Portfolio Choice with CRRA Utility (Merton-Samuelson)

Merton (1969) and Samuelson (1969) study the optimal portfolio choice of a consumer with Constant Relative Risk Aversion utility $\mathrm{u}(c) = (1-\rho)^{-1}c^{1-\rho}$. This consumer has assets at the end of period t equal to a_t and is deciding how much to invest in a risky asset with a lognormally distributed return factor $\mathfrak{R}_{t+1}$, $\log \mathfrak{R}_{t+1} = \mathfrak{r}_{t+1} \sim \mathcal{N}(\mathfrak{r} - \sigma_{\mathfrak{r}}^2/2, \sigma_{\mathfrak{r}}^2)$,[1] versus a riskfree asset that earns return factor $\mathsf{R} = e^{\mathsf{r}}$. Importantly, the consumer is assumed to have no labor income and to face no risk except from the investment in the risky asset.[2,3]

Both papers consider a multiperiod optimization problem, but here we examine the problem of a consumer in a period t which is the second-to-last period of life (the insights, and even the formulas, carry over to the multiperiod case).

If the period-t consumer invests proportion ς in the risky asset, spending all available resources in the last period of life $t+1$ will yield:

$$\begin{aligned} c_{t+1} &= \left(\mathsf{R}(1-\varsigma) + \mathfrak{R}_{t+1}\varsigma\right) a_t \\ &= \underbrace{\left(\mathsf{R} + (\mathfrak{R}_{t+1} - \mathsf{R})\varsigma\right)}_{\equiv \mathbb{R}_{t+1}} a_t \end{aligned} \tag{1}$$

where $\mathbb{R}_{t+1}$ is the realized arithmetic[4] return factor for the portfolio.

For mathematical analysis (especially under the assumption of CRRA utility) it would be convenient if we could approximate the realized arithmetic portfolio return factor by the realized geometric return factor $\mathbb{R}_{t+1} = \mathsf{R}^{1-\varsigma}\mathfrak{R}_{t+1}^{\varsigma}$, because then the logarithm of the return factor would be $_{t+1} = \mathsf{r}(1-\varsigma) + \mathfrak{r}_{t+1}\varsigma = \mathsf{r} + (\mathfrak{r}_{t+1} - \mathsf{r})\varsigma = \mathsf{r} + \phi_{t+1}\varsigma$ and the realized 'portfolio excess return' would be simply $_{t+1} - \mathsf{r} = \varsigma\phi_{t+1}$. Unfortunately, for ς values well away from 0 and 1 (that is, for any *interesting* values of portfolio shares), the geometric mean is a badly biased approximation to the arithmetic mean when the variance of the risky asset is substantial.

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[1] MathFacts ELogNorm tells us that a variable with this lognormal distribution has an expected return factor of $\mathbb{E}_t[e^{\mathfrak{r}_{t+1}}] = e^{\mathfrak{r}} = \mathfrak{R}$ (where variables like $\mathfrak{R}$ without a subscript are the time-invariant mean).

[2] A common interpretation is that this is the problem of a retired investor who expects to receive no further labor income. Note however that *all* risks other than the returns from financial investments have been ruled out; for example, health expense risk is not possible in this model, though recent research has argued such risk is important later in life.

[3] Riskless labor income can trivially be added to the problem, because its risklessness means that (in the absence of liquidity constraints) it is indistinguishable from a lump sum of extra current wealth with a value equal to the present discounted value (using the riskless rate) of the (riskless) future labor income. Of course, in practice, labor income is not riskless, but when labor income is risky the problem no longer has the tidy analytical solution we present here and must be solved numerically. See **?** for an introduction to numerical solution methods.

[4] Google "arithmetic geometric mean wiki" for a refresher on the difference between arithmetic and geometric means.

? point out that a much better approximation is obtained by assuming

$$_{t+1} = \mathsf{r} + \varsigma\phi_{t+1} + \varsigma\sigma_{\mathfrak{r}}^2/2 - \varsigma^2\sigma_{\mathfrak{r}}^2/2. \tag{2}$$

To see one virtue of this approximation, note (using NormTimes and SumNormsIsNorm) that since the mean and variance of $\phi_{t+1}\varsigma$ are respectively $\varsigma(\mathfrak{r} - \sigma_{\mathfrak{r}}^2/2 - \mathsf{r})$ and $\varsigma^2\sigma_{\mathfrak{r}}^2$, LogELogNormTimes implies that

$$\log \mathbb{E}_t[e^{\phi_{t+1}\varsigma}] = \varsigma(\mathfrak{r} - \mathsf{r} - \sigma_{\mathfrak{r}}^2/2) + \varsigma^2\sigma_{\mathfrak{r}}^2/2 \tag{3}$$

which means that exponentiating then taking the expectation then taking the logarithm of (2) gives

$$\begin{aligned}
\log \mathbb{E}_t[e^{t+1}] &= \log e^{\mathsf{r}} + \log \mathbb{E}_t[e^{\varsigma\phi_{t+1}}] + \log e^{\varsigma\sigma_{\mathfrak{r}}^2/2 - \varsigma^2\sigma_{\mathfrak{r}}^2/2} && (4)\\
&= \mathsf{r} + \varsigma(\mathfrak{r} - \mathsf{r} - \sigma_{\mathfrak{r}}^2/2) + \varsigma^2\sigma_{\mathfrak{r}}^2/2 + \varsigma\sigma_{\mathfrak{r}}^2/2 - \varsigma^2\sigma_{\mathfrak{r}}^2/2 && (5)\\
\log \mathbb{E}_t[e^{t+1}] - \mathsf{r} &= \varsigma(\mathfrak{r} - \mathsf{r}) && (6)
\end{aligned}$$

or, in words: The expected excess portfolio return is equal to the proportion invested in the risky asset times the expected return of the risky asset (we use the word 'return' always to mean the logarithm of the corresponding 'factor'; and when not explicitly specified, we always take the expectation before taking the log; if we wanted to refer to $\mathbb{E}_t[_{t+1}]$ we would call it the expected log portfolio return (to distinguish it from the expected portfolio return $\log \mathbb{E}[e^{t+1}]$).

Under these assumptions, the expectation as of date t of utility at date $t+1$ is:

$$\begin{aligned}
\mathbb{E}_t[\mathrm{u}(c_{t+1})] &\approx (1-\rho)^{-1}\,\mathbb{E}_t\left[\left(a_t e^{\mathsf{r}} e^{\varsigma\phi_{t+1} + \varsigma(1-\varsigma)\sigma_{\mathfrak{r}}^2/2}\right)^{1-\rho}\right]\\
&\approx (1-\rho)^{-1}\,\mathbb{E}_t\left[(a_t\mathsf{R})^{1-\rho}\left(e^{\varsigma\phi_{t+1} + \varsigma(1-\varsigma)\sigma_{\mathfrak{r}}^2/2}\right)^{1-\rho}\right]\\
&\approx (1-\rho)^{-1}(a_t\mathsf{R})^{1-\rho}\,\mathbb{E}_t\left[e^{(\varsigma\phi_{t+1} + \varsigma(1-\varsigma)\sigma_{\mathfrak{r}}^2/2)(1-\rho)}\right]\\
&\approx \underbrace{(1-\rho)^{-1}(a_t\mathsf{R})^{1-\rho}}_{\text{constant} < 0}\,\underbrace{e^{(1-\rho)\varsigma(1-\varsigma)\sigma_{\mathfrak{r}}^2/2}\,\mathbb{E}_t\left[e^{\varsigma\phi_{t+1}(1-\rho)}\right]}_{\text{excess return utility factor}} && (7)
\end{aligned}$$

where the first term is a negative constant under the usual assumption that relative risk aversion $\rho > 1$.

Our foregoing assumptions imply that $\varsigma(1-\rho)\phi_{t+1} \sim \mathcal{N}(\varsigma(1-\rho)(\phi - \sigma_{\mathfrak{r}}^2/2), (\varsigma(1-\rho))^2\sigma_{\mathfrak{r}}^2)$ (again using LogELogNormTimes). With a couple of extra lines of derivation we can show that the log of the expectation in (7) is

$$\begin{aligned}
\log \mathbb{E}_t\left[e^{\varsigma\phi_{t+1}(1-\rho)}\right] &= (1-\rho)\varsigma\phi - (1-\rho)\varsigma\sigma_{\mathfrak{r}}^2/2 + ((1-\rho)\varsigma)^2\sigma_{\mathfrak{r}}^2/2\\
&= (1-\rho)\varsigma\phi - (1-\rho)\varsigma(1 - \varsigma(1-\rho))\sigma_{\mathfrak{r}}^2/2\\
&= (1-\rho)\varsigma\phi - (1-\rho)\varsigma(1-\varsigma)\sigma_{\mathfrak{r}}^2/2 - \rho(1-\rho)\varsigma^2\sigma_{\mathfrak{r}}^2/2. && (8)
\end{aligned}$$

Substitute from (8) for the log of the expectation in (7) and note that the resulting expression simplifies because it contains $(1-\rho)\varsigma\sigma_{\mathfrak{r}}^2/2 - (1-\rho)\varsigma\sigma_{\mathfrak{r}}^2/2 = 0$; thus the log of the 'excess return utility factor' in (7) is

$$-(\rho - 1)\varsigma\phi - (\rho - 1)(-\rho\varsigma^2\sigma_{\mathfrak{r}}^2/2) \tag{9}$$

and the ς that minimizes the log will also minimize the level; minimizing this when $\rho > 1$ is equivalent to maximizing the terms multiplied by $-(\rho-1)$, so our problem reduces to

$$\max_{\varsigma} \quad \varsigma\phi - \rho\varsigma^2\sigma_{\mathfrak{r}}^2/2$$

with FOC

$$\begin{aligned} \phi - \varsigma\rho\sigma_{\mathfrak{r}}^2 &= 0 \\ \varsigma &= \left(\frac{\phi}{\rho\sigma_{\mathfrak{r}}^2}\right). \end{aligned} \tag{10}$$

Equation (10) says[5] that the consumer allocates more of his portfolio to the high-risk, high-return asset when

1. the amount ϕ by which the risky asset's return exceeds the riskless return is greater
2. the consumer is less risk averse (ρ is lower)
3. riskiness $\sigma_{\mathfrak{r}}^2$ is less

If there is no excess return, nothing will be put in the risky asset. Similarly, if risk aversion or the variance of the risk is infinity, again nothing will be put in the risky asset.[6]

A final interesting question is what the expected rate of return on the consumer's portfolio will be once the portfolio share in risky assets has been chosen optimally. Note first that (6) implies that

$$\log \mathbb{E}_t[e^{t+1-\mathsf{r}}] = \varsigma\phi \tag{11}$$

while the variance of the log of the excess return factor for the portfolio is $\sigma^2 = \varsigma^2\sigma_{\mathfrak{r}}^2$.

Substituting the solution (10) for ς into (11), we have

$$\begin{aligned} \varsigma\phi &= \left(\frac{\phi^2}{\rho\sigma_{\mathfrak{r}}^2}\right) \\ &= (\phi/\sigma_{\mathfrak{r}})^2/\rho \end{aligned} \tag{12}$$

which is an interesting formula for the excess return of the optimally chosen portfolio because the object $\phi/\sigma_{\mathfrak{r}}$ (the excess return divided by the standard deviation) is a well-known tool in finance for evaluating the tradeoff between risk and return (the 'Sharpe ratio'). Equation (12) says that the consumer will choose a portfolio that earns an excess return that is directly related to the (square of the) Sharpe ratio and inversely related to the risk aversion coefficient. Higher reward (per unit of risk) convinces the consumer to take the risk necessary to earn higher returns; but higher risk aversion convinces the investor to sacrifice return for safety.

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[5] This expression differs slightly from that derived by **?**, because we adjust the mean logarithmic return of the risky investment for its variance in order to keep the mean return factor constant, which makes comparisons of alternative levels of risk more transparent.

[6] See the appendix for a figure showing the quality of the approximation.

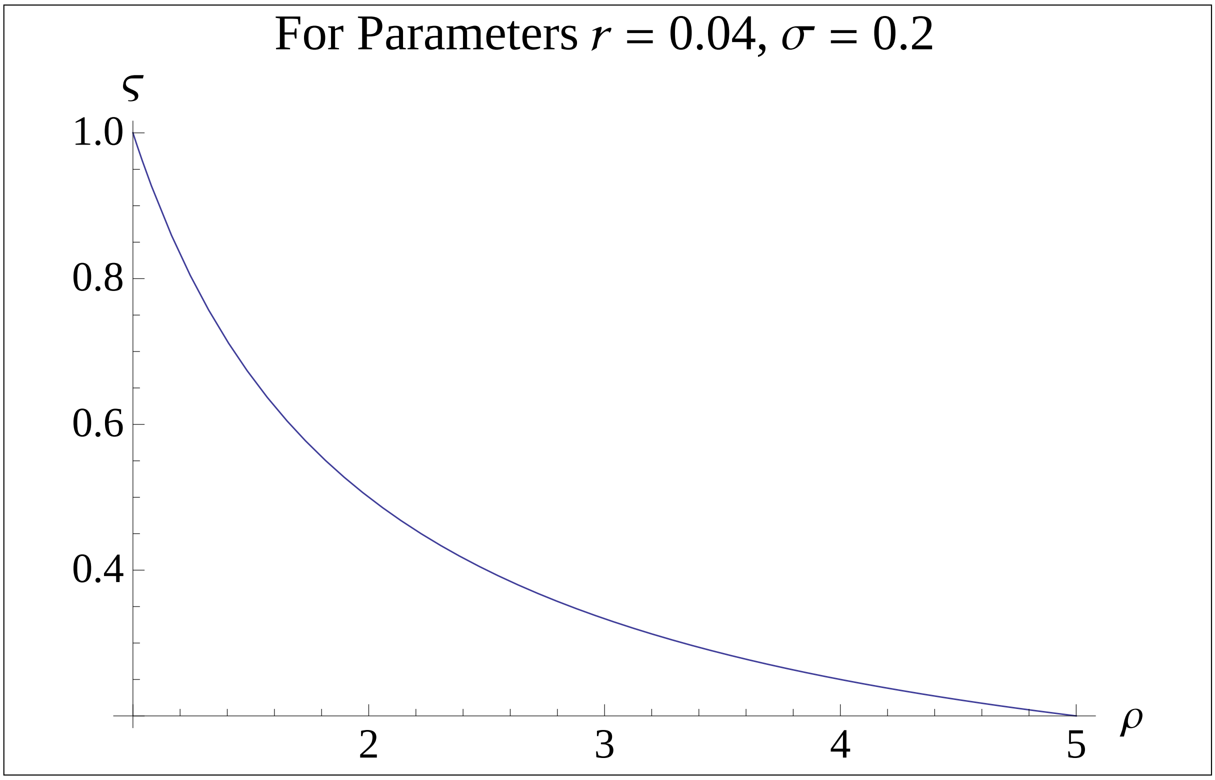
Finally, we can ask what effect an exogenous increase in the risk of the risky asset has on the endogenous riskiness of the portfolio once the consumer has chosen optimally. The answer is surprising: The variance of the optimally-chosen portfolio is

$$\varsigma^2\sigma_{\mathfrak{r}}^2 = \left(\frac{\phi}{\rho\sigma_{\mathfrak{r}}^2}\right)^2 \sigma_{\mathfrak{r}}^2 \tag{13}$$

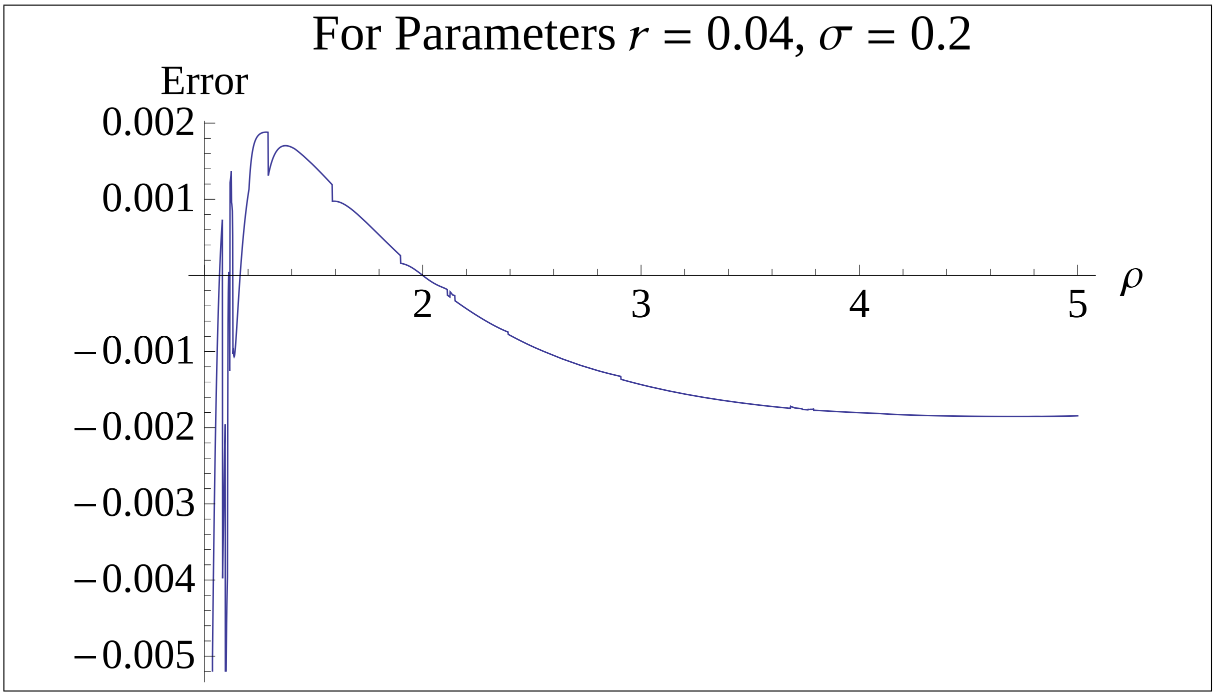
$$= \left(\frac{(\phi/\rho)^2}{\sigma_{\mathfrak{r}}^2}\right) \tag{14}$$

which is actually *smaller* when $\sigma_{\mathfrak{r}}^2$ is larger. Upon reflection, maybe this makes sense. Imagine that the consumer had adjusted his portfolio share in the risky asset downward just enough to return the portfolio's riskiness to its original level that it had before the increase in risk. The consumer would now be bearing the same degree of risk but for a lower (mean) rate of return (because of his partial reduction in exposure to the risky asset). It makes intuitive sense that the consumer will not be satisfied with this "same riskiness, lower return" outcome and therefore that the undesirableness of the risky asset must have increased enough to make him want to hold even less than the amount that would return his portfolio's riskiness to its original value.

Figure 1 The Risky Portfolio Share ς and Relative Risk Aversion ρ



(a) The Approximate Risky Portfolio Share ς Declines as Relative Risk Aversion ρ Increases



(b) The Approximation Error for the Portfolio Share in Risky Assets ς Is Small

Note: The approximation error is computed by solving for the exactly optimal portfolio share numerically. See the `Portfolio-CRRA-Derivations.nb` Mathematica notebook for details.

References

MERTON, ROBERT C. (1969): "Lifetime Portfolio Selection under Uncertainty: The Continuous Time Case," *Review of Economics and Statistics*, 50, 247–257.

SAMUELSON, PAUL A. (1969): "Lifetime Portfolio Selection By Dynamic Stochastic Programming," *Review of Economics and Statistics*, 51, 239–46.