\nu}$ is well defined.

Definition 3.4 Suppose that $\mathcal{M}$ is an $\mathcal{L}_{\mathcal{A}}$-structure, τ is an $\mathcal{L}_{\mathcal{A}}$-term, and ν and μ are $\mathcal{M}$-assignments.

- Then, ν and μ *agree on the free variables of* τ if and only if for all variables x_i, if x_i appears in τ, then $\nu(x_i) = \mu(x_i)$.
- Similarly, ν and μ *agree on the free variables of* φ if and only if for all variables x_i, if x_i appears freely in φ, then $\nu(x_i) = \mu(x_i)$.

Lemma 3.5 *Suppose that $\mathcal{M} = (M, I)$ is an $\mathcal{L}_{\mathcal{A}}$-structure and τ is an $\mathcal{L}_{\mathcal{A}}$-term. Suppose that ν and μ are $\mathcal{M}$-assignments which agree on the free variables of τ. Then $\overline{\mu}(\tau) = \overline{\nu}(\tau)$*

Proof. We prove Lemma 3.5 by induction on the length of τ. If τ has length 1, then by Theorem 2.5, unique readability for terms, there is an i such that either τ is $\langle x_i \rangle$ and $\overline{\mu}(\tau) = \mu(x_i) = \nu(x_i) = \overline{\nu}(\tau)$ or τ is $\langle c_i \rangle$ and $\overline{\mu}(\tau) = I(c_i) = \overline{\nu}(\tau)$. In either case, the lemma is verified. Now, suppose that τ has length greater than 1 and assume the lemma for all terms which are shorter than τ. Again by Theorem 2.5, τ has the form $F_i(\tau_1, \ldots, \tau_n)$, and we use the inductive hypothesis as follows.

$$\begin{aligned} \overline{\mu}(\tau) &= \overline{\mu}(F_i(\tau_1, \ldots, \tau_n)) \\ &= I(F_i)(\overline{\mu}(\tau_1), \ldots, \overline{\mu}(\tau_n)) \\ &= I(F_i)(\overline{\nu}(\tau_1), \ldots, \overline{\nu}(\tau_n)) && \text{(by induction)} \\ &= \overline{\nu}(F_i(\tau_1, \ldots, \tau_n)) \\ &= \overline{\nu}(\tau) \end{aligned}$$

Thus, $\overline{\mu}(\tau) = \overline{\nu}(\tau)$ as required. $\square$

Definition 3.6 Suppose that $\mathcal{M} = (M, I)$ is an $\mathcal{L}_\mathcal{A}$-structure and ν is an $\mathcal{M}$-assignment. We define the *satisfaction relation*,

$$(\mathcal{M}, \nu) \models \varphi$$

by recursion on the length of φ as follows.

Atomic cases. Suppose that φ is an atomic formula.

(1) Suppose that $\varphi = P_i(\tau_1 \dots \tau_n)$ where $n = \pi(P_i)$ and where $\tau_1, \dots, \tau_n$ are terms. Then

$$(\mathcal{M}, \nu) \models \varphi \text{ if and only if } \langle \overline{\nu}(\tau_1), \dots, \overline{\nu}(\tau_n) \rangle \in I(P_i).$$

(2) Suppose that $\varphi = (\sigma \mathrel{\hat{=}} \tau)$ where σ and τ are terms. Then

$$(\mathcal{M}, \nu) \models \varphi \text{ if and only if } \overline{\nu}(\sigma) = \overline{\nu}(\tau).$$

Inductive cases. Suppose that φ is not an atomic formula.

(3) Suppose that $\varphi = (\neg \psi)$. Then

$$(\mathcal{M}, \nu) \models \varphi \text{ if and only if } (\mathcal{M}, \nu) \not\models \psi.$$

Here, we use $(\mathcal{M}, \nu) \not\models \psi$ to indicate that it is not the case that $(\mathcal{M}, \nu) \models \psi$.

(4) Suppose that $\varphi = (\psi_1 \rightarrow \psi_2)$. Then

$$(\mathcal{M}, \nu) \models \varphi \text{ if and only if}$$
$$\text{either} (\mathcal{M}, \nu) \not\models \psi_1 \text{ or } (\mathcal{M}, \nu) \models \psi_2.$$

(5) Suppose that $\varphi = (\forall x_i \psi)$. Then $(\mathcal{M}, \nu) \models \varphi$ if and only if for all $\mathcal{M}$-assignments μ, if ν and μ agree on the free variables of φ, then $(\mathcal{M}, \mu) \models \psi$. Since x_i is bound in φ, the values of these μ's on x_i range over all of M.

By Theorem 2.13, unique readability for formulas, $(\mathcal{M}, \nu) \models \varphi$ is well defined for all $\mathcal{M}$-assignments ν and $\mathcal{L}_\mathcal{A}$-formulas φ. We will sometimes say that $(\mathcal{M}, \nu)$ *satisfies* φ to indicate $(\mathcal{M}, \nu) \models \varphi$.

Theorem 3.7 *Suppose that $\mathcal{M} = (M, I)$ is an $\mathcal{L}_\mathcal{A}$-structure, φ is an $\mathcal{L}_\mathcal{A}$-formula, and ν and μ are $\mathcal{M}$-assignments which agree on the free variables of φ. Then,*

$$(\mathcal{M}, \nu) \models \varphi \;\leftrightarrow\; (\mathcal{M}, \mu) \models \varphi.$$

Proof. We proceed by induction on the length of φ. We now suppose that the length of φ is n and that the theorem holds for all $\mathcal{L}_\mathcal{A}$-formulas of length less than n. To be precise, we can assume the following.

Induction hypothesis. *For all ψ, ν_1, and μ_1, if ψ is an $\mathcal{L}_{\mathcal{A}}$-formula of length less than n, ν_1 is an $\mathcal{M}$-assignment, μ_1 is an $\mathcal{M}$-assignment, and ν_1 and μ_1 agree on the free variables of ψ, then $(\mathcal{M}, \nu_1) \models \psi$ if and only if $(\mathcal{M}, \mu_1) \models \psi$.*

To ground the induction, note that there are no formulas of length 0, so the induction hypothesis holds when n is equal to 1.

First consider the case in which φ is an atomic formula. There are two subcases.

First, φ could be instance of a predicate, say $\varphi = P_i(\tau_1 \dots \tau_n)$, where $\tau_i, \dots, \tau_n$ are $\mathcal{L}_{\mathcal{A}}$-terms and $n = \pi(P_i)$. Then, every variable occurring in φ is necessarily a free variable of φ. By Lemma 3.5, for each $i \leq n$, $\overline{\nu}(\tau_i) = \overline{\mu}(\tau_i)$. By definition,

$$(\mathcal{M}, \nu) \models P_i(\tau_1 \dots \tau_n) \ \leftrightarrow \ \langle \overline{\nu}(\tau_1), \dots, \overline{\nu}(\tau_n) \rangle \in I(P_i);$$

and similarly

$$(\mathcal{M}, \mu) \models P_i(\tau_1 \dots \tau_n) \ \leftrightarrow \ \langle \overline{\mu}(\tau_1), \dots, \overline{\mu}(\tau_n) \rangle \in I(P_i);$$

Thus since,

$$\langle \overline{\nu}(\tau_1), \dots, \overline{\nu}(\tau_n) \rangle = \langle \overline{\mu}(\tau_1), \dots, \overline{\mu}(\tau_n) \rangle,$$

it follows that

$$(\mathcal{M}, \nu) \models \varphi \ \leftrightarrow \ (\mathcal{M}, \mu) \models \varphi.$$

Second, φ could assert an equality, say $\varphi = (\tau \mathrel{\hat{=}} \sigma)$, where σ and τ are terms. Again, every variable occurring in φ is necessarily a free variable of φ. Then, by Lemma 3.5, $\overline{\nu}(\tau) = \overline{\mu}(\tau)$ and $\overline{\nu}(\sigma) = \overline{\mu}(\sigma)$. It follows from the definition of satisfaction as above that $(\mathcal{M}, \nu) \models (\tau \mathrel{\hat{=}} \sigma)$ if and only if $(\mathcal{M}, \mu) \models (\tau \mathrel{\hat{=}} \sigma)$

This finishes the case that φ is an atomic formula, and now we consider the three cases in which φ is not atomic.

First, φ could be a negation, say $\varphi = (\neg\psi)$. Then, φ and ψ have the same free variables, and so we may apply the induction hypotheses as follows.

$$\begin{aligned}
(\mathcal{M}, \nu) \models \varphi \ &\leftrightarrow \ (\mathcal{M}, \nu) \not\models \psi && \text{(by definition)} \\
&\leftrightarrow \ (\mathcal{M}, \mu) \not\models \psi && \text{(by induction)} \\
&\leftrightarrow \ (\mathcal{M}, \mu) \models \varphi. && \text{(by definition)}
\end{aligned}$$

Second, ψ could be an implication, say $\varphi = (\psi_1 \to \psi_2)$. In this case,

$$\begin{aligned}
&\{x_j : x_j \text{ is a free variable of } \varphi\} = \\
&\qquad \{x_j : x_j \text{ is a free variable of } \psi_1\} \cup \{x_j : x_j \text{ is a free variable of } \psi_2\}.
\end{aligned}$$

We apply the induction hypothesis to conclude that $(\mathcal{M}, \nu) \models \psi_1$ if and only if $(\mathcal{M}, \mu) \models \psi_1$ and $(\mathcal{M}, \nu) \models \psi_2$ if and only if $(\mathcal{M}, \mu) \models \psi_2$. By definition,

$(\mathcal{M},\nu) \models \varphi$ if and only if, either $(\mathcal{M},\nu) \not\models \psi_1$ or $(\mathcal{M},\nu) \models \psi_2$, and $(\mathcal{M},\mu) \models \varphi$ if and only if, either $(\mathcal{M},\mu) \not\models \psi_1$ or $(\mathcal{M},\mu) \models \psi_2$. Thus (by the induction hypothesis), $(\mathcal{M},\nu) \models \varphi$ if and only if $(\mathcal{M},\mu) \models \varphi$.

Finally, φ could be obtained by quantification, say $\varphi = (\forall x_i \psi)$. By definition, $(\mathcal{M},\nu) \models \varphi$ if and only if for all $\mathcal{M}$-assignments ρ, if ρ and ν agree on the free variables of φ, then $(\mathcal{M},\rho) \models \psi$. Similarly, $(\mathcal{M},\mu) \models \varphi$ if and only if for all $\mathcal{M}$-assignments ρ, if ρ and μ agree on the free variables of φ, then $(\mathcal{M},\rho) \models \psi$. By assumption, μ and ν agree on the free variables of φ. Therefore (trivially) for all $\mathcal{M}$-assignments ρ, ρ and μ agree on the free variables of φ if and only if ρ and ν agree on the free variables of φ. But then, $(\mathcal{M},\nu) \models \varphi$ if and only if $(\mathcal{M},\mu) \models \varphi$

This completes the proof of the theorem. $\square$

Definition 3.8 (1) If $\mathcal{M} = (M, I)$ is an $\mathcal{L}_{\mathcal{A}}$-structure, $\tau = \tau(x_1, \ldots, x_n)$ is an $\mathcal{L}_{\mathcal{A}}$-term, and $a_1, \ldots, a_n$ are elements of M, then

$$\tau[a_1, \ldots, a_n]$$

indicates $\overline{\nu}(\tau)$, where ν is any $\mathcal{M}$-assignment such that for all $i \leq n$, $\nu(x_i) = a_i$.

(2) If $\mathcal{M} = (M, I)$ is an $\mathcal{L}_{\mathcal{A}}$-structure, $\varphi = \varphi(x_1, \ldots, x_n)$ is an $\mathcal{L}_{\mathcal{A}}$-formula, and $a_1, \ldots, a_n$ are elements of M, then

$$\mathcal{M} \models \varphi[a_1, \ldots, a_n]$$

indicates that $(\mathcal{M},\nu) \models \varphi$, where ν is any $\mathcal{M}$-assignment such that for all $i \leq n$, $\nu(x_i) = a_i$.

By Lemma 3.5 and Theorem 3.7, the definitions of $\tau[a_1, \ldots, a_n]$ and of the relation $\mathcal{M} \models \varphi[a_1, \ldots, a_n]$ given above do not depend on the choice of ν.

In particular, if φ is a sentence, with no free variables, then we write $\mathcal{M} \models \varphi$ or $\mathcal{M} \not\models \varphi$.

3.3 Substitution and the satisfaction relation

Definition 3.9 (1) If τ is a term and $\tau_1, \ldots, \tau_n$ are terms, we write $\tau(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n)$ to indicate the term obtained by simultaneously, for each i substituting the term τ_i for each occurrence of x_i in τ.

(2) If $\varphi(x_1, \ldots, x_n)$ is a formula and $\tau_1, \ldots, \tau_n$ are terms, we write $\varphi(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n)$ to indicate the formula obtained by simultaneously, for each i substituting τ_i for each free occurrence of x_i in φ.

Lemma 3.10 *(1) For any term $\tau(x_1, \ldots, x_n)$ and for any sequence of terms $\tau_1, \ldots, \tau_n$, $\tau(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n)$ is a term.*

(2) For any formula $\varphi(x_1, \ldots, x_n)$ and any sequence of terms $\tau_1, \ldots, \tau_n$, $\varphi(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n)$ is a formula.

The proof of Lemma 3.10 is by induction, first on the length of terms to prove (1) and then on the length of formulas using (1) to prove (2).

Example 3.11 We think of $\varphi(x_1;\tau)$ as saying that φ holds of τ. However, blind substitution can have unintended results. Consider the following formula φ.

$$\varphi = \varphi(x_1) = (\forall x_2(x_1 \hat{=} x_2))$$

Now, we substitute x_2 for x_1.

$$\varphi(x_1;x_2) = (\forall x_2(x_2 \hat{=} x_2))$$

There is a substantial difference between the two formulas. Every structure satisfies $\varphi(x_1;x_2)$, but for every structure $\mathcal{M}$ and every $a \in M$, $\mathcal{M} \models \varphi[a]$ if and only if $M = \{a\}$.

Definition 3.12 Suppose φ is a formula, x_i is a free variable of φ, and τ is a term. The term τ is *free for x_i in φ* if for each variable x_j occurring in τ, no free occurrence of x_i in φ is within the scope of an occurrence of $\forall x_j$.

Theorem 3.13 (Substitution) *Let $\mathcal{M} = (M, I)$ be an $\mathcal{L}_\mathcal{A}$-structure and ν be an $\mathcal{M}$-assignment.*

(1) If $\tau(x_1,\dots,x_n)$ is an $\mathcal{L}_\mathcal{A}$-term and $\tau_1,\dots,\tau_n$ are $\mathcal{L}_\mathcal{A}$-terms, then

$$\overline{\nu}(\tau(x_1,\dots,x_n;\tau_1,\dots,\tau_n)) = \tau[b_1,\dots,b_n],$$

where for each $i \le n$, $b_i = \overline{\nu}(\tau_i)$.

(2) If $\varphi(x_1,\dots,x_n)$ is an $\mathcal{L}_\mathcal{A}$-formula, $\tau_1,\dots,\tau_n$ are $\mathcal{L}_\mathcal{A}$-terms, and for each $i \le n$, τ_i is free for x_i in φ, then

$$(\mathcal{M},\nu) \models \varphi(x_1,\dots,x_n;\tau_1,\dots,\tau_n) \;\leftrightarrow\; \mathcal{M} \models \varphi[b_1,\dots,b_n],$$

where for each $i \le n$, $b_i = \overline{\nu}(\tau_i)$.

Proof. The two parts are proven by induction on the lengths of τ and φ, respectively. We leave the proof of the first to the reader and present the proof of the second.

So, assume that (1) is proven, that φ is an $\mathcal{L}_\mathcal{A}$-formula, $\tau_1,\dots,\tau_n$ are $\mathcal{L}_\mathcal{A}$-terms such that for each $i \le n$, τ_i is free for x_i in φ, and that (2) holds for every formula of length less than that of φ. For each $i \le n$, let b_i denote $\overline{\nu}(\tau_i)$.

When φ is an atomic $\mathcal{L}_\mathcal{A}$-formula, the claim follows directly from the definitions and (1).

Now suppose that φ is not an atomic $\mathcal{L}_\mathcal{A}$-formula. We must prove, using the induction hypothesis, that $(\mathcal{M},\nu) \models \varphi(x_1,\dots,x_n;\tau_1,\dots,\tau_n)$ if and only if $\mathcal{M} \models \varphi[b_1,\dots,b_n]$.

There are three cases to consider.

First, φ could be a negation, say $\varphi = (\neg\psi)$. Then, the free variables of φ are exactly the same as those of ψ, and for each $i \leq n$, τ_i is free for x_i in ψ. Thus,

$$\varphi(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n) = (\neg\psi(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n)).$$

By the induction hypothesis,

$$(\mathcal{M}, \nu) \models \psi(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n) \ \leftrightarrow \ \mathcal{M} \models \psi[b_1, \ldots, b_n].$$

But then by definition of the satisfaction of a negation,

$$(\mathcal{M}, \nu) \models \varphi(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n) \ \leftrightarrow \ \mathcal{M} \models \varphi[b_1, \ldots, b_n].$$

Second, φ could be an implication, say $\varphi = (\psi_1 \to \psi_2)$. In this case, the free variables of φ are those variables which are free in at least one of ψ_1 or ψ_2. Further, as in the previous case, $\varphi(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n)$ is equal to $(\psi_1(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n) \to \psi_2(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n))$. So again we may apply the induction hypothesis to obtain the following equivalences.

$$\begin{aligned}
(\mathcal{M}, \nu) \models \psi_1(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n) \ &\leftrightarrow \ \mathcal{M} \models \psi_1[b_1, \ldots, b_n] \\
(\mathcal{M}, \nu) \models \psi_2(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n) \ &\leftrightarrow \ \mathcal{M} \models \psi_2[b_1, \ldots, b_n]
\end{aligned}$$

By definition, $(\mathcal{M}, \nu)$ satisfies $\varphi(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n)$ if and only if either $(\mathcal{M}, \nu)$ does not satisfy $\psi_1(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n)$ or it does satisfy $\psi_2(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n)$. Similarly, $\mathcal{M}$ satisfies $\varphi[b_1, \ldots, b_n]$ if and only if either $\mathcal{M}$ does not satisfy $\psi_1[b_1, \ldots, b_n]$ or it does satisfy $\psi_2[b_1, \ldots, b_n]$. Thus, we have the required equivalence.

$$(\mathcal{M}, \nu) \models \varphi(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n) \ \leftrightarrow \ \mathcal{M} \models \varphi[b_1, \ldots, b_n]$$

Finally, φ could be obtained by quantification, say $\varphi = (\forall x_i \psi)$.

First, we may assume that n is greater than or equal to i, since our notation $\varphi(x_1, \ldots, x_n)$ merely indicates that the free variables of φ are a subset of $\{x_1, \ldots, x_n\}$. Second, for each j such that x_j does not appear freely in φ, $\varphi(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n)$ does not depend on the value of τ_j. Consequently, we may assume that each such τ_j is equal to $\langle x_j \rangle$. In particular, we may assume that τ_i is $\langle x_i \rangle$.

By definition, $(\mathcal{M}, \nu)$ satisfies $\varphi(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n)$ if and only if Condition 1 holds.

Condition 1. For all $\mathcal{M}$-assignments μ, if μ and ν agree on the free variables of $\varphi(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n)$, then

$$(\mathcal{M}, \mu) \models \psi(x_1, \ldots, x_n; \tau_1, \ldots, \tau_n).$$

Now, we can apply the inductive hypothesis in the conclusion of Condition 1 and see that it is equivalent to Condition 2.

Condition 2. For all $\mathcal{M}$-assignments μ, if μ and ν agree on the free variables of $\varphi(x_1,\ldots,x_n;\tau_1,\ldots,\tau_n)$, then

$$\mathcal{M} \models \psi[\overline{\mu}(\tau_1),\ldots,\overline{\mu}(\tau_n)]$$

We will show Condition 2 is equivalent to the following one, Condition 3.

Condition 3. For all $\mathcal{M}$-assignments ρ, if for each j such that x_j appears freely in φ, $\rho(x_j) = \overline{\nu}(\tau_j)$, then $(\mathcal{M},\rho) \models \psi$

We give separate proofs for the implications between the two conditions.

First, suppose that Condition 2 holds. To prove Condition 3, suppose that ρ is an $\mathcal{M}$-assignment such that for each x_j such that j appears freely in φ, $\rho(x_j) = \overline{\nu}(\tau_j)$. Let μ_ρ be the $\mathcal{M}$-assignment defined as follows.

$$\mu_\rho(x_j) = \begin{cases} \nu(x_j), & \text{if } x_j \text{ occurs freely in} \\ & \qquad \varphi(x_1,\ldots,x_n;\tau_1,\ldots,\tau_n); \\ \rho(x_j), & \text{otherwise.} \end{cases}$$

By Condition 2, $\mathcal{M} \models \psi[\overline{\mu_\rho}(\tau_1),\ldots,\overline{\mu_\rho}(\tau_n)]$. Because each τ_j was free for x_j in φ, if x_j occurs freely in φ, then all of the variables which occur in τ_j occur freely in $\varphi(x_1,\ldots,x_n;\tau_1,\ldots,\tau_n)$. We may apply Lemma 3.5 to conclude that for each such j, $\overline{\mu_\rho}(\tau_j) = \overline{\nu}(\tau_j)$ and, since $\rho(x_j) = \overline{\nu}(\tau_j)$, $\overline{\mu_\rho}(\tau_j) = \rho(x_j)$. Since x_i does not occur freely in $\varphi(x_1,\ldots,x_n;\tau_1,\ldots,\tau_n)$ and $\tau_i = \langle x_i\rangle$, $\overline{\mu_\rho}(\tau_i) = \rho(x_i)$. If a variable occurs freely in ψ, then either it occurs freely in φ or it is equal to x_i. Consequently, for each variable x_j such that x_j appears freely in ψ, $\rho(x_j) = \overline{\mu_\rho}(\tau_j)$. Since $\mathcal{M} \models \psi[\overline{\mu_\rho}(\tau_1),\ldots,\overline{\mu_\rho}(\tau_n)]$, it follows from Theorem 3.7 that $(\mathcal{M},\rho) \models \psi$, as required.

Now, suppose that Condition 3 holds. To prove Condition 2, suppose that μ is an $\mathcal{M}$-assignment which agrees with ν on the free variables of $\varphi(x_1,\ldots,x_n;\tau_1,\ldots,\tau_n)$. Since each τ_j is free for x_j in φ, μ and ν agree on all of the variables that appear in any τ_j such that x_j occurs freely in φ. Applying Lemma 3.5, if x_j appears freely in φ, then $\overline{\mu}(\tau_j) = \overline{\nu}(\tau_j)$. Now, define ρ_μ as follows.

$$\rho_\mu(x_j) = \begin{cases} \overline{\mu}(\tau_j), & \text{if } x_j \text{ occurs freely in } \varphi; \\ \mu(x_j), & \text{otherwise.} \end{cases}$$

By Condition 3, $(\mathcal{M},\rho_\mu) \models \psi$. Since all of the free variables of ψ are included in $\{x_1,\ldots,x_n\}$, $\mathcal{M} \models \psi[\rho_\mu(x_1),\ldots,\rho_\mu(x_n)]$. By the earlier remarks, if x_j does not appear freely in φ then $\tau_j = \langle x_j\rangle$. Consequently, $\langle \rho_\mu(x_1),\ldots,\rho_\mu(x_n)\rangle$ is equal to $\langle \overline{\mu}(\tau_1),\ldots,\overline{\mu}(\tau_n)\rangle$. Thus, $\mathcal{M} \models \psi[\overline{\mu}(\tau_1),\ldots,\overline{\mu}(\tau_n)]$ as required.

By Theorem 3.7, Condition 3 is equivalent to $\mathcal{M} \models \varphi[b_1,\ldots,b_n]$, where for each i, b_i is equal to $\overline{\nu}(\tau_i)$. Therefore, we have the desired equivalence

$$(\mathcal{M},\nu) \models \varphi(x_1,\ldots,x_n;\tau_1,\ldots,\tau_n) \;\leftrightarrow\; \mathcal{M} \models \varphi[b_1,\ldots,b_n]$$

where for each i, b_i is equal to $\overline{\nu}(\tau_i)$.

This completes the final case (and hence the proof). $\square$

3.3.1 Exercises

(1) Let $\mathcal{A}$ be a language with one binary relation symbol. Give an example of a sentence φ in this language and $\mathcal{L}_\mathcal{A}$-structures $\mathcal{M}_1$ and $\mathcal{M}_2$ such that $\mathcal{M}_1 \models \varphi$ and $\mathcal{M}_2 \not\models \varphi$.

(2) Do there exist an $\mathcal{L}_\mathcal{A}$-structure $\mathcal{M}$, an $\mathcal{M}$-assignment ν, and an $\mathcal{L}_\mathcal{A}$-formula φ such that $(\mathcal{M}, \nu) \models \varphi$ and $(\mathcal{M}, \nu) \models (\neg\varphi)$? Do there exist such $\mathcal{M}$ and ν such that $(\mathcal{M}, \nu) \not\models \varphi$ and $(\mathcal{M}, \nu) \not\models (\neg\varphi)$?

(3) Suppose that $A_1, \ldots, A_n$ are propositional symbols, that θ is a propositional tautology, and that $\varphi_1, \ldots, \varphi_n$ are $\mathcal{L}_\mathcal{A}$ formulas. Let ψ be the result of substituting for each i, the formula φ_i for each occurrence of the propositional symbol A_i in θ. Prove that for every $\mathcal{L}_\mathcal{A}$-structure $\mathcal{M}$ and every $\mathcal{M}$-assignment ν, $(\mathcal{M}, \nu) \models \psi$.

(4) Give an example of an $\mathcal{A}$ and an $\mathcal{L}_\mathcal{A}$-formula φ such that

a) φ is a sentence,

b) there is at least one $\mathcal{L}_\mathcal{A}$-structure $\mathcal{M}$ such that $\mathcal{M} \models \varphi$,

c) and for all $\mathcal{L}_\mathcal{A}$-structures $\mathcal{M}$, if $\mathcal{M} \models \varphi$, then the universe of $\mathcal{M}$ is infinite.

4

The logic of first order structures

4.1 Isomorphisms between structures

We begin with the fundamental notion of isomorphism for $\mathcal{L}_{\mathcal{A}}$-structures. Notice that if $\mathcal{M} = (M, I)$ is an $\mathcal{L}_{\mathcal{A}}$-structure, then $\mathcal{L}_{\mathcal{A}}$ is uniquely specified by the domain of I.

Definition 4.1 Suppose that $\mathcal{M} = (M, I)$ and $\mathcal{N} = (N, J)$ are $\mathcal{L}_{\mathcal{A}}$-structures. A bijection $e : M \to N$ defines an isomorphism between $\mathcal{M}$ and $\mathcal{N}$ if and only if the following conditions hold.

(1) For each constant symbol c_i in the domain of I,

$$e(I(c_i)) = J(c_i).$$

(2) For each function symbol F_i in the domain of I, if $n = \pi(F_i)$ then for each $\langle a_1, \ldots, a_{n+1} \rangle \in M^{n+1}$,

$$I(F_i)(a_1, \ldots, a_n) = a_{n+1} \leftrightarrow J(F_i)(e(a_1), \ldots, e(a_n)) = e(a_{n+1}).$$

(3) For each predicate symbol P_i in the domain of I, if $n = \pi(P_i)$ then for each $\langle a_1, \ldots, a_n \rangle \in M^n$,

$$\langle a_1, \ldots, a_n \rangle \in I(P_i) \leftrightarrow \langle e(a_1), \ldots, e(a_n) \rangle \in J(P_i).$$

When $\mathcal{M}$ is equal to $\mathcal{N}$, we say that e is an *automorphism.*

For any structure $\mathcal{M}$, the identity function $e : x \mapsto x$ is an example, though a trivial one, of an automorphism of $\mathcal{M}$. It follows directly from Definition 4.1 that the inverse of an isomorphism is also an isomorphism and that the composition of two isomorphisms is also an isomorphism.

Theorem 4.2 *Suppose that $e : M \to N$ is an isomorphism of $\mathcal{L}_{\mathcal{A}}$-structures $\mathcal{M} = (M, I)$ and $\mathcal{N} = (N, J)$. Suppose that ν is an $\mathcal{M}$-assignment. Then, the composition of e and ν, $e \circ \nu$, is an $\mathcal{N}$-assignment, and for each $\mathcal{L}_{\mathcal{A}}$-formula φ,*

$$(\mathcal{M}, \nu) \models \varphi \leftrightarrow (\mathcal{N}, e \circ \nu) \models \varphi.$$

Proof. Clearly, $e \circ \nu$ is an $\mathcal{N}$-assignment. It maps each variable symbol x_i to an element of M via ν and then maps that element of M to an element of N via e. Consequently, $e \circ \nu$ maps the variable symbols to N, as required.

Next, we note that the isomorphism e preserves the interpretation of terms. In other words, if τ is an $\mathcal{L}_{\mathcal{A}}$-term, then $e(\overline{\nu}(\tau)) = \overline{e \circ \nu}(\tau)$. The proof is by induction on the length of terms. In the atomic cases,

$$\begin{aligned} e(\overline{\nu}(\langle c_i \rangle)) &= e(I(c_i)) \\ &= J(c_i) \qquad \text{(since } e \text{ is an isomorphism)} \\ &= \overline{e \circ \nu}(\langle c_i \rangle). \end{aligned}$$

and

$$\begin{aligned} e(\overline{\nu}(\langle x_i \rangle)) &= e(\nu(x_i)) \\ &= e \circ \nu(x_i) \\ &= \overline{e \circ \nu}(\langle x_i \rangle). \end{aligned}$$

Assuming the claim for all shorter terms, consider $F_i(\tau_1, \ldots, \tau_n)$.

$$\begin{aligned} e(\overline{\nu}(F_i(\tau_1, \ldots, \tau_n))) &= e(I(F_i)(\overline{\nu}(\tau_1), \ldots, \overline{\nu}(\tau_n))) \\ &= J(F_i)(e(\overline{\nu}(\tau_1)), \ldots, e(\overline{\nu}(\tau_n))) \\ &\qquad \text{(since } e \text{ is an isomorphism)} \\ &= J(F_i)(\overline{e \circ \nu}(\tau_1), \ldots, \overline{e \circ \nu}(\tau_n)) \\ &\qquad \text{(by induction)} \\ &= \overline{e \circ \nu}(F_i(\tau_1, \ldots, \tau_n) \end{aligned}$$

Finally, we verify the statement of the theorem, by induction on the length of formulas.

The atomic cases follow from the above observation about terms.

$$\begin{aligned} &(\mathcal{M}, \nu) \models P_i(\tau_1 \ldots \tau_n) \;\leftrightarrow \\ &\quad \leftrightarrow\; \langle \overline{\nu}(\tau_1), \ldots, \overline{\nu}(\tau_n) \rangle \in I(P_i) \\ &\qquad\qquad \text{(by definition)} \\ &\quad \leftrightarrow\; \langle e(\overline{\nu}(\tau_1)), \ldots, e(\overline{\nu}(\tau_n)) \rangle \in J(P_i) \\ &\qquad\qquad \text{(since } e \text{ is an isomorphism)} \\ &\quad \leftrightarrow\; \langle \overline{e \circ \nu}(\tau_1), \ldots, \overline{e \circ \nu}(\tau_n)) \rangle \in J(P_i) \\ &\qquad\qquad \text{(by the observation on terms)} \\ &\quad \leftrightarrow\; (\mathcal{N}, e \circ \nu) \models P_i(\tau_1 \ldots \tau_n) \end{aligned}$$

The case of an equality uses the fact that an isomorphism is injective.

$$\begin{aligned} (\mathcal{M}, \nu) \models \tau_1 \hat{=} \tau_n &\leftrightarrow \overline{\nu}(\tau_1) = \overline{\nu}(\tau_2) && \text{(by definition)} \\ &\leftrightarrow e(\overline{\nu}(\tau_1)) = e(\overline{\nu}(\tau_2)) && \text{(as } e \text{ is injective)} \\ &\leftrightarrow \overline{e \circ \nu}(\tau_1) = \overline{e \circ \nu}(\tau_2) && \text{(as above)} \\ &\leftrightarrow (\mathcal{N}, e \circ \nu) \models (\tau_1 \hat{=} \tau_2) && \text{(by definition)} \end{aligned}$$

Now, we consider the propositional connectives.

$$\begin{aligned}
(\mathcal{M},\nu)\models(\neg\psi) &\leftrightarrow (\mathcal{M},\nu)\not\models\psi && \text{(by definition)}\\
&\leftrightarrow (\mathcal{N},e\circ\nu)\not\models\psi && \text{(by induction)}\\
&\leftrightarrow (\mathcal{N},e\circ\nu)\models(\neg\psi) && \text{(by definition)}
\end{aligned}$$

The analysis of implication is similar.

Finally, we consider applications of quantification, say $\varphi=(\forall x_i\psi)$. By definition, $(\mathcal{M},\nu)\models(\forall x_i\psi)$ if and only if for every $\mathcal{M}$-assignment μ, if ν and μ agree on the free variables of φ, then $(\mathcal{M},\mu)\models\psi$. Since e is surjective, for every $\mathcal{N}$-assignment μ^* which agrees with $e\circ\nu$ on the free variables of φ, there is an $\mathcal{M}$-assignment μ which agrees with ν on the free variables of φ such that $e\circ\mu=\mu^*$. By induction, for each such μ and μ^*, $(\mathcal{M},\mu)\models\psi \leftrightarrow (\mathcal{N},\mu^*)\models\psi$. Thus, if $(\mathcal{M},\nu)\models(\forall x_i\psi)$, then $(\mathcal{N},e\circ\nu)\models(\forall x_i\psi)$. Similarly, if $(\mathcal{M},\nu)\not\models(\forall x_i\psi)$, then let μ be an $\mathcal{M}$-assignment such that ν and μ agree on the free variables of φ and $(\mathcal{M},\mu)\not\models\psi$. Then, $e\circ\mu$ is such an $\mathcal{N}$-assignment for $e\circ\nu$, and so $(\mathcal{N},e\circ\nu)\not\models(\forall x_i\psi)$. □

4.1.1 Exercises

(1) Let $A=\{F_1\}$ be the alphabet with one unary function symbol. Give examples of different infinite $\mathcal{L}_A$-structures $\mathcal{M}=(M,I)$ with the following properties.
  a) $\mathcal{M}$ has no nontrivial automorphisms.
  b) $\mathcal{M}$ has a countably infinite set of automorphisms.
  c) For each element a of M there are only finitely many b's in M such that there is an automorphism f of $\mathcal{M}$ with $f(a)=b$. However, there are uncountably many automorphisms of $\mathcal{M}$.

Theorem 4.2 suggests the following definition.

Definition 4.3 Suppose that $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}_{\mathcal{A}}$-structures. Then $\mathcal{M}$ and $\mathcal{N}$ are *elementarily equivalent* if and only if for each $\mathcal{L}_{\mathcal{A}}$-sentence φ

$$\mathcal{M}\models\varphi \leftrightarrow \mathcal{N}\models\varphi.$$

We write $\mathcal{M}\equiv\mathcal{N}$ that these structures are elementarily equivalent.

4.1.2 Exercises

(1) Characterize the collection of automorphisms of the integers $\mathbb{Z}$ with the binary relation $<$.

(2) Suppose that $\mathcal{A}$ is finite and that $\mathcal{M}$ is a finite $\mathcal{L}_{\mathcal{A}}$-structure. Prove that there is an $\mathcal{L}_{\mathcal{A}}$-sentence φ such that for every $\mathcal{L}_{\mathcal{A}}$-structure $\mathcal{N}$, if $\mathcal{N}\models\varphi$ then $\mathcal{N}\cong\mathcal{M}$.

(3) Suppose that $\mathcal{M}$ and $\mathcal{N}$ are finite $\mathcal{L}_\mathcal{A}$-structures. Prove that the following are equivalent.
   a) $\mathcal{M} \cong \mathcal{N}$
   b) $\mathcal{M} \equiv \mathcal{N}$

4.2 Substructures and elementary substructures

Definition 4.4 Suppose that $\mathcal{M} = (M, I)$ and $\mathcal{N} = (N, J)$ are $\mathcal{L}_\mathcal{A}$-structures. $\mathcal{M}$ is a *substructure* of $\mathcal{N}$ if and only if $M \subseteq N$ and the following conditions hold.

(1) If c_i is a constant symbol of $\mathcal{L}_\mathcal{A}$ then $I(c_i) = J(c_i)$.
(2) If F_i is a function symbol of $\mathcal{L}_\mathcal{A}$ with $n = \pi(F_i)$, then $I(F_i)$ is the restriction of $J(F_i)$ to M^n.
(3) If P_i is a predicate symbol of $\mathcal{L}_\mathcal{A}$ with $n = \pi(P_i)$, then $I(P_i)$ is equal to $J(P_i) \cap M^n$.

We will write $\mathcal{M} \subseteq \mathcal{N}$ to indicate that $\mathcal{M}$ is a substructure of $\mathcal{N}$.

Theorem 4.5 *Suppose that $\mathcal{M} = (M, I)$ and $\mathcal{N} = (N, J)$ are $\mathcal{L}_\mathcal{A}$-structures with $M \subseteq N$. Then the following are equivalent.*

(1) $\mathcal{M}$ is a substructure of $\mathcal{N}$.
(2) For all atomic $\mathcal{L}_\mathcal{A}$-formulas, φ and for all $\mathcal{M}$-assignments ν,

$$(\mathcal{M}, \nu) \models \varphi \;\leftrightarrow\; (\mathcal{N}, \nu) \models \varphi.$$

Proof. The theorem follows directly from the definitions. □

The equivalence given in Theorem 4.5 suggests the following definition.

Definition 4.6 Suppose that $\mathcal{M}$ and $\mathcal{N}$ are $\mathcal{L}_\mathcal{A}$-structures with $\mathcal{M} \subseteq \mathcal{N}$. $\mathcal{M}$ is an *elementary substructure* of $\mathcal{N}$, if and only if for all $\mathcal{L}_\mathcal{A}$-formulas φ and for all $\mathcal{M}$-assignments ν,

$$(\mathcal{M}, \nu) \models \varphi \;\leftrightarrow\; (\mathcal{N}, \nu) \models \varphi.$$

We write $\mathcal{M} \preceq \mathcal{N}$ to indicate that $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$.

4.2.1 Exercise

(1) Let $\mathcal{A} = \emptyset$ and let $\mathcal{N}$ be the $\mathcal{L}_\mathcal{A}$ structure whose universe is $\mathbb{N}$, the natural numbers. Show that for every infinite subset S of $\mathbb{N}$, the $\mathcal{L}_\mathcal{A}$-structure with universe S is an elementary substructure of $\mathcal{N}$.

4.3 Definable sets and Tarski's Criterion

Suppose that $\mathcal{N}$ is an $\mathcal{L}_\mathcal{A}$-structure. The problem of constructing elementary substructures of $\mathcal{N}$ looks difficult because the criterion for success involves truth in the substructure to be constructed and, in particular, anticipating quantification over the whole substructure while still in the process of its construction.

Tarski's Theorem below gives an elegant characterization of when a substructure of $\mathcal{N}$ is an elementary substructure. Tarski's criterion is given in terms of definable sets.

Definition 4.7 Suppose that $\mathcal{M} = (M, I)$ is an $\mathcal{L}_{\mathcal{A}}$-structure.

(1) Suppose that $X \subseteq M$. A set $Y \subseteq M^n$ is *definable in $\mathcal{M}$ with parameters from X* if and only if there are elements $b_1, \dots, b_m$ of X and an $\mathcal{L}_{\mathcal{A}}$-formula φ such that the following conditions hold.
   a) φ has $n + m$ free variables $x_{j_1}, \dots, x_{j_n}$ and $x_{k_1}, \dots, x_{k_m}$.
   b) For each $\langle a_1, \dots, a_n \rangle \in M^n$, $\langle a_1, \dots, a_n \rangle \in Y$ if and only if there is an $\mathcal{M}$-assignment ν such that
      i. for each $i \leq n$, $\nu(x_{j_i}) = a_i$,
      ii. for each $i \leq m$, $\nu(x_{k_i}) = b_i$,
      iii. and $(\mathcal{M}, \nu) \vDash \varphi$.

(2) A set $Y \subseteq M^n$ is *definable* in $\mathcal{M}$ without parameters if and only if it is definable with parameters from $\emptyset$.

When n is equal to 1, we identify M^1 with M and speak of definable subsets of M.

Lemma 4.8 *Suppose that $\mathcal{M} = (M, I)$ is an $\mathcal{L}_{\mathcal{A}}$-structure and that $Y \subseteq M^n$ is definable in $\mathcal{M}$ with parameters from X. Then, there are $b_1, \dots, b_m$ in X and there is a formula $\varphi = \varphi(x_1, \dots, x_{n+m})$ such that*

$$Y = \{\langle a_1, \dots, a_n \rangle : \mathcal{M} \vDash \varphi[a_1, \dots, a_n, b_1, \dots, b_m]\}$$

Proof. We will leave the proof of Lemma 4.8 to the Exercises. □

Theorem 4.9 *Suppose that $\mathcal{M} = (M, I)$ is an $\mathcal{L}_{\mathcal{A}}$-structure and that $X \subseteq M$. Suppose that $Y \subset M^n$ is definable in $\mathcal{M}$ with parameters from X and that $e : M \to M$ is an automorphism of $\mathcal{M}$.*

If for each $b \in X$, $e(b) = b$, then

$$Y = \{\langle e(a_1), \dots, e(a_n) \rangle : \langle a_1, \dots, a_n \rangle \in Y\}.$$

Proof. Let $b_1, \dots, b_m$ be elements of X and let $\varphi = \varphi(x_1, \dots, x_{n+m})$ be an $\mathcal{L}_{\mathcal{A}}$-formula such that for all $a_1, \dots, a_n$ in M,

$$\langle a_1, \dots, a_n \rangle \in Y \;\leftrightarrow\; \mathcal{M} \vDash \varphi[a_1, \dots, a_n, b_1, \dots, b_m].$$

Suppose that $\langle a_1, \dots, a_n \rangle \in Y$, then we can apply Theorem 4.2 to conclude that $\mathcal{M} \vDash \varphi[e(a_1), \dots, e(a_n), e(b_1), \dots, e(b_m)]$. Since e fixes all of the elements of X, $\mathcal{M} \vDash \varphi[e(a_1), \dots, e(a_n), b_1, \dots, b_m]$ and so $\langle e(a_1), \dots, e(a_n) \rangle \in Y$. To verify the reverse inclusion, suppose that $\langle c_1, \dots, c_n \rangle \in Y$, that is $\mathcal{M} \vDash \varphi[c_1, \dots, c_n, b_1, \dots, b_m]$. Since e is an automorphism, e is surjective. Let $a_1, \dots, a_n$ be elements of

M such that for each i less than or equal to n, $e(a_i) = c_i$. Consequently, $\mathcal{M} \models \varphi[e(a_1), \ldots, e(a_n), b_1, \ldots, b_m]$. Applying Theorem 4.2 in the other direction, $\mathcal{M} \models \varphi[a_1, \ldots, a_n, b_1, \ldots, b_m]$ and so $\langle a_1, \ldots, a_n \rangle \in Y$. Thus, there is a sequence $\langle a_1, \ldots, a_n \rangle \in Y$ such that $\langle c_1, \ldots, c_n \rangle$ is equal to $\langle e(a_1), \ldots, e(a_n) \rangle$, as required.

Remark 4.10 Definability within a structure is one of the central concepts in Mathematical Logic. In the next section, we shall consider the problem of classifying the definable sets of various specific structures. In many cases, the analysis requires that careful attention be paid to parameters.

Example 4.11 Suppose that $\mathcal{A} = \emptyset$, so that $\mathcal{L}_\mathcal{A}$ is the trivial language. Suppose that M is a nonempty set. Then, $\mathcal{M} = (M, \emptyset)$ is an $\mathcal{L}_\mathcal{A}$-structure. Further *any* bijection $e : M \to M$ defines an isomorphism of $\mathcal{M}$ to $\mathcal{M}$. We can use Theorem 4.2 to prove the following.

(1) Suppose that $A \subset M$. Then A is definable in $\mathcal{M}$ without parameters if and only if $A = \emptyset$ or $A = M$.
(2) Suppose that $A \subset M$. Then A is definable in $\mathcal{M}$ from parameters if and only if A is finite or $M \setminus A$ is finite.

To verify the first claim, suppose that φ is an $\mathcal{L}_\mathcal{A}$-formula and x_1 is the only free variable in φ. If there is no m in $\mathcal{M}$ such that $\mathcal{M} \models \varphi[m]$, then φ defines $\emptyset$ in $\mathcal{M}$. Otherwise, suppose that $m \in M$ and $\mathcal{M} \models \varphi[m]$. If n is another element of M, then the function e from M to M obtained by transposing m and n is a bijection from M to M, and therefore an isomorphism from $\mathcal{M}$ to $\mathcal{M}$. By Theorem 4.2, since $\mathcal{M} \models \varphi[m]$ we also have $\mathcal{M} \models \varphi[e(m)]$, that is $\mathcal{M} \models \varphi[n]$. Consequently, if φ defines a nonempty set, then that set is all of M.

We leave the proof of the second claim to the Exercises.

Theorem 4.12 (Tarski) *Suppose that $\mathcal{M} = (M, I)$ and $\mathcal{N} = (N, J)$ are $\mathcal{L}_\mathcal{A}$-structures, and $\mathcal{M}$ is a substructure of $\mathcal{N}$. The following are equivalent.*

(1) $\mathcal{M} \preceq \mathcal{N}$
(2) $\mathcal{M} \subseteq \mathcal{N}$ *and for each nonempty set $A \subseteq N$, if A is definable in $\mathcal{N}$ with parameters from M, then $A \cap M \neq \emptyset$.*

Proof. The easier direction is the implication from (1) to (2). Suppose that $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$, and suppose that A is a nonempty set which is definable in $\mathcal{N}$ with parameters from M. Let $\varphi(x_1, \ldots, x_{1+n})$ be a $\mathcal{L}_\mathcal{A}$-formula and let $m_1, \ldots, m_n$ be elements of M such that for all $a \in N$, $a \in A \leftrightarrow \mathcal{N} \models \varphi[a, m_1, \ldots, m_n]$. Since A is not empty, $\mathcal{N} \not\models (\forall x_1(\neg\varphi))[m_1, \ldots, m_n]$. Since $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$, $\mathcal{M} \not\models (\forall x_1(\neg\varphi))[m_1, \ldots, m_n]$. Fix m in M so that $\mathcal{M} \models \varphi[m, m_1, \ldots, m_n]$. Since $\mathcal{M}$ is an elementary substructure of $\mathcal{N}$, $\mathcal{N} \models \varphi[m, m_1, \ldots, m_n]$, and so m is an element of A. Thus, $A \cap M$ is not empty, as required.

Now, we prove the implication from (2) to (1). Suppose that $\mathcal{M} \subseteq \mathcal{N}$ and for each nonempty set $A \subseteq N$, if A is definable in $\mathcal{N}$ with parameters from M, then $A \cap M \neq \emptyset$. We prove by induction on the length of formulas φ that for all $\mathcal{M}$-assignments ν, $(\mathcal{M}, \nu) \models \varphi$ if and only if $(\mathcal{N}, \nu) \models \varphi$. Note that since $M \subseteq N$, every $\mathcal{M}$-assignment is also an $\mathcal{N}$-assignment.

The atomic cases follow from Theorem 4.5, and the propositional cases follow directly from the inductive hypothesis. We consider only the case when φ is $(\forall x_i \psi)$.

Suppose that ν is an $\mathcal{M}$-assignment and $(\mathcal{N}, \nu) \models \varphi$. Then, for every $\mathcal{N}$-assignment μ which agrees with ν on the free variables of φ, $(\mathcal{N}, \mu) \models \psi$. In particular, for every $\mathcal{M}$-assignment μ which agrees with ν on the free variables of φ, $(\mathcal{N}, \mu) \models \psi$. By induction, for these $\mathcal{M}$-assignments, $(\mathcal{N}, \mu) \models \psi$ if and only if $(\mathcal{M}, \mu) \models \psi$. Consequently, for every $\mathcal{M}$-assignment μ which agrees with ν on the free variables of φ, $(\mathcal{M}, \mu) \models \psi$, and $(\mathcal{M}, \nu) \models \varphi$ as required.

Now, suppose that ν is an $\mathcal{M}$-assignment and $(\mathcal{N}, \nu) \not\models (\forall x_i \psi)$. Then there is an $\mathcal{N}$-assignment μ which agrees with ν on the free variables of $(\forall x_i \psi)$ such that $(\mathcal{N}, \mu) \not\models \psi$. Let $x_{k_1}, \dots, x_{k_m}$ be the free variables of φ and let Y be the subset of N defined in $\mathcal{N}$ by the formula $\neg\psi$ using x_i for the elements of Y and $x_{k_1}, \dots, x_{k_m}$ for the parameters $\nu(x_{k_1}), \dots, \nu(x_{k_m})$. Since $(\mathcal{N}, \mu) \not\models \psi$, $\mu(x_i)$ is an element of Y and so Y is not empty.

Now, we use the assumption (2). By (2), $Y \cap M$ is not empty, and we let b be an element of $Y \cap M$. Consequently, if ρ is an $\mathcal{M}$-assignment such that ρ agrees with ν on the free variables of $(\forall x_i \psi)$ and $\rho(x_i) = b$, then $(\mathcal{N}, \rho) \models \neg\psi$. By induction, $(\mathcal{M}, \rho) \models \neg\psi$ and so $(\mathcal{M}, \nu) \not\models (\forall x_1 \psi)$. Taking the contrapositive of the above, if $(\mathcal{M}, \nu) \models (\forall x_1 \psi)$ then $(\mathcal{N}, \nu) \models (\forall x_1 \psi)$, as required. $\square$

4.3.1 Exercises

(1) Let $\mathcal{N}$ be the structure with universe $\mathbb{N}$, interpreting constants for 0 and 1, and functions for $+$ and $\times$. Show that if $\mathcal{M} \subseteq \mathcal{N}$, then $\mathcal{M} = \mathcal{N}$.

(2) Let $\mathcal{L}_A$ be the language with one unary predicate symbol P. Let $\mathcal{M}$ be the finite structure (M, I) such that $M = \{a, b, c, d, e\}$ and $I(P) = \{a, b\}$. In other words, $\mathcal{M}$ interprets P as holding of a and b and as not holding of c, d, or e.

   a) Which subsets of M which are definable in $\mathcal{M}$ without parameters?
   b) Which subsets of M are are definable in $\mathcal{M}$ with parameters?

(3) Suppose that T is a set of sentences and that there is an $\mathcal{N} = (N, J)$ such that $\mathcal{N} \models T$ and N is infinite. Show that there is an $\mathcal{M} = (M, I)$ and an element a of M such that $\mathcal{M} \models T$ and a is not definable in $\mathcal{M}$ without parameters.

(4) Prove Lemma 4.8.

(5) Prove the second claim of Example 4.11.

4.4 Dense orders

We consider the following example and show how, by constructing automorphisms one can in this case easily determine which sets are definable.

Suppose $\mathcal{L}_{\mathcal{A}}$ has only one 2-place predicate symbol. Thus $\mathcal{L}_{\mathcal{A}}$-structures are naturally of the form (M, P), where $M \neq \emptyset$ and $P \subseteq M \times M$. We consider the $\mathcal{L}_{\mathcal{A}}$-structure $(\mathbb{R}, <)$, given by the set of real numbers with the usual order. Suppose that $X \subset \mathbb{R}$ is finite. Define for reals a and b, $a \sim_X b$ if and only if there exists a bijection $e : \mathbb{R} \to \mathbb{R}$ such that e is an automorphism of the $\mathcal{L}_{\mathcal{A}}$-structure $(\mathbb{R}, <)$, such that $e(a) = b$ and such that for all $t \in X$, $e(t) = t$.

The relation $\sim_X$ is an equivalence relation on $\mathbb{R}$. That is to say that for all t_1, t_2, t_3 in $\mathbb{R}$ the following conditions hold.

(1) $t_1 \sim_X t_1$; since the identity map $x \mapsto x$ is an automorphism.
(2) If $t_1 \sim_X t_2$ then $t_2 \sim_X t_1$; since the inverse of an automorphism is an automorphism.
(3) if $t_1 \sim_X t_2$ and $t_2 \sim_X t_3$ then $t_1 \sim_X t_3$; since the composition of automorphisms is an automorphism.

For each $t \in \mathbb{R}$ let

$$[t]_X = \{w \in \mathbb{R} : w \sim_X t\}$$

be the equivalence class of t.

Definition 4.13 (1) A set $I \subseteq \mathbb{R}$ is an *interval* if for all a, b, c in $\mathbb{R}$, if $a \leq b \leq c$ and $\{a, c\} \subseteq I$ then $b \in I$.
(2) If $I \subseteq \mathbb{R}$ is an interval, then a real number a is an *endpoint* of I if and only if either a is the greatest lower bound of I or a is the least upper bound of I.
(3) If $c < d$ are real numbers, then we use the following notation to represent intervals with endpoints c and d.

$$(c, d) = \{x : c < x < d\} \tag{1}$$
$$[c, d] = \{x : c \leq x \leq d\} \tag{2}$$

Lemma 4.14 *Suppose that $X \subset \mathbb{R}$ is finite. Then for each $a \in \mathbb{R}$, $[a]_X$ is an interval. Further*

(1) if $a \in X$ then $[a]_X = \{a\}$,
(2) and if $a \notin X$ then $[a]_X$ is the maximum interval $I \subseteq \mathbb{R}$ such that $a \in I$ and $I \cap X = \emptyset$.

Proof. If a is an element of X, then any automorphism of $(\mathbb{R}, \leq)$ which fixes all of the elements of X must fix a. But then for all b, if $a \sim_X b$ then $a = b$. In other words, $[a]_X = \{a\}$.

Otherwise, let I be the maximum interval such that $a \in I$ and $I \cap X = \emptyset$. To show that I is equal to $[a]_X$, let b be an element of I. Let (c, d) be a subinterval of I such that a and b belong to (c,d) and $[c,d]$ is a subset of I. We may assume that a is less than b, since otherwise we can consider the inverse of the map constructed as below to send b to a. First, we define an order preserving bijection e_0 from $[c,d]$ to itself so that e_0 maps a to b.

$$e_0(x) = \begin{cases} c + \frac{b-c}{a-c}(x-c), & \text{if } x \in [c,a]; \\ b + \frac{d-b}{d-a}(x-a), & \text{if } x \in [a,d]. \end{cases}$$

The function e_0 consists of stretching the interval $[c,a]$ to match $[c,b]$ and compressing $[a,d]$ to match $[b,d]$. Then, we extend e_0 to an order preserving bijection of $\mathbb{R}$ by mapping every real number not in $[c,d]$ to itself. The resulting function, e is an automorphism of $\mathbb{R}$ which shows that $a \sim_X b$. □

Theorem 4.15 *Suppose that $X \subseteq \mathbb{R}$ and that $A \subseteq \mathbb{R}$. Then the following are equivalent.*

(1) A is definable in $(\mathbb{R},<)$ with parameters from X.

(2) A is a finite union of intervals I such that the endpoints of I belong to X.

Proof. We will take the implication from (2) to (1) as being self-evident, and we will prove the implication from (1) to (2).

Suppose that A is definable in $(\mathbb{R},<)$ with parameters from X. Let φ be a formula in the first order language with $\leq$, let $a_1,\dots,a_n$ be elements of X, and suppose that for all real numbers b,

$$b \in A \;\leftrightarrow\; (\mathbb{R},<) \models \varphi[b,a_1,\dots,a_n].$$

We first show that for each b, if $b \in A$ then $[b]_X \subseteq A$. So, suppose that b and c are real numbers, $b \in A$, and $b \sim_X c$. By Lemma 4.14, there is an automorphism e of $(\mathbb{R},\leq)$ such that e maps b to c and e fixes the elements of $a_1,\dots,a_n$. By Theorem 4.9, $b \in A$ if and only if $e(b) \in A$. Consequently, $b \in A$ implies $c \in A$, as required.

But then, A is a union of $\sim_{\{a_1,\dots,a_n\}}$ equivalence classes. Each of these classes is an interval, and since $\{a_1,\dots,a_n\}$ is finite, there are only finitely many of them. Theorem 4.15 follows immediately. □

By applying Tarski's Theorem 4.12, we can characterize the elementary substructures of $(\mathbb{R},<)$.

Corollary 4.16 *Let $\mathcal{R} = (\mathbb{R},<)$. Suppose that $M \subseteq \mathbb{R}$ and that $\mathcal{M} = (M,<_M)$ is the induced substructure of $\mathcal{R}$. Then the following are equivalent.*

(1) $\mathcal{M} \preceq \mathcal{R}$.

(2) $(M,<_M)$ is a dense total order without endpoints.

Proof. The implication from (1) to (2) is direct: being a dense total order without endpoints is a first order property of $\mathcal{R}$ which must apply to any of its elementary substructures.

We now prove the implication from (2) to (1). Suppose that $(M, <_M)$ is a dense total order without endpoints. We will apply Tarski's Criterion to show that $\mathcal{M} \preceq \mathcal{R}$. Let $\{m_1, \ldots, m_n\}$ be a finite subset of M, and let A be a nonempty subset of $\mathbb{R}$ which is definable in $\mathcal{R}$ using parameters from $\{m_1, \ldots, m_n\}$. It is sufficient to show that $A \cap M$ is not empty.

By Lemma 4.15, A is a finite union of intervals I in $\mathcal{R}$ whose endpoints belong to $\{m_1, \ldots, m_n\}$. Let I be a nonempty such interval. If I is a singleton $\{m_i\}$, then $m_i \in (A \cap M)$. Secondly, there could be $m_i < m_j$ such every real number between m_i and m_j belongs to I. Since $\mathcal{M}$ is dense and m_i and m_j are elements of M, there is an $m \in M$ such that $m_i < m < m_j$. Then, $m \in (A \cap M)$ as required. Finally, I could be an unbounded interval. Since M has no endpoints, there must be an element of m in I in this case as well. □

Another corollary is the following version of Theorem 4.15 but for the structure $(\mathbb{Q}, <)$.

Theorem 4.17 *Suppose that $X \subseteq \mathbb{Q}$ and that $A \subseteq \mathbb{Q}$. Then the following are equivalent.*

(1) A is definable in $(\mathbb{Q}, <)$ with parameters from X.

(2) A is a finite union of intervals I such that the endpoints of I belong to X.

Proof. Given a formula $\varphi(x_1, \ldots, x_{n+1})$ and parameters $q_1, \ldots, q_n$ from X, we can evaluate φ in $(\mathbb{R}, <)$ relative to $q_1, \ldots, q_n$. The set A so defined is a finite union of intervals I such that the endpoints of I belong to $\{q_1, \ldots, q_n\}$. The equivalence between satisfying φ relative to $q_1, \ldots, q_n$ and belonging to the finite union of intervals is a first order property of $(\mathbb{R}, <)$. Since $(\mathbb{Q}, <)$ is a dense total order without endpoints, $(\mathbb{Q}, <) \prec (\mathbb{R}, <)$. Thus, the equivalence between satisfying φ relative to $q_1, \ldots, q_n$ and belonging to the finite union of intervals is satisfied by $(\mathbb{Q}, <)$. □

4.5 Countable sets

Definition 4.18 A set A is *countable* if either it is empty or there is a surjective map from $\mathbb{N}$ to A.

In particular, every finite set is countable, and every subset of $\mathbb{N}$ is countable. Intuitively, the countable sets are those sets whose size is less than or equal to the size of $\mathbb{N}$.

Theorem 4.19 *Suppose that $\langle A_i : i \in \mathbb{N}\rangle$ is a countable sequence of countable sets. Then $A = \cup\{A_i \mid i \in \mathbb{N}\}$ is a countable set.*

Proof. If all of the A_i's are empty, then their union is empty and hence countable. Otherwise, we may assume that none of the A_i's are empty, since discarding the empty A_i's does not change the value of A. Similarly, we may assume that there are infinitely many A_i's by allowing sets to appear more than once.

Fix a sequence of functions, $\langle f_i : i \in \mathbb{N} \rangle$ so that for each i, f_i is a surjection from $\mathbb{N}$ to A_i.

A side remark. To make sense of the above sentence, we must appeal to the axiom of choice (AC). AC is the assertion that if F is a set of nonempty sets, then there is a function c with domain F such that for each element x in F, $c(x) \in x$. In other words, c chooses an element from each element of F. In our application, we are given that for each i, there is at least one function mapping $\mathbb{N}$ onto A_i. In fact, if A_i has more than one element then there are infinitely many distinct such functions. We use the axiom of choice to choose particular countings of the A_i's.

Returning to our proof, let a be an element of A. Define $f : \mathbb{N} \to A$ as follows.

$$f(n) = \begin{cases} f_i(j), & \text{if } n = 2^i 3^j; \\ a, & \text{otherwise.} \end{cases}$$

f is well defined since every element of $\mathbb{N}$ is uniquely factored as a product of prime numbers. If b is an element of A, then there is an i such that $b \in A_i$ and hence there is a j such that $f_i(j) = b$. But then, $f(2^i 3^j) = b$. Consequently, f is a surjection. $\square$

Even though $\mathcal{N}$ is infinite, there are sets whose size is not less than or equal to the size of $\mathcal{N}$.

Theorem 4.20 (Cantor) *The set of real numbers is not countable.*

Proof. We show first that the set $\mathcal{P}(\mathbb{N})$ of all subsets of $\mathbb{N}$ is not countable.

Suppose that

$$f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$$

Define

$$A = \{k \in \mathbb{N} \mid k \notin f(k)\}.$$

We claim that A is not in the range of f. Suppose toward a contradiction that $f(i) = A$. Then $i \in A$ if and only if $i \notin A$ which is a contradiction. This proves that f is not a surjection.

Thus $\mathcal{P}(\mathbb{N})$ is uncountable. Finally we show that set of real numbers is uncountable by producing a function

$$g; \mathcal{P}(\mathbb{N}) \to \mathbb{R}$$

which is one to one. Define

$$g(A) = \sum_{i=1}^{\infty} \epsilon_i^A 3^{-i}$$

where for each i, $\epsilon_i^A = 1$ if $i \in A$ and 0 otherwise. It follows that for A, B in $\mathcal{P}(\mathbb{N})$, if $A \neq B$ then $g(A) \neq g(B)$ and so g is one to one as required. Thus the range of g is uncountable and so the set of real numbers is uncountable. $\square$

4.5.1 Exercises

(1) Show that $\mathbb{Q}$, the set of rational numbers, is countable.
(2) Show that there is a bijection between $\mathcal{P}(\mathbb{N})$ and $\mathbb{R}$.
(3) Show that if $\mathcal{M} = (M, I)$ is an $\mathcal{L}_\mathcal{A}$-structure and X is a countable subset of M, then the collection of all sets $A \subseteq M$ such that A is definable in $\mathcal{M}$ with parameters from X is countable. (Hint: Show that there are only countably many formulas and countably many finite sequences from X.)

4.6 The Lowenheim-Skolem Theorem

The (Downward) Lowenheim-Skolem Theorem is an important application of Tarski's Theorem.

Definition 4.21 If $\mathcal{M} = (M, I)$ is an $\mathcal{L}_\mathcal{A}$-structure, then we say $\mathcal{M}$ *is countable* to indicate that M is a countable set.

Theorem 4.22 (Lowenheim-Skolem) *Suppose that $\mathcal{N}$ is an $\mathcal{L}_\mathcal{A}$-structure. Then there exists an elementary substructure $(M, I) \preceq \mathcal{N}$ such that M is countable.*

Proof. By Tarski's criterion, it suffices to find a countable set $M \subseteq N$ such that for each nonempty set $A \subseteq N$, if A is definable in the structure $\mathcal{N}$ from parameters in M, then $A \cap M \neq \emptyset$.

We construct a countable sequence $\langle M_k : k \in \mathbb{N}\rangle$ of countable subsets of N such that

(1.1) $M_0 = \emptyset$,
(1.2) for each $k \in \mathbb{N}$, $M_k \subseteq M_{k+1}$,
(1.3) and for each $k \in \mathbb{N}$, if $A \subseteq N$ is definable in $\mathcal{N}$ with parameters from M_k and $A \neq \emptyset$ then $A \cap M_{k+1} \neq \emptyset$.

Given M_k, we can find M_{k+1} as follows. Fix a counting $\langle A_i : i \in \mathbb{N}\rangle$ of the collection of subsets of N which are definable in $\mathcal{N}$ using parameters from M_k. (See the Exercises at the end of the previous section.) For each i such that A_i is not empty, let a_i be an element of A_i. Let M_{k+1} be $M_k \cup \{a_i : i \in \mathbb{N}\}$. M_{k+1} is a union of two countable sets and hence is countable. By construction, it satisfies (2) and (3) relative to M_k. Note that we have used the Axiom of Choice to choose the a_i's from the A_i's.

By another use of the Axiom of Choice, it follows that there is a sequence $\langle M_k : k \in \mathbb{N}\rangle$ which satisfies. (1)–(3). Let $M = \cup\{M_k : k \in \mathbb{N}\}$. By Theorem 4.19, M is countable.

Suppose that $A \subseteq N$ and A is definable in the structure $\mathcal{N}$ from parameters in M. Then since $M = \cup\{M_k : k \in \mathbb{N}\}$, it follows (by (2)) that for sufficiently large $k \in \mathbb{N}$, A is definable in the structure $\mathcal{N}$ from parameters in M_k. Therefore, if $A \neq \emptyset$, then $M \cap A \neq \emptyset$. Finally, for each constant symbol c_i of $\mathcal{L}_\mathcal{A}$, $J(c_i) \in M$

and and for each function symbol F_i of $\mathcal{L}_\mathcal{A}$, and for each $(a_1, \ldots, a_n) \in M^n$ where $n = \pi(F_i)$, $J(F_i)(a_1, \ldots, a_n) \in M$.

Thus, there exists I such that the structure $\mathcal{M} = (M, I)$ is a substructure of $\mathcal{N}$. By Tarski's Theorem, $\mathcal{M} \preceq \mathcal{N}$ and so $\mathcal{M}$ is a countable elementary substructure of $\mathcal{N}$. □

In the proof of Theorem 4.22, we could have started by setting M_0 to be any given countable set. Thus, our proof establishes the stronger statement given below.

Theorem 4.23 *Suppose that $\mathcal{N} = (N, J)$ is an $\mathcal{L}_\mathcal{A}$-structure and X is a countable subset of N. Then there exists an elementary substructure $(M, I) \preceq \mathcal{N}$ such that M is countable and $X \subseteq M$.*

4.7 Arbitrary dense total orders

Suppose that $\mathcal{M} = (M, <)$ is a dense total order without endpoints. Must $\mathcal{M}$ be elementarily equivalent to $(\mathbb{Q}, <)$, and can one characterize the subsets of M which are definable in $\mathcal{M}$?

Our analysis of the definable sets of the structure $\mathcal{R}$ made essential use of the existence of automorphisms. One can construct examples of structures $\mathcal{M} = (M, <)$ which are dense orders without endpoints and with the additional property that if

$$e : M \to M$$

is a bijection which defines an automorphism of the structure $\mathcal{M}$ then e is the identity. In fact one can construct $\mathcal{M}$ as a substructure of $\mathcal{R}$. So one cannot hope to use the method of automorphisms to directly analyze the definable sets of an arbitrary dense order without endpoints.

We begin with a characterization of the countable dense total orders without endpoints.

Theorem 4.24 *Suppose that $\mathcal{M} = (M, \leq_M)$ is a countable dense total order without endpoints. Then $\mathcal{M}$ and $(\mathbb{Q}, \leq)$ are isomorphic.*

Proof. Let $m_1, m_2, \ldots$ be a counting of M, and let $q_1, q_2, \ldots$ be a counting of $\mathbb{Q}$. (See the Exercises in the previous section.)

We build a function $f : M \to \mathbb{Q}$ by recursion, specifying finitely many of the values of f during each stage. Let f_n denote the finite function defined by the end of stage n. We will ensure that f_n preserves order: for all x and y in the domain of f_n, $x <_M y$ if and only if $f_n(x) < f_n(y)$.

To start, let f_0 be the function with empty domain. In other words, we will not have specified any of the values of f at the beginning of step 1.

During step $n+1$, we extend f_n as follows. First, let $a_1, \ldots, a_k$ be the domain of f_n written in increasing order with respect to $<_M$. For each i less than or equal to k, we let $f_{n+1}(a_i)$ be equal to $f_n(a_i)$.

Second, if m_{n+1} is not in the domain of f_n, then we define $f_{n+1}(m_{n+1})$ so as to preserve order. If m_{n+1} is less than a_1, then using the fact that $\mathbb{Q}$ has no least element, we let j be the least integer such that q_j is less than $f(a_1)$ and let $f_{n+1}(m_{n+1}) = q_j$; if there is an i such that $a_i <_M m_{n+1} <_M a_{i+1}$, then using the fact that $\mathbb{Q}$ is dense, we let j be the least integer such that $f_{n+1}(a_i) < q_j < f_{n+1}(a_{i+1})$ and let $f_{n+1}(m_{n+1}) = q_j$; if m_{n+1} is greater than a_k, then using the fact that $\mathbb{Q}$ has no greatest element, we let j be the least integer such that q_j is greater than $f_{n+1}(a_k)$ and let $f_{n+1}(m_{n+1}) = q_j$

We complete step $n+1$ as follows. If q_{n+1} is not in the range of f_{n+1}, even after we have defined f_{n+1} on m_{n+1}, then we find the least j such that m_j has the same order theoretic properties relative to $a_1, \dots, a_k$ and m_{n+1} as q_{n+1} has relative to $f_{n+1}(a_1), \dots, f_{n+1}(a_k)$ and $f_{n+1}(M_{n+1})$. The only properties of $\mathbb{Q}$ that we used of in the above were that it has no least or greatest elements, that has no endpoints, and it is totally ordered. $\mathcal{M}$ shares these properties, and so we can find m_j in M just as we found q_j in $\mathbb{Q}$.

Let f be the function defined by $f(m) = q$ if and only if there is an n such that $f_n(m) = q$. f is well defined since each f_n extends all of the functions defined at steps before n. Every element of M belongs to the domain of f, since for every $m \in M$ there is an n such that $m = m_n$. Then either m was added to the domain of f during an earlier step, or we add m to the domain of f by setting a value for f_n at m_n during step n. Similarly, every $q \in \mathbb{Q}$ is in the range of f. f preserves order by construction, hence f is injective and for all x and y in M, $x <_M y$ if and only if $f(x) < f(y)$. Thus, $f : M \to \mathbb{Q}$ is an isomorphism.□

Theorem 4.25 *Suppose that $\mathcal{M}$ is a dense total order without endpoints. Then, the following conditions hold.*

(1) $\mathcal{M} \equiv (\mathbb{Q}, \leq)$.

(2) If $X \subseteq M$ and $A \subseteq M$ is definable in the structure $\mathcal{M}$ with parameters from X, then A is a finite union of intervals with endpoints in X.

Proof. By the Downward Lowenheim-Skolem Theorem there exists an elementary substructure

$$(M_0, <_0) = \mathcal{M}_0 \preceq \mathcal{M}$$

such that M_0 is countable. But then, $\mathcal{M}_0 \cong (\mathbb{Q}, <)$ and so $\mathcal{M} \equiv (\mathbb{Q}, <)$. This proves (1).

We now prove (2). In fact (2) follows from (1) (why?) but we shall prove (2) more directly. Fix $X \subseteq M$ and $A \subseteq M$ such that A is definable in $\mathcal{M}$ with parameters from X. Let $\varphi(x_1, x_2, \dots, x_n)$ be a formula and let $a_2, \dots, a_n$ be elements of X such that

$$A = \{a \in M : \mathcal{M} \vDash \varphi[a, a_2, \dots, a_n]\}.$$

We prove that A is a union of intervals with endpoints from $\{a_2, \dots, a_n\}$.

Assume toward a contradiction that this fails. By Theorem 4.23, Choose $\mathcal{M}_0 = (M_0, I_0)$ so that $\{a_2, \ldots, a_n\} \subseteq M_0$ and so that $\mathcal{M}_0$ is a countable elementary substructure $\mathcal{M}$. Thus, since $\mathcal{M}_0 \preceq \mathcal{M}$,

$$A \cap M_0 = \{a \in M_0 : \mathcal{M}_0 \models \varphi[a, a_2, \ldots, a_n]\}$$

and $A \cap M_0$ is not a union of intervals of $\mathcal{M}_0$ with endpoints from $\{a_2, \ldots, a_n\}$. But $\mathcal{M}_0 \cong (\mathbb{Q}, <)$ and this contradicts Theorem 4.17. □

Thus we have managed to analyze the definable sets in an arbitrary structure $\mathcal{M} = (M, <)$ which is a dense order without endpoints. The analysis succeeds by using automorphisms of countable elementary substructures.

What about the definable subsets of the structure

$$(\mathbb{N}, <)?$$

We shall eventually show that if $A \subseteq \mathbb{N}$ is definable from parameters in the structure, $(\mathbb{N}, <)$, then A is either finite or the complement of A is finite. There are no automorphisms of the structure $(\mathbb{N}, <)$ except for the trivial automorphism (given by the identity function). However automorphisms *can* be used to analyze the structure $(\mathbb{N}, <)$ by first constructing a structure $(M, <)$ such that

$$(\mathbb{N}, <) \prec (M, <)$$

and then using automorphisms of $(M, <)$ to show that if $A \subset M$ is definable in the structure $(M, <)$ (from parameters) then A is a finite union of intervals.

This suggests the general problem of given an $\mathcal{L}_{\mathcal{A}}$-structure $\mathcal{M}$ under what circumstances must there exist an $\mathcal{L}_{\mathcal{A}}$-structure $\mathcal{N}$ such that

$$\mathcal{M} \prec \mathcal{N}$$

and $\mathcal{M} \neq \mathcal{N}$. The only requirement is that $\mathcal{M}$ be infinite. We shall prove this as a corollary of the the Gödel Completeness Theorem which is the topic of the next chapter.

4.7.1 Exercises

(1) Let $A = \{P_1\}$ be the alphabet with one unary predicate symbol. For each of i equal to 1 or 2, suppose that $\mathcal{M}_i = (M_i, I_i)$ is an A structure such that M_i, $I_i(P_1)$, and $M_i \setminus I_i(P_1)$ are all infinite. Here $M_i \setminus I_i(P_1)$ consists of those elements of M_i which are not in $I_i(P_1)$.
Show that $\mathcal{M}_1 \equiv \mathcal{M}_2$.

5

The Gödel Completeness Theorem

5.1 The notion of proof

Definition 5.1 A *validity* is an $\mathcal{L}$ formula φ which is satisfied in any and every interpretation. That is, for all $\mathcal{M}$ and all ν, $(\mathcal{M}, \nu) \vDash \varphi$.

Validities are the most uninteresting $\mathcal{L}$ formulas, since their being satisfied by a particular $(\mathcal{M}, \nu)$ provides no information whatsoever about $(\mathcal{M}, \nu)$. But the set of validities, which can be regarded as the set of first order absolute truths, is a fascinating set. In this chapter, we will give a syntactic characterization of this set, describing it in terms of pure logic. We will show that a first order $\mathcal{L}$ formula is valid if and only if it can be proven.

The formal notion of proof involves specifying the *logical axioms*. Every logical axiom is valid, and there will be a straightforward algorithm to determine whether any given $\mathcal{L}$ formula is a logical axiom.

Some of the logical axioms involve the deduction of instances of a formula φ from the hypothesis $\forall x_i \varphi$. Others involve deducing that τ_1 has the property asserted by φ from the hypothesis that $\tau_1 \hat{=} \tau_2$ and τ_2 has that property. To state these axioms, we have to extend our notation from Definitions 3.9 and 3.12.

Definition 5.2 Suppose that φ is an $\mathcal{L}$ formula and that τ is a term.

(1) Suppose that x_i is a free variable of φ.
   a) The term τ is *substitutable* for x_i if and only if every variable x_j of τ is free for x_i in φ.
   b) If τ is substitutable for x_i in φ, then $\varphi(x_i; \tau)$ denotes the $\mathcal{L}$ formula obtained by substituting τ for each free occurrence of x_i in φ. Similarly, $\varphi(x_{i_1}, \ldots, x_{i_n}; \tau_{i_1}, \ldots, \tau_{i_n})$ denotes the $\mathcal{L}$ formula obtained by simultaneously substituting each τ_{i_j} for the free occurrences of x_{i_j} in φ.

(2) Suppose that c_i is a constant symbol.
   a) The term τ is *substitutable* for c_i if and only if for every variable x_j of τ, no occurrence of c_i in φ is within the scope of an occurrence of $\forall x_j$.
   b) If τ is substitutable for c_i in φ, then $\varphi(c_i; \tau)$ denotes the $\mathcal{L}$ formula obtained by substituting τ for each occurrence of c_i in φ. Similarly, $\varphi(c_{i_1}, \ldots, c_{i_n}; \tau_{i_1}, \ldots, \tau_{i_n})$ denotes the $\mathcal{L}$ formula obtained by simultaneously substituting each τ_{i_j} for the occurrences of c_{i_j} in φ.

Definition 5.3 The set of logical axioms, denoted Δ, is the smallest set of $\mathcal{L}$ formulas which satisfies the following closure properties.

(1) (Instances of Propositional Tautologies) Suppose that φ_1, φ_2 and φ_3 are $\mathcal{L}$ formulas. Then each of the following $\mathcal{L}$ formulas is a logical axiom:
(Group I axioms)
a) $((\varphi_1 \to (\varphi_2 \to \varphi_3)) \to ((\varphi_1 \to \varphi_2) \to (\varphi_1 \to \varphi_3)))$
b) $(\varphi_1 \to \varphi_1)$
c) $(\varphi_1 \to (\varphi_2 \to \varphi_1))$
(Group II axioms)
a) $(\varphi_1 \to ((\neg\varphi_1) \to \varphi_2))$
(Group III axioms)
a) $(((\neg\varphi_1) \to \varphi_1) \to \varphi_1)$
(Group IV axioms)
a) $((\neg\varphi_1) \to (\varphi_1 \to \varphi_2))$
b) $(\varphi_1 \to ((\neg\varphi_2) \to (\neg(\varphi_1 \to \varphi_2))))$

(2) Suppose that φ is an $\mathcal{L}$ formula, τ is a term, and that τ is substitutable for x_i in φ. Then
$$((\forall x_i \varphi) \to \varphi(x_i; \tau)) \in \Delta.$$

(3) Suppose that φ_1 and φ_2 are $\mathcal{L}$ formulas. Then
$$((\forall x_i(\varphi_1 \to \varphi_2)) \to ((\forall x_i \varphi_1) \to (\forall x_i \varphi_2))) \in \Delta.$$

(4) Suppose that φ is an $\mathcal{L}$ formula and that x_i is not a free variable of φ. Then
$$(\varphi \to (\forall x_i \varphi)) \in \Delta.$$

(5) For every variable x_i, $(x_i \hat{=} x_i) \in \Delta$.

(6) Suppose that φ_1 and φ_2 are $\mathcal{L}$ formulas and that x_j is substitutable for x_i in φ_1 and in φ_2.
$$\begin{aligned}&\text{If } \varphi_2(x_i; x_j) = \varphi_1(x_i; x_j),\\ &\qquad \text{then } ((x_i \hat{=} x_j) \to (\varphi_1 \to \varphi_2)) \in \Delta.\end{aligned}$$

(7) Suppose that $\varphi \in \Delta$. Then $(\forall x_i \varphi) \in \Delta$.

Definition 5.4 Suppose that Γ is a set of $\mathcal{L}$ formulas and that φ is an $\mathcal{L}$ formula. Then
$$\Gamma \vdash \varphi$$
if and only if there exists a finite sequence $\langle \varphi_1, \ldots, \varphi_n \rangle$ of $\mathcal{L}$ formulas such that

(1) $\varphi_1 \in \Gamma \cup \Delta$,
(2) $\varphi_n = \varphi$,

(3) for each $i \leq n$, either
   a) $\varphi_i \in \Gamma \cup \Delta$, or
   b) there exist $i_0 < i$ and $i_1 < i$ such that φ_{i_1} is equal to $(\varphi_{i_0} \rightarrow \varphi_i)$. This rule of inference is called *modus ponens*.

$\langle \varphi_1, \ldots, \varphi_n \rangle$ is called a *deduction* of φ_n from Γ.

When $\Gamma \vdash \varphi$, we say that Γ *proves* φ or that *there is a proof of* φ *from* Γ.

5.2 Soundness

Definition 5.5 Suppose that Γ is a set of $\mathcal{L}$ formulas.

(1) Γ is *consistent* if and only if for every φ, if $\Gamma \vdash \varphi$, then $\Gamma \nvdash (\neg\varphi)$.
(2) Γ is *satisfiable* if and only if there exists a structure $\mathcal{M}$ and an $\mathcal{M}$-assignment ν such that $(\mathcal{M}, \nu) \models \Gamma$.

We can now state the Gödel Completeness Theorem.

Theorem 5.6 (Gödel Completeness) *For any set of* $\mathcal{L}$ *formulas* Γ*, the following conditions are equivalent.*

(1) Γ *is consistent.*
(2) Γ *is satisfiable.*

The implication from satisfiability to consistency can be expressed heuristically—if $(\mathcal{M}, \nu)$ satisfies Γ, then $(\mathcal{M}, \nu)$ satisfies all of the deductive consequences of Γ. We check this implication in the following theorem.

Theorem 5.7 (Soundness) *Suppose that* Γ *is a set of* $\mathcal{L}$ *formulas, that* φ *is an* $\mathcal{L}$ *formula, and that* $\Gamma \vdash \varphi$*. Suppose that* $\mathcal{M}$ *is a structure and that* ν *is an* $\mathcal{M}$*-assignment such that* $(\mathcal{M}, \nu) \models \Gamma$*. Then,* $(\mathcal{M}, \nu) \models \varphi$*.*

Proof. We show by induction on n that if $\langle \varphi_1, \ldots, \varphi_n \rangle$ is a deduction from Γ, then for each i less than or equal to n, $\mathcal{M} \models \varphi_i$. We assume that the claim holds for every i less than n, and we check that it holds for n.

If $\varphi_n \in \Gamma$, then since $(\mathcal{M}, \nu) \models \Gamma$, $(\mathcal{M}, \nu) \models \varphi_n$.

If there are i and j less than n such that φ_j is equal to $(\varphi_i \rightarrow \varphi_n)$, then by induction $(\mathcal{M}, \nu) \models \varphi_i$ and $(\mathcal{M}, \nu) \models (\varphi_i \rightarrow \varphi_n)$. By the definition of satisfaction $(\mathcal{M}, \nu) \models \varphi_n$.

It remains to consider the case in which $\varphi_n \in \Delta$. For this, we must consider each of the clauses in Definition 5.3.

Clause 1. If φ_n is one of the propositional tautologies of Clause 1, then $(\mathcal{M}, \nu) \models \varphi_n$ by the definition of satisfaction for the logical connectives. (See Exercise 3 on page 37.)

Clause 2. If φ_n is obtained by Clause 2, then it has the form

$$((\forall x_i \varphi) \to \varphi(x_i; \tau)),$$

where τ is substitutable for x_i in φ. If $(\mathcal{M}, \nu) \not\models (\forall x_i \varphi)$, then trivially $(\mathcal{M}, \nu) \models \varphi_n$, so we may assume that $(\mathcal{M}, \nu) \models (\forall x_i \varphi)$. Then for every $\mathcal{M}$-assignment μ which agrees with ν on the free variables of $(\forall x_i \varphi)$, $(\mathcal{M}, \mu) \models \varphi$. In particular, if μ agrees with ν on all of the variables except for x_j and $\mu(x_j) = \overline{\nu}(\tau)$, then $(\mathcal{M}, \mu) \models \varphi$. By the Substitution Theorem 3.13, $(\mathcal{M}, \nu) \models \varphi(x_i; \tau)$, and so $(\mathcal{M}, \nu) \models \varphi_n$.

Clause 3. If φ_n is obtained by Clause 3, then it has the form

$$((\forall x_i (\psi_1 \to \psi_2)) \to ((\forall x_i \psi_1) \to (\forall x_i \psi_2))).$$

If $(\mathcal{M}, \nu) \not\models (\forall x_i (\psi_1 \to \psi_2))$ or if $(\mathcal{M}, \nu) \not\models (\forall x_i \psi_1)$, then $(\mathcal{M}, \nu) \models \varphi_n$. Otherwise, for every $\mathcal{M}$-assignment μ which agrees with ν on the free variables of $(\forall x_i (\psi_1 \to \psi_2))$, $(\mathcal{M}, \mu) \models (\psi_1 \to \psi_2)$ and $(\mathcal{M}, \mu) \models \psi_1$. Consequently, for every such μ, $(\mathcal{M}, \mu) \models \psi_2$. Since every free variable of $(\forall x_i \psi_2)$ is also free in $(\forall x_i (\psi_1 \to \psi_2))$, for every $\mathcal{M}$-assignment μ which agrees with ν on the free variables of $(\forall x_i \psi_2)$, $(\mathcal{M}, \mu) \models \psi_2$. It follows that $(\mathcal{M}, \nu) \models (\forall x_i \psi_2)$, and hence that $(\mathcal{M}, \nu) \models \varphi_n$.

Clause 4. If φ_n is obtained by Clause 4, then it has the form

$$(\psi \to (\forall x_i \psi))$$

where x_i is not free in ψ. If $(\mathcal{M}, \nu) \not\models \psi$, then $(\mathcal{M}, \nu) \models \varphi_n$. Assume that $(\mathcal{M}, \nu) \models \psi$. Then, by Theorem 3.7, for every $\mathcal{M}$-assignment μ, if ν and μ agree on the free variables of ψ, then $(\mathcal{M}, \mu) \models \psi$. Since x_i is not free in ψ, the variables which occur freely ψ also occur freely in $(\forall x_i \psi)$. Thus, if ν and μ agree on the free variables of $(\forall x_i \psi)$, then $(\mathcal{M}, \mu) \models \psi$. It follows that $(\mathcal{M}, \nu) \models (\forall x_i \psi)$.

Clause 5. If φ_n is obtained by Clause 5, then it has the form $(x_i \mathrel{\hat{=}} x_i)$, which is satisfied in every $(\mathcal{M}, \nu)$.

Clause 6. If φ_n is obtained by Clause 6, then it has the form

$$((x_i \mathrel{\hat{=}} x_j) \to (\psi_1 \to \psi_2)),$$

where ψ_1 and ψ_2 are $\mathcal{L}$ formulas such that x_j is substitutable for x_i in ψ_1 and in ψ_2 and such that $\psi_2(x_i; x_j) = \psi_1(x_i; x_j)$. If $(\mathcal{M}, \nu) \not\models (x_i \mathrel{\hat{=}} x_j)$ or $(\mathcal{M}, \nu) \not\models \psi_1$, then $(\mathcal{M}, \nu) \models \varphi_n$. Thus, we may assume that $(\mathcal{M}, \nu) \models (x_i \mathrel{\hat{=}} x_j)$ and $(\mathcal{M}, \nu) \models \psi_1$. Since $(\mathcal{M}, \nu) \models (x_i \mathrel{\hat{=}} x_j)$, $\nu(x_i) = \overline{\nu}(\langle x_j \rangle)$ and we can apply the Substitution Theorem 3.13 to the $\mathcal{L}$ formula obtained by substituting the term $\langle x_j \rangle$ for the variable x_i in ψ_1. Thus, $(\mathcal{M}, \nu) \models \psi_1(x_i; x_j)$. Since $\psi_2(x_i; x_j) = \psi_1(x_i; x_j)$, $(\mathcal{M}, \nu) \models \psi_2(x_i; x_j)$. Again noting that $\nu(x_i) = \overline{\nu}(\langle x_j \rangle)$, we may apply Theorem 3.13 and conclude from $(\mathcal{M}, \nu) \models \psi_2(x_i; x_j)$ that $(\mathcal{M}, \nu) \models \psi_2$. Consequently, $(\mathcal{M}, \nu) \models \varphi_n$.

Clause 7. If φ_n is obtained by Clause 7, then it has the form $(\forall x_i \psi)$, where $\psi \in \Delta$. By induction, for every $\mathcal{M}$-assignment μ, $(\mathcal{M}, \mu) \models \psi$. Consequently, $(\mathcal{M}, \nu) \models (\forall x_i \psi)$, as required. $\square$

Corollary 5.8 *Suppose that Γ is a set of $\mathcal{L}$ formulas. If Γ is satisfiable, then Γ is consistent.*

5.2.1 Exercises

(1) Show that for every pair of $\mathcal{L}$ formulas φ and ψ, $\{\varphi, (\neg\varphi)\} \vdash \psi$.

5.3 Deduction and generalization theorems

Theorem 5.9 (Deduction) *Suppose that Γ is a set of $\mathcal{L}$ formulas and that φ_1 and φ_2 are $\mathcal{L}$ formulas. Then*

$$\Gamma \cup \{\varphi_1\} \vdash \varphi_2 \textit{ if and only if } \Gamma \vdash (\varphi_1 \to \varphi_2).$$

Proof. We first verify the implication from right to left. Suppose that $\langle \theta_1, \ldots, \theta_n \rangle$ is a deduction from Γ of $(\varphi_1 \to \varphi_2)$. In particular, θ_n is equal to $(\varphi_1 \to \varphi_2)$. Then,

$$\langle \theta_1, \ldots, \theta_{n-1}, (\varphi_1 \to \varphi_2), \varphi_1, \varphi_2 \rangle$$

is a deduction of φ_2 from Γ, as required.

For the implication from left to right, we proceed by induction on the length of deductions to show that if $\Gamma \cup \{\varphi_1\} \vdash \varphi_2$, then $\Gamma \vdash (\varphi_1 \to \varphi_2)$. So, let $\langle \theta_1, \ldots, \theta_n \rangle$ be a deduction from $\Gamma \cup \{\varphi_1\}$. Assume that j is less than or equal to n and that for each i less than j, $\Gamma \vdash (\varphi_1 \to \theta_i)$. By Definition 5.4, either θ_j is equal to φ_1, $\theta_j \in \Gamma \cup \Delta$, or there are i_1 and i_2 less than j such that θ_{i_2} is equal $(\theta_{i_1} \to \theta_j)$.

If θ_j is equal to φ_1, then we apply Clause 6 of Definition 5.3. On trivial grounds, x_1 is free for x_1 in φ_1 and $\varphi_1(x_1; x_1) = \varphi_1(x_1; x_1)$. Consequently, $((x_1 \mathrel{\hat{=}} x_1) \to (\varphi_1 \to \varphi_1))$ is an element of Δ. By Clause 5, $(x_1 \mathrel{\hat{=}} x_1)$ is also an element of Δ. Thus,

$$\begin{aligned}&\langle((x_1 \mathrel{\hat{=}} x_1) \to (\varphi_1 \to \varphi_1),\\ &\ (x_1 \mathrel{\hat{=}} x_1),\\ &\ (\varphi_1 \to \varphi_1))\rangle\end{aligned}$$

is a deduction from Γ of $(\varphi_1 \to \varphi_1)$.

In fact, $(\varphi_1 \to \varphi_1)$ is an instance of a propositional tautology, and any such can be deduced using only Clause 1 and modus ponens. We give a second deduction of $(\varphi_1 \to \varphi_1)$ of this sort.

$$\begin{array}{ll}
\langle(\varphi_1 \to ((\varphi_1 \to \varphi_1) \to \varphi_1)), & \text{Clause 1a} \\
\ ((\varphi_1 \to ((\varphi_1 \to \varphi_1) \to \varphi_1)) \to & \\
\qquad ((\varphi_1 \to (\varphi_1 \to \varphi_1)) \to (\varphi_1 \to \varphi_1))), & \text{Clause 1b} \\
\ ((\varphi_1 \to (\varphi_1 \to \varphi_1)) \to (\varphi_1 \to \varphi_1))), & \text{Modus ponens} \\
\ (\varphi_1 \to (\varphi_1 \to \varphi_1)), & \text{Clause 1a} \\
\ (\varphi_1 \to \varphi_1)\rangle & \text{Modus ponens}
\end{array}$$

If θ_j is an element of $\Gamma \cup \Delta$, then we apply Clause 1a of Definition 5.3. Namely, $(\theta_j \to (\varphi_1 \to \theta_j))$ is an element of Δ. But then $\langle \theta_j, (\theta_j \to (\varphi_1 \to \theta_j)), (\varphi_1 \to \theta_j)\rangle$ is a deduction from Γ. The first two elements of the sequence belong to $\Gamma \cup \Delta$ and the third element of the sequence is obtained from its predecessors by an application of modus ponens.

If there are i_1 and i_2 less than j such that θ_{i_2} is equal $(\theta_{i_1} \to \theta_j)$, then we apply Clause 1b of Definition 5.3 as follows. We may assume that Γ proves $(\varphi_1 \to \theta_{i_1})$ by means of the deduction $\langle \alpha_1, \ldots, \alpha_n\rangle$ and that Γ proves $(\varphi_1 \to (\theta_{i_1} \to \theta_j))$ by means of the deduction $\langle \beta_1, \ldots, \beta_m\rangle$. By Clause 1b, $((\varphi_1 \to (\theta_{i_1} \to \theta_j)) \to ((\varphi_1 \to \theta_{i_1}) \to (\varphi_1 \to \theta_j)))$ is an element of Δ. Consequently,

$$\begin{array}{l}
\langle \alpha_1, \ldots, \alpha_{n-1}, (\varphi_1 \to \theta_{i_1}), \\
\quad \beta_1, \ldots, \beta_{m-1}, (\varphi_1 \to (\theta_{i_1} \to \theta_j)), \\
\quad ((\varphi_1 \to (\theta_{i_1} \to \theta_j)) \to ((\varphi_1 \to \theta_{i_1}) \to (\varphi_1 \to \theta_j))), \\
\quad ((\varphi_1 \to \theta_{i_1}) \to (\varphi_1 \to \theta_j)), (\varphi_1 \to \theta_j)\rangle
\end{array}$$

is a deduction of $(\varphi_1 \to \theta_j)$ from Γ, as required. □

Theorem 5.10 (Generalization) *Suppose that Γ is a set of $\mathcal{L}$ formulas, that φ is an $\mathcal{L}$ formula, and that $\Gamma \vdash \varphi$. Suppose that x_i is a variable and that x_i is not a free variable of any formula in Γ. Then, $\Gamma \vdash (\forall x_i \varphi)$.*

Proof. Once again we go by induction on the lengths of deductions. Let $\langle \theta_1, \ldots, \theta_n\rangle$ be a deduction from Γ of φ, and assume that for each j less than n, $\Gamma \vdash (\forall x_i \theta_j)$.

First, consider the case when φ is an element of Γ. We can apply Clause 4 of Definition 5.3 and conclude that $(\varphi \to (\forall x_i \varphi))$ is an element of Δ. Then, $\langle \varphi, (\varphi \to (\forall x_i \varphi)), (\forall x_i \varphi)\rangle$ is a deduction of $(\forall x_i \varphi)$ from Γ, as required.

Next, consider the case when there are i_1 and i_2 less than n such that θ_{i_2} is equal to $(\theta_{i_1} \to \varphi)$. By induction, Γ proves $(\forall x_i \theta_{i_1})$ and $(\forall x_i(\theta_{i_1} \to \varphi))$. By Clause 3 of Definition 5.3,

$$((\forall x_i(\theta_{i_1} \to \varphi)) \to ((\forall x_i \theta_{i_1}) \to (\forall x_i \varphi)))$$

is an element of Δ. By concatenating deductions as in the proof of the previous theorem, it follows that $\Gamma \vdash (\forall x_i \varphi)$, as required. □

The next theorem requires two lemmas.

Lemma 5.11 *Suppose that φ is an $\mathcal{L}$ formula, x_i is free for x_j in φ, and x_i does not occur freely in $(\forall x_j \varphi)$. Then*

$$\emptyset \vdash ((\forall x_j \varphi) \to (\forall x_i \varphi(x_j; x_i)))$$

Proof. Since x_i is free for x_j in φ, we may apply Clause 2 of Definition 5.3 to conclude that $((\forall x_j \varphi) \to \varphi(x_j; x_i))$ is an element of Δ. By the Deduction Theorem 5.9, $\{(\forall x_j \varphi)\} \vdash \varphi(x_j; x_i)$. Since x_i does not occur freely in $(\forall x_j \varphi)$, we can apply the Generalization Theorem 5.10 to conclude that $\{(\forall x_j \varphi)\} \vdash (\forall x_i \varphi(x_j; x_i))$. By the Deduction Theorem again, $\emptyset \vdash ((\forall x_j \varphi) \to (\forall x_i \varphi(x_j; x_i)))$, as required. □

Lemma 5.12 *Suppose that Γ is a set of $\mathcal{L}$ formulas and that the constant symbol c_i does not occur in any formula of Γ. Suppose that $\langle \theta_1, \ldots, \theta_m \rangle$ is a deduction from Γ and that the variable x_j does not occur in any of the formulas θ_n for which $n \leq m$. Then $\langle \theta_1(c_i; x_j), \ldots, \theta_m(c_i; x_j) \rangle$ is a deduction from Γ.*

Proof. Note, if c_i does not occur in φ, then $\varphi(c_i; x_j) = \varphi$. By assumption c_i does not occur in any formula of Γ, so for each $\varphi \in \Gamma$, $\varphi(c_i; x_j) = \varphi$.

It can be verified by inspection of Definition 5.3 that if φ is a logical axiom and x_j does not occur in φ, then $\varphi(c_i; x_j)$ is a logical axiom.

Finally, if φ_1 and φ_2 are $\mathcal{L}$ formulas then

$$(\varphi_1 \to \varphi_2)(c_i; x_j) = (\varphi_1(c_i; x_j) \to \varphi_2(c_i; x_j)).$$

It follows by induction on $n \leq m$, that $\langle \theta_1(c_i; x_j), \ldots, \theta_n(c_i; x_j) \rangle$ is a proof from Γ. □

Theorem 5.13 (Constants) *Suppose that Γ is a set of $\mathcal{L}$ formulas, that φ is an $\mathcal{L}$ formula, and that $\Gamma \vdash \varphi$. Suppose that c_i is a constant and that c_i does not occur in any formula of Γ. Let x_j be a variable which is substitutable for c_i in φ and which does not occur freely in φ. Then the following conditions hold.*

(1) $\Gamma \vdash (\forall x_j \varphi(c_i; x_j))$.

(2) There is a deduction $\langle \varphi_1, \ldots, \varphi_n \rangle$ of $(\forall x_j \varphi(c_i; x_j))$ from Γ such that
   - *a) for each $m \leq n$, c_i does not occur in φ_m,*
   - *b) and for each $m < n$ and each c_k, if c_k occurs in φ_m, then either c_k occurs in $(\forall x_j \varphi(c_i; x_j))$ or c_k occurs in some formula of Γ.*

Proof. Let $\langle \theta_1, \ldots, \theta_n \rangle$ be a deduction of φ from Γ. Let x_{j_1} be a variable such that for each m less than or equal to n, x_{j_1} does not appear in θ_m. Let Γ_0 be $\{\theta_m : m \leq n\} \cap \Gamma$. By Lemma 5.12, $\langle \theta_1(c_i; x_{j_1}), \ldots, \theta_n(c_i; x_{j_1}) \rangle$ is a deduction of $\varphi(c_i; x_{j_1})$ from Γ_0. By the Generalization Theorem, $\Gamma_0 \vdash (\forall x_{j_1} \varphi(c_i; x_{j_1}))$. Since

x_j is substitutable for c_i in φ and does not occur freely in φ, x_j is substitutable for x_{j_1} and does not occur freely in $(\forall x_{j_1}\varphi(c_i; x_{j_1}))$. By Lemma 5.11,

$$\emptyset \vdash ((\forall x_{j_1}\varphi(c_i; x_{j_1})) \to (\forall x_j \varphi(c_i; x_{j_1})(x_{j_1}; x_j))).$$

Of course, $\varphi(c_i; x_{j_1})(x_{j_1}; x_j)$ is equal to $\varphi(c_i; x_j)$ and so

$$\emptyset \vdash ((\forall x_{j_1}\varphi(c_i; x_{j_1})) \to (\forall x_j \varphi(c_i; x_j))).$$

Thus,

$$\Gamma_0 \vdash (\forall x_{j_1}\varphi(c_i; x_{j_1})) \text{ and}$$
$$\Gamma_0 \vdash ((\forall x_{j_1}\varphi(c_i; x_{j_1})) \to (\forall x_j \varphi(c_i; x_j)))$$

and so $\Gamma_0 \vdash (\forall x_j \varphi(c_i; x_j))$.

Now, we verify the second claim. Let $\langle \theta_1, \dots, \theta_n \rangle$ be a deduction of $(\forall x_j \varphi(c_i; x_j))$ from Γ. By an application of Lemma 5.12, if m is less than n, c_k occurs in θ_m, and c_k does not occur in φ or in any element of Γ_0, then taking x_{k_1} so that x_{k_1} does not appear in any of the θ_i's, $\langle \theta_1(c_m; x_{k_1}), \dots, \theta_n(c_m; x_{k_1}) \rangle$ is a deduction of φ from Γ_0. By sequential application of this observation, we may assume that for each $m < n$ and each c_k, if c_k occurs in φ_m, then either c_k occurs in φ or c_k occurs in some formula of Γ. $\square$

5.3.1 Exercises

(1) Suppose that $\Gamma \cup \{(\neg\varphi)\}$ is not consistent. Show that $\Gamma \vdash \varphi$. (This is a technical formulation of the legitimacy of proofs by contradiction.)

(2) Suppose that $\mathcal{M}$ is an $\mathcal{L}$-structure and ν is an $\mathcal{M}$-assignment. Show that $\{\varphi : (\mathcal{M}, \nu) \models \varphi\}$ is maximally consistent.

5.4 The Henkin property

Definition 5.14 We use the notation $(\exists x_i \varphi)$ to represent the $\mathcal{L}$ formula $(\neg(\forall x_i(\neg\varphi)))$.

By inspection of Definition 3.6, $(\mathcal{M}, \nu) \models (\exists x_i \varphi)$ if and only if there is an $\mathcal{M}$-assignment μ which agrees with ν on the free variables of $(\exists x_i \varphi)$ such that $(\mathcal{M}, \mu) \models \varphi$.

To complete the proof the Gödel Completeness Theorem, we must develop the machinery to show that if Γ is consistent then Γ is satisfiable.

Definition 5.15 Suppose that Γ is a consistent set of $\mathcal{L}$ formulas. Γ is *maximally consistent* if and only if for any $\mathcal{L}$ formula φ, either $\varphi \in \Gamma$ or $\Gamma \cup \{\varphi\}$ is not consistent.

Lemma 5.16 *Suppose that Γ is a maximally consistent set of $\mathcal{L}$ formulas. Then for each $\mathcal{L}$ formula φ, either $\varphi \in \Gamma$ or $(\neg\varphi) \in \Gamma$.*

Proof. Suppose that $(\neg\varphi)$ does not belong to Γ. By the maximality of Γ, $\Gamma \cup \{(\neg\varphi)\}$ is inconsistent. By the exercise at the end of the previous section, for any $\mathcal{L}$ formula θ, $\Gamma \cup \{(\neg\varphi)\} \vdash \theta$. Consequently, letting θ be $(\neg(x_1 \mathrel{\hat{=}} x_1))$, $\Gamma \cup \{(\neg\varphi)\} \vdash (\neg(x_1 \mathrel{\hat{=}} x_1))$. By the Deduction Theorem 5.9, $\Gamma \vdash ((\neg\varphi) \to (\neg(x_1 \mathrel{\hat{=}} x_1)))$. Applying Clause 1 in the definition of Δ, $\Gamma \vdash (((\neg\varphi) \to (\neg(x_1 \mathrel{\hat{=}} x_1))) \to ((x_1 \mathrel{\hat{=}} x_1) \to \varphi))$. Two applications of modus ponens yield $\Gamma \vdash \varphi$. Now, since $\Gamma \vdash \varphi$, any deduction from $\Gamma \cup \{\varphi\}$ can be converted into a deduction from Γ by replacing each instance of φ with a deduction of φ from Γ. Thus, for each θ, if $\Gamma \cup \{\varphi\} \vdash \theta$ then $\Gamma \vdash \theta$. Since Γ is consistent, $\Gamma \cup \{\varphi\}$ is also consistent. Since no proper superset of Γ is consistent, $\varphi \in \Gamma$, as required. $\square$

Definition 5.17 A set of $\mathcal{L}$ formulas Γ has the *Henkin Property* if and only if for each $\mathcal{L}$ formula φ and for each variable x_i, if $(\exists x_i\varphi) \in \Gamma$ then there exists a constant c_j such that $\varphi(x_i; c_j) \in \Gamma$.

The following application of Tarski's Theorem motivates the definition of the Henkin property.

Theorem 5.18 *Suppose that $\mathcal{M} = (M, I)$ is a structure and ν is an $\mathcal{M}$-assignment such that*

$$\{\nu(x_i) : i \in \mathbb{N}\} \subseteq \{I(c_i) : i \in \mathbb{N}\}.$$

Let $\Gamma = \{\varphi : (\mathcal{M}, \nu) \models \varphi\}$. Then the following conditions hold.

(1) Γ is maximally consistent.

(2) Γ has the Henkin property if and only if there exists an elementary substructure $(M_0, I_0) \preceq \mathcal{M}$ such that

$$M_0 = \{I(c_i) : i \in \mathbb{N}\}.$$

Proof. We begin with the first claim. For the consistency of Γ, note that every formula in Γ is satisfied by $(\mathcal{M}, \nu)$. By the Soundness Theorem 5.7, if $\Gamma \vdash \varphi$, then $(\mathcal{M}, \nu) \models \varphi$. By the definition of satisfaction, if $(\mathcal{M}, \nu)$ satisfies φ, then $(\mathcal{M}, \nu)$ does not satisfy $(\neg\varphi)$. Consequently, if $\Gamma \vdash \varphi$, then $\Gamma \nvdash (\neg\varphi)$, and so Γ is consistent.

For the maximality of Γ, suppose that $\varphi \notin \Gamma$. Then $(\mathcal{M}, \nu) \not\models \varphi$, hence $(\mathcal{M}, \nu) \models (\neg\varphi)$ and so $(\neg\varphi) \in \Gamma$. Thus, φ and $(\neg\varphi)$ can be deduced from $\Gamma \cup \{\varphi\}$, showing that it is not consistent. Since Γ is consistent and no proper extension of Γ is consistent, Γ is maximally consistent.

Now, we consider the second claim.

For the implication from left to right, suppose that Γ has the Henkin property. We will show that M_0 satisfies Tarski's Criterion.

Let $m_1, \dots, m_n$ be elements of M_0 and suppose that $A \subseteq M$ is definable in $\mathcal{M}$ from these elements as follows.

$$a \in A \leftrightarrow \mathcal{M} \models \varphi[a, m_1, \dots, m_n]$$

We must show that $A \cap M_0$ is not empty.

Since each element of M_0 is in the range of I applied to the set of constant symbols, we fix $c_{i_1}, \ldots, c_{i_n}$ so that for each $j \leq n$, $I(c_{i_j}) = m_j$. By the Substitution Theorem,

$$a \in A \ \leftrightarrow \ \mathcal{M} \models \varphi(x_1, \ldots, x_n; c_{i_1}, \ldots, c_{i_n})[a].$$

Since A is not empty,

$$(\mathcal{M}, \nu) \models (\exists x_0 \varphi(x_1, \ldots, x_n; c_{i_1}, \ldots, c_{i_n})).$$

Then, $(\exists x_0 \varphi(x_1, \ldots, x_n; c_{i_1}, \ldots, c_{i_n}))$ is an element of Γ, and by the Henkin property, there is a c_{i_0} such that

$$\varphi(x_1, \ldots, x_n; c_{i_1}, \ldots, c_{i_n})(x_0; c_{i_0}) \in \Gamma.$$

Note that

$$\varphi(x_1, \ldots, x_n; c_{i_1}, \ldots, c_{i_n})(x_0; c_{i_0}) = \varphi(x_0, \ldots, x_n; c_{i_0}, \ldots, c_{i_n}).$$

Consequently,

$$\mathcal{M} \models \varphi(x_0, \ldots, x_n; c_{i_0}, c_{i_1}, \ldots, c_{i_n}),$$

and so

$$\mathcal{M} \models \varphi[I(c_{i_0}), I(c_{i_1}), \ldots, I(c_{i_n})].$$

By the above, each $I(c_{i_j})$ is equal to m_j, so

$$\mathcal{M} \models \varphi[I(c_{i_0}), m_1, \ldots, m_n].$$

$I(c_{i_0})$ is the desired element of $M_0 \cap A$.

For the implication from right to left, suppose that $\mathcal{M}_0$ is an elementary substructure of $\mathcal{M}$. To check that Γ has the Henkin property, suppose that $(\exists x_i \varphi) \in \Gamma$ and that the variables of φ are included in $x_1, \ldots, x_n$. Let $c_{k_1}, \ldots, c_{k_{i-1}}, c_{k_{i+1}}, \ldots, c_{k_n}$ be constant symbols of $\mathcal{L}_\mathcal{A}$ such that for each j less then or equal to n other than i, $\nu(x_j) = I(c_{k_j})$. Since $(\exists x_i \varphi) \in \Gamma$, by the definition of Γ,

$$(\mathcal{M}, \nu) \models (\exists x_i \varphi),$$

the Substitution Theorem 3.13 implies that

$$\mathcal{M} \models (\exists x_i \varphi(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n; c_{k_1}, \ldots, c_{k_{i-1}}, c_{k_{i+1}}, \ldots, c_{k_n})).$$

Now, $\mathcal{M}_0 \preceq \mathcal{M}$, so

$$\mathcal{M}_0 \models (\exists x_i \varphi(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n; c_{k_1}, \ldots, c_{k_{i-1}}, c_{k_{i+1}}, \ldots, c_{k_n})).$$

Then, there is an $\mathcal{M}_0$ assignment μ such that

$$(\mathcal{M}_0, \mu) \models \varphi(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n; c_{k_1}, \ldots, c_{k_{i-1}}, c_{k_{i+1}}, \ldots, c_{k_n}).$$

By the fact that every element of M_0 is the denotation of a constant symbol, there is a c_{k_i} such that $\mu(x_i) = I_0(c_{k_i})$. But then, by applying the Substitution Theorem 3.13 again,

$$\mathcal{M}_0 \models \varphi(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n; c_{k_1}, \ldots, c_{k_{i-1}}, c_{k_{i+1}}, \ldots, c_{k_n})(x_i; c_{k_i}).$$

The order of substitution does not matter, and so

$$\mathcal{M}_0 \models \varphi(x_i; c_{k_i})(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n; c_{k_1}, \ldots, c_{k_{i-1}}, c_{k_{i+1}}, \ldots, c_{k_n}).$$

But then, by application of Theorem 3.13, $(\mathcal{M}, \nu) \models \varphi(x_i; c_{k_i})$ and so $\varphi(x_i; c_{k_i}) \in \Gamma$, as required. $\square$

5.4.1 Exercises

(1) Suppose that Γ is a maximally consistent set of $\mathcal{L}$ formulas. Define the relation $\sim_\Gamma$ between terms by $\tau_1 \sim_\Gamma \tau_2$ if and only if $(\tau_1 \hat{=} \tau_2) \in \Gamma$. Show that $\sim_\Gamma$ is an equivalence relation. That is to say that it is reflexive, symmetric, and transitive.

Lemma 5.19 *Suppose that φ is an $\mathcal{L}$ formula with no quantifiers and that $\tau_1, \ldots, \tau_n$ and $\sigma_1, \ldots, \sigma_n$ are terms. Then, for any sequence of variables $x_{m_1}, \ldots, x_{m_n}$,*

$$\{(\tau_i \hat{=} \sigma_i) : i \leq n\} \cup \{\varphi(x_{m_1}, \ldots, x_{m_n}; \tau_1, \ldots, \tau_n)\} \vdash \\ \varphi(x_{m_1}, \ldots, x_{m_n}; \sigma_1, \ldots, \sigma_n).$$

Proof. We prove Lemma 5.19 by induction on n. Suppose that it holds for all $\mathcal{L}$ formulas with no quantifiers and for all m less than n. Since we are substituting terms for all of the x_{m_j}'s, we may assume that none of these variables occur in any of the τ_i's or σ_i's. By induction

$$\{(\tau_i \hat{=} \sigma_i) : i \leq n\} \cup \{\varphi(x_{m_1}, \ldots, x_{m_{n-1}}; \tau_1, \ldots, \tau_{n-1})\} \vdash \\ \varphi(x_{m_1}, \ldots, x_{m_{n-1}}; \sigma_1, \ldots, \sigma_{n-1})$$

By the deduction theorem,

$$\{(\tau_i \hat{=} \sigma_i) : i \leq n\} \vdash \\ \begin{pmatrix} \varphi(x_{m_1}, \ldots, x_{m_{n-1}}; \tau_1, \ldots, \tau_{n-1}) \rightarrow \\ \varphi(x_{m_1}, \ldots, x_{m_{n-1}}; \sigma_1, \ldots, \sigma_{n-1}) \end{pmatrix}$$

Since x_{m_n} does not appear in any of the hypotheses,

$$\{(\tau_i \mathrel{\hat{=}} \sigma_i) : i \leq n\} \vdash \left(\forall x_{m_n} \left(\begin{array}{c}\varphi(x_{m_1}, \ldots, x_{m_{n-1}}; \tau_1, \ldots, \tau_{n-1}) \to \\ \varphi(x_{m_1}, \ldots, x_{m_{n-1}}; \sigma_1, \ldots, \sigma_{n-1})\end{array}\right)\right)$$

Then Clause 2 applies and so

$$\{(\tau_i \mathrel{\hat{=}} \sigma_i) : i \leq n\} \vdash \left(\begin{array}{c}\varphi(x_{m_1}, \ldots, x_{m_{n-1}}; \tau_1, \ldots, \tau_{n-1})(x_{m_n}; \sigma_n) \to \\ \varphi(x_{m_1}, \ldots, x_{m_{n-1}}; \sigma_1, \ldots, \sigma_{n-1})(x_{m_n}; \sigma_n)\end{array}\right).$$

Using the fact that the variables x_{m_j} do not appear in any of the τ_i's or σ_i's, we can rewrite this formula as

$$\{(\tau_i \mathrel{\hat{=}} \sigma_i) : i \leq n\} \vdash \left(\begin{array}{c}\varphi(x_{m_1}, \ldots, x_{m_n}; \tau_1, \ldots, \tau_{n-1}, \sigma_n) \to \\ \varphi(x_{m_1}, \ldots, x_{m_n}; \sigma_1, \ldots, \sigma_n)\end{array}\right)$$

Let x_k be a variable which does not appear in any of the τ_i's or σ_i's and does not appear in φ. By Clause 6 in Definition 5.3,

$$\emptyset \vdash \left((x_{m_n} \mathrel{\hat{=}} x_k) \to \left(\begin{array}{c}\varphi(x_{m_1}, \ldots, x_{m_{n-1}}; \tau_1, \ldots, \tau_{n-1}) \to \\ \varphi(x_{m_1}, \ldots, x_{m_{n-1}}; \tau_1, \ldots, \tau_{n-1})(x_{m_n}; x_k)\end{array}\right)\right).$$

By Clauses 7 and then 2,

$$\emptyset \vdash \left((x_{m_n} \mathrel{\hat{=}} x_k) \to \left(\begin{array}{c}\varphi(x_{m_1}, \ldots, x_{m_{n-1}}; \tau_1, \ldots, \tau_{n-1}) \to \\ \varphi(x_{m_1}, \ldots, x_{m_{n-1}}; \tau_1, \ldots, \tau_{n-1})(x_{m_n}; x_k)\end{array}\right)\right)(x_{m_n}, x_k; \tau_n, \sigma_n)$$

Making the substitutions,

$$\emptyset \vdash \left((\tau_n \mathrel{\hat{=}} \sigma_n) \to \left(\begin{array}{c}\varphi(x_{m_1}, \ldots, x_{m_n}; \tau_1, \ldots, \tau_n) \to \\ \varphi(x_{m_1}, \ldots, x_{m_n}; \tau_1, \ldots, \tau_{n-1}, \sigma_n)\end{array}\right)\right)$$

By the deduction theorem,

$$\{(\tau_n \mathrel{\hat{=}} \sigma_n)\} \vdash \left(\begin{array}{c}\varphi(x_{m_1}, \ldots, x_{m_n}; \tau_1, \ldots, \tau_n) \to \\ \varphi(x_{m_1}, \ldots, x_{m_n}; \tau_1, \ldots, \tau_{n-1}, \sigma_n)\end{array}\right).$$

Combining the previous two paragraphs,

$$\{(\tau_n \mathrel{\hat{=}} \sigma_n)\} \cup \{\varphi(x_{m_1}, \ldots, x_{m_n}; \tau_1, \ldots, \tau_n)\} \vdash \varphi(x_{m_1}, \ldots, x_{m_n}; \sigma_1, \ldots, \sigma_n)$$

as required. □

We now prove a special case of the Gödel Completeness Theorem.

Theorem 5.20 *Suppose that* Γ *is a maximally consistent set of* $\mathcal{L}$ *formulas with the Henkin property. Then* Γ *is satisfiable.*

Proof. Our proof is naturally divided into parts. First we must define a model $\mathcal{M}$ and an assignment ν for that model. Then we must verify that $(\mathcal{M}, \nu)$ satisfies Γ.

Defining $\mathcal{M}$ and ν. We say that two constants c_i and c_j are *equivalent mod* Γ, written $c_i \sim_\Gamma c_j$, if and only if $(c_i \mathrel{\hat{=}} c_j) \in \Gamma$. As was stated in the exercise at the end of the previous section, $\sim_\Gamma$ is an equivalence relation on the set of all constant symbols. That is to say that for all i, j, and k,

(1.1) $c_i \sim_\Gamma c_i$,
(1.2) if $c_i \sim_\Gamma c_j$ then $c_j \sim_\Gamma c_i$,
(1.3) and if $c_i \sim_\Gamma c_j$ and $c_j \sim_\Gamma c_k$ then $c_i \sim_\Gamma c_k$.

For each $i \in \mathbb{N}$, let

$$[c_i]_\Gamma = \{c_j : j \in \mathbb{N} \text{ and } c_i \sim_\Gamma c_j\}.$$

$[c_i]_\Gamma$ is the equivalence class of c_i under the equivalence relation $\sim_\Gamma$. This set of equivalence classes is the universe of our model. Let

$$M = \{[c_i]_\Gamma : i \in \mathbb{N}\}.$$

We now define our $\mathcal{M}$-assignment ν. For each variable x_i, chose j so that $(x_i \mathrel{\hat{=}} c_j) \in \Gamma$ and let $\nu(x_i) = [c_j]_\Gamma$.

To see that ν is well defined, we must show that for each x_i there is at least one c_j such that $(x_i \mathrel{\hat{=}} c_j) \in \Gamma$. Further, we must show that for any two constant symbols c_{j_1} and c_{j_2}, if $(x_i \mathrel{\hat{=}} c_{j_1}) \in \Gamma$ and $(x_i \mathrel{\hat{=}} c_{j_2}) \in \Gamma$ then $c_{j_1} \sim_\Gamma c_{j_2}$.

For the first of these claims, consider the formula $(\exists x_{i+1}(x_i \mathrel{\hat{=}} x_{i+1}))$. If it is not an element of Γ, then by Lemma 5.16 $(\forall x_{i+1}(\neg(x_i \mathrel{\hat{=}} x_{i+1})))$ is an element of Γ. But then x_i is substitutable for x_{i+1} in $(\neg(x_i \mathrel{\hat{=}} x_{i+1}))$, and so $\Gamma \vdash (\neg(x_i \mathrel{\hat{=}} x_i))$. But $(x_i \mathrel{\hat{=}} x_i) \in \Delta$ and so $\Gamma \vdash (x_i \mathrel{\hat{=}} x_i)$. Thus, Γ is not consistent, contrary to assumption. Thus, $(\exists x_{i+1}(x_i \mathrel{\hat{=}} x_{i+1})) \in \Gamma$. Since Γ has the Henkin property, there exists a constant c_j such that $(x_i \mathrel{\hat{=}} c_j) \in \Gamma$. Thus, for each x_i, there is a c_j as required by the definition of ν.

The second claim follows from Lemma 5.19. Thus, ν is well defined.

We next define the interpretation map I.

(2.1) Suppose c_i is a constant. Then

$$I(c_i) = [c_i]_\Gamma.$$

(2.2) Suppose that P_i is a predicate symbol and that $n = \pi(P_i)$. Then

$$I(P_i) = \{\langle [c_{k_1}]_\Gamma, \ldots, [c_{k_n}]_\Gamma \rangle \in M^n : P_i(c_{k_1} \ldots c_{k_n}) \in \Gamma\}.$$

(2.3) Suppose that F_i is a function symbol and that $n = \pi(F_i)$. Then

$$I(F_i)([c_{k_1}]_\Gamma, \dots, [c_{k_n}]_\Gamma) = [c_{k_{n+1}}]_\Gamma$$

if and only if

$$(F_i(c_{k_1}, \dots, c_{k_n}) \hat{=} c_{k_{n+1}}) \in \Gamma.$$

The proofs that $I(P_i)$ and $I(F_i)$ are well defined are analogous to the proof that ν is well defined.

Claim 5.21 *For any term τ, $\overline{\nu}(\tau) = [c_i]_\Gamma$ if and only if $(\tau \hat{=} c_i) \in \Gamma$.*

Proof. We prove Claim 5.21 by induction on the length of τ. If τ has length 1 then for some $k \in \mathbb{N}$, $\tau = \langle x_k \rangle$ or $\tau = \langle c_k \rangle$. If $\tau = \langle x_k \rangle$ then $\overline{\nu}(\tau) = \nu(x_k)$ and the claim follows by the fact that ν is well defined. If $\tau = \langle c_k \rangle$ then $\overline{\nu}(\tau) = I(c_k) = [c_k]_\Gamma$ and the claim follows from the definition of $\sim_\Gamma$.

Now suppose that τ has length $n > 1$ and that:

Induction Hypothesis: *If σ is a term of length less than n, then for all constants c_k, $\overline{\nu}(\sigma) = [c_k]_\Gamma$ if and only if $(\sigma \hat{=} c_k) \in \Gamma$.*

Since τ has length > 1, there are terms $\tau_1, \dots, \tau_m$ and a function symbol F_i such that $\tau = F_i(\tau_1, \dots, \tau_m)$, where $m = \pi(F_i)$. By the definition of $\overline{\nu}$,

$$\overline{\nu}(\tau) = I(F_i)(\overline{\nu}(\tau_1), \dots, \overline{\nu}(\tau_m)).$$

Let $c_{j_1}, \dots, c_{j_m}$ be constants such that for each $k \le m$, $\overline{\nu}(\tau_k) = [c_{j_k}]_\Gamma$. Thus,

$$\overline{\nu}(\tau) = I(F_i)([c_{j_1}]_\Gamma, \dots, [c_{j_m}]_\Gamma).$$

By the definition of $I(F_i)$, for each constant symbol c_s,

$$I(F_i)([c_{j_1}]_\Gamma, \dots, [c_{j_m}]_\Gamma) = [c_s]_\Gamma$$

if and only if

$$(F_i(c_{j_1}, \dots, c_{j_m}) \hat{=} c_s) \in \Gamma.$$

Consequently, $\overline{\nu}(\tau) = [c_s]_\Gamma$ if and only if $(F_i(c_{j_1}, \dots, c_{j_m}) \hat{=} c_s) \in \Gamma$.

By the induction hypothesis, for each k less than or equal to m, $(\tau_k \hat{=} c_{j_k})$ is an element of Γ. Therefore, by Lemma 5.19,

$$(F_i(\tau_1, \dots, \tau_n) \hat{=} F_i(c_{j_1}, \dots, c_{j_m})) \in \Gamma.$$

We can conclude that

$$(F_i(c_{j_1}, \dots, c_{j_m}) \hat{=} c_s) \in \Gamma \text{ if and only if } (\tau \hat{=} c_s) \in \Gamma.$$

Consequently, $\overline{\nu}(\tau) = [c_s]_\Gamma$ if and only if $(\tau \hat{=} c_s) \in \Gamma$. This completes the inductive step and so proves the claim. □

We now prove that for each formula φ, $\varphi \in \Gamma$ if and only if $(\mathcal{M}, \nu) \models \varphi$.

We first reduce to the case in which φ is a sentence. Suppose that φ is a formula, x_i is a variable and that $\nu(x_i) = [c_j]_\Gamma$. Suppose that x_k is a variable not occurring in φ. Thus, $((x_i \hat{=} x_k) \to (\varphi \to \varphi(x_i; x_k)))$ is a logical axiom, and by application of Clause 7 and then 2,

$$\emptyset \vdash ((x_i \hat{=} c_j) \to (\varphi \to \varphi(x_i; c_j))),$$

since $\varphi(x_i; x_k)(x_k; c_j) = \varphi(x_i; c_j)$. It follows that if $\nu(x_i) = [c_j]_\Gamma$, then $\Gamma \vdash (\varphi \to \varphi(x_i; c_j))$. In a similar way, if $\nu(x_i) = [c_j]_\Gamma$, then $\Gamma \vdash (\varphi(x_i; c_j) \to \varphi)$: use the above argument for $(\neg\varphi)$ and then apply Clause 1. Consequently, $\Gamma \vdash (\varphi \leftrightarrow \varphi(x_i; c_j))$ and so $(\varphi \leftrightarrow \varphi(x_i; c_j)) \in \Gamma$.

Let n be large enough so that all the free variables of φ belong to $\{x_1, \ldots, x_n\}$. For each $k \leq n$ let m_k be such that $\nu(x_k) = c_{m_k}$. By the above analysis,

$$(\varphi \leftrightarrow \varphi(x_1, \ldots, x_n; c_{m_1}, \ldots, c_{m_n})) \in \Gamma.$$

Notice that $\varphi(x_1, \ldots, x_n; c_{m_1}, \ldots, c_{m_n})$ is a sentence. Thus, for each formula φ there exists a sentence φ^* such that $(\varphi \leftrightarrow \varphi^*) \in \Gamma$.

Claim 5.22 *For every sentence φ, $\varphi \in \Gamma$ if and only if $\mathcal{M} \models \varphi$.*

Proof. We proceed by induction on the length of φ.

We first suppose that φ is a sentence and that φ is an atomic formula. There are two subcases.

First, φ could be of the form $(\tau_1 \hat{=} \tau_2)$, where τ_1 and τ_2 are terms. Let c_{i_1} be a constant such that $\overline{\nu}(\tau_1) = [c_{i_1}]_\Gamma$ and let c_{i_2} be a constant such that $\overline{\nu}(\tau_2) = [c_{i_2}]_\Gamma$. By Claim 5.21, $(\tau_1 \hat{=} c_{i_1})$ and $(\tau_2 \hat{=} c_{i_2})$ are elements of Γ. Lemma 5.19 applies, and so $(\tau_1 \hat{=} \tau_2) \in \Gamma$ if and only if $(c_{i_1} \hat{=} c_{i_2})$ is an element of Γ. Further $(c_{i_1} \hat{=} c_{i_2})$ is an element of Γ if and only if $[c_{i_1}]_\Gamma = [c_{i_2}]_\Gamma$. Consequently, $(\tau_1 \hat{=} \tau_2) \in \Gamma$ if and only if $\mathcal{M} \models (\tau_1 \hat{=} \tau_2)$, as required.

The second subcase is that $\varphi = P_i(\tau_1 \ldots \tau_n)$, where $\tau_1, \ldots, \tau_n$ are terms and $n = \pi(P_i)$. By definition,

$$(\mathcal{M}, \nu) \models \varphi \text{ if and only if } \langle \overline{\nu}(\tau_1), \ldots, \overline{\nu}(\tau_n) \rangle \in I(P_i).$$

For each $k \leq n$, let c_{j_k} be a constant such that $\overline{\nu}(\tau_k) = [c_{j_k}]_\Gamma$. Thus,

$$\langle \overline{\nu}(\tau_1), \ldots, \overline{\nu}(\tau_n) \rangle \in I(P_i) \text{ if and only if } \langle [c_{j_1}]_\Gamma, \ldots, [c_{j_n}]_\Gamma \rangle \in I(P_i).$$

By the definition of $I(P_i)$,

$$\langle [c_{j_1}]_\Gamma, \ldots, [c_{j_n}]_\Gamma \rangle \in I(P_i) \text{ if and only if } P_i(c_{j_1}, \ldots, c_{j_n}) \in \Gamma.$$

By Claim 5.21, for each $k \leq n$, $(\tau_k \hat{=} c_{j_k}) \in \Gamma$ and so we can apply Lemma 5.19 to conclude that

$$P_i(c_{j_1} \ldots c_{j_n}) \in \Gamma \text{ if and only if } P_i(\tau_1 \ldots \tau_n) \in \Gamma.$$

Thus, $\mathcal{M} \models \varphi$ if and only if $\varphi \in \Gamma$. This finishes the case in which φ is an atomic formula.

We now suppose that the length of φ is N, φ is not an atomic formula and that the following condition holds.

Induction Hypothesis: *Suppose that ψ is a sentence of length less than N. Then $\mathcal{M} \models \psi$ if and only if $\psi \in \Gamma$.*

There are three subcases.

Negation. Suppose that $\varphi = (\neg\psi)$. Since φ is a sentence, so is ψ. Consequently, the induction hypothesis applies, and so $\mathcal{M} \models \psi$ if and only if $\psi \in \Gamma$. Therefore $\mathcal{M} \models (\neg\psi)$ if and only if $\mathcal{M} \not\models \psi$, if and only if $\psi \notin \Gamma$, if and only if $(\neg\psi) \in \Gamma$ (as Γ is maximally consistent).

Implication. Suppose that $\varphi = (\psi_1 \to \psi_2)$. Again, the induction hypothesis applies and we obtain both: $\mathcal{M} \models \psi_1$ if and only if $\psi_1 \in \Gamma$ and $\mathcal{M} \models \psi_2$ if and only if $\psi_2 \in \Gamma$. By definition, $\mathcal{M} \models \varphi$ if and only if either $\mathcal{M} \not\models \psi_1$ or $\mathcal{M} \models \psi_2$. Since Γ is maximally consistent, $\varphi \in \Gamma$ if and only if either $(\neg\psi_1) \in \Gamma$ or $\psi_2 \in \Gamma$. Thus, $\mathcal{M} \models \varphi$ if and only if $\varphi \in \Gamma$.

Quantification. Suppose that $\varphi = (\forall x_i \psi)$. By the Henkin property, $(\exists x_i(\neg\psi)) \in \Gamma$ if and only if for some constant c_j, $(\neg\psi)(x_i; c_j) \in \Gamma$. By a straightforward deduction, $(\forall x_i \psi) \in \Gamma$ if and only if $(\exists x_i(\neg\psi)) \notin \Gamma$, if and only if for every constant c_j, $(\neg\psi)(x_i; c_j) \notin \Gamma$. But $(\neg\psi)(x_i; c_j)$ is equal to $(\neg\psi(x_i; c_j))$, and so $(\forall x_i \psi) \in \Gamma$ if and only if for every constant c_j, $\psi(x_i; c_j) \in \Gamma$.

By definition, $\mathcal{M} \models \varphi$ if and only if for all $\mathcal{M}$-assignments μ, $(\mathcal{M}, \mu) \models \psi$. (Since φ is a sentence, every $\mathcal{M}$-assignment agrees with every other one on the free variables of φ.)

Suppose that μ is an $\mathcal{M}$-assignment. Let c_j be a constant such that $\mu(x_i) = [c_j]_\Gamma$. Then, since x_i is the only free variable of ψ and since $I(c_j) = \mu(x_i)$, it follows that $(\mathcal{M}, \mu) \models \psi$ if and only if $\mathcal{M} \models \psi(x_i; c_j)$. Thus, the condition that

for all $\mathcal{M}$-assignments μ, $(\mathcal{M}, \mu) \models \psi$

is equivalent to the one that

for all constants c_j, $\mathcal{M} \models \psi(x_i; c_j)$.

By the induction hypothesis, for each constant c_j, $\mathcal{M} \models \psi(x_i; c_j)$ if and only if $\psi(x_i; c_j) \in \Gamma$.

Thus, $\varphi \in \Gamma$ if and only if for each constant c_j, $\psi(x_i; c_j) \in \Gamma$, if and only if for each constant c_j, $\mathcal{M} \models \psi(x_i; c_j)$, if and only if $\mathcal{M} \models \varphi$.

This completes the final case, and so we have proved that for each sentence φ, $\mathcal{M} \models \varphi$ if and only if $\varphi \in \Gamma$. □

By Claim 5.22, Γ is satisfiable. □

5.5 Extensions of consistent sets of $\mathcal{L}$ formulas

Unfortunately there are consistent sets, Γ, of $\mathcal{L}$ formulas which cannot be extended to maximally consistent sets with the Henkin property. The difficulty is the Henkin property. However with certain restrictions on the sets Γ one can in fact extend Γ to a maximally consistent set with the Henkin property.

Theorem 5.23 *Suppose that Γ is a consistent set of $\mathcal{L}$ formulas and that there are infinitely many constants, c_i, which do not occur in any formula of Γ. Then there is a set of formulas Σ such that*

(1) $\Gamma \subseteq \Sigma$,
(2) Σ *is maximally consistent,*
(3) and Σ *has the Henkin property.*

Proof. Let $\langle c_{n_i} : i \in \mathbb{N}\rangle$ enumerate the constants which do not occur in any formula of Γ. Let $\langle \varphi_i : i \in \mathbb{N}\rangle$ be an enumeration of all $\mathcal{L}$ formulas which satisfies

(1.1) for each formula φ there exist distinct positive integers i_0 and i_1 such that $\varphi = \varphi_{i_0} = \varphi_{i_1}$,
(1.2) no constant in the set, $\{c_{n_k} : k \geq i\}$, occurs in φ_i.

Define by induction on $i \in \mathbb{N}$ an increasing sequence $\langle \Sigma_i : i \in \mathbb{N}\rangle$ of sets of formulas as follows.

(2.1) $\Sigma_0 = \Gamma$.
(2.2) a) If $\varphi_i \notin \Sigma_i$ and if $\Sigma_i \cup \{\varphi_i\}$ is consistent, then $\Sigma_{i+1} = \Sigma_i \cup \{\varphi_i\}$.
b) If $\varphi_i \in \Sigma_i$ and if φ_i is an existential formula $\varphi_i = (\exists x_j \psi)$, then $\Sigma_{i+1} = \Sigma_i \cup \{\psi(x_j; c_{n_i})\}$.
c) Otherwise, $\Sigma_{i+1} = \Sigma_i$.

Claim 5.24 *For each i, the following properties hold.*

(3.1) $\Gamma \subseteq \Sigma_i$.
(3.2) $\Sigma_i \subseteq \Sigma_{i+1}$.
(3.3) Σ_i *is consistent.*
(3.4) No constant in the set, $\{c_{n_k} : k \geq i\}$ *occurs in any formula of* Σ_i.
(3.5) $\varphi_i \in \Sigma_{i+1}$ *or* $\Sigma_i \cup \{\varphi_i\}$ *is not consistent.*

Proof. Proceed by induction on i. The only subtle point is to show that Σ_{i+1} is consistent when it is defined by means of case 2(b). We give the argument for this case below.

Suppose that Claim 5.24 holds for i and that $\Sigma_{i+1} = \Sigma_i \cup \{\psi(x_j; c_{n_j})\}$ as specified in case 2(b). Suppose for a contradiction that Σ_{i+1} is not consistent. Then, $\Sigma_i \cup \{\psi(x_j; c_{n_i})\} \vdash (\neg(x_1 \mathrel{\hat{=}} x_1))$, as every formula can be derived from an inconsistent set. By the Deduction Theorem 5.9, $\Sigma_i \vdash (\psi(x_j; c_{n_i}) \to (\neg(x_1 \mathrel{\hat{=}} x_1)))$ and so $\Sigma_i \vdash (\neg\psi(x_j; c_{n_i}))$. By the Theorem on Constants 5.13, $\Sigma_i \vdash (\forall x_j(\neg\psi(x_; c_{n_i})))(c_{n_i}; x_j)$. Making the substitution, $\Sigma_i \vdash (\forall x_j(\neg\psi))$. But since case 2(b) applied, $(\exists x_j \psi) \in \Sigma_i$ and so $(\neg(\forall x_j(\neg\psi))) \in \Sigma_i$. Thus, Σ_i is not consistent, contrary to assumption. Thus, Σ_{i+1} is consistent, as required. □

Let $\Sigma = \cup\{\Sigma_i : i \in \mathbb{N}\}$. By the first four items in Claim 5.24, $\Gamma \subseteq \Sigma$ and Σ is consistent. Further, since every $\mathcal{L}$ formula appears in the list $\langle \varphi_i : i \in \mathbb{N}\rangle$, the fifth item in Claim 5.24 implies that Σ is maximally consistent.

Finally, we verify that Σ has the Henkin property. Suppose that $(\exists x_j \psi) \in \Sigma$. By our choice of the enumeration $\langle \varphi_i : i \in \mathbb{N} \rangle$, there exist $i_0 < i_1$ such that $\varphi_{i_0} = \varphi_{i_1} = (\exists x_j \psi)$. Since $(\exists x_j \psi) \in \Sigma$, $\Sigma_{i_0} \cup \{(\exists x_j \psi)\}$ is consistent. By case 2(a) in the definition of Σ_{i+1} from Σ_i, $(\exists x_j \psi) \in \Sigma_{i_0+1} \subseteq \Sigma_{i_1}$. Thus, case 2(b) applies in the definition of Σ_{i_1+1}, and so $\psi(x_j; c_{n_{i_1}}) \in \Sigma_{i_1+1}$, as required to verify the Henkin property. □

As an immediate corollary of Theorem 5.23 we obtain the following special case of the Gödel Completeness Theorem.

Theorem 5.25 *Suppose that Γ is a consistent set of $\mathcal{L}$ formulas and that there are infinitely many constants, c_i, which do not occur in any formula of Γ. Then Γ is satisfiable.*

Proof. By Theorem 5.23 there exists a set of $\mathcal{L}$ formulas Σ containing Γ such that Σ is maximally consistent and such that Σ has the Henkin property. By Theorem 5.20, Σ is satisfiable. Therefore Γ is satisfiable. □

5.5.1 Exercises

(1) Show that the Compactness Theorem for $\mathcal{L}_0$ follows from Theorem 5.25.

5.6 The Gödel Completeness Theorem

We fix some notation. Suppose that φ is an $\mathcal{L}$ formula. Let φ^* be the formula obtained from φ by substituting c_{2i} for c_i for each constant c_i occurring in φ. Similarly, if Γ is a set of $\mathcal{L}$ formulas, let Γ^* denote the set $\{\varphi^* : \varphi \in \Gamma\}$.

The following is a corollary of the theorem on constants.

Lemma 5.26 *Suppose that Γ is a consistent set of $\mathcal{L}$ formulas. Then, Γ^* is also consistent.*

Proof. Note that if φ is a formula and every constant which occurs in φ is in the set $\{c_{2i} : i \in \mathbb{N}\}$, then $\varphi = \psi^*$ for some formula ψ. Of course ψ is unique and obtained from φ by substituting, for each i, c_i for c_{2i} in φ.

We assume toward a contradiction that Γ^* is not consistent. Let $\langle \varphi_1, \ldots, \varphi_n \rangle$ be a deduction from Γ^* such that $\varphi_n = (\neg(x_1 \mathrel{\hat{=}} x_1))$. By the Theorem on Constants, Theorem 5.13, there is a deduction $\langle \psi_1^*, \ldots, \psi_m^* \rangle$ from Γ^* such that if c_i is a constant symbols which appears in one of the ψ_k^*'s then i is even. Thus for each $k \leq m$ there exists a formula ψ_k such that $\psi_k^* = (\psi_k)^*$.

By a straightforward induction on m, $\langle \psi_1, \ldots, \psi_m \rangle$ is a proof from Γ. But $\psi_m = \varphi_n = (\neg(x_1 \mathrel{\hat{=}} x_1))$, and so Γ is not consistent. This contradicts our assumption that Γ is consistent, and therefore Γ^* is consistent.

Lemma 5.27 *Suppose that Γ is a set of $\mathcal{L}$ formulas and Γ^* is satisfiable. Then Γ is satisfiable.*

Proof. Let $A_{\mathcal{L}^*}$ be the alphabet of Γ^*. Let $\mathcal{M}^* = (M, I^*)$ be a structure and let ν be an $\mathcal{M}^*$-assignment such that

$$(\mathcal{M}^*, \nu) \vDash \Gamma^*.$$

Let $\mathcal{M} = (M, I)$ be the structure obtained from $\mathcal{M}^*$ by changing I^* to produce I such that

(1.1) $I(F_k) = I^*(F_k)$ for all function symbols, F_k,
(1.2) $I(P_k) = I^*(P_k)$ for all function symbols, P_k,
(1.3) $I(c_k) = I^*(c_{2k})$ for all $k \in \mathbb{N}$.

Since $\mathcal{M}$ and $\mathcal{M}^*$ have the same universe, for each function μ, μ is an $\mathcal{M}$-assignment if and only if μ is an $\mathcal{M}^*$-assignment.

For each term τ^* in $A_{\mathcal{L}^*}$, let τ be the term obtained from τ^* by substituting, for each $i \in \mathbb{N}$, c_{2i} for c_i in τ^*.

Now consider an arbitrary $\mathcal{M}$-assignment μ. Thus μ is both an $\mathcal{M}$-assignment and an $\mathcal{M}^*$-assignment. Let $(\overline{\mu})^{\mathcal{M}}$ denote the extension of μ to terms as calculated relative to the structure $\mathcal{M}$, and let $(\overline{\mu})^{\mathcal{M}^*}$ denote the extension of μ to terms as calculated relative to the structure $\mathcal{M}^*$.

It is easily verified by induction on the length of τ that for all terms τ,

$$(\overline{\mu})^{\mathcal{M}}(\tau) = (\overline{\mu})^{\mathcal{M}^*}(\tau^*).$$

We claim that for all formulas φ and for all $\mathcal{M}$-assignments μ,

$$(\mathcal{M}, \mu) \vDash \varphi \text{ if and only if } (\mathcal{M}^*, \mu) \vDash \varphi^*.$$

This is proved by induction on the length of φ. We leave the details as an exercise.

Since $(\mathcal{M}, \nu) \vDash \Gamma^*$, it follows that $(\mathcal{M}^*, \nu) \vDash \Gamma$, and so Γ is satisfiable. □

With these two lemmas and the results of the previous sections, of the Gödel Completeness Theorem is immediate.

Theorem 5.28 (Gödel Completeness) *A set of $\mathcal{L}$ formulas is consistent if and only if it is satisfiable.*

Proof. By soundness, if Γ is satisfiable then Γ is consistent. Therefore it suffices to prove that if Γ is consistent then Γ is satisfiable.

By Lemma 5.26, since Γ is consistent, Γ^* is consistent. By Theorem 5.25, Γ^* is satisfiable. By Lemma 5.27, Γ is satisfiable. □

The Gödel Completeness Theorem is often succinctly reformulated as follows.

Theorem 5.29 *For any set of $\mathcal{L}$ formulas Γ and any $\mathcal{L}$ formula φ,*

$$\Gamma \vDash \varphi \textit{ if and only if } \Gamma \vdash \varphi.$$

Here $\Gamma \vDash \varphi$ is used to express the condition that φ is satisfied whenever Γ is satisfied.

5.7 The Craig Interpolation Theorem

The Completeness Theorem states that if φ is satisfied whenever Γ is satisfied then there is a proof of φ from Γ. In this section, we generalize the Completeness Theorem to the restricted languages $\mathcal{L}_\mathcal{A}$. Of course one can fairly easily convince oneself that the proof of the Completeness Theorem we have given works for the languages $\mathcal{L}_\mathcal{A}$. We shall take a slightly different approach for it will reveal some additional interesting features of our formal notion of proof, this approach culminates with the statement of the Craig Interpretation Theorem.

Suppose that

$$\mathcal{A} \subseteq (6i+2 : i \in \mathbb{N}\} \cup \{6i+3 : i \in \mathbb{N}\} \cup \{6i+4 : i \in \mathbb{N}\}) .$$

is a first order alphabet.

Definition 5.30 Suppose that Γ is a set of $\mathcal{L}_\mathcal{A}$-formulas and that φ is an $\mathcal{L}_\mathcal{A}$-formula. Then $\Gamma \vdash_{\mathcal{L}_\mathcal{A}} \varphi$ if and only if there exists a proof $\langle \varphi_1, \ldots, \varphi_n \rangle$ of φ from Γ such that for each $i \leq n$, φ_i is an $\mathcal{L}_\mathcal{A}$-formula.

Eliminating predicate symbols. Suppose that φ is a formula and that P_i is a predicate symbol. Let $[\varphi]_{P_i}$ denote the formula defined as follows by induction on the length of φ:

(1) If φ is an atomic formula, then

$$[\varphi]_{P_i} = \begin{cases} (\tau_1 \mathrel{\hat{=}} \tau_1), & \text{if } \varphi = P_i(\tau_1 \ldots \tau_n), \text{ where } n = \pi(P_i); \\ \varphi, & \text{otherwise.} \end{cases}$$

(2) $[(\neg\psi)]_{P_i} = (\neg[\psi]_{P_i})$.
(3) $[(\psi_1 \rightarrow \psi_2)]_{P_i} = ([\psi_1]_{P_i} \rightarrow [\psi_2]_{P_i})$.
(4) $[(\forall x_i \psi)]_{P_i} = (\forall x_i [\psi]_{P_i})$.

Thus $[\varphi]_{P_i}$ is obtained from φ by replacing every instance of P_i by a trivial formula.

Lemma 5.31 (Predicates) *Suppose that Γ is a set of formulas and that P_i is a predicate symbol which does not occur in any formula of Γ. Suppose that $\langle \psi_1, \ldots, \psi_m \rangle$ is a proof from Γ. Then $\langle [\psi_1]_{P_i}, \ldots, [\psi_m]_{P_i} \rangle$ is a proof from Γ.*

Proof. Note that if P_i does not occur in φ, then $[\varphi]_{P_i} = \varphi$. Thus, for each $\varphi \in \Gamma$, $[\varphi]_{P_i} = \varphi$.

By inspection of Definition 5.3, if φ is a logical axiom then $[\varphi]_{P_i}$ is a logical axiom.

Finally, if φ_1 and φ_2 are formulas then

$$[(\varphi_1 \rightarrow \varphi_2)]_{P_i} = ([\varphi_1]_{P_i} \rightarrow [\varphi_2]_{P_i}).$$

It follows by induction on $n \leq m$, that $\langle [\psi_1]_{P_i}, \ldots, [\psi_n]_{P_i} \rangle$ is a proof from Γ. $\square$

Eliminating function symbols. Suppose that τ is a term and that F_i is a function symbol. Let $[\tau]_{F_i}$ denote the term defined by induction on the length of τ as follows.

(1) $[x_j]_{F_i} = x_j$, $[c_j]_{F_i} = c_j$;
(2) Suppose $\tau = F_j(\tau_1 \dots \tau_m)$, where $m = \pi(F_j)$. Then

$$[\tau]_{F_i} = \begin{cases} F_j([\tau_1]_{F_i}, \dots, [\tau_m]_{F_i}), & \text{if } F_i \neq F_j; \\ [\tau_1]_{F_i}, & \text{otherwise.} \end{cases}$$

Let $[\varphi]_{F_i}$ denote the formula defined as follows by induction on the length of φ.

(1) If φ is an atomic formula and $\varphi = (\tau_1 \hat{=} \tau_2)$, then $[\varphi]_{F_i} = ([\tau_1]_{F_i} \hat{=} [\tau_2]_{F_i})$.
(2) If φ is an atomic formula and $\varphi = P_j(\tau_1 \dots \tau_n)$, where $n = \pi(P_j)$, then $[\varphi]_{F_i} = P_j([\tau_1]_{F_i}, \dots, [\tau_n]_{F_i})$.
(3) $[(\neg\psi)]_{F_i} = (\neg[\psi]_{F_i})$.
(4) $[\psi_1 \to \psi_2]_{F_i} = ([\psi_1]_{F_i} \to [\psi_2]_{F_i})$.
(5) $[(\forall x_i \psi)]_{F_i} = (\forall x_i [\psi]_{F_i})$.

Thus $[\varphi]_{F_i}$ is obtained from φ by replacing those terms which express application of F_i by simpler terms which do not refer to F_I.

Lemma 5.32 (Functions) *Suppose that Γ is a set of formulas and that F_i is a function symbol which does not occur in any formula of Γ. Suppose that $\langle \psi_1, \dots, \psi_m \rangle$ is a proof from Γ. Then $\langle [\psi_1]_{F_i}, \dots, [\psi_m]_{F_i} \rangle$ is a proof from Γ.*

Proof. The proof is quite similar to that of the Lemma on Predicates 5.31.

Note that if F_i does not occur in φ then $[\varphi]_{F_i} = \varphi$. Thus for each $\varphi \in \Gamma$, $[\varphi]_{F_i} = \varphi$. By inspection of Definition 5.3, if φ is a logical axiom then $[\varphi]_{F_i}$ is a logical axiom. Finally, if φ_1 and φ_2 are formulas, then $[(\varphi_1 \to \varphi_2)]_{F_i} = ([\varphi_1]_{F_i} \to [\varphi_2]_{F_i})$.

It follows by induction on $n \leq m$, that $\langle [\psi_1]_{F_i}, \dots, [\psi_n]_{F_i} \rangle$ is a proof from Γ. $\square$

As a corollary we obtain:

Theorem 5.33 *Suppose that Γ is a set of $\mathcal{L}_\mathcal{A}$-formulas and that φ is an $\mathcal{L}_\mathcal{A}$-formula. Then*

$\Gamma \vdash \varphi$ *if and only if* $\Gamma \vdash_{\mathcal{L}_\mathcal{A}} \varphi$.

Proof. This is an immediate corollary of Lemma 5.12 the Lemmas on Constants 5.12, on Predicates 5.31, and on Functions 5.32.

The implication from right to left is immediate, so we have only to prove that if $\Gamma \vdash \varphi$ then $\Gamma \vdash_{\mathcal{L}_\mathcal{A}} \varphi$. Let $\langle \psi_1, \dots, \psi_m \rangle$ be a proof from Γ of φ.

Let $\langle \mathcal{A}_k : k \leq n \rangle$ be an increasing sequences of sets such that for all $k < n$,

(1.1) $\mathcal{A}_k \subseteq (\{6i+2 : i \in \mathbb{N}\} \cup \{6i+3 : i \in \mathbb{N}\} \cup \{6i+4 : i \in \mathbb{N}\})$,
(1.2) $\mathcal{A}_{k+1} \backslash \mathcal{A}_k$ contains at most one element,
(1.3) $\mathcal{A}_0 = \mathcal{A}$,
(1.4) for each $j \leq m$, ψ_j is an $\mathcal{L}_{\mathcal{A}_k}$-formula.

Thus by condition (4), $\Gamma \vdash_{\mathcal{L}_n} \varphi$. Observe that for each $k < n$ if $\Gamma \vdash_{\mathcal{L}_{k+1}} \varphi$, then $\Gamma \vdash_{\mathcal{L}_k} \varphi$. This follows from Lemma 5.12 if $A_{\mathcal{L}_{k+1}} \setminus A_{\mathcal{L}_k}$ contains only a constant symbol; it follows from Lemma 5.31 if $A_{\mathcal{L}_{k+1}} \setminus A_{\mathcal{L}_k}$ contains only a predicate symbol; and it follows from Lemma 5.32 if $A_{\mathcal{L}_{k+1}} \setminus A_{\mathcal{L}_k}$ contains only a predicate symbol.

Thus (by reverse induction), $\Gamma \vdash_{\mathcal{L}} \varphi$. □

Theorem 5.34 (Gödel Completeness Theorem for $\mathcal{L}_{\mathcal{A}}$) *Suppose that Γ is a set of $\mathcal{L}_{\mathcal{A}}$-formulas and that φ is an $\mathcal{L}_{\mathcal{A}}$-formula. Then*

$$\Gamma \vdash_{\mathcal{L}_{\mathcal{A}}} \varphi \textit{ if and only if } \Gamma \vDash \varphi.$$

Proof. By Theorem 5.29, $\Gamma \vDash \varphi$ if and only if $\Gamma \vdash \varphi$. By Theorem 5.34, $\Gamma \vdash \varphi$ if and only if $\Gamma \vdash_{\mathcal{L}_{\mathcal{A}}} \varphi$. Thus, $\Gamma \vDash \varphi$ if and only if $\Gamma \vdash_{\mathcal{L}} \varphi$, as required. □

The three lemmas on Constants, on Functions, and on Predicates can be generalized and combined into a single theorem, the Craig Interpolation Theorem, which we cite below without proof.

Suppose that Γ is a set of formulas. Let $\mathcal{A}_\Gamma$ be the set of constant symbols, predicate symbols and function symbols which occur in some formula of Γ. Thus $\mathcal{A}_\Gamma$ is the minimum set, $\mathcal{A}$, such that each formula of Γ is an $\mathcal{L}_{\mathcal{A}}$-formula.

Theorem 5.35 (Craig Interpolation Theorem) *Suppose that φ_1 and φ_2 are formulas, Γ is a set of formulas and that*

$$\Gamma \vdash (\varphi_1 \to \varphi_2).$$

Suppose that

$$\mathcal{A}_{\{\varphi_1\}} \cap \mathcal{A}_{\{\varphi_2\}} \subseteq \mathcal{A}_\Gamma.$$

Then there is a formula ψ such that:
(1) ψ is an $\mathcal{L}_{\mathcal{A}_\Gamma}$-formula,
(2) $\Gamma \vdash (\varphi_1 \to \psi)$,
(3) $\Gamma \vdash (\psi \to \varphi_2)$.

5.7.1 Exercise

(1) Suppose that φ and ψ are propositional formulas and that φ logically implies ψ. Show that there is a propositional formula θ such that every sentence symbol in θ appears both in φ and in ψ and such that φ logically implies θ and θ logically implies ψ. (This is the propositional form of the Interpolation Theorem.)